AIMS METHODS RESULTS

TITLE

AST5220

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ABSTRACT

Key words. KEYWORDS

1. Introduction

mention here that we use Planck 2018

mention that theory is from the website

2. Milestone I

The evolution of the universe is governed by the interplay between different energy components, including radiation, matter, and dark energy. Understanding how these components influence the expansion history is essential for predicting the large-scale structure of the Universe and the Cosmic Microwave Background (CMB) fluctuations. This milestone focuses on modeling the background evolution of the universe using the Friedmann equations, which describe how the Hubble parameter H, and thus time and distance measures, evolve with redshift. I will implement a numerical framework that takes in cosmological parameters and computes such key background quantities, and use this to fit to measurements of supernova luminosity distances. word differently? My approach involves numerically solving the Friedmann equations and interpolating the results using splines to extract meaningful physical insights.

By completing this milestone, the goal is to establish a robust computational framework that will serve as the foundation for later stages of the project, where I will analyze perturbations and extract information about the CMB anisotropies. The results from this milestone will therefore be essential for constraining cosmological parameters using observational data.

2.1. Theory

As mentioned in section 1, I will adopt the best-fit Planck 2018 cosmology cite as my fiducial model, which includes the following parameters:

$$
\begin{aligned}
h &= 0.67, \\
T_{\mathrm{CMB0}} &= 2.7255\,\mathrm{K}, \\
N_{\mathrm{eff}} &= 3.046, \\
\Omega_{b0} &= 0.05, \\
\Omega_{\mathrm{CDM0}} &= 0.267, \\
\Omega_{k0} &= 0.
\end{aligned}
$$

Here, h is the dimensionless Hubble constant, which is related to the commonly presented Hubble constant through

$$H_0 = 100h\,\mathrm{km\,s^{-1}\,Mpc^{-1}}. \tag{1}$$

Furthermore, T_{CMB0} is the present day value of the CMB temperature, and N_{eff} is the effective number of relativistic degrees of freedom. These quantities are necessary to compute the photon and neutrino density parameters, which are given by

$$\Omega_{\gamma 0} = g\frac{\pi^2}{30}\frac{(k_{\mathrm{B}}T_{\mathrm{CMB0}})^4}{\hbar^3 c^5}\frac{8\pi G}{3H_0^2}, \tag{2}$$

$$\Omega_{\nu 0} = \frac{7}{8}N_{\mathrm{eff}}\left(\frac{4}{11}\right)^{1/3}\Omega_{\gamma 0}, \tag{3}$$

where $g = g_\gamma = g_\nu = 2$, since photons and neutrinos both have 2 internal polarization states. Together with the baryon (Ω_{b0}), cold dark matter (Ω_{CDM0}) and curvature (Ω_{k0}) density parameters, we then get the dark energy density parameter from

$$\Omega_{\Lambda 0} = 1 - \left(\Omega_{k0} + \Omega_{b0} + \Omega_{\mathrm{CDM0}} + \Omega_{\gamma 0} + \Omega_{\nu 0}\right), \tag{4}$$

since the density parameters must all sum to unity.

TODO: explain critical density, what the density parameters actually represent, equations for the evolutions of the different parameters stated in the project, etc.

Throughout this work, I will frequently refer to matter and radiation density parameters

$$\Omega_m = \Omega_b + \Omega_{\mathrm{CDM}}, \tag{5}$$

$$\Omega_r = \Omega_\gamma + \Omega_\nu, \tag{6}$$

since baryonic and cold dark matter are effectively non-relativistic, while neutrinos have such small mass that they can be treated as radiation. correct? As such, the matter density $\Omega_m \propto a^{-3}$, which corresponds to dilution of density in an expanding volume, while $\Omega_r \propto a^{-4}$, as radiation also gets redshifted as the Universe expands. explain more thoroughly?

2.1.1. Evolution of the Universe and the Hubble parameter

The expansion of the Universe is governed by General Relativity, with the large-scale dynamics described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. Assuming a homogeneous and isotropic universe, the metric is given by

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right], \tag{7}$$

where $a(t) = 1/(1+z)$ is the scale factor, with z being the cosmological redshift. The constant k determines the curvature of the

Universe ($k = 0$ for a flat universe, $k > 0$ for a closed universe, and $k < 0$ for an open universe). explain scaling of a?

The evolution of $a(t)$ is governed by the Friedmann equation, which is derived from Einstein's field equations: cite notes?

