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SOME PROPERTIES OF FIBONACCI NUMBERS

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Abstract. In this paper we study the Fibonacci numbers and derive some interesting properties and recurrence relations. We also define period of a Fibonacci sequence modulo an integer, m and derive certain interesting properties related to them.

1. Preliminaries

We begin with the following famous results without proof.

Lemma 1.1 (Euclid)**.** *If $ab \equiv 0 \pmod p$ with a, b two integers and p a prime, then either $p|a$ or $p|b$.*

Remark 1.2. *In particular, if $gcd(a, b) = 1$, p divides only one of the numbers a, b.*

Theorem 1.3 (Fermat's Little Theorem)**.** *If p is a prime and $n \in \mathbb{N}$ relatively prime to p, then $n^{p-1} \equiv 1 \pmod p$.*

The following is an easy exercise from [1].

Lemma 1.4. $(5/p) = 1$ *if and only if* $p \equiv 1, 9, 11,$ *or* $19 \pmod{20}$.

Remark 1.5. *Clearly* $5k + 2 \not\equiv 1, 9, 11, 19 \pmod{20}$.

Theorem 1.6.

$$(5/5k + 2) = -1.$$

Theorem 1.7 (Euler)**.** *Let p be an odd prime and $\gcd(a, p) = 1$. Then a is a quadratic nonresidue of p if $a^{\frac{p-1}{2}} \equiv -1 \pmod p$.*

For proofs the reader is suggested to see [1] or [2].

Theorem 1.8. *If $x^2 \equiv 1 \pmod p$ with p a prime, then either $x \equiv 1 \pmod p$ or $x \equiv p - 1 \pmod p$.*

Proof. If $x^2 \equiv 1 \pmod p$ with p a prime, then we have

$$x^2 - 1 \equiv 0 \pmod p$$

$$(x - 1)(x + 1) \equiv 0 \pmod p$$

$x - 1 \equiv 0 \pmod p$ or $x + 1 \equiv 0 \pmod p$. It is equivalent to say that $x \equiv 1 \pmod p$ or $x \equiv -1 \equiv p - 1 \pmod p$. □

Let p a prime number such that $p = 5k + 2$ with k an odd positive integer. From Fermat's Little Theorem we have

$$\left(5^{\frac{5k+1}{2}}\right)^2 \equiv 1 \pmod{5k + 2}.$$

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From Theorem 1.8, we have either $5^{\frac{5k+1}{2}} \equiv 1 \pmod{5k+2}$ or $5^{\frac{5k+1}{2}} \equiv 5k+1 \pmod{5k+2}$. Moreover, we can observe that

$$5(2k+1) \equiv 1 \pmod{5k+2}.$$

Definition 1.9. *Let p be an odd prime and let $\gcd(a,p) = 1$. The Legendre symbol (a/p) is defined to be equal to 1 if a is a quadratic residue of p and is equal to -1 is a is a quadratic non residue of p.*

Theorem 1.10.

$$5^{\frac{5k+1}{2}} \equiv 5k+1 \pmod{5k+2}$$

where $5k+2$ is a prime.

The proof of Theorem 1.10 follows very easily from Lemma 1.4, Remark 1.5 and Theorems 1.7 and 1.6.

We fix the notation $[[1,n]] = \{1,2,\dots,n\}$ throughout the rest of the paper. We now have the following properties.

Property 1.11.

$$\binom{5k+1}{2l+1} \equiv 5k+1 \pmod{5k+2},$$

with $l \in [[0, \lfloor \frac{5k}{2} \rfloor]]$ and $5k+2$ is a prime.

Proof. Notice that for $l = 0$ the property is obviously true.

