OLYMPIAD GEOMETRY

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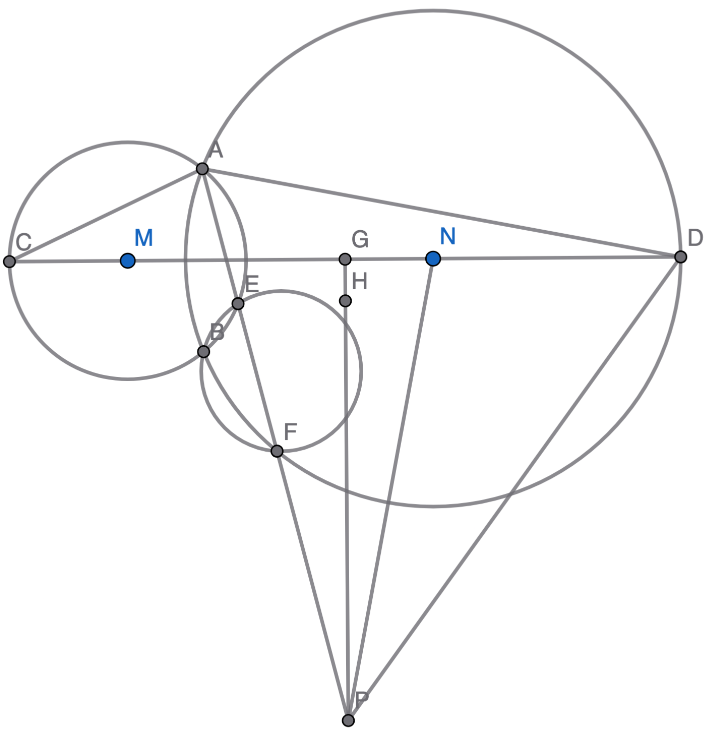
Abstract. This document is a compilation of my attempts at bashing IMO geometry problems using algebraic tools.

Contents

1. **IMO 2025 Problem 2**

Let Ω and Γ be circles with centres M and N, respectively, such that the radius of Ω is less than the radius of Γ. Suppose circles Ω and Γ intersect at two distinct points A and B. Line MN intersects Ω at C and Γ at D, such that points C, M, N and D lie on the line in that order. Let P be the circumcentre of triangle ACD. Line AP intersects Ω again at $E \neq A$. Line AP intersects Γ again at $F \neq A$. Let H be the orthocentre of triangle PMN. Prove that the line through H parallel to AP is tangent to the circumcircle of triangle BEF.



Solution: Suppose $AB = x, CD = y, \angle ACD = C$ and $\angle ADC = D$. Since $AB \perp CD$, it is clear that we have the following relation between these quantities:

$$y = \frac{x}{2}(\cot C + \cot D) = \frac{x \sin(C+D)}{2 \sin C \sin D}.$$

We start by computing the circumradius R of the $\triangle BEF$. To this end, we note that $\angle AEB = \pi - 2C$, which implies that $\angle BEF = 2C$. Similarly, we have $\angle BFE = 2D$. Next, as P is the circumcenter of $\triangle CAD$, we have $\angle APD = 2C$. So, $\angle PAD = \frac{\pi}{2} - C$. This implies that:

$$\angle BAE = \angle BAD - \angle EAD = (\frac{\pi}{2} - D) - (\frac{\pi}{2} - C) = C - D.$$

In $\triangle ABE$, by the sine rule, we have:

$$BE = AB\frac{\sin(\angle BAE)}{\sin(\angle AEB)} = x\frac{\sin(C-D)}{\sin(\pi - 2C)} = x\frac{\sin(C-D)}{\sin 2C}.$$

Then, the circumradius R of $\triangle BEF$ is given by:

$$R = \frac{BE}{2\sin(\angle BFE)} = x\frac{\sin(C-D)}{2\sin 2D \sin 2C}.$$

Next, we compute the distance of the line EF from the center of the circle BEF. Again by the sine rule, we have $EF = 2R\sin(\angle EBF) = 2R\sin(\pi - 2C - 2D) = 2R\sin(2C+2D)$. Hence, the distance of EF from the center is given by:

$$\sqrt{R^2 - R^2\sin^2(2C+2D)} = R|\cos(2C+2D)|.$$

Then, the distance of a tangent line to the circle BEF that is parallel to EF from the line EF is given by:

$$R \pm R|\cos(2C+2D)| = 2R\cos^2(C+D) \text{ or } 2R\sin^2(C+D).$$

In order to show that the parallel to EF that passes through H is parallel to the circle BEF, it suffices to compute the distance of this parallel line from EF and to verify that it is equal to one of the 2 quantities above. The distance of this parallel line from EF is equal to $HP\sin(\angle HPF)$. So, to prove the required claim, it suffices to prove the equality:

$$HP\sin(\angle HPF) = 2R\cos^2(C+D).$$

Note that $\angle HPF = \angle BAE = C - D$ since $AB \parallel PG$, since both are perpendicular to CD.

Next, note that $\angle PCD = \angle PCA - C = \frac{\pi}{2} - D - C$. Also, PG bisects CD and so $CG = \frac{y}{2}$ Therefore, $PG = CG\tan(\angle PCD) = \frac{y}{2}\cot(C+D)$.

Also, by the sine rule for $\triangle ABC$, we have that the radius CM of the circle ABC is equal to: $CM = \frac{AB}{2\sin(\angle ACB)} = \frac{x}{2\sin 2C}$. Hence,

$$\begin{aligned}
MG &= CG - CM \\
&= \frac{y}{2} - \frac{x}{2\sin 2C} \\
&= \frac{x\sin(C+D)}{4\sin C\sin D} - \frac{x}{2\sin 2C} \\
&= \frac{x}{4}\Big(\frac{\sin(C+D)\cos C - \sin D}{\sin C\cos C\sin D}\Big) \\
&= \frac{x}{4}\Big(\frac{\cos(C+D)\sin C}{\sin C\cos C\sin D}\Big) \\
&= \frac{x}{4}\Big(\frac{\cos(C+D)}{\cos C\sin D}\Big).
\end{aligned}$$

Similarly, we have $NG = \frac{x}{4}\big(\frac{\cos(C+D)}{\cos D\sin C}\big)$.

