AST2000 - Part 5
Satellite Launch

Candidates 15361 & 15384
(Dated: December 4, 2022)

We have studied the method of the Hohmann transfer orbit in order for us to be able to travel from our home planet Doofenshmirtz to our destination planet Buttercup. We simulated our planned trajectory, but when the time came for the actual interplanetary travel, we found that we had to make some spontaneous decisions. However, we managed to arrive at our destination, enter a stable orbit, and take some beautiful pictures along the way.

I. INTRODUCTION

We are finally going to travel toward our chosen destination! By simulating our spacecraft's trajectory, we will plan the journey towards our destination planet, Buttercup. Because of miscalculations, inaccurate assumptions and general errors that may occur when simulating the trajectory, we have to keep in mind that it may need adjustments as we are travelling. Fortunately, our spacecraft is able to change its angular orientation without using any fuel, so this shouldn't be too difficult.

As we eventually enter an orbit around Buttercup, we will need to stabilize our orbit by performing minor acceleration and/or deceleration maneuvers. We really don't have time, nor do we want to wait, to see if our orbit is successfully stabilized by actually letting our spacecraft perform multiple orbits. Thus, we will attempt to simulate a few orbits and calculate our spacecraft's angular and tangential velocity over time, as well as properties of each orbit to see if they're similar.

II. METHOD

Due to the gravitational pull from the other astronomical bodies in our solar system, our spacecraft will be affected by these forces and we have to include them in our calculations to better our simulation of its trajectory through space. Meteors, moons and other smaller bodies will be ignored and we will only look at the larger masses such as our sun and the planets. Using the superposition principle, the net gravitational force working on the spacecraft can be written as

$$\mathbf{F}_{net} = \mathbf{F}_s + \sum_i \mathbf{F}_i \tag{1}$$

Where $\mathbf{F}_s$ is the gravitational pull from the star and $\mathbf{F}_i$ s the gravitational pull from planet i. Newton's Gravitational Law tells us that the force from a mass M on a mass m is given by

$$\mathbf{F} = \frac{GMm}{r^3}\mathbf{r} \tag{2}$$

Which gives us a more specific equation

$$m\ddot{\mathbf{r}} = -G\frac{mM_s}{|\mathbf{r}|^3}\mathbf{r} - \sum_{i=1}^{N} G\frac{mM_i}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i) \tag{3}$$

Where M_s is the mass of our sun, M_i is the mass of planet i, m is the mass of our spacecraft and G is the gravitational constant. Both relative to our sun, $\mathbf{r}$ is the spacecraft's position and $\mathbf{r}_i$ is the position of planet i. Because the direction of the forces are *pulling* the spacecraft towards the respective bodies, the forces in the equation are negative. To find the spacecraft's gravitational acceleration, we need to divide (3) with the mass of the spacecraft:

$$\mathbf{a} = \ddot{\mathbf{r}} = -G\frac{M_s}{|\mathbf{r}|^3}\mathbf{r} - \sum_{i=1}^{N} G\frac{M_i}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i) \tag{4}$$

Having our wonderful research team supply us with the masses of all the astronomical bodies in our planetary system, we can calculate the spacecraft's acceleration. By once again using the trusty leap frog integration method

$$\begin{aligned} v_h &= v_i + a_i\frac{\Delta t}{2} \\ r_{i+1} &= r_i + v_h\Delta t \\ v_{i+1} &= v_h + a_{i+1}\frac{\Delta t}{2} \end{aligned}$$

we can use the acceleration (4) to simulate the spacecraft's position and velocity during its travel through our solar system.

To further plan our upcoming journey, the two most important tasks ahead of us are:

1. Planning the time when we want to perform our launch, and where we want to do so
2. Planning when and it which direction we want to perform the boosts that will happen *during* the journey

Since each boost takes up such a short amount of time in comparison to the rest of the journey, we will ignore this time and rather look at the boost as an instant acceleration. This means that after a boost the rocket will have a new velocity, but remain at the same position. Between each boost the rocket will move freely and only be affected by the gravitational force from the planets and our sun. This means that the fuel will only be used in the launch and the boosts.

To plan our trajectory and how we're going to travel from our home planet to Buttercup in the most efficient way possible, we'll be studying an orbit transfer technique called the *Hohmann transfer orbit*. This is a fuel-saving method used to transfer an object from one orbit to another. In our case, we want to transfer our spacecraft from its orbit around our home planet to the eventual orbit around Buttercup.

To better explain how this method works, we will be using Figure 1 which consists of two circular orbits and one elliptical.

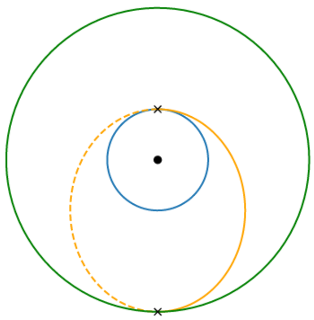

Figure 1. Hohmann transfer orbit

The principle of the method can be described by how Newton explained gravity: If an object is launched horizontally at a low velocity, it will be "taken" by gravity and fall back to the home planet. If the velocity is too high, it will drift into space. If the velocity is just right, it will orbit the planet. This principle is used in the Hohmann transfer orbit. If we increase the velocity enough, the spacecraft will drift away from its orbit around our home planet and follow an elliptical curve. Our goal is for this elliptical *transfer orbit*'s aphelion to stretch out towards Buttercup's orbit. Thus, we need to figure out the necessary boosts when going from our initial circular orbit to the transfer orbit, and from the transfer orbit to a circular orbit around Buttercup, as well as the optimal angular alignment α between our launch site and the assumed coordinates of our destination point. The idea is illustrated in Figure 2.

