AST2000 - Part 9
General Relativity

Candidates 15361 & 15384
(Dated: December 16, 2022)

EXERCISE 1

Introduction

We will be studying the *gravitational Doppler effect*, which is different from the Doppler effect caused by relative velocities that we've studied earlier. The aforementioned effect affects how observers at different distances from a massive body observes incoming radiation due to its gravitational field. We already know that for an observer in a strong gravitational field, time runs at a different rate than for an observer far away. In this exercise we aim to see if there's a connection between this effect and the gravitational Doppler effect, and from there attempt to learn more about the latter by calculating the Doppler shifts caused by bodies with different masses.

The Situation

We have a shell observer equipped with a laser pen at distance r from a massive object. They point the laser pen radially outwards, and we have a far-away observer receiving the laser. Normally, we define the wavelength as observed by the rest frame as λ_0, but here we will denote it using λ_{shell} to specify that the shell observer is emitting the light beam. The wavelength observed by the far-away observer is λ. Instead of using the wavelengths in our calculations, we will mostly use the time intervals Δt_{shell} and Δt, which are the measured time intervals between two peaks in the electromagnetic radiation emitted from the laser pen for the two observers respectively. This gives us a new way to define their measured frequencies and wavelengths, which are shown in the table below.

Observer	Frequency ν	Wavelength λ
Shell	$1/\Delta t_{\text{shell}}$	Δt_{shell}
Far Away	$1/\Delta t$	Δt

Method

In the presence of a massive body, spacetime is deformed, which according to general relativity affects the shell observer's time interval Δt_{shell}. Furthermore, we know from Lecture Notes 2C [4] that the shell observer can be approximated as a *local intertial frame* around the massive object when looking at sufficiently small time- and space intervals, while the far-away observer cannot. What this tells us is that the shell observer can use Lorentz geometry, while the far-away observer must use *Schwarzschild geometry*. In the latter, we use polar coordinates, as we usually are interested in radial distances. We'll therefore redefine the Lorentz line element that we're used to using polar coordinates:

$$\Delta s^2 = \Delta t_{\text{shell}}^2 - \Delta r_{\text{shell}}^2 - r^2\Delta\phi_{\text{shell}}^2 \tag{1}$$

Where we have marked the intervals with "shell" to specify that these are the intervals measured by the shell observer. Furthermore, the Schwarzschild line element is defined in the following way:

$$\Delta s^2 = \left(1 - \frac{2M}{r}\right)\Delta t^2 - \frac{\Delta r^2}{\left(1-\frac{2M}{r}\right)} - r^2\Delta\phi^2 \tag{2}$$

Since the shell observer is at rest relative to itself, Δr_{shell} and $\Delta\phi_{\text{shell}}$ must be zero. We therefore get $\Delta s^2 = \Delta t_{\text{shell}}^2$. We can also see this from the fact that the time interval Δt_{shell} is the time interval measured on the shell observer's clock, which is its proper time interval $\Delta\tau$. When the observer is at rest, we know that this is equal to Δs. In the case we'll be studying, the shell observer is also at rest relative to the far-away observer. This means that the intervals Δr and $\Delta\phi$ that the far-away observer measures for the shell observer must be zero as well. Thus, we get

$$\Delta s^2 = \left(1 - \frac{2M}{r}\right)\Delta t^2$$

Since the observed wavelengths are the same as the time intervals measured by the two observers, we want to find a relation between the two to make future calculations easier. To do this, we remember that because the path in spacetime Δs is a conserved quantity, we must demand that (1) and (2) are equal. Now that we've eliminated all the space intervals in these expressions, it should be an easy task to find the relation we're looking for. We get

$$\Delta t = \frac{\Delta t_{\text{shell}}}{\sqrt{\left(1-\frac{2M}{r}\right)}} \tag{3}$$

Furthermore, we can use this relation to derive the following formula for the gravitational Doppler effect (see the "Specific Questions" section):

$$\frac{\Delta\lambda}{\lambda_{\text{shell}}} = \frac{1}{\sqrt{\left(1-\frac{2M}{r}\right)}} - 1 \tag{4}$$

By making a first order Taylor expansion around $2M/r = 0$, we found that the formula above can be approximated in the following way for $r \gg 2M$:

$$\frac{\Delta\lambda}{\lambda_{\text{shell}}} = \frac{M}{r} \tag{5}$$

To further study how the magnitude of the gravitational Doppler effect increases with the mass M of the body causing it, we will attempt use our derived formulas to calculate the Doppler shifts caused by the Earth, the Sun, and a hypothetical black hole at the center of a quasar. We want to calculate the significance of the redshift measured by a far-away observer for the radiation sent out by a shell observer at rest on the Sun's surface. Furthermore, we want to see if the light received from the sun is significantly blueshifted due to the Earth's gravitational field. At last, we will look at how gravity affects an observer right outside a black hole's event horizon to visualize how drastically its enormous mass affects the gravitational Doppler shift in comparison to smaller stellar bodies.

Conclusion

We were able to derive the gravitational Doppler formula, and can from this formula alone safely conclude that an observer far away from a massive body measures a redshift in its emitted light, while a shell observer right outside the massive body measures incoming radiation as blueshifted. Furthermore, we used our calculations to decipher how the magnitude of the Doppler effect changes as the body's mass M increases. Not surprisingly, we found that the blueshift measured by an observer on Earth's surface due to its gravitational field was negligible. The measured for an observer far away from the Sun was larger, not suprisingly as the sun is much more massive than the Earth, but this Doppler shift was still well below 1 nanometer. When studying an observer right outside a black hole's event horizon, we found that all incoming radiation was blueshifted so much so that visible light was shifted towards the shorter wavelengths in the UV-specter. This definitely puts it into perspective how incredibly massive and dense a black hole really is.

Specific Questions

We want to derive a formula for the gravitational Doppler shift. Since the observed wavelengths λ and λ_{shell} are the same as the time intervals Δt and Δt_{shell}, it immediately follows from (3) that we get the following relation for the observed wavelengths:

$$\lambda = \frac{\lambda_{\text{shell}}}{\sqrt{\left(1 - \frac{2M}{r}\right)}} \tag{6}$$

We want to get this on the form we're used to, namely $\Delta\lambda/\lambda_0$, where $\Delta\lambda = \lambda - \lambda_0$ and $\lambda_0 = \lambda_{\text{shell}}$ in this case. If we solve (6) for λ_{shell}, we get

$$\lambda_{\text{shell}} = \lambda\sqrt{1 - \frac{2M}{r}} \tag{7}$$

which gives us

$$\Delta\lambda = \lambda - \lambda_{\text{shell}} = \lambda\left(1 - \sqrt{1 - \frac{2M}{r}}\right) \tag{8}$$

Inserted into (6), we get

$$\begin{aligned}
\frac{\lambda}{\lambda_{\text{shell}}} &= \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}} \\
\frac{\Delta\lambda / \left(1 - \sqrt{1 - \frac{2M}{r}}\right)}{\lambda_{\text{shell}}} &= \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}} \\
\frac{\Delta\lambda}{\lambda_{\text{shell}}} &= \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}} - 1 \ \square
\end{aligned}$$

Now, if $r \gg 2M$, we see that the gravitational Doppler shift tends to zero. By defining a substitute variable $x = 2M/r$, we can make a first order Taylor expansion around $x = 0$ for (4) to derive an approximate formula for large r:

$$\begin{aligned}
T_f(x) &= f(0) + f'(0)(x - 0) \\
&= 0 + \left(-\frac{1}{2}\frac{1}{(1-0)^{3/2}}(-1)\right)x \\
&= \frac{1}{2}x \\
&= \frac{M}{r} \ \square
\end{aligned} \tag{9}$$

EXERCISE 2

Introduction

In this exercise we will study the motion of an object moving close to a black hole in hopes of unveiling properties of angular momentum in general relativity. To do this, we will study the Schwarzschild line element and principle of maximum aging when there's curvature in spacetime. Our goal is to see if there is a connection between the path that an object in free float takes in curved spacetime and the behaviour of its angular momentum.