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \tag{8}$$

Here, $H = \dot{a}/a$ is the Hubble parameter, and ρ denotes the total energy density of the universe. TODO: derive this we arrive at

$$H = H_0\sqrt{(\Omega_{b0} + \Omega_{\mathrm{CDM}0})a^{-3} + (\Omega_{\gamma 0} + \Omega_{\nu 0})a^{-4} + \Omega_{k0}a^{-2} + \Omega_{\Lambda 0}}. \tag{9}$$

TODO: present 2nd eq. when deriving t_acc

TODO: continuity eq. explain scaling of components

When integrating from the very early universe, it becomes quite numerically challenging to use the scale factor a as time parameter, as it quickly tends to vanishing orders of magnitude as we move further back in time. rephrase?, I will therefore adopt the logarithmic time coordinate

$$x = \log a, \tag{10}$$

instead, which implies that $x = 0$ today and $x = -\infty$ at the Big Bang. Expressed in terms of the matter and radiation density parameters, we can equivalently write the Hubble parameter as

$$H = H_0\sqrt{\Omega_{m0}e^{-3x} + \Omega_{r0}e^{-4x} + \Omega_{k0}e^{-2x} + \Omega_{\Lambda 0}}. \tag{11}$$

A commonly used rescaled version of the Hubble parameter is the conformal Hubble parameter:

$$\mathcal{H} = aH = H_0\sqrt{\Omega_{m0}e^{-x} + \Omega_{r0}e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0}e^{2x}}. \tag{12}$$

This naturally appears when rewriting cosmological equations in terms of the conformal time η, which I will present below. explain why? I will focus more on this version of the Hubble parameter, and it will become useful to know its first and second derivatives with respect to x, specifically in order to verify numerical stability. explain why? After some tedious calculation, we find that these are

$$\begin{aligned}\frac{d\mathcal{H}}{dx} &= \frac{H_0}{2}\frac{-\Omega_{m0}e^{-x} - 2\Omega_{r0}e^{-2x} + 2\Omega_{\Lambda 0}e^{2x}}{\sqrt{\Omega_{m0}e^{-x} + \Omega_{r0}e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0}e^{2x}}}, \\ &= -\frac{H_0^2}{2\mathcal{H}}\left(\Omega_{m0}e^{-x} + 2\Omega_{r0}e^{-2x} - 2\Omega_{\Lambda 0}e^{2x}\right), \end{aligned} \tag{13}$$

$$\begin{aligned}\frac{d^2\mathcal{H}}{dx^2} &= \frac{H_0}{2}\left(\frac{\Omega_{m0}e^{-x} + 4\Omega_{r0}e^{-2x} + 4\Omega_{\Lambda 0}e^{2x}}{\sqrt{\Omega_{m0}e^{-x} + \Omega_{r0}e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0}e^{2x}}}\right. \\ &\qquad \left. -\frac{1}{2}\frac{\left(\Omega_{m0}e^{-x} + 2\Omega_{r0}e^{-2x} - 2\Omega_{\Lambda 0}e^{2x}\right)^2}{\left(\Omega_{m0}e^{-x} + \Omega_{r0}e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0}e^{2x}\right)^{3/2}}\right), \\ &= \frac{H_0^2}{\mathcal{H}}\left[\frac{1}{2}\Omega_{m0}e^{-x} + 2\Omega_{r0}e^{-2x} + 2\Omega_{\Lambda 0}e^{2x} - \frac{1}{H_0^2}\left(\frac{d\mathcal{H}}{dx}\right)^2\right]. \end{aligned} \tag{14}$$

2.1.2. Conformal time and distance measures

TODO: explain horizon thing The conformal time η is an alternative time coordinate useful in cosmology, defined by:

$$d\eta = \frac{dt}{a}.$$

Substituting $dt = da/Ha$, we obtain:

$$\frac{d\eta}{dx} = \frac{c}{\mathcal{H}}.$$

This equation can be numerically integrated to obtain $\eta(x)$, which describes how far light has traveled since the Big Bang.

In cosmology, three main distance measures are used:

1. Comoving distance (used to measure separations at fixed cosmological time):

$$\chi = \eta_0 - \eta.$$

2. Angular diameter distance (relates an object's physical size to its angular size):

$$d_A = a\chi.$$

3. Luminosity distance (used in supernova observations):

$$d_L = d_A(1+z)^2.$$

For a flat universe ($k = 0$), the comoving distance is simply:

$$\chi(x) = \int_x^{x_0}\frac{c\,dx'}{H(x')}.$$

These distances are essential for comparing theoretical predictions with observational data, such as the Hubble diagram from supernova measurements.