Hence,

$$\tan(\angle PMG) = \frac{PG}{MG} = \frac{\frac{y}{2}\cot(C+D)}{\frac{x}{4}\big(\frac{\cos(C+D)}{\cos C\sin D}\big)} = \cot C.$$

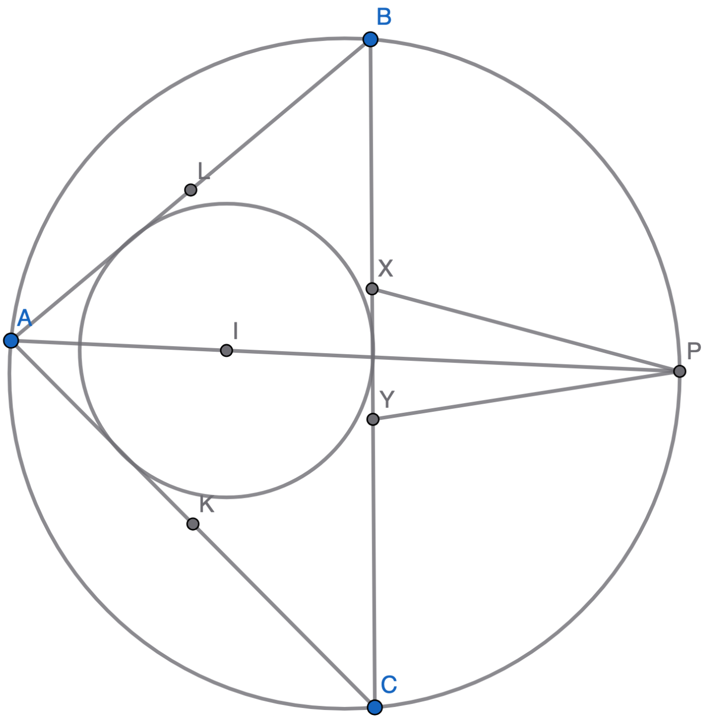
Furthermore, $\angle NHG = \pi - \angle PHG = \pi - (\pi - \angle PMG) = \angle PMG$. Therefore, $HG = NG\cot(\angle NHG) = \frac{x}{4}\big(\frac{\cos(C+D)}{\cos D\cos C}\big)$. Hence, we have:

$$\begin{aligned}
HP &= PG - HG \\
&= \frac{x\cos(C+D)}{4\sin D\sin C} - \frac{x\cos(C+D)}{4\cos D\cos C} \\
&= \frac{x\cos^2(C+D)}{4\sin C\sin D\cos C\cos D} \\
&= \frac{x\cos^2(C+D)}{\sin 2C\sin 2D}.
\end{aligned}$$

Thus, $HP\sin(\angle HPF) = \frac{x\cos^2(C+D)}{\sin 2C\sin 2D}\sin(C-D) = 2R\cos^2(C+D)$, completing the proof.

2. IMO 2024 Problem 4

Let ABC be a triangle with $AB < AC < BC$. Let the incentre and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively. Prove that $\angle KIL + \angle YPX = \pi$.



Solution: Let $I = (0, 0)$ and the radius of the circle $\omega_1 = 1$. Let $B = (1, b)$ and $C = (1, c)$ for some $b > 0$ and $c < 0$. In order to show that $\angle KIL + \angle YPX = \pi$, it suffices to show that $\tan(\angle KIL) = -\tan(\angle YPX)$.

Let the slope of the line AB be m. Then the equation of AB is:

$$\frac{y-b}{x-1} = m.$$

As AB is tangent to ω_1, its distance from I should be 1. Thus,

$$\frac{|m-b|}{\sqrt{1+m^2}} = 1 \implies m = \frac{b^2-1}{2b}.$$

Hence, the equation of AB is:

$$\frac{y-b}{x-1} = \frac{b^2-1}{2b}.$$

Similarly, the equation of AC is:

$$\frac{y-c}{x-1} = \frac{c^2-1}{2c}.$$

The intersection of these lines gives the coordinates of point A:

$$A = \Big(\frac{1-bc}{1+bc}, \frac{b+c}{1+bc}\Big).$$

Then, we compute:

$$K = \frac{A+C}{2} = \Big(\frac{1}{1+bc}, \frac{bc^2+2c+b}{2(1+bc)}\Big),$$
$$L = \frac{A+B}{2} = \Big(\frac{1}{1+bc}, \frac{b^c c+2b+c}{2(1+bc)}\Big).$$

Hence,

$$\text{Slope of } IK = \frac{bc^2+2c+b}{2},$$
$$\text{Slope of } IL = \frac{b^2c+2b+c}{2}.$$

Therefore,

$$\begin{aligned}\tan(\angle KIL) &= \frac{\frac{bc^2+2c+b}{2} - \frac{b^2c+2b+c}{2}}{1+\big(\frac{bc^2+2c+b}{2}\big)\big(\frac{b^2c+2b+c}{2}\big)} \\ &= \frac{2(c-b)(bc+1)}{4+(bc^2+2c+b)(b^2c+2b+c)}.\end{aligned}$$

Next, we compute the coordinates of P. As AI bisects $\angle BAC$, we have that $PB = PC$. Thus, $P = (r, \frac{b+c}{2})$ for some r. Also, since A, I and P are colinear, we have:

$$\text{Slope of } AI = \text{Slope of } PI \implies \frac{1-bc}{b+c} = \frac{2r}{b+c}.$$

This gives that $r = \frac{1-bc}{2}$ implying that $P = (\frac{1-bc}{2}, \frac{b+c}{2})$.