The Situation

The object we will study moves through three points in spacetime, with coordinates (t_1, ϕ_1), (t_2, ϕ_2) and (t_3, ϕ_3), as illustrated in Figure 1. All three of the time coordinates are fixed, as well as the angles ϕ_1 and ϕ_3, while ϕ_2 is unknown. When the object moves from (t_1, ϕ_1) to (t_2, ϕ_2), it experiences a proper time interval $\Delta\tau_{12}$ that we assume to be so small that we can approximate the object's orbit's radius r_A as constant throughout this change $\Delta\phi_{12}$. The same holds for the proper time interval $\Delta\tau_{23}$, where the orbit's radius is denoted by r_B.

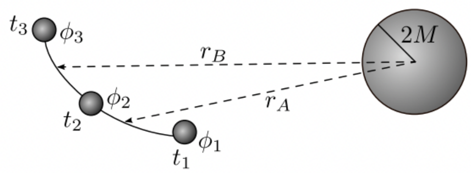

Figure 1: An object passing through three points in spacetime close to a black hole.

We plan to use the fact that the radii's are constant for the respective time proper intervals to help us find what the angle ϕ_2 must be, and then what this can tell us about the object's angular momentum.

Method

To understand how we can use proper time intervals to study angular momentum, we will first take a closer look at the principle of maximum aging in general relativity. We know from Lecture Notes 2B [3] that if an object is moving in *free float*, no external forces work on it. In flat spacetime, the principle of maximum aging tells us that this object will follow the path that leads to the longest proper time interval $\Delta\tau$, or in other words, the longest path in spacetime Δs. When there are massive and dense bodies like white dwarfs, neutron stars and black holes in close viscinity, spacetime is no longer considered flat because their mass is significant enough to curve it (see Figure 2). When this happens, the object will instead follow the path that leads to the center of this massive object, as this is then longest path in spacetime.

In many of these cases, we have to use Schwarzschild geometry instead of Lorentz geometry for observers significantly far away from the object, as we no longer can approximate the observer as a local inertial frame. For the object passing through the three points, we can use the Schwarzschild line element to find an expression for the change in its proper time coordinates $\Delta\tau_{13}$. Since this time interval is the sum of $\Delta\tau_{12}$ and $\Delta\tau_{23}$, and we defined the orbital radii r_A and r_B as constant during these proper time intervals, this should be an easy task. From here, we can use the principle of maximum aging in general relativity to find what path the object will follow from the starting and ending points. Since we know that the object will follow the path where $\Delta s = \Delta\tau$ is at its maximum, we can find this by taking the derivative

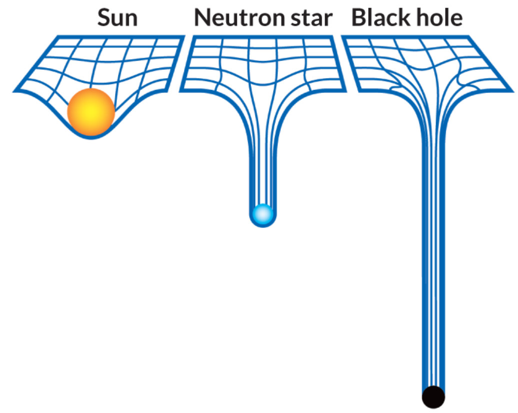

Figure 2: Curvature in spacetime due to the Sun, a neutron star and a black hole [7].

of our newfound expression for $\Delta\tau_{13}$ with respect to the angle ϕ_2.

When we find the derivative of $\Delta\tau_{13}$, we really find the sum of the derivatives of $\Delta\tau_{12}$ and $\Delta\tau_{23}$. To find the ϕ_2 that gives the longest proper time interval, we must demand that $d(\Delta\tau_{13})/d\phi_2$ is zero, which means that $d(\Delta\tau_{12})/d\phi_2$ must be equal to $-d(\Delta\tau_{23})/d\phi_2$ and vice versa. We will attempt to use this to find how the angle ϕ changes as the proper time τ passes by, which hopefully can help us study the properties of angular momentum in general relativity.

Conclusion

When performing the calculations, we found that the change in angle $d\phi$ in a proper time interval $d\tau$ times the radius of the trajectory r powered by two is a conserved quantity. We recognize this as the classical spin divided by the object's mass. This is a crucial result of our derivations, as this shows that an object in free float will follow a path where its angular momentum per mass is constant. This is very similar to Newtonian mechanics, where the object's angular momentum remains unchanged if there is no torque working on it from external forces.

EXERCISE 6

Introduction

A rocket has launched from a shell around a black hole, moving both tangentially and radially inward. Suddenly the rocket engines stop working, and we wish to find out whether or not the rocket will be "swallowed" by the black hole. In order for us to determine the onboard astronaut's fate, we will attempt to study the general relativistic gravitational potential around a black hole. If we find that all hope really is lost for the astronaut, we want to use this study to decipher how the intense forces affect them on their way toward the singularity.

The Situation

The rocket with mass m is launched from a shell at $r = R = 20M$, where M is the black hole's mass. The launch is directed at an angle $\theta = 167°$ off of the radial vector going from out of the black hole's singularity (see Figure 3). After the launch, its absolute velocity is $v = 0.993c$ until the engines stop working. Because of the launch direction, this velocity has both a radial and a tangential component, which means that the rocket has an angular momentum L.

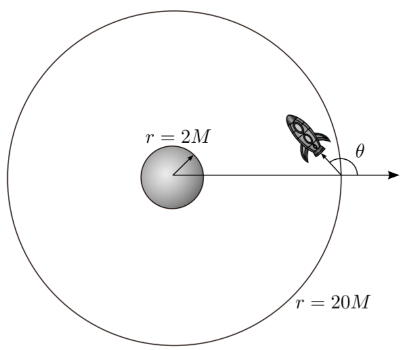

Figure 3: The rocket launch direction from the shell at distance R from the black hole's singularity.

In Newtonian mechanics we're used to approximating gravity as a force, although we know from general relativity that this really is inaccurate. In reality, gravity is curvature of spacetime due to the presence of mass. However, this is hard to visualize, and we will therefore attempt to draw the usual arrows representing gravitational forces on the astronaut once we've learned more than the black hole's gravitational potential, to determine how the astronaut would experience a journey toward the singularity.