We also have

$$\binom{5k+1}{2l+1} = \frac{(5k+1)5k(5k-1)\dots(5k-2l+1)}{(2l+1)!}.$$

Or,

$$\begin{aligned} 5k &\equiv -2 \pmod{5k+2}, \\ 5k-1 &\equiv -3 \pmod{5k+2}, \\ &\vdots \\ 5k-2l+1 &\equiv -(2l+1) \pmod{5k+2}. \end{aligned}$$

Multiplying these congruences we get

$$5k(5k-1)\dots(5k-2l+1) \equiv (2l+1)! \pmod{5k+2}.$$

Therfore

$$(2l+1)!\binom{5k+1}{2l+1} \equiv (5k+1)(2l+1)! \pmod{5k+2}.$$

Since $(2l+1)!$ and $5k+2$ are relatively prime, we obtain

$$\binom{5k+1}{2l+1} \equiv 5k+1 \pmod{5k+2}.$$

□

Property 1.12.

$$\binom{5k}{2l+1} \equiv 5k-2l \equiv -2(l+1) \pmod{5k+2}$$

with $l \in [[0, \lfloor \frac{5k}{2} \rfloor]]$ and $5k+2$ is a prime.

Proof. Notice that for $l = 0$ the property is obviously true.

We have
$$\binom{5k}{2l+1} = \frac{5k(5k-1)\dots(5k-2l+1)(5k-2l)}{(2l+1)!}.$$
Or
$$\begin{gathered} 5k \equiv -2 \pmod{5k+2}, \\ 5k-1 \equiv -3 \pmod{5k+2}, \\ \vdots \\ 5k-2l+1 \equiv -(2l+1) \pmod{5k+2}. \end{gathered}$$
Multiplying these congruences we get
$$5k(5k-1)\dots(5k-2l+1) \equiv (2l+1)! \pmod{5k+2}.$$
Therfore
$$(2l+1)!\binom{5k}{2l+1} \equiv (2l+1)!(5k-2l) \pmod{5k+2}.$$
Since $(2l+1)!$ and $5k+2$ are relatively prime, we obtain
$$\binom{5k+1}{2l+1} \equiv 5k-2l \equiv 5k+2-2-2l \equiv -2(l+1) \pmod{5k+2}.$$
□

2. Formulas for the Fibonacci numbers

The Fibonacci sequence (F_n) is defined by $F_0 = 0, F_1 = 1, F_{n+2} = F_n + F_{n+1}$.

The first Fibonacci numbers are

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13,\ F_8 = 21,\ F_9 = 34,\ F_{10} = 55,$

$F_{11} = 89,\ F_{12} = 144,\ F_{13} = 233,\ F_{14} = 377,\ F_{15} = 610,\ F_{16} = 987,\ F_{17} = 1597,\ F_{18} = 2584, \dots$

From the definition of the Fibonacci sequences it can be established the formula for the *nth* Fibonacci number
$$F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$
where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

From binomial theorem, we have for $a \neq 0$ and $n \in \mathbb{N}$
$$(a+b)^n - (a-b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k (1-(-1)^k) = 2 \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} a^{n-(2l+1)} b^{2l+1},$$
$$(a+b)^n - (a-b)^n = 2a^n \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} \left(\frac{b}{a}\right)^{2l+1}. \tag{2.1}$$
We set
$$\begin{gathered} a+b = \varphi = \tfrac{1+\sqrt{5}}{2}, \\ a-b = 1-\varphi = \tfrac{1-\sqrt{5}}{2}. \end{gathered}$$
So
$$a = \frac{1}{2},\ b = \frac{2\varphi-1}{2} = \frac{\sqrt{5}}{2}.$$

Thus

$$\frac{b}{a} = \sqrt{5}.$$

We get from (2.1)

$$\varphi^n - (1-\varphi)^n = \frac{\sqrt{5}}{2^{n-1}} \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} 5^l.$$

Thus we have

Theorem 2.1.

$$F_n = \frac{1}{2^{n-1}} \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} 5^l.$$

