

Contents

<i>List of Figures</i>	xix
<i>Preface</i>	xxi

A GENERAL INTRODUCTION	1
1 Rules of Logic	3
1.1 Sentences	3
1.2 Binary Relations on Sentences	3
1.3 Logical Equivalence and Implication	7
1.4 Predicate Logic	11
1.4.1 The necessity for using predicates	11
1.4.2 Quantifiers	12
1.5 Rules of Inference	13
1.5.1 Rules of inference for propositional logic	14
1.5.2 Rules of inference for quantifiers	15
1.5.3 Combining rules of the previous two types	16
1.5.4 Proofs using sequences	16
2 Sets and Functions	18
2.1 Set Theory	18
2.1.1 Fundamentals	18
2.1.2 Basic operations on sets	20
2.1.3 Subsets	23
2.1.4 Product sets	23
2.2 Functions	24
2.2.1 Definition	24
2.2.2 Image and pre-image of sets	25
2.2.3 Binary operations	28
2.3 Countable and Uncountable Sets	30

2.4	Sets with Relations	32
2.4.1	Sets with equivalence relations	33
2.4.2	Sets with order relations	34
3	Algebraic Structures	37
3.1	What are Algebraic Structures?	37
3.2	Metric Space	38
3.3	Group	40
3.4	Ring	40
3.4.1	General considerations	40
3.4.2	Modular numbers	45
3.4.3	Algebraic integers	47
3.5	Field	49
3.6	Vector Space	52
3.7	Algebra	54
3.8	Boolean Algebra	55
3.9	Arithmetic	56
3.10	Grassmann Arithmetic	57
3.10.1	Grassmannian numbers	57
3.10.2	Grassmannian matrices	59
3.11	Measure Space	60
3.11.1	σ -algebra	60
3.11.2	Measure space	61
3.11.3	Lebesgue integral	62
B	VECTOR SPACES	65
4	Basics	67
4.1	Definition and Examples	67
4.2	Linear Independence of Vectors	68
4.3	Dual Space	69
4.4	Subspaces	71
4.5	Vector Spaces with Extra Structure	72
4.5.1	Normed vector spaces	72
4.5.2	Inner product spaces	74
4.6	Further Properties of Inner Product Spaces	75
4.6.1	Properties of the null vector	76
4.6.2	Cauchy–Schwarz inequality	76
4.6.3	Equivalence between vectors and linear maps	78
4.7	Orthonormal Basis	80
4.7.1	Finding an orthonormal basis	80

4.7.2	Vector components in an orthonormal basis	81
4.7.3	Completeness of a basis	82
4.8	Hilbert Spaces	83
4.8.1	Sequences and their convergence	83
4.8.2	Banach space	84
4.8.3	Hilbert space	85
4.9	Tensors	86
5	Operators on Vector Spaces	88
5.1	Definition and Basic Properties	88
5.2	Matrices	89
5.3	Adjoint Operator	91
5.4	Some Special Kinds of Matrices	93
5.4.1	Hermitian matrices	93
5.4.2	Unitary matrices	94
5.5	Change of Basis	95
5.6	Invariants of Basis Change	99
5.6.1	Trace	99
5.6.2	Determinant	100
5.7	Eigenvalues and Eigenvectors	103
5.7.1	Definition	103
5.7.2	How to find eigenvalues and eigenvectors	103
5.7.3	Some general theorems regarding eigenvectors	105
5.7.4	About eigensystem of Hermitian matrices	109
5.7.5	About eigensystem of unitary matrices	111
5.8	Diagonalization of a Matrix	111
5.9	Normal Matrices	115
5.10	Matrices Which Are Not Normal	120
5.11	Antiunitary Operators	125
6	Infinite Dimensional Vector Spaces	126
6.1	Examples	126
6.2	Linear Independence and Basis	127
6.3	Inner Product and Norm	130
6.4	Orthogonal Polynomials	131
6.4.1	The procedure	132
6.4.2	Examples of polynomials	133
6.5	Operators	136
6.5.1	Types of operators	136
6.5.2	Eigenvalues and eigenvectors	137
6.5.3	Orthogonal polynomials as eigenfunctions	138
6.5.4	Bounded and unbounded operators	143