Method

To be able to determine if the rocket can avoid being swallowed by the black hole, we want to study the gravitational potential around it. We that an object's gravitational potential per mass around a body with mass M is defined as

$$U = -G\frac{M}{r} \tag{10}$$

Where G is the gravitational constant and r is the object's distance from the body's center of mass. What's interesting in this expression is the ratio M/r, which we can sketch to further study whether or not the astronaut will be "captured" by the black hole.

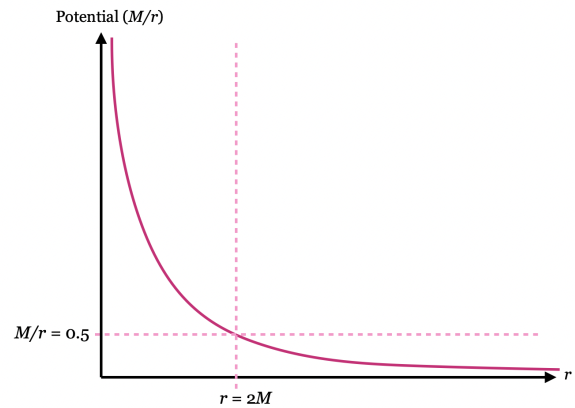

Figure 4: Sketch of the gravitational potential's magnitude $-U/G$ around the black hole as a function of distance r from the singularity.

As evident in Figure 4, the gravitational potential tends to $-\infty$ as we move closer to the singularity. We know that the gravitational potential around a massive body is the amount of work needed to help a gravitationally bound object at distance r from the massive body move toward a reference point where $U = 0$. The most natural way to define this reference point is infinitely far away from the massive body, where the object can move freely. Furthermore, we know that after entering the event horizon, where $r = 2M$, a free falling object can no longer escape the black hole.

Now, how can we use our knowledge of the black hole's gravitational potential to determine whether or not the free falling object will be captured by it? It certainly seems that since only gravity is working on it, it has be drawn to the hole, and eventually enter the event horizon. If we were to have sketched $-M/r$ instead of M/r, the curve in Figure 4 would have looked like a steep hill instead. We could then imagine that if an object were to get too close to the black hole, it would fall down this hill and then be unable to get up again. However, Newtonian mechanics are not valid in this case because of the black hole's enormous mass, and we need to find the black hole's potential in another way. Usually we're familiar with measuring an object with mass m's total energy in a body with mass M's gravitational field as the sum of its kinetic and potential energy:

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r} \tag{11}$$

v denotes the object's absolute velocity relative to the massive body. Here, the gravitational potential is the *effective potential* around the massive body. However, from Lecture Notes 2D [5] we know that we can redefine this in general relativity to take the object's *angular momentum per mass* into consideration. This is an conserved quantity defined as

$$\frac{L}{m} = R\gamma_{\text{shell}}v_{\text{shell}}\sin\theta \tag{12}$$

which we can use to define the general relativistic effective potential around the black hole:

$$V_{\text{eff}}(r) = \sqrt{\left(1 - \frac{2M}{r}\right)\left[1 + \frac{(L/m)^2}{r^2}\right]} \tag{13}$$

Not surprisingly, this version of the effective potential naturally behaves different from the gravitational potential alone. Figure 5 illustrates this.

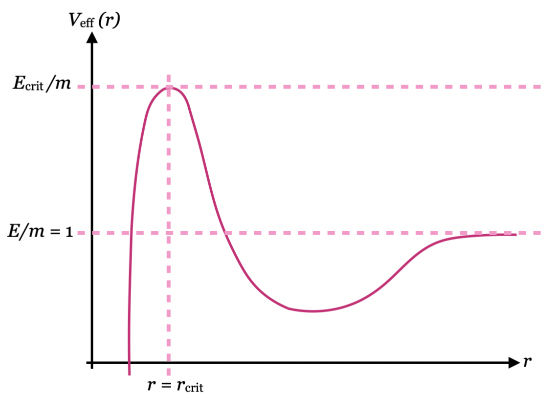

Figure 5: Sketch of the relativistic effective potential V_{eff} per mass as a function of distance r from the black hole's singularity.

We can see the peak in effective potential around the black hole is the total energy per mass an object needs to have in order to enter the event horizon. If the object's energy per mass is less than 1, it will fall into orbit close to the black hole, and alternate between two effective potential states. If its effective potential is between 1 and E_{crit}/m, it will get close to the black hole, but eventually swung away from it and move infinitely far away. If its effective potential is more than E_{crit}/m, it will get too close, surpass r_{crit}, and get swallowed by the black hole.

We have found that the rocket's total energy per mass can be defined in the following way:

$$\frac{E}{m} = \sqrt{1 - \frac{2M}{R}}\gamma_{\text{shell}} \tag{14}$$

where the velocity v_{shell} in γ_{shell} is the rocket's velocity as observed by the shell observers at $r = R$ before the engine stops working. We also found that this is a conserved quantity when the rocket is in free float, which it is after the engines stop working. Therefore, if the rocket's energy per mass is larger than the maximum effective potential needed to enter the black hole, it will be absorbed unless the engines start working again. To calculate this maximum, we need to first find r_{crit}. We found that the distances from the black hole where the effective potential is at its maximum and minimum are

$$r_{\text{extremum}} = \frac{(L/m)^2}{2M}\left(1 \pm \sqrt{1 - \frac{12M^2}{(L/m)^2}}\right) \tag{15}$$

We see from Figure 5 that the effective potential has its maximum closer to the singularity than its minimum, which means that we must find r_{crit} by using (-) in (15). This gives us

$$r_{\text{crit}} \approx 3M$$

Using (12), we find that the rocket's angular momentum and total energy per mass is

$$\frac{L}{m} \approx 37.824 \text{ m}$$
$$\frac{E}{m} \approx 8.032 \text{ m}$$

which we use to plot the effective potential per mass m as a function of distance r from the black hole's singularity expressed using M (see Figure 6). Now that we have found the rocket's angular momentum, total energy per mass and a way to calculate the effective potential, we should be able to determine the astronaut's fate.

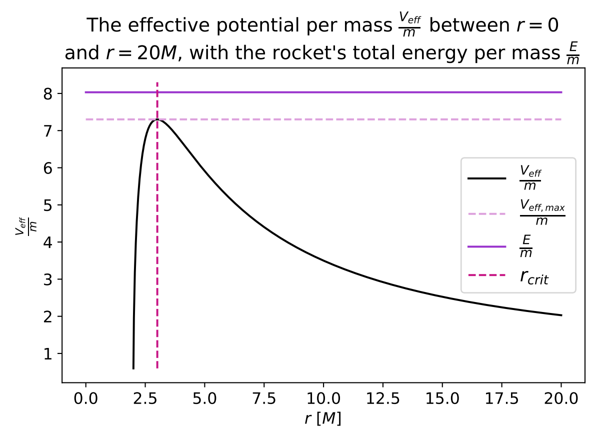

Figure 6: Plot of the effective potential V_{eff} per mass between the $r = R$ and $r = 0$ along with the rocket's energy per mass.

