

## 1 The response of materials to electromagnetic waves

The electromagnetic waves create polarizations and magnetizations in a material. But are the responses of the medium instantaneous? In most materials, the time domain displacement field  $\vec{D}$  is directly and mostly linearly proportional to the applied electric field  $\vec{E}$ , and is a function of the material in which the field propagates. Due to the mass of the electrons in the medium that introduce a certain inertia in the response,  $\vec{D}$  does not vary instantaneously with  $\vec{E}$ , but instead is a function of the entire time history of how  $\vec{E}$  excited the medium. A somewhat general form for  $\vec{D}$  can therefore be written in the following form:

$$\vec{D}(\vec{r}, t) = \int_{-\infty}^t \kappa(\vec{r}; t, t') \vec{E}(\vec{r}, t') dt' \quad (1)$$

where  $\kappa(\vec{r}; t, t')$  is called the local response function. We assume here that the polarization that sets in a medium depends on the local fields – an assumption that can be violated at small lengthscales due to correlations in the polarization over a given volume of the material. For stationary processes,  $\kappa(\vec{r}; t, t') = \kappa(\vec{r}; t - t')$ , i.e., all physical quantities depend only on the elapsed time intervals. We further note that the response function  $\kappa$  can be non-zero only for  $t' < t$  and can be defined to be zero for  $t' > t$ . This permits the upper limit of the above integral to be made infinity and the integral becomes a convolution. Frequency domain displacement field and electric field can be defined such as

$$\vec{E}(\vec{r}, t) = \int_{-\infty}^{\infty} \vec{E}(\vec{r}, \omega) e^{-i\omega t} d\omega \quad (2)$$

$$\vec{D}(\vec{r}, t) = \int_{-\infty}^{\infty} \vec{D}(\vec{r}, \omega) e^{-i\omega t} d\omega \quad (3)$$

Now using these definitions for the temporal field along with the convolution theorem (asked in the assignment), we obtain

$$\vec{D}(\vec{r}, \omega) = \epsilon_0 \epsilon(\vec{r}, \omega) \vec{E}(\vec{r}, \omega), \quad (4)$$

i.e., the fields in the frequency domain are related by a simple linear relation. The frequency-dependent dielectric function is related to the linear response function by

$$\epsilon(\vec{r}, \omega) = \frac{1}{\epsilon_0} \int_{-\infty}^{\infty} d\tau \kappa(\vec{r}, \tau) e^{i\omega\tau}. \quad (5)$$

This relation indicates that the dielectric permittivity is dispersive, i.e., function of the frequency  $\omega$ . The dielectric function is complex as well. The imaginary parts indicate the

response that are  $\pi/2$  out of phase (in quadrature) to the applied fields. The dispersive nature arises from the inertia of the dipoles in a causal medium (due to the mass of the electrons), which defines a material polarization that does not respond instantaneously to the applied fields, but depends on its time history as we have seen. This causes the phase lag in the response giving both in-phase and out-of-phase components of the response. At extremely high frequencies, for example x-rays or  $\gamma$ -rays, the matter cannot even respond and the electronic matter is almost transparent leading to the limit

$$\lim_{\omega \rightarrow \infty} \varepsilon(\omega) \rightarrow 1.$$

A similar analysis also holds true for the magnetic permeability  $\mu(\vec{r}, \omega)$ , which can be space and frequency dependent.

## 2 Dielectric response of a bunch of dipoles

We begin by noting that an applied electric field causes charge separation of the positively charged nuclei and the negatively charged electrons in an atom or molecules. Thus a dipole moment is generated and to a good approximation dominates over the other multipole moments. The induced dipole moments can be determined by the displacements of the charges from their equilibrium positions. The atoms or molecules may additionally have a permanent dipole moment in which case the equilibrium positions of the positive and negative charges do not coincide. We will ignore the motion of the nuclei due to their comparatively large mass.

Let us make a model for a dielectric medium. We will consider the electrons in an atom or molecule to be bound to its equilibrium position through an elastic restoring force. This is a not a very bad assumption despite the great complicated structure of the atom or molecule. Further, note that the displacement caused by terrestrial light fields are very small – a very small fraction of the atomic size. For small displacements about the equilibrium, most potentials may be modelled as harmonic oscillator potential (the quadratic term is the lowest order term). For example, if you consider that the electronic charge distribution to be a uniform spherically symmetric cloud of charge density  $\rho_0$ , the electric field inside the cloud will go as  $\rho_0 r / 3\epsilon_0$ , i.e., linearly with the distance from the centre. If the positive charge at the centre is displaced by a distance  $r$ , it will feel a restoring force linear in the displacement. The electronic cloud will equivalently also feel a linear restoring force. If the mass of the electron is  $m$ , we can then write an equation of motion of a time harmonic