Finally, we find coordinates of X and Y. The slope of the line through X tangent to ω is the same as the slope of AC, which is equal to $\frac{c^2-1}{2c}$. Hence, the equation of the tangent line is

$$y - \frac{c^2-1}{2c}x = \alpha$$

for some α. For this to be tangent to ω_1, its distance from I should be 1. Therefore,

$$\frac{|\alpha|}{\sqrt{1+(\frac{c^2-1}{2c})^2}} = 1 \implies \alpha = \pm\frac{c^2+1}{2c}.$$

Thus, the equation of the tangent to ω through the point X is:

$$y - \frac{c^2-1}{2c}x = -\frac{c^2+1}{2c}.$$

(We choose the negative sign, since the positive sign corresponds to the line AC.) Intersecting this tangent line with the line BC, which is given by $x = 1$, we get that $X = (1, -\frac{1}{c})$. Similarly, we get that $Y = (1, -\frac{1}{b})$.

So, we can compute:

$$\text{Slope of } PX = \frac{\frac{b+c}{2}+\frac{1}{c}}{\frac{1-bc}{2}-1} = -\frac{bc+c^2+2}{c(1+bc)},$$
$$\text{Slope of } PY = \frac{\frac{b+c}{2}+\frac{1}{b}}{\frac{1-bc}{2}-1} = -\frac{bc+b^2+2}{b(1+bc)}.$$

Hence,

$$\begin{aligned}\tan(\angle YPX) &= \frac{-\frac{bc+b^2+2}{b(1+bc)} + \frac{bc+c^2+2}{c(1+bc)}}{1+\left(\frac{bc+b^2+2}{b(1+bc)}\right)\left(\frac{bc+c^2+2}{c(1+bc)}\right)} \\ &= \frac{-bc(b+c)(1+bc) - 2c(1+bc) + bc(b+c)(1+bc) + 2b(1+bc)}{bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2)} \\ &= \frac{2(b-c)(1+bc)}{bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2)}.\end{aligned}$$

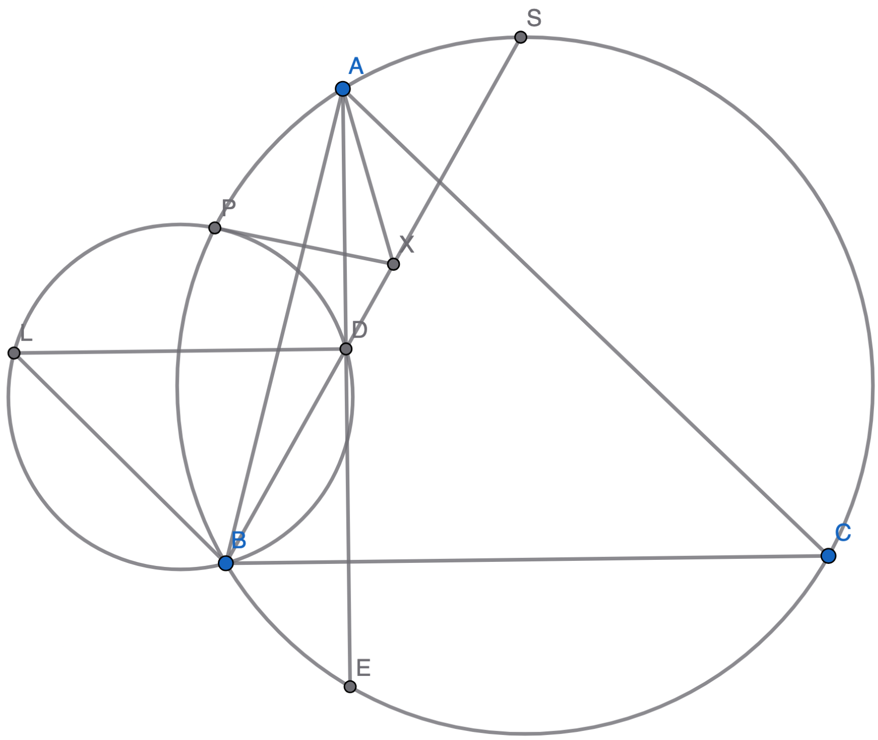
Therefore, from our expressions for $\tan(\angle KIL)$ and $\tan(\angle YPX)$, it follows that the equality $\tan(\angle KIL) = -\tan(\angle YPX)$ is equivalent to the following algebraic identity:

$$4 + (bc^2 + 2c + b)(b^2c + 2b + c) = bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2),$$

which is easily verified.

3. IMO 2023 Problem 2

Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$. Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.



Solution: We have $\angle SBC = \angle SCB = \frac{\pi}{2} - \frac{A}{2}$. Also, $\angle CBE = \angle CAE = \frac{\pi}{2} - C$. Thus, $\angle LBD = \pi - \angle SBE = \pi - (\angle SBC + \angle CBE) = \frac{\pi}{2} + \frac{C-B}{2}$.

Let $\angle PBS = t$. In ω, sine rule gives:

$$\begin{aligned}\frac{PD}{\sin t} &= \frac{LD}{\sin(\angle LBD)} \\ &= \frac{LD}{\sin(\frac{\pi}{2} + \frac{C-B}{2})} \\ &= \frac{LD}{\cos(\frac{B-C}{2})}.\end{aligned}$$

Next, we have $(\angle PAD + \angle DAC) + \angle PBC = \pi$. Thus,

$$\begin{aligned}\angle PAD &= \pi - \angle PBC - \angle DAC \\ &= \pi - (t + \frac{B+C}{2}) - (\frac{\pi}{2} - C)\end{aligned}$$

$$= \frac{\pi}{2} - t + \frac{C-B}{2}.$$

Similarly, we have $(\angle DPA + \angle DPB) + \angle ACB = \pi$, Thus,

$$\begin{aligned}
\angle DPA &= \pi - \angle DPB - \angle ACB \\
&= \pi - \angle DLB - C \\
&= \pi - (\frac{\pi}{2} - \angle LED) - C \\
&= \pi - (\frac{\pi}{2} - C) - C \\
&= \frac{\pi}{2}.
\end{aligned}$$