For instance,

$$n = 1: \ F_1 = \frac{1}{2^0} \sum_{l=0}^{0} \binom{1}{2l+1} 5^l = 1,$$

$$n = 2: \ F_2 = \frac{1}{2} \sum_{l=0}^{0} \binom{2}{2l+1} 5^l = 1,$$

$$n = 3: \ F_3 = \frac{1}{2^2} \sum_{l=0}^{1} \binom{3}{2l+1} 5^l = \frac{1}{4} \left\{ \binom{3}{1} + \binom{3}{3} \times 5 \right\} = \frac{1}{4}(3+5) = \frac{8}{4} = 2,$$

$$n = 4: \ F_4 = \frac{1}{2^3} \sum_{l=0}^{1} \binom{4}{2l+1} 5^l = \frac{1}{8} \left\{ \binom{4}{1} + \binom{4}{3} \times 5 \right\} = \frac{1}{8}(4+20) = \frac{24}{8} = 3,$$

$$n = 5: \ F_5 = \frac{1}{2^4} \sum_{l=0}^{2} \binom{5}{2l+1} 5^l = \frac{1}{16} \left\{ \binom{5}{1} + \binom{5}{3} \times 5 + \binom{5}{5} \times 5^2 \right\} = \frac{1}{16}(5+50+25) = \frac{80}{16} = 5,$$

$$n = 6: \ F_6 = \frac{1}{2^5} \sum_{l=0}^{2} \binom{6}{2l+1} 5^l = \frac{1}{32} \left\{ \binom{6}{1} + \binom{6}{3} \times 5 + \binom{6}{5} \times 5^2 \right\} = \frac{1}{32}(6+100+150) = \frac{256}{32} = 8.$$

Property 2.2.

$$F_{k+2} = 1 + \sum_{i=1}^{k} F_i.$$

Proof. We have

$$\begin{gathered} F_0 = 0, \\ F_1 = 1, \\ F_2 = F_0 + F_1, \\ F_3 = F_1 + F_2, \\ \vdots \\ F_{k+2} = F_k + F_{k+1}. \end{gathered}$$

Adding these equalities we get

$$F_{k+2} = 1 + F_1 + F_2 + \ldots + F_k = 1 + \sum_{i=1}^{k} F_i.$$

□

Theorem 2.3.

$$F_{k+l} = F_l F_{k+1} + F_{l-1} F_k$$

with $k \in \mathbb{N}$ *and* $l \geq 2$. *(*$l \geq 2$*).*

Proof. We prove the result by induction on l.

From the definition of the Fibonacci sequence, we konw that $F_{k+2} = F_{k+1} + F_k = F_2 F_{k+1} + F_1 F_k$ since $F_1 = F_2 = 1$.

Let assume that: $F_{k+i} = F_i F_{k+1} + F_{i-1} F_k$ with $i = 2, 3, \ldots, l-1, l$.

Then, we have

$$F_{k+l+1} = F_{k+l} + F_{k+l-1}.$$

From our assumption, we have

$$F_{k+l} = F_l F_{k+1} + F_{l-1} F_k$$

and

$$F_{k+l-1} = F_{l-1} F_{k+1} + F_{l-2} F_k.$$

Thus we get

$$F_{k+l+1} = F_l F_{k+1} + F_{l-1} F_k + F_{l-1} F_{k+1} + F_{l-2} F_k,$$
$$F_{k+l+1} = (F_l + F_{l-1}) F_{k+1} + (F_{l-1} + F_{l-2}) F_k.$$

Using the recurrence relation of the Fibonacci sequence, we obtain

$$F_{k+l+1} = F_{l+1} F_{k+1} + F_l F_k.$$

□

3. Properties of the Fibonacci numbers

Property 3.1. $F_n \equiv 0 \pmod 2$ *if and only if* $n \equiv 0 \pmod 3$.

Proof. We prove by induction that for $k \in \mathbb{N}$ we have

$$F_{3k} \equiv 0 \pmod 2$$

We have $F_0 = 0 \equiv 0 \pmod 2$.

We assume that $F_{3k} \equiv 0 \pmod 2$.

From Theorem 2.3 we have

$$F_{3(k+1)} = F_{3k+3} = F_3 F_{3k+1} + F_2 F_{3k}.$$

Since $F_3 = 2 \equiv 0 \pmod 2$ and we assumed that $F_{3k} \equiv 0 \pmod 2$, we have

$$F_{3(k+1)} \equiv 0 \pmod 2.$$

We again prove by induction for $k \in \mathbb{N}$

$$F_{3k+1} \equiv 1 \pmod 2.$$

We have $F_1 = 1 \equiv 1 \pmod 2$.

