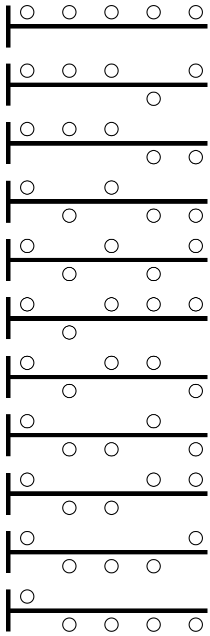

Initial position	10101
After 1st step	10110
" 2nd "	10111
" 3rd "	11000
" 4th "	11001
" 5th "	11010
" 6th "	11011
" 7th "	11100
" 8th "	11101
" 9th "	11110
" 10th "	11111

necessitate no less than 768614, 336404, 564650 steps, and would require nearly 55000, 000000 years work—assuming of course that no mistakes were made.

THE EIGHT QUEENS PROBLEM*. The determination of the number of ways in which eight queens can be placed on a chess-board—or more generally, in which n queens can be placed on a board of n^2 cells—so that no queen can take any other was proposed originally by Nauck in 1850.

In 1874 Dr S. Günther† suggested a method of solution by means of determinants. For, if each symbol represents the corresponding cell of the board, the possible solutions for a board of n^2 cells are given by

* On the history of this problem see W. Ahrens, *Mathematische Unterhaltungen und Spiele*, Leipzig, 1901, chap. IX—a work issued subsequent to the third edition of this book.

† Grunert's *Archiv der Mathematik und Physik*, 1874, vol. LVI pp. 281–292.

those terms, if any, of the determinant

$$\begin{vmatrix} a_1 & b_2 & c_3 & d_4 & \cdots\cdots & \\ \beta_2 & a_3 & b_4 & c_5 & \cdots\cdots & \\ \gamma_3 & \beta_4 & a_5 & b_6 & \cdots\cdots & \\ \delta_4 & \gamma_5 & \beta_6 & a_7 & \cdots\cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & a_{2n-3} & b_{2n-2} \\ \cdots & \cdots & \cdots & \cdots & \beta_{2n-2} & a_{2n-1} \end{vmatrix}$$

in which no letter and no suffix appears more than once.

The reason is obvious. Every term in a determinant contains one and only one element out of every row and out of every column: hence any term will indicate a position on the board in which the queens cannot take one another by moves rook-wise. Again in the above determinant the letters and suffixes are so arranged that all the same letters and all the same suffixes lie along bishop's paths: hence, if we retain only those terms in each of which all the letters and all the suffixes are different, they will denote positions in which the queens cannot take one another by moves bishop-wise. It is clear that the signs of the terms are immaterial.

In the case of an ordinary chess-board the determinant is of the 8th order, and therefore contains 8!, that is, 40320 terms, so that it would be out of the question to use this method for the usual chess-board of 64 cells or for a board of larger size unless some way of picking out the required terms could be discovered.

A way of effecting this was suggested by Dr J.W.L. Glaisher* in 1874, and as far as I am aware the theory remains as he left it. He showed that if all the solutions of n queens on a board of n^2 cells were known, then all the solutions of a certain type for $n+1$ queens on a board of $(n+1)^2$ cells could be deduced, and that all the other solutions of $n+1$ queens on a board of $(n+1)^2$ cells could be obtained without difficulty. The method will be sufficiently illustrated by one instance of its application.

It is easily seen that there are no solutions when $n = 2$ and $n = 3$. If $n = 4$ there are two terms in the determinant which give solutions, namely, $b_2c_5\gamma_3\beta_6$ and $c_3\beta_2b_6\gamma_5$. To find the solutions when $n = 5$,

* *Philosophical Magazine*, London, December, 1874, series 4, vol. XLVIII, pp. 457–467.

