1 Introduction

The corresponding spin-up and spin-down normalized eigenfunctions are

$$\psi_{n=1,j=\frac{1}{2},\uparrow} = \frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}}\sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}}(2mZ\alpha r)^{\gamma-1}e^{-mZ\alpha r}\begin{bmatrix} 1 \\ 0 \\ \frac{i(1-\gamma)}{Z\alpha}\cos\theta \\ \frac{i(1-\gamma)}{Z\alpha}\sin\theta e^{i\phi} \end{bmatrix}$$

$$\psi_{n=1,j=\frac{1}{2},\downarrow} = \frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}}\sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}}(2mZ\alpha r)^{\gamma-1}e^{-mZ\alpha r}\begin{bmatrix} 1 \\ 0 \\ \frac{i(1-\gamma)}{Z\alpha}\sin\theta e^{-i\phi} \\ \frac{-i(1-\gamma)}{Z\alpha}\cos\theta \end{bmatrix}$$

checking the Normalization

$$N = \int_\tau \psi^\dagger_{n=1,j=\frac{1}{2},\uparrow}\psi_{n=1,j=\frac{1}{2},\uparrow}d\tau$$

$$= \int_{r=0}^{\infty}\int_{\theta=-\pi}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}}\sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}}(2mZ\alpha r)^{\gamma-1}e^{-mZ\alpha r}\begin{bmatrix} 1 & 0 & \frac{-i(1-\gamma)}{Z\alpha}\cos\theta & \frac{-i(1-\gamma)}{Z\alpha}\sin\theta e^{-i\phi} \end{bmatrix}$$
$$\frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}}\sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}}(2mZ\alpha r)^{\gamma-1}\begin{bmatrix} 1 \\ 0 \\ \frac{i(1-\gamma)}{Z\alpha}\cos\theta \\ \frac{i(1-\gamma)}{Z\alpha}\sin\theta e^{i\phi} \end{bmatrix} r^2 sin\theta dr d\theta d\phi$$

$$= \int_{r=0}^{\infty}\int_{\theta=-\pi}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\frac{(2mZ\alpha)^{3}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}(2mZ\alpha r)^{2\gamma-2}e^{-2mZ\alpha r}\left(1^2+0^2+\frac{(1-\gamma)^2}{Z^2\alpha^2}\cos^2\theta+\frac{(1-\gamma)^2}{Z^2\alpha^2}\sin^2\theta\right)r^2\sin\theta dr d\theta d\phi$$

$$= \int_{r=0}^{\infty}\int_{\theta=-\pi}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\frac{(2mZ\alpha)^{3+2\gamma-2}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}r^{2\gamma-2}e^{-2mZ\alpha r}\left(1+\tfrac{(1-\gamma)^2}{Z^2\alpha^2}\right)r^2\sin\theta dr d\theta d\phi$$

$$= \frac{(2mZ\alpha)^{3+2\gamma-2}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\left[\int_{r=0}^{\infty}\int_{\theta=-\pi}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}r^{2\gamma}e^{-2mZ\alpha r}\sin\theta dr d\theta d\phi+\int_{r=0}^{\infty}\int_{\theta=-\pi}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\frac{(1-\gamma)^2}{Z^2\alpha^2}r^{2\gamma}e^{-2mZ\alpha r}\sin\theta dr d\theta d\phi\right]$$

$$= \frac{(2mZ\alpha)^{2\gamma+1}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\left[\int_{r=0}^{\infty}r^{2\gamma}e^{-2mZ\alpha r}dr\int_{\theta=-\pi}^{\theta=\pi}\sin\theta d\theta\int_{\phi=0}^{\phi=2\pi}d\phi+\frac{(1-\gamma)^2}{Z^2\alpha^2}\int_{r=0}^{\infty}r^{2\gamma}e^{-2mZ\alpha r}dr\int_{\theta=-\pi}^{\theta=\pi}\sin\theta d\theta\int_{\phi=0}^{\phi=2\pi}d\phi\right]$$

$$= \frac{(2mZ\alpha)^{2\gamma+1}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\left[4\pi\int_{r=0}^{\infty}r^{2\gamma}e^{-2mZ\alpha r}dr+4\pi\frac{(1-\gamma)^2}{Z^2\alpha^2}\int_{r=0}^{\infty}r^{2\gamma}e^{-2mZ\alpha r}dr\right]$$

$$= (2mZ\alpha)^{2\gamma+1}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\left[\int_{r=0}^{\infty} r^{2\gamma}e^{-2mZ\alpha r}dr + \frac{(1-\gamma)^2}{Z^2\alpha^2}\int_{r=0}^{\infty} r^{2\gamma}e^{-2mZ\alpha r}dr\right] \tag{1}$$

