AST2000 - Part 8
Special Relativity

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EXERCISE 1 - PART 1

Introduction

In this exercise we will let two spaceships fire light beams at each other, and then observe what happens from from the first light beam is emitted until they both have been hit. We will study the outcome in two different frames of reference that move with a constant velocity $v = 0.602c$ relative to one another, where c is the light speed. We will do this in hopes of finding out more about how relative velocities close to the light speed affect how events occur in the eyes of different observers.

The Situation

There is an observer situated at an equal distance between the two spaceships, as well as an observer situated on the planet. In the spaceship frame, which we'll use primed coordinates for, the shapeships and the observer between them all stand still while the planet moves with the constant velocity v to the left. In the planet frame, which uses unprimed coordinates, both spaceships and the observer between them move with v to the right. We let S_1 denote the left-most spaceship, S_2 denote the right-most spaceship, and M denote the observer in the middle. At $t = t' = 0$, they have the following coordinates:

	Planet frame		Spaceship frame	
	x	t	x'	t'
S_1	0	0	0	0
S_2	L	0	L	0
M	$L/2$	0	$L/2$	0

We will study a series of events in both frames of reference to see what plays out similar and what doesn't. We hope that this will be able to tell us more about how observers with different velocities percieve elapsed time and simultaneous events. The events we wish to study are the following:

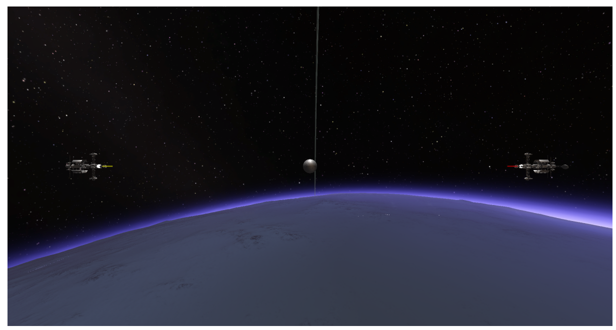
A - S_1 emits light beam L_1

B - S_2 emits light beam L_2

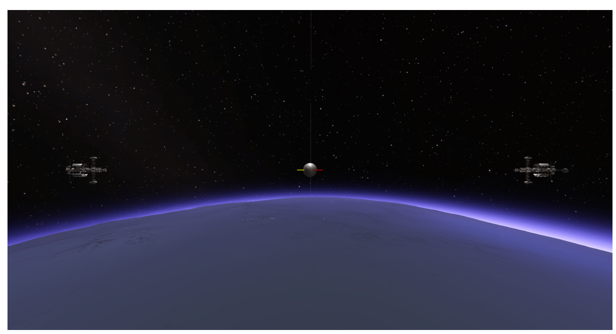
C - L_2 hits S_1

D - L_1 hits S_2

Because neither of the spaceships have any velocities relative to observer M, she observes events A and B simultaneously at $t' = 0$, as apparent in the video taken from her frame of reference (see Figure 1a). However, we are not yet sure if this is the case in the planet frame. What we do know is that the light beams must cross at M in the spaceship frame, as evident in Figure 1b, since they are fired at an equal distance from her, and both of them travel at the speed of light.



(a) The spaceships emit their light beams.



(b) The light beams crossing at M.

Figure 1: The spaceships emitting their light beams simultaneously (a), and the light beams crossing at M (b), both in the spaceship frame.

Method

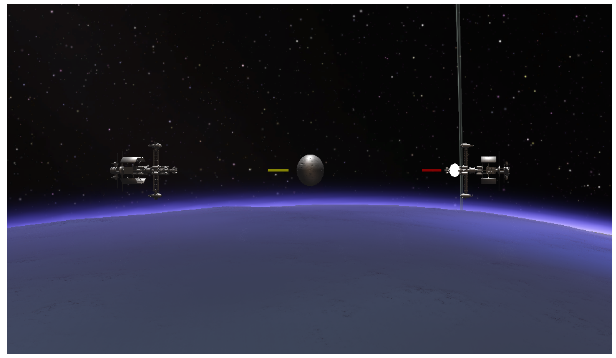
We know that the speed of light is the same in all reference systems, which implies that whenever the light beams are in the same position, all observers must see that they are in this exact position. This is because the light from both of the beams have an equal distance they need to travel to reach any observer. It also follows from this property that whenever two events occur simulatenously for an observer, they won't necessarily occur simultaneously for all other observers. This is because other observers' positions relative to the two events may be different. To further explain this, let's imagine that two events E_1 and E_2 happen simultaneosly for observer O_1. Now, if an observer O_2 were to stand closer to the position of event E_1 than O_1 does, but further away from the position of E_2 than O_1, the light coming from E_1 will hit O_2's eye before the light from E_2. Thus, E_1 will happen before E_2 for O_2, even though they happen simultaneously for O_1.

In the planet frame, the spaceships, as well as M, all travel with the velocity v to the right. Since the spaceships fire light beams at each other, one of the light beams will move in the same direction as v, while the other one will move in the opposite direction. Because of this, the light beam that moves in the same direction as the spaceships will appear to "chase" its target, while the other one will appear to "meet" its target. Since M is right in the middle of the spaceships, the one that appears to chase M will therefore have to be fired first in order to reach her at the same time as the one meeting her. The same will naturally affect which explosions occur first in the planet frame, as the light beam chasing its target will still have a longer way to go than the light beam meeting its target. After they cross at M, the light beams' respective targets are, as we know, the opposite spaceship of the one they were emitted from.