Glaisher proceeded thus. In this case Günther's determinant is

$$\begin{vmatrix} a_1 & b_2 & c_3 & d_4 & e_5 \\ \beta_2 & a_3 & b_4 & c_5 & d_6 \\ \gamma_3 & \beta_4 & a_5 & b_6 & c_7 \\ \delta_4 & \gamma_5 & \beta_6 & a_7 & b_8 \\ \varepsilon_5 & \delta_6 & \gamma_7 & \beta_8 & a_9 \end{vmatrix}$$

To obtain those solutions (if any) which involve a_9 it is sufficient to append a_9 to such of the solutions for a board of 16 cells as do not involve a. As neither of those given above involves an a we thus get two solutions, namely, $b_2c_5\gamma_3\beta_6a_9$ and $c_3\beta_2b_6\gamma_5a_9$. The solutions which involve a_1, e_5 and ε_5 can be written down by symmetry. The eight solutions thus obtained are all distinct; we may call them of the first type.

The above are the only solutions which can involve elements in the corner squares of the determinant. Hence the remaining solutions are obtainable from the determinant

$$\begin{vmatrix} 0 & b_2 & c_3 & d_4 & 0 \\ \beta_2 & a_3 & b_4 & c_5 & d_6 \\ \gamma_3 & \beta_4 & a_5 & b_6 & c_7 \\ \delta_4 & \gamma_5 & \beta_6 & a_7 & b_8 \\ 0 & \delta_6 & \gamma_7 & \beta_8 & 0 \end{vmatrix}$$

If, in this, we take the minor of b_2 and in it replace by zero every term involving the letter b or the suffix 2 we shall get all solutions involving b_2. But in this case the minor at once reduces to $d_6a_5\delta_4\beta_8$. We thus get one solution, namely, $b_2d_6a5\delta_4\beta_8$. The solutions which involve β_2, δ_4, δ_6, β_8, b_8, d_6, and d_4 can be obtained by symmetry. Of these eight solutions it is easily seen that only two are distinct: these may be called solutions of the second type.

Similarly the remaining solutions must be obtained from the determinant

$$\begin{vmatrix} 0 & 0 & c_3 & 0 & 0 \\ 0 & a_3 & b_4 & c_5 & 0 \\ \gamma_3 & \beta_4 & a_5 & b_6 & c_7 \\ 0 & \gamma_5 & \beta_6 & a_7 & 0 \\ 0 & 0 & \gamma_7 & 0 & 0 \end{vmatrix}$$

If, in this, we take the minor of c_3, and in it replace by zero every term involving the letter c or the suffix 3, we shall get all the solutions which involve c_3. But in this case the minor vanishes. Hence there is no solution involving c_3, and therefore by symmetry no solutions which involve γ_3, γ_7, or c_3. Had there been any solutions involving the third element in the first or last row or column of the determinant we should have described them as of the third type.

Thus in all there are ten and only ten solutions, namely, eight of the first type, two of the second type, and none of the third type.

Similarly, if $n = 6$, we obtain no solutions of the first type, four solutions of the second type, and no solutions of the third type; that is, four solutions in all. If $n = 7$, we obtain sixteen solutions of the first type, twenty-four solutions of the second type, no solutions of the third type, and no solutions of the fourth type; that is, forty solutions in all. If $n = 8$, we obtain sixteen solutions of the first type, fifty-six solutions of the second type, and twenty solutions of the third type, that is, ninety-two solutions in all.

It will be noticed that all the solutions of one type are not always distinct. In general, from any solution seven others can be obtained at once. Of these eight solutions, four consist of the initial or fundamental solution and the three similar ones obtained by turning the board through one, two, or three right angles; the other four are the reflexions of these in a mirror: but in any particular case it may happen that the reflexions reproduce the originals, or that a rotation through one or two right angles makes no difference. Thus on boards of 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2 cells there are respectively 1, 2, 1, 6, 12, 46, 92 fundamental solutions; while altogether there are respectively 2, 10, 4, 40, 92, 352, 724 solutions.