2.1.3. Key cosmological epochs

Though it also has been verified by numerous observations, based on the expression for the Hubble parameter it is not hard to see that the Universe must have gone through phases where its energy budget was (or will be) dominated by radiation, matter and dark energy, separately, in that order. This implies that there must have been a point in time where the Universe was equal amounts of radiation and matter (matter and dark energy), if we neglect the curvature and dark energy (radiation). At radiation-matter equality (rm) we have

$$\Omega_{r,rm} = \Omega_{m,rm} \quad \Leftrightarrow \quad \Omega_{r0}e^{-4x_{rm}} = \Omega_{m0}e^{-3x_{rm}}, \tag{15}$$

and thus

$$x_{rm} = \log\left(\frac{\Omega_{r0}}{\Omega_{m0}}\right). \tag{16}$$

Using that $x = \log a$ and $z = 1/a - 1$ this gives us an expression for the redshift at rm:

$$z_{rm} = \frac{\Omega_{m0}}{\Omega_{r0}} - 1. \tag{17}$$

Similarly, at matter-dark energy equality ($m\Lambda$) we have

$$\Omega_{m0}e^{-3x_{m\Lambda}} = \Omega_{\Lambda 0} \quad \Leftrightarrow \quad x_{m\Lambda} = \frac{1}{3}\log\left(\frac{\Omega_{m0}}{\Omega_{\Lambda 0}}\right), \tag{18}$$

and thus

$$z_{m\Lambda} = \left(\frac{\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3} - 1. \tag{19}$$

To ensure that the numerical results presented in this work agree with analytical expectations, it will be beneficial to have approximate expressions for the cosmic and conformal times in the different regimes. The cosmic time is related to our time variable x through

$$\frac{dt}{dx} = \frac{dt}{da}\frac{da}{dx} = \frac{a}{\dot{a}} = \frac{1}{H}. \tag{20}$$

Thus, for a universe dominated by a single component with equation of state w_i the cosmic time t is given by

$$t = \int_0^t dt' = \int_{-\infty}^{x} \frac{dx'}{H_0\sqrt{\Omega_{i0}e^{-3(1+w_i)x'}}}. \tag{21}$$

When the Universe transitions between an era where its energy density is dominated by some component ρ_j to some other component ρ_i, we may neglect all other components and compute an approximate expression for the cosmic time as function of x by writing

$$t_i(x) \approx t_{j,i} + \int_{x_{j,i}}^{x} \frac{dx'}{H_0\sqrt{\Omega_{i0}e^{-3(1+w_i)x'}}}, \tag{22}$$

where $x_{j,i}$ and $t_{j,i}$ correspond to their values when $\rho_j = \rho_i$. For radiation we simply have $x_{j,i} = -\infty$ and thus $t_{j,i} = 0$, since the very early Universe was filled with relativistic particles, hence

$$t_r(x) = \int_{-\infty}^{x} \frac{dx'}{H_0\sqrt{\Omega_{r0}e^{-4x'}}} = \frac{1}{2H_0\sqrt{\Omega_{r0}e^{-4x}}}. \tag{23}$$

We see that radiation-matter equality occurs at

$$t_{rm} = t_r(x_{rm}) = \frac{\Omega_{r0}^{3/2}}{2H_0\Omega_{m0}^2}, \tag{24}$$

and for matter it then follows

$$\begin{aligned} t_m(x) &\approx t_{rm} + \int_{x_{rm}}^{x} \frac{dx'}{H_0\sqrt{\Omega_{m0}e^{-3x'}}}, \\ &= \frac{\Omega_{r0}^{3/2}}{2H_0\Omega_{m0}^2} + \frac{2}{3H_0}\left[\frac{1}{\sqrt{\Omega_{m0}e^{-3x}}} - \frac{\Omega_{r0}^{3/2}}{H_0\Omega_{m0}^2}\right], \\ &= \frac{1}{3H_0}\left[\frac{2}{\sqrt{\Omega_{m0}e^{-3x}}} - \frac{\Omega_{r0}^{3/2}}{2\Omega_{m0}^2}\right], \end{aligned} \tag{25}$$

with matter-dark energy equality occuring at

$$t_{m\Lambda} = \frac{1}{3H_0}\left[\frac{2}{\sqrt{\Omega_{\Lambda 0}}} - \frac{\Omega_{r0}^{3/2}}{2\Omega_{m0}^2}\right]. \tag{26}$$

Lastly, for dark energy we have

$$\begin{aligned} t_\Lambda(x) &\approx t_{m\Lambda} + \int_{x_{m\Lambda}}^{x} \frac{dx'}{H_0\sqrt{\Omega_{\Lambda 0}}}, \\ &= \frac{1}{3H_0}\left[\frac{2}{\sqrt{\Omega_{\Lambda 0}}} - \frac{\Omega_{r0}^{3/2}}{2\Omega_{m0}^2}\right] + \frac{1}{H_0\sqrt{\Omega_{\Lambda 0}}}\left[x - \frac{1}{3}\log\left(\frac{\Omega_{m0}}{\Omega_{\Lambda 0}}\right)\right], \\ &= \frac{1}{H_0\sqrt{\Omega_{\Lambda 0}}}\left[x + \frac{2}{3} - \frac{1}{3}\log\left(\frac{\Omega_{m0}}{\Omega_{\Lambda 0}}\right) - \frac{\sqrt{\Omega_{\Lambda 0}}\Omega_{r0}^{3/2}}{6\Omega_{m0}^2}\right]. \end{aligned} \tag{27}$$

From these derived expressions, it is obvious that $a \propto t^{1/2}$ in the radiation dominated era, $a \propto t^{2/3}$ in the matter era and $a \propto e^{H_0\sqrt{\Omega_{\Lambda 0}}t}$ in the dark energy era, which is the expected result. word different?