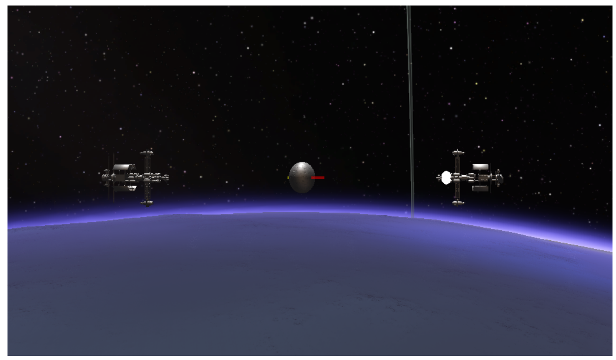
Conclusion

Because we could see from the video taken from the spaceships' frame of reference that L_1 and L_2 cross by M, we knew that this also had to be the case in the planet frame. Furthermore, since L_1 moves in the same direction as S_1, S_2 and M in the planet frame, while L_2 moves in the opposite direction, we found that L_1 would have to have been fired first in order for it to be able to reach M by the time L_2 got there. Since L_2 and S_1 continue to move toward each other while L_1 chases S_2, it was apparent that event C had to happen before event D, as L_1 needs to travel further in order to reach its target. We see that this goes hand in hand with the fact that the light from event A needs to travel a shorter distance than the light from event B in order to reach the planet observer's eyes, and they therefore see event A unfolding before event B. The same holds for events C and D, of course.

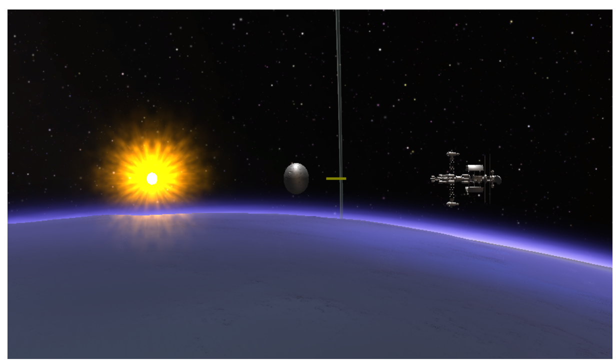
To conclude, we found that event A happens first in the planet frame, then B, then C, and then D at last. When looking at the video taken from the planet frame, we could see that this is in fact the case (see Figure 2). Thus, we can confirm that what happens simultaneously in one frame of reference, not necessarily happens simultaneously in all other frames of reference.



(a) Event B happening after A



(b) The light beams crossing at M



(c) Event C happening before D

Figure 2: S_2 emitting its light beam (a), the light beams crossing at M (b), and S_2's light beam hitting S_1 (c), all in the planet frame.

EXERCISE 1 - PART 2

Introduction

We will continue to study the two spaceships that fire light beams at each other, but now we want to find out more about how and why time intervals run different in the two frames of reference. To do this, we will compare the spaceship frame observations with the corresponding ones made in the planet frame. Why does the observers' velocities relative to one another seemingly affect how the events play out?

The Situation

We have derived expressions for the positions of S_1, S_2, L_1, L_2 and M in the planet frame as functions of the time t, velocity v of the entire spaceship frame, and the distance L between the two spaceships. In our expressions we also used the time stamps t_A and t_C for event A and C in the planet frame, as well as t_M, which marks when the light beams L_1 and L_2 meet at M. Since we knew that the light beams crossed at M simultaneously in both frames of reference, we get the following expressions for t_A and t_C:

$$t_A = t_M - \frac{L/2}{1-v}$$
$$t_C = t_M + \frac{L/2}{1+v}$$

In turn, this gives us the following expressions for the positions of S_1, S_2, L_1, L_2 and M:

$$x_1 = vt$$
$$x_2 = L + vt$$
$$x_{L_1} = t - t_A(1-v)$$
$$x_{L_2} = t_M(1+v) - t + \frac{L}{2}$$
$$x_M = \frac{L}{2} + vt$$

Method

We know that event A happens at the exact same time as the origo event, when S_1's position is $x_1 = x'_t = 0$ and $t = t' = 0$, in the spaceship frame. As we can see, the origo event happens simultaneously in both frames of reference, which means that event A necessarily would have to have happened at the same time as the origo event in the planet frame as well. This gives us the following relation:

$$t_A = 0 \rightarrow t_M = \frac{L/2}{1-v}$$

Which immediately lets us rewrite some of the expressions above in a much easier way:

$$t_A = 0$$
$$t_C = \frac{L}{1-v^2}$$
$$x_{L_1} = t$$
$$x_{L_2} = \frac{L}{2}\left(1 + \frac{1+v}{1-v}\right) - t$$

It is now easy for us to see that the time elapsed between events A and C in the two frames can be expressed in the following way:

$$\Delta t = \frac{L}{1-v^2}, \qquad \text{in the planet frame}$$
$$\Delta t' = L, \qquad \text{in the spaceship frame}$$

where the elapsed time in the spaceship frame comes from the fact that the light beam emitted from S_2 at $t'_B = 0$ has to travel a distance L with the speed of light before reaching S_1. These time intervals are obviously not the same, but why is that? We see that the time interval for the planet observer is larger than for the spaceship observer, since $v < 1$. Could this have something to do with the fact that the planet observer is moving with the velocity v relative to the spaceships, while the spaceship observer has no velocity relative to them? We aim to find out more about why these time intervals happen to be different by comparing the expressions above.

Conclusion

When comparing the elapsed time we found for each system, we saw that they had the following relation:

$$\Delta t = \frac{\Delta t'}{1-v^2} \tag{1}$$

This immediately looks familiar, as the formula for time dilation in special relativity is

$$\Delta t = \frac{\Delta t'}{\sqrt{1-v^2}} \tag{2}$$

Which means our expression is missing a factor $\sqrt{1-v^2}$. We recognize this as $1/\gamma$, where γ is the *Lorentz factor*. The reason for this is that in in special relativity, not only time is relative, but position as well. In fact, this was very apparent, as the position of the spaceships relative to the observer was different for the two frames of reference. However, we did not acount for this in our calculations, and that's where things went wrong.

EXERCISE 4

Introduction

In this exercise we will study three spaceships travelling with different velocities with respect to a space station that stands still, in order to further look into how relative velocity affects time and distance. We will draw *wordlines* for the space station and the three spaceships into *spacetime diagrams*, where position is plotted against time. We will create a spacetime diagram for each of the reference systems, and compare these to see how relative velocity affects the worldlines. Furthermore, we'll mostly focus on the diagram belonging to the space station frame, and analyze this to learn more about time dilation in special relativity, which we discussed at the end of the last exercise.

