AST2000 - Part 4
Onboard Orientation Software

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We have studied stereographic projection, image analysis and transformation between coordinate systems to learn more about how our spacecraft can orient itself after completing a launch from our home planet. Furthermore, using the spacecraft's onboard spectrograph, we were able to find a relation between measured Doppler shifts from two reference stars and our spacecraft's velocity relative to our sun. We also used the trilateration method by comparing measured distances to find the spacecraft's correct coordinates after launch. Our software was successfully developed, and our spacecraft will now be able to orient itself in space after completing a long awaited rocket launch.

I. INTRODUCTION

After eventually launching our spacecraft, it will have to be able to orient itself in space for us to be able to travel to our destination planet. In this study we are going to attempt to develop software that calculates the angular orientation, velocity and position of our spacecraft when in space. By studying image analysis, we want to find a way for our spacecraft to calculate its angular orientation by comparing a picture it takes immediately after launch to a series of 360 reference pictures taken by a satellite orbiting our home planet at this very moment.

For our spacecraft to successfully determine its velocity relative to our sun immediately after launch, we will study the relation between relative radial velocity and Doppler shift. Our spacecraft is equipped with a spectrograph that measures the wavelengths of light received by two reference stars far away, and we will try to compare this to the expected spectral line at rest in order to find our spacecraft's radial velocity relative to the two stars. By knowing their Doppler shifts at our sun, we wish to use transformation between coordinate systems to find our cartesian velocity relative to our sun. Finally, we need to know our spacecraft's coordinates when in space, which we'll attempt to calculate using measured distances and applying the *trilateration* method.

II. METHOD

When we eventually launch our spacecraft, we want it's built in software to be able to determine the angle it's facing relative to the solar system x-axis. In order for us to develop this software, we will first have to study the principles behind *stereographic projection*, which relates spherical coordinates (θ, ϕ) into planar coordinates (X, Y). Thus, the stereographic projection essentially flattens a spherical surface into straight lines (see Figure 1). Because the spacecraft's position relative to bodies outside of our solar system is so far away, we will assume that the night sky surrounding us is a constant sphere. The satellite orbiting our home planet and our spacecraft's onboard camera are both using spherical coordinates to generate pictures. Furthermore, a camera's

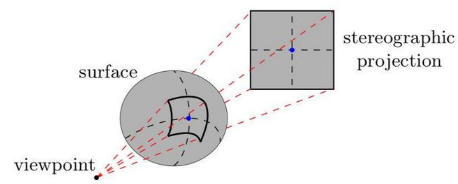

Figure 1. Illustration of stereographic projection

field of view is defined as the maximum angular width of the pictures it generates, defined in the following way:

$$\begin{aligned} \alpha_\phi &= \phi_{max} - \phi_{min} \\ \alpha_\theta &= \theta_{max} - \theta_{min} \end{aligned} \tag{1}$$

By using the principles of stereographc projection, we're then able to define the maximum and minimum width and height of a picture in planar coordinates X, Y (see Appendix A):

$$\begin{aligned} X_{max/min} &= \pm\frac{2\sin(\alpha_\phi/2)}{1+\cos(\alpha_\phi/2)} \\ Y_{max/min} &= \pm\frac{2\sin(\alpha_\theta/2)}{1+\cos(\alpha_\theta/2)} \end{aligned} \tag{2}$$

To make sure that our software is able to correctly analyze pictures using stereographic projection, we will attempt to decompose and then recreate the picture shown in Figure 2. This is a part of the total spherical picture of the night sky taken by the orbiting satellite. In this spherical picture, each pixel contains RGB values describing its color, and they all have their own spherical coordinate that they're centered about. We will assume that our spacecraft's angular orientation only changes in the ϕ-direction, and that we therefore only need to analyze the part of the spherical picture centered about $\theta = 90°$. To develop the software that'll hopefully be able to determine the angle that our spacecraft is facing immediately after launch, we'll attempt to compare the RGB values of the pixels within the photo taken by our onboard camera to the RGB values of 3600 flattened images constructed

