Department of Physics, Indian Institute of Technology, Kanpur

PHY-224A: OPTICS

Lecture -2: Review of electromagnetic theory

Date: 02 Aug. 2016, **Time :1700 hours** **Instructor: S.A. Ramakrishna**

1 The Maxwell Equations

We will begin with an acceptance that light is an electromagnetic wave. It is characterized by the electric and magnetic field vectors $\vec{E}(\vec{r})$ and $\vec{B}(\vec{r})$. These fields are coupled to each other through induction and to matter via the currents and charge densities induced in matter. To account for the polarization (bound) charge and the magnetization (bound) currents that in the materials, we introduce the $\vec{D}$ and $\vec{H}$ fields. The Maxwell equations relate all these together as

$$\nabla \cdot \vec{D} = \rho(\vec{r}) \tag{1}$$

$$\nabla \cdot \vec{B} = 0 \tag{2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\nabla \times \vec{H} = \vec{J}(\vec{r}) + \frac{\partial \vec{D}}{\partial t} \tag{4}$$

where the sources are the free current densities $\vec{J}(\vec{r})$ and the free charge densities $\rho(\vec{r})$. We have a conservation of charge whereby these densities are related as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0. \tag{5}$$

The material polarization and magnetization fields are supplemented by a description of the materials through some parameters. If a linear response is assumed of the materials, we can write

$$\vec{J} = \sigma \vec{E} \qquad (\text{Ohm's law}), \tag{6}$$

$$\vec{D} = \varepsilon_0 \varepsilon \vec{E} \qquad (\varepsilon \text{ is the relative dielectric permittivity}), \tag{7}$$

$$\vec{B} = \mu_0 \mu \vec{H} \qquad (\mu \text{ is the relative magnetic permeability}). \tag{8}$$

We call these as the *constitutive relations*. All the above parameters σ, ε and μ, in general, depend on the frequency of the radiation.

2 The Wave equation

We will now proceed to deduce the wave equation in free space, where it can be taken that $\rho = 0$, $\vec{J} = 0$, $\varepsilon = 1$ and $\mu = 1$. Taking the curl of Equation (3), and using Equation (1), we obtain

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial(\nabla \times \vec{B})}{\partial t}. \tag{9}$$

Using Equation (4) and and Equation (1), we obtain that

$$\nabla^2 \vec{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}. \tag{10}$$

This is the classical wave equation[1], only we note that $\vec{E}$ is a vector. The speed of these waves in vacuum or free space is identified to be c, where $c^2 = 1/(\varepsilon_0 \mu_0)$ and depends only on fundamental constants. Thus, the speed of light in free space is itself a fundamental constant.

We note that the the solutions of this equation can be written in the form of $\vec{E}(\vec{k} \cdot \vec{r} \pm \omega t)$, where $\vec{k}$ is a constant vector and ω is a constant number. The student is asked to verify these solutions to the wave equation. We find that $\vec{k} = (k_x, k_y, k_z$ and ω will need to satisfy a relationship

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}. \tag{11}$$

The vector $\vec{k}$ is taken to indicate the direction of propagation of the wave.

2.1 Spherical waves

If we have spherical symmetry (one preferred origin and isotropy of space), then in spherical coordinates, the equation reduces to

$$\nabla^2 E = \frac{1}{r} \frac{\partial^2}{\partial r^2}(rE) = \frac{1}{v^2} \frac{\partial^2 E}{\partial t^2}. \tag{12}$$

We deal with a scalar wave equation here for convenience. Setting $U = rE$ and noting that it reduces to the familiar one-dimensional wave equation for $U(\vec{r}, t)$, we note that the solution is

$$E(\vec{r}, t) = \frac{1}{r} U(\vec{k} \cdot \vec{r} \pm \omega t). \tag{13}$$

[1] The classical wave equation in one spatial and one temporal dimension is written as

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.$$

We note that the solutions are of form $f(x - vt)$ and $f(x + ct)$. The student should explicitly verify this. We note that the former corresponds to a wave form that continuously shifts forward in time and the latter shifts backwards in time. Note that a fiducial feature on the wave would shift forward or backward at the speed v, which is, hence, the speed of the wave. Note that the wave forms do not change their shapes in time and only move in time.