The Situation

In all frames of reference, the spaceships and the space station start out in the origin (see Figure 3). Two of the spaceships, which we'll call ship 1 and ship 2, move with constant velocities relative to the space station. The third spaceship, ship 3, accelerates until it catches up with ship 2, which is the fastest ship. Right after ship 3 reaches ship 2, it decelerates until it ultimately stops in the space station frame.

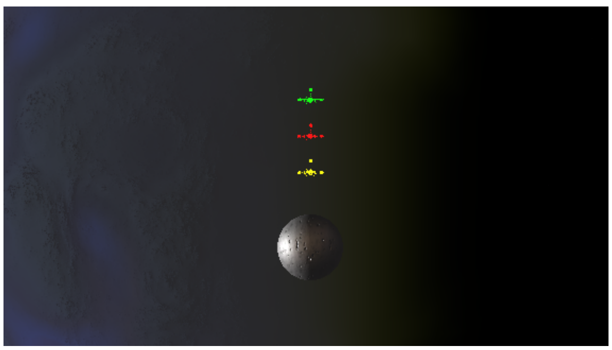


Figure 3: All the ships are aligned with the space station in all frames of reference at $t = 0$.

The space station, as well as ship 2 and ship 3, are equipped with clocks that tick for each millisecond that passes by. We will define two events that we aim to look more into:

1 - When all spaceships are aligned

2 - When ship 3 catches up with ship 2

Figure 4: Ship 3 catches up with ship 2 in the space station frame.

Event 1 occurs at $x = 0$ and $t = 0$. The clock at the space station measures 10 milliseconds between the two events, while the clock on ship 2 measures only 8. We don't know how many milliseconds were measured on ship 3. Why do the clocks measure different amounts of ticks? And will the amount of ticks measured on the clock onboard ship 3 be affected by the fact that it accelerates in the time period between the two events?

Method

First, we wish to draw worldlines for the space station and the three ships in the spacetime diagrams for the space station, ship 1 and ship 2, to see how they differ in reference systems moving with constant velocities relative to each other.

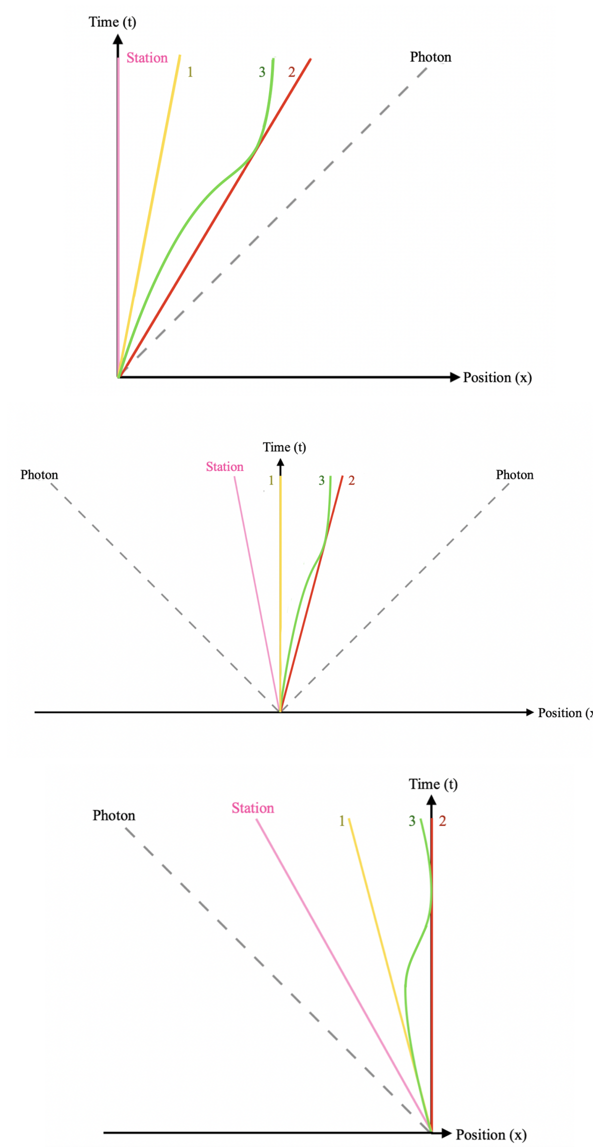


Figure 5: Spacetime diagrams for the space station frame (upper diagram), ship 1 frame (middle diagram) and ship 2 frame (lower diagram).

Now, since ship 3 accelerates and decelerates in the space station's, ship 1's and ship 2's frames of reference, they will naturally all have to accelerate and decelerate in ship 3's frame of reference. To explain, when ship 3 is accelerating relative to the other ships and the station, they all decelerate relative to ship 3. This also counts for when ship 3 is decelerating, which then makes it look like the others are accelerating in this frame. Figure 6 illustrates this.

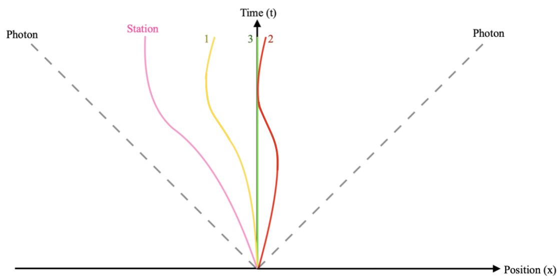


Figure 6: Spacetime diagram for the ship 3 frame.