We will attempt to use mechanical energy to calculate the boosts and optimal angle α. When the spacecraft is in the elliptic transfer orbit, its potential energy will increase and its kinetic energy will decrease. The velocity will therefore be larger after the first boost, and will decrease as the spacecraft has to slow down in order to change its angular orientation. When travelling in space, our spacecraft is moving in our sun's gravitational field.

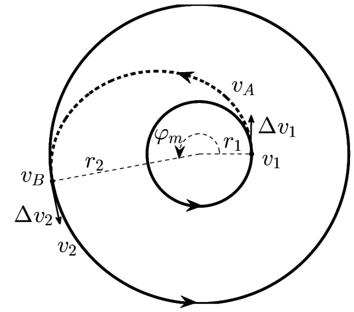

Figure 2. Illustration of the necessary boosts and the optimal angle α, here denoted by φ_m, between the launch site and Buttercup's position.

Thus, we can use the following expression when calculating our spacecraft's gravitational potential energy during the upcoming journey:

$$U = -\frac{GM_s m}{r} \tag{5}$$

M_s and m are our sun's and spacecraft's masses, respectively, and r is the mean distance between the two. Furthermore, its kinetic energy while still on or orbiting around our home planet is given by

$$E_k = \frac{mv_0^2}{2} \tag{6}$$

where v_0 is the velocity of the system consisting of our spacecraft and our home planet relative to our sun. The total mechanical energy at this point in the journey can then be written as

$$E = E_k + U = \frac{mv_0^2}{2} - \frac{GM_s m}{r_1} \tag{7}$$

where we have replaced r with r_1 in the expression for the distance between our spacecraft and our sun, as these still are approximately equal. In Appendix A we were able to use this to find that the velocity v_0 is given by

$$v_0 = \sqrt{\frac{GM_s}{r_1}} \tag{8}$$

Which, when applied to (7), gives us

$$E = -\sqrt{\frac{GM_s m}{r_1}} \tag{9}$$

For a small body orbiting a much larger body in an elliptical orbit, its total energy equals half its potential energy at its average distance from the large body. When our spacecraft is moving in the elliptical transfer orbit, this average distance is the orbit's semi-major axis a, which can be defined in the following way (see Figure 1 for illustration):

$$a = \frac{r_1 + r_2}{2} \tag{10}$$

where r_2 is the mean distance between our sun and the destination planet. With this expression, we can rewrite (9) in the following way

$$E = -\frac{GM_s m}{r_1 + r_2} \tag{11}$$

To find how we will need to boost our spacecraft to get it from the first orbit to the second, we will calculate the difference between the spacecraft's velocity before and after the respective boosts. In Appendix B we derived the following expressions with the help of (9):

$$\Delta v_1 = \sqrt{\frac{GM_s}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right) \tag{12}$$

$$\Delta v_2 = \sqrt{\frac{GM_s}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right) \tag{13}$$

Δv_1 is the first boost, when we go from the orbit around the home planet to the elliptical orbit, while Δv_2 is the second boost, necessary when transferring from the elliptical orbit to the orbit around our destination planet. The optimal angle α between the launch site and our destination has to be specific in order for the transfer to work out properly. In Appendix D we found that this is given by

$$\alpha = \pi\left(1 - \frac{\sqrt{\left(\frac{r1}{r2} + 1\right)^3}}{2\sqrt{2}}\right) \tag{14}$$

When the angle between its position and Buttercup is α, our home planet's position will be at an angle ϕ_1 off of the solar system x-axis. We know that since our home planet's Cartesian positional coordinates will be

$$x_p = r_1 \cos\phi_1$$
$$y_p = r_1 \sin\phi_1$$

when positioned at the angle ϕ_1, its velocity components will be

$$v_{x,p} = -v_p \sin\phi_1$$
$$v_{y,p} = v_p \cos\phi_1$$

where v_p is its absolute velocity. Because all our rocket launches are directed radially outwards, we can therefore use ϕ_1 to find the optimal launch site coordinates relative to our home planet:

$$x = -R\sin\phi_1$$
$$y = R\cos\phi_1$$

where R is its radius. This will then let us boost in the same direction that our home planet is moving when we launch, which hopefully will aid us in entering the elliptical transfer orbit.

Our last step in planning the transfer is then find at what angle ϕ_2 we need to perform the boost Δv_2 at. In Appendix C 1 we derived the following expression for a planet's angular velocity ω while in orbit around its sun:

$$\omega = \sqrt{\frac{GM_s}{r^3}} \tag{15}$$

where M_s is the mass of the sun and r is its distance from the sun. Furthermore, we derived the following expression for the time T it will take for the spacecraft to transfer between its orbit around our home planet to the orbit around Buttercup in Appendix C 2):

$$T = \frac{\pi}{2}\sqrt{\frac{(r_1 + r_2)^3}{2GM_s}} \tag{16}$$

To transfer from the elliptical orbit to the circular orbit around our destination, we want to boost our spacecraft in a way that redirect our direction of motion towards Buttercup. To do this, we will attempt to boost in the angle

$$\phi_2 = \omega_2 T \tag{17}$$

off of the solar system x-axis, as this is the angle that Buttercup has travelled during the time T. Here, ω_2 is Buttercup's angular velocity as defined in (15), with $r = r_2$.

