Relativistic Rockets

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Abstract

Two starships that uniformly accelerate in intergalactic space may be unable to exchange radio messages with one another, whilst two stationary starships in a uniform gravitational field can always do so. This apparent conflict with Einstein's principle of the equivalence between acceleration and gravity is resolved in the present note. July 16, 2012

1 A quandary

Suppose two starships, s_a and s_b, are idling, at a distance b light-years apart, at rest with respect to one another, far away from all gravitating masses. At time $t = 0$ both ships begin to move, s_a directly towards s_b, while s_b moves directly away from s_a. Both ships maintain a constant acceleration g, as measured in their instantaneous rest-frames, which means that the crews feel their normal weights throughout the voyage.

At time t the captain of s_a sends a radio signal to s_b. Will the captain of s_b receive the message? Not necessarily! If the signal is sent off at the beginning of the voyage, $t = 0$, then the signal from s_a will never arrive at s_b if $b > \frac{c^2}{g} \approx$ one light-year. If the signal is sent off at a considerably later time, the signal will fail to arrive even if the starships are initially much less than one light-year apart.

Consider next another configuration, in which the two starships are situated in a uniform gravitational field, and their reaction drives are working as before, but the strength of the field is just such as to compensate the acceleration, so that the starships remain forever at rest. Or we might imagine the ships to be nailed to a cosmic tree, with their drives shut down. The nails prevent the starships from falling into the gravity well, and the crew feel their accustomed weight. What is now the fate of a radio signal sent from s_a? Although the electromagnetic radiation will be influenced by the gravitational field, the signal will not fail to reach s_b in a finite time, no matter how far s_b is ahead of s_a.

There appears to be a difficulty, for did we not learn from Einstein that gravitation and acceleration are equivalent? Should not the two configurations we have just considered — the first, in which the starships move with a constant proper acceleration, g, and the second, in which they are at rest in an environment in which the acceleration of gravity is g — be two ways of looking at basically the same situation? How then can a radio signal fail to be exchanged between s_a and s_b in one configuration, and not in the other?

The purpose of this note is to resolve the apparent conflict. In a nutshell, general relativity does *not* tell us that a uniformly accelerating frame of reference is equivalent to a uniform gravitational field, but only that infinitesimal neighbourhoods of space-time points enjoy such equivalence. This important point was stressed in an illuminating paper of Desloge (1989). Configuration

1 is not the same as configuration 2 in empirically detectable ways, the difference between the fates of the radio messages being but one way. In Sect. 2 we analyze the accelerating space ships in free space, which can be done in the framework of special relativity; and in Sect. 3 we use general relativity to treat the problem of the stationary space ships in a uniform gravitational field. In Sect. 4 we look again at the accelerating space ships, but now in a general relativistic context. The concept of a rigid frame in general relativity is introduced in Sect. 5, and this throws new light on the resolution of the original quandary.

2 Two starships in intergalactic space

At time $t = 0$ spaceship s_a is at rest, and we take the origin of the x-axis to be its initial position. From this time onwards s_a accelerates, and, at some particular later time, suppose its velocity is v. At this time the instantaneous rest-frame of s_a is defined by the familiar Lorentz transformation:

$$\begin{aligned} d\overline{x} &= \gamma(dx - v\,dt) \\ d\overline{t} &= \gamma\,(dt - v\,dx)\;, \end{aligned} \tag{1}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}}\,.$$

The units are such that the speed of light is unity, $c = 1$.

2.1 Setting up the equations: It should be understood that $\overline{x}$ and $\overline{t}$ refer to an inertial frame with respect to which s_a has velocity 0 for just one instant. From (1) we see that

$$\frac{d\overline{x}}{dt} = \gamma\left(\frac{dx}{dt} - v\right) \tag{2}$$

$$\frac{d\overline{t}}{dt} = \gamma\left(1 - v\frac{dx}{dt}\right)\,, \tag{3}$$

and on dividing equation (2) by (3), we find

$$\frac{d\overline{x}}{d\overline{t}} = \frac{\frac{dx}{dt} - v}{1 - v\frac{dx}{dt}}\,, \tag{4}$$

the familiar formula for the relativistic composition of velocities. Next we differentiate (4) with respect to $\overline{t}$:

$$\frac{d^2\overline{x}}{d\overline{t}^2} = \frac{dt}{d\overline{t}}\left[\frac{\frac{d^2x}{dt^2}}{1 - v\frac{dx}{dt}} + \frac{(\frac{dx}{dt} - v)v\frac{d^2x}{dt^2}}{(1 - v\frac{dx}{dt})^2}\right]\,. \tag{5}$$

Note that v has been treated as a constant: it is the velocity that defines the instantaneous Lorentz transformation (1).