As mentioned in the introduction, we want to focus primarily on the spacetime diagram for the space station (the upper diagram in Figure 5). We know that the station and ships 2 and 3 are equipped with clocks, and that the time elapsed between events 1 and 2 are 10 milliseconds in the space station frame, while it is 8 milliseconds in the ship 2 frame. We do not yet know how much time passes by in the ship 3 frame. Before we try finding this out, we drew dots on the worldlines of the space station and ship 2 for each tick on their respective clocks (see Figure 7)

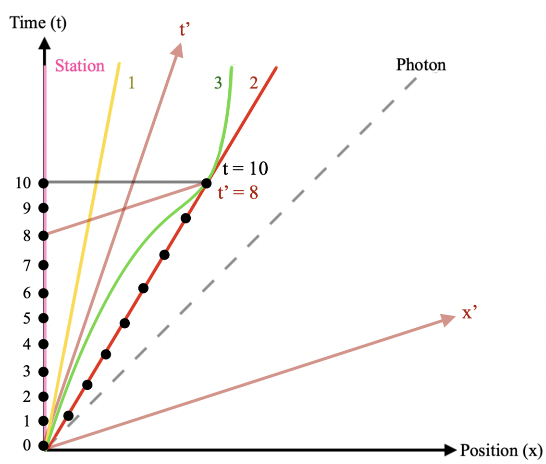


Figure 7: Spacetime diagram in the reference frame of the space station with dots drawn on the worldlines of the station and ship 2.

The faded lines drawn from the last dot on ship 2's worldline to the 10 millisecond mark (faded black) and 8 millisecon mark (faded red) on the worldline belonging to the space station is there to visualize that when 8 milliseconds have passed in ship 2's frame, 10 milliseconds have passed in the space station frame. We have drawn on the axes of ship 2's spacetime diagram in the faded red color, to show that the aforementioned line is parallell with this reference system's positional axis, and that these axes are closer to the photon line in the space station frame. This tells us that relative to the space station, time runs faster in the ship 2 frame. If ship 2 were to travel at the speed of light, the time- and position axes would be on top of each other, because $v = \Delta t/\Delta x = 1$.

Now, how many ticks could have gone by in ship 3? To decipher this, we'll have to look more into something called *the principle of maximal aging*. We know from Lecture Notes 2B [1] that an object in *free float*, meaning that no external forces work on it, will have the worldline between two events that corresponds to the longest *proper time interval*. A reference system's proper time interval is the time interval an observer in this system will measure on a clock that is moving along with it. To illustrate, the proper time interval of the space station frame is 10 milliseconds, while the proper time interval of the ship 2 frame is 8 milliseconds. Now, if ship 2 is moving with constant velocity, we know from Newton's first law of motion that this means no external forces are working on it. For ship 3 however, Newton's second law of motion tells us that since it accelerates, an external force must be working on it.

Conclusion

First of all, we saw that when switching between reference systems that move with constant velocities relative to one another, the distance between each worldline stays the same. As evident from Figure 5, the only difference between the three spacetime diagrams was that the worldline of the reference system belonging to the respective diagram is positioned at the time-axis. This is not surprising, as this observer is standing still in its own frame of reference, and is therefore not moving along the position-axis.

When studying the spacetime diagram for the space station, it became clearer why the elapsed time between the two events was longer for the space station than for ship 2. Since ship 2 is moving with a constant velocity relative to the space station standing still, the time and position axes belonging to ship 2's frame close in on each other in the space station frame. This makes the time elapsed when travelling between events shorter for an observer on board ship 2 than for an observer at the space station. This is not surprising, when recalling the formula for time dilation that we presented in the last exercise:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2}} \tag{3}$$

Here, Δt denotes the time interval measured by the space station observer, while $\Delta t'$ is for the observer on board ship 2. Since $\sqrt{1-v^2} < 1$, we see that $\Delta t > \Delta t'$.

Furthermore, the principle of maximal aging tells us that ship 2 must follow the worldline with the longest proper time interval between events 1 and 2. One should keep in mind that the worldline belonging to the space station is not a path between the two events, as the space station is only present for the second event in time, not space. Therefore, ship 2's worldline has the longest proper time interval, meaning that ship 3 measures less than 8 ticks on its clock. With this in mind, we were able to sketch the clock on board ship 3's ticks on its worldline (see Figure 8). We've drawn on the last dot to mark approximately when and where the 8$^{\text{th}}$ tick happens on ship 3's clock, and to show that the observers on board the space station and ship 2 respectively measure more than 10 and 8 milliseconds at this point.

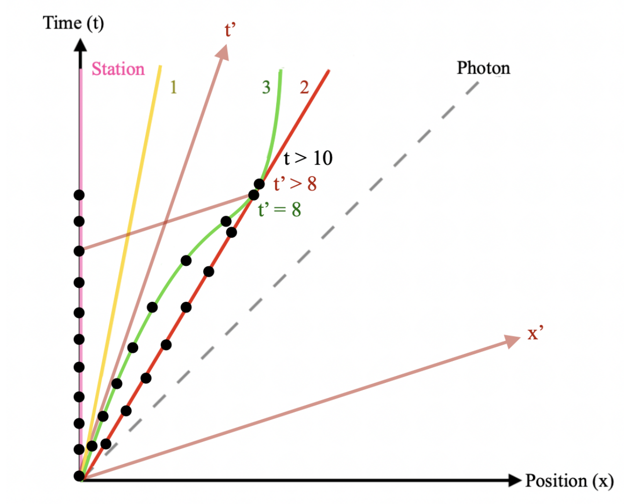


Figure 8: Same diagram with dots drawn on ship 3 as well.

EXERCISE 7

Introduction

In this exercise we will study what happens when two spaceships with equal masses collide. One of these spaceships is made solely from antimatter, so that when they collide and explode, all their matter is converted into a vast amount of energetic photons, essentially light particles, all with the same wavelength. We have two frames of reference, where each of the observers have been equipped with wavelength detectors to measure the energy levels of the photons emitted in the explosion. We wish to study the concept of the relativistic *momenergy*, and see how this affects the Doppler shift between the wavelengths measured by each of the observers after the spaceships collide.

