TITLE

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ABSTRACT

Aims. AIMS
Methods. METHODS
Results. RESULTS

Key words. KEYWORDS

1. Introduction

2. Milestone I

Introduction

- Make a test function in the BackgroundCosmology class to check that the density parameters today sum up to unity.
- Make a plot showing that $\eta'\mathcal{H}/c = 1$ for all x to test η.
- Compute analytical expressions for the matter-radiation equality, matter-dark energy equality (see exercises), and the time when the Universe starts accelerating. These should be functions of x, z and t.
- Compute numerical values for these times, and present them in a table.
- Compute numerical values for the age of the Universe today $t(x = 0)$ in Gigayears, and the conformal time today $\eta(x = 0)/c$.
- Compute analytical expressions for $\mathcal{H}'/\mathcal{H}$ and $\mathcal{H}''/\mathcal{H}$ in the radiation-, matter-, and dark energy dominated eras (see exercises).
- Alternatively, solve the system by setting Ω_{r0} to 1 and the rest to zero up to the matter-radiation equality, then do the same for $\Omega_m 0$ up to the matter-dark energy equality, and then the same for $\Omega_{\Lambda 0}$.
- Either way it is done, plot analytical expectations vs. numerical results for $\mathcal{H}'/\mathcal{H}$ and $\mathcal{H}''/\mathcal{H}$.
- Make a plot showing $a(t)$ vs. t along with analytical expectations in the radiation-, matter- and dark energy dominated eras (compute these expressions the same way as for the last points).
- Do the neutrino stuff if time and fun.

2.1. Theory

The Hubble parameter is

$$H = H_0\sqrt{\left(\Omega_{b0} + \Omega_{\mathrm{CDM}0}\right)a^{-3} + \left(\Omega_{\gamma 0} + \Omega_{\nu 0}\right)a^{-4} + \Omega_{k0}a^{-2} + \Omega_{\Lambda 0}}. \tag{1}$$

TEXT The scaled Hubble parameter is given by

$$\mathcal{H} = aH, \tag{2}$$

Thus, if we define $\Omega_{m0} = \Omega_{b0} + \Omega_{\mathrm{CDM}0}$ and $\Omega_{r0} = \Omega_{\gamma 0} + \Omega_{\nu 0}$, and use our chosen time variable $x = \log a$ instead of a, the expressions (1) and (2) can equivalently be written as

$$H = H_0\sqrt{\Omega_{m0}e^{-3x} + \Omega_{r0}e^{-4x} + \Omega_{k0}e^{-2x} + \Omega_{\Lambda 0}}, \tag{3}$$

$$\mathcal{H} = H_0\sqrt{\Omega_{m0}e^{-x} + \Omega_{r0}e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0}e^{2x}}. \tag{4}$$

The first and second derivatives of $\mathcal{H}$ with respect to x are then correct expressions?

$$\begin{aligned}\frac{d\mathcal{H}}{dx} &= \frac{H_0}{2}\frac{-\Omega_{m0}e^{-x} - 2\Omega_{r0}e^{-2x} + 2\Omega_{\Lambda 0}e^{2x}}{\sqrt{\Omega_{m0}e^{-x} + \Omega_{r0}e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0}e^{2x}}},\\ &= -\frac{H_0^2}{2\mathcal{H}}\left(\Omega_{m0}e^{-x} + 2\Omega_{r0}e^{-2x} - 2\Omega_{\Lambda 0}e^{2x}\right),\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{d^2\mathcal{H}}{dx^2} &= \frac{H_0}{2}\left(\frac{\Omega_{m0}e^{-x} + 4\Omega_{r0}e^{-2x} + 4\Omega_{\Lambda 0}e^{2x}}{\sqrt{\Omega_{m0}e^{-x} + \Omega_{r0}e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0}e^{2x}}}\right.\\ &\qquad \left. + \frac{1}{2}\frac{\Omega_{m0}e^{-x} + 2\Omega_{r0}e^{-2x} - 2\Omega_{\Lambda 0}e^{2x}}{\left(\Omega_{m0}e^{-x} + \Omega_{r0}e^{-2x} + \Omega_{k0} + \Omega_{\Lambda 0}e^{2x}\right)^{3/2}}\right),\\ &= \frac{H_0^2}{2\mathcal{H}}\left(\Omega_{m0}e^{-x} + 4\Omega_{r0}e^{-2x} + 4\Omega_{\Lambda 0}e^{2x} - \frac{1}{2\mathcal{H}}\frac{d\mathcal{H}}{dx}\right).\end{aligned} \tag{6}$$