Conclusion

Studying Figure 6, we can definitely conclude that the rocket will surpass r_{crit} and be absorbed by the black hole. This was also evident in our calculation of the rocket's total energy per mass E/m was larger than the maximum effective potential, which we found to be approximately

$$\frac{V_{\text{eff,max}}}{m} \approx 7.302 \text{ m}$$

We were interested in seeing what would happen to the astronaut if it enters the event horizon and moves closer to the singularity It is evident from Figures 4 and 5 that the gravitational potential rapidly tends to $-\infty$ after the rocket has surpassed r_{crit}, and it will therefore need large amounts of energy to be able to escape. This must mean that the "force" of gravity rapidly increases toward the singularity. The forces working on the astronaut's legs must then be significantly larger than the forces working on their head, if we imagine them falling towards the singularity with their feet first. Because of this large difference, the astronaut will be stretched out and elongated as it gets closer to the singularity (see Figure 7). This effect is also called *spaghettification*.

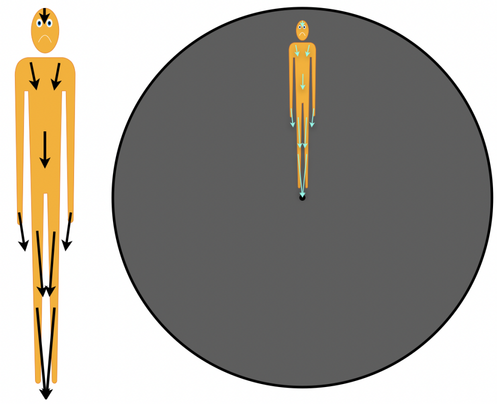

Figure 7: Sketch of the gravitational forces working on the astronaut as it closes in on the event horizon, and how this affects them.

EXERCISE 7

Introduction

The poor astronaut enters the event horizon and is close to accepting their fate, when the rocket engines suddenly start working again. They know that they never had a chance to escape while in free float, but maybe there's hope now? To determine whether or not this is true, the astronaut fires off two light beams, one backwards, directed radially out of the event horizon, and one forward, towards the singularity. Will shell observers at $r > 2M$ be able to observe these light beams? And what about the astronaut themself?

The Situation

We knew that we no longer could use Lorentz geometry for the astronaut once they were inside the event horizon, since they would be accelerated toward the singularity at such a large rate that we couldn't approximate them as a local inertial frame anymore, even when we made time and space intervals infinitesimal. Therefore, we found a way to rewrite the Schwarzschild line elements in terms of the astronaut's wristwatch time dt' instead of Schwarzschild time dt before they entered the black hole, as we were able to derive the following relation between the two:

$$dt = dt' - \frac{\sqrt{2M/r}dr}{\left(1 - \frac{2M}{r}\right)} \tag{16}$$

We did this so that we could use the shell observers outside the black hole, which use Lorentz geometry, as help in our attempt to determine whether or not the astronaut would have any chance of escaping if the rocket engines were to start working.

Thanks to the shell observers' properties, we were able to find that the velocities of the emitted light beams' as measured by the astronaut is

$$v' = \frac{dr}{dt'} = -\sqrt{\frac{2M}{r}} \pm 1 \tag{17}$$

For the light beam fired toward the singularity, we have -1 in the expression above, while we have $+1$ for the light beam fired toward the event horizon. It's valuable information for us to see how the astronaut observes the light beams' velocities when they're emitted from different distances r from the singularity. Therefore, we'll attempt to use (17) to decipher how and if the astronaut themself, but also the shell observers, measure the light beams when they're emmited close to and inside the event horizon.

By reminiscing back to our special relativity experiments, where we drew worldlines for different spaceships moving with both constant and varying velocities relative to a space station, we'll attempt to use Figure 8 to determine the light beams' trajectories. This illustrates the rocket's worldline relative to the black hole's singularity, and the start of the light beams' worldlines when emitted at different distances r. The astronaut is marked by the balls at the different positions in spacetime when they would have emitted the light beams. Their respective worldlines are marked by the arrows.

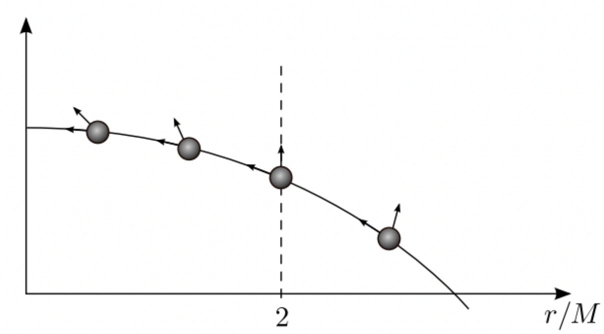

Figure 8: The rocket's worldline as it moves toward the center of the black hole, and the beginning of the light beams' worldlines (arrows) when emitted at different distances r from the singularity.

Method

We see from (17) that if the astronaut fires the light beams when $r > 2M$, still outside the event horizon, they will measure that the light beam fired toward the black hole will move with $v > c$, while the other one will move with $v < c$. This is because, as we know, the black hole's gravitational pull is so immense close to and inside the event horizon that we no longer can approximate the astronaut as being a local inertial frame, even though we've made the space and time intervals infinitesimal. We know that a far-away observer will observe that time runs slower for an observer closer to a source of gravity. This is also true for the astronaut at distance r from the black hole's singularity. They will measure that time moves slower for the light beam moving toward the singularity, and faster for the light beam moving away from the singularity. Because the light beams still are travelling the same distances, the astronaut will therefore measure their velocities as different from the light speed.

Now, if the astronaut fires the light beams when they've just entered the event horizon, we get $dr/dt' = -2$ for the light beam fired forwards and $dr/dt' = 0$ for the light beam fired backwards. To be able to understand the reason for this, we have to remember that gravity is not really a force, but curvature in spacetime caused by the presence of mass. Let's start by imagining how this affects the path of the light beam fired backwards out of the black hole. Normally this light beam would follow a straight path away from the astronaut which is parallell to their line of sight, but because of the black hole's enormous mass, this path now curves so much that the light beam falls back toward the astronaut.

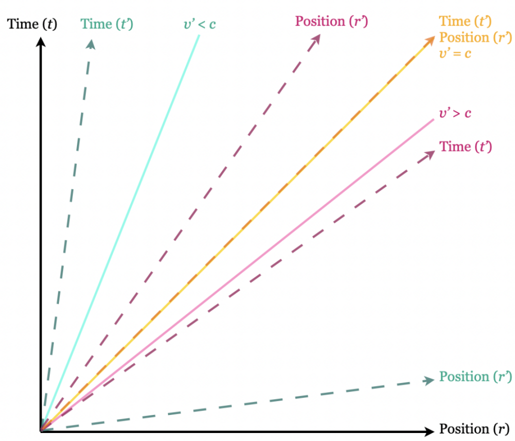

Figure 9: Worldlines and corresponding axes for an object travelling with $v < c$, light travelling with $v = c$ and an object travelling with $v > c$ relative to an observer in flat spacetime.

