Indian Institute of Technology, Kanpur
Department of Physics

PHY-224A: OPTICS
Lecture -17: Interference of light - 4

Date: 22 Sept. 2014, **Time :1700 hours** **Instructor: S.A. Ramakrishna**

1 Interference of multiple beams

Previously, we considered the interference of only two beams. This is because, in most cases of reflection from a slab of refractive index not very different from the surrounding medium ($n = 1.5$ for glass, $n = 1.33$ for water, $n = 1.38$ for some oils), only two of the reflections are comparably large to have good fringe visibility – a situation that is violated when the index contrast is larger or for larger angles. Also, stacking up several layers of materials can substantially modify the conditions for interference due to reflection from several layers. In this section, we will consider the interference that results when several sources are involved.

1.1 Interference due to multiple slits

Let us generalise the Youngs double slit experiment to multiple slits. As before, we will assume that the slits are of very small dimension and that the slits act as line sources (point source if they were replaced by apertures). As shown in the Fig. **??**, the paths for light from each slit differs from the next one by a certain amount. Let the paths be for light from the various slits to the point P on the screen be s_j, where the label j runs from 0 to N, there being N slits. Using the same co-ordinate system as before, the point P is located at (x, y, z_0) and the coordinates of the j^{th} slit would be $(x - id, y, 0)$. Hence we can write that

$$s_j^2 - s_0^2 = [x^2 + y^2 + z_0^2] - [(x - jd)^2 + y^2 + z_0^2] = 2jdx + j^2d^2$$

For $d \ll x \ll z_0$, we can neglect the second term and we obtain,

$$s_0 - s_j \simeq j\frac{d\ x}{z_0} \tag{1}$$

The electromagnetic field at the point P is given by the sum of the fields emanating from the various slits

$$E(P) = \sum_{j=0}^{N} E_0 e^{iks_j} = E_0 e^{iks_0} \sum_{j=0}^{N} e^{-ikjd\ x/z_0}$$

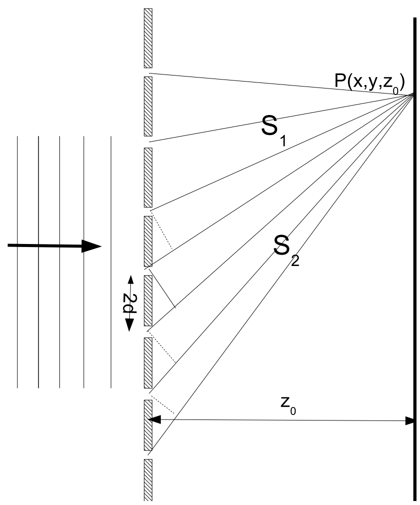

Figure 1: A schematic diagram of the interference due to multiple slits, showing the associated path lengths.

where E_0 is a reference field that is transmitted by each of the slits. The geometric series is easily summed and we obtain

$$E(P) = E_0 e^{iks_0} \left[\frac{1 - e^{-ik(N+1)d\ x/z_0}}{1 - e^{-ikd\ x/z_0}} \right]$$

Noting that $x/z_0 \simeq \theta$, the angle of scattering (or observation), and setting $\Phi = kd\theta$, we have

$$E(P) = E_0 e^{iks_0} e^{-iN\Phi} \left[\frac{\sin[(N+1)\Phi/2]}{\sin(\Phi/2)} \right]$$

Hence the intensity at the point P appears as

$$I(x, y, z_0) = \frac{c\varepsilon_0}{2} |E_0|^2 \left[\frac{\sin[(N+1)\Phi/2]}{\sin(\Phi/2)} \right]^2 \tag{2}$$

Let us understand the properties of the above field. For$\Phi \rightarrow 0$, the intensity $I(P) = N^2 I_o$, where I_0 is the intensity due to a single slit. The intensity goes up quadratically with the number of slits at points where $\Phi = 2m\pi$ where m is an integer. The intensity for other values of ϕ drastically reduces as N becomes larger and larger. Also note that width of the peaks at $\Phi = m\pi$ also become narrower with N, with the full width of the peak being $\delta\Phi = 4m\pi/(N+!)$.