Hence, we have that $PD = AD\sin(\angle PAD) = AD\sin(\frac{\pi}{2} - t + \frac{C-B}{2}) = AD\cos(\frac{C-B}{2} - t)$. Inserting this into the above sine rule equation, we get:

$$\begin{aligned}
\frac{AD\cos(\frac{C-B}{2} - t)}{\sin t} &= \frac{LD}{\cos(\frac{B-C}{2})} \\
\implies AD\cos\left(\frac{C-B}{2}\right)\cot t + AD\sin\left(\frac{C-B}{2}\right) &= \frac{LD}{\cos(\frac{B-C}{2})} \\
\implies \cot t = \frac{LD}{AD\cos^2(\frac{B-C}{2})} + \tan\left(\frac{B-C}{2}\right).
\end{aligned}$$

(1) First, suppose PX is tangent to the circle ω. Then, $\angle DPX = \angle DBP = t$. So, by the sine rule in $\triangle PDX$ we get:

$$\begin{aligned}
\frac{DX}{\sin t} &= \frac{PX}{\sin PDX} \\
&= \frac{PX}{\sin PLB}.
\end{aligned}$$

Now, $\angle PLB = \angle PLD + \angle BLD = \angle PBD + \angle BLD = \frac{\pi}{2} + t - C$. Hence,

$$\frac{PX}{DX} = \frac{\sin(\frac{\pi}{2} + t - C)}{\sin t} = \frac{\cos(t-C)}{\sin t} = \cot t\cos C + \sin C.$$

Computing the power of the point X with respect to ω, we get $PX^2 = BX \cdot DX$. Hence,

$$\frac{BX}{DX} = \frac{BX}{PX}\cdot\frac{PX}{DX} = \left(\frac{PX}{DX}\right)^2 = (\cot t\cos C + \sin C)^2.$$

(2) Now, suppose AX bisects $\angle BAC$. Then, $\angle AXB = \pi - \angle XAB - \angle XBA = \pi - \frac{A}{2} - (B - \frac{B+C}{2}) = \frac{\pi}{2} + C$. Also, $\angle DAX = \angle DAC - \angle XAC = \frac{\pi}{2} - C - \frac{A}{2}$. So, applying the sine rule in $\triangle ADX$, we get:

$$\begin{aligned}
\frac{DX}{\sin(\angle DAX)} &= \frac{AD}{\sin(\angle AXD)} \\
\implies DX = AD\frac{\sin(\frac{\pi}{2} - C - \frac{A}{2})}{\sin(\frac{\pi}{2} + C)} &= AD\frac{\cos(C + \frac{A}{2})}{\cos C}.
\end{aligned}$$

Next, applying the sine rule in $\triangle BAX$, we get:

$$\begin{aligned}
\frac{BX}{\sin(\angle BAX)} &= \frac{BA}{\sin(\angle AXB)} \\
\implies BX = AB\frac{\sin(\frac{A}{2})}{\sin(\frac{\pi}{2} + C)} &= \frac{c\sin(\frac{A}{2})}{\cos C},
\end{aligned}$$

where we suppose $AB = c$. Combining the two expressions above, we get that:

$$\frac{BX}{DX} = \frac{c\sin(\frac{A}{2})}{AD\cos(C + \frac{A}{2})}.$$

We have obtained expressions for $\frac{BX}{DX}$ in both of the above cases. Thus, to prove the required claim, it suffices to prove the equality:

$$(\cot t\cos C+\sin C)^2=\frac{c\sin(\frac{A}{2})}{AD\cos(C+\frac{A}{2})}.$$

To that end, we first compute AD. Note that $\angle ADB=\pi-\angle ABD-\angle BAD=\pi-(B-\frac{B+C}{2})-(\frac{\pi}{2}-B)=\pi-\frac{A}{2}$. Then, using the sine rule in $\triangle BAD$, we get:

$$AD=AB\frac{\sin(\angle ABD)}{\sin(\angle ADB)}=\frac{c\sin(\frac{B-C}{2})}{\sin(\pi-\frac{A}{2})}=\frac{c\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}.$$

This implies that:

$$\frac{c\sin(\frac{A}{2})}{AD\cos(C+\frac{A}{2})}=\frac{\sin^2(\frac{A}{2})}{\cos(C+\frac{A}{2})\sin(\frac{B-C}{2})}=\frac{\sin^2(\frac{A}{2})}{\sin^2(\frac{B-C}{2})}$$

since $\frac{B-C}{2}=\frac{\pi}{2}-(C+\frac{A}{2})$. Thus, we are reduced to proving the equality:

$$\cot t\cos C+\sin C=\frac{\sin(\frac{A}{2})}{\sin(\frac{B-C}{2})}.$$

Recall that $\cot t=\frac{LD}{AD\cos^2(\frac{B-C}{2})}+\tan\left(\frac{B-C}{2}\right)$. To compute this, we need to find LD. Note that $LD=DE\tan(\angle LED)=DE\tan C=(AE-AD)\tan C$. Applying the sine rule in circle Ω, we get:

$$AE=AB\frac{\sin(\angle ABE)}{\sin(\angle ACB)}=\frac{c\sin(90+B-C)}{\sin C}=\frac{c\cos(B-C)}{\sin C}=c\cot C\cos B+c\sin B.$$

Therefore,

$$\begin{aligned}LD&=\tan C(AE-AD)\\&=c\tan C\Big(\cot C\cos B+\sin B-\frac{\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}\Big)\\&=c\cos B+c\tan C\sin B-c\frac{\tan C\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{LD}{AD\cos^2(\frac{B-C}{2})}&=\frac{\cos B+\tan C\sin B-\frac{\tan C\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}}{\frac{\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}\cos^2(\frac{B-C}{2})}\\&=\frac{\sin(\frac{A}{2})(\cos B+\tan C\sin B)-\tan C\sin(\frac{B-C}{2})}{\sin(\frac{B-C}{2})\cos^2(\frac{B-C}{2})}.\end{aligned}$$

