PHY 224A: Optics, **(2013-14, Semester -I)**

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Lecture – 5 by S.A. Ramakrishna

1 Relating to the refractive index

In early optical studies, it was customary to work with the refractive index. We consider Maxwell's equations inside a charge-free and current-free material medium,

$$\nabla \cdot \vec{D} = 0, \tag{1}$$

$$\nabla \cdot \vec{B} = 0, \tag{2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{3}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}, \tag{4}$$

and time harmonic solutions to them so that we can write the constitutive relations between fields of the same frequency as $\vec{D}(\omega) = \varepsilon_0 \varepsilon(\omega) \vec{E}(\omega)$ and $\vec{B}(\omega) = \mu_0 \mu(\omega) \vec{H}(\omega)$. Hence the Maxwell's equations can be re-written for the frequency components as

$$\nabla \cdot \vec{D} = 0, \tag{5}$$

$$\nabla \cdot \vec{B} = 0, \tag{6}$$

$$\nabla \times \vec{E} = i\omega\mu_0\mu(\omega)\vec{H}(\vec{r},\omega), \tag{7}$$

$$\nabla \times \vec{H} = -i\omega\varepsilon_0\varepsilon(\omega)\vec{E}(\vec{r},\omega). \tag{8}$$

Taking once again the curl of one of the curl equations we obtain,

$$\begin{aligned}
\nabla \times (\nabla \times \vec{E}) &= \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \\
&= \varepsilon(\omega)\mu(\omega) \, \frac{\omega^2}{c^2} \vec{E}(\vec{r},\omega) \\
&\Rightarrow \nabla^2 \vec{E}(\vec{r},\omega) + \varepsilon(\omega)\mu(\omega) \, \frac{\omega^2}{c^2} \vec{E}(\vec{r},\omega) = 0
\end{aligned} \tag{9}$$

where we assumed a homogenous medium to set $\nabla \cdot \vec{E} = 0$ and also identified $c^2 = 1/(\varepsilon_0 \mu_0)$. This last equation is called the Helmholtz equation. Note that the dispersion for a plane, time-harmonic wave in the material medium is

$$k^2 = \varepsilon(\omega)\mu(\omega) \, \frac{\omega^2}{c^2},$$

which can be refreshingly different from that in free space. From the analogy to the case of the wave equation in vacuum, we immediately identify the phase velocity of the mono-frequency wave as

$$v = \frac{c}{\sqrt{(\varepsilon\mu)}} = \frac{c}{n},$$

where the quantity n is the refractive index and is related to the constitutive electromagnetic parameters by

$$n^2 = \varepsilon(\omega)\mu(\omega). \tag{10}$$

This is called the Maxwell relation and makes the connection between the two seemingly disparate fields of Electromagnetics (ε and μ) and traditional Optics (n). A lot of modern developments in Optics (often called photonics for some unfathomable reasons) are due to this changed dispersion. Due to the complex calues nature of the material parameters, the wavector k now becomes complex: $k = k_r + ik_i$. So the structure of the plane wave propagating along the z-axis looks like:

$$\exp[i(k_r z - \omega t)]\ \exp(-k_i z)$$

Thus, the wave begins to decay with distance that corresponds to the absorption of the wave with propagating distance. In this case, the power flow and the energy denisty also decays exponentially.

2 Wave motion – what moves and at what rate?

As I have often insisted, when we consider a pure time-harmonic plane wave (single frequency, single direction), we are looking at wave that extends over all space, that has existed from infinite time in the past and will continue to flow until infinite time in the future. Then the question is, what moves? The motion is really that of the constant phase surfaces.

Consider the scalar wave

$$V(\vec{r}, t) = a(\vec{r})e^{i[\phi(\vec{r}) - \omega t]},$$

where $\phi(\vec{r})$ is some scalar function. For this wave, $\phi(\vec{r}) =$ constant, defines the constant phase surfaces. To trace the motion of these surfaces, let us look at the condition at two points, $(\vec{r}, t)$ and $(\vec{r} + \delta\vec{r}, t + \delta t)$. The phase is the same if , and only if,

$$\phi(\vec{r} + \delta\vec{r}) - \omega(t + \delta t) = \phi(\vec{r}) + \delta\vec{r} \cdot \nabla g - \omega t - \omega\delta t = \phi(\vec{r}) - \omega t,$$

where we have assumed the first order term only in the infinitesimal $\delta\vec{r}$. Hence we obtain

$$v_{\mathrm{ph}} = \left|\frac{\delta\vec{r}}{\delta t}\right| = \frac{\omega}{|\nabla\phi|} \tag{11}$$

as the phase velocity for a wave with an arbitrary wave front. The student is asked to verify whether this expression works for a time harmonic plane wave. Note that the derivative of the phase function comes up. This means that the phase velocity could really be anything depending on the phase structure of the wave. For example, it could be very small if $|\nabla\phi|$ were very large. It could be literally infinite is the derivate were zero. The phase velocity indicates only the motion of the phase planes and nothing else!