This is a very interesting result as the intensity at selected points grows quadratically with the number of slits – this is entirely due to the coherence in the field across all the slits. Also

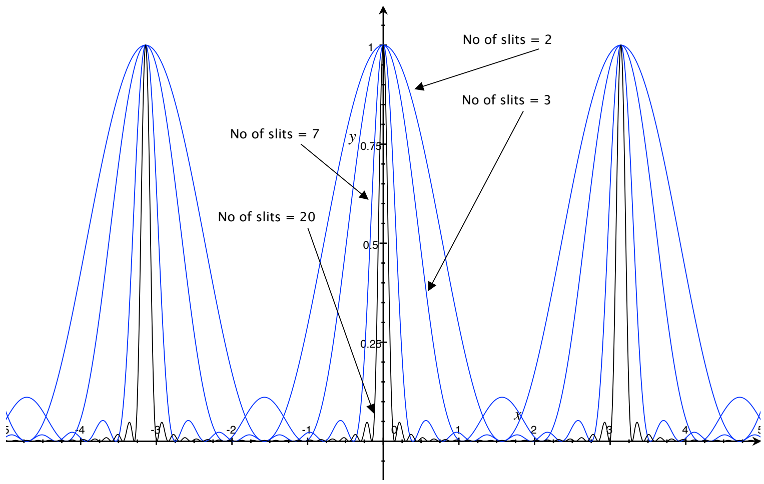

Figure 2: The plot of $I(P)/N^2$ for number slits = 2, 4, 7 and 20.

note that the peaks have a very narrow angular spread as N increases. This is at the heart of angular selectivity using a diffraction grating that we will analyse later. Now let us understand what happens as we have an infinity of point sources ($N \to \infty$) or in other words, we have a comb of Dirac δ functions. It turns out that the above expression can be shown to tend to a sum of Dirac δ functions in Φ (Problem for the assignment).

1.2 Multiple beam interference from a plane parallel plate

Consider a plane parallel plate of large transverse extent (essentially taken to be infinite). The plate has a refractive index n (or equivalently dielectric permittivity of ε and magnetic permeability μ). Now we will label the various regions of space and the corresponding Fresnel coefficients for reflection and transmission at the two interfaces as in Fig. 3.

Now the amplitude transmission coefficient can be expressed as a sum of fields travelling along the different possible paths as

$$
\begin{aligned}
\mathcal{T} &= t_{21}e^{ik_zd}t_{32} + t_{21}e^{ik_zd}r_{32}e^{ik_zd}r_{12}e^{ik_zd}t_{32} \\
&+ t_{21}e^{ik_zd}r_{32}e^{ik_zd}r_{12}e^{ik_zd}r_{32}e^{ik_zd}r_{12}e^{ik_zd}t_{32} + \cdots \qquad (3) \\
&= t_{21}e^{ik_zd}t_{32}\left[1 + r_{32}e^{ik_zd}r_{12}e^{ik_zd} + (r_{32}e^{ik_zd}r_{12}e^{ik_zd})^2 + \cdots\right] \qquad (4) \\
&= \frac{t_{21}t_{32}e^{ik_zd}}{1 - r_{32}r_{12}e^{2ik_zd}} \qquad (5)
\end{aligned}
$$

$$
\mathcal{R} = r_{21} + t_{21}e^{ik_zd}r_{32}e^{ik_zd}t_{12} + t_{21}e^{ik_zd}r_{32}e^{ik_zd}r_{12}e^{ik_zd}t_{32}
$$

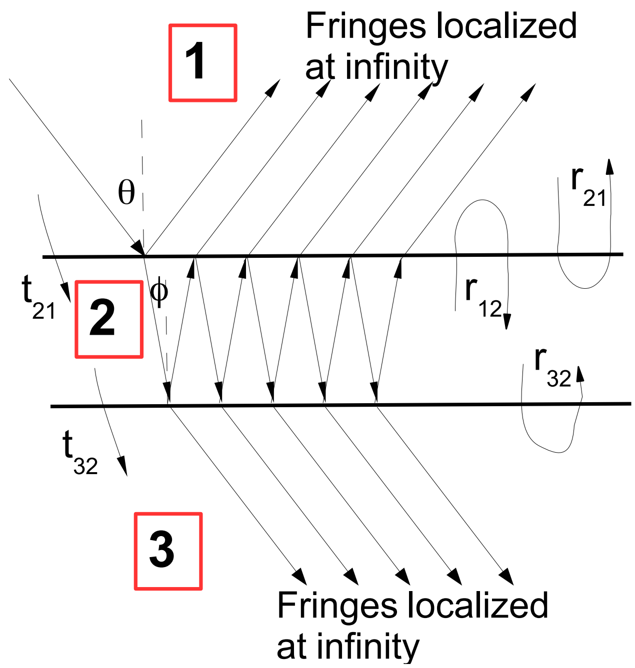

Figure 3: Schematic of the multiple reflections that occur at the interfaces of a parallel slab. The Fresnel coefficients r_{ij} and t_{ij} for the remittances at the interfaces are shown in the figure.