Now x, and its derivatives with respect to t, change with time, and at the particular time for which we wrote the Lorentz transformation (1), we know that $\frac{dx}{dt} = v$, at which point Eq.(3) reduces to

$$\frac{d\overline{t}}{dt} = \gamma\left(1 - v^2\right) = \gamma^{-1}\,. \tag{6}$$

Eq.(4) entails $\frac{d\overline{x}}{d\overline{t}} = 0$, which confirms the fact that, at time t the instantaneous rest-frame of s_a is indeed $(\overline{t}, \overline{x})$. Moreover, Eq.(5) reduces to

$$\frac{d^2\overline{x}}{d\overline{t}^2} \quad = \quad \gamma \, \frac{\frac{d^2x}{dt^2}}{1-v^2} = \gamma^3 \, \frac{d^2x}{dt^2} \, . \tag{7}$$

The second derivative on the left-hand side of this equation is the acceleration with respect to the instantaneous rest-frame of s_a, and this gives rise to the weight that the crew feels: it is to be set equal to a constant, say $\frac{d^2\overline{x}}{d\overline{t}^2} = g$. So

$$\frac{d^2x}{dt^2} = \frac{g}{\gamma^3} = g\left(1-v^2\right)^{\frac{3}{2}} \, .$$

We have set $c = 1$; and to simplify the formulae still further, we adjust the units such that $g = 1$, so

$$\frac{d^2x}{dt^2} = \left(1-v^2\right)^{\frac{3}{2}} \, . \tag{8}$$

It is easy to reinstate the units at the end: the unit of time is $\frac{c}{g}$, about a year; and the unit of distance is $\frac{c^2}{g}$, about a light-year.

2.2 Integrating the equations: Since

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx} \, ,$$

we can integrate (8) readily:

$$x = \int v dv \left(1-v^2\right)^{-\frac{3}{2}} = \left(1-v^2\right)^{-\frac{1}{2}} - 1 + x_0 \, , \tag{9}$$

the last term, x_0, being an integration constant. A little algebra produces

$$\frac{dx}{dt} = v = \frac{\sqrt{(1+x-x_0)^2-1}}{1+x-x_0} \, , \tag{10}$$

which can in turn be integrated:

$$t = \int dx \, \frac{1+x-x_0}{\sqrt{(1+x-x_0)^2-1}} = \sqrt{\left(1+x-x_0\right)^2-1} \, , \tag{11}$$

or equivalently,

$$x = \sqrt{1+t^2} - 1 + x_0 \, . \tag{12}$$

By differentiating x with respect to t, we obtain

$$v = \frac{dx}{dt} = \frac{t}{\sqrt{1+t^2}} \, , \tag{13}$$

and some more algebra yields

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \sqrt{1+t^2} \, . \tag{14}$$

Thus we can rewrite Eq.(12) succinctly in the form

$$x = \gamma - 1 + x_0 \, . \tag{15}$$

Note that, at time $t = 0$, $x = x_0$ and $v = 0$.

2.3 Light signals: The position of starship s_a at $t = 0$ is the origin, so this starship's trajectory is given by $x_a(t) = \gamma(t) - 1$. Suppose that the captain of s_a sends a light — or radio — signal to s_b at time t_0, as measured in the original inertial frame. The signal starts at the position $x_a(t_0)$ and proceeds in the direction of s_b, with velocity c. Suppose the light has reached the point $x_\lambda(t)$ at time t. Clearly

$$x_\lambda(t) = \gamma(t_0) - 1 + t - t_0 \,. \tag{16}$$

If the ship s_b starts a distance b in the lead, it will maintain this same distance in front of s_a, in the original inertial frame. The position of s_b at time t is in fact

$$x_b(t) = \gamma(t) - 1 + b \,. \tag{17}$$

To see whether the signal will overtake s_b, we must examine the sign of

$$\begin{aligned} x_b(t) - x_\lambda(t) &= [\gamma(t) - 1 + b] - [\gamma(t_0) - 1 + t - t_0] \\ &= b + [\gamma(t) - t] - [\gamma(t_0) - t_0] \,. \end{aligned} \tag{18}$$

Now

$$\gamma(t) - t = \frac{1}{\sqrt{1+t^2} + t} \,, \tag{19}$$

which decreases monotonically from 1 to 0 as t goes from 0 to ∞. So if

$$b > \gamma(t_0) - t_0 = \frac{1}{\sqrt{1+t_0^2} + t_0} \,, \tag{20}$$

then the signal will never be received.

