Department of Physics, Indian Institute of Technology, Kanpur

PHY-224A: OPTICS

Lecture -6: Polarization of light

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1 Some optical devices based on reflection / refraction

Using our simple understanding of reflection /refraction at an interface, we will now design some optical devices that let us manipulate the polarization of light.

- **A multi-interface polarizer and attenuator :** Fist we will utlize the Brewster angle for a device Consider several parallel plates of glass held together as inf Fig. 1 and a beam of light incident on the stack at an angle θ_i. We note that as θ_i nears the Brewster angle, S-polarized light would suffer from large reflection, while P-polarized light would principally transmit. Suppose the ratios of the transmitted intensities at one air-glass interface were T_p/T_s, where $T_p > T_s$), then after transmission through n-plates or 2n interfaces, the ratio of the transmitted intensities would be $(T_p/T_s)^{2N}$. For a typical angle of 60^o, the transmittivities, T_p and T_s, have values of about $> 99\%$ and 80% respectively. Thus, with about 8 plates, we obtain a transmittance ratio > 30 and we have mostly P-polarized light that is transmitted. Such techniques become very important for high intensity lasrs where many materials behave in a nonlinear manner or get damaged and designing attenuators or polarizers can be difficult with thin films used as attenuating filters. This system can also be used as an angle variable attentuator for the S-polarized light. To prevent a beam shift, two sets of stacks are used as shown in fig. 1.

- **Fresnel's Rhomb:** We will now use total internal reflection to produce circularly polarized light from linearly polarized light. Circularly polarized light can be thought of as a superposition of two orthogonal states of linear polarized light, but with a phase difference of $\pi/2$ between them. Let us consider the phase difference between $r_s = e^{i\phi_s}$ and $r_p = e^{i\phi_p}$ in total internal reflection inside a medium of refractive index n, where we can write

$$\tan\left(\frac{\phi_p}{2}\right) = -\frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i}, \tag{1}$$

$$\tan\left(\frac{\phi_s}{2}\right) = -\frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i}. \tag{2}$$

Then the phase difference is $\delta = \phi_s - \phi_p$ and we obtain

$$\tan\left(\frac{\delta}{2}\right) = \tan\left(\frac{\phi_s}{2} - \frac{\phi_s}{2}\right) = \frac{\cos\theta_i\sqrt{\sin^2\theta_i - n^2}}{\sin^2\theta_i}. \tag{3}$$

We should now analyse this expression. At grazing incidence $\theta_i \sim \pi/2$, and the critical angle $\sin\theta_i = n$, this expression becomes zero. Hence it goes through a maximum in between. At this angle, the phase difference between the S- and P-polarized light will be a maximum. For (crown) glass, $n \simeq 1.52$ and this $\delta_{\max} = 45^o 56'$. So one cannot get a phase difference of $\pi/2$ in a single reflection. Hence, one can design to have two total internal reflections as shown in the Fig. 2, in which case, one needs only a $\delta = 45^o$. There are then two angles of incidence $48^o 37'$ and $54^o 37'$ at which we can achieve this. If linearly polarized light is enters the rhomb so that is twice reflected at this angle, the the output light will be circularly polarized. By receiprocity, circularly polarized light can be converted into linearly polarized in this device. Thus, we have linear device that allows us to convert one kind of polarization into another.

2 Vectorial superpositions of waves

Let us consider superpositions of plane waves with polarizations in different directions. Let the kinematic phase be $\tau = \vec{k} \cdot \vec{r} - \omega t$, so that the variation of τ can result by change of either t at a fixed point in space or change of position at a fixed time. Now let us also fix the direction of propagation of the wave to be along the z-axis so that the electric field vectors then can have either an x or a y component. In general, we can have a superposition of two plane waves, one with its electric field along the x direction only and another with the electric field only along the y direction, so that

$$\vec{E}(\tau) = E_{0x}\hat{x}e^{i\tau} + E_{0y}\hat{y}e^{i\tau}, \tag{4}$$

where E_{0x} and E_{0y} are are complex valued coefficients in general. Suppose they have the same phase, then the resulting electric field always lies in the same plane for all values of τ. Such a wave is called a plane polarized wave. When the value of $E_{0y} = 0$, the electric field is always along $\hat{x}$ and we will call this as x-polarized light. In the other case of $E_{0x} = 0$, the light is said to be y-polarized as the electric field always is along the y-direction.

When the phases of these coefficients of superposition are unequal, the electric field vector will not be in the same direction for different τ. As a special case, let us consider $E_{0y} = -iE_{0x}$, where E_{0x} is real. To understand the direction of the field, consider the real part of the field (remember that while we calculate using the complex valued phasors for linear operations, we agreed to take the real value of the field for all purposes),

$$\begin{aligned}
\vec{E}(\tau) &= \mathrm{Re}\left[E_{0x}\hat{x}e^{i\tau} + e^{(-i\pi/2)}E_{0y}\hat{y}e^{i\tau}\right] \\
&= E_{0x}[\cos(\tau)\hat{x} + \cos(\tau - \pi/2)\hat{y}], \\
&= E_{0x}[\hat{x}\cos(\tau) + \hat{y}\sin(\tau)].
\end{aligned} \tag{5}$$

