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A TRANSPARENT DERIVATION OF THE RELATIVISTIC ROCKET EQUATION

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ABSTRACT

The rocket equation derived for the case of a rocket approaching relativistic speeds looks drastically different from the classical rocket equation. The classical rocket equation for the mass ratio of the rocket is a simple exponential in terms of the mission ΔV over the exhaust velocity w, while the relativistic rocket equation is a complicated function of the mission ΔV raised to the power of c/2w. Since they look so completely different, the student assumes the derivation must be different, and perhaps too complicated for the student to understand. In this paper, the relativistic rocket equation is derived side-by-side with the classical (non-relativistic) rocket equation. The resulting side-by-side derivation is found to be more transparent and more easily understood, since the student can see, at what point and how, the relativistic case deviates from the non-relativistic case, and why the end result for the relativistic mass ratio is a power of a complicated function of ΔV rather than a simple exponential in ΔV.

BACKGROUND

While preparing a chapter on Advanced Space Propulsion Concepts for a textbook due out this year[1], I included a derivation of the relativistic rocket equation taken from the usual primary[2] and secondary[3] sources. In an attempt to keep the derivation as simple and as short as possible for the page-limited chapter, I came up with a structure for the derivation where I derived the non-relativistic and relativistic rocket equations side-by-side. I found the resulting presentation to be more transparent than the normal derivation, since the student can see at what point and how the relativistic case deviates from the non-relativistic case, and why the end results of the two derivations look so different.

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RELATIVISTIC MECHANICS

Travel at significant fractions of the speed of light requires the use of equations for relativistic mechanics instead of classical mechanics[4]. In relativistic mechanics, even the fundamental properties of time, distance, and mass vary with the relative velocity of the observer. For example, the time T measured on a rocket moving at a velocity v is shorter than the time $\mathcal{T}$ measured by a stationary observer, by the relation:

$$T = (1 - v^2/c^2)^{1/2}\ \mathcal{T} \quad , \tag{1}$$

where c=299.8 Mm/s is the speed of light.

In the same manner, a distance D on a rocket moving at velocity v, as measured by a stationary observer, is shorter than the distance $\mathcal{D}$ measured when the rocket is at rest by:

$$D = (1 - v^2/c^2)^{1/2}\ \mathcal{D} \quad . \tag{2}$$

While the mass M of a moving rocket is greater than the rest mass $\mathcal{M}$ of the rocket by the relation:

$$M = \frac{\mathcal{M}}{(1 - v^2/c^2)^{1/2}} \quad . \tag{3}$$

Another result of relativistic mechanics is that nothing can go faster than the speed of light. If a rocket is moving at a velocity v, and shoots a projectile forward with a velocity w with respect to the rocket, then the velocity u of the projectile as seen by the stationary observer is not u=w+v, but instead is:

$$u = \frac{w + v}{1 + wv/c^2} \quad . \tag{4}$$

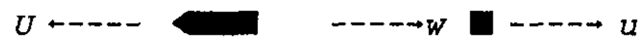
This equation always produces a velocity for u that is less than the speed of light.

DERIVATION OF THE RELATIVISTIC ROCKET EQUATION

The rocket equation for the mass ratio of a relativistic rocket[2] is quite different-looking than the classical mass ratio. The following derivation shows <u>why</u> it is so different.

In the classical derivation, a rocket with an initial mass M ejects an amount dm of reaction mass at an exhaust velocity with respect to the rocket of w (assumed to be a constant). In the center of mass of the system, the resultant velocity of the rocket is U and the velocity of the reaction mass is u. In the relativistic case, the masses are replaced with their relativistic equivalents given by Equation (3), which vary with the velocities U and u.

Classical Masses	M	dm
Relativistic Masses	$\dfrac{\mathcal{M}}{\sqrt{1-U^2/c^2}}$	$\dfrac{d\mathcal{M}}{\sqrt{1-u^2/c^2}}$



Below, the derivation of the classical rocket equation (left set of equations) is compared with the relativistic derivation (right set of equations). If doing this in a classroom, it would be useful to use two adjacent blackboards, and carry out the two derivations, step-by-step, side-by-side, stopping after each phase to comment on the similarities and the differences.

From the law of conservation of linear momentum we obtain the relation between the change in the total linear momentum d(MU) of the spacecraft due to the expulsion of the small increment of reaction mass dm at the velocity u:

$$d(MU) = u\cdot dm \qquad\qquad d\left[\frac{\mathcal{M}}{(1-U^2/c^2)^{1/2}}\cdot U\right] = u\cdot\frac{d\mathcal{M}}{(1-u^2/c^2)^{1/2}} \quad (5)$$

From the law of conservation of mass (mass-energy in the relativistic case) we obtain the relations:

$$dM = -\,dm \qquad\qquad d\left[\frac{\mathcal{M}}{(1-U^2/c^2)^{1/2}}\cdot c^2\right] = -\,\frac{d\mathcal{M}}{(1-u^2/c^2)^{1/2}}\cdot c^2 \quad (6)$$