To attempt to explain this, we can again look back at how we made spacetime diagrams for observers in special relativity. We know that for an object with mass that creates a negligible curve in spacetime and constant velocity v relative to an observer that is situated in flat spacetime, the angle between their worldline and the time-axis is between 0° and 45°. We can draw on the moving object's spacetime diagram's corresponding time and position axes, and see that these are closer together in the observer's diagram (see Figure 9). Photons' worldlines are at a 45° angle from the time axis, which means that the axes corresponding to their spacetime diagram are on top of each other. This is why the proper time interval of light is zero. If the angle of an object's worldline is more than 45°, the two axes switch place meaning that the object travels with a velocity larger than the speed of light relative to the observer. We know that this is physically impossible, even for light when the observer and the light beam both are situated in flat spacetime. This is the case in the light green area in Figure 10.

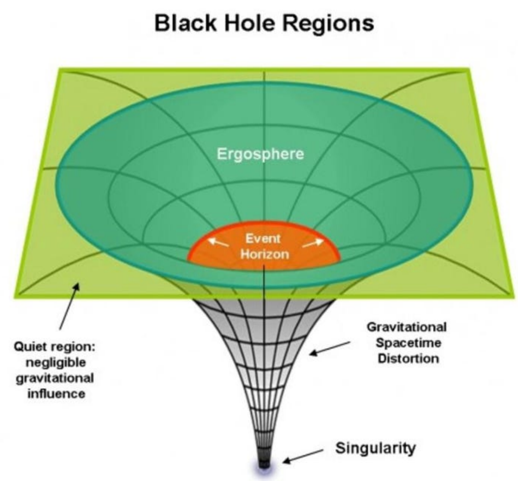

Figure 10: The regions of the curvature in spacetime caused by a black hole's mass [1].

Here, we can imagine the time axis as being perpendicular to the ground, meaning that anything can move around freely no matter how small their velocity is. When the curvature at a point in spacetime gets to 45°, we can imagine that the time axis is parallell to the ground, meaning that an object has to travel with the speed of light to move up from the curve, (or in other words, escape the gravitational pull of the object causing the curve). As we move further into the black hole, the curve grows steeper, and we can imagine the time axis now becoming more horizontal until the ground is vertical and they once again are perpendicular. In these cases, we can see from the spacetime diagrams that an object has to travel with a velocity larger than the speed of light to move up from the curve, which is impossible.

We see from (17) that the light beams' observed velocities both tend to $-\infty$, meaning that they seem to travel faster away from the astronaut and toward the singularity the closer they all get to it. To be able to interpret the result of this, we can once again consider the spacetime diagrams, as well as the curvature of spacetime visualized in Figure 10. We see that since photons move along the "ground", and this grows more vertical the closer we get to the singularity, the light beams soon appear to be able to travel an infinite distance in no period of time. This is because they simply fall straight down the hole, and they always move faster than an object with mass, so they're further ahead than the astronaut.

To further study how the light beams will move when emitted from different distances from the black hole's singularity, we will start by interpreting the sketch in Figure 8. We see from the curve of the worldline that the rocket's velocity also grows infinitely large as it gets closer to the singularity, and that the light beam emitted forwards always follows the same worldline as the rocket in the beginning. What's more interesting is the light beam that the astronaut attempts to fire backwards out of the black hole, which we can see follows a worldline closer and closer to the rocket itself as they both near the singularity. In flat spacetime, this light beam's worldline would be perpendicular to the rocket's when fired by the astronaut, as it moves away from them. However, because the spacetime curvature already is significant before they enter the horizon, the emitted light beam is immediately slowed down once it is fired, as we also could see in (17). As expected, this effect grows larger and larger the closer they get, and when fired at $r/M = 2$ the beginning of the light beam's worldline is vertical, meaning that its velocity relative to both the astronaut and the singularity is zero. After entering the event horizon, we see that the start of its worldline is pointed towards lower values of r, meaning that it moves towards the singularity as well.

Conclusion

It is obvious that shell observers at $r > 2M$ won't be able to observe the light beams emitted from the astronaut when they have crossed the event horizon, as light no longer can move up the spacetime curvature caused by the black hole. Even when fired from outside the event horizon, the light beam fired forwards will always move straight towards the singularity with increasing speed. From Figure 8, it's evident that the light beam fired backwards, although moving slower relative to the astronaut, is able to escape onto flat spactime. Thus, shell observers will only be able to measure the latter light beam's velocity when they are fired at $r > 2M$.

If the astronaut emits the light beams at $r < 2M$, the one fired forwards will naturally move straight toward the black hole right away, and the astronaut will never actually observe it. This is exactly like if you point a laser pen out into vacuum, its light moves away from you until it meets an object, and you therefore aren't able to observe it unless you can see this object. We saw in Figure 8 that after entering the event horizon, the light beam fired backwards immediately heads towards the singularity as well. This is also evident in (17), as its velocity is negative when $r < 2M$.

When emitting the backwards light beam at $r = 2M$, the light beam will stand still relative to the astronaut, before moving down towards the singularity and away from them. Therefore, none of the light beam will never meet their eyes when emitted at or after $r = 2M$, and the astronaut can not actually measure their velocities. As there's no chance of the light beams escaping when emitted from inside the event horizon, we can safely conclude that the astronaut has no chance of escaping out of the black hole once they've entered either, even though their rocket can accelerate up to velocities very close to the light speed.

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SMILEY-FACE TASKS

EXERCISE 1

Task 4

We find that the mass of the Sun in meters is

$$M_{\mathrm{m}} = \frac{G}{c^2} M_{\mathrm{kg}} \approx 1476.674 \text{ m}$$

which gives us the ratio

$$\frac{\Delta\lambda}{\lambda_{\mathrm{shell}}} \approx \frac{M_{\mathrm{m}}}{R} \approx 2.122573 \times 10^{-6}$$

Where R is the Sun's radius in meters. If we calculate this ratio using the formula necessary for small r, we get

$$\frac{\Delta\lambda}{\lambda_{\mathrm{shell}}} = \frac{1}{\sqrt{\left(1 - \frac{2M_{\mathrm{m}}}{R}\right)}} - 1 \approx 2.122579 \times 10^{-6}$$

We see that this difference is in magnitude 10^{-12}, which is insignificant. Thus, the ratio M/r is an accurate approximation.

The redshift measured by a far-away observer is

$$\frac{\Delta\lambda}{\lambda_{\mathrm{shell}}} \lambda_{\mathrm{max}} \approx 0.001061 \text{ nm}$$

which is insignificant when considering that $\lambda_{\mathrm{max}} = 500$ nm. Thus, we can conclude that the apparent color of the Sun is the same for the far-away observer as for the shell observer at the Sun's surface.

We find that the mass of the Earth in meters is

$$M_{\mathrm{m,Earth}} = \frac{G}{c^2} M_{\mathrm{kg,Earth}} \approx 0.00443 \text{ m}$$

which gives us the ratio

$$\frac{\Delta\lambda}{\lambda_{\mathrm{shell}}} \approx \frac{M_{\mathrm{m,Earth}}}{R_{\mathrm{Earth}}} \approx 6.961 \times 10^{-10}$$

Since we found that the redshift of the light sent from the shell observer at the Sun's surface that an observer at Earth's distance would measure is so insignificant, we don't really need to take this into account. Thus, we can calculate the blueshift measured by a shell observer at the Earth's surface from light sent by a far-away observer of the Earth with wavelength λ_{max}:

$$\frac{\Delta\lambda}{\lambda_{\mathrm{shell}}} \lambda_{\mathrm{max}} \approx 3.481 \times 10^{-7} \text{ nm}$$

As we can see, this blueshift is even smaller than the redshift previously calculated, making it negligible. We can therefore conclude that the apparent color of the sun is the same for an observer on the Earth.