Let us assume that $F_{3k+1} \equiv 1 \pmod 2$.

From Theorem 2.3 we have

$$F_{3(k+1)+1} = F_{3k+4} = F_4 F_{3k+1} + F_3 F_{3k}.$$

Since $F_{3k} \equiv 0 \pmod 2$ and $F_4 = 3 \equiv 1 \pmod 2$, using the assumption $F_{3k+1} \equiv 1 \pmod 2$, it follows that

$$F_{3(k+1)+1} \equiv 1 \pmod 2.$$

From $F_{3k} \equiv 0 \pmod 2$ and $F_{3k+1} \equiv 1 \pmod 2$, using the recurence relation of the Fibonacci sequence, we also have

$$F_{3k+2} = F_{3k+1} + F_{3k} \equiv 1 \pmod 2.$$

Thus the property is established. $\square$

Corollary 3.2. *If $p = 5k + 2$ is a prime which is strictly greater than 5 ($k \in \mathbb{N}$ and k odd), then $F_p = F_{5k+2}$ is an odd number.*

In order to prove this assertion, it suffices to remark that p is not divisible by 4.

Property 3.3.

$$F_{5k} \equiv 0 \pmod 5$$

with $k \in \mathbb{N}$.

Proof. We again prove this result by induction.

We have $F_0 = 0 \equiv 0 \pmod 5$.

Notice also that $F_5 = 5 \equiv 0 \pmod 5$.

Let assume that $F_{5k} \equiv 0 \pmod 5$.

From Theorem 2.3 we have

$$F_{5(k+1)} = F_{5k+5} = F_5F_{5k+1} + F_4F_{5k}.$$

Since $F_5 \equiv 0 \pmod 5$ and we assumed that $F_{5k} \equiv 0 \pmod 5$, we obtain

$$F_{5(k+1)} \equiv 0 \pmod 5.$$

$\square$

Property 3.4.

$$F_n \geq n$$

with $n \in \mathbb{N}$ and $n \geq 5$.

Proof. We prove this by induction.

We have $F_5 = 5 \geq 5$.

Let assume that $F_i \geq i$ with $i = 5, 6, \ldots, n$.

From Property 2.2 using the assumption above, we have

$$F_{n+1} = 1 + \sum_{i=0}^{n-1} F_i \geq 1 + \sum_{i=0}^{n-1} i.$$

$$F_{n+1} \geq 1 + \frac{n(n-1)}{2}.$$

Again

$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2} > 0.$$

So, from $F_{n+1} \geq 1 + \frac{n(n-1)}{2}$, we obtain

$$F_{n+1} \geq n + 1.$$

$\square$

4. Characterization of numbers $5k+2$ which are primes with $k \in \mathbb{N}$ and k odd

Let $p = 5k+2$ be a prime number with k a non-zero positive integer which is odd. Notice that in this case, $5k \pm 1$ is an even number and so:

$$\left\lfloor \frac{5k \pm 1}{2} \right\rfloor = \frac{5k \pm 1}{2}.$$

Property 4.1.

$$F_{5k+2} \equiv 5k+1 \pmod{5k+2}$$

with $k \in \mathbb{N}$ and k odd such that $5k+2$ is prime.

Proof. From Theorems 1.10 and 2.1 we have

$$2^{5k+1} F_{5k+2} = \sum_{l=0}^{\frac{5k+1}{2}} \binom{5k+2}{2l+1} 5^l \equiv 5^{\frac{5k+1}{2}} \equiv 5k+1 \pmod{5k+2}$$

where we used the fact that $\binom{5k+2}{2l+1}$ is divisible by $5k+2$ for $l = 0, 1, \ldots, \frac{5k-1}{2}$.