6.6	Self-adjoint Operators	145
6.6.1	Adjoint of an operator	145
6.6.2	Problem with self-adjointness	147
6.6.3	Self-adjoint extensions	149
C	GROUP THEORY	153
7	General Properties of Groups	155
7.1	Introduction	155
7.2	Relevance of Groups for Physics	156
7.3	Some Preliminary Properties of Groups	156
7.4	Group Composition Table	158
7.5	Representations of Groups	160
7.5.1	Definitions	160
7.5.2	An example	161
7.5.3	Determinants of representation matrices	164
7.5.4	Reducibility	164
7.5.5	Unitary representations	167
7.6	Representations from Other Representations	168
7.6.1	Complex conjugation	168
7.6.2	Kronecker products	169
7.7	Conjugacy Classes	171
7.8	Groups from Other Groups	172
7.8.1	Subgroups	172
7.8.2	Cosets and normal subgroups	175
7.8.3	Quotient groups	177
7.8.4	Direct product groups	178
7.8.5	Semidirect product groups	179
7.9	Simple Groups	181
7.10	Maps from Groups	182
7.10.1	Kernel of a homomorphism	182
7.10.2	Exact sequences of maps	185
8	Finite Groups	187
8.1	Groups with Fewer than Six Elements	187
8.1.1	Group with two elements	187
8.1.2	Group with three elements	188
8.1.3	Group with four elements	188
8.1.4	Group with five elements	189
8.2	Generators of Finite Groups	191
8.3	Presentation of Finite Groups	192
8.4	Subgroups of Finite Groups	195

8.5	Order of an Element	198
8.6	Finite Groups with Six or More Elements	199
8.6.1	Groups with six elements	199
8.6.2	Larger finite groups	202
8.7	Realization of the Groups	208
8.8	Permutation Groups	211
8.8.1	Notations for permutations	211
8.8.2	Transpositions and parity	214
8.8.3	Generators for permutation groups	216
8.8.4	Groups of even permutations	217
8.9	Conjugacy Classes	217
8.9.1	Interpretation	217
8.9.2	A few generalities about conjugacy classes	218
8.9.3	Conjugacy classes for the groups S_n	221
8.9.4	Conjugacy classes for the groups A_n	223
8.9.5	Conjugacy classes for the groups D_n	225
9	Representation of Finite Groups	227
9.1	Example: The S_3 Group	227
9.2	Representation by Unitary Matrices	228
9.3	Schur's Lemmas	231
9.4	The Great Orthogonality Theorem	235
9.5	Character of a Representation	237
9.5.1	Motivation and definition	237
9.5.2	Some characters are easy to find	238
9.5.3	Orthogonality of irrep characters	238
9.5.4	Basis-invariants other than characters	239
9.6	Reducible Representations	240
9.6.1	Reduction of a reducible representation	240
9.6.2	Test of reducibility	241
9.7	Regular Representation	242
9.7.1	Defining the representation	242
9.7.2	Decomposition of the representation into irreps	242
9.8	Orthogonality of Class Characters	243
9.9	Number and Dimensions of Irreps	246
9.10	Representation of Abelian Groups	250
9.11	Constructing the Character Table	251
9.11.1	From orthogonality relations to characters: the example of S_3	251
9.11.2	A few theorems to make the task easier	252
9.11.3	Simplification of the procedure: the example of S_4	257
9.11.4	Some general comments	259
9.11.5	Character tables of subgroups of S_n	261

9.12	Uses of the Character Table	267
9.12.1	Finding normal subgroups	267
9.12.2	Decomposition of representations under subgroups	268
9.12.3	Kronecker products of representations	271
9.13	Self-conjugate Representations	274
9.14	Representations of Product Groups	277
9.15	Induced Representations from Subgroups	277
10	Symmetries of Regular Geometrical Objects	281
10.1	Symmetries of Regular Polygons	281
10.1.1	Representations of D_n for odd n	281
10.1.2	Representations of D_n for even n	284
10.2	Symmetries of Polyhedrons	285
10.2.1	Enumerating polyhedrons	285
10.2.2	Useful formulas from 3-dimensional geometry	286
10.2.3	Symmetries of a tetrahedron	288
10.2.4	Symmetries of a cube and an octahedron	290
10.2.5	Symmetries of an icosahedron and a dodecahedron	294
10.3	Crystal Structures	299
10.3.1	Point group and space group	299
10.3.2	Enumeration of point groups	300
10.4	Coxeter Groups	308
11	Countably Infinite Groups	311
11.1	Definition and Examples	311
11.2	The Infinite Cyclic Group \mathbb{Z}	312
11.2.1	Presentation and representations	312
11.2.2	\mathbb{Z}^n and its subgroups	313
11.3	$SL(2, \mathbb{Z})$ and Modular Group	316
11.4	Braid Groups	321
12	General Properties of Lie Groups	326
12.1	Generators	326
12.2	Algebra	329
12.3	Subalgebra, Ideal, Simple Algebra	330
12.4	General Comments on Representations	334
12.5	Normalization of Generators	337
12.6	Properties of Structure Constants	339
12.7	Cartan–Killing Form	342
12.8	Differential Representation	345
12.9	Matrix Representations	346

