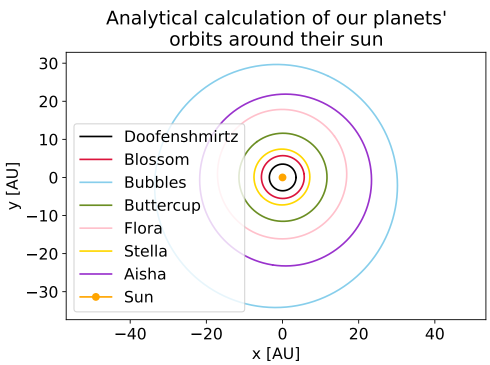
AST2000 - Part 2
Planetary Orbits

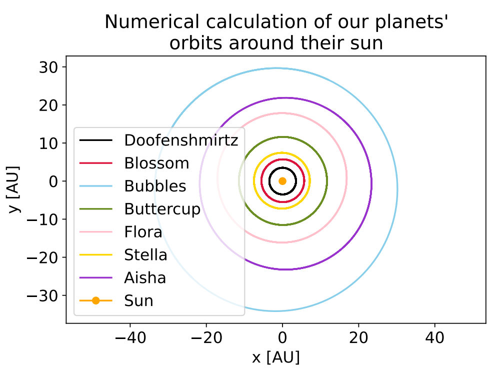
Candidates 15361 & 15384
(Dated: November 30, 2022)

In this part of the project we would like to be judged only on how well we have accomplished the challenges, including our code, results, discussions, conclusions etc. We have satisfyingly simulated the planetary orbits in our solar system, and used Kepler's laws to analyze said orbits. We compared Kepler's version of his 3$^{\text{rd}}$ law to Newton's version, and saw that there were significant differences between the two versions. We simulated how the planets in our solar system play a role on the sun's orbit around the center of mass, and both constructed and analyzed radial velocity and light curves.

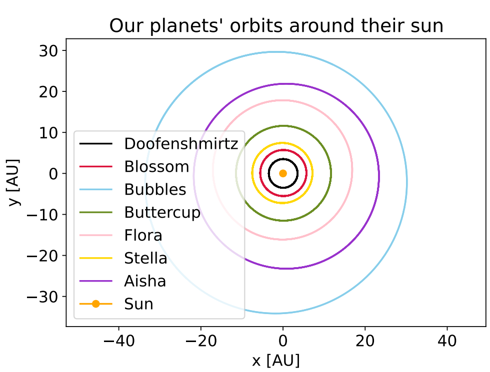
I. RESULTS



(a) Analytical



(b) Numerical



(c) Numerical on top of analytical

Figure 1: Analytical (a) versus numerical (b) calculation of the planetary orbits, both alone and together (c)

When calculating the planetary orbits analytically and numerically, we wanted to compare the results. Figure 1 shows the analytically calculated orbits as well as the numerical, with them plotted on top of each other in (c). These seemed identical, so we wanted to really study the difference by plotting the numerically calculated orbits as white dotted lines on top of the analytical orbits. Figure 2 shows the resulting plot, along with multiple spots where we zoomed in to see at what degree they were aligned.

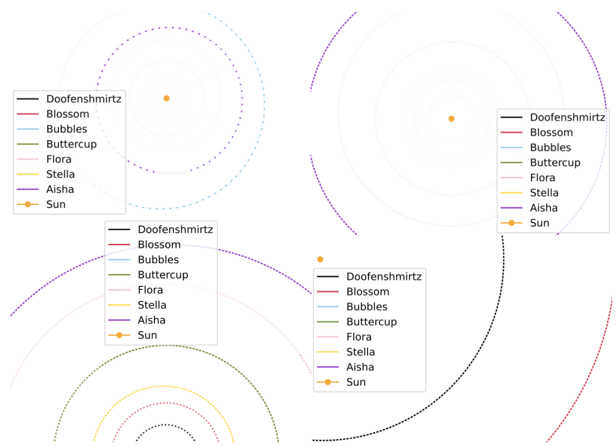


Figure 2: The numerical orbits plotted over the analytical orbits as white dotted lines.

When checking if our simulated orbits were consistent with those of our research team, we found that we were well below the acceptable margin of 1%. The biggest relative deviation was for our home planet Doofenshmirtz, which drifted approximately 0.033% off of its actual position in our simulation.

To analyze the simulated planetary orbits, we primarily wanted to see if they were consistent with Kepler's laws. We compared two areas swept out by each planet, one were they were close to their orbit's aphelion, and one where they were close to the perihelion. The difference in these areas multiplied by 10 000, as well as the relative error in percentage is displayed in Table I. Also displayed in the table is the distance each planet travelled when sweeping out these areas, and their mean velocity.