$$
\begin{aligned}
&+ \quad t_{21}e^{ik_zd}r_{32}e^{ik_zd}r_{12}e^{ik_zd}r_{32}e^{ik_zd}t_{12} + \cdots && (6)\\
&= \quad r_{21} + t_{21}e^{ik_zd}r_{32}e^{ik_zd}t_{12}\left[1 + r_{32}e^{ik_zd}r_{12}e^{ik_zd} + (r_{32}e^{ik_zd}r_{12}e^{ik_zd})^2 + \cdots\right] && (7)\\
&= \quad r_{21} + \frac{t_{21}t_{12}r_{32}e^{2ik_zd}}{1 - r_{32}r_{12}e^{2ik_zd}} && (8)\\
&= \quad \frac{r_{21} - r_{32}e^{2ik_zd}}{1 - r_{32}r_{12}e^{2ik_zd}} && (9)
\end{aligned}
$$

where the equality, $t_{21}t_{12} - r_{21}r_{12} = 1$ has been used. If we set $r_{21} = -r_{12} = \sqrt{R}$ and $\delta = 2k_zd$, then we have

$$\mathcal{R} = \frac{(1 - e^{i\delta})\sqrt{R}}{1 - Re^{i\delta}},$$

from which we deduce that the reflected intensity is

$$|\mathcal{R}|^2 = \frac{2 - 2\cos\delta}{1 + R^2 - 2R\cos\delta} = \frac{4R\sin^2(\delta/2)}{(1 - R)^2 + 4R\sin^2(\delta/2)}\frac{F}{1 + F\sin^2(\delta/2)}, \tag{10}$$

where $F = 4R/(1 - R)^2$.

Similarly the transmitted intensity can be calculated as

$$|\mathcal{T}|^2 = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2(\delta/2)} = \frac{1}{1+F\sin^2(\delta/2)}. \tag{11}$$

Let us analyse the transmittance. A maximum or minimum occurs depending on the value of *delta*. For $\delta = 2m\pi$ where m is an integer, a maximum occurs and this corresponds to $m = (2k_z d)/(2\pi) = (2nd\cos\phi)/\lambda$ where ϕ is the angle the rays make to the normal in the slab medium. Thus, depending on the value of F, transmittances can have broad (for small F) or narrow features (for large F). Note that a large reflectivity ($R \to 1$) will result in a large value of F. For such rage values, the transmittance as a function of thickness (d) or incidence angle (θ) or the wavelength λ) consists of a series of sharp resonances where the transmittance is close to one, and it will be close to zero otherwise. Similarly, the reflectivity will be close to zero for these resonance values and will be close to zero away from the resonant values.

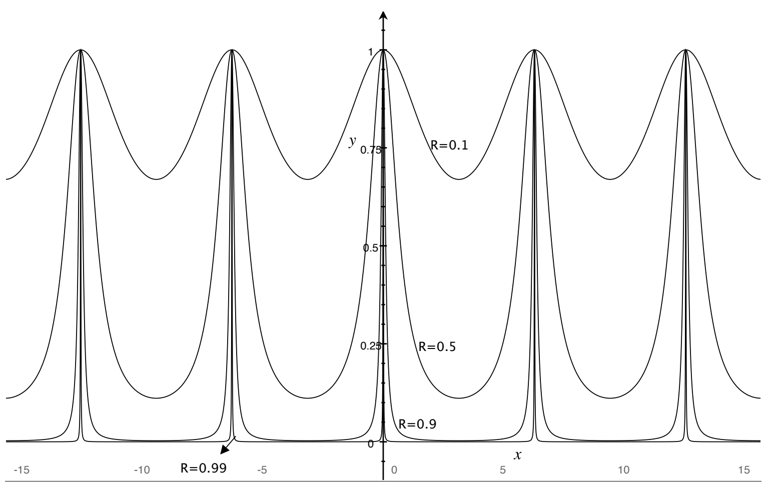

Figure 4: The plot of the intensity transmitted across a slab versus *delta* for $R = 0.1,\ 0.5,\ 0.9,\ 0.99$

The sharpness of the peaks are measured by the full-width at half maximum (FWHM) - (deduce it – problem for assignment). The ratio of the FWHM and the separation of the peaks is called the *Finesse* ($\mathcal{F}$). It can be shown that

$$\mathcal{F} = \frac{\pi}{2}\sqrt{F}$$

(show this as your assignment problem).