In other words, this is the field for a point source, which is steadily radiating with frequency ω. Such solutions are called spherical waves and the amplitude of these waves typically decays as $1/r$ with distance.

2.2 Plane waves

We are used to thinking about waves that oscillate harmonically with a well-defined frequency. Such solutions can be written as

$$E(\vec{r}, t) = A\cos(\vec{k}\cdot\vec{r} \pm \omega t + \delta). \tag{14}$$

After all, the function E of equation (10) could have been any function at all and could be a harmonic function as well. We note that the addition of a phase δ is arbitrary. This is because $\sin(\vec{k}\cdot\vec{r} \pm \omega t)$ is also a solution, and so is a superposition of the sine and the cosine functions. A change of $\vec{k}\cdot\vec{r} \to \vec{k}\cdot\vec{r} + \lambda$, where $\vec{k} = 2\pi/\lambda\hat{k}$, does not change the phase, as $\lambda/v = 2\pi/\omega$. Thus the waves repeat their amplitude periodically in space with a period of λ. Similarly, the wave amplitude repeats periodically in time at each point with a period of $2\pi/\omega$. Now the dispersion relation $k^2 = \omega^2/c^2$ involving the wave vector and a the angular frequency becomes recognizable to us immediately. This predicts that in free space, the magnitude of the wave-vector and the angular frequency are linearly proportional or that the dispersion is *linear*. Now the surfaces with constant phase are infinite planes to which the vector $\vec{k}$ is perpendicular to. These are called the constant phase surfaces. Due to this, waves of the kind given by Equation (14) are called *time harmonic plane waves*. We should remember that these waves are infinitely extended in the transverse direction (with respect to the propagation direction), are infinitely extended in the propagating direction and also have always been present at time $t \to -\infty$ and will always be present at $t \to \infty$.

2.3 Phase velocity

Now we will consider the motion of the constant phase surfaces. consider an arbitrary wave $E(\vec{r}, t) = A(\vec{r})\exp[-i(\omega t - g(\vec{r})]$, where $A(\vec{r})$ and $g(\vec{r})$ are some scalar functions. The surfaces with $g(\vec{r})$ = constant are the constant phase surfaces. Now $\omega t - g(\vec{r})$ = constant for $(\vec{r}, t)$ and $(\vec{r} + \Delta\vec{r}, t + \Delta t)$, if and only if,

$$\omega(t + \Delta t) - g(\vec{r} + \Delta\vec{r}) \simeq \omega t + \omega\Delta t - g(\vec{r}) - \nabla g \cdot \Delta\vec{r} = \omega t - g(\vec{r}),$$

where an expansion of $g(\vec{r})$ upto first order in $\Delta\vec{r}$ has been carried out. This implies that the speed with which the constant phase surface moves in space is

$$v_{\text{ph}} = \lim_{\Delta t \to 0} \frac{|\Delta\vec{r}|}{\Delta t} = \frac{\omega}{\hat{n}\cdot\nabla g(\vec{r})}. \tag{15}$$

where $\hat{n}$ is the direction along the Δr. The minimum value of this occurs when $\hat{n}$ is along the normal to the constant phase surface, i.e., ∇g. The phase velocity is conventionally taken to be this minimum value, which is the rate at which the constant phase surfaces move. Thus, the phase velocity of a wave depends on the details of its phase surface. For a plane wave with a plane for a constant phase surface, this phase velocity reduces to $v_{\mathrm{ph}} = \omega/k$ as $g(\vec{r}) = \vec{k} \cdot \vec{r}$ in that case. For an infinitely extended plane wave, the phase velocity is the most important characteristic of motion that can be properly talked about. It turns out, as the frequency of optical waves is large, unless the phase structure occurs on the scale of the wavelength, the phase velocity is nearly equal to ω/c, even for waves whose constant phase surfaces are not quite plane.