With units reinstated, (20) reads

$$b > \frac{c^2}{g} \left[\sqrt{1 + \frac{g^2 t_0^2}{c^2}} + \frac{g t_0}{c} \right]^{-1} \,, \tag{21}$$

which reduces to $b > \frac{c^2}{g}$ in the case $t_0 = 0$.

3 Two starships in a uniform field

We will first calculate the trajectory of a freely moving material body in a uniform gravitational field, according to general relativity. The space-time is not Minkowskian, i.e. the relation between proper time, τ, and t and x is no longer given by $d\tau^2 = dt^2 - dx^2$. Consider the form

$$d\tau^2 = \alpha^2 \, dt^2 - dx^2 \,, \tag{22}$$

where α is some function of x only. At a particular value of x, we can define the local inertial frame by means of the transformation

$$d\bar{t} = \alpha \, dt \qquad\qquad d\bar{x} = dx \,, \tag{23}$$

so that (22) takes on the familiar Minkowski form, $d\tau^2 = d\bar{t}^2 - d\bar{x}^2$. However, according to general relativity, a freely moving body does not travel with constant velocity with respect to such a local inertial frame, rather it traces out a trajectory that is called a geodesic, defined by the following differential equation:

$$\frac{d^2 x}{d\tau^2} + \Gamma^1_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \,, \tag{24}$$

with summation convention over μ and ν. The Christoffel symbol is

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\, g^{\delta\lambda}\left(\frac{\partial g_{\nu\delta}}{\partial x^{\mu}} + \frac{\partial g_{\mu\delta}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\delta}}\right) .$$

Here the covariant metric tensor is defined through

$$d\tau^2 = g_{\mu\nu}\, dx^{\mu} dx^{\nu} ,$$

where x^0 is t, and x^1 is x. The contravariant tensor is the inverse, that is,

$$g^{\lambda\mu} g_{\mu\nu} = \delta^{\lambda}_{\nu} .$$

For our form (22), clearly $g_{00} = \alpha^2$ and $g_{11} = -1$, the nondiagonal terms being zero. It is straightforward to check that the only nonzero Christoffel symbol contributing to Eq.(24) is

$$\Gamma^{1}_{00} = \alpha \frac{d\alpha}{dx} ,$$

so the geodesic equation becomes

$$\frac{d^2x}{d\tau^2} + \alpha \frac{d\alpha}{dx}\left(\frac{dt}{d\tau}\right)^2 = 0 ,$$

which can be rewritten

$$\frac{dt}{d\tau}\frac{d}{dt}\left(\frac{dt}{d\tau}\frac{dx}{dt}\right) + \alpha\frac{d\alpha}{dx}\left(\frac{dt}{d\tau}\right)^2 = 0 . \tag{25}$$

From Eq.(22) we have

$$\frac{dt}{d\tau} = 1 \Bigg/ \sqrt{\alpha^2 - \left(\frac{dx}{dt}\right)^2} ,$$

and, using this in (25) we find, after some algebra, that

$$\frac{d^2x}{dt^2} - \frac{2}{\alpha}\frac{d\alpha}{dx}\left(\frac{dx}{dt}\right)^2 + \alpha\frac{d\alpha}{dx} = 0. \tag{26}$$

In order to gain some insight into the meaning of Eq.(26), note that

$$\frac{d}{dt}\frac{1}{\alpha}\frac{dx}{dt} = \frac{1}{\alpha}\frac{d^2x}{dt^2} - \frac{1}{\alpha^2}\frac{d\alpha}{dx}\left(\frac{dx}{dt}\right)^2 ,$$

so that Eq.(26) can be rewritten in the form

$$\frac{1}{\alpha}\frac{d}{dt}\frac{1}{\alpha}\frac{dx}{dt} - \frac{1}{\alpha}\frac{d\alpha}{dx}\left(\frac{1}{\alpha}\frac{dx}{dt}\right)^2 + \frac{1}{\alpha}\frac{d\alpha}{dx} = 0 . \tag{27}$$

In (23) we saw that $(d\bar{t}, d\bar{x}) = (\alpha dt, dx)$ corresponds to the local inertial system at x, so the term $\frac{1}{\alpha}\frac{d}{dt}\frac{1}{\alpha}\frac{dx}{dt}$ in (27) is the local acceleration of a falling body, while $\frac{1}{\alpha}\frac{dx}{dt}$ is its local velocity. Suppose that a body is momentarily held at rest at x, so $\frac{1}{\alpha}\frac{dx}{dt} = 0$ for an instant, and that the body is then released. According to (27), its initial local acceleration is equal to $-\frac{1}{\alpha}\frac{d\alpha}{dx}$, so this must be the effective gravitational field strength.