From Fermat's little theorem, we have

$$2^{5k+1} \equiv 1 \pmod{5k+2}.$$

We get $F_{5k+2} \equiv 5k+1 \pmod{5k+2}$. □

Property 4.2.

$$F_{5k+1} \equiv 1 \pmod{5k+2}$$

with $k \in \mathbb{N}$ and k odd such that $5k+2$ is prime.

Proof. From Theorem 2.1 and Property 1.11 we have

$$2^{5k} F_{5k+1} = \sum_{l=0}^{\lfloor \frac{5k}{2} \rfloor} \binom{5k+1}{2l+1} 5^l \equiv (5k+1) \sum_{l=0}^{\lfloor \frac{5k}{2} \rfloor} 5^l \pmod{5k+2}.$$

We have

$$\sum_{l=0}^{\lfloor \frac{5k}{2} \rfloor} 5^l = \frac{5^{\lfloor \frac{5k}{2} \rfloor + 1} - 1}{4}.$$

We get from the above

$$2^{5k+2} F_{5k+1} \equiv (5k+1) \left\{ 5^{\lfloor \frac{5k}{2} \rfloor + 1} - 1 \right\} \pmod{5k+2}.$$

Since k is an odd positive integer, there exists a positive integer m such that $k = 2m+1$. It follows that

$$\left\lfloor \frac{5k}{2} \right\rfloor = 5m+2.$$

Notice that $5k+2 = 10m+7$ is prime, implies that $k \neq 5$ and $k \neq 11$ or equivalently $m \neq 2$ and $m \neq 5$. Other restrictions on k and m can be given.

From Theorem 1.10 we have

$$5^{5m+3} \equiv 10m+6 \pmod{10m+7}.$$

We can rewritte $\sum_{l=0}^{\lfloor \frac{5k}{2} \rfloor} 5^l = \frac{5^{\lfloor \frac{5k}{2} \rfloor + 1} - 1}{4}$ as

$$\sum_{l=0}^{\lfloor \frac{5k}{2} \rfloor} 5^l = \frac{5^{5m+3} - 1}{4}.$$

Moreover, we have

$$(5k+1)\left\{5^{\lfloor \frac{5k}{2} \rfloor + 1} - 1\right\} = (10m+6)\left\{5^{5m+3} - 1\right\}.$$

Or,

$$(10m+6)\left\{5^{5m+3} - 1\right\} \equiv 5^{5m+3}\left\{5^{5m+3} - 1\right\} \equiv 5^{10m+6} - 10m - 6 \pmod{10m+7}.$$

We have

$$(10m+6)\left\{5^{5m+3} - 1\right\} \equiv 5^{10m+6} + 1 \pmod{10m+7}.$$

From Fermat's little theorem, we have $5^{10m+6} \equiv 1 \pmod{10m+7}$. Therefore

$$(10m+6)\left\{5^{5m+3} - 1\right\} \equiv 2 \pmod{10m+7},$$

or equivalently

$$(5k+1)\left\{5^{\lfloor \frac{5k}{2} \rfloor + 1} - 1\right\} \equiv 2 \pmod{5k+2}.$$

It follows that

$$2^{5k+2} F_{5k+1} \equiv 2 \pmod{5k+2}.$$

Since 2 and $5k+2$ are relatively prime

$$2^{5k+1} F_{5k+1} \equiv 1 \pmod{5k+2}.$$

From Fermat's little theorem, we have $2^{5k+1} \equiv 1 \pmod{5k+2}$. Therefore

$$F_{5k+1} \equiv 1 \pmod{5k+2}.$$

□

Property 4.3.

$$F_{5k} \equiv 5k \pmod{5k+2}$$

with $k \in \mathbb{N}$ and k odd such that $5k+2$ is prime.