forcing due to the electric field of the wave as

$$m\ddot{r} + m\gamma\dot{r} + m\omega_0^2 r = -eE_0 e^{-i\omega t}. \quad (6)$$

Here  $r$  is the displacement,  $\omega_0$  is the resonance angular frequency of the harmonic potential that confines the electron,  $\omega$  is the angular frequency of light that forces the electron to oscillate and  $m\gamma\dot{r}$  is a phenomenological viscous damping force on the electron due to all the inelastic processes. This force represents all the processes by which the electron can lose energy like collisions and includes the radiation emitted by the accelerating electron. Using  $r \sim r_0 \exp(-i\omega t)$  as a trial solution to indicate the forced oscillations of the electron, we obtain

$$r_0 = \frac{-eE_0/m}{\omega_0^2 - \omega(\omega + i\gamma)}.$$

Noting that the total dipole moment due to each displaced electron is  $\vec{p} = e\vec{r}$ , the polarization field  $\vec{P}$  is given by the vectorial sum of all the dipoles in a unit volume. Assuming one dipole per molecule or atom and a number density of  $N$  atoms /molecules per unit volume, we get for the amplitudes,

$$\vec{P} = N\vec{p} = \frac{Ne^2 E/m}{\omega_0^2 - \omega(\omega + i\gamma)} = \epsilon_0 \chi_e \vec{E}, \quad (7)$$

where  $\chi_e$  is the dielectric susceptibility. Hence we obtain for the dielectric permittivity

$$\epsilon(\omega) = 1 + \chi_e(\omega) = 1 + \frac{Ne^2/m\epsilon_0}{\omega_0^2 - \omega(\omega + i\gamma)}. \quad (8)$$

Here the quantity  $(Ne^2/m\epsilon_0) = f^2$  is called the oscillator strenght of the resonance.

An atomic or molecular system may have many resonances. Actually each absorption line corresponds to a resonance. In that case, the response may be written as a sum over several resonances, each one with its own oscillator strenght, frequency and damping, as

$$\epsilon(\omega) = 1 + \sum_j \frac{f_j^2}{\omega_j^2 - \omega(\omega + i\gamma_j)} \quad (9)$$

The real and imaginary parts for a single resonance are plotted in Fig. ???. Note the large changes in the real part of the permittivity at frequencies close to the resonance and the peak in the imaginary part of the permittivity at the resonance.

We point out that our present model strictly holds only for a dilute gas of the polarizable objects. In a dense material medium with a much larger concentration, the fields that arise due to nearby polarized objects affect the polarization at any given point. So our expression will need to be corrected for that - which you will do in higher level courses.

The model presented here is due to the famous Dutch physicist H.A. Lorentz, who developed it in the early part of the twentieth century. Although one can give an adequate

description of dispersion only by a quantum mechanical treatment, a simplified description becomes possible by using only a few basic results concerning the properties of atoms and molecules. The resulting expression for the dielectric permittivity is very general and works wonderfully well for many materials as long as one does not ask very probing questions as to what happens at frequencies close to the line center.

### Assignment -3

1. Plot the real and imaginary parts of the dielectric permittivity as a function of the scaled frequency  $\omega/\omega_0$  for various values of the scaled damping parameter  $\gamma/\omega_0 = 0.01, 0.1, 0.5, 1, 5, 10$  assuming some suitable value of  $f/\omega_0 \sim 1$  or, 10.
2. How will the dispersion appear for a lossless medium ( $\gamma = 0$ ). Plot the dielectric permittivity in such a situation. Identify the point at which the permittivity undergoes a zero crossing.
3. Plot the dispersion for a double resonance with  $\omega/\omega_1 = 1$  and  $\omega/\omega_1 = 1.5$  and other suitable parameters.
4. Prove that the full width at half maximum (FWHM) of the resonance peak in the imaginary part is directly proportional to  $\gamma$ .
5. Consider a metal: the valence electrons are free and not bound to any atom. Obtain the dielectric permittivity function for the metal. The frequency below which the permittivity is negative is called the plasma frequency (assume  $\gamma$  is small while doing this).
6. Consider an alternative way of modelling a metal as a medium with a finite conductivity ( $\sigma$ ) and induced currents being proportional to the applied fields via Ohm's law  $\vec{J} = \sigma \vec{E}$ . Starting from Maxwell's equations, obtain an equivalent dielectric permittivity for the metal.