Having identified the gravitational field strength as $-\frac{1}{\alpha}\frac{d\alpha}{dx}$, we can now determine what α must be, in order that this field be static and uniform, i.e. the same at all values of t and x. Clearly

$$\alpha = \mathrm{e}^x \tag{28}$$

does the job, producing a uniform field of strength -1, that is, unit force per unit mass in the downwards direction.

The main purpose of the above analysis was to identify this gravitational field, and to require its uniformity. Let us now consider our starships s_a and s_b, in the second configuration, in which they are held at rest in the uniform gravitational field. We choose the position of s_a to be the origin, and s_b to be at a distance b, against the direction of the field, i.e. in the direction of positive x. Suppose the captain of s_a sends a radio or light signal towards s_b. How long will the light take to arrive?

Light follows a null geodesic, $d\tau = 0$, or $\alpha^2\,dt^2 - dx^2 = 0$, so for a positive velocity,

$$\frac{dx}{dt} = \alpha = \mathrm{e}^x\,,$$

and since the signal starts at $t = 0 = x$, its trajectory between the two starships is

$$x = -\log(1-t)\,,$$

and the time taken for the signal to arrive is

$$t_b = 1 - \mathrm{e}^{-b}\,.$$

Unlike the situation in the first configuration, in which the starships accelerate uniformly in intergalactic space, the signal always arrives after a finite time. Indeed, if the captain of s_b reflects the light signal back to s_a, the trajectory of this returning signal is

$$x = -\log(t - 1 + 2\,\mathrm{e}^{-b})\,;$$

and this means that it arrives at time $2(1-\mathrm{e}^{-b})$. Thus the 'light distance' of s_b from s_a is $1-\mathrm{e}^{-b}$, which is never more than 1 (light-year).

3.1 Freely falling starships As a variant on the above scenario, suppose the cosmic nails break at $t = 0$, and the two starships fall pell-mell into the gravitational abyss. This is effectively the situation envisaged by Desloge in his subsection V.D.

Suppose s_a and s_b are in free fall, so

$$\begin{aligned} x_a &= -\tfrac{1}{2}\log(1+t^2) \\ x_b &= -\tfrac{1}{2}\log(\mathrm{e}^{-2b}+t^2)\,. \end{aligned}$$

At $t = 0$, $x_a = 0$ and $x_b = b$, while $dx_a/dt = 0 = dx_b/dt$. Suppose a light signal is emitted from s_a at time t_1, and define v by

$$t_1 = \sinh v\,.$$

Let t_2 be the time that the light reaches s_b, where it is reflected, arriving back at s_a at t_3. We find

$$\begin{aligned} t_2 &= \mathrm{e}^{-b}\sinh(v+b) \\ t_3 &= \sinh(v+2b)\,. \end{aligned}$$

The proper time of s_a is given by

$$\begin{aligned} d\tau^2 &= \mathrm{e}^{2x}dt^2 - dx_a^2 \\ &= \left[\mathrm{e}^{2x} - \left(\frac{dx_a}{dt}\right)^2\right] dt^2 \\ &= \frac{dt^2}{(1+t^2)^2}\,, \end{aligned}$$

and so

$$\tau = \int \frac{dt}{1+t^2} = \tan^{-1} t\,,$$

if we require $\tau = 0$ when $t = 0$.

We have then, in terms of the proper times of s_a for emission and reception of the light,

$$t_1 = \tan\tau_1 = \sinh v \qquad\qquad t_3 = \tan\tau_3 = \sinh(v+2b)$$

Define, with Desloge,

$$T = \tfrac{1}{2}(\tau_3 + \tau_1) \qquad\qquad R = \tfrac{1}{2}(\tau_3 - \tau_1)$$

$$\tau_1 = T - R \qquad\qquad \tau_3 = T + R\,.$$

Hence

$$\tan(T-R) = \sinh v \qquad\qquad \tan(T+R) = \sinh(v+2b)\,.$$

After some trigonometric gymnastics we eliminate v and obtain

$$\sin R = \tanh b\,\cos T\,,$$

which is equivalent to, but simpler than Eq. 16 of Desloge. One sees that, as t_1 and t_3 tend to ∞, τ_1 and τ_3 tend to $\pi/2$, so T tends to $\pi/2$, thus $\cos T$ tends to 0, and so R also tends to 0, as Desloge says.