Planet	ΔA [AU^2]		dr [AU]		$\overline{v}$ [AU/yr]	
	diff	rel err	aphel	perihel	aphel	perihel
0	3.45	0.036	0.543	0.555	5.587	5.710
1	2.58	0.018	0.510	0.534	4.367	4.569
2	183.42	0.053	2.001	2.387	1.713	2.043
3	14.16	0.040	0.601	0.614	3.076	3.144
4	5.11	0.008	0.712	0.789	2.438	2.701
5	0.49	0.002	0.594	0.626	3.804	4.010
6	26.30	0.018	1.241	1.363	2.125	2.332

Table I: ΔA denotes the difference in area swept out by each planet close to its aphelion and close to its perihelion. dr and $\overline{v}$ denote the distance travelled and the mean velocity.

When checking if the orbits were consistent with Kepler's 3$^{\text{rd}}$ law, we saw that there was a large difference in the calculated orbital periods when using his original version

$$P^2 = a^3 \tag{1}$$

versus Newton's version

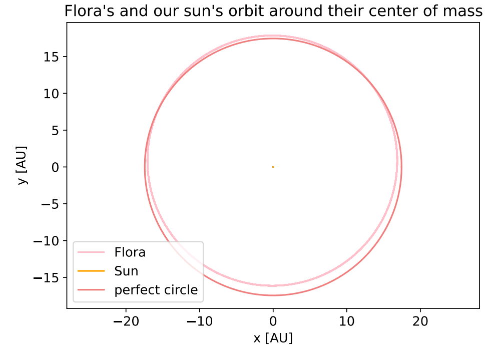
$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3 \tag{2}$$

Newton's version is the most accurate one, as Kepler's version does not take the gravitational pull affecting the planets into consideration. Fortunately for us, our numerically calculated orbital periods are much closer to those calculated analytically using Newton's version than those using Kepler's version, as evident in Table II.

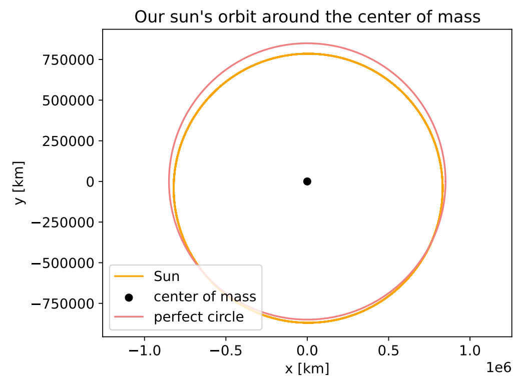
Planet	Kepler	Newton	Numerical
0	6.554	3.896	3.896
1	13.337	7.928	7.928
2	180.651	107.345	107.390
3	39.328	23.379	23.379
4	70.056	41.639	41.646
5	19.863	11.808	11.808
6	107.245	63.752	63.753

Table II: Each planet's orbital period in years calculated using Kepler's 3$^{\text{rd}}$ law, Newton's version, and our simulation.

When simulating a two-body system, we chose planet 4, which is Flora, as this is the planet with the second largest mass, and it's not too far from the sun (see Figure 1). Both Flora's and our sun's simulated orbits are shown in Figure 3, with the latter shown alone in (b), since it's minuscule in comparison to Flora's, and therefore is barely visible in (a). Both orbits are compared to perfect circles with radius equal to their mean distance from the center of mass for illustrative purposes.



(a) Flora and our sun



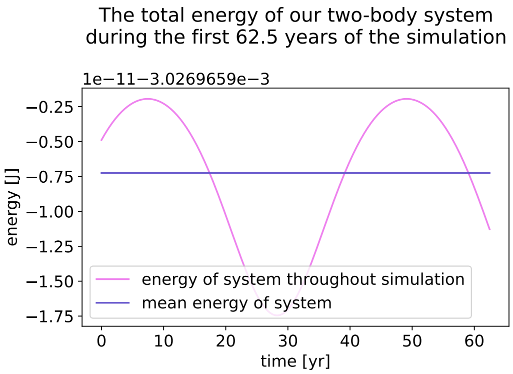
(b) Our sun alone

Figure 3: Flora's and our sun's obits around their common center of mass compared to perfect circles.

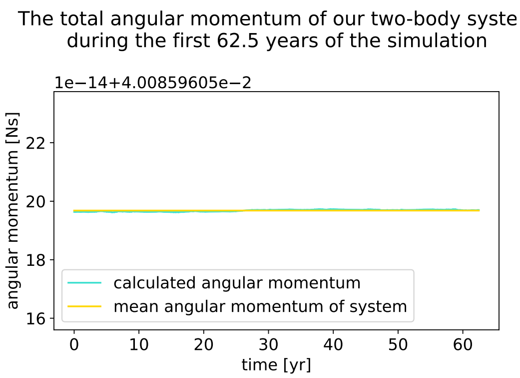
To visualize the orbits' degree of stability, we simulated their orbits for 20 times our home planet's orbital period, and zoomed in on their orbits to see if they changed with time, which they didn't (see Figure 4).