This implies that:

$$\begin{aligned}\cot t&=\frac{LD}{AD\cos^2(\frac{B-C}{2})}+\tan\left(\frac{B-C}{2}\right)\\&=\frac{\sin(\frac{A}{2})(\cos B+\tan C\sin B)-\tan C\sin(\frac{B-C}{2})+\sin^2(\frac{B-C}{2})\cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2})\cos^2(\frac{B-C}{2})}.\end{aligned}$$

Next,

$$\begin{aligned}\cos C\cot t&=\frac{\sin(\frac{A}{2})(\cos B\cos C+\sin C\sin B)-\sin C\sin(\frac{B-C}{2})+\cos C\sin^2(\frac{B-C}{2})\cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2})\cos^2(\frac{B-C}{2})}\\&=\frac{\sin(\frac{A}{2})\cos(B-C)-\sin C\sin(\frac{B-C}{2})+\cos C\sin^2(\frac{B-C}{2})\cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2})\cos^2(\frac{B-C}{2})}.\end{aligned}$$

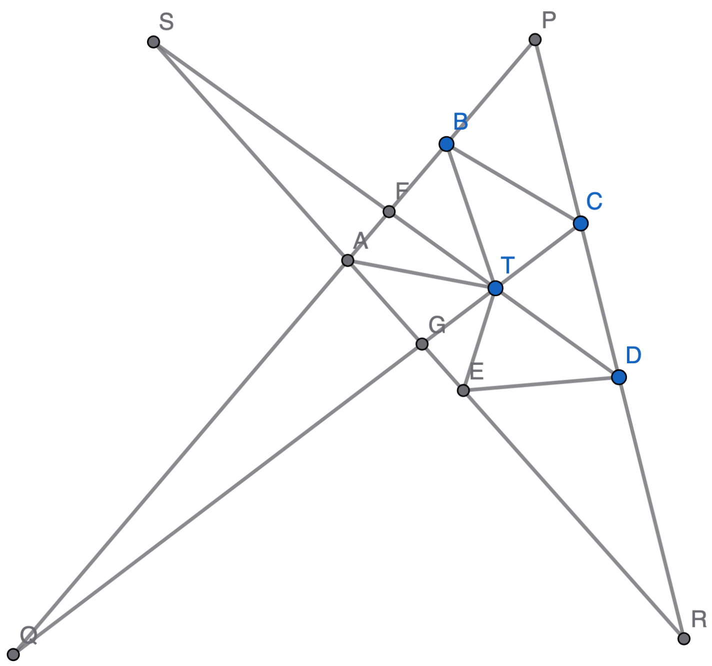
Finally,

$$\begin{aligned}
\sin C + \cos C \cot t &= \frac{\sin C \sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2}) + \sin(\frac{A}{2}) \cos(B-C) - \sin C \sin(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) - \sin C \sin^3(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin^2(\frac{B-C}{2}) \cos(C + \frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin^2(\frac{B-C}{2}) \cos(\frac{B+C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin(\frac{A}{2}) \sin^2(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2})(\cos(B-C) + \sin^2(\frac{B-C}{2}))}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2})}{\sin(\frac{B-C}{2})},
\end{aligned}$$

completing the proof.

4. **IMO 2022 Problem 4**

Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD, TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.



Solution: Suppose $TB = TD = s$ and $TC = TE = r$. We normalize $BC = DE = 1$. As we have $\triangle BTC \cong \triangle DTE$, we have the following equalities of angles:

$$\alpha := \angle BTC = \angle DTE$$

$$\beta := \angle TCB = \angle TED$$

$$\gamma := \angle CBT = \angle EDT.$$

Let $\angle DTC = \phi$ and $\angle ABT = \angle TEA = \theta$. Next, let $\angle TDC = m$ and $\angle TCD = n$. Finally, let $\angle FAT = f$ and $\angle GAT = g$. Note that $\angle BTE = 2\pi - (\angle BTC + \angle DTE + \angle DTC) = 2\pi - 2\alpha - 2\phi$. Thus, we have the equality:

$$f + g = 2\alpha + 2\phi - 2\theta.$$

As the angles at vertex B add to π, we have $\angle CBP = \pi - \gamma - \theta$. Similar consideration at vertex C gives that $\angle BCP = \pi - \beta - n$. Thus, $\angle P = \pi - (\angle CBP + \angle BCP) = \theta + n - \alpha$. Similarly,

$\angle R = \theta + m - \alpha$. Hence, by the sine rule in $\triangle APR$, we get:

$$\frac{AP}{AR} = \frac{\sin(\theta + m - \alpha)}{\sin(\theta + n - \alpha)}.$$

Next, $\angle Q = \pi - (\angle P + \angle QCP) = \alpha - \theta$.

By exactly the same argument, we get that $\angle S = \alpha - \theta$. By the sine rule in $\triangle QAT$ and $\triangle SAT$, we have:

$$AQ = \frac{\sin(\angle ATQ)}{\angle Q} AT$$
$$AS = \frac{\sin(\angle ATS)}{\angle S} AT,$$

and so, $\frac{AQ}{AS} = \frac{\sin(\angle ATQ)}{\sin(\angle ATS)}$. Note that $\angle ATQ = \angle TAF - \angle Q = f - \alpha + \theta$. Similarly, $\angle ATS = g - \alpha + \theta$. So,

$$\frac{AQ}{AS} = \frac{\sin(f - \alpha + \theta)}{\sin(g - \alpha + \theta)}.$$

In order to show that P, S, Q and R are concylic, it suffices to show that:

$$\begin{aligned}
& AP \cdot AQ = AR \cdot AS \\
\iff & \frac{AP}{AR} = \frac{AS}{AQ} \\
\iff & \frac{\sin(\theta + m - \alpha)}{\sin(\theta + n - \alpha)} = \frac{\sin(g - \alpha + \theta)}{\sin(f - \alpha + \theta)} \\
\iff & \frac{\sin(\theta + m - \alpha) + \sin(\theta + n - \alpha)}{\sin(\theta + m - \alpha) - \sin(\theta + n - \alpha)} = \frac{\sin(g - \alpha + \theta) + \sin(f - \alpha + \theta)}{\sin(g - \alpha + \theta) - \sin(f - \alpha + \theta)} \\
\iff & \frac{\tan(\theta - \alpha + \frac{m+n}{2})}{\tan(\frac{m-n}{2})} = \frac{\tan(\theta - \alpha + \frac{f+g}{2})}{\tan(\frac{g-f}{2})} \\
\iff & \frac{\tan(\theta - \alpha + \frac{\pi - 2\phi}{2})}{\tan(\frac{m-n}{2})} = \frac{\tan(\theta - \alpha + \frac{2\alpha + 2\phi - 2\theta}{2})}{\tan(\frac{g-f}{2})} \\
\iff & \frac{\cot(\phi - \theta + \alpha)}{\tan(\frac{m-n}{2})} = \frac{\tan(\phi)}{\tan(\frac{g-f}{2})} \\
\iff & \tan\left(\frac{g-f}{2}\right)\cot(\phi - \theta + \alpha) = \tan\left(\frac{m-n}{2}\right)\tan(\phi) \\
\iff & \tan\left(\frac{g-f}{2}\right)\cot(\frac{g+f}{2}\Big) = \tan\left(\frac{m-n}{2}\right)\cot(\frac{m+n}{2}\Big) \\
\iff & \frac{\sin(g) - \sin(f)}{\sin(g) + \sin(f)} = \frac{\sin(m) - \sin(n)}{\sin(m) + \sin(n)} \\
\iff & \frac{\sin(g)}{\sin(f)} = \frac{\sin(m)}{\sin(n)}.
\end{aligned}$$

Note that in $\triangle CDT$ by the sine rule:

$$\frac{\sin(m)}{\sin(n)} = \frac{TC}{TD} = \frac{r}{s}.$$

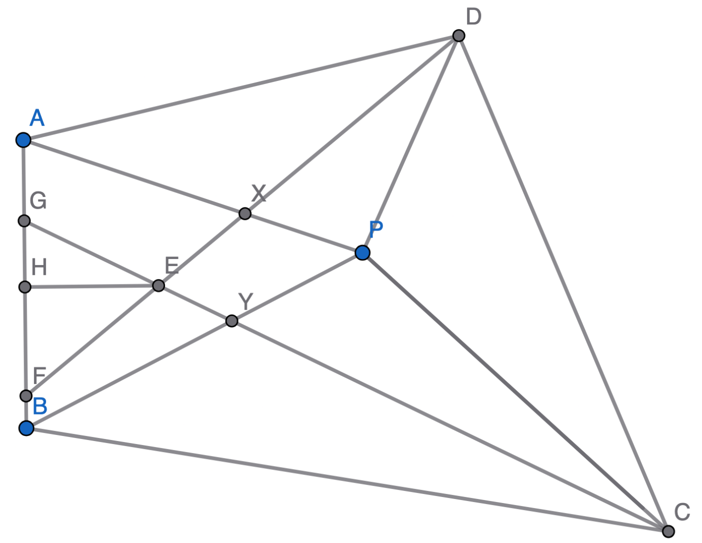
On the other hand, by the sine rule in $\triangle BAT$ and $\triangle EAT$, we have:

$$\frac{\sin(g)}{\sin(f)} = \frac{\frac{TE}{TA}\sin(\angle TEA)}{\frac{TB}{TA}\sin(\angle TBA)} = \frac{TE\sin(\theta)}{TB\sin(\theta)} = \frac{r}{s},$$

completing the proof.

5. IMO 2020 Problem 1

Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold: $\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC$. Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.



Solution: Let DF and CG be the bisectors of $\angle ADP$ and $\angle PCB$ respectively. Let $BP = a$, $AP = b$ and R be the circumradius of $\triangle PAB$. Let $\angle PBA = 2x$ and $\angle PAB = 2y$. Draw $EH \perp AB$. It suffices to show that H is the mid-point of AB.

Since AD bisects $\angle ADP$, we have by the sine rule in $\triangle ADP$:

$$\frac{AX}{XP} = \frac{AD}{DP} = \frac{\sin(\angle APD)}{\sin(\angle PAD)} = \frac{\sin 3x}{\sin x}.$$

Thus, $AX = b\frac{\sin 3x}{\sin x + \sin 3x} = b\frac{\sin 3x}{2\sin 2x \cos x} = R\frac{\sin 3x}{\cos x}$, by the sine rule in $\triangle APB$.

Next, $\angle AXF = \angle XAD + \angle XDA = \angle XAD + \frac{1}{2}\angle PDA = x + \frac{\pi}{2} - 2x = \frac{\pi}{2} - x$. This implies that $\angle XFA = \pi - \angle FAX - \angle AXF = \frac{\pi}{2} + x - 2y$. Then, by the sine rule in $\triangle AFX$, we get:

$$\begin{aligned} AF &= AX\frac{\sin(\angle AXF)}{\sin(\angle XFA)} \\ &= R\frac{\sin 3x}{\cos x}\frac{\cos x}{\cos(2y - x)} \\ &= R\frac{\sin 3x}{\cos(2y - x)} \end{aligned}$$

$$= R\frac{\sin((2x+2y)-(2y-x))}{\cos(2y-x)}$$
$$= R(\sin(2x+2y)-\cos(2x+2y)\tan(2y-x)).$$

Similarly, we have $BG = R(\sin(2x+2y)-\cos(2x+2y)\tan(2x-y))$. Finally, $AB = 2R\sin(\angle APB)) = 2R\sin(\pi-2x-2y) = 2R\sin(2x+2y)$. Thus, we have:

$$GF = AF+BG-AB = -R\cos(2x+2y)(\tan(2y-x)+\tan(2x-y)) = -R\cos(2x+2y)\frac{\sin(x+y)}{\cos(2y-x)\cos(2x-y)}.$$