2.1.1. Important time stamps

At radiation-matter equality (rm) we have

$$\Omega_{r,rm} = \Omega_{m,rm} \quad \Leftrightarrow \quad \Omega_{r0}e^{-4x_{rm}} = \Omega_{m0}e^{-3x_{rm}}, \tag{7}$$

and thus

$$x_{rm} = \log\left(\frac{\Omega_{r0}}{\Omega_{m0}}\right). \tag{8}$$

Using that $x = \log a$ and $z = 1/a - 1$ this gives us an expression for the redshift at rm:

$$z_{rm} = \frac{\Omega_{m0}}{\Omega_{r0}} - 1. \tag{9}$$

Furthermore, to compute the age of the universe at rm we may use that

$$t_{rm} = \int_0^{t_{rm}} dt = \int_{-\infty}^{x_{rm}} \frac{dt}{dx}dx = \int_{-\infty}^{x_{rm}} \frac{dx}{H}, \tag{10}$$

and that the Universe is radiation dominated before this point. This lets us ignore the matter and dark energy terms in the expression for H, although the former component plays an increasingly important role as we reach t_{rm}. valid? include matter term? Thus, we have

$$t_{rm} = \int_{-\infty}^{x_{rm}} \frac{dx}{H_0\sqrt{\Omega_{r0}e^{-4x}}} = \frac{e^{2x_{rm}}}{2H_0\sqrt{\Omega_{r0}}} = \frac{\Omega_{r0}^{3/2}}{2H_0\Omega_{m0}^2}. \tag{11}$$

Following a similar logic, at matter-dark energy equality ($m\Lambda$) we have

$$\Omega_{m0}e^{-3x_{m\Lambda}} = \Omega_{\Lambda 0} \quad \Leftrightarrow \quad x_{m\Lambda} = \frac{1}{3}\log\left(\frac{\Omega_{m0}}{\Omega_{\Lambda 0}}\right), \tag{12}$$

and thus

$$z_{m\Lambda} = \left(\frac{\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3} - 1. \tag{13}$$

Since the Universe is matter dominated between rm and $m\Lambda$ we may ignore the radiation and dark energy terms to obtain

$$\begin{aligned} t_{m\Lambda} &= t_{rm} + \int_{x_{rm}}^{x_{m\Lambda}} \frac{dx}{H_0\sqrt{\Omega_{m0}e^{-3x}}}, \\ &= \frac{\Omega_{r0}^{3/2}}{2H_0\Omega_{m0}^2} + \frac{2\left(e^{3x_{m\Lambda}/2} - e^{3x_{rm}/2}\right)}{3H_0\sqrt{\Omega_{m0}}}, \\ &= \frac{\Omega_{r0}^{3/2}}{2H_0\Omega_{m0}^2} + \frac{2}{3H_0\sqrt{\Omega_{\Lambda 0}}} - \frac{2\Omega_{r0}^{3/2}}{3H_0\Omega_{m0}^2}, \\ &= \frac{1}{3H_0}\left(\frac{2}{\sqrt{\Omega_{\Lambda 0}}} - \frac{\Omega_{r0}^{3/2}}{2\Omega_{m0}^2}\right). \end{aligned} \tag{14}$$

correct?

We know that the Universe starts to accelerate after $\ddot{a}$ goes from being negative to positive. Assuming that this happens well after the radiation dominated era, the second Friedmann equation reads as

$$0 = -\frac{4\pi G}{3}\left[\rho_m(x_{\text{acc}}) - 2\rho_\Lambda(x_{\text{acc}})\right]. \tag{15}$$

Using that the density parameters at arbitrary x can be written as

$$\Omega_i(x) = \frac{\Omega_{i0}}{e^{nx}H^2(x)/H_0^2}, \tag{16}$$

where $n = 3$ for matter and $n = 0$ for dark energy, we can rewrite eq. (15) to get

$$\Omega_{m0}e^{-3x_{\text{acc}}} = 2\Omega_{\Lambda 0} \quad \Leftrightarrow \quad x_{\text{acc}} = \frac{1}{3}\log\left(\frac{\Omega_{m0}}{2\Omega_{\Lambda 0}}\right). \tag{17}$$