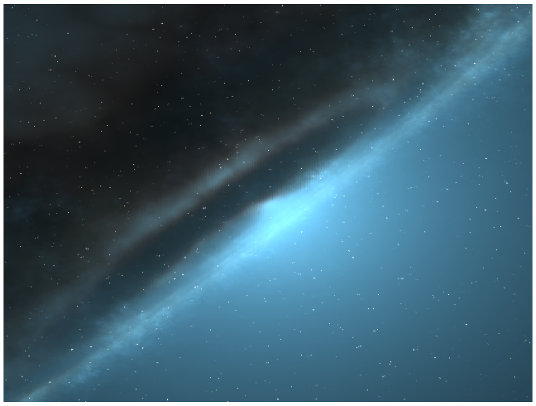

Figure 2. The picture taken by the orbiting satellite that we will attempt to recreate using our software.

from the spherical picture using stereographic projection. The i^{th} flattened image will then be centered about $\phi = i$, where $i = 0.0°, 0.1°, 0.2°, ..., 359.9°$.

After completing our rocket launch, we will also have to be able to determine our spacecraft's velocity relative to the sun in order for us to travel in the planned direction. If our spacecraft is able to rotate using our newly developed software, it should be able to point its camera at any star in the night sky. Our spacecraft is equipped with a spectrograph that we'll use to measure the Doppler shift in the H_α spectral line, a spectral line present in the received light from stars, which has a wavelength $\lambda_0 = 656.3$ nm at rest. From Lecture Notes 1D [1], we know that the formula for non-relativistic Doppler shift is

$$\Delta\lambda = \frac{v_r}{c}\lambda_0 \tag{3}$$

where v_r is the radial velocity of the object we're measuring the spectral line of, and c is the speed of light. If we solve this expression for v_r, we get

$$v_r = c\frac{\Delta\lambda}{\lambda_0} \tag{4}$$

We aim to use this formula to find the radial velocity of two reference stars relative to us and our sun, respectively, and then use this to find our spacecraft's velocity relative to our sun. These stars are positioned at angles ϕ_1 and ϕ_2 off of the solar system x-axis, and our wonderful research team was kind enough to supply us with the Doppler shifts measured from these stars' spectral lines at our sun. As soon as we're in space, we'll use our spectrograph to measure their Doppler shifts as seen from our spacecraft as well.

In the usual Cartesian coordinates, we have unit vectors

$$\hat{x} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \hat{y} = \begin{pmatrix}0\\1\end{pmatrix} \tag{5}$$

Since we don't know our spacecraft's Cartesian velocity components relative to our sun after launch, we will attempt to use the angles ϕ_1 and ϕ_2 that our reference stars are positioned at relative to the solar system x-axis to find them. If we introduce the (ϕ_1, ϕ_2) coordinate system, we get the unit vectors:

$$\hat{u_1} = \begin{pmatrix}\cos\phi_1\\ \sin\phi_1\end{pmatrix}, \quad \hat{u_2} = \begin{pmatrix}\cos\phi_2\\ \sin\phi_2\end{pmatrix} \tag{6}$$

By using (4) to find the stars' radial velocities $v_{1,\text{sun}}$ and $v_{2,\text{sun}}$ relative to our sun, and $v_{1,\text{sc}}$ and $v_{2,\text{sc}}$ relative to our spacecraft, we can find the spacecraft's (ϕ_1, ϕ_2) velocity relative to our sun in the following way:

$$\begin{aligned} v_{\phi_1} &= v_{1,\text{sun}} - v_{1,\text{sc}} \\ v_{\phi_2} &= v_{2,\text{sun}} - v_{2,\text{sc}} \end{aligned} \tag{7}$$

Then, we can find our spacecraft's (x, y) relative to the sun using our knowledge of change of coordinates matrices:

$$\begin{pmatrix}v_x\\ v_y\end{pmatrix} = \frac{1}{\sin(\phi_2 - \phi_0)}\begin{pmatrix}\sin\phi_2 & -\sin\phi_1\\ -\cos\phi_2 & \cos\phi_1\end{pmatrix}\begin{pmatrix}v_{\phi_1}\\ v_{\phi_2}\end{pmatrix} \tag{8}$$

Finally, we'll need to find a way to calculate our spacecraft's positional coordinates as it travels through space. Our spacecraft is equipped with an onboard radar array which lets it measure its distance from the different bodies in our solar system when in space, taking their respective radii into consideration when doing so. To determine our spacecraft's position, we will take use of the trilateration method, which is illustrated in Figure 3.