Note that $\angle GFE = \frac{\pi}{2}+x-2y$ and $\angle FGE = \frac{\pi}{2}+y-2x$. Thus, we have $\angle GEF = x+y$. Hence, by the sine rule in $\triangle GFE$, we get:

$$EF = GF\frac{\sin(\angle FGE)}{\sin(\angle GEF)} = -R\cos(2x+2y)\frac{\sin(x+y)}{\cos(2y-x)\cos(2x-y)}\frac{\cos(2x-y)}{\sin(x+y)} = -R\frac{\cos(2x+2y)}{\cos(2y-x)}.$$

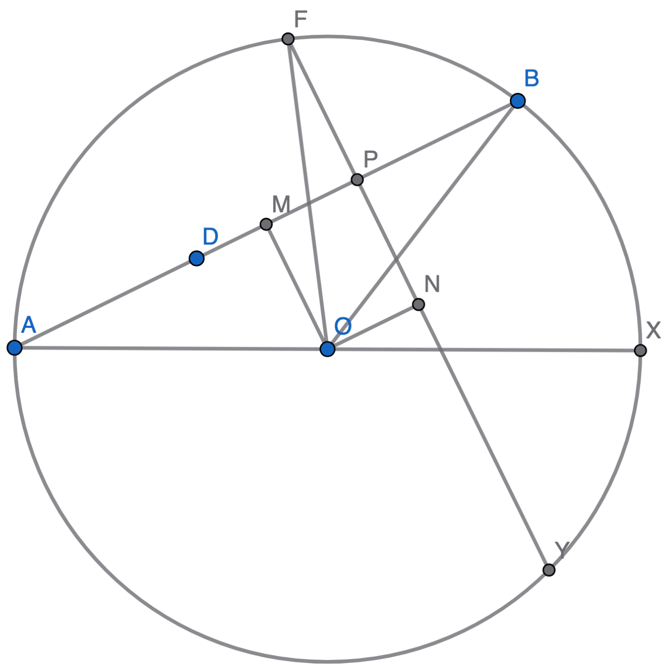
Next,

$$FH = EF\cos(\angle HFE) = -R\frac{\cos(2x+2y)}{\cos(2y-x)}\sin(2y-x) = -R\cos(2x+2y)\tan(2y-x).$$

Finally, $AH = AF-FH = (R(\sin(2x+2y)-\cos(2x+2y)\tan(2y-x)))-(-R\cos(2x+2y)\tan(2y-x)) = R\sin(2x+2y) = \frac{1}{2}AB$, completing the proof.

6. IMO 2018 Problem 1

Let Γ be the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line).



Solution: Let O be center of Γ and AO meets Γ at X. Let FY be the perpendicular bisector of BD and draw $OM \perp AB$ and $ON \perp FY$. Let $P = FY \cap AB$. Let $\angle BOX = 2\beta$, $AD = 2d$ and R be the radius of Γ. We assume that $0 < 2\beta < \pi$.

We start by observing that $\angle BAO = \angle ABO = \beta$. Since FY is perpendicular to both AB and ON, we have $ON \parallel AB$. Therefore, $\angle NOX = \angle BOA = \beta$.

Next, since FY and OM are both perpendicular to AB, we have that $ONMP$ is a rectangle. Therefore, $ON = MP = MB - PB = \frac{1}{2}AB - \frac{1}{2}DB = \frac{1}{2}AD = d$. Thus, $\angle FON = \cos^{-1}\left(\frac{ON}{OF}\right) = \cos^{-1}\left(\frac{d}{R}\right)$. Hence, $\angle FOX = \beta + \cos^{-1}\left(\frac{d}{R}\right)$.

Similarly, suppose the point C is chosen on Γ such that $\angle COX = 2\gamma$ and $-\pi < 2\gamma < 0$. Then, we will have that $\angle GOX = \gamma - \cos^{-1}\left(\frac{d}{R}\right)$.

Thus, $\angle FOG = \beta - \gamma + 2\cos^{-1}\left(\frac{d}{R}\right)$. This implies that $\angle OFG = \angle OGF = \frac{\pi}{2} - \frac{\beta-\gamma}{2} - \cos^{-1}\left(\frac{d}{R}\right)$. Hence, the angle that the line FG makes with AX is equal to:

$$\angle OFG + \angle FOX = \left(\frac{\pi}{2} - \frac{\beta - \gamma}{2} - \cos^{-1}\left(\frac{d}{R}\right)\right) + \left(\beta + \cos^{-1}\left(\frac{d}{R}\right)\right) = \frac{\pi}{2} + \frac{\beta + \gamma}{2}.$$

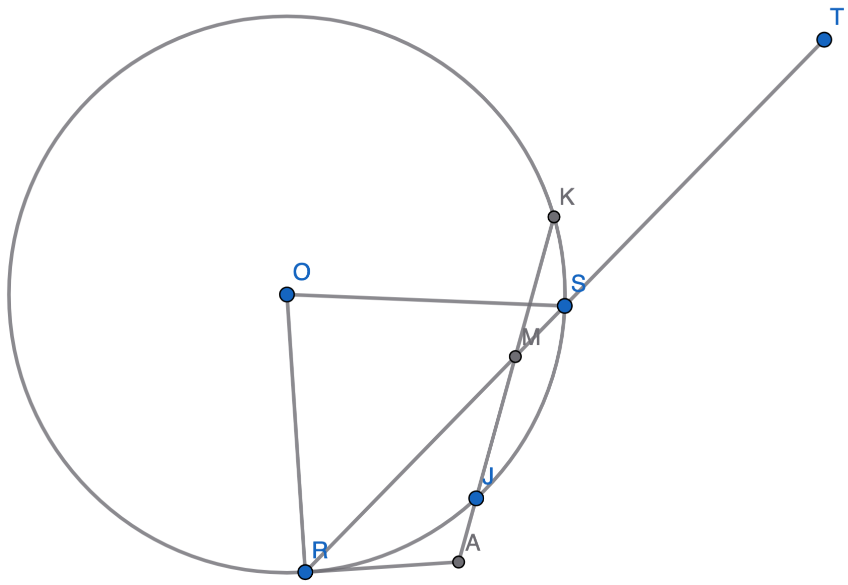
But then, exactly by a similar argument, we have $\angle DOE = \beta - \gamma$ and $\angle ADE = \angle AED = \frac{\pi}{2} - \frac{\beta-\gamma}{2}$. Thus, the angle that the line DE makes with AX is equal to:

$$\angle ADE + \angle DAX = \frac{\pi}{2} - \frac{\beta - \gamma}{2} + \beta = \frac{\pi}{2} + \frac{\beta + \gamma}{2}.$$

As the two angles above are equal, the claim stands proven.