Proof. From Theorem 2.1 and Property 1.12 we have

$$2^{5k-1} F_{5k} = \sum_{l=0}^{\frac{5k-1}{2}} \binom{5k}{2l+1} 5^l \equiv \sum_{l=0}^{\frac{5k-1}{2}} (5k-2l) 5^l \pmod{5k+2}.$$

Also

$$\sum_{l=0}^{\frac{5k-1}{2}} (5k-2l) 5^l = \frac{5\left[3 \times 5^{\frac{5k-1}{2}} - (2k+1)\right]}{8}.$$

So

$$2^{5k+2} F_{5k} \equiv 5\left(3 \times 5^{\frac{5k-1}{2}} - (2k+1)\right) \pmod{5k+2}.$$

Moreover since $k = 2m+1$, we have

$$3 \times 5^{\frac{5k-1}{2}} - (2k+1) = 3 \times 5^{5m+2} - (4m+3).$$

Since $5^{5m+3} \equiv 10m+6 \pmod{10m+7}$, we have

$$3 \times 5^{5m+3} \equiv 30m+18 \pmod{10m+7},$$

$$3 \times 5^{5m+3} \equiv 40m + 25 \pmod{10m + 7}.$$

Consequently

$$3 \times 5^{5m+3} \equiv 8m + 5 \pmod{10m + 7}$$

which implies

$$3 \times 5^{5m+2} - (4m + 3) \equiv 4m + 2 \pmod{10m + 7},$$

or equivalently for $k = 2m + 1$

$$3 \times 5^{\frac{5k-1}{2}} - (2k + 1) \equiv 2k \pmod{5k + 2},$$

$$2^{5k+2} F_{5k} \equiv 2 \times 5k \pmod{5k + 2}.$$

Since 2 and $5k + 2$ are relatively prime

$$2^{5k+1} F_{5k} \equiv 5k \pmod{5k + 2}.$$

From Fermat's little theorem, we have $2^{5k+1} \equiv 1 \pmod{5k + 2}$. Therefore

$$F_{5k} \equiv 5k \pmod{5k + 2}$$

□

5. Periods of the Fibonacci sequence modulo a positive integer

Notice that $F_1 = F_2 \equiv 1 \pmod{m}$ with m an integer which is greater than 2.

Definition 5.1. *The Fibonacci sequence (F_n) is periodic modulo a positive integer m which is greater than 2 $(m \geq 2)$, if there exists at least a non-zero integer ℓ_m such that:*

$$F_{1+\ell_m} \equiv F_{2+\ell_m} \equiv 1 \pmod{m}$$

The number ℓ_m is called a period of the Fibonacci sequence (F_n) modulo m.

Remark 5.2. *For $m \geq 2$ we have $l_m \geq 2$. Indeed, ℓ_m cannot be equal to 1 since $F_3 = 2$.*

From Theorem 2.3 we have

$$F_{2+\ell_m} = F_{\ell_m} F_3 + F_{\ell_m - 1} F_2 \equiv 2F_{\ell_m} + F_{\ell_m - 1} \pmod{m}.$$

Since $F_{\ell_m} + F_{\ell_m - 1} = F_{1+\ell_m}$, we get

$$F_{2+\ell_m} \equiv 2F_{\ell_m} + F_{\ell_m - 1} \equiv F_{\ell_m} + F_{1+\ell_m} \equiv F_{\ell_m} + F_{2+\ell_m} \pmod{m}.$$

Therefore we have the following

Property 5.3.

$$F_{\ell_m} \equiv 0 \pmod{m}$$

Moreover, from Theorem 2.3 we have

$$F_{1+\ell_m} = F_{\ell_m} F_2 + F_{\ell_m - 1} F_1 \equiv F_{\ell_m} + F_{\ell_m - 1} \equiv F_{\ell_m - 1} \pmod{m}.$$

Since $F_{1+\ell_m} \equiv 1 \pmod{m}$, we obtain the following

Property 5.4.