The Situation

The frame 1 observer is at rest on our planet, while the frame 2 observer follows closely behind the spaceship made of normal matter, which we'll call A. We'll call the ship made of antimatter B, and we'll let unprimed coordinates denote frame 1 and primed coordinates denote frame 2. We have found the ships' momenergy four-vectors in both frames of reference, which contain their relativistic energies, as well as their relativistic momenta in the x-, y- and z-directions. Since we only will be looking at the ships' movement in the x-direction, their momenergy vectors only have two components. Their velocities, relativistic energies and relativistic momenta in the two frames of reference are displayed in the table below.

	Frame 1			Frame 2		
	v	E	p_x	v'	E'	p'_x
A	v_A	γm	$\gamma m v_A$	0	m	0
B	$-v_A$	γm	$-\gamma m v_A$	$-2v_A$	$\gamma^2 m(1+v_A^2)$	$-2\gamma^2 m v_A$

We use γ instead of γ_A and γ_B, as these are the same in this situation, because the difference in velocity between the two frames is $|\Delta v| = v_A$ for both A and B. Figure 9 shows what the two observers see right before the two spaceships collide.

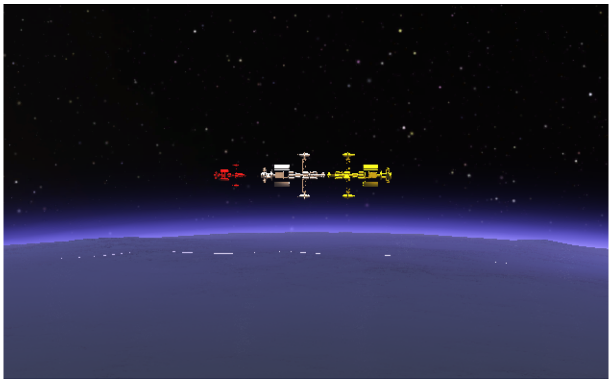
Method

Photons come from electromagnetic radiation, such as gamma rays, microwaves and light, and they have both particle- and wavelike properties. When approximated as elementary particles, they are still massless and travel at the speed of light. Because of this, we had to use a different approach when finding the momenergy of photons. We found this to be (see the "Specific Questions" section):

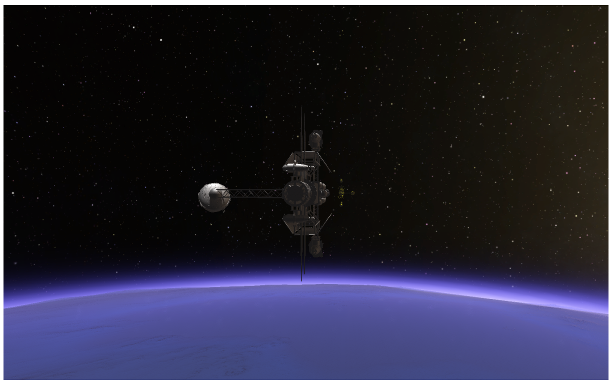
$$P_\mu = (E, E, 0, 0) \tag{4}$$

For a photon moving in the positive x-direction. We will use this newfound expression for the momenergy of the photons created in the explosion to study how the relative velocities of the spaceships affect their observed wavelength in the two frames.

We want to find out more about the properties of the photons emitted in the explosion. To do this, we will use the conservation of momenergy in the collision, as both of the ships' masses are completely converted into photons. Let us first assume that when the spaceships collide, only two photons are emitted in the explosion. In frame 1, the two spaceships move toward each other with the exact same velocity. Since their relativistic energies are positive quantities which are only related to their masses and the absolute value of their velocities, these will be equal. Because of conservation of momenergy,



(a) Frame 1



(b) Frame 2

Figure 9: What the observers in the planet frame (a) and the spaceship frame (b) see right before the spaceships collide.

the two photons emitted in the collision must have the same energies as the spaceship they come from. Since this is unrelated to the direction of motion, these will also be equal in frame 1. However, the momentum of the photons, which is denoted by $\pm E$, will depend on if they move in the positive or negative x-direction, as expected from our knowledge of normal vectorized momentum.

Now, what if the two photons would have been emitted with an angle θ off the x-axis? Their momenergy would then be

$$P_\mu^\gamma = (E, \pm E\cos\theta, \pm E\sin\theta, 0) \tag{5}$$

instead, assuming that they have no movement in the z-direction. As we can see, their energies are still unrelated to the direction of motion. Because of conservation of momenergy, these will still have to be equal in the planet frame. It also follows from this that they will move in the opposite direction, as apparent in (5).

Realistically, enormous amounts of photons are emitted in the explosion. We will now assume that all the photons emitted in the explosion have the same energy. Once again we can use the conservation of momenergy to argue that the sum of all the photons' relative energies and momenta must be equal to the sum of the spaceships' relative energies and momenta. Since the spaceships' momenta cancel each other out in the planet frame, so must the photons'. This means that for each photon emitted in a random direction, there must be a photon emitted in the completely opposite direction to cancel out its momentum. Since the relative momentum is determined by the energy of the photon, their energies also have to be equal, just like we assumed.

Now, how will the explosion look for the two observers? We expect there to be a large Doppler shift in the measured wavelengths because of the huge difference in the observers' velocities relative to the collision. For velocities $v \ll c$, we're used to using the following formula

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} \tag{6}$$

with SI-units for calculating Doppler shifts measured between an observer at rest and an observer moving with a radial velocity v relative to the source that they're measuring the wavelength of. However, when this velocity grows sufficiently close to the light speed c, this formula is no longer accurate. We found that when v is over $0.5c$, which is the colliding spaceships' radial velocities relative to each other, we have to use the following formula instead:

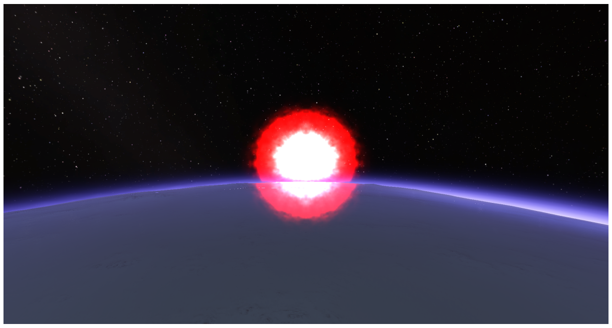
$$\frac{\Delta\lambda}{\lambda} = \left(\sqrt{\frac{1+v}{1-v}} - 1\right) \tag{7}$$

Where we here use relativistic units. By Taylor expanding this expression around $v = 0$, one can fortunately see that the normal Doppler shift formula does not collide with the relativistic formula for sufficiently low velocities.