The radial integration,

$$I_n = \int_{r=0}^{\infty} r^{2\gamma}e^{-2mZ\alpha r}dr$$

let's assume,

$$p = 2mZ\alpha r$$
$$dp = 2mZ\alpha dr$$

$$I_n = \int_{p=0}^{\infty}\left(\frac{p}{2mZ\alpha}\right)^{2\gamma} e^{-p}\frac{dp}{2mZ\alpha}$$

$$I_n = \frac{1}{(2mZ\alpha)^{2\gamma+1}}\int_{p=0}^{\infty} p^{2\gamma}e^{-p}dp$$

$$I_n = \frac{1}{(2mZ\alpha)^{2\gamma+1}}\Gamma(2\gamma+1)$$

Put the value in the integration we will get,

$$N = (2mZ\alpha)^{2\gamma+1}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\left[\frac{1}{(2mZ\alpha)^{2\gamma+1}}\Gamma(2\gamma+1) + \frac{(1-\gamma)^2}{Z^2\alpha^2}\frac{1}{(2mZ\alpha)^{2\gamma+1}}\Gamma(2\gamma+1)\right]$$

$$N = \left[\frac{1+\gamma}{2} + \frac{(1-\gamma^2)(1-\gamma)}{2Z^2\alpha^2}\right]$$

We know the relation,

$$\gamma = \sqrt{1-Z^2\alpha^2}$$
$$1-\gamma^2 = Z^2\alpha^2$$

After putting the value of the Normalization like this,

$$N = \left[\frac{1+\gamma}{2} + \frac{1-\gamma}{2}\right]$$

$$N = 1$$

The expectation value is

$$\langle r\rangle = \langle\psi|r|\psi\rangle$$

$$= \int_{r=0}^{\infty}\int_{\theta=-\pi}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}}\sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}}(2mZ\alpha r)^{\gamma-1}e^{-mZ\alpha r}\begin{bmatrix}1 & 0 & \frac{-i(1-\gamma)}{Z\alpha}\cos\theta & \frac{-i(1-\gamma)}{Z\alpha}\sin\theta e^{-i\phi}\end{bmatrix}$$
$$\frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}}\sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}}(2mZ\alpha r)^{\gamma-1}r\begin{bmatrix}1\\ 0\\ \frac{i(1-\gamma)}{Z\alpha}\cos\theta\\ \frac{i(1-\gamma)}{Z\alpha}\sin\theta e^{i\phi}\end{bmatrix} r^2 sin\theta dr d\theta d\phi$$

$$= \int_{r=0}^{\infty}\int_{\theta=-\pi}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \frac{(2mZ\alpha)^3}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}(2mZ\alpha r)^{2\gamma-2}e^{-2mZ\alpha r}r\left(1^2+0^2+\frac{(1-\gamma)^2}{Z^2\alpha^2}\cos^2\theta+\frac{(1-\gamma)^2}{Z^2\alpha^2}\sin^2\theta\right)r^2\sin\theta dr d\theta d\phi$$

$$= \int_{r=0}^{\infty}\int_{\theta=-\pi}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \frac{(2mZ\alpha)^{3+2\gamma-2}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}r^{3+2\gamma-2}e^{-2mZ\alpha r}\left(1+\tfrac{(1-\gamma)^2}{Z^2\alpha^2}\right)r^2\sin\theta dr d\theta d\phi$$

$$= \frac{(2mZ\alpha)^{2\gamma+1}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\left[\int_{r=0}^{\infty}\int_{\theta=-\pi}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} r^{2\gamma+1}e^{-2mZ\alpha r}\sin\theta dr d\theta d\phi+\int_{r=0}^{\infty}\int_{\theta=-\pi}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\frac{(1-\gamma)^2}{Z^2\alpha^2}r^{2\gamma+1}e^{-2mZ\alpha r}\sin\theta dr d\theta d\phi\right]$$