Following an analogous approach for the conformal time, it is straight-forward to show that since

$$\eta_i(x) \approx \eta_{j,i} + \int_{x_{j,i}}^{x} \frac{c\,dx'}{H_0\sqrt{\Omega_{i0}e^{-(1+3w_i)x'}}}, \tag{28}$$

we have the following approximate expressions:

$$\eta_r(x) = \frac{c}{H_0\sqrt{\Omega_{r0}e^{-2x}}}, \tag{29}$$

$$\eta_m(x) = \frac{c}{H_0}\left[\frac{2}{\sqrt{\Omega_{m0}e^{-x}}} - \frac{\sqrt{\Omega_{r0}}}{\Omega_{m0}}\right], \tag{30}$$

$$\eta_\Lambda(x) = -\frac{c}{H_0}\left[\frac{1}{\sqrt{\Omega_{\Lambda 0}e^{2x}}} + \frac{\sqrt{\Omega_{r0}}}{\Omega_{m0}} - \frac{3}{\Omega_{m0}^{1/3}\Omega_{\Lambda 0}^{1/6}}\right], \tag{31}$$

with the conformal equality times:

$$\eta_{rm} = \frac{c}{H_0}\frac{\sqrt{\Omega_{r0}}}{\Omega_{m0}}, \tag{32}$$

$$\eta_{m\Lambda} = \frac{c}{H_0}\left[\frac{2}{\Omega_{m0}^{1/3}\Omega_{\Lambda 0}^{1/6}} - \frac{\sqrt{\Omega_{r0}}}{\Omega_{m0}}\right]. \tag{33}$$

To test for numerical stability and accuracy, it is essential to compute analytical expressions that can be used for comparison. Obviously, in an era dominated by component i we have

$$\mathcal{H}_i \approx H_0\sqrt{\Omega_{i0}e^{-(1+3w_i)}}, \tag{34}$$

and in the radiation dominated era we thus have

$$\left(\frac{d\mathcal{H}}{dx}\right)_r = -H_0\sqrt{\Omega_{r0}}e^{-x} = -\mathcal{H}_r \quad\Leftrightarrow\quad \left(\frac{1}{\mathcal{H}}\frac{d\mathcal{H}}{dx}\right)_r = -1, \tag{35}$$

$$\left(\frac{d^2\mathcal{H}}{dx^2}\right)_r = H_0\sqrt{\Omega_{r0}}e^{-x} = \mathcal{H}_r \quad\Leftrightarrow\quad \left(\frac{1}{\mathcal{H}}\frac{d^2\mathcal{H}}{dx^2}\right)_r = 1. \tag{36}$$

Similarly, we have

$$\left(\frac{d\mathcal{H}}{dx}\right)_m = -\frac{\mathcal{H}_m}{2} \quad\Leftrightarrow\quad \left(\frac{1}{\mathcal{H}}\frac{d\mathcal{H}}{dx}\right)_m = -\frac{1}{2}, \tag{37}$$

$$\left(\frac{d^2\mathcal{H}}{dx^2}\right)_m = \frac{\mathcal{H}_m}{4} \quad\Leftrightarrow\quad \left(\frac{1}{\mathcal{H}}\frac{d^2\mathcal{H}}{dx^2}\right)_m = \frac{1}{4}. \tag{38}$$

in the matter dominated era, and

$$\mathcal{H}_\Lambda = \left(\frac{d\mathcal{H}}{dx}\right)_\Lambda = \left(\frac{d^2\mathcal{H}}{dx^2}\right)_\Lambda = H_0\sqrt{\Omega_{\Lambda 0}}e^{x}, \tag{39}$$

in the dark energy dominated era. From the latter it is obvious that

$$\left(\frac{1}{\mathcal{H}}\frac{d\mathcal{H}}{dx}\right)_\Lambda = \left(\frac{1}{\mathcal{H}}\frac{d^2\mathcal{H}}{dx^2}\right)_\Lambda = 1. \tag{40}$$