After we have developed a reasonable plan for how our spacecraft will travel through space, we can launch it. We knew from modelling the rocket's engine that it used a problematic amount of fuel during one launch. Thankfully, when preparing for our journey by making adjustments to our software, our research team at the University of Oslo was able to help us cut down its mass loss rate to one third of its original amount. Thus, our rocket's parameters are now

$$F \approx 469.33 \text{ kN}$$
$$mlr \approx 37.85 \text{ kg/s}$$
$$m_{\text{fuel}} = 4.5 \times 10^4 \text{ kg}$$

where F, mlr and m_{fuel} are its trust force, mass loss rate and initial fuel mass, respectively.

We have to be aware of the likelyhood of the errors that might occur while we're on our journey. We have chosen to ignore the gravitational pull from smaller astronomical bodies in our solar system, but they will still be present and may cause our spacecraft to drift from the planned trajectory and prevent us from arriving at our destination. To regulate these errors we will have to boost our spacecraft in a way that gets us back on our planned path. By comparing our trajectory to the planned trajectory that we made through our simulation, we can determine the boost necessary for our spacecraft to get back on track. When doing this, we will attempt to use the orientation software we developed in our last study to compare the two trajectories.

Our main goal in this study is to be able to place our spacecraft in a stable orbit around Buttercup so that we

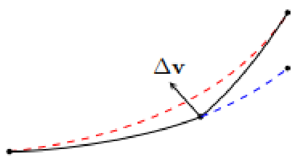

Figure 3. Using boost to correct the trajectory

can analyze its atmosphere and plan our eventual landing accordingly. By using small boosts to direct ourselves back towards the planned trajectory, we will hopefully be able to succeed in this task. In order for us to stabilize the orbit, we'll have to perform an *orbital injection maneuver*. The appropriate distance l from Buttercup for this to be possible is given by

$$l = |\mathbf{r}|\sqrt{\frac{M_p}{10M_s}} \tag{18}$$

where M_B is the mass of Buttercup, M_s is the mass of our sun, and $|\mathbf{r}|$ is the distance between our spacecraft and our sun. We will attempt to perform the orbital injection maneuver by performing a boost $(\Delta\mathbf{v})_{\text{inj}}$ given by

$$(\Delta\mathbf{v})_{\text{inj}} = \pm\mathbf{e}_\theta v_{\text{stable}} + \mathbf{v}_0 \tag{19}$$

where $\mathbf{e}_\theta$ is the tangential unit vector, $\mathbf{v}_0$ is our spacecraft's current velocity and v_{stable} is defined as

$$v_{\text{stable}} = \sqrt{\frac{GM_B}{r}} \tag{20}$$

Where M_B is Buttercup's mass and r is the distance between our spacecraft and the center of Buttercup. The sign of $\mathbf{e}_\theta v_{\text{stable}}$ will depend on whether our spacecraft traverses its orbit around Buttercup counterclockwise (+) or clockwise (-). To check if our orbit is successfully stabilized, we will once again be using the leap frog method, this time to attempt simulate our spacecraft's orbit three times using our knowledge of Buttercup's mass and the distance between us. We want to see if our radial and angular velocities v_r and v_θ change consistently. We will also use these velocities, along with the trajectory during the three orbital periods, to calculate the semi-major axis, semi-minor axis, eccentricity, orbital period, apoapsis and periapsis of each of the simulated orbits. If these values are consistent, we've most likely entered a stable orbit, given our simulation is accurate.

III. RESULTS

When simulating our spacecraft's travel towards Buttercup, we aimed to use the Hohmann transfer orbit. We got good use of the optimal angular alignment, which we found to be

$$\alpha \approx 85.36°$$

When they were optimally aligned, approximately six and a half months after we simulated the planetary orbits, our home planet's position was at an angle

$$\phi_1 \approx 48.58°$$

off the solar system's x-axis. We therefore found that our optimal launch site coordinates relative to our home planet were

$$x = -4728.86 \text{ km}$$
$$y = 4171.99 \text{ km}$$

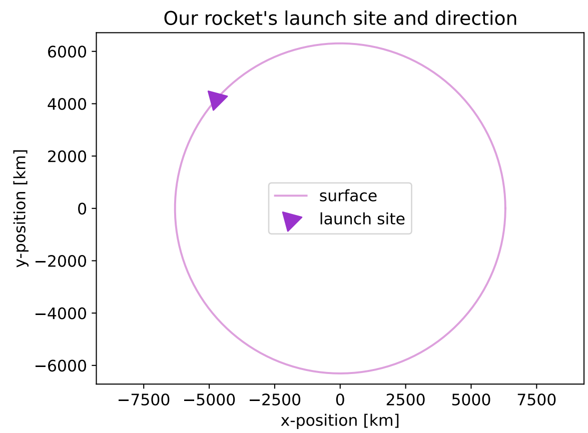

Figure 4. The spot on our home planet that we ended up launching our rocket from. The triangle points in the launch direction.

