TITLE

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ABSTRACT

Aims. AIMS
Methods. METHODS
Results. RESULTS

Key words. KEYWORDS

1. Introduction

2. Milestone I

Introduction

- Make a test function in the BackgroundCosmology class to check that the density parameters today sum up to unity.
- Make a plot showing that $\eta' \mathcal{H}/c = 1$ for all x to test η.
- Compute analytical expressions for the matter-radiation equality, matter-dark energy equality (see exercises), and the time when the Universe starts accelerating. These should be functions of x, z and t.
- Compute numerical values for these times, and present them in a table.
- Compute numerical values for the age of the Universe today $t(x = 0)$ in Gigayears, and the conformal time today $\eta(x = 0)/c$.
- Compute analytical expressions for $\mathcal{H}'/\mathcal{H}$ and $\mathcal{H}''/\mathcal{H}$ in the radiation-, matter-, and dark energy dominated eras (see exercises).
- Alternatively, solve the system by setting Ω_{r0} to 1 and the rest to zero up to the matter-radiation equality, then do the same for $\Omega_m 0$ up to the matter-dark energy equality, and then the same for $\Omega_{\Lambda 0}$.
- Either way it is done, plot analytical expectations vs. numerical results for $\mathcal{H}'/\mathcal{H}$ and $\mathcal{H}''/\mathcal{H}$.
- Make a plot showing $a(t)$ vs. t along with analytical expectations in the radiation-, matter- and dark energy dominated eras (compute these expressions the same way as for the last points).
- Do the neutrino stuff if time and fun.

2.1. Theory

The Hubble parameter is

$$H = H_0 \sqrt{\left(\Omega_{b0} + \Omega_{\mathrm{CDM}0}\right) a^{-3} + \left(\Omega_{\gamma 0} + \Omega_{\nu 0}\right) a^{-4} + \Omega_{k0} a^{-2} + \Omega_{\Lambda 0}}. \tag{1}$$

TEXT The scaled Hubble parameter is given by

$$\mathcal{H} = aH, \tag{2}$$

Thus, if we define $\Omega_{m0} = \Omega_{b0} + \Omega_{\mathrm{CDM}0}$ and $\Omega_{r0} = \Omega_{\gamma 0} + \Omega_{\nu 0}$, and use our chosen time variable $x = \log a$ instead of a, the expressions (1) and (2) can equivalently be written as

$$H = H_0 \sqrt{\Omega_{m0} e^{-3x} + \Omega_{r0} e^{-4x} + \Omega_{k0} e^{-2x} + \Omega_{\Lambda 0}}, \tag{3}$$

$$\mathcal{H} = H_0 \sqrt{\Omega_{m0} e^{-x} + \Omega_{r0} e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0} e^{2x}}. \tag{4}$$

The first and second derivatives of $\mathcal{H}$ with respect to x are then correct expressions?

$$\frac{d\mathcal{H}}{dx} = \frac{H_0}{2} \frac{-\Omega_{m0} e^{-x} - 2\Omega_{r0} e^{-2x} + 2\Omega_{\Lambda 0} e^{2x}}{\sqrt{\Omega_{m0} e^{-x} + \Omega_{r0} e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0} e^{2x}}},$$
$$= -\frac{H_0^2}{2\mathcal{H}} \left(\Omega_{m0} e^{-x} + 2\Omega_{r0} e^{-2x} - 2\Omega_{\Lambda 0} e^{2x}\right), \tag{5}$$

$$\frac{d^2\mathcal{H}}{dx^2} = \frac{H_0}{2} \left(\frac{\Omega_{m0} e^{-x} + 4\Omega_{r0} e^{-2x} + 4\Omega_{\Lambda 0} e^{2x}}{\sqrt{\Omega_{m0} e^{-x} + \Omega_{r0} e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0} e^{2x}}} - \frac{1}{2} \frac{\left(\Omega_{m0} e^{-x} + 2\Omega_{r0} e^{-2x} - 2\Omega_{\Lambda 0} e^{2x}\right)^2}{\left(\Omega_{m0} e^{-x} + \Omega_{r0} e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0} e^{2x}\right)^{3/2}} \right),$$
$$= \frac{H_0^2}{\mathcal{H}} \left[\frac{1}{2} \Omega_{m0} e^{-x} + 2\Omega_{r0} e^{-2x} + 2\Omega_{\Lambda 0} e^{2x} - \frac{1}{H_0^2} \left(\frac{d\mathcal{H}}{dx}\right)^2 \right]. \tag{6}$$

Derive expression using w instead?