Figure 4: Zoomed in on Flora's and the sun's orbits in the two-body system.

We plotted the change in energy and angular momentum during the first three of these orbital periods, along with the mean energy and momentum of the system (see Figure 5), to see if these were conserved to a reasonable degree.

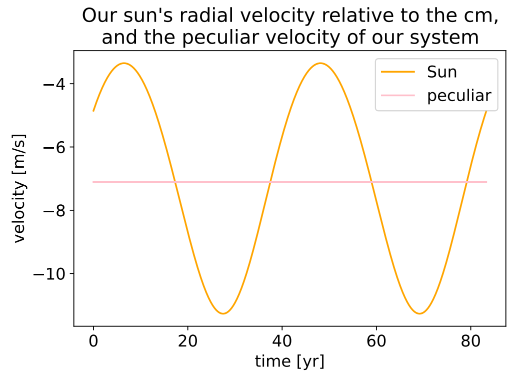


(a) Energy

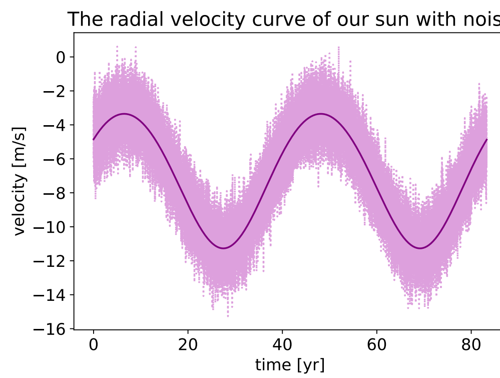


(b) Angular momentum

Figure 5: The total energy (a) and angular momentum (b) in the two-body system consisting of Flora and our sun.



(a) Radial velocity with peculiar velocity



(b) Constructed radial velocity curve

Figure 6: Our sun's radial velocity plotted with the peculiar velocity of our two-body system (a), and the radial velocity curve we designed (b).

We found that the relative error of the variation in our two-body system's energy was $5.12 \times 10^{-7}\%$, while the relative error of the variation in angular momentum was $2.84 \times 10^{-12}\%$.

When designing our radial velocity curve, we chose the following parameters

$$
\begin{aligned}
v_{pec} &= -1.5 \times 10^{-3} \text{ AU/yr} \\
i &= 90^\circ
\end{aligned}
$$

We added some Gaussian noise and constructed the curve shown in Figure 6b.

In Figure 7a, we have plotted the noisy data received from the group we collaborated with, along with the approximate peculiar velocity of their two-body system, which we found to be

$$v_{pec} = -5 \times 10^{-6} \text{ AU/yr} \approx -0.024 \text{ m/s}$$

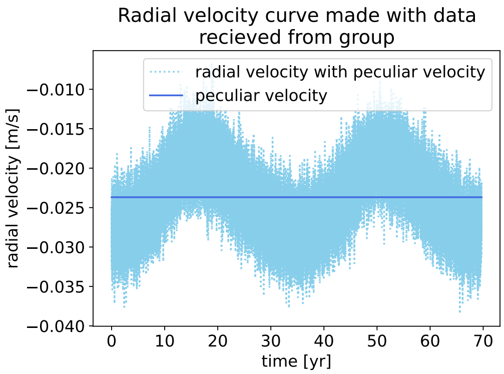
After subtracting v_{pec}, we managed to model the radial velocity curve shown in Figure 7b using the least-squares method.

Thanks to the least-squares method, we were able to determine the following

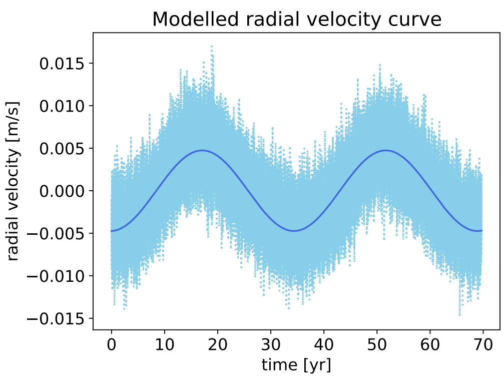
$$
\begin{aligned}
v_r &\approx 0.005 \text{ m/s} \\
P &\approx 34.57 \text{ yr} \\
t_0 &\approx 17.06 \text{ yr} \\
m &\approx 2.786 \times 10^{24} \text{ kg}
\end{aligned}
$$

Where v_r, P and t_0 are the estimated values for the group's sun's maximum radial velocity, revolution period around the center of mass and first peak in radial velocity, respectively. m is the estimated mass of the planet they used in their two-body orbit, which had an actual mass of 2.778×10^{24}. Thus, the relative error of our estimated mass was 0.29%.