When launching from this position, we actually didn't need to use the speed boost

$$\Delta v_1 \approx 1.350 \text{ AU/yr}$$

that we calculated and originally meant to use when attempting to enter the elliptical transfer orbit. We ran a simulation where we let our spacecraft follow the transfer orbit we entered without performing this speed boost, for illustrative purposes.

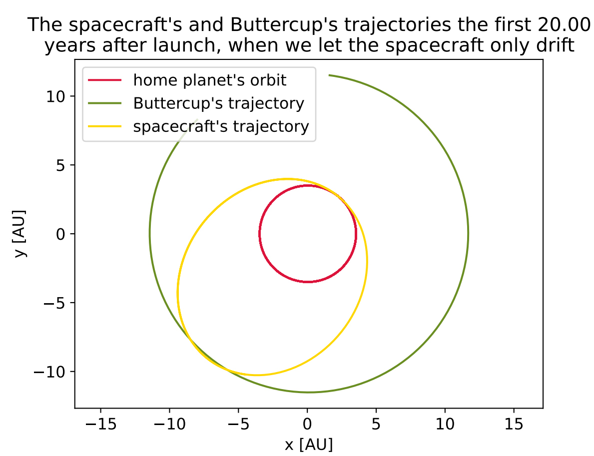

Figure 5. The spacecraft's simulated trajectory without any speed boosts.

We restarted our simulation because we wanted to use the speed boost

$$\Delta v_2 \approx 0.989 \text{ AU/yr}$$

that we calculated using the Hohmann method. We calculated that the angle our spacecraft should perform this speed boost at, after staying in the transfer orbit for approximately $T \approx 6.15$ years, was

$$\phi_2 \approx 94.64^\circ$$

off of the solar system x-axis. When doing this, we still weren't close enough to perform an orbital injection maneuver.

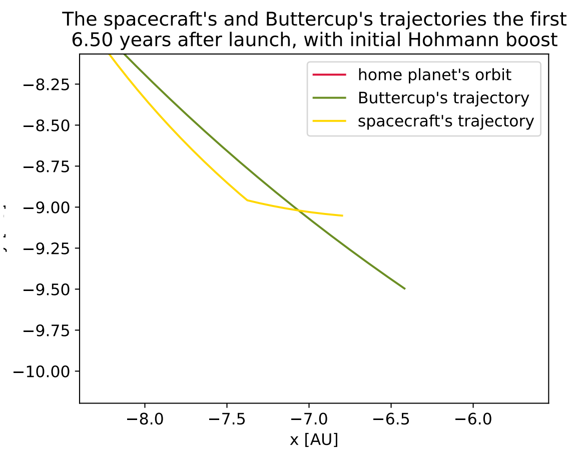

Figure 6. Close-up of the spacecraft's simulated trajectory after performing the planned speed boost Δv_2.

As apparent in Figure 6, we were close, but we boosted in the wrong direction and therefore went off track. By restarting the simulation a few times, and trying out a selection of other boosts each time, we were finally able to get close enough to Buttercup to perform an orbital injection maneuver. To do this, we had to perform the boosts in the x- and y-directions displayed in Table I:

t	Δv_x	Δv_y
6.15	0.975	-0.596
6.80	-0.295	0.211

Table I. The boosts our spacecraft performed during the simulation

Where t of course is the amount of years passed since launch, and the boosts are in astronomical units per year. The spacecraft's simulated trajectory with these boosts is displayed in Figure 7 and 8.

As we can see, our spacecraft was close enough to perform an orbital injection maneuver after approximately 7.17 years. Its distance from Buttercup was then

$$l = 0.00348 \text{ AU} \approx 520600.6 \text{ km}$$

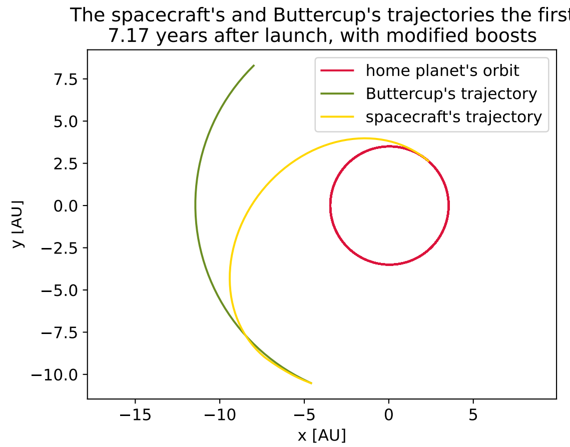

Figure 7. The spacecraft's simulated trajectory after performing speed boosts 6.15 and 6.80 years after launch.

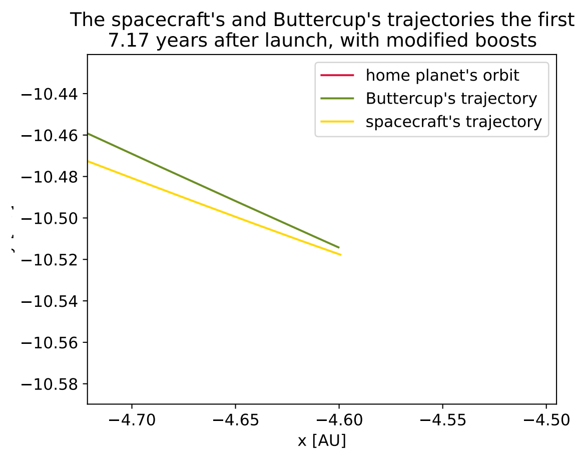

Figure 8. Close-up of the spacecraft's simulated trajectory after performing speed boosts 6.15 and 6.80 years after launch.