Thus,it seen that for different values of τ, the electric field vector points in different di- rections. Actually, the electric field vector rotates continuously with increase in τ in the counter-clockwise direction. This implies that in a snap-shot of the electromagnetic wave at any particular time, the electric field vector in this case behaves in a manner such that with increase in z, the electric field direction rotates in a manner given by that of a right-handed screw. In other words, the electric- field vector describes a right-handed helix and it is called *right-circular polarized light.* One can also view this by fixing the location (z plane and looking at the electric field vector in time. In this case, it will appear as if the electric field is rotating in the clock-wise sense. It is easy to note that a choice of $E_{0y} = +iE_{0x}$ would have have caused the electric field vector to rotate in the opposite sense. The polarization of the wave in that case would be called *left circularly polarized light.*

3 Jones Vectors and Matrices

We can represent the states of polarization by 2×1 vectors and the action of optical elements that affect the states of polarization by 2×2 matrices that act on these vectors. Thus, the entire matrix algebra can then be conveniently used to compute the states of polarization at any point. This formalism was introduced by R. Clark Jones in 1941 and is, hence, named after him as the *Jones matrix formalism.*

We have already noted that it is only the relative phase difference between the$x$- and$y$-polarized waves that matters to fix the state of polarization. Hence, we write the total electric field as

$$\vec{E}(\tau) = E_{\text{eff}}(A\hat{x} + Be^{i\delta}\hat{y})e^{i\tau}, \tag{6}$$

where we have

$$E_{\text{eff}} = [E_{0x}^2 + E_{0y}^2]^{1/2}e^{i\delta_x}, \qquad A = \frac{E_{0x}}{|E_{\text{eff}}|}, \qquad B = \frac{E_{0y}}{|E_{\text{eff}}|}.$$

Here$A$ and B are non-dimensionless numbers normalized to unity. While the amplitude of the vector E_{eff} carries information on the intensity,

$$I = \frac{1}{2}c\varepsilon_0|E_{\text{eff}}|^2 \tag{7}$$

The unit vector contains all the information on the polarization state of light. The vector, referred to as the *Jones vector*, is conveniently written as a column vector as

$$\begin{pmatrix} A \\ Be^{i\delta} \end{pmatrix} \tag{8}$$

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Explicitly, we will write the forms of the vectors for some common states of polarization as

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{X} - \text{linear polarized wave}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{Y} - \text{linear polarized wave}$$

$$\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} \qquad \text{linearly polarized wave at an angle } \phi \text{ to x} - \text{axis}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \text{right} - \text{circularly polarized wave}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ +i \end{bmatrix} \qquad \text{left} - \text{circularly polarized wave}$$

2.2 Polarizing elements and Jones matrices

The most common optical elements that are used to change the state of polarization are the polarizer and wave-plates. A polarizer essentially has a defined axis and it only allows through that component of light that has polarization parallel to that axis. The other orthogonal component is rejected – either absorbed or reflected. Thus, if the polarizer is oriented with its transmission axis along the x-direction, only the x-linearly polarized component comes through an the y-component is rejected. Then, we can represent that polarizer by a 2X2 matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

so that if this acts on any Jones vector corresponding to any state of polarization, the output will consist of the x-polarized component only:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ Be^{i\delta} \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix}.$$

Similarly, the Jones matrix for a polarizer with its transmission axis along the y-direction is seen to be

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note the effect of two mutually crossed polarizers on any state of polarizers

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ Be^{i\delta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

Now what about a polarizer with a transmission axis that is oriented at some angle ϕ from the x-axis (lab axes)? Note that we know the form of the Jones matrix in the polarizer axis frame. We need to re-write it in the lab axes. Note that if we can find the components of the electric fields or the Jones Vector in the polarizer axes, we can write the Jones vector for the output light easily in the polarizer axes. Then it is a simple transformation back to the lab frame.

Let U_l be the Jones vector in the lab axes and U_p be the Jones vector in the polarizer axes for the input light. Let P_p be the matrix for the transmission through the polarizer in the polarzer frame. Finally, let V_l be the Jones vector in the lab axes and V_p be the Jones vector in the polarizer axes for the output light. The output Jones vector in the polarizer axes can be written as

$$V_p = P_p \cdot U_p \tag{10}$$

The Jones vectors in the lab axes and the polarizer axes are related through a simple rotation matrix:

$$R = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}, \tag{11}$$

so that

$$U_p = R \cdot U_l, \tag{12}$$

$$V_p = R \cdot V_l. \tag{13}$$

Note that the rotation matrices are orthonormal, i.e., $R^{-1}R = RR^{-1} = 1$. Also, rotational by $-\phi$ should yield the inverse, i.e., $R(-\phi) = R^{-1}(\phi)$. Hence, we have

$$R \cdot V_l = P_p \cdot R \cdot U_l \quad \Rightarrow \quad V_l = R^{-1}\ P_p\ R \cdot U_l$$

Thus, we obtain the form of the Jones matrix for the polarizer in the lab axes (P_l) in terms of the rotation matrix and the Jones matrix in the polarizer axes (P_p) as

$$P_l = R^{-1} \cdot P_p \cdot R \tag{14}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} \cos^2\phi & \cos\phi\ \sin\phi \\ \cos\phi\ \sin\phi & \sin^2\phi \end{bmatrix} \tag{16}$$

In general, this is the transformation rule for a second ranked tensor under a rotation. This formalism is general and can be applied to the rotation of any polarizing optical element. The rule of transforming the vectors (Jones vector) which in turn depended on the way electric fields transform under a rotation was crucial to developing this form.