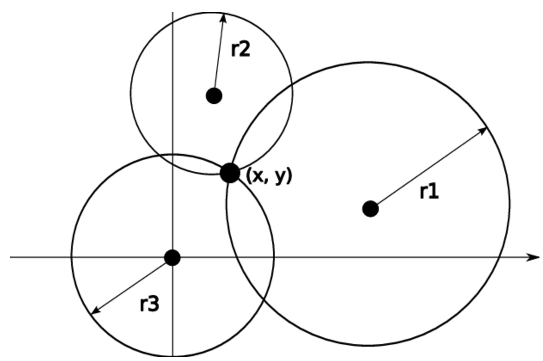

Figure 3. Illustration of the trilateration method

The main principle of this method is that we create three circles that each represent a celestial body within our solar system, one of which will necessarily have to be our sun. The circles' centres are the respective bodies' positional vectors, and the circles' radii are their distances to the spacecraft. If we were to only use two circles, one representing a chosen planet, and one representing the sun, we would have two possible positions for the spacecraft. By adding a third circle, we only have one possible position, as the point where all three of these circles intersect is the spacecraft's positional vector relative to the sun. We plan to take use of the planet trajectories, which we correctly calculated in our study of the planetary orbits within our solar system, to check what Blossom and Flora's positional coordinates relative to our sun are when the distances are measured. Then, by applying the trilateration method, we'll hopefully be able to find our spacecraft's coordinates as well.

To test our software in a realistic situation, we'll simulate a rocket launch identical to one of the launches we simulated in our last study where we prepared for our journey ahead. Thus, we'll launch at present time from the far east side on Doofenshmirtz' equator with the following launch parameters:

$$F \approx 469.33 \text{ kN}$$
$$mlr \approx 37.85 \text{ kg/s}$$
$$m_{\text{fuel}} = 4.5 \times 10^4 \text{ kg}$$

Where F, mlr and m_{fuel} are the rocket's trust force, mass loss rate and initial fuel mass, respectively. After finding what angle our spacecraft is facing, as well as its Cartesian velocity and positional coordinates relative to our sun, we'll share the results with our research team at the University of Oslo to check if our software is working as expected.

III. RESULTS

We wanted to test our software to see if it was reliable when analyzing pictures taken by the orbiting satellite. We found that the reference picture `sample0000.png` has 640 pixels in the horizontal direction and 480 pixels in the vertical direction, which means that it consists of a total of 307200 pixels. When attempting to use our software to decompose the picture and then recreate it, we were able to create the picture shown in Figure 4.

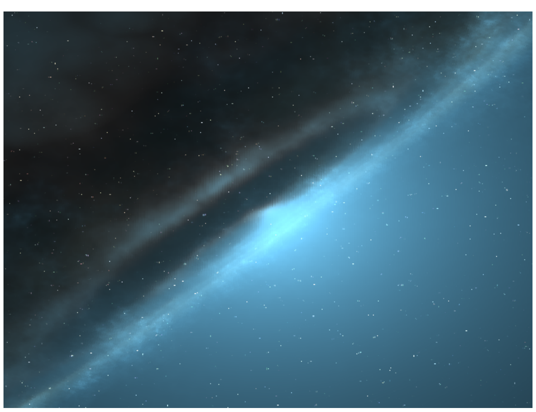

Figure 4. Recreation of the picture `sample0000.png` taken by the satellite orbiting our home planet.

With our software being able to decompose pictures, we were then ready to test its ability to find out at which angle the satellite was facing when taking a set of reference pictures. Our software's approximated angles compared to the actual angles for each reference picture are displayed in Table I.