12.10 Casimir Invariants	353
12.10.1 Defining a second-order invariant	353
12.10.2 Connection with normalization constants	354
12.10.3 $C_2^{(R)}$ for Kronecker product representations	355
12.10.4 Casimir invariant for direct product groups	356
12.10.5 Higher order Casimir invariants	356
13 Rotations and Translations	358
13.1 Translation Group and Its Representation	358
13.1.1 The group and the algebra	358
13.1.2 Differential representation	359
13.1.3 Unitary matrix representation	360
13.1.4 Non-unitary matrix representation	361
13.2 Rotation Algebra	362
13.2.1 Generators	362
13.2.2 Differential representation	363
13.2.3 Algebra	365
13.3 Some Properties of the Group	366
13.4 Matrix Representations of Rotation Algebra	369
13.4.1 The fundamental and the adjoint	369
13.4.2 New representations through Kronecker products	370
13.4.3 Spinor representations	372
13.5 Casimir Invariants for the Representations	379
13.6 States in the Matrix Representations	381
13.6.1 Basis states in a single irrep	381
13.6.2 States in Kronecker products of irreps	383
13.7 Representation of the Group Elements	385
13.7.1 Parametrizing rotation	385
13.7.2 Wigner matrices	388
13.7.3 Combining two rotations	390
13.8 The Group O(3)	391
14 Unitary Groups and Their Representations	393
14.1 Counting Parameters of U(N)	393
14.2 Representations of U(1)	395
14.3 Representations of SU(2)	396
14.4 Decomposition of SU(2) Irreps under U(1)	397
14.5 SU(N) Algebra and Its Representations	398
14.5.1 A few general characteristics	398
14.5.2 Fundamental representation and its complex conjugate	399
14.5.3 The algebra	402
14.5.4 Invariant tensors	403
14.5.5 Kronecker products of representations	404

14.6 Young Tableaux	406
14.7 Representation Matrices	409
14.8 Decomposition of Irreps under Subgroups	411
14.8.1 Decomposition for $SU(N) \supseteq SU(M)$	411
14.8.2 Decomposition for $SU(N) \supseteq SO(M) \times SU(N-M) \times U(1)$	413
14.8.3 Decomposition for $SU(MN) \supseteq SU(M) \times SU(N)$	414
14.8.4 Decomposition for $SU(N) \times SU(N) \supseteq SU(N)$	414
14.9 Casimir Invariants for $SU(N)$	415
14.9.1 The quadratic invariant	415
14.9.2 Higher order invariants	417
15 Orthogonal Groups and Their Representations	420
15.1 Definition of Orthogonal Groups	420
15.2 Lessons from the Determinant	421
15.3 Parameter Count and Fundamental Representation	422
15.3.1 Parameter count	422
15.3.2 Fundamental representation and algebra	423
15.4 Tensorial Representations	424
15.4.1 Invariant tensors	424
15.4.2 Representations from Kronecker products	424
15.5 Spinorial Representations	425
15.5.1 New invariants	425
15.5.2 Prescription for constructing the Γ -matrices	428
15.5.3 Non-uniqueness of the Γ -matrices	432
15.5.4 The matrix C	434
15.5.5 A few properties of the Γ -matrices	436
15.5.6 Basic spinor representation	441
15.5.7 Higher spinor representations	444
15.5.8 Kronecker products involving spinor representations	445
15.6 The Special Case of $SO(4)$	449
15.7 Decomposition of Irreps under Subgroups	450
15.7.1 Decomposition for $SO(N) \supseteq SO(M)$	450
15.7.2 Decomposition for $SO(N) \supseteq SO(M) \times SO(N-M)$	451
15.7.3 Decomposition for $SU(N) \supseteq SO(N)$	452
15.7.4 Decomposition for $SO(2M) \supseteq SU(M)$	453
15.8 Casimir Invariants	455
15.8.1 The second-order invariant	455
15.8.2 The higher order invariants	457
16 Parameter Space of Lie Groups	458
16.1 Parameter Space	458
16.1.1 Parameter space for $U(1)$	458
16.1.2 Parameter space for $SO(3)$	458