Task 5

We have $\lambda_{\mathrm{shell}} = 600$nm for a shell observer situated at the exact radius where the radiation is emitted from, and $\Delta\lambda = 2150$ nm $- 600$ nm $= 1550$ nm. By using the formula

$$\frac{\Delta\lambda}{\lambda_{\mathrm{shell}}} = \frac{M}{r} \tag{18}$$

we find that the light must be emitted at a distance

$$r = \frac{\lambda_{\mathrm{shell}}}{\Delta\lambda} M \approx 0.387M$$

from the center of the black hole. However, this is definitely not a case where $r \gg 2M$, which means we need to use the equation

$$\frac{\Delta\lambda}{\lambda_{\mathrm{shell}}} = \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}} - 1 \tag{19}$$

Solving this for r, we get

$$r = 2\frac{\left(\frac{\Delta\lambda}{\lambda_{\mathrm{shell}}} + 1\right)^2}{\left(\frac{\Delta\lambda}{\lambda_{\mathrm{shell}}} + 1\right)^2 - 1} M \approx 2.167M$$

This distance from the singularity of a black hole corresponds to around where its accretion disk is situated. Since the redshift measured by an observer far away is so tremendous, there must be a very large source of gravity at the center of the quasar. We saw in Task 4 that the reshift measured for the Sun's radiation due to its gravitation by a far-away observer was minuscule in comparison, meaning that this source must have an enormous mass in comparison to the Sun. Combined with the fact that the radiation comes from a distance to the center fo the quasar that is so close to the horizon of a hypothetical black hole, we can conclude that the likelyhood of this source of gravity being a black hole is very large.

Task 6

For a shell observer living at a shell at $r = 2.01M$, we need to use (19) to find the blueshift that the shell observer at this distance from the black hole observes for incoming radiation. We find the following ratio:

$$\frac{\Delta\lambda}{\lambda_{\mathrm{shell}}} = \frac{1}{\sqrt{\left(1 - \frac{2M}{2.01M}\right)}} - 1 \approx 13.177 \tag{20}$$

To find the observed wavelength of incoming radiation for the shell observer, we remember that $\Delta\lambda = \lambda - \lambda_{\mathrm{shell}}$, and solve (20) for λ_{shell}:

$$\frac{\lambda - \lambda_{\text{shell}}}{\lambda_{\text{shell}}} \approx 13.177$$
$$\lambda_{\text{shell}} \approx \frac{\lambda}{14.177}$$

We see that radiation from stars and other celestial objects is strongly blueshifted for the shell observer. Visible light has wavelengths from approximately 400nm to 750nm. This means that the shell observer will measure these wavelengths as going from around 28nm to 53nm. Thus, the observer, or their telescope, needs to be able to observe the part of the UV-spectrum with shorter wavelengths, as this spectrum goes from 10nm to 400nm.

EXERCISE 2

Task 1

From Lecture Notes 2C [4] we know that the Schwarzschild line element is defined in the following way:

$$\Delta s^2 = \Delta\tau^2 = \left(1 - \frac{2M}{r}\right)\Delta t^2 - \frac{\Delta r^2}{\left(1 - \frac{2M}{r}\right)} - r^2\Delta\phi^2 \quad (21)$$

This gives us that the proper time intervals $\Delta\tau_{12} = \tau_2 - \tau_1$ and $\Delta\tau_{23} = \tau_3 - \tau_2$ are given by

$$\Delta\tau_{12} = \sqrt{\left(1 - \frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}^2}{1 - \frac{2M}{r_A}} - r_A^2\Delta\phi_{12}^2}$$
$$\Delta\tau_{23} = \sqrt{\left(1 - \frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}^2}{1 - \frac{2M}{r_B}} - r_B^2\Delta\phi_{23}^2}$$

Since $\Delta\tau_{13} = \Delta\tau_{12} + \Delta\tau_{23}$, we get

$$\Delta\tau_{13} = \sqrt{\left(1 - \frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}^2}{1 - \frac{2M}{r_A}} - r_A^2\Delta\phi_{12}^2} + \sqrt{\left(1 - \frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}^2}{1 - \frac{2M}{r_B}} - r_B^2\Delta\phi_{23}^2} \quad \square \quad (22)$$

Task 2

To find at what ϕ_2 we get the longest proper time interval $\Delta\tau_{13}$, we need to take the derivative of $\Delta\tau_{13}$ as defined in (22) with respect to ϕ_2. We remember that we can rewrite $\Delta\phi_{12}$ and $\Delta\phi_{23}$ in the following way:

$$\Delta\phi_{12} = \phi_2 - \phi_1$$
$$\Delta\phi_{23} = \phi_3 - \phi_2$$

Furthermore, to make the next calculations easier to follow, we'll define the following constants:

$$C_1 = \left(1 - \frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}^2}{1 - \frac{2M}{r_A}}$$
$$C_2 = \left(1 - \frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}^2}{1 - \frac{2M}{r_B}}$$

as these are independent of ϕ_2 and will remain as they are throughout the derivation because of the square root. Thus, we get

$$\frac{\mathrm{d}\left(\Delta\tau_{13}(\phi_2)\right)}{\mathrm{d}\phi_2} = \frac{\left(-2r_A^2\left(\phi_2 - \phi_1\right)\right) \times 1}{2\sqrt{C_1 - r_A^2\left(\phi_2 - \phi_1\right)^2}} + \frac{\left(-2r_B^2\left(\phi_3 - \phi_2\right)\right) \times (-1)}{2\sqrt{C_2 - r_B^2\left(\phi_3 - \phi_2\right)^2}}$$

For this to be zero, giving us the longest possible proper time interval, we need to demand the following:

$$-\frac{\left(-2r_A^2\left(\phi_2 - \phi_1\right)\right) \times 1}{2\sqrt{C_1 - r_A^2\left(\phi_2 - \phi_1\right)^2}} = \frac{\left(-2r_B^2\left(\phi_3 - \phi_2\right)\right) \times (-1)}{2\sqrt{C_2 - r_B^2\left(\phi_3 - \phi_2\right)^2}}$$
$$\frac{r_A^2\left(\phi_2 - \phi_1\right)}{\sqrt{C_1 - r_A^2\left(\phi_2 - \phi_1\right)^2}} = \frac{r_B^2\left(\phi_3 - \phi_2\right)}{\sqrt{C_2 - r_B^2\left(\phi_3 - \phi_2\right)^2}}$$
$$\frac{r_A^2\Delta\phi_{12}}{\Delta\tau_{12}} = \frac{r_B^2\Delta\phi_{23}}{\Delta\tau_{23}} \quad \square \quad (23)$$

If we let all intervals defined become infinitesimal, replacing Δ with d, we see that for the longest proper time interval, we have

$$r^2\frac{d\phi}{d\tau} = constant$$

where r and $d\tau$ are the orbital radius and proper time interval between ϕ and $\phi + d\phi$, respectively. The principle of maximum aging tells us that an object in free float always will follow the path through spacetime with the longest possible proper time interval. An orbiting object, or an object "falling" towards a black hole, are considered as being in free float in general relativity as gravity is not really a force, but spacetime geometry in the viscinity of masses. Thus, this quantity must be conserved in the cases we study.