$$F_{\ell_m - 1} \equiv 1 \pmod{m}.$$

Besides, using the recurrence relation of the Fibonacci sequence, from Property 5.3 we get

$$F_{\ell_m - 2} + F_{\ell_m - 1} = F_{\ell_m} \equiv 0 \pmod{m}.$$

Using Property 5.4 we obtain

Property 5.5.
$$F_{\ell_m-2} \equiv m-1 \pmod{m}.$$

Remark 5.6. *From Theorem 2.3 we have for* $m \geq 2$
$$F_{2m} = F_{m+m} = F_m F_{m+1} + F_{m-1}F_m = F_m(F_{m+1} + F_{m-1})$$
and
$$F_{2m+1} = F_{(m+1)+m} = F_m F_{m+2} + F_{m-1}F_{m+1} = F_m(F_m + F_{m+1}) + F_{m-1}(F_{m-1} + F_m),$$
$$F_{2m+1} = F_m(2F_m + F_{m-1}) + F_{m-1}(F_{m-1} + F_m) = 2F_m^2 + 2F_m F_{m-1} + F_{m-1}^2 = F_m^2 + F_{m+1}^2.$$
From this we get
$$F_{2m+2} = F_{2m+1} + F_{2m} = F_m^2 + F_{m+1}^2 + F_m(F_{m+1} + F_{m-1}),$$
$$F_{2m+3} = F_3 F_{2m+1} + F_2 F_{2m} = 2(F_m^2 + F_{m+1}^2) + F_m(F_{m+1} + F_{m-1}),$$
$$F_{2m+4} = F_{2m+3} + F_{2m+2} = 3(F_m^2 + F_{m+1}^2) + 2F_m(F_{m+1} + F_{m-1}).$$

Theorem 5.7. *A period of the Fibonacci sequence modulo* $5k+2$ *with* $5k+2$ *a prime and* k *odd is given by*
$$\ell_{5k+2} = 2(5k+3).$$

Proof. Using the recurrence relation of the Fibonacci sequence, and from Properties 4.1, 4.2 and 4.3 we have
$$F_{5k+3} = F_{5k+2} + F_{5k+1} \equiv 5k+2 \equiv 0 \pmod{5k+2}.$$
Taking $m = 5k+2$ prime (k odd) in the formulas of F_{2m+3} and F_{2m+4}, we have
$$F_{10k+7} = 2(F_{5k+2}^2 + F_{5k+3}^2) + F_{5k+2}(F_{5k+3} + F_{5k+1}) \equiv 2(5k+1)^2 + 5k + 1 \pmod{5k+2},$$
$$F_{10k+7} \equiv 50k^2 + 20k + 2 + 5k + 1 \equiv 10k(5k+2) + (5k+2) + 1 \pmod{5k+2},$$
$$F_{10k+7} \equiv 1 + (10k+1)(5k+2) \equiv 1 \pmod{5k+2},$$
and
$$F_{10k+8} = 3(F_{5k+2}^2 + F_{5k+3}^2) + 2F_{5k+2}(F_{5k+3} + F_{5k+1}) \equiv 3(5k+1)^2 + 2(5k+1) \pmod{5k+2},$$
$$F_{10k+8} \equiv 75k^2 + 30k + 3 + 10k + 2 \equiv 15k(5k+2) + 2(5k+2) + 1 \pmod{5k+2},$$
$$F_{10k+8} \equiv 1 + (15k+2)(5k+2) \equiv 1 \pmod{5k+2}.$$
Thus
$$F_{10k+7} \equiv F_{10k+8} \equiv 1 \pmod{5k+2},$$
or equivalently
$$F_{1+2(5k+3)} \equiv F_{2+2(5k+3)} \equiv 1 \pmod{5k+2}.$$
We deduce that a period of the Fibonacci sequence modulo $5k+2$ with $5k+2$ a prime is $\ell_{5k+2} = 2(5k+3)$. □

Remark 5.8. *We can observe that*
$$F_{5k-1} = F_{5k+1} - F_{5k} \equiv 1 - 5k \equiv 3 \equiv F_4 \pmod{5k+2},$$
$$F_{5k-2} = F_{5k} - F_{5k-1} \equiv 5k - 3 \equiv 5k - F_4 \pmod{5k+2},$$
$$F_{5k-3} = F_{5k-1} - F_{5k-2} \equiv 6 - 5k \equiv 8 \equiv F_6 \pmod{5k+2},$$
$$F_{5k-4} = F_{5k-2} - F_{5k-3} \equiv 5k - 11 \equiv 5k - (F_4 + F_6) \pmod{5k+2}.$$

Using induction we can show the following two properties.