Move to the top/bottom?

Connect to derivations above?

To test for numerical stability and accuracy, it is essential to compute analytical expressions that can be used for comparison. In the radiation dominated era we have

$$\mathcal{H}_r \approx H_0 \sqrt{\Omega_{r0}} e^{-x}, \tag{7}$$

and thus

$$\left(\frac{d\mathcal{H}}{dx}\right)_r = -H_0 \sqrt{\Omega_{r0}} e^{-x} = -\mathcal{H}_r \quad \Leftrightarrow \quad \left(\frac{1}{\mathcal{H}} \frac{d\mathcal{H}}{dx}\right)_r = -1, \tag{8}$$

$$\left(\frac{d^2\mathcal{H}}{dx^2}\right)_r = H_0 \sqrt{\Omega_{r0}} e^{-x} = \mathcal{H}_r \quad \Leftrightarrow \quad \left(\frac{1}{\mathcal{H}} \frac{d^2\mathcal{H}}{dx^2}\right)_r = 1. \tag{9}$$

Similarly, we have

$$\mathcal{H}_m = H_0\sqrt{\Omega_{m0}e^{-x}}, \tag{10}$$

$$\left(\frac{d\mathcal{H}}{dx}\right)_m = -\frac{\mathcal{H}_m}{2} \quad \Leftrightarrow \quad \left(\frac{1}{\mathcal{H}}\frac{d\mathcal{H}}{dx}\right)_m = -\frac{1}{2}, \tag{11}$$

$$\left(\frac{d^2\mathcal{H}}{dx^2}\right)_m = \frac{\mathcal{H}_m}{4} \quad \Leftrightarrow \quad \left(\frac{1}{\mathcal{H}}\frac{d^2\mathcal{H}}{dx^2}\right)_m = \frac{1}{4}. \tag{12}$$

in the matter dominated era, and

$$\mathcal{H}_\Lambda = \left(\frac{d\mathcal{H}}{dx}\right)_\Lambda = \left(\frac{d^2\mathcal{H}}{dx^2}\right)_\Lambda = H_0\sqrt{\Omega_{\Lambda 0}}e^{x} \tag{13}$$

in the dark energy dominated era. From the latter it is obvious that

$$\left(\frac{1}{\mathcal{H}}\frac{d\mathcal{H}}{dx}\right)_\Lambda = \left(\frac{1}{\mathcal{H}}\frac{d^2\mathcal{H}}{dx^2}\right)_\Lambda = 1. \tag{14}$$

only calculate x and z here? do it different? At radiation-matter equality (rm) we have

$$\Omega_{r,rm} = \Omega_{m,rm} \quad \Leftrightarrow \quad \Omega_{r0}e^{-4x_{rm}} = \Omega_{m0}e^{-3x_{rm}}, \tag{15}$$

and thus

$$x_{rm} = \log\left(\frac{\Omega_{r0}}{\Omega_{m0}}\right). \tag{16}$$

Using that $x = \log a$ and $z = 1/a - 1$ this gives us an expression for the redshift at rm:

$$z_{rm} = \frac{\Omega_{m0}}{\Omega_{r0}} - 1. \tag{17}$$

Furthermore, to compute the age of the universe at rm we may use that

$$t_{rm} = \int_0^{t_{rm}} dt = \int_{-\infty}^{x_{rm}} \frac{dt}{dx}dx = \int_{-\infty}^{x_{rm}} \frac{dx}{H}, \tag{18}$$

and that the Universe is radiation dominated before this point. This lets us ignore the matter and dark energy terms in the expression for H, although the former component plays an increasingly important role as we reach t_{rm}. valid? include matter term? Thus, we have

$$t_{rm} = \int_{-\infty}^{x_{rm}} \frac{dx}{H_0\sqrt{\Omega_{r0}e^{-4x}}} = \frac{e^{2x_{rm}}}{2H_0\sqrt{\Omega_{r0}}} = \frac{\Omega_{r0}^{3/2}}{2H_0\Omega_{m0}^2}. \tag{19}$$

Following a similar logic, at matter-dark energy equality ($m\Lambda$) we have

$$\Omega_{m0}e^{-3x_{m\Lambda}} = \Omega_{\Lambda 0} \quad \Leftrightarrow \quad x_{m\Lambda} = \frac{1}{3}\log\left(\frac{\Omega_{m0}}{\Omega_{\Lambda 0}}\right), \tag{20}$$

and thus

$$z_{m\Lambda} = \left(\frac{\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3} - 1. \tag{21}$$