Task 3

We know from the celestial mechanics studied in Lecture Notes 1B [2] that the angular velocity of an orbiting body is

$$v_\phi = r\frac{d\phi}{dt}$$

For an observer positioned at a shell with distance r from the center of mass is then defined as

$$v_{\phi,\text{shell}} = r\frac{d\phi_{\text{shell}}}{dt_{\text{shell}}} \tag{24}$$

Since a shell observer can use Lorentz geometry for small time intervals, we also get

$$dt_{\text{shell}} = \frac{d\tau}{\sqrt{1 - v_{\text{shell}}}} = \gamma_{\text{shell}} d\tau$$

where v_{shell} is the shell observer's absolute velocity relative to a far-away observer. Inserting this into (24), we get

$$\begin{aligned} v_{\phi,\text{shell}} &= r\frac{d\phi_{\text{shell}}}{d\tau\gamma_{\text{shell}}} \\ \gamma_{\text{shell}} r v_{\phi,\text{shell}} &= r^2\frac{d\phi_{\text{shell}}}{d\tau} \quad \square \end{aligned} \tag{25}$$

Task 4

We have now defined the relativistic formula for angular momentum per mass, L/m, in two seperate ways. The non-relativistic formula is defined as

$$\frac{L}{m} = rv_\phi = r^2\frac{d\phi}{dt} \tag{26}$$

We see that when $v_{\text{shell}} \ll 1$, the quantity γ_{shell} tends to 1. In addition to this, we also know that the difference between the proper time time interval $d\tau$ and the time interval dt_{shell} measured by an observer far away becomes insignificantly small when their relative velocities are much smaller than the light speed. Thus, we get

$$\gamma_{\text{shell}} r v_{\phi,\text{shell}} \approx r v_{\phi,\text{shell}} = r^2\frac{d\phi_{\text{shell}}}{dt_{\text{shell}}} \approx r^2\frac{d\phi}{d\tau} \quad \square \tag{27}$$

EXERCISE 6

Task 2

We know from Lecture Notes 2D [5] that the observed total energy of a free falling object can be defined in the following way:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \tag{28}$$

Now, when time intervals become sufficiently small, like when we use the infinitesimal dt, the shell observer can use Lorentz geometry. We know from Lecture Notes 2C [4] that we can write $d\tau$ in the following way

$$\frac{dt}{d\tau} = \frac{dt}{dt_{\text{shell}}}\frac{dt_{\text{shell}}}{d\tau} = \frac{1}{\sqrt{1-\frac{2M}{r}}}\frac{dt_{\text{shell}}}{d\tau} \tag{29}$$

Furthermore, we saw in Exercise 2, Task 3 that

$$\frac{dt_{\text{shell}}}{d\tau} = \gamma_{\text{shell}} \tag{30}$$

when using Lorentz geometry. By inserting (29) and (30) into (28), we find that the rocket's total energy per mass at the moment its engine stops working, as observed by a shell observer, can be written as

$$\frac{E}{m} = \sqrt{1 - \frac{2M}{R}}\gamma_{\text{shell}} \quad \square \tag{31}$$

Since this is a conserved quantity, this expression is also valid when the rocket's distance from the singularity is $r < R$.

Task 3

From the same Lecture Notes we know that the general relativistic expression for the effective potential is

$$V_{\text{eff}}(r) = \sqrt{\left(1 - \frac{2M}{r}\right)\left[1 + \frac{(L/m)^2}{r^2}\right]} \tag{32}$$

To find at what distances from the singularity where the effective potential is at its maximum and minimum, we have to find where the derivative of (32) with regards to the distance r is zero. Let's first find the derivative:

$$\begin{aligned} \frac{\mathrm{d}\left(V_{\text{eff}}(r)\right)}{\mathrm{d}r} &= \frac{\mathrm{d}\left(\sqrt{\left(1-\frac{2M}{r}\right)\left[1+\frac{(L/m)^2}{r^2}\right]}\right)/\mathrm{d}r}{2\sqrt{\left(1-\frac{2M}{r}\right)\left[1+\frac{(L/m)^2}{r^2}\right]}} \\ &= \frac{M - \frac{(L/m)^2}{r} + \frac{3M(L/m)^2}{r^2}}{r^2\sqrt{\left(1-\frac{2M}{r}\right)\left[1+\frac{(L/m)^2}{r^2}\right]}} \end{aligned}$$

For this expression to be zero, we only need to focus on the numerator. As long as r is neither zero nor $2M$, this should be fine. We'll demand

$$\begin{aligned} M - \frac{(L/m)^2}{r} + \frac{3M(L/m)^2}{r^2} &= 0 \\ r^2 - \frac{(L/m)^2}{M}r + 3\left(\frac{L}{m}\right)^2 &= 0 \end{aligned}$$

Solving this for r, we get

$$\begin{aligned} r_{\text{extremum}} &= \frac{\frac{(L/m)^2}{M} \pm \sqrt{\frac{(L/m)^4}{M^2} - 12\left(\frac{L}{m}\right)^2}}{2} \\ &= \frac{(L/m)^2}{2M}\left(1 \pm \sqrt{1 - \frac{12M^2}{(L/m)^2}}\right) \quad \square \end{aligned} \tag{33}$$

We can see from Figure 5 that the critical distance r_{crit} where the effective potential is at its maximum is closer to $r = 0$ than the distance where it seems to have its minimum. Therefore, we must get the maximum effective potential when there's a minus sign in (33), and the minimum when there's a plus sign.

Task 4

Since we can use Lorentz geometry for the shell observers when using infinitesimal time intervals, we have

$$\frac{d\phi_{\text{shell}}}{d\tau} = \frac{d\phi_{\text{shell}}}{dt_{\text{shell}}}\frac{dt_{\text{shell}}}{d\tau} = \frac{d\phi_{\text{shell}}}{dt_{\text{shell}}}\gamma_{\text{shell}}$$

Furthermore, we know that

$$v_{\phi,\text{shell}} = R\frac{d\phi_{\text{shell}}}{dt_{\text{shell}}} = v_{\text{shell}}\sin\theta$$

at the moment when the rocket is launched. In Exercise 2, Task 2 we showed that $r^2 d\phi/d\tau = L/m$ is a conserved quantity. Thus, we can express the rocket's angular momentum per mass as observed by the shell observers in the following way:

$$\frac{L}{m} = R\gamma_{\text{shell}}v_{\text{shell}}\sin\theta \;\square \tag{34}$$

Even though $r < R$ as it approaches the black hole.