Picture	Approximated ϕ	Actual ϕ
`sample0000.png`	0.0°	0.0°
`sample0200.png`	20.0°	20.0°
`sample0435.png`	43.5°	43.5°
`sample0911.png`	91.1°	91.1°
`sample1400.png`	140.0°	140.0°
`sample1900.png`	190.0°	190.0°

Table I. The angles that the satellite was facing when taking the respective reference pictures.

We wanted to test if our spacecraft was able to measure its own velocity after launching from our home planet. We first tested our software in the case where the Doppler shifts from each of the reference stars measured by the onboard spectrograph were $\Delta\lambda_1 = \Delta\lambda_2 = 0$, and found that the spacecraft's (ϕ_1, ϕ_2) velocity relative to our sun was

$$v_{\phi_1} = 0.072 \text{ AU/yr}$$
$$v_{\phi_2} = 0.491 \text{ AU/yr}$$

Since the stars' Doppler shifts at the sun is

$$\Delta\lambda_{\text{sun},1} = 7.51 \times 10^{-4} \text{ nm}$$
$$\Delta\lambda_{\text{sun},2} = 5.09 \times 10^{-3} \text{ nm}$$

we were able to calculate that our spacecraft's (x, y) velocity relative to our sun would've been

$$v_x = 0.471 \text{ AU/yr}$$
$$v_y = 0.141 \text{ AU/yr}$$

Immediately after we completed our launch simulation, we were able to decipher that the spacecraft was rotated $\phi = 161.0°$ off of the solar system x-axis. What our spacecraft's view would have been in this situation is shown in Figure 5.

When using the spectrograph to measure the reference stars' Doppler shifts at the spacecraft after launch, we found that they were

$$\Delta\lambda_1 = -5.45 \times 10^{-2} \text{ nm}$$
$$\Delta\lambda_2 = -2.76 \times 10^{-2} \text{ nm}$$

Since their Doppler shifts at our sun are the same, we found that the spacecraft's (ϕ_1, ϕ_2) velocity relative to our sun was

$$v_{\phi_1} = 5.328 \text{ AU/yr}$$
$$v_{\phi_2} = 3.146 \text{ AU/yr}$$

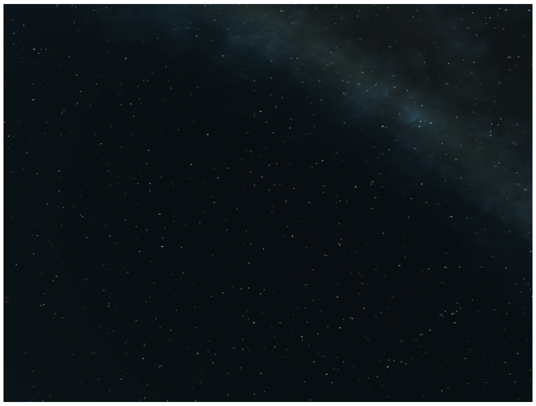

Figure 5. What our spacecraft's view would be immediately after reaching escape velocity.

Thus, our software was able to calculate the spacecraft's (x, y) velocity relative to our sun:

$$v_x = 1.956 \text{ AU/yr}$$
$$v_y = 5.664 \text{ AU/yr}$$

Finally, we wanted to test our trilateration software. Our research team informed us that the spacecraft's distances d to Blossom, Flora and our sun after the launch simulation were:

$$d_B = 2.311 \text{ AU}$$
$$d_F = 14.798 \text{ AU}$$
$$d_s = 3.541 \text{ AU}$$

By using our correctly calculated planetary trajectories from our study of the planetary orbits within our solar system, we were able to determine Blossom's positional coordinates relative to our sun after the launch simulation:

$$x_B = 5.635 \text{ AU}$$
$$y_B = 0.978 \text{ AU}$$

As well as Flora's positional coordinates:

$$x_F = 13.652 \text{ AU}$$
$$y_F = 10.805 \text{ AU}$$

Which let us calculate the spacecraft's coordinates:

$$x = 3.541 \text{ AU}$$
$$y = 2.225 \times 10^{-4} \text{ AU}$$

When checking with our research team whether or not our software was successful in manually orienting our spacecraft after launch, we were pleased to hear that all of our calculations were satisfactory.