Hence in order to convey a signal, very often, we envision a pulse of light, which can be an amplitude modulation. We know that the pulse will move undistorted in free space. But, will it move undistorted inside a material medium? Note that a pulse or an amplitude motion indicates that there are waves with several frequencies involved. Due to dispersion in the material parameters, they will all not propagate with the same phase velocity. This implies that the different waves will go out of phase with each other as they propagate and the coefficients of (Fourier) superposition of the waves will change with distance. This simply means that the shape of the pulse will change as it propagates.

To understand this better, consider a superposition of two co-propagating plane time harmonic waves with only slightly differing frequencies,

$$E(z,t) = E_1 e^{i(k_1 z - \omega_1 t)} + E_2 e^{i(k_2 z - \omega_2 t)} \tag{12}$$

where $k_1^2 = \varepsilon(\omega_1)\omega_1^2/c^2$ and $k_2^2 = \varepsilon(\omega_2)\omega_2^2/c^2$ (where $\mu = 1$ is assumed). It is well known that such a superposition will produce a "beat" pattern. For simplicity, we will also assume that $E_1 \simeq E_2$. Then we get

$$\begin{aligned} E(z,t) &= E_1(e^{i(k_1 z - \omega_1 t)} + E_2 e^{i(k_2 z - \omega_2 t)}) \\ &= E_1 \exp\left\{\left[i\frac{k_1+k_2}{2}z - \frac{\omega_1+\omega_2}{2}t\right]\right\} \times 2\cos\left[\frac{k_1-k_2}{2}z - \frac{\omega_1-\omega_2}{2}t\right] \end{aligned} \tag{13}$$

Define $\Delta k = (k_1 - k_2)/2$ and $\Delta omega = (\omega_1 - \omega_2)/2$. The first factor of the expression above represents the fast varying component at the average frequency, while the second cosine term represents a slowly moving envelope whose motion can be percieved. So we now ask for the rate at which this envelope moves and require $\phi = \Delta k z - \Delta\omega t =$ constant. We obtain the speed at which a recognizable (fiducial) point on the envelope moves is given by

$$\Delta k \left(\frac{dz}{dt}\right) - \Delta\omega = 0$$

or

$$v_g = \lim_{\Delta k \to 0} \frac{\Delta\omega}{\Delta k}$$

is the group velocity. Generalizing to vector notation in higher dimensions, we obtain

$$\vec{v}_g = \nabla_{\vec{k}}\omega(\vec{k})\Big|_{\vec{k}_0} \tag{14}$$

This group velocity is representive of the rate at which the envelope moves if it is evaluated at the centre frequency. Note that the dispersion gives a relationship between the wave-vector and the frequency. It can be assumed to be an invertible function.

Using the dispersion in a homogenous isotropic medium $k^2 = \varepsilon(\omega)\omega^2/c^2$, the group velocity is also often conveniently written in the following manner:

$$v_g = \frac{c}{n(\omega_0) + \omega_0 \left.\frac{dn}{d\omega}\right|_{\omega_0}} \tag{15}$$

This definition for the group velocity can be generalized to arbitrarily shaped pulse forms. But a couple of cautionary notes are required:

- The group velocity is well defined, if and only if, the superposed waves are propagating in the same direction.
- The group velocity is meaningful to describe the motion of a fiducial point on a pulse or waveform only if the bandwidth involved is small compared to the width (or scale) of the dispersion. Otherwise, the pulse rapidly deforms with propagation and could well become unrecognizable. Even the breakup of the pulse cannot be ruled out.

Assignment -4

1. Write down the dispersion relation for the wavevector of a plane time harmonic wave inside a metal and compare it to the momentum-energy relationship of a relativistic particle.

2. Obtain the real and imaginary parts of the wavevector for a dissipative medium in the limit of small losses (imaginary part of permittivity begin much smaller than the real part). Check if there is a direct relation between the imaginary part of k and γ, the damping coefficient in the Lorentz model.

3. Using the Poynting vector, obtain the flow of power with distance for a plane wave propagating in an absorptive medium. From that obtain the energy per unit volume per unit time (power dissipation) in the medium.

4. Deduce Equation 15 above from the from the definition of the group velocity.

5. For a typical single resonance dispersion of the previous assignment, obtain and plot the group velocity as a function of the frequency.