Department of Physics, Indian Institute of Technology, Kanpur

PHY-224A: OPTICS

Lecture -3: Polarization, plane waves,and superposition of waves

Date: 03 Aug. 2016, Time :1700 hours Instructor: S.A. Ramakrishna

1 Time harmonic, plane wave solutions

Let us return to the Maxwell equations. We now know that plane-wave time harmonic waves are good solutions to the to the wave equations. Since both sines and cosine functions are good solutions, a superposition of the two is also a solution and we will prefer to use to complex superpositions of these functions as

$$\exp[\pm i(\vec{k}\cdot\vec{r} \pm \omega t)] = \cos(\vec{k}\cdot\vec{r} \pm \omega t) \pm i \sin(\vec{k}\cdot\vec{r} \pm \omega t).$$

Now the exasperated student might always ask, what do we mean by a complex electric and magnetic field? What does an imaginary part of the field mean? The answer is that the electric field at any point will only have to be real. Afterall, an electron will feel an actual force applied on it by the electric and magnetic fields. we will only use this notation, known as the *phasor notation* for convenience. We will agree to always take the real part of the complex valued expression whenever we need to ask what is the actual value of the field. It turns out to be easy to calculate with this notation as long as only linear quantities are involved and we are solving linear equations. Of course, we could have as well agreed to take the imaginary part as the actual field. This is only a matter of practice whereby the real part is taken. Take the imaginary part if you want to, but always take only the imaginary part then.

Let us assume solutions such as $\vec{E} = E_0 \exp[i(\vec{k}\cdot\vec{r} - \omega t)]\hat{e}$ and $\vec{B} = B_0 \exp[i(\vec{k}\cdot\vec{r} - \omega t)]\hat{b}$ and substitute them into the Maxwell equations for free space to get

$$i\vec{k}\cdot\hat{e} = 0 \tag{1}$$

$$i\vec{k}\cdot\hat{b} = 0 \tag{2}$$

$$i\vec{k}\times E_0\hat{e} = i\omega B_0\hat{b} \tag{3}$$

$$i\vec{k}\times B_0\hat{b} = -i\omega\varepsilon_0\mu_0 E_0\hat{e} \tag{4}$$

From the above, we obtain that the electric and magnetic fields are perpendicular to the wavevector (direction of propagation) and that they are mutually perpendicular to each other. . The triad of unit vectors $(\hat{e}, \hat{b}, \hat{k})$ forms a *right-handed system* - electromagnetism seems to have a preference for right-handedness!! Further, we can also quick ascertain that $k^2 = \varepsilon_0\mu_0\omega^2$ is the dispersion for the wave.

Using the equations above, one can quickly relate the amplitudes of the electric field and magnetic field amplitudes of the time harmonic plane wave. Noting that

$$\vec{H} = \frac{1}{\mu_0 \omega}\vec{k} \times \vec{E} = \frac{1}{\mu_0 c}\hat{k} \times \vec{E} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\hat{k} \times \vec{E}, \tag{5}$$

and

$$\vec{E} = \frac{1}{\varepsilon_0 c}\hat{k} \times \vec{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\hat{k} \times \vec{H}, \tag{6}$$

we obtain that

$$\sqrt{\mu_0}|\vec{H}| = \sqrt{\varepsilon_0}|\vec{E}|. \tag{7}$$

The ratio of the amplitudes of the electric and magnetic fields is known as the wave impedance. In free space, this impedance (denoted by Z_0) is given by $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ and has a value of $Z_0 = 376.73031 \cdots \Omega$.

The other thing is to put down aspects of energy associated with a plane wave. teh energy density in free space is given by the classical expression

$$\mathcal{U} = \frac{\varepsilon_0}{2}|E|^2 + \frac{\mu_0}{2}|H|^2. \tag{8}$$

The Poynting vector, that denotes the rate of energy transfer in space is given by

$$\vec{S} = \vec{E} \times \vec{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}}|E|^2\hat{k} = \sqrt{\frac{\mu_0}{\varepsilon_0}}|E|^2\hat{k}. \tag{9}$$

The magnitude of this vector indicates the power flow per unit area. The student is cautioned to be careful with the phasor notation while dealing with nonlinear terms like the energy density or flow. It is best to first take the real parts of the fields and then compute the quadratic terms. Otherwise, some of the phase components would usually cancel out giving rise to incorrect calculations.