$$= \frac{(2mZ\alpha)^{2\gamma+1}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\left[\int_{r=0}^{\infty} r^{2\gamma+1}e^{-2mZ\alpha r}dr\int_{\theta=-\pi}^{\theta=\pi}\sin\theta d\theta\int_{\phi=0}^{\phi=2\pi}d\phi+\frac{(1-\gamma)^2}{Z^2\alpha^2}\int_{r=0}^{\infty} r^{2\gamma+1}e^{-2mZ\alpha r}dr\int_{\theta=-\pi}^{\theta=\pi}\sin\theta d\theta\int_{\phi=0}^{\phi=2\pi}d\phi\right]$$

$$= \frac{(2mZ\alpha)^{2\gamma+1}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\left[4\pi\int_{r=0}^{\infty} r^{2\gamma+1}e^{-2mZ\alpha r}dr+4\pi\frac{(1-\gamma)^2}{Z^2\alpha^2}\int_{r=0}^{\infty} r^{2\gamma+1}e^{-2mZ\alpha r}dr\right]$$

$$= (2mZ\alpha)^{2\gamma+1}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\left[\int_{r=0}^{\infty} r^{2\gamma+1}e^{-2mZ\alpha r}dr+\frac{(1-\gamma)^2}{Z^2\alpha^2}\int_{r=0}^{\infty} r^{2\gamma+1}e^{-2mZ\alpha r}dr\right] \tag{2}$$

The radial integration,

$$I_n = \int_{r=0}^{\infty} r^{2\gamma+1}e^{-2mZ\alpha r}dr$$

let's assume,

$$p = 2mZ\alpha r$$
$$dp = 2mZ\alpha dr$$

$$I_n = \int_{p=0}^{\infty}\left(\frac{p}{2mZ\alpha}\right)^{2\gamma+1}e^{-p}\frac{dp}{2mZ\alpha}$$

$$I_n = \frac{1}{(2mZ\alpha)^{2\gamma+2}}\int_{p=0}^{\infty}p^{2\gamma}e^{-p}dp$$

$$I_n = \frac{1}{(2mZ\alpha)^{2\gamma+2}}\Gamma(2\gamma+2)$$

Put the value in the integration we will get,

$$\langle r\rangle = (2mZ\alpha)^{2\gamma+1}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\left[\frac{\Gamma(2\gamma+1)}{(2mZ\alpha)^{2\gamma+2}}+\frac{(1-\gamma)^2}{Z^2\alpha^2}\frac{\Gamma(2\gamma+2)}{(2mZ\alpha)^{2\gamma+2}}\right]$$

$$\langle r\rangle = \left[\frac{1+\gamma}{2\Gamma(1+2\gamma)}\frac{\Gamma(2\gamma+2)}{2mZ\alpha} + \frac{(1-\gamma^2)}{2Z^2\alpha^2}\frac{(1-\gamma)}{2mZ\alpha}\frac{\Gamma(2\gamma+2)}{\Gamma(1+2\gamma)}\right]$$

$$\langle r\rangle = \left[\frac{1+\gamma}{2\Gamma(1+2\gamma)}\frac{(1+2\gamma)\Gamma(2\gamma+1)}{2mZ\alpha} + \frac{(1-\gamma^2)}{2Z^2\alpha^2}\frac{(1-\gamma)}{2mZ\alpha}\frac{(1+2\gamma)\Gamma(2\gamma+1)}{\Gamma(1+2\gamma)}\right]$$

$$\langle r\rangle = \left[\frac{(1+\gamma)(1+2\gamma)}{4mZ\alpha} + \frac{(1-\gamma)(1+2\gamma)}{4mZ\alpha}\right]$$

$$\langle r\rangle = \frac{2(1+2\gamma)}{4mZ\alpha}$$

$$\langle r\rangle = \frac{(1+2\gamma)}{2mZ\alpha} \tag{3}$$

we know ,

$$\gamma = \sqrt{1-Z^2\alpha^2}$$
$$a = \frac{1}{m\alpha}$$

put the values of a and γ in equation (1)

$$\langle r\rangle = \frac{a}{2Z}(1+2\sqrt{1-Z^2\alpha^2})$$

The expectation value of r for the hydrogen atom in the ground state is

$$\langle r\rangle_{gs} = \frac{a}{2}(1+2\sqrt{1-\alpha^2})$$