When simulating a launch with the exact same parameters and initial conditions as the one we were planning to execute, it took approximately 15 minutes. We found that our rocket would have 11067 kg fuel left after performing this launch, and then calculated that it should have about 4800 kg fuel left after performing our planned boosts.

When actually launching our spacecraft, we completed the launch within 15 minutes, just as expected, and our spacecraft had approximately 11065 kg fuel left. However, when we tried to use our results from the simulated trajectory, we quickly found that these were quite unrealistic. We therefore has to improvise, and use other methods when aiming our rocket towards Buttercup.

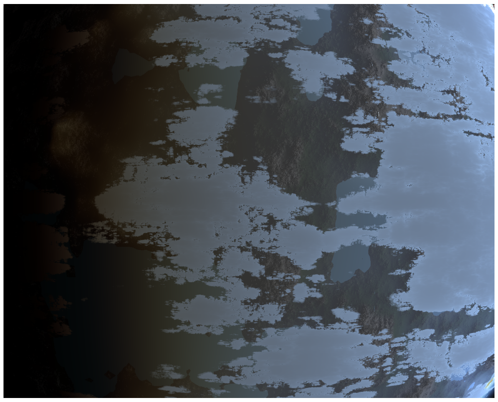

Figure 9. Picture taken of our home planet Doofenshmirtz' surface immediately after we reached escape velocity.

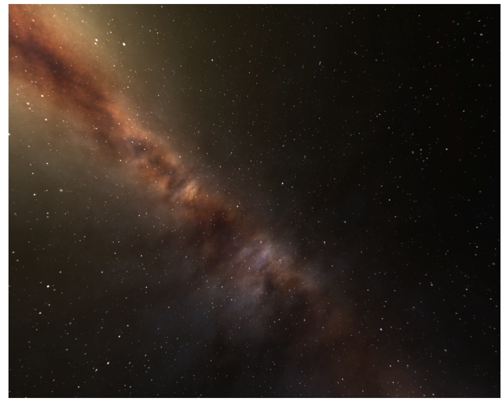

Figure 10. Our spacecraft's view in its direction of motion after coasting freely for three years.

Table II shows our spacecraft's and Buttercup's positions and velocities after launch. S denotes the spacecraft, B denotes Buttercup, and all values are measured in astronomical units.

	x	y	v_x	v_y
S	2.334	2.645	-5.754	4.909
B	-7.996	8.283	-2.224	-2.190

Table II. Our spacecraft's and Buttercup's position and velocity relative to the sun right after launch

After analyzing their position and velocity components, we chose to give our spacecraft a boost of

$$\Delta v_x = -0.7 \text{ AU/yr}$$
$$\Delta v_y = 0.3 \text{ AU/yr}$$

Which used approximately 2744 kg of our fuel. After these boosts we let our spacecraft coast freely for three years, to avoid using unnecessary amounts of fuel. Fortunately for us, we were much closer to Buttercup by this time, as evident in Table III

	x	y	v_x	v_y
S	-11.478	-0.303	-2.477	-2.354
B	-11.445	-0.181	0.061	-3.140

Table III. Our spacecraft's and Buttercup's position and velocity relative to the sun three years after launch

Now that we were this close, we wanted to even out our velocity components compared to Buttercup's. We gave our spacecraft the following boosts

$$\Delta v_x = 2.55 \text{ AU/yr}$$
$$\Delta v_y = -0.75 \text{ AU/yr}$$

Which drained our rocket of approximately 4750 kg fuel. Even though this is a lot, we took the chance considering how close we were. We let our spacecraft coast freely for another 2.1 years, before performing a small boost of 0.2 AU/yr in the negative x-direction as well as in the negative y-direction, to even out their velocity components. This used another 456 kg of fuel.

After coasting for 2 years, we had the coordinates shown in Table IV

	x	y	v_x	v_y
S	-4.78918	-10.42850	3.36664	-1.36889
B	-4.78894	-10.42786	2.83649	-1.32874

Table IV. Our spacecraft's and Buttercup's position and velocity relative to the sun after performing three separate boosts and travelling for about 7.64 years

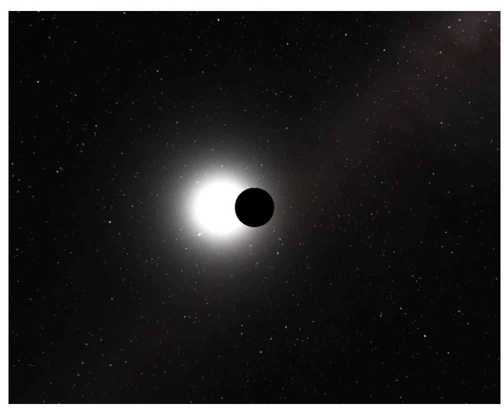

Figure 11. Our spacecraft's view of Buttercup after performing three separate boosts and travelling for about 7.64 years.