The following collection of fundamental solutions may interest the reader. The positions on the board of the queens are indicated by digits: the first digit represents the number of the cell occupied by the queen in the first column reckoned from one end of the column, the second digit the number in the second column, and so on. Thus on a board of 4^2 cells the solution 3142 means that one queen is on the 3rd square of the first column, one on the 1st square of the second column, one on the 4th square of the third column, and one on the 2nd square of the fourth column. If a fundamental solution gives rise to only four solutions the number which indicates it is placed in curved brackets, ( ); if it gives rise to only two solutions the number which indicates it is placed in square brackets, [ ]; the other fundamental solutions give rise to eight solutions each.

On a board of 4^2 cells there is 1 fundamental solution: namely, [3142].

On a board of 5^2 cells there are 2 fundamental solutions: namely, 14253, [25314].

On a board of 6^2 cells there is 1 fundamental solution: namely, (246135).

On a board of 7^2 cells there are 6 fundamental solutions: namely, 1357246, 3572461, (5724613), 4613572, 3162574, (2574136).

On a board of 8^2 cells there are 12 fundamental solutions: namely, 25713864, 57138642, 71386425, 82417536, 68241753, 36824175, 64713528, 36814752, 36815724, 72418536, 26831475, (64718253). The arrangement in this order is due to Mr Oram. It will be noticed that the 10th, 11th, and 12th solutions somewhat resemble the 4th, 6th, and 7th respectively. The 6th solution is the only one in which no three queens are in a straight line.

On a board of 9^2 cells there are 46 fundamental solutions; one of them is 248396157. On a board of 10^2 cells there are 92 fundamental solutions; these were given by Dr A. Pein*; one of them is $2468t13579$, where t stands for ten. On a board of 11^2 cells there are 341 fundamental solutions; these have been given by Dr T.B. Sprague†: one of them is $15926t37e48$. I may add that for a board of n^2 cells there is always a symmetrical solution of the form $246 \ldots n135 \ldots (n-1)$, when $n = 6m$ or $n = 6m + 4$, Also Mr Oram has shown that for a board of n^2 cells, when n is a prime, cyclical arrangements of the n natural numbers, other than in their natural order, will give solutions; see, for instance, the solution quoted above.

The puzzle in the form of a board of 36 squares is sold in the streets of London for a penny, a small wooden board being ruled in the manner shown in the diagram and having holes drilled in it at the points marked by dots. The object is to put six pins into the holes so that no two are connected by a straight line.

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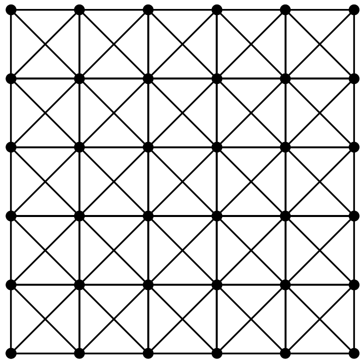
* *Aufstellung von n Königinnen auf einem Schachbrett von n^2 Feldern*, Leipzig, 1889.

† *Proceedings of the Edinburgh Mathematical Society*, vol. XVII, 1898–9, pp. 43–68.



Other Problems with Queens. Captain Turton called my attention to two other problems of a somewhat analogous character, neither of which, as far as I know, has been published elsewhere, or solved otherwise than empirically.

The first of these is to place eight queens on a chess-board so as to command the fewest possible squares. Thus, if queens are placed on cells 1 and 2 of the second column, on cell 2 of the sixth column, on cells 1, 3, and 7 of the seventh column, and on cells 2 and 7 of the eighth column, eleven cells on the board will not be in check; the same number can be obtained by other arrangements. Is it possible to place the eight queens so as to leave more than eleven cells out of check? I have never succeeded in doing so, nor in showing that it is impossible to do it.