When designing our light curve, which is shown in Figure 8, we found that it took approximately 3.45 hours from Flora started eclipsing our sun until it was fully eclipsed. The relative flux was then 0.9973.



(a) Radial velocity curve with peculiar velocity



(b) Modelled radial velocity curve

Figure 7: The radial velocity curve we managed to make from the other group's noisy data.

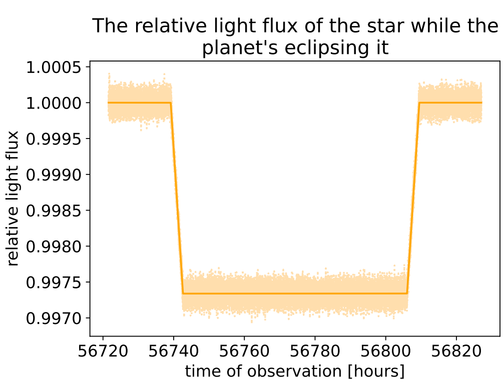


Figure 8: Our constructed light curve as Flora's eclipsing our sun with added noise.

After exchanging data with the other group, we quickly saw that it would be difficult to construct an approximate light curve, because of the large amounts of noise in their data. Therefore, we were not able to determine the radius and density of the planet eclipsing their sun, which was

$$R = 5134.05 \text{ km}$$

$$\rho = 4899.45 \text{ kg/m}^3$$

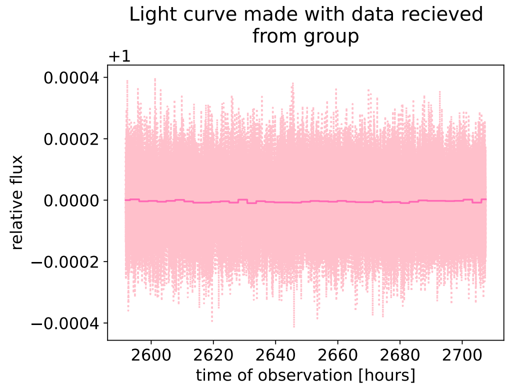
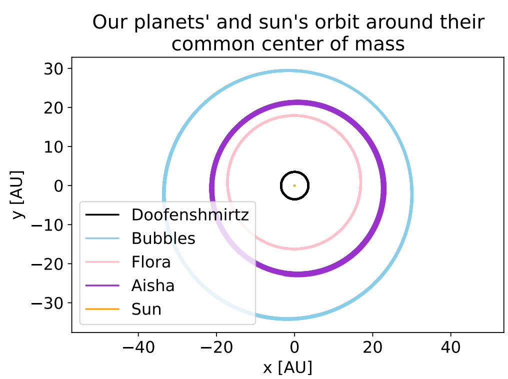
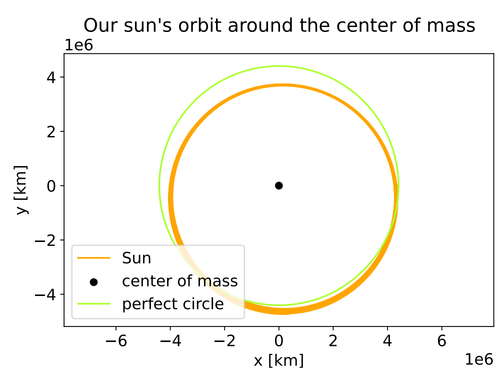


Figure 9: The light curve we managed to construct from the group's noisy data.

We chose to simulate our multiple-bodied system with planet 0, 2, and 6, which are Doofenshmirtz, Bubbles and Aisha, in addition to Flora. Like with the two-body system, we simulated their orbits and plotted them together (see Figure 10a). We also plotted our sun's orbit alone, compared to a perfect circle (see Figure 10b).



(a) Planets and sun



(b) Sun alone

Figure 10: The planets' and our sun's obits around their common center of mass compared to a perfect circle (a). Due to the planets' orbit being much larger than our sun's, the latter is shown alone in (b).

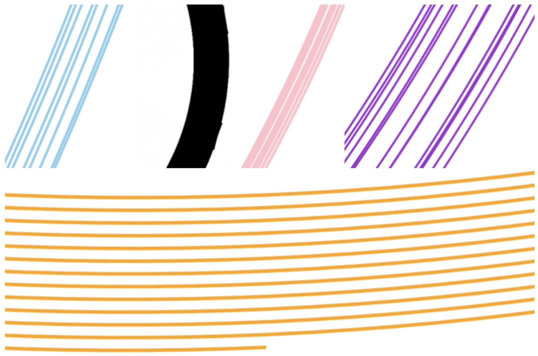
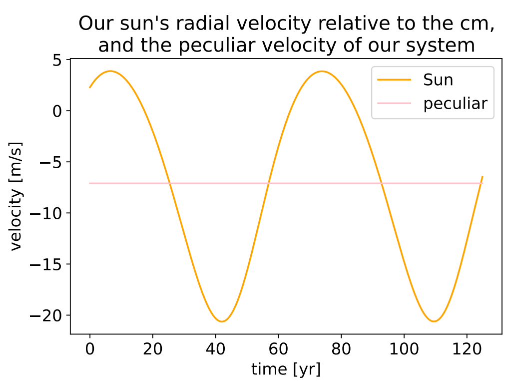