IV. DISCUSSION

As our all our results were satisfactory, there's not much for us to discuss. However, we will mention that our angular orientation software has a flaw that may appear troublesome when eventually launching our spacecraft and travelling through space. For our spacecraft to be able to calculate at which angle it's facing in the solar system coordinate system, we compare a picture taken by the camera onboard to the 3600 pictures taken by the satellite orbiting our home planet. This means that it only can decipher which angle it's facing with a precision of one decimal. Fortunately for us, our research team has uploaded our software to the orbiting satellite, which means that the manual orientation process is only necessary right after launch. After that, the spacecraft will be able to orient itself at any point during the journey, so as long as we correctly calculate its angle the first time, this minor lack of precision hopefully won't lead to any troubles.

V. CONCLUSION

We have studied stereographic projection and image analysis, transformation between coordinate systems, the relation between Doppler shift and radial velocity, as well as the trilateration method for finding relative coordinates in space. We managed to decompose and recompose a reference image taken by the satellite orbiting our home planet, and were successful in applying this process when developing software that calculates the spacecraft's angular orientation in space by analyzing 360 flat pictures.

We found a way to convert wavelength data measured by our spacecraft's spectrograph from two reference stars, positioned at angles ϕ_1 and ϕ_2) relative to the solar system x-axis, to their radial velocities relative to us. This let us find our radial velocity in (ϕ_1, ϕ_2)-coordinates, and then use transformation between coordinate systems to find our (x, y) velocity relative to our sun. By utilizing measured distances between our spacecraft and Blossom, Flora and our sun respectively, we were able to use the trilateration method to correctly calculate its positional coordinates after a completed launch simulation. All of our orientation software was deemed satisfactory by our research team, and we're now feeling ready for the next step of our journey, the actual rocket launch and interplanetary travel.

REFERENCES

[1] Hansen F. K. *Lecture Notes 1D*. URL: https://www.uio.no/studier/emner/matnat/astro/AST2000/h22/undervisningsmateriell/lecture_notes/part1d.pdf.

Appendix A: Limits for X and Y

We have the following ranges for values of θ and ϕ:

$$0 \leq \theta \leq \pi, \qquad 0 \leq \phi \leq 2\pi$$

As well as the following coordinate ranges:

$$\begin{aligned} -\frac{\alpha_\theta}{2} &\leq \theta - \theta_0 \leq \frac{\alpha_\theta}{2} \\ -\frac{\alpha_\phi}{2} &\leq \phi - \phi_0 \leq \frac{\alpha_\phi}{2} \end{aligned}$$

And the following definitions of α_θ and α_ϕ:

$$\begin{aligned} \alpha_\theta &= \theta_{\max} - \theta_{\min} \\ \alpha_\phi &= \phi_{\max} - \phi_{\min} \end{aligned}$$

This means that we have

$$\begin{aligned} \theta_{\max} - \theta_0 &= \frac{\alpha_\theta}{2}, \qquad \theta_{\min} - \theta_0 = -\frac{\alpha_\theta}{2} \\ \phi_{\max} - \phi_0 &= \frac{\alpha_\phi}{2}, \qquad \phi_{\min} - \phi_0 = -\frac{\alpha_\phi}{2} \end{aligned}$$

Which we can rewrite in the following way:

$$\begin{aligned} \theta_{\max/\min} &= \pm\frac{\alpha_\theta}{2} \\ \phi_{\max/\min} &= \pm\frac{\alpha_\phi}{2} \end{aligned} \tag{A1}$$

We also have the following expressions for X and Y:

$$\begin{aligned} X &= \kappa \sin\theta \sin(\phi - \phi_0) \\ Y &= \kappa(\sin\theta_0 \cos\theta - \cos\theta_0 \sin\theta \cos(\phi - \phi_0)) \end{aligned} \tag{A2}$$

Where

$$\frac{2}{\kappa} = 1 + \cos\theta_0 \cos\theta + \sin\theta_0 \sin\theta \cos(\phi - \phi_0) \tag{A3}$$