The other problem is to place m queens (m being less than 5) on a chess-board so as to command as many cells as possible. For instance, four queens can be placed in several ways on the board so as to command 58 cells besides those on which the queens stand, thus leaving only 2 cells which are not commanded: *ex. gr.* this is effected if the queens are placed on cell 5 of the third column, cell 1 of the fourth column, cell 6 of the seventh column, and cell 2 of the eighth column; or on cell 1 of the first column, cell 7 of the third column, cell 3 of the fifth column, and cell 5 of the seventh column. A similar problem is to determine the minimum number and the position of queens which can be placed on a board of n^2 cells so as to occupy or command every cell. It would seem that, even with the additional restriction that no queen shall be able to take any other queen, there are no less than ninety-one typical solutions in which five queens can be placed on a chess-board so as to command every cell*.

Extension to other Chess Pieces. Analogous problems may be proposed with other chess-pieces. For instance, questions as to the maximum number of knights which can be placed on a board of n^2 cells so that no knight can take any other, and the minimum number of knights which can be placed on it so as to occupy or command every cell have been propounded†.

Similar problems have also been proposed for k kings placed on a chess-board of n^2 cells‡. It has been asserted that, if $k = 2$, the number of ways in which two kings can be placed on a board so that they may not occupy adjacent squares is $\frac{1}{2}(n-1)(n-2)(n^2+3n-2)$. Similarly, if $k = 3$, the number of ways in which three kings can be placed on a board so that no two of them occupy adjacent squares is said to be $\frac{1}{6}(n-1)(n-2)(n^4+3n^3-20n^2-30n+132)$.

THE FIFTEEN SCHOOL-GIRLS PROBLEM. This problem—which was first enunciated by Mr T.P. Kirkman, and is sometimes known as *Kirkman's problem*§—consists in arranging fifteen things in different sets of triplets. It is usually presented in the form that a school-mistress was in the habit of taking her girls for a daily walk.

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* *L'Intermédiaire des mathématiciens*, Paris, 1901, vol. VIII, p. 88.

† *Ibid.*, March, 1896, vol. III, p. 58; 1897, vol. IV, p. 15, 254; and 1898, vol. V, p. 87.

‡ *Ibid.*, June, 1901, p. 140.

§ It was published first in the *Lady's and Gentleman's Diary* for 1850, p. 48, and has been the subject of numerous memoirs. Among these I may single out the papers by A. Cayley in the *Philosophical Magazine*, July, 1850, series 3, vol. XXXVII, pp. 50–53; by T.P. Kirkman in the *Cambridge and Dublin Mathematical Journal*, 1850, vol. V, p. 260; by R.R. Anstice, *Ibid.*, 1852, vol. VII, pp. 279–292; by B. Pierce, *Gould's Journal*, Cambridge, U.S., 1860, vol. VI, pp. 169–174; by T.P. Kirkman, *Philosophical Magazine*, March, 1862, series 4, vol. XXIII, pp. 198–204; by W.S.B. Woolhouse in the *Lady's Diary* for 1862, pp. 84–88, and for 1863, pp. 79–90, and in the *Educational Times Reprints*, 1867, vol. VIII, pp. 76–83; by J. Power in the *Quarterly Journal of Mathematics*, 1867, vol. VIII, pp. 236–251; by A.H. Frost, *Ibid.*, 1871, vol. XI, pp. 26–37; by E. Carpmael in the *Proceedings of the London Mathematical Society*, 1881, vol. XII, pp. 148–156; by Lucas in his *Récréations*, vol. II, part vi; by A.C. Dixon in the *Messenger of Mathematics*, Cambridge, October, 1893, vol. XXIII, pp. 88–89; and by W. Burnside, *Ibid.*, 1894, vol. XXIII, pp. 137–143. It has also, since the issue of my third edition, been discussed by W. Ahrens in his *Mathematische Unterhaltungen und Spiele*, Leipzig, 1901, chapter xiv.