Task 6

By approximating the rocket's angular momentum as zero, and assuming that the black hole it's falling into has the same mass as the supermassive black hole at the center of the Milky Way, we can calculate how long it will take on the astronaut's wristwatch from they enter the horizon until they reach the singularity. To do this, we first remember the following expression from Lecture Notes 2D [5]:

$$\frac{dr}{d\tau} = \sqrt{\left(\frac{E}{m}\right)^2 - \left(1-\frac{2M}{r}\right)\left[1+\frac{(L/m)^2}{r^2}\right]} \tag{35}$$

By replacing the infinitesimal intervals with Δr and $\Delta\tau$, inserting $L/m = 0$ and (28) for E/m, and solving for $\Delta\tau$, we get

$$\Delta\tau = \frac{\Delta r}{\sqrt{\left(1-\frac{2M}{R}\right)\left(\gamma_{\text{shell}}^2 - 1\right)}} \tag{36}$$

Inserting our numbers, with $\Delta r = R = 20M$, and dividing the result by c to get the time interval in seconds, we get

$$\begin{aligned}\Delta\tau &= \frac{20\times 4\times 10^6 M_\odot}{\sqrt{\frac{9}{10}\left(\frac{1}{(1-(0.993)^2)}-1\right)}}\frac{1}{c}\\ &\approx 49.41 \text{ s}\end{aligned}$$

EXERCISE 7

Task 1

From Lecture Notes 2C [4] we know that the Lorentz line element in polar coordinates is

$$\Delta s^2 = \Delta\tau^2 = \Delta t^2 - \Delta r^2 - r^2\Delta\phi^2 \tag{37}$$

Since the astronaut is only moving in the radial direction, we get

$$(dt')^2 = dt_{\text{shell}}^2 - dr_{\text{shell}}^2 \tag{38}$$

where we've used that the proper time interval $\Delta\tau$ here is the infinitesimal time interval dt' measured on the astronaut's wristwatch. We also let Δt and Δr denote the infinitesimal time and position intervals measured for the astronaut by the shell observers. From the same Lecture Notes we know that we can rewrite (38) in the following way:

$$\begin{aligned}dt' &= -\frac{dr_{\text{shell}}^2}{dt'} + \frac{dt_{\text{shell}}^2}{dt'}\\ &= -v_{\text{shell}}\gamma_{\text{shell}}dr_{\text{shell}} + \gamma_{\text{shell}}dt_{\text{shell}} \;\square\end{aligned} \tag{39}$$

Task 2

Lecture Notes 2C [4] tells us that we can relate the time and position intervals measured by the shell observers to the corresponding Schwarzschild intervals in the following way:

$$dt_{\text{shell}} = \sqrt{\left(1-\frac{2M}{r}\right)}dt \tag{40}$$

$$dr_{\text{shell}} = \frac{dr}{\sqrt{\left(1-\frac{2M}{r}\right)}} \tag{41}$$

Applying this to (39), we get

$$dt' = -\frac{v_{\text{shell}}\gamma_{\text{shell}}dr}{\sqrt{\left(1-\frac{2M}{r}\right)}} + \gamma_{\text{shell}}\sqrt{\left(1-\frac{2M}{r}\right)}dt \;\square \tag{42}$$

Task 3

In Lecture Notes 2D [5] we found the following expression for the shell velocity v_{shell} of a falling spaceship starting with velocity $v = 0$ far from a black hole:

$$v_{\text{shell}} = -\sqrt{\frac{2M}{r}} \tag{43}$$

Remembering that γ_{shell} is defined in the following way:

$$\gamma_{\text{shell}} = \frac{1}{\sqrt{1-v_{\text{shell}}^2}} \tag{44}$$

We can rewrite (42):

$$dt' = \sqrt{\frac{2M}{r}}\frac{1}{\sqrt{(1-\frac{2M}{r})}}\frac{dr}{\sqrt{(1-\frac{2M}{r})}} + \frac{\sqrt{(1-\frac{2M}{r})}}{\sqrt{(1-\frac{2M}{r})}}dt$$
$$= \frac{\sqrt{2M/r}dr}{(1-\frac{2M}{r})} + dt \tag{45}$$

Which gives us the following expression for the infinitesimal Schwarzschild time interval:

$$dt = dt' - \frac{\sqrt{2M/r}dr}{(1-\frac{2M}{r})} \quad \square \tag{46}$$

Task 4

We define the infinitesimal Schwarzschild line element

$$ds^2 = d\tau^2 = \left(1-\frac{2M}{r}\right)dt^2 - \frac{dr^2}{(1-\frac{2M}{r})} - r^2 d\phi^2 \tag{47}$$

By substituting dt in 47 with 46, we get

$$ds^2 = d\tau^2$$
$$= \left(1-\frac{2M}{r}\right)\left(dt' - \frac{\sqrt{2M/r}dr}{(1-\frac{2M}{r})}\right)^2 - \frac{dr^2}{(1-\frac{2M}{r})} - r^2 d\phi^2$$
$$= \left(1-\frac{2M}{r}\right)(dt')^2 - 2\sqrt{\frac{2M}{r}}dr dt' - dr^2 - r^2 d\phi^2 \quad \square \tag{48}$$

Task 5

We know from Lecture Notes 2E [6] that the radial and tangential velocity components that a far-away observer observes for a photon moving in the Schwarzschild spacetime can be defined in the following way:

$$v_r = \frac{dr}{dt} = \pm\left(1-\frac{2M}{r}\right)\sqrt{1-\left(1-\frac{2M}{r}\right)\frac{(L/E)^2}{r^2}} \tag{49}$$

$$v_\phi = r\frac{d\phi}{dt} = \pm\frac{L/E}{r}\left(1-\frac{2M}{r}\right) \tag{50}$$

Where L and E are the photon's angular momentum and energy, respectively. In our case, we know that the light beams only move radially, so $d\phi/dt = 0$ and $L = 0$. Thus, we get

$$\frac{dr}{dt} = \pm\left(1-\frac{2M}{r}\right) \tag{51}$$

By inserting (46) into (51), we can find the observed radial velocities of the two light beams from the falling astronaut's frame of reference:

$$\frac{dr}{dt' - \frac{\sqrt{2M/r}dr}{(1-\frac{2M}{r})}} = \pm\left(1-\frac{2M}{r}\right)$$
$$dr = \pm\left(1-\frac{2M}{r}\right)\left(dt' - \frac{\sqrt{2M/r}dr}{(1-\frac{2M}{r})}\right)$$
$$= \pm\left(1-\frac{2M}{r}\right)dt' \mp \sqrt{\frac{2M}{r}}dr$$
$$dr\left(1 \pm \sqrt{\frac{2M}{r}}\right) = \pm\left(1-\frac{2M}{r}\right)dt'$$
$$\frac{dr}{dt'} = \pm\frac{(1-\frac{2M}{r})}{\left(1\pm\sqrt{\frac{2M}{r}}\right)}$$
$$= \pm\left(1 \mp \sqrt{\frac{2M}{r}}\right)$$
$$= -\sqrt{\frac{2M}{r}} \pm 1 \quad \square \tag{52}$$

We see that the light beam moving towards the singularity has the radial velocity

$$\frac{dr}{dt'} = -\sqrt{\frac{2M}{r}} - 1$$

while the light beam moving away from the singularity has the radial velocity

$$\frac{dr}{dt'} = -\sqrt{\frac{2M}{r}} + 1$$