2.1.4. Onset of acceleration

Later on, when we will analyze the observed CMB power spectrum, it will be interesting to know when the expansion of the Universe started to accelerate. It is a well known fact that the expansion rate (governed by $\dot{a}$) is increasing as of today, and that we are in the early stage of a dark energy dominated era. We have the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i \rho_i\,(1+3w_i)\,, \tag{41}$$

where the sum runs over all components (matter, radiation, etc.). The onset of acceleration occurs when $\ddot{a}$ switches sign, i.e., when

$$\sum_i \rho_i\,(1+3w_i) = 0. \tag{42}$$

Assuming that this happens well after the radiation dominated era, we can approximate this as

$$\rho_m(x_{\rm acc}) - 2\rho_\Lambda(x_{\rm acc}) = 0 \tag{43}$$

Using that the density parameters at arbitrary x can be written as present this further up

$$\Omega_i(x) = \frac{\Omega_{i0}}{e^{3(1+w_i)}H^2(x)/H_0^2}, \tag{44}$$

we can rewrite eq. (43) to get

$$\Omega_{m0}e^{-3x_{\rm acc}} = 2\Omega_{\Lambda0} \quad\Leftrightarrow\quad x_{\rm acc} = \frac{1}{3}\log\left(\frac{\Omega_{m0}}{2\Omega_{\Lambda0}}\right). \tag{45}$$

This corresponds to a redshift

$$z_{\rm acc} = \left(\frac{2\Omega_{\Lambda0}}{\Omega_{m0}}\right)^{1/3} - 1. \tag{46}$$

Obviously, $t_{\rm acc} < t_{m\Lambda}$, so we can make the same approximation this time, hence

$$t_{\rm acc} = t_m(x_{\rm acc}) = \frac{1}{3H_0}\left[\sqrt{\frac{2}{\Omega_{\Lambda0}}} - \frac{\Omega_{r0}^{3/2}}{2\Omega_{m0}^2}\right], \tag{47}$$

with the conformal time being

$$\eta_{\rm acc} = \eta_m(x_{\rm acc}) = \frac{c}{H_0}\left[\frac{2^{5/6}}{\Omega_{m0}^{1/3}\Omega_{\Lambda0}^{1/6}} - \frac{\sqrt{\Omega_{r0}}}{\Omega_{m0}}\right]. \tag{48}$$

maybe do this derivation more thorough/different

2.2. Implementation

Our numerical methods must accurately compute $H(x)$, $\mathcal{H}(x)$, and $\eta(x)$ to ensure physical accuracy.

TODO: briefly explain code, spline

TODO: explain chi2 thing here?

2.3. Results

2.3.1. Important time stamps

2.3.2. Tests of numerical stability

2.3.3. Time evolution

maybe rename / split into fewer sections

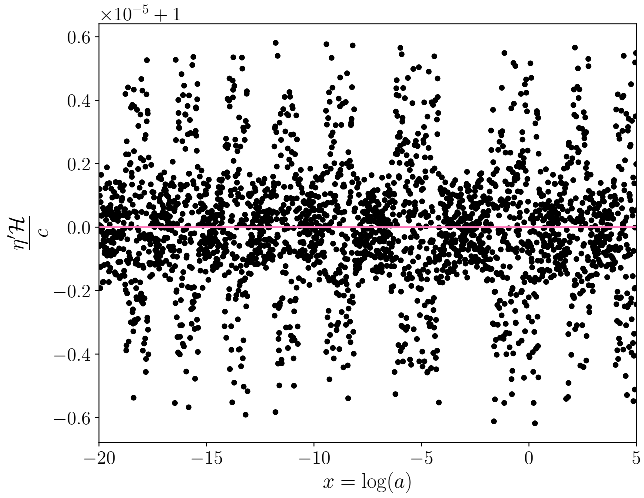

Fig. 1. caption.

2.3.4. Supernova fitting

3. Milestone II

3.1. Theory

3.2. Implementation

3.3. Results

4. Milestone III

4.1. Theory

4.2. Implementation

4.3. Results

5. Milestone IV

5.1. Theory

5.2. Implementation

5.3. Results

6. Conclusions

Sanderson & Curtin (Accessed: October 2023)

References

Sanderson, D. C. & Curtin, D. R. Accessed: October 2023, Armadillo C++ Library", https://arma.sourceforge.net/docs.html

Table 1. CAPTION

	Radiation-matter equality	Onset of acceleration	Matter-dark energy equality	Present day values
x	-8.13	-0.49	-0.26	0
z	3400.33	0.63	0.29	0
t	65.38 kyr	8.33 Gyr	11.78 Gyr	13.86 Gyr
η/c	444.75 Myr	40.22 Gyr	45.20 Gyr	46.32 Gyr