We were finally in orbit at this point, but it was rather elliptical. We therefore tried to coast for about 13 hours, until we were by the orbit's aphelion. At this point, our velocity component in the x-direction was the one that differed the most, and we therefore tried to even this out.

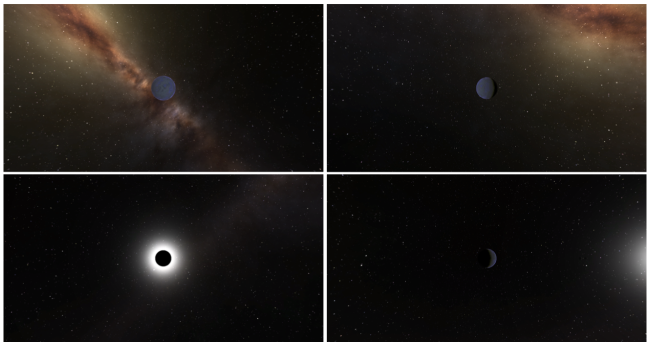

Figure 12. Different pictures taken of Buttercup after we had entered our stable orbit.

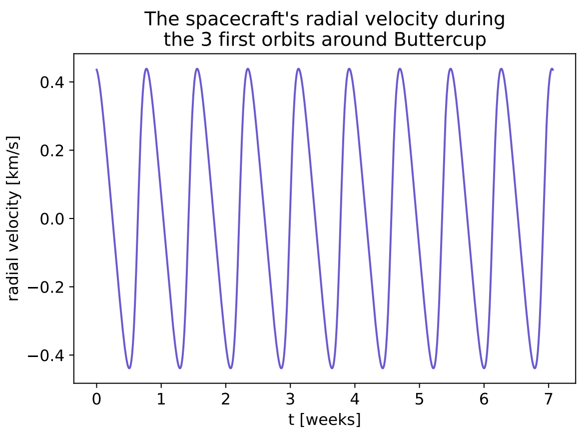

Figure 13. Our spacecraft's radial velocity during the first 3 orbits around Buttercup after entering a stable orbit.

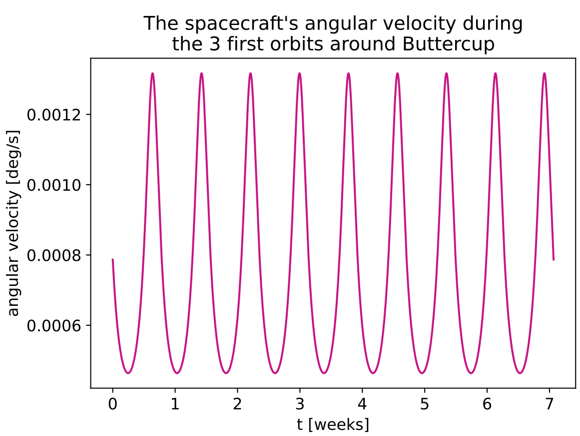

Figure 14. Our spacecraft's angular velocity during the first 3 orbits around Buttercup after entering a stable orbit.

We ran a simulation of the three first orbits after entering and gathered the following values

$$
\begin{aligned}
a &= 126038.03 \text{ km} \\
b &= 121864.01 \text{ km} \\
\epsilon &= 0.26 \\
P &= 5.475 \text{ days} \\
\text{apoapsis} &= 158205.50 \text{ km} \\
\text{periapsis} &= 93870.55 \text{ km}
\end{aligned}
$$

Where a, b, ϵ and P are the semi-major axis, semi-minor axis, eccentricity and revolution period, respectively. Fortunately, these values were constant for each of the three orbits, indicating a stable orbit.

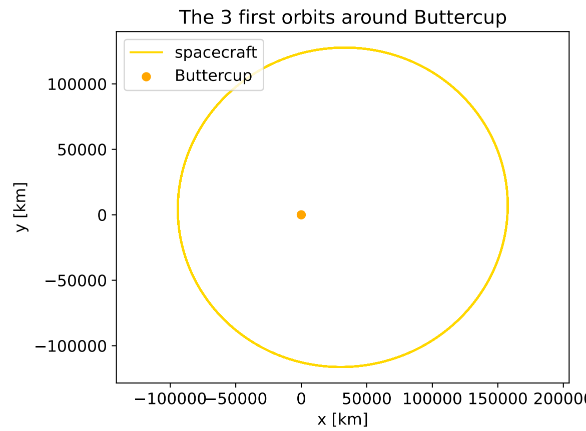

Figure 15. Our spacecraft's trajectory the first 3 orbits around Buttercup after entering a stable orbit.

IV. DISCUSSION

Although our simulation went relatively well, with only a few adjustments needed when trying to use the Hohmann transfer orbit method, it was sadly quite unrealistic. As we saw, we had to take a very different route when actually travelling towards Buttercup. The reason for this could of course be round-off errors in our simulation, but it could also be miscalculations when using the rather complex Hohmann method to find out at what point in time and how we should travel from Doofenshmirtz to Buttercup.