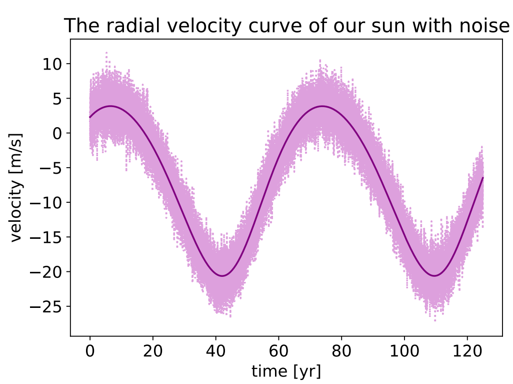
Figure 11: Zoomed in on each of the planets', as well as the sun's, orbits in the multiple-body system.

It was apparent from the lines' thickness that these orbits were not as stable as when we simulated the two-body system. To visualize this instability, we zoomed in on their orbits to see just how much they went off track (see Figure 11).

Once again we designed a radial velocity curve for our sun, where we chose the same peculiar velocity v_{pec} and inclination angle i as the last time (see Figure 12b)



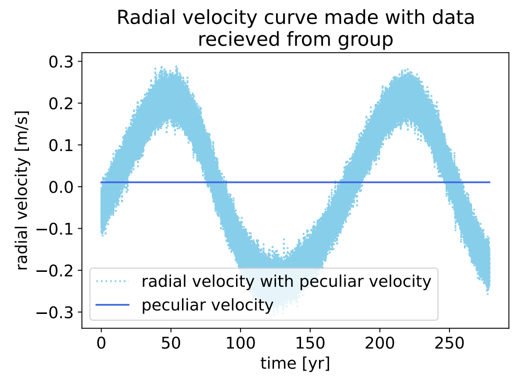
(a) Radial velocity with peculiar velocity



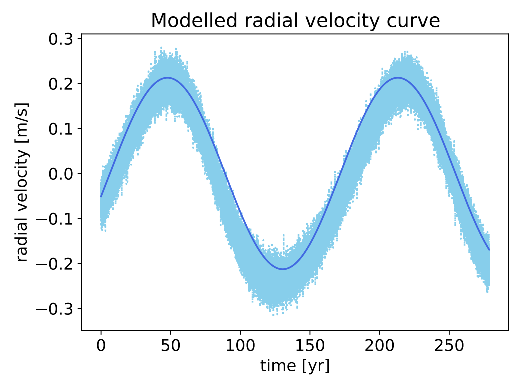
(b) Constructed radial velocity curve

Figure 12: Our sun's radial velocity plotted with the peculiar velocity of our multiple-body system (a), and the radial velocity curve we designed for our sun (b).

When analyzing the other group's radial velocity curve made from their multiple-bodied system (see Figure 13), we found that they had used the same peculiar velocity as in their two-body system.



(a) Radial velocity curve with peculiar velocity



(b) Modelled radial velocity curve

Figure 13: The radial velocity curve we managed to make from the other group's noisy data associated with their multiple-bodied system.

By once again using the least-squares method, we determined the following values

$$v_r \approx 0.213 \text{ m/s}$$
$$P \approx 165.43 \text{ yr}$$
$$t_0 \approx 47.69 \text{ yr}$$

In an effort to try to resolve how many planets they used in their multiple-bodied system, we found that they were using between 44 and 45 planets, when in reality they were using 4, just like us.

II. DISCUSSION

As most of our results in this study is based off of our simulations, they are all very prone to round-off errors and human mistakes leading to miscalculations. For example, the small deviation in Doofenshmirtz' simulated orbit was most likely due to round-off errors, as we used time steps of approximately 204.78 minutes when using the leap frog integration method, which may have been too large. Fortunately for us, this deviation was well below the acceptable margin, so the errors weren't too problematic in this case.

