Midsemester Examination

ME 321: Advanced Mechanics of Solids

17 September 2018, 8 - 10 a.m.

Problem 1: Do the following. Notation: a - scalar, a - vector, $\boldsymbol{a}$ - tensor.

i. Show that if $\hat{\mathsf{n}}$ is a unit vector, then any vector a can be written as $(\mathsf{a}\cdot\hat{\mathsf{n}})\hat{\mathsf{n}} + \hat{\mathsf{n}}\times\mathsf{a}\times\hat{\mathsf{n}}$.

ii. Show that $\boldsymbol{\nabla}(r^n\mathsf{r}) = r^n\mathbf{1} + nr^{n-2}\mathsf{r}\otimes\mathsf{r}$, where r is the position vector with respect to the origin.

iii. The rotational part of the deformation is given by $\boldsymbol{R} = \left\{\boldsymbol{\nabla}\mathsf{u} - (\boldsymbol{\nabla}\mathsf{u})^T\right\}/2$. Show that $\boldsymbol{R} = -\mathbf{1}\times(\boldsymbol{\nabla}\times\mathsf{u})/2$.

iv. Let the *fourth*-order unit tensor $\mathcal{I}$ be such that

$$\mathcal{I}:\boldsymbol{A} = \boldsymbol{A}, \quad \text{i.e.} \quad I_{ijkl}A_{lk} = A_{ij},$$

for all second-order tensors $\boldsymbol{A}$. Then show that $\mathcal{I} = \hat{\mathsf{e}}_i\otimes\hat{\mathsf{e}}_j\otimes\hat{\mathsf{e}}_j\otimes\hat{\mathsf{e}}_i$, for any orthogonal triad $\{\hat{\mathsf{e}}_i\}$. Then find the components I_{ijkl} of $\mathcal{I}$ in terms of the Krönecker delta function.

Problem 2: *Linear Momentum Balance* (LMB). We have discussed that we hypothesize that for continuum materials of our interest, surfaces interact with other surfaces / external environment through traction (force per unit area) distributions. Moment distributions are ignored / not permitted, in contrast to ESO202 where they were allowed. Do the following:

a. Show that the LMB for such continuum materials may be written in indical notation as

$$\sigma_{ij,j} + \rho b_i = \rho a_i.$$

b. Show that the traction vector t acting on a surface with normal $\hat{\mathsf{n}}$ passing through a point P may be related to the stress $\boldsymbol{\sigma}$ at P by

$$\mathsf{t} = \boldsymbol{\sigma}\cdot\hat{\mathsf{n}}.$$

c. Now consider a complex continuum material (e.g. granular materials like sand) wherein it is useful to permit moment distributions across surfaces. Does angular momentum balance (see next problem also) hold for such materials?

d. Write down the LMB and the traction-stress relation in indical notation for materials in which moment distributions are permitted.

Problem 3: *Angular Momentum Balance* (AMB). Show that AMB leads to the proof of the stress being symmetric, i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$. Make sure that you fill in all steps of the derivation that were skipped in class.

Problem 4: *Boundary Condtitions.* Consider the old fashioned uniaxial tensile test shown in Fig. 1. The body being investigated is the solid cube $ABCDEFGH$. Gravity acts downwards in the $-\hat{e}_3$ direction. On the bottom surface $x_3 = -L$ (i.e. $ABCD$) we attach (stick) a *rigid* block of mass m, as shown. On the top surface the cube is built into (i.e. clamped to) the *rigid* ceiling. Do the following:

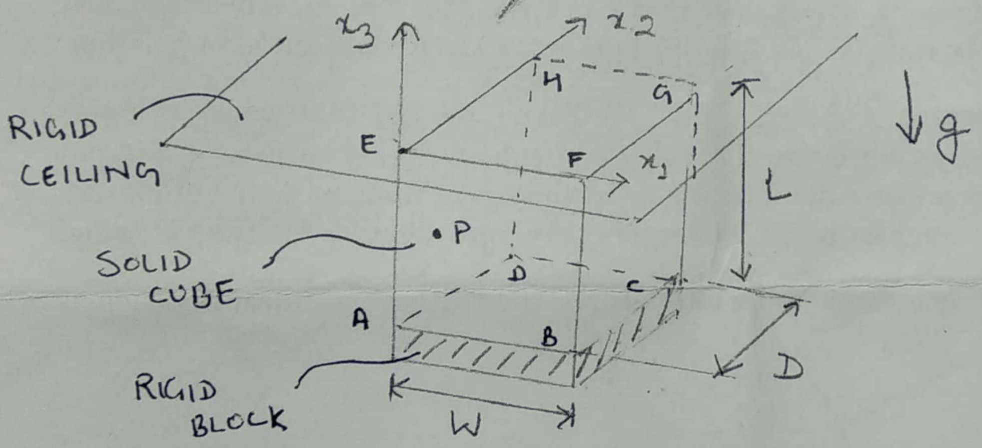
a. Draw the free body diagram of the cube $ABCDEFGH$ and the rigid block separately. Pay careful attention to which forces are present / absent.

b. Write the boundary conditions on the top surface.

c. Write the boundary conditions on the four side surfaces.

d. Write the boundary conditions on the bottom surface. The rigid block can*not* deform.

e. Investigate the stress tensor σ at point P on the surface $x_2 = 0$ (i.e. $ABFE$). Using the boundary conditions on $x_2 = 0$ compute as many components of σ at P as possible.



Problem 5: *The Compatibility Equation.* The strain tensor S at location r and time t is related to the displacement field $\mathsf{u}(\mathsf{r}, t)$ by

$$S = \frac{1}{2}\left\{\nabla \mathsf{u} + (\nabla \mathsf{u})^T\right\}. \tag{1}$$

Do the following:

a. Suppose, at a given time, we know the strain tensor field $S(\mathsf{r}, t)$ for all locations r. Is it always possible to integrate (1) to find the displacement field u? In other words, does a solution u always exist such that it satisfies (1) for any given S?

b. Show that a necessary condition for (1) to be solvable is that $\nabla \times S \times \nabla = 0$.
Hint: Recall that $S \times \nabla$ means that the curl operation is done from the right. At the same time, it is easy to show that $\nabla \times S \times \nabla = -\nabla \times (\nabla \times S)^T$.