As soon as we entered space, we quickly realized that our plan wasn't going to work the way we had hoped, and we therefore had to make some spontaneous decisions. All the boosts our spacecraft performed during our travel is displayed in table V. Our strategy when deciding how to boost was to analyze our position and velocity and compare this to Buttercup's position and velocity. After launch, we saw that our spacecraft's x-coordinate was positive while Buttercup's was negative (see Table II). Since the v_x-component was negative for them both, we saw it fit to give our spacecraft an additional boost in the negative x-direction, to try to catch up to Buttercup.

t	Δv_x	Δv_y
0.536	-0.700	0.300
3.536	2.550	-0.750
5.636	-0.200	-0.200
7.638	-0.265	0.000

Table V. The boosts our spacecraft performed during the actual travel

Even though our v_y-component was positive while Buttercup's was negative, we still chose to boost our spacecraft a small amount in the positive y-direction as well, since Buttercup's y-coordinate was much larger. To avoid using too much fuel, we let our spacecraft drift for three years to see where we were heading. Fortunately for us, we had already gotten much closer by then. Our strategy then changed to boost our spacecraft in a way that would even out its velocity components with that of Buttercup, to hopefully get on the same path as it, as we saw this working in our simulation. This was the case in the actual travel as well, so we kept attempting to even out their velocity components by performing a set of small boosts with a couple years in between. Thankfully, this ensured us a seemingly stable orbit.

We simulated the three next orbits using our current position and velocity relative to Buttercup, combined with our masses, to see just how stable the orbit we've entered actually is. As we've seen, our simulations are seldom flawless, as there are multiple factors like other celestial bodies with gravitational pull playing a role as well. However, our next step will be to lower this orbit enough to be able to analyze Buttercup's atmosphere, and we'll then need to stabilize it yet again. Therefore, a flaw in this simulation won't be too dramatic, as we probably won't be staying in our current orbit for long.

V. CONCLUSION

We were unsuccessful in using the Hohmann transfer method when travelling towards Buttercup, although our simulations went rather smoothly. However, we did manage to successfully launch our rocket, perform a manual orientation using our impressive software, and enter a seemingly stable orbit in the end by making some improvised, yet rational decisions during our travel through the solar system. In addition to this, we also avoided using too much fuel on our way, so we'll have some left to use when we lower our orbit to analyze Buttercup's atmosphere and eventually perform a landing.

Appendix A: Derivation of the spacecraft's velocity v_0 while it's still orbiting our sun along with our home planet

While orbiting our sun along with our home planet, they're both moving with a velocity v_0 relative to our sun. There are small variations in our spacecraft's velocity since it orbits around our home planet as well, but these are almost insignificant and we will therefore ignore them. The system consisting of our home planet and our spacecraft is affected by two forces: the gravitational pull F_G from our sun, and the fictional *centripetal force* F_C. When still in a stable orbit, these forces are equal to each other:

$$\frac{M_p v_0^2}{r_1} = \frac{GM_s M_p}{r_1^2} \tag{A1}$$

where we have included our spacecraft's mass in our home planet's mass M_p because it's negligible in comparison. Solving (A1) for v_0, we get

$$v_0 = \sqrt{\frac{GM_s}{r_1}} \quad \square \tag{A2}$$

Appendix B: Derivation of the necessary boosts Δv_1 and Δv_2

1. Δv_1

The first boost Δv_1 that our spacecraft will perform is immediately after launch, for it to transfer from our home planet's orbit around the sun to the elliptical transfer orbit. The spacecraft's total mechanical energy immediately after entering the elliptical orbit can be written in two ways:

$$E = \frac{mv_1^2}{2} - \frac{GM_s m}{r_1} \tag{B1}$$

since its distance r from our sun is approximately r_1 still. However, it can also be written as

$$E = -\frac{GM_s m}{r_1 + r_2} \tag{B2}$$

since now has entered the elliptical transfer orbit. m denotes its mass and v_1 is its velocity after performing the speed boost Δv_1. Solving the equation (B1) = (B2) for v_1, we get

$$\begin{aligned} \frac{mv_1^2}{2} - \frac{GM_s m}{r_1} &= -\frac{GM_s m}{r_1 + r_2} \\ v_1 &= \sqrt{2GM_s \left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right)} \end{aligned} \tag{B3}$$

Thus, to achieve v_1, our spacecraft must perform the boost

$$\begin{aligned} \Delta v_1 &= v_1 - v_0 \\ &= \sqrt{2GM_s \left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right)} - \sqrt{\frac{GM_s}{r_1}} \\ &= \sqrt{\frac{GM_s}{r_1}} \left(\sqrt{\left(2 - \frac{2r_1}{r_1 + r_2}\right)} - 1\right) \\ &= \sqrt{\frac{GM_s}{r_1}} \left(\sqrt{\left(\frac{2(r_1 + r_2)}{r_1 + r_2} - \frac{2r_1}{r_1 + r_2}\right)} - 1\right) \\ &= \sqrt{\frac{GM_s}{r_1}} \left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right) \quad \square \end{aligned} \tag{B4}$$