When comparing our results to what was expected from Kepler's laws, we were pleased to see that they were consistent in most cases, as all the planets' mean velocities were larger by the perihelion than the aphelion (see Table I). The relative errors of the difference in the areas sweeped out during equal time periods was well below 1% for all of the planets, which indicated that our there were no major flaws in our calculations. We believed that for planets 0 (Doofenshmirtz), 2 (Bubbles) and 3 (Buttercup), where the relative errors are the largest, this could be because of how elliptical their orbits are. To look more into this, we requested our wonderful research team for some data describing exactly this, and were able to gather the following eccentricity values:

Planet no.	Planet name	Eccentricity
0	Doofenshmirtz	0.01105101
1	Blossom	0.02276295
2	Bubbles	0.08812752
3	Buttercup	0.01088058
4	Flora	0.05121254
5	Stella	0.02648055
6	Aisha	0.04672344

Table III: Each planet's orbit's eccentricity.

We noticed that Bubbles, which had the largest relative error, also is the planet with the most elliptical orbit. However, Buttercup's orbit is the most circular of them all, while having the second largest relative error. In addition to this, we noticed that Flora's orbit is the second most elliptical, while its relative error is the second smallest. This made us doubt our theory, and we therefore went over to believe that the main reason for the errors are round-off errors in our simulations.

As mentioned in Section I, the main flaw of Kepler's 3$^{\text{rd}}$ law was that he did not include the force of gravity in his formula. This is probably the reason why the analytically calculated orbital periods using Kepler's original version differ so much from those calculated using Newton's version (see Table II). Fortunately for us, our numerically calculated orbital periods barely differed from those we found using Newton's method.

We believe that the error in the planetary mass we estimated by analyzing the other group's radial velocity curve probably was a result of the amount of noise in their data. It could of course also be because of our curve-fitting software not being sufficiently accurate, or maybe a combination of the two. Our software used the least-squares approximation method to pick out values for v_r, P and t_0 that fit best with the noisy data we were supplied with. From these values, we attempted to estimate their planet's mass. When supplying our software with possible values, we determined intervals where a good fit most likely would be included, and then divided these intervals into 50 possible values. Naturally, if we divided the main intervals into even more values, we would probably have found even better estimates for v_r, P and t_0. This would of course affect the accuracy of our estimation of their planet's mass as well. However, we chose to use 50 to avoid overworking our software.

Comparing our light curve with the one we got in exchange, it is obvious that the size ratio between their planet and their sun is much bigger than in our case. In fact, our sun is about 19.4 times the size of Flora, while their sun is about 403.4 times the size of their chosen planet. As evident in Figure 8, the relative flux when Flora was eclipsing our sun was small enough for the noise to not completely overshadow the curve. This was not the case when looking at the other group's data (see Figure 9), as the difference in flux was undetectable. Because of this, it was impossible for our software to calculate an estimate of their planet's radius and density. The only thing that was evident from their data was that its radius had to be minuscule compared to that of their sun, unless its density were to be unrealistically low.

As evident in Figure 11, our simulation of the orbits in our multiple-body system may have been subject to some miscalculations. We tried some different approaches when simulating these orbits, and we had to make a major simplification in order for our sun's orbit to appear more realistic. At first, we attempted to simulate all the orbits by calculating the gravitational force they all were affected by from each other. Our sun then immediately flew straight out of the solar system, so we had to come up with a better way. We tried to calculate it all using conservation of momentum instead, but then the planets flew out of the solar system instead. However, we did notice that when using gravity to calculate the orbits, the planetary orbits seemed more accurate, while the solar orbit seemed more accurate when using conservation of momentum. Therefore, we chose to simulate the planetary by calculating the gravitational pull from each of the bodies in the system on each of the planets. For the solar orbit, we found that we had to simulate the location of the planets' common center of mass and the gravitational pull from our sun working on it. We then continuously calculated the planetary center of mass' velocity to find its momentum as time passed by. From this, we were able to simulate our sun's velocity and trajectory around the total system's center of mass. Obviously, these simplifications are not very accurate, and we believe that this most likely is the reason why the orbits appear more unstable than they are in real life.

When analyzing the other researchers' data that they gathered from their simulation of their multiple-bodied system, it was apparent that their sun's orbit must be rather unstable as well (see Figure 13). Because of the unnatural tilt of the curve constructed from their noisy data, the modelled radial velocity curve doesn't line too well up with the data in all places. This is because our software constructs a sinus wave using parameters gathered using the least-squares method. This curve will obviously we symmetric with an amplitude, phase and frequency that doesn't change with time. Of course, another very possible source of error here is that we still didn't want to check more than 50 values to spare our software.

When trying to resolve how many planets the other researchers included in their multiple-bodied system, we believe that the inaccuracies in the use of the least-squares method may have played a role. This is because we took use of the estimated maximum radial velocity of their sun in the multiple-bodied system to calculate an estimate of its maximum momentum. We then assumed this to be equal to the momentum of all the planets in the system combined, and compared this to their sun's momentum in the two-body system. Obviously, these are also simplifications that are prone to many simulation and calculation errors. We were therefore not too surprised when our estimate turned out to be completely off the rails.