Since the Universe is matter dominated between rm and $m\Lambda$ we may ignore the radiation and dark energy terms to obtain

$$\begin{aligned} t_{m\Lambda} &= t_{rm} + \int_{x_{rm}}^{x_{m\Lambda}} \frac{dx}{H_0\sqrt{\Omega_{m0}e^{-3x}}}, \\ &= \frac{\Omega_{r0}^{3/2}}{2H_0\Omega_{m0}^2} + \frac{2\left(e^{3x_{m\Lambda}/2} - e^{3x_{rm}/2}\right)}{3H_0\sqrt{\Omega_{m0}}}, \\ &= \frac{\Omega_{r0}^{3/2}}{2H_0\Omega_{m0}^2} + \frac{2}{3H_0\sqrt{\Omega_{\Lambda 0}}} - \frac{2\Omega_{r0}^{3/2}}{3H_0\Omega_{m0}^2}, \\ &= \frac{1}{3H_0}\left(\frac{2}{\sqrt{\Omega_{\Lambda 0}}} - \frac{\Omega_{r0}^{3/2}}{2\Omega_{m0}^2}\right). \end{aligned} \tag{22}$$

correct?

We know that the Universe starts to accelerate after $\ddot{a}$ goes from being negative to positive. Assuming that this happens well after the radiation dominated era, the second Friedmann equation reads as

$$0 = -\frac{4\pi G}{3}\left[\rho_m(x_{\text{acc}}) - 2\rho_\Lambda(x_{\text{acc}})\right]. \tag{23}$$

Using that the density parameters at arbitrary x can be written as

$$\Omega_i(x) = \frac{\Omega_{i0}}{e^{nx}H^2(x)/H_0^2}, \tag{24}$$

where $n = 3$ for matter and $n = 0$ for dark energy, we can rewrite eq. (23) to get

$$\Omega_{m0}e^{-3x_{\text{acc}}} = 2\Omega_{\Lambda 0} \quad \Leftrightarrow \quad x_{\text{acc}} = \frac{1}{3}\log\left(\frac{\Omega_{m0}}{2\Omega_{\Lambda 0}}\right). \tag{25}$$

This corresponds to a redshift

$$z_{\text{acc}} = \left(\frac{2\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3} - 1. \tag{26}$$

Obviously, $t_{\text{acc}} < t_{m\Lambda}$, so we can make the same approximation this time, hence

$$t_{\text{acc}} = t_{rm} + \int_{x_{rm}}^{x_{\text{acc}}} \frac{dx}{H_0\sqrt{\Omega_{m0}e^{-3x}}} = \frac{1}{3H_0}\left(\sqrt{\frac{2}{\Omega_{\Lambda 0}}} - \frac{\Omega_{r0}^{3/2}}{2\Omega_{m0}^2}\right). \tag{27}$$

maybe do this derivation more thorough/different

The cosmic time is related to our time variable x through

$$\frac{dt}{dx} = \frac{dt}{da}\frac{da}{dx} = \frac{a}{\dot{a}} = \frac{1}{H}. \tag{28}$$

Thus, for a universe dominated by a single component with equation of state w_i the cosmic time t is given by

$$t = \int_0^t dt' = \int_{-\infty}^{x} \frac{dx'}{H_0\sqrt{\Omega_{i0}e^{-3(1+w_i)x'}}}. \tag{29}$$

When the Universe transitions between an era where its energy density is dominated by some component ρ_j to some other component ρ_i, we may neglect all other components and compute an approximate expression for the cosmic time as function of x by writing

$$t_i(x) \approx t_{j,i} + \int_{x_{j,i}}^{x} \frac{dx'}{H_0\sqrt{\Omega_{i0}e^{-3(1+w_i)x'}}}, \tag{30}$$

where $x_{j,i}$ and $t_{j,i}$ correspond to their values when $\rho_j = \rho_i$. For radiation we simply have $x_{j,i} = -\infty$ and thus $t_{j,i} = 0$, since the very early Universe was filled with relativistic particles, hence

$$t_r(x) = \int_{-\infty}^{x} \frac{dx'}{H_0\sqrt{\Omega_{r0}e^{-4x'}}} = \frac{1}{2H_0\sqrt{\Omega_{r0}e^{-4x}}}. \tag{31}$$