2. Δv_2

In Appendix A we derived an expression for our spacecraft's velocity v_0 relative to our sun when still in orbit around our home planet. This formula is also valid when it's orbiting around the destination planet, but we now get

$$v_{0,2} = \sqrt{\frac{GM_s}{r_2}} \tag{B5}$$

We also know that our spacecraft's total mechanical energy right before it enters the stable orbit around the destination planet can be written as

$$E = \frac{mv_2^2}{2} - \frac{GM_s m}{r_2} \tag{B6}$$

since its distance r from our sun is approximately the same as our destination planet's distance r_2 from our sun at this point, but also

$$E = -\frac{GM_s m}{r_1 + r_2} \tag{B7}$$

as it's still technically in the elliptical transfer orbit. v_2 denotes our spacecraft's velocity at this point, and is derived by solving (B6) = (B7) in the same way as in B 1. We get

$$v_2 = \sqrt{2GM_s \left(\frac{1}{r_2} - \frac{1}{r_1 + r_2}\right)} \tag{B8}$$

Thus, we find the necessary boost Δv_2 to enter a stable orbit around our destination planet in the same way as we found Δv_1:

$$\begin{aligned}\Delta v_2 &= v_{0,2} - v_2 \\ &= \sqrt{\frac{GM_s}{r_2}} - \sqrt{2GM_s\left(\frac{1}{r_2} - \frac{1}{r_1+r_2}\right)} \\ &= \sqrt{\frac{GM_s}{r_2}}\left(1 - \sqrt{\left(2 - \frac{2r_2}{r_1+r_2}\right)}\right) \\ &= \sqrt{\frac{GM_s}{r_2}}\left(1 - \sqrt{\left(\frac{2(r_1+r_2)}{r_1+r_2} - \frac{2r_2}{r_1+r_2}\right)}\right) \\ &= \sqrt{\frac{GM_s}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1+r_2}}\right) \quad \square\end{aligned} \tag{B9}$$

Appendix C: Derivation of the time T it takes to transfer between the circular orbits

1. Destination planet's angular velocity ω

We know from Appendix A that the centripetal force F_C working on an orbiting body can be written as

$$F_C = \frac{mv^2}{r} = mr\omega^2 \tag{C1}$$

where m is the mass of the orbiting body, and v and ω are its Cartesian and angular velocities respectively. r denotes the distance between the orbiting body in the gravitational center it orbits around.

When a planet is orbiting its sun in a stable orbit, we know from Newton's second law of motion that the sum of the centripetal force and the gravitational force from the sun must be zero. Thus, we get:

$$\begin{aligned}F_C &= F_G \\ M_p r\omega^2 &= \frac{GM_sM_p}{r^2} \\ \omega &= \sqrt{\frac{GM_s}{r^3}} \quad \square\end{aligned} \tag{C2}$$

where we have renamed m to M_p to show that the orbiting body is the planet.

2. Transfer time T

The orbiting planet's angular velocity can also be written as a function of the orbital period P:

$$\omega = \frac{2\pi}{P} \tag{C3}$$

which gives us the following relation between (C2) and (C3):

$$\frac{4\pi^2}{P^2} = \frac{GM_s}{r^3} \tag{C4}$$

Solving this for the orbital period gives us:

$$P^2 = \frac{4\pi^2 r^3}{GM_s} \tag{C5}$$

Because r_1 is the mean distance between our home planet and our sun, while r_2 is the mean distance between our destination planet and our sun, we can use the geometry of an ellipse to find that

$$a = \frac{r_1+r_2}{2} \tag{C6}$$

where a is the elliptical transfer orbit's semi-major axis. This relation is illustrated well in Figure 1. Replacing r with a in (C5), we find the orbital period for the elliptical transfer orbit:

$$\begin{aligned}P^2 &= \frac{4\pi^2 a^3}{GM_s} \\ &= \frac{4\pi^2}{GM_s}\left(\frac{r_1+r_2}{2}\right)^3\end{aligned} \tag{C7}$$

The Hohmann transfer orbit only requires the spacecraft to travel approximately half the elliptical orbit, which gives us $T = P/2$, where T is the time between each boost. The final time it takes to transfer is therefore:

$$\begin{aligned}T &= \frac{\sqrt{\frac{4\pi^2}{GM_s}\left(\frac{r_1+r_2}{2}\right)^3}}{2} \\ &= \frac{\pi}{2}\sqrt{\frac{(r_1+r_2)^3}{2GM_s}} \quad \square\end{aligned} \tag{C8}$$

Appendix D: Derivation of the optimal angular alignment angle α

From Appendix C 1, we know that our destination planet's angular velocity is

$$\omega_2 = \sqrt{\frac{GM_s}{r_2^3}} \tag{D1}$$

where r_2 is its mean distance from our sun. Therefore, this planet has travelled the distance

$$\phi_2 = \omega_2 T \tag{D2}$$

after our spacecraft has completed half of the elliptical transfer orbit. In this time period, our spacecraft will have orbited an angle π. Thus, for our spacecraft to arrive at its destination without having to perform an unnecessary amount of boosts, the angle α between our home planet and the destination planet at the time of launch will have to be

$$\alpha = \pi - \phi_2 \tag{D3}$$

so that it arrives at the same position as our destination after the time T. This angle is what is called the *optimal angular alignment* between our home planet and the destination planet. By combining (D2) with (D1) and (C8), we can derive a more suitable expression for α:

$$\begin{aligned}\alpha &= \pi - \omega_2 T \\ &= \pi - \sqrt{\frac{GM_s}{r_2^3}}\frac{\pi}{2}\sqrt{\frac{(r_1+r_2)^3}{2GM_s}} \\ &= \pi\left(1 - \frac{\sqrt{\left(\frac{r1}{r2}+1\right)^3}}{2\sqrt{2}}\right) \quad \square\end{aligned} \tag{D4}$$