III. CONCLUSION

We found that we were successful in simulating and studying the planetary orbits within our solar system by comparing our results to those gathered by our research team. We also used these results to see if they were consistent with Kepler's 2nd law, and Newton's version of his 3rd law, which they were, at least to a reasonable degree when considering typical minor errors that often occur during simulations.

We were able to analyze how the gravitational pull from a large planet in our solar system, namely Flora, affected our sun, by ignoring the other planets and simulating their trajectories when approximated as a two-body system. By gathering data from this simulation, we managed to construct a radial velocity curve for our sun. We exchanged this velocity curve with two other aspiring researches from another solar system, and tried to analyze their data. Thanks to our amazing software utilizing the least-squares method, we managed to find out at which peculiar velocity their system travelled with, subtract this and model a radial velocity curve from their noisy data. This made it possible for us to calculate a relatively accurate estimate of the planetary mass in their system.

After making a few simplifications, we also managed to create a light curve describing the relative flux we would receive at our home planet by our sun while Flora's eclipsing it. We once again exchange this curve with the distant researchers, in hopes of determining the radius and density of the planet eclipsing their sun. Sadly, because of their large sun literally outshining their planet, we were unsuccessful in this task, as well as in creating an approximate light curve from their data.

We repeated our study of the effect gravity from smaller celestial bodies in our solar system has on our sun, by including more planets in our system, and found that this indeed affected the steadiness of the multiple bodies' orbits. Constructing yet another radial velocity curve and exchanging our data with the other researchers, we also observed how this affected both our and their sun's radial velocities. We were unsuccessful in resolving how many planets they had included in their system, but found that this most likely was due to a combination of miscalculations, noisy data and unrealistic simplifications.

REFERENCES

[1] Hansen F. K. *Lecture Notes 1B*. URL: https://www.uio.no/studier/emner/matnat/astro/AST2000/h22/undervisningsmateriell/lecture_notes/part1b.pdf.

Appendix A: Proving Kepler's laws

1. The 2nd law

As we know, the length covered by an arc within a circle can be written as

$$s = r\Delta\theta \tag{A1}$$

Where r is the radius of the path's curvature, and $\Delta\theta$ is the angle covered by the arc. Since the planetary orbits are elliptical, r will change with time, so $\mathbf{r}(t)$ and $\mathbf{r}(t+\Delta t)$ will be different. If we let Δt become infinitesimal, then denoted by $\mathrm{d}t$, Δr will tend to zero. We can then approximate s as

$$s = r\mathrm{d}\theta \tag{A2}$$

Where $\mathrm{d}\theta$ is the infinitesimal angle covered by the arc sweeped out during $\mathrm{d}t$. We also know that the area of a right triangle is defined as

$$A = \frac{b}{2}h \tag{A3}$$

where b is the baseline and h is the height of the triangle. Since the height of the area sweeped out by a planet in its orbit during an infinitesimal time period $\mathrm{d}t$ will be r, and the baseline of this area is defined as in (A2), this area can be written as

$$\mathrm{d}A = \frac{1}{2}r^2\mathrm{d}\theta \tag{A4}$$

From Lecture Notes 1B [1], we know that the angular momentum per mass h is defined as

$$h = |\vec{h}| = |\vec{r} \times \dot{\vec{r}}| = r^2\dot{\theta} = r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{A5}$$

and that this magnitude is conserved, meaning

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0 \tag{A6}$$

Combined with (A4), this means that the total area sweeped out by a planet in orbit per time is

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\\ &= \frac{1}{2}r^2\dot{\theta}\\ &= \frac{1}{2}h\\ &= constant \ \square\end{aligned} \tag{A7}$$

2. Newton's version of the 3rd law

By integrating (A7) for a full orbital time period P, we get

$$\begin{aligned}\int_0^P \frac{\mathrm{d}A}{\mathrm{d}t}\mathrm{d}t &= \int_0^P \frac{1}{2}h\mathrm{d}t\\ \int_0^P \mathrm{d}A &= \int_0^P \frac{1}{2}h\mathrm{d}t\\ A &= \frac{1}{2}hP\\ P &= 2\pi\frac{ab}{h}\end{aligned} \tag{A8}$$

Since the area swept out during an entire orbital period P is the area $A = \pi ab$ of the elliptical orbit, where a is the semi-major axis and b is the semi-minor axis.