Conclusion

It was apparent that whatever direction a photon is headed, there will have to be a photon headed in the complete opposite direction. This is because all the photons emitted in the explosion will have to cancel out each other's relativistiv momenta for the momenergy to be conserved in the collision. Because their relativistiv momenta is determined by their respective energies, all their energies also have to be equal.

With these results in mind, we could successfully determine the observed wavelength of the photons in the collision from each frame of reference. In frame 1, where none of the spaceships are moving toward the observer, the observed wavelength was approximately 630 nanometers, which corresponds to the color red. In frame 2 however, where B moves toward the observer with a radial velocity of over half the speed of light, the observed wavelength was as low as 436 nanonemeters, which corresponds to purple.



(a) Frame 1

(b) Frame 2

Figure 10: The explosions as seen from the planet frame (a) and the spaceship frame (b).

We know that the shorter the wavelength, the more energetic the photon is. As we found earlier, spaceship B's observed energy in frame 2 is bigger than in frame 1 because it moves toward the observer. Because of the conservation of momenergy, the photons emitted in the explosion also have to appear more energetic.

Specific Questions

We want to derive an expression for the momenergy four-vector of a photon. For particles and objects with mass, we have defined its energy as $E = \gamma m$. For photons, this relation implies that they don't have energy, which is clearly not the case. Thankfully, we have the following relation for the energy of a photon:

$$E = \frac{h}{\lambda} = h\nu \tag{8}$$

Where h is Planck's constant, λ is the photon's wavelength, and ν is the frequency of the photon. Thus, by using relativistic units, we have an expression for the time-component of the photons momenergy. But what about their momentum? We take a look at the momenergy four-vector $P_\mu = (E, \vec{p}) = (E, p, 0, 0)$. By using properties of the four-vector, we find that its length P is given by

$$P = \sqrt{P_\mu P^\mu} = \sqrt{E^2 - p^2} \tag{9}$$

When the energy and momentum are expressed in the usual relativistic way, we have the following relation:

$$P = \sqrt{\gamma^2 m^2 - \gamma^2 m^2 v^2} = \gamma m \sqrt{1 - v^2} = m \tag{10}$$

Since a photon is massless, we get

$$\begin{aligned} \sqrt{E^2 - p^2} &= m \\ E^2 - p^2 &= 0 \\ p^2 &= E^2 \\ p &= \pm E \end{aligned} \tag{11}$$

Where the sign naturally depends on which way the photon is headed, just like with the usual expression for momentum. For a photon headed in the positive x-direction, we can then define its momenergy four-vector as

$$P_\mu = (E, p, 0, 0) = (E, E, 0, 0) = \left(\frac{h}{\lambda}, \frac{h}{\lambda}, 0, 0\right) \tag{12}$$

Since h and the wavelength λ are positive quantities, we see that $E \geq 0$ also counts for photons.

REFERENCES

[1] Hansen F. K. *Lecture Notes 2B*. URL: https://www.uio.no/studier/emner/matnat/astro/AST2000/h22/undervisningsmateriell/lecture_notes/part2b.pdf.

SMILEY-FACE TASKS

EXERCISE 1 - PART 2

Task 1

We let x_1, x_M and x_{L_1} denote the positions of left-most spaceship's, the observer in the middle and the light beam emitted from the left-most spaceship, respectively. Since the left-most spaceship moves with constant velocity v in the positive direction in the planet frame, and starts out at $x = 0$, we get the following equation for x_1:

$$x_1 = vt \tag{13}$$

Furthermore, the observer in the middle moves with the same velocity as the left-most spaceship in this frame, but starts out in $x = L/2$. We get the following:

$$x_M = \frac{L}{2} + vt \tag{14}$$

Since the light beam is emitted from the left-most spaceship at $t = t_A$, it will start out in the position $x_1(t_A) = vt_A$. After t_A, it will move along the x-axis in the positive direction with the speed of light. We get

$$\begin{aligned} x_{L_1} &= vt_A + (t - t_A) \\ &= t - t_A(1 - v) \end{aligned} \tag{15}$$

Task 2

Since the position of the middle observer equals the position of the light beam emitted from the left-most spaceship at $t = t_M$, we get:

$$\begin{aligned} x_M(t_M) &= x_{L1}(t_M) \\ \frac{L}{2} + vt_M &= t_M - t_A(1 - v) \\ t_A(1 - v) &= t_M(1 - v) - \frac{L}{2} \\ t_A &= t_M - \frac{L/2}{1 - v} \quad \square \end{aligned}$$

Since the origo events happen simultaneously in both reference systems, and the origo event and event A happen simultaneously in the spaceship frame, they'll naturally have to happen simultaneously in the planet frame as well.