We see that radiation-matter equality occurs at

$$t_{rm} = t_r(x_{rm}) = \frac{\Omega_{r0}^{3/2}}{2H_0\Omega_{m0}^2}, \tag{32}$$

and for matter it then follows

$$\begin{aligned} t_m(x) &\approx t_{rm} + \int_{x_{rm}}^{x} \frac{dx'}{H_0\sqrt{\Omega_{m0}e^{-3x'}}}, \\ &= \frac{\Omega_{r0}^{3/2}}{2H_0\Omega_{m0}^2} + \frac{2}{3H_0}\left[\frac{1}{\sqrt{\Omega_{m0}e^{-3x}}} - \frac{\Omega_{r0}^{3/2}}{H_0\Omega_{m0}^2}\right], \\ &= \frac{1}{3H_0}\left[\frac{2}{\sqrt{\Omega_{m0}e^{-3x}}} - \frac{\Omega_{r0}^{3/2}}{2\Omega_{m0}^2}\right], \end{aligned} \tag{33}$$

with matter-dark energy equality occuring at

$$t_{m\Lambda} = \frac{1}{3H_0}\left[\frac{2}{\sqrt{\Omega_{\Lambda0}}} - \frac{\Omega_{r0}^{3/2}}{2\Omega_{m0}^2}\right] \tag{34}$$

Lastly, for dark energy we have

$$\begin{aligned} t_\Lambda(x) &\approx t_{m\Lambda} + \int_{x_{m\Lambda}}^{x} \frac{dx'}{H_0\sqrt{\Omega_{\Lambda0}}}, \\ &= \frac{1}{3H_0}\left[\frac{2}{\sqrt{\Omega_{\Lambda0}}} - \frac{\Omega_{r0}^{3/2}}{2\Omega_{m0}^2}\right] + \frac{1}{H_0\sqrt{\Omega_{\Lambda0}}}\left[x - \frac{1}{3}\log\left(\frac{\Omega_{m0}}{\Omega_{\Lambda0}}\right)\right], \\ &= \frac{1}{H_0\sqrt{\Omega_{\Lambda0}}}\left[x + \frac{2}{3} - \frac{1}{3}\log\left(\frac{\Omega_{m0}}{\Omega_{\Lambda0}}\right) - \frac{\sqrt{\Omega_{\Lambda0}}\Omega_{r0}^{3/2}}{6\Omega_{m0}^2}\right]. \end{aligned} \tag{35}$$

comment that the dependencies between a and t are as expected

These approximate expressions are interesting to have when analyzing the numerical results, as they point us toward the correct solution. We can derive analogous expressions for the conformal time η, and it is straight-forward to show that since

$$\eta_i(x) \approx \eta_{j,i} + \int_{x_{j,i}}^{x} \frac{c\,dx'}{H_0\sqrt{\Omega_{i0}e^{-(1+3w_i)x'}}}, \tag{36}$$

we have rephrase paragraph .

$$\eta_r(x) = \frac{c}{H_0\sqrt{\Omega_{r0}e^{-2x}}}, \tag{37}$$

$$\eta_m(x) = \frac{c}{H_0}\left[\frac{2}{\sqrt{\Omega_{m0}e^{-x}}} - \frac{\sqrt{\Omega_{r0}}}{\Omega_{m0}}\right], \tag{38}$$

$$\eta_\Lambda(x) = -\frac{c}{H_0}\left[\frac{1}{\sqrt{\Omega_{\Lambda0}e^{2x}}} + \frac{\sqrt{\Omega_{r0}}}{\Omega_{m0}} - \frac{3}{\Omega_{m0}^{1/3}\Omega_{\Lambda0}^{1/6}}\right], \tag{39}$$

with the conformal equality times:

$$\eta_{rm} = \frac{c}{H_0}\frac{\sqrt{\Omega_{r0}}}{\Omega_{m0}}, \tag{40}$$

$$\eta_{m\Lambda} = \frac{c}{H_0}\left[\frac{2}{\Omega_{m0}^{1/3}\Omega_{\Lambda0}^{1/6}} - \frac{\sqrt{\Omega_{r0}}}{\Omega_{m0}}\right]. \tag{41}$$

2.2. Implementation

2.3. Results

3. Conclusions

Sanderson & Curtin (Accessed: October 2023)

References

Sanderson, D. C. & Curtin, D. R. Accessed: October 2023, Armadillo C++ Library", https://arma.sourceforge.net/docs.html