In Lecture Notes 1B [1], we find that h also can be expressed as

$$\begin{aligned}h &= \sqrt{GMp}\\ &= \sqrt{G(m_1+m_2)a(1-e^2)}\end{aligned} \tag{A9}$$

Where G is the gravitational constant, M is the combined mass of the orbiting planet (m_1) and the sun (m_2). e defined as an ellipse's eccentricity, and is defined as

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

which gives us a new expression for b

$$b = a\sqrt{1-e^2} \tag{A10}$$

By combining (A9) and (A10) with (A8), we get

$$\begin{aligned}P &= 2\pi\frac{a^2\sqrt{1-e^2}}{\sqrt{G(m_1+m_2)a(1-e^2)}}\\ &= 2\pi\frac{a^{3/2}}{\sqrt{G(m_1+m_2)}}\end{aligned} \tag{A11}$$

Which, by powering both sides of (A11) by two, gives us Newton's version of Kepler's 3rd law:

$$P^2 = \frac{4\pi^2}{G(m_1+m_2)}a^3 \ \square \tag{A12}$$

Appendix B: Deriving expressions for the total energy and angular momentum of a two-body system

1. Total energy

For two objects to be bound by gravity, their total potential energy must be larger in magnitude than their total kinetic energy. The potential gravitational energy of a system of this kind is defined as

$$U = -G\frac{m_1 m_2}{r} \tag{B1}$$

where r is the distance between the two. As we can see, this quantity is always negative. We define the reduced mass $\hat{\mu}$ in the following way:

$$\hat{\mu} = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M} \tag{B2}$$

which, combined with (B1), gives us a new way to express U:

$$U = -\frac{GM\hat{\mu}}{r} \tag{B3}$$

The objects' kinetic energies relative to one another is

$$K = \frac{1}{2}mv^2 \tag{B4}$$

where m denotes the mass of the object we want to calculate the kinetic energy of, and v is its velocity relative to the other object. This quantity is always positive, which means that the total energy of the system must be negative in all frames of reference.

We know from Lecture Notes 1B [1] that the positions of the bodies in a two-body system seen from their common center of mass can be written as

$$\begin{aligned} \vec{r_1}^{\,\mathrm{CM}} &= -\frac{\hat{\mu}}{m_1}\vec{r} \\ \vec{r_2}^{\,\mathrm{CM}} &= \frac{\hat{\mu}}{m_2}\vec{r} \end{aligned} \tag{B5}$$

where $\vec{r}$ is m_2's positional vector relative to m_1. This gives us the following expressions for their velocities relative to the center of mass:

$$\begin{aligned} \vec{v_1}^{\,\mathrm{CM}} &= -\frac{\hat{\mu}}{m_1}\vec{v} \\ \vec{v_2}^{\,\mathrm{CM}} &= \frac{\hat{\mu}}{m_2}\vec{v} \end{aligned} \tag{B6}$$

By combining (B6) and (B2), we can derive an expression for the total kinetic energy of the system as seen from the center of mass:

$$\begin{aligned} K_{tot}^{\mathrm{CM}} &= K_1^{\mathrm{CM}} + K_2^{\mathrm{CM}} \\ &= \frac{1}{2}m_1 \left|\left|\vec{v_1}^{\,\mathrm{CM}}\right|\right|^2 + \frac{1}{2}m_2 \left|\left|\vec{v_2}^{\,\mathrm{CM}}\right|\right|^2 \\ &= \frac{1}{2}\left(m_1\left(\frac{\hat{\mu}}{m_1}v\right)^2 + m_2\left(\frac{\hat{\mu}}{m_2}v\right)^2\right) \\ &= \frac{1}{2}\left(\hat{\mu}^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right)v^2 \\ &= \frac{1}{2}\hat{\mu}v^2 \end{aligned} \tag{B7}$$

Since the observed potential energy is the same for all points of reference, we find that the total energy of the two-body system as seen from the center of mass can be written as

$$\begin{aligned} E &= K_{tot}^{\mathrm{CM}} + U \\ &= \frac{1}{2}\hat{\mu}v^2 - G\frac{m_1 m_2}{r} \ \square \end{aligned} \tag{B8}$$

2. Angular momentum

The angular momentum per mass of a one-body system, like a planet orbiting a star, is defined as

$$\begin{aligned} \vec{h} &= \vec{r} \times \dot{\vec{r}} \\ &= \vec{r} \times \frac{\vec{p}}{m} \end{aligned} \tag{B9}$$

where m is the mass of the orbiting body. In this system, $\vec{r}$ and $\dot{\vec{r}}$ are its position and velocity vectors relative to the body it orbits around, since the latter is seen as the system's center of mass. We can use this to express $\vec{h}$ relative to the center of mass of a two-body system by exchanging m with the system's reduced mass $\hat{\mu}$:

$$\vec{h} = \vec{r} \times \frac{\vec{p}}{\hat{\mu}} \tag{B10}$$

This gives us the final expression for the angular momentum of the two-body system as seen from the center of mass:

$$\begin{aligned} \vec{P} &= \hat{\mu}\vec{h} \\ &= \vec{r} \times \hat{\mu}\frac{\vec{p}}{\hat{\mu}} \\ &= \vec{r} \times \hat{\mu}\dot{\vec{r}} \ \square \end{aligned} \tag{B11}$$