Task 3

Relative to the observer on the planet, the right-most spaceship is also moving with a constant velocity v in the positive x-direction. We let x_2 denote the position of the right-most spaceship, which is at $x = L$ at $t = 0$: We get

$$x_2 = L + vt \tag{16}$$

We know that the light beam emitted from the right-most spaceship is positioned at $x_{L2} = x_{L1} = L/2 + vt_M$ when they cross in the middle, because they're both at $x = L/2$ in the spaceships' frame of reference, while this entire reference system has moved a distance $\Delta x = vt_M$ in the planet frame during the time it took for them to cross. The light beam emitted from the right-most spaceship travels in the negative direction, and we therefore get

$$\begin{aligned} x_{L_2} &= \frac{L}{2} + vt_M - (t - t_M) \\ &= t_M(1 + v) - t + \frac{L}{2} \end{aligned} \tag{17}$$

Since we know that $x_1(t_C) = x_{L_2}(t_C)$, we can solve this for t_C:

$$\begin{aligned} x_1(t_C) &= x_{L_2}(t_C) \\ vt_C &= t_M(1 + v) - t_C + \frac{L}{2} \\ t_C(1 + v) &= t_M(1 + v) + \frac{L}{2} \\ t_C &= t_M + \frac{L/2}{1 + v} \quad \square \end{aligned}$$

Task 4

In the planet frame, we get

$$\begin{aligned} \Delta t &= t_C - t_A \\ &= t_M + \frac{L/2}{1 + v} - \left(t_M - \frac{L/2}{1 - v}\right) \\ &= \frac{L/2}{1 + v} + \frac{L/2}{1 - v} \\ &= \frac{L/2(1 - v) + L/2(1 + v)}{(1 + v)(1 - v)} \\ &= \frac{L}{1 - v^2} \end{aligned} \tag{18}$$

In the spaceship frame, both light beams are emitted at the same time. Since the light beam emitted from the right-most spceship must travel a distance L to reach the left-most spaceship, we get $\Delta t' = Lv_{L_2} = L$

EXERCISE 4

Task 2

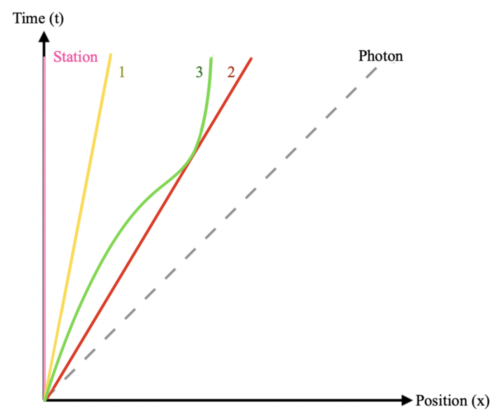


Figure 11: Spacetime diagram in the reference frame of the space station

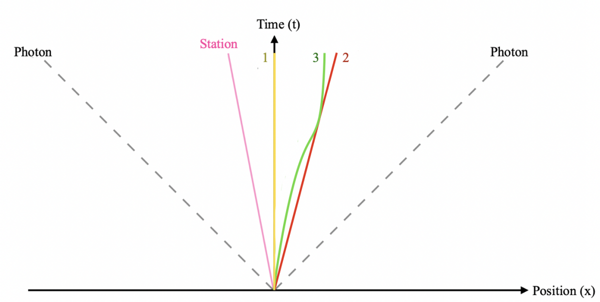


Figure 12: Spacetime diagram in the reference frame of ship 1

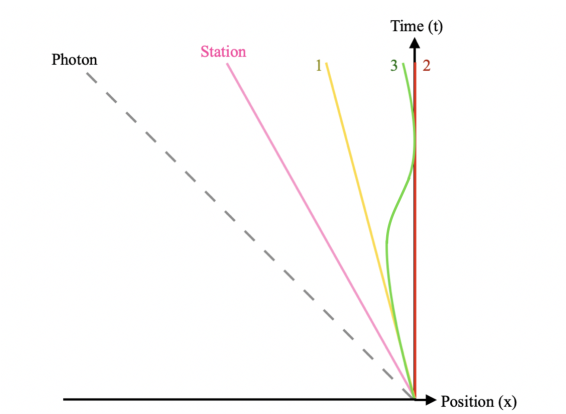


Figure 13: Spacetime diagram in the reference frame of ship 2

Task 5

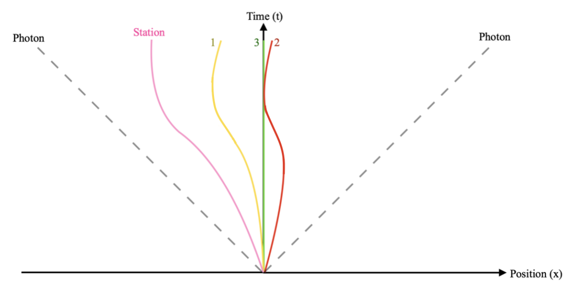


Figure 14: Spacetime diagram in the reference frame of ship 3

EXERCISE 7

Task 1

We know that the frame 1 observer is at rest on the planet and using unprimed coordinates, while the spaceships A and B travel with velocities v_A and $v_B = -v_A$ relative to frame 1. The frame 2 observer is close behind spaceship A, so this observer and A have no velocity in regards to each other, meaning $v'_A = 0$. We wish to find v'_B:

$$\begin{aligned} v'_B &= v_B - v_A \\ &= -v_A - v_A \\ &= -2v_A \end{aligned} \tag{19}$$

Task 2

We find the momenergy four-vectors in our frame of reference, where they have velocities v_A and v_B:

$$\begin{aligned} P_\mu(A) &= (E(A), \vec{p}(A)) = \gamma_A(m, mv_A, 0, 0) \\ P_\mu(B) &= (E(B), \vec{p}(B)) = \gamma_B(m, -mv_A, 0, 0) \end{aligned} \tag{20}$$

Where we have used the fact that $v_B = -v_A$ in this frame. We cut out p_y and p_z from now one, since these will stay zero as we only have movement in the x-direction.