From the law of addition of velocities, with the relativistic version of the law given by Equation (4), with v = -U, we obtain the relation:

$$u = w - U \qquad\qquad u = \frac{w - U}{1 - wU/c^2} \quad (7)$$

We now substitute into Equation (5) the relation given in Equation (7) for u and the relation given in Equation (6) for dm in the classical case and for $d\mathcal{M}/(1-u^2/c^2)^{1/2}$ in the relativistic case. In the relativistic case for Equation (6), c^2 is a constant, and can be taken out from under the derivative and cancelled. The result is an equation combining the three laws expressed in Equations (5), (6), and (7) in terms of the constant exhaust velocity w, the spacecraft mass M (rest mass $\mathcal{M}$ for the relativistic case), and the spacecraft velocity U.

$$d(MU) = (U-w)\,dM \qquad\qquad d\left[\frac{\mathcal{M}U}{(1-U^2/c^2)^{1/2}}\right] = \left(\frac{U-w}{1-wU/c^2}\right) d\left[\frac{\mathcal{M}}{(1-U^2/c^2)^{1/2}}\right] \quad (8)$$

The derivatives can now be expanded using the well known relation that d(ABC)=BCdA+ACdB+ABdC. To simplify calculations in the relativistic case, we will also use the fact that:

$$d\left[\frac{1}{(1-U^2/c^2)^{1/2}}\right] = \frac{UdU}{(c^2-U^2)\,(1-U^2/c^2)^{1/2}} \tag{9}$$

By expanding the derivatives in Equation (8), using Equation (9) in the relativistic equation, and multiplying the relativistic equation through by $(1-U^2/c^2)^{1/2}$ to eliminate a common factor, we obtain:

$$MdU+UdM = (U-w)\,dM \qquad \mathcal{M}dU+Ud\mathcal{M}+\frac{\mathcal{M}U^2dU}{c^2-U^2} = \left(\frac{U-w}{1-wU/c^2}\right)\left[d\mathcal{M}+\frac{\mathcal{M}UdU}{c^2-U^2}\right] \tag{10}$$

Combining terms in dU and dM ($d\mathcal{M}$ in the relativistic case), we obtain:

$$wdM = -MdU \qquad \left(U-\frac{U-w}{1-wU/c^2}\right)d\mathcal{M} = -\left[\mathcal{M}+\frac{\mathcal{M}U}{c^2-U^2}\left(U-\frac{U-w}{1-wU/c^2}\right)\right]dU \tag{11}$$

Amazingly enough, the complicated expressions in the relativistic case reduce considerably, and Equation (11) simplifies to:

$$\frac{dM}{M} = -\frac{dU}{w} \qquad \frac{d\mathcal{M}}{\mathcal{M}} = -\frac{dU}{w(1-U^2/c^2)} \tag{12}$$

With these equations side-by-side on the blackboard, the student can now see, that although the calculations for the relativistic case have been complicated, the two derivations have produced nearly the same result so far. The next step is to integrate Equation 12. Since the relativistic equation has the velocity U appearing in the denominator, the integration of the relativistic version of the equation produces a more complicated expression than the integration of the classical version. [The integral for the relativistic equation can be found in standard mathematical handbooks[5], or can be checked by simply differentiating the answer given in Equation (13) using the fact that d(ln x)=dx/x.] Integrating Equation(12) and setting the integration constant equal to the initial mass M_i or $\mathcal{M}_i$ of the rocket, we obtain:

$$\ln\frac{M}{M_i} = -\frac{U}{w} \qquad \ln\frac{\mathcal{M}}{\mathcal{M}_i} = -\frac{c}{2w}\ln\left[\frac{1+U/c}{1-U/c}\right] \tag{13}$$

or alternatively:

$$M = M_i\, e^{-\frac{U}{w}} \qquad \mathcal{M} = \mathcal{M}_i\left[\frac{1+U/c}{1-U/c}\right]^{-\frac{c}{2w}} \tag{14}$$

It is right here, in this integration step, that the student sees how and why the two derivations start to look significantly different from each other. This is also the time to have the student plot these two separate equations on the same piece of graph paper for increasing values of U. The students will find that although the two equations look quite different, the numerical results they give start to diverge significantly only after U exceeds c/4.

If the initial mass of the rocket is M_i when the rocket is at rest and U=0, and the final mass of the rocket is M_f when the rocket has reached the mission velocity ΔV, then the rocket mass ratio $R=M_i/M_f$ obtained from using these boundary conditions on Equation (14) is:

$$R = e^{\frac{\Delta V}{w}} \qquad\qquad R = \left[\frac{1 + \Delta V/c}{1 - \Delta V/c}\right]^{\frac{c}{2w}} \tag{15}$$

These are the classical and relativistic rocket equations. For the relativistic case, there is a maximum exhaust velocity for the reaction mass that is given by:[2]

$$w = [e(2 - e)]^{1/2}\, c \quad , \tag{16}$$

where e is the fuel mass fraction converted into kinetic energy of the reaction mass. I was not able to improve on that derivation from a presentation point of view.

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[5] CRC Handbook of Mathematical Science, 5th Edition (1975); Integral #61 on page 391.