Property 5.9. *Let $5k+2$ be a prime with k odd. Then, we have*

$$F_{5k-(2l+1)} \equiv F_{2(l+2)} \pmod{5k+2}$$

with l a positive integer such that $l \leq \lfloor \frac{5k-1}{2} \rfloor$.

Property 5.10. *Let $5k+2$ be a prime with k odd. Then, we have*

$$F_{5k-2l} \equiv 5k - \sum_{i=0}^{l-1} F_{2(i+2)} \pmod{5k+2}$$

with $l \geq 1$ such that $l \leq \lfloor \frac{5k}{2} \rfloor$.

Remark 5.11. *We can notice that*

$$F_{5k+4} = F_{5k+3} + F_{5k+2} \equiv F_{5k+2} \equiv 5k+1 \pmod{5k+2},$$
$$F_{5k+5} = F_{5k+4} + F_{5k+3} \equiv F_{5k+4} \equiv 5k+1 \pmod{5k+2},$$
$$F_{5k+6} = F_{5k+5} + F_{5k+4} \equiv 10k+2 \equiv 5k \pmod{5k+2},$$

and for $l \geq 1$ we have

$$F_{5k+3l+2} = F_{3l+2}F_{5k+1} + F_{3l+1}F_{5k} \equiv F_{3l+2} + 5kF_{3l+1} \pmod{5k+2},$$

$$F_{5k+3l+2} \equiv F_{3l}+(5k+1)F_{3l+1} \equiv F_{3l}-F_{3l+1}+(5k+2)F_{3l+1} \equiv F_{3l}-F_{3l+1} \equiv -F_{3l-1} \pmod{5k+2}.$$

Furthermore, we have for $l \geq 1$

$$F_{5k+3l+1} = F_{3l+1}F_{5k+1} + F_{3l}F_{5k} \equiv F_{3l+1} + 5kF_{3l} \pmod{5k+2},$$

$$F_{5k+3l+1} \equiv F_{3l-1}+(5k+1)F_{3l} \equiv F_{3l-1}-F_{3l}+(5k+2)F_{3l} \equiv F_{3l-1}-F_{3l} \equiv -F_{3l-2} \pmod{5k+2}.$$

Besides, we have for $l \geq 1$

$$F_{5k+3l} = F_{3l}F_{5k+1} + F_{3l-1}F_{5k} \equiv F_{3l} + 5kF_{3l-1} \pmod{5k+2},$$

$$F_{5k+3l} \equiv F_{3l-2}+(5k+1)F_{3l-1} \equiv F_{3l-2}-F_{3l-1}+(5k+2)F_{3l-1} \equiv F_{3l-2}-F_{3l-1} \equiv -F_{3l-3} \pmod{5k+2}.$$

Finally we can state the following property, the proof of which follows from the above remark and by using induction.

Property 5.12. $F_{5k+n} \equiv -F_{n-3} \pmod{5k+2}$.

Theorem 5.13. *Let $5k+2$ be a prime with k an odd positive number and let n a positive integer. Then, we have:*

$$F_{n(5k+3)} \equiv 0 \pmod{5k+2}$$

Proof. The proof of the theorem will be done by induction. We have $F_0 \equiv 0 \pmod{5k+2}$. Moreover, we know that $F_{5k+3} \equiv 0 \pmod{5k+2}$. Let assume that $F_{n(5k+3)} \equiv 0 \pmod{5k+2}$. We have:

$$F_{(n+1)(5k+3)} = F_{n(5k+3)+5k+3} = F_{5k+3}F_{n(5k+3)+1} + F_{5k+2}F_{n(5k+3)}$$

Since $F_{5k+3} \equiv 0 \pmod{5k+2}$, using the assumption $F_{n(5k+3)} \equiv 0 \pmod{5k+2}$, we deduce that:

$$F_{(n+1)(5k+3)} \equiv 0 \pmod{5k+2}$$

It achieves the proof of the theorem. □

Corollary 5.14. *If $5k+3|m$, then $F_m \equiv 0 \pmod{5k+2}$.*

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