Task 3

We find the momenergy four-vectors in frame 2 by utilizing their transformation properties and applying this to $P_\mu(A)$ and $P_\mu(B)$. First we find $P'_\mu(A)$:

$$\begin{aligned}
P'_\mu(A) &= C_{\mu\nu}P_\nu(A) \\
&= \gamma_A(\gamma_{rel}E_A - v_{rel}\gamma_{rel}p_{x,A}, \gamma_{rel}p_{x,A} - v_{rel}\gamma_{rel}E_A) \\
&= {\gamma'_A}^2(m - mv_A^2, mv_A - mv_A) \\
&= \gamma_A^2(m(1 - v_A^2), 0) \\
&= (m, 0)
\end{aligned} \tag{21}$$

Where $\gamma_{rel} = \gamma'_A = \gamma_A$, because this is only defined by the relative velocity v_{rel} between the two frames, which is v_A. This also counts for $P'_\mu(B)$:

$$\begin{aligned}
P'_\mu(B) &= C_{\mu\nu}P_\nu(B) \\
&= \gamma_B(\gamma_{rel}E_B - v_{rel}\gamma_{rel}p_{x,B}, \gamma_{rel}p_{x,B} - v_{rel}\gamma_{rel}E_B) \\
&= {\gamma'_B}^2(m + mv_A^2, -mv_A - mv_A) \\
&= \gamma_A^2(m(1 + v_A^2), -2mv_A)
\end{aligned} \tag{22}$$

Where we in each corresponding equation have used $v'_A = 0$ and $v'_B = -2v_A$ that we found in Task 1.

Task 8

We find the following values from the videos:

$$\begin{aligned}
m &= 10^6 \text{ kg} \\
v &= 0.262183c \\
N &= 5.90745 \times 10^{41}
\end{aligned}$$

Where m, v, and N are the spaceships' rest masses, their velocities in the planet frame, and the number of photons emitted from the explosion. Because of conservation of energy, we get the following relation between the energy of a photon emitted from the explosion and the energies of the spaceships:

$$E = \frac{E_A + E_B}{N} = \frac{2E_A}{N} = \gamma_{rel}\frac{2m}{N} \tag{23}$$

As we could see in Task 3, the energies of the spaceships were the same in both frames. We will have to multiply the photon energy we calculate with c to convert it to SI-units. We get

$$\begin{aligned}
E &= \frac{1}{\sqrt{1 - 0.262183^2}}\frac{2 \times 10^6 \text{ kg}}{5.90745 \times 10^{41}} \times \left(3 \times 10^8 \text{ m/s}\right) \\
&= 1.05 \times 10^{-27} \text{ J}
\end{aligned}$$

We use this result to find the wavelength of the photons observed in the planet frame:

$$\begin{aligned}
\lambda_1 &= \frac{h}{E} \\
&= \frac{6.62607015 \times 10^{34} \text{ Js}}{1.05 \times 10^{-27} \text{ J}} \\
&\approx 630 \text{ nm}
\end{aligned}$$

Task 9

We see from the wavelength that the color of the explosion as seen from the planet frame is red. This fits well with what we see from the video.

Task 10

As we know, we can find the energy of a photon in spaceship A's frame of reference by using the transformation properties of four-vectors. We know that the four-vector of a ph0ton travelling in the positive x-direction is

$$P_\mu^\gamma = (E, E, 0, 0) \tag{24}$$

Once again we will ignore the y and z components. We use (24) to find E' for a photon in this same direction:

$$\begin{aligned}
E' &= \gamma_{rel}E - v_{rel}\gamma_{rel}p_x \\
&= \gamma E - v\gamma E \\
&= E\gamma(1 - v)
\end{aligned} \tag{25}$$

For a photon travelling in the negative x-direction, we will have $v = -v$, which gives us $E' = E\gamma(1 + v)$ instead. The general expression for the energies of the photons travelling along the x-axis will therefore be

$$E' = E\gamma(1 \pm v) \quad \square \tag{26}$$

Task 11

First, we use (26) to derive an expression for the wavelength λ' observed in the spaceship frame:

$$\begin{aligned}
\lambda' &= \frac{h}{E'} \\
&= \frac{h}{E\gamma(1 \pm v)} \\
&= \frac{h}{E}\frac{\sqrt{1 - v^2}}{1 \pm v} \\
&= \lambda\left(\frac{\sqrt{1 - v^2}}{1 \pm v}\right)
\end{aligned} \tag{27}$$

This gives us an expression for the Doppler shift:

$$\begin{aligned}
\Delta\lambda &= \lambda - \lambda' \\
&= \lambda\left(1 - \frac{\sqrt{1 - v^2}}{1 \pm v}\right) \\
&= \lambda\left(1 - \frac{\sqrt{(1 - v)(1 + v)}}{1 \pm v}\right) \\
&= \lambda\left(1 - \sqrt{\frac{1 - v}{1 + v}}\right) \\
&= \lambda\left(\sqrt{\frac{1 + v}{1 - v}} - 1\right)
\end{aligned} \tag{28}$$

Which gives us the relativistiv Doppler formula:

$$\frac{\Delta\lambda}{\lambda} = \left(\sqrt{\frac{1+v}{1-v}} - 1\right) \quad \square \tag{29}$$

Task 12

We want to use (29) to find the Doppler shift between the two frames of reference. Inserting our values, we get

$$\begin{aligned}\Delta\lambda &= \lambda\left(\sqrt{\frac{1+v}{1-v}} - 1\right) \\ &= 630 \times 10^{-9}\ \mathrm{m}\left(\sqrt{\frac{1+0.262183}{1-0.262183}} - 1\right) \\ &\approx 436\ \mathrm{nm}\end{aligned}$$

Which corresponds to a violet color. This fits with what we see on the video of the spaceship frame.

Task 13

We want to see if the relativistiv Doppler formula is consistent with the normal Doppler formula $\Delta\lambda/\lambda = v/c$ for small v. To do this, we make a first-order Taylor expansion around $v = 0$ for the relativistic Doppler formula defined in (29), which we'll call $f(v)$ for now:

$$\begin{aligned}T_f(v) &= f(0) + f'(0)(v-0) \\ &= 0 + \frac{1}{(1-v)^2\frac{1+v}{1-v}}(0)v \\ &= \frac{v}{c}\end{aligned} \tag{30}$$

Where the last step comes from the fact that v in the relativistic Doppler formula only is given as a fraction of the light speed c, with no units. Thus, we find that the Taylor-expansion gives us the normal Doppler formula for low velocities:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{v}{c} \tag{31}$$

As we can see, for $v \ll c$, the two formulas can be considered consistent.