For time harmonic waves, we note that the energy density and flow oscillate at twice the frequency due to the quadratic nature. These fast oscillations, particularly for optical fields cannot be measured in real time by any detector. Most detectors measure a quatntity that is proportional to the time average of these quantities over the detector response time, which is typically many periods longs. Hence the time averaged Poynting vector for a time harmonic wave is

$$\langle\vec{S}\rangle_t = \sqrt{\frac{\varepsilon_0}{\mu_0}}\langle|E|^2\rangle_t\hat{k} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{E_0^2}{2}\hat{k} = \frac{1}{2}c\varepsilon_0 E_0^2, \tag{10}$$

where E_0 is the electric field amplitude of the time harmonic wave. Further, the student's attention is drawn to the fact that the intensity does not depend on the frequency of the wave, although we do know that the individual photon energy is given by $\hbar\omega$. Thus, a wave of higher frequency would correspond to more number of photons than a wave of lower frequency and same energy.

1.1 assignment

Calculate the intensity and the electric field amplitudes associated with

1. Sunlight: We know that the solar flux consists of roughly 1kW over a 1 m^2 area. If you assume that the sunlight was due to a single time harmonic wave , what is the corresponding electric field amplitude?

2. He-Ne laser: Consider the He-Ne laser used by you in the laboratory with about 3 mW or 5mW outputs. Assuming that the beam consisted of a rectangular spot of 1mm width, and with constant intensity over it, compute the intensity and electric field amplitude of the wave.

3. A femtosecond laser: Consider an ultrashort laser pulse lasting about 10 fs and with about 1 J energy. If the laser is focused to a square spot of 100 μm side and assuming as before a uniform intensity, calculating the peak electric field due to the laser at the focal spot. Compare this to the electric field of the nucleas on the 1s electron in a Pb atom (you can assume the Bohr model).

2 Non-instantaneous response of material media

Let us ask ourselves what happens to a wave inside a material medium. The first thing to note that real charges and currents in medium are now induced by the applied wave. Hence, the wave in the medium consists of the original wave and a wave that occurs due to the re-radiation by these induced charges and currents. Hence the question arises as to what is the polarization or magnetization in a meidum? Can we assume that the constitutive equations such as $\vec{P} = \varepsilon\chi_e\vec{E}$ are valid in time? We would suspect that since real charges have to move and real currents have to flow, the polarization at a given time would not be equal to the electric field applied at that particular time alone. Thus, we do not have instantaneous response, but a time-delayed response. In other words, a time harmonic wave incident on a medium would elicit responses from the medium both in-phase and in quadrature (at $\pi/2$ phase angle) at the same frequency (forced response). Thus, a complex valued response function would be able to represent this response. Hence, we see that the constitutive relations would be valid, if they represented the amplitudes of time harmonic fields. The material response parameters like ε and μ will, of course, change with frequency, and will have real and imaginary parts. While we will rigorously prove these aspects later, let us assume them for the present discussion.

The reason to concentrate on time harmonic waves is because any pulse or waveform of any shape can be constructed out of periodic waveforms. This construction goes by what is called *Fourier Analysis* and we will soon have a crash course of that. Right now, let us ask what happens to a time harmonic wave in a material medium? Since the material parameters are different, the rate at which the time harmonic wave would propagate would depend on its frequency. Thus, a general waveform or pulse consisting of different time harmonic waves would result in each component going at its own pace and the pulse form would, in general, change. Again, this will be investigate this in detail a bot later.

In spite of these explanations of the non-instantaenous response, I will for the present proceed to assume an almost instantaneous response in media whereby we can characterize the media by non-dispersive (constant) real parameters $\varepsilon\varepsilon_0$ and $\mu\mu_0$, where ε and μ are called the relative material permittivity and the permeability respectively. These parameters will replace the free space permittivity and permeability in the field equations and the wave equation. The justification for this is that we will consider the wave to be at frequencies that are very different from the characteristic frequencies of the medium. In that case, the medium offers only a feeble polarization response, which can be encapsulated in a real number for the response parameter. Then we note that the dispersion becomes

$$k_x^2 + k_y^2 + k_z^2 = \varepsilon\mu\frac{\omega^2}{c^2}. \tag{11}$$

The quantity $n = \sqrt{\varepsilon\mu}$ is called the refractive index, with which you would be very familiar. We will justify all these assumptions a bit later in a rigorous manner and I will ask you to bear with me for the present with these caveats.