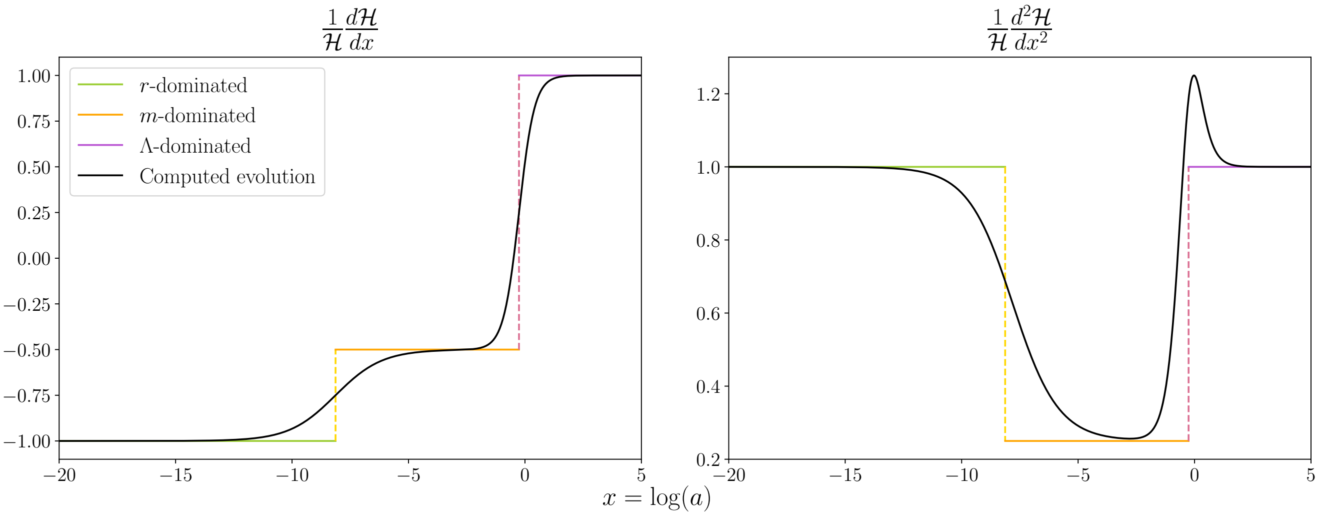

Fig. 2. caption .

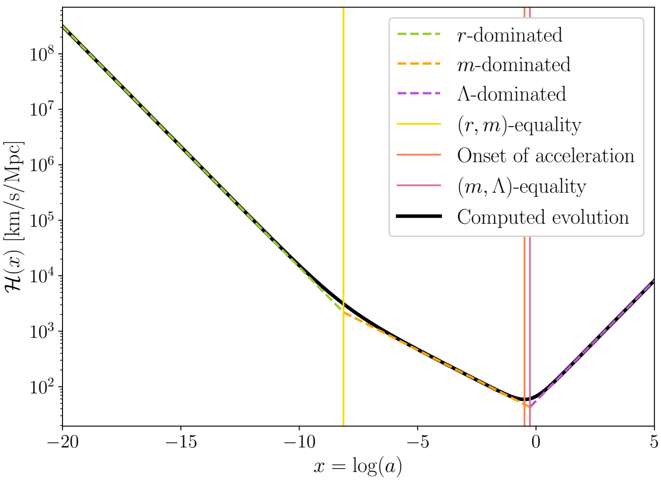

Fig. 3. caption .