4 In the wake of a starship

We return now to configuration 1, in which the starships are accelerating uniformly in free space, with the aim of clarifying still further the difference between the two situations. Since no gravitating masses are present, the space-time is Minkowskian:

$$d\tau^2 = dt^2 - dx^2\,. \tag{29}$$

Consider the following nonlinear transformation of coordinates, due to Rindler — see in this connection Chapter 3 of 't Hooft (2003), Chapter 5 of Johnson (2007) and pages 25-27 and 103-105 of Narlikar (2010):

$$\begin{aligned} x' &= \sqrt{(1+x)^2 - t^2} - 1 \\ t' &= \tfrac{1}{2}\log\frac{1+x+t}{1+x-t}\,, \end{aligned} \tag{30}$$

with the inverse

$$\begin{aligned} x &= (1+x')\cosh t' - 1 \\ t &= (1+x')\sinh t'\,. \end{aligned} \tag{31}$$

In terms of these new coordinates, we find that

$$d\tau^2 = (1 + x')^2 dt'^2 - dx'^2 \,. \tag{32}$$

Note that this has the same form as Eq.(22), in terms of the primed coordinates, with

$$\alpha(x') = 1 + x' \,.$$

The general solution (12) of Eq.(8) that was obtained in Sect. 2 had the form $x = \sqrt{1 + t^2} - 1 + x_0$, where $x_0 = 0$ corresponds to the accelerated trajectory of starship s_a, and $x_0 = b$ to that of s_b. On making the Rindler transformation (31), we find the trajectory in terms of the primed variables to be

$$x' = \sqrt{1 + x_0^2 \sinh^2 t'} + x_0 \cosh t' - 1 \,. \tag{33}$$

For s_a this reduces to $x'_a = 0$ for all t', so the origin of Rindler space keeps precise pace with this starship. Starship s_b, on the other hand, follows the trajectory $x'_b = \sqrt{1 + b^2 \sinh^2 t'} + b \cosh t' - 1$, so it moves further and further away from s_a, as time t' passes.

Let us check, in Rindler coordinates, that the light signal from s_a will never reach s_b, if b is too large. In Rindler space, the null trajectory, $d\tau = 0$, traced out by light, is governed by

$$\frac{dx'}{dt'} = 1 + x' \,,$$

and if the signal is sent out at time t'_0, the relevant solution for the trajectory of the signal is

$$x'_\lambda = \mathrm{e}^{t' - t'_0} - 1 \,.$$

Now for very large t', $x'_b \approx b\,\mathrm{e}^{t'}$, and so $x'_b - x'_\lambda$ always remains negative if $b > \mathrm{e}^{-t'_0}$. One readily shows that this condition is equivalent to the inequality (20) that we obtained in Sect. 2, when working in the Minkowski frame.

5 Resolution of the quandary

We have seen that configurations 1 and 2 can both be described by coordinate systems of the form

$$d\tau^2 = \alpha^2 \, dt^2 - dx^2 \,,$$

in which starship s_a remains at the origin for all time. Such coordinate systems are called *rigid*, for the following reason. The trajectory of light obeys the differential equation $\frac{dx}{dt} = \alpha(x)$, and so the time taken for light to go from a point x_1 to another point x_2 is

$$T = \int_{x_1}^{x_2} \frac{dx}{\alpha(x)} \,.$$

If there is a mirror at x_2 that reflects the light back to x_1, the total time for the return light journey is $2T$. The significant point is that this depends only on x_1 and x_2, and not on time, i.e. not on when the light is emitted in the first place. So, as measured in the rigid frame, the 'light distances' between points remains fixed in time, whence the nomenclature.

The rigid spaces describing the two configurations are quite different — for the accelerating starships in intergalactic space, $\alpha(x) = 1 + x$, and for the starships nailed down in the uniform gravity well, $\alpha(x) = e^x$. As we have seen, in the first case a radio message from s_a can fail to arrive at s_b, but in the second case it will eventually arrive, no matter how far away the spaceships are from one another.

Acknowledgement

I would like to thank Peter Pijl for having brought this interesting problem to my notice.

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