16.1.3	Parameter space for SU(2)	459
16.1.4	Parameter space for translation groups	460
16.2	Compactness	460
16.3	Haar Measure	463
16.3.1	Formulation	463
16.3.2	Examples	466
16.4	Compact Spaces and Unitary Representations	468
16.5	Orthogonality Relations	470
17	Representations of the Lorentz Group	472
17.1	Definition of the Lorentz Group	472
17.2	Fundamental Representation and Algebra	474
17.2.1	The generators	474
17.2.2	Fundamental representation	474
17.2.3	Algebra	476
17.2.4	Differential representation of the algebra	478
17.3	Tensor Representations of the Algebra	479
17.3.1	General comments	479
17.3.2	Scalar representation	481
17.3.3	Vector representation	481
17.3.4	Rank-2 tensor representations	483
17.4	Spinor Representations	485
17.4.1	Dirac matrices	485
17.4.2	Representation of Dirac matrices	487
17.4.3	Reducibility of the representation	488
17.4.4	Basic spinor representations	489
17.4.5	Spinors	490
17.4.6	Higher spinor representations	491
17.5	Casimir Invariants	494
17.6	Extended Lorentz Group	495
17.6.1	Group elements	495
17.6.2	Irreducible representations	496
17.7	Poincaré Algebra	497
17.7.1	The algebra including translations	497
17.7.2	Casimir invariants	499
17.7.3	Representations of the algebra	499
17.8	Lorentz Algebra in Other Dimensions	500
18	Roots and Weights	503
18.1	Rank of an Algebra	503
18.2	Root Vectors	504
18.3	Weight Vectors	509
18.4	Simple Roots	512

18.5 Dynkin Diagrams	517
18.6 Cartan Matrix	524
18.7 Weyl Group	525
18.8 Dynkin Index for Irreps	527
18.8.1 Finding all weights of an irrep	527
18.8.2 Dimension of an irrep	530
18.8.3 Complex conjugate of an irrep	531
18.8.4 Kronecker products of irreps	532
18.9 Subalgebras and Their Representations	533
18.9.1 Regular and special subalgebras	533
18.9.2 Finding maximal regular subalgebras	536
18.9.3 Finding maximal special subalgebras	538
18.9.4 Projection matrices	539
18.9.5 Decomposition of irreps in subalgebras	545
19 Some Other Groups and Algebras	548
19.1 Symplectic Groups	548
19.2 Exceptional Compact Groups	552
19.2.1 G_2	552
19.2.2 F_4	554
19.2.3 E_6	555
19.2.4 E_7 and E_8	558
19.3 Special Linear Groups	558
19.4 Euclidean Groups	562
19.4.1 Transformations and algebra	562
19.4.2 ISO(2)	562
19.4.3 ISO(3)	565
19.4.4 ISO(4)	566
19.5 Heisenberg–Weyl Group	566
19.5.1 The algebra	566
19.5.2 The group	569
19.5.3 Oscillator algebra	569
19.6 Finding Casimir Invariants	572
19.7 Conformal Group	577
19.8 Algebras with Infinite Number of Generators	582
19.8.1 Virasoro algebra	582
19.8.2 Kac–Moody algebra	587
19.8.3 Central extensions	587
D TOPOLOGY	595
20 Continuity of Functions	597
20.1 Continuity and Metric Spaces	597

20.2 Open Sets in Metric Spaces	598
20.3 Closed Sets in Metric Spaces	601
20.4 Redefining Continuity	604
21 Topological Spaces	607
21.1 Definition and Examples	607
21.1.1 Open sets	607
21.1.2 Associated definitions	609
21.2 Metrizable Topological Spaces	610
21.3 Continuous, Open and Closed Functions	612
21.4 New Topological Spaces from Old Ones	615
21.4.1 Product topology	615
21.4.2 Induced topology on a subset	617
21.4.3 Quotient topology	619
21.5 Homeomorphic Spaces	623
21.6 Connected Spaces	625
21.7 Compact Spaces	629
22 Homotopy Theory	631
22.1 Paths	631
22.2 Multiplication of Paths	632
22.3 Homotopy of Paths	634
22.4 Fundamental Group	638
22.5 Examples of Fundamental Groups	639
22.6 Finding Fundamental Groups by Triangulation	642
22.6.1 The algorithm	642
22.6.2 Examples	645
22.7 Parameter Spaces for Groups	649
22.8 Higher Homotopy Groups	650
23 Homology	655
23.1 Simplexes and Chains	655
23.2 Cycles and Boundaries	657
23.3 Homology Groups	659
23.3.1 Definition	659
23.3.2 Some general comments	660
23.3.3 Homology groups of D^2	661
23.3.4 Homology groups of S^2	664
23.4 Homology Groups from Δ -complexes	666
23.4.1 The basic idea	666
23.4.2 Homology groups of S^2 , once again	667

23.4.3 Homology groups of T^2	668
23.4.4 Homology groups of P^2	669
23.5 Betti Numbers and Torsion	671
E APPENDICES	675
A Books and Papers	677
B Answers to Selected Exercises	679
C Index	687