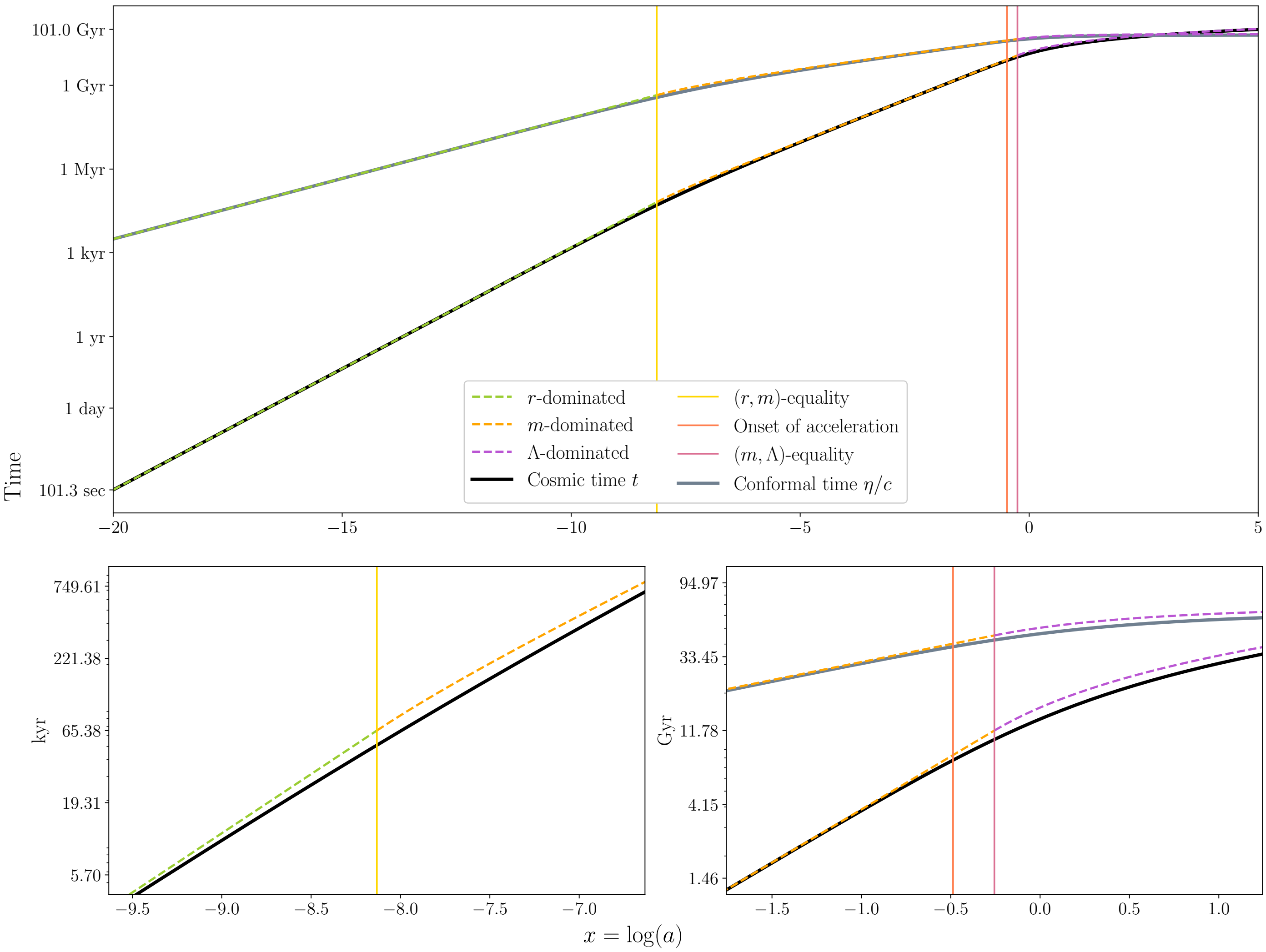

Fig. 4. caption .

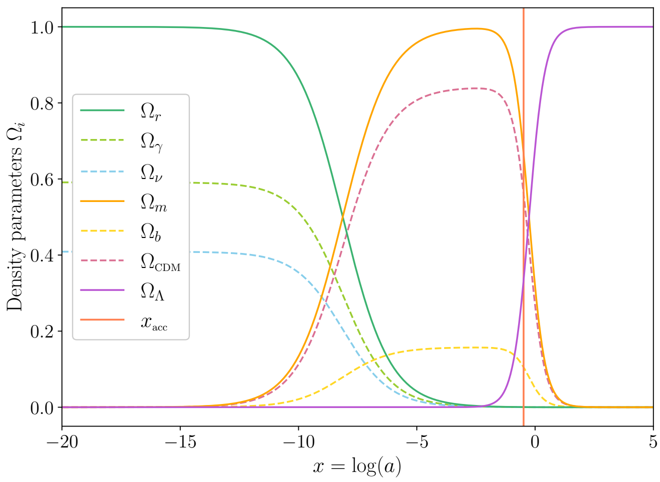

Fig. 5. caption .

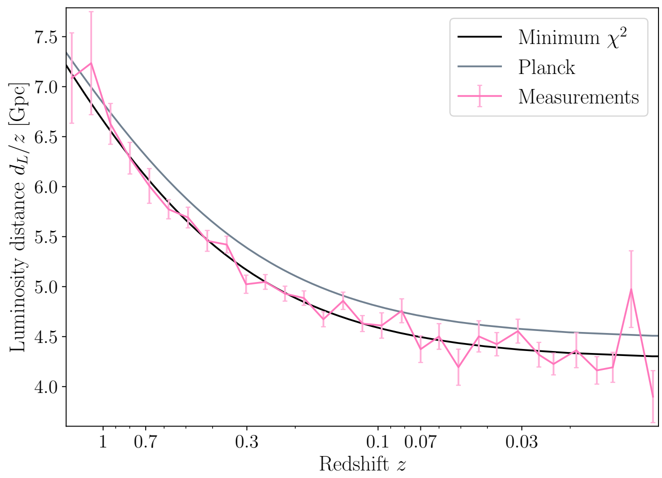

Fig. 6. caption .