1. $X_{\max/\min}$

We'll first look at the points where $X = X_{\max/\min}$ and $Y = 0$. We'll have to demand

$$\begin{aligned} \kappa \sin\theta \sin(\phi - \phi_0) &= X_{\max/\min} \\ \kappa(\sin\theta_0 \cos\theta - \cos\theta_0 \sin\theta \cos(\phi - \phi_0)) &= 0 \end{aligned}$$

Which is achieved when we have $\theta = \theta_0$ and $\phi = \phi_{\max/\min}$. Combined with (A3), we get

$$\begin{aligned} \frac{2}{\kappa} &= 1 + \cos^2\theta_0 + \sin^2\theta_0 \cos(\phi_{\max/\min} - \phi_0) \\ &= 1 + \cos(\phi_{\max/\min} - \phi_0) \\ &= 1 + \cos\left(\pm\frac{\alpha_\phi}{2}\right) \\ &= 1 + \cos\left(\frac{\alpha_\phi}{2}\right) \end{aligned}$$

Where we have used that $\theta_0 = \pi/2$ and $\cos -x = \cos x$. This gives us the following expression for κ:

$$\kappa = \frac{2}{1 + \cos(\alpha_\phi/2)}$$

Which, when applied to (A2) gives us an expression for $X_{\max/\min}$:

$$\begin{aligned} X_{\max/\min} &= \kappa \sin\theta_0 \sin(\phi_{\max/\min} - \phi_0) \\ &= \frac{2\sin(\pm\alpha_\phi/2)}{1 + \cos(\alpha_\phi/2)} \\ &= \pm\frac{2\sin(\alpha_\phi/2)}{1 + \cos(\alpha_\phi/2)} \quad \square \end{aligned} \tag{A4}$$

Where we have used that $\sin(-x) = -\sin x$.

2. $Y_{\max/\min}$

If we now look at the points where $X = 0$ and $Y = Y_{\max/\min}$, we have to demand

$$\begin{aligned} \kappa \sin\theta \sin(\phi - \phi_0) &= 0 \\ \kappa(\sin\theta_0 \cos\theta - \cos\theta_0 \sin\theta \cos(\phi - \phi_0)) &= Y_{\max/\min} \end{aligned}$$

Which is achieved when we have $\theta = \theta_{\max/\min}$ and $\phi = \phi_0$. Combined with (A3), we get

$$\begin{aligned} \frac{2}{\kappa} &= 1 + \cos\theta_0 \cos\theta_{\max/\min} + \sin\theta_0 \sin\theta_{\max/\min} \cos(0) \\ &= 1 + \cos\theta_{\max/\min} - \theta_0 \\ &= 1 + \cos\left(\frac{\alpha_\theta}{2}\right) \end{aligned}$$

Where we have used that $\theta_0 = \pi/2$ and $\cos(\phi_0 - \phi_0) = \cos 0 = 1$, as well as the following trigonometric identity:

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

With $\theta_{\max/\min}$ as u og θ_0 as v. This gives us the following expression for κ:

$$\kappa = \frac{2}{1 + \cos(\alpha_\theta/2)}$$

Which, when applied to (A2), gives us an expression for $Y_{\max/\min}$:

$$\begin{aligned} Y_{\max/\min} &= \kappa\left(\sin\theta_0 \cos\theta_{\max/\min} - \cos\theta_0 \sin\theta_{\max/\min}\right) \\ &= \frac{2}{1 + \cos(\alpha_\theta/2)} \sin\left(\pm\frac{\alpha_\theta}{2}\right) \\ &= \pm\frac{2\sin(\alpha_\theta/2)}{1 + \cos(\alpha_\theta/2)} \quad \square \end{aligned} \tag{A5}$$

Where we again have used that $\cos(\phi_0 - \phi_0) = \cos 0 = 1$ and that $\sin -x = -\sin x$, as well as the following trigonometric identity:

$$\sin u - v = \sin u \cos v - \cos u \sin v$$

With θ_0 as u og $\theta_{\max/\min}$ as v.