This corresponds to a redshift

$$z_{\text{acc}} = \left(\frac{2\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3} - 1. \tag{18}$$

Obviously, $t_{\text{acc}} < t_{m\Lambda}$, so we can make the same approximation this time, hence

$$t_{\text{acc}} = t_{rm} + \int_{x_{rm}}^{x_{\text{acc}}} \frac{dx}{H_0\sqrt{\Omega_{m0}e^{-3x}}} = \frac{1}{3H_0}\left(\sqrt{\frac{2}{\Omega_{\Lambda 0}}} - \frac{\Omega_{r0}^{3/2}}{2\Omega_{m0}^2}\right). \tag{19}$$

maybe do this derivation more thorough

2.1.2. Approximate expressions

To test for numerical stability and accuracy, it is essential to compute analytical expressions that can be used for comparison. In the radiation dominated era we have

$$\mathcal{H}_r \approx H_0\sqrt{\Omega_{r0}}e^{-x}, \tag{20}$$

and thus

$$\left(\frac{d\mathcal{H}}{dx}\right)_r = -H_0\sqrt{\Omega_{r0}}e^{-x} = -\mathcal{H}_r \quad \Leftrightarrow \quad \left(\frac{1}{\mathcal{H}}\frac{d\mathcal{H}}{dx}\right)_r = -1, \tag{21}$$

$$\left(\frac{d^2\mathcal{H}}{dx^2}\right)_r = H_0\sqrt{\Omega_{r0}}e^{-x} = \mathcal{H}_r \quad \Leftrightarrow \quad \left(\frac{1}{\mathcal{H}}\frac{d^2\mathcal{H}}{dx^2}\right)_r = 1. \tag{22}$$

Similarly, we have

$$\mathcal{H}_m = H_0\sqrt{\Omega_{m0}e^{-x}}, \tag{23}$$

$$\left(\frac{d\mathcal{H}}{dx}\right)_m = -\frac{\mathcal{H}_m}{2} \quad \Leftrightarrow \quad \left(\frac{1}{\mathcal{H}}\frac{d\mathcal{H}}{dx}\right)_m = -\frac{1}{2}, \tag{24}$$

$$\left(\frac{d^2\mathcal{H}}{dx^2}\right)_m = \frac{\mathcal{H}_m}{4} \quad \Leftrightarrow \quad \left(\frac{1}{\mathcal{H}}\frac{d^2\mathcal{H}}{dx^2}\right)_m = \frac{1}{4}. \tag{25}$$

in the matter dominated era, and

$$\mathcal{H}_\Lambda = \left(\frac{d\mathcal{H}}{dx}\right)_\Lambda = \left(\frac{d^2\mathcal{H}}{dx^2}\right)_\Lambda = H_0\sqrt{\Omega_{\Lambda 0}}e^{x} \tag{26}$$

in the dark energy dominated era. From the latter it is obvious that

$$\left(\frac{1}{\mathcal{H}}\frac{d\mathcal{H}}{dx}\right)_\Lambda = \left(\frac{1}{\mathcal{H}}\frac{d^2\mathcal{H}}{dx^2}\right)_\Lambda = 1. \tag{27}$$

Furthermore, we can compute the expected evolution of the scale factor $a(t)$ or just x? and compare this to the numerical solution. In the different regimes we have

$$\left(\frac{da}{dt}\right)_r \approx H_0\frac{\sqrt{\Omega_{r0}}}{a} \quad \Rightarrow \quad a \propto t^{1/2}, \tag{28}$$

$$\left(\frac{da}{dt}\right)_m \approx H_0\frac{\sqrt{\Omega_{m0}}}{a^{1/2}} \quad \Rightarrow \quad a \propto t^{2/3}, \tag{29}$$

$$\left(\frac{da}{dt}\right)_\Lambda \approx H_0\sqrt{\Omega_{\Lambda 0}}\,a \quad \Rightarrow \quad a \propto e^{H_0\sqrt{\Omega_{\Lambda 0}}t}, \tag{30}$$

where I have used that $\mathcal{H} = aH = \dot{a}$. do more thorough?

2.2. Implementation

2.3. Results

3. Conclusions

Sanderson & Curtin (Accessed: October 2023)

References

Sanderson, D. C. & Curtin, D. R. Accessed: October 2023, Armadillo C++ Library", https://arma.sourceforge.net/docs.html