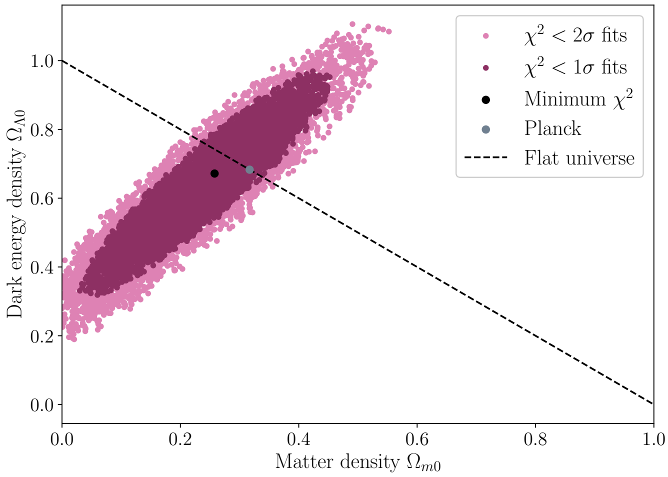

Fig. 7. caption .

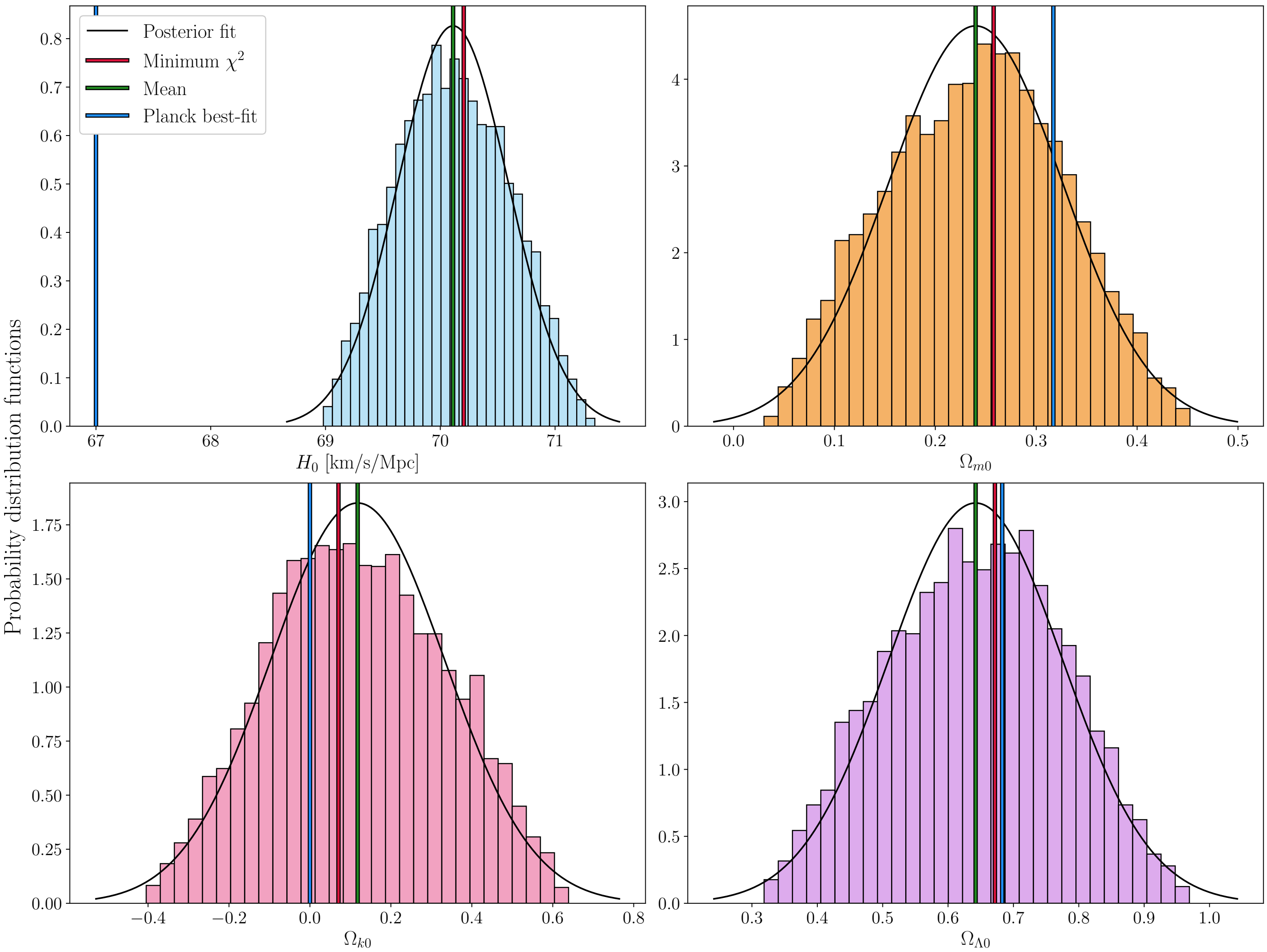

Fig. 8. caption .