7. IMO 2017 Problem 4

Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.



Solution: Let $M = RT \cap AK$ and let $\angle ROS = 2\alpha$ and $\angle RMA = \gamma$. Finally, let $RM = x$ and $RS = d$. To prove the required claim, we need to show that $KT^2 = KJ \cdot KA$.

To that end, we compute the power of the point M with respect to Ω and Γ. First, note that $MJ \cdot MA = MS \cdot MT$. Thus, $MJ = \frac{(d-x)(2d-x)}{MA}$. Next, $MJ \cdot MK = MR \cdot MS$ and so $MK = \frac{x(d-x)}{MJ} = \frac{x}{2d-x}MA$. This implies that $KJ = MK + MJ = \frac{x}{2d-x}MA + \frac{(d-x)(2d-x)}{MA}$. Finally, $KA = MK + MA = \frac{x}{2d-x}MA + MA = \frac{2d}{2d-x}MA$.

Next, in $\triangle MRA$, we have $\angle RMA = \gamma$ and $\angle MRA = \alpha$. So, $\angle RAM = \pi - \alpha - \gamma$. Hence, by the sine rule, $MA = x\frac{\sin\alpha}{\sin(\alpha+\gamma)}$ and $RA = x\frac{\sin\gamma}{\sin(\alpha+\gamma)}$. Next, since AR is tangent to Ω, we have $AR^2 = AJ \cdot AM$. Thus,

$$AR^2 = (AM - MJ) \cdot AK$$

$$AR^2 = \left(MA - \frac{(d-x)(2d-x)}{MA}\right)\frac{2d}{2d-x}MA$$

$$AR^2 = \frac{2d}{2d-x}MA^2 - 2d(d-x)$$

$$x^2 \frac{\sin^2\gamma}{\sin^2(\alpha+\gamma)} = x^2 \frac{2d}{2d-x} \frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)} - 2d(d-x)$$

$$-2d(d-x) = x^2 \frac{\sin^2\gamma}{\sin^2(\alpha+\gamma)} - x^2 \frac{2d}{2d-x} \frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)}.$$

Finally, we can compute the required expressions.

$$\begin{aligned} KJ \cdot KA &= \Big(\frac{x}{2d-x} MA + \frac{(d-x)(2d-x)}{MA}\Big) \frac{2d}{2d-x} MA \\ &= \frac{2dx}{(2d-x)^2} MA^2 + 2d(d-x) \\ &= \frac{2dx^3}{(2d-x)^2} \frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)} + 2d(d-x). \end{aligned}$$

Next, by the cosine rule in $\triangle MKT$, we get:

$$\begin{aligned} KT^2 &= MK^2 + MT^2 - 2MK \cdot MT \cos(\angle KMT) \\ &= \frac{x^4}{(2d-x)^2} \frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)} + (2d-x)^2 - 2x^2 \frac{\sin\alpha\cos\gamma}{\sin(\alpha+\gamma)}. \end{aligned}$$

Therefore,

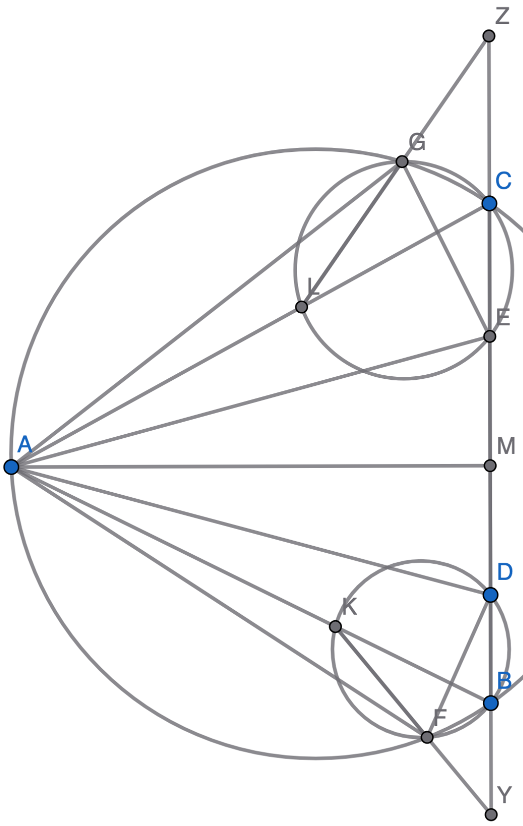
$$\begin{aligned} KJ \cdot KA - KT^2 &= \frac{x^3 \sin^2\alpha}{(2d-x)\sin^2(\alpha+\gamma)} + 2x^2 \frac{\sin\alpha\cos\gamma}{\sin(\alpha+\gamma)} - 2d(d-x) - x^2 \\ &= \frac{x^3 \sin^2\alpha}{(2d-x)\sin^2(\alpha+\gamma)} + 2x^2 \frac{\sin\alpha\cos\gamma}{\sin(\alpha+\gamma)} + x^2 \frac{\sin^2\gamma}{\sin^2(\alpha+\gamma)} - x^2 \frac{2d}{2d-x} \frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)} - x^2 \\ &= 2x^2 \frac{\sin\alpha\cos\gamma}{\sin(\alpha+\gamma)} + x^2 \frac{\sin^2\gamma}{\sin^2(\alpha+\gamma)} - x^2 \frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)} - x^2 \\ &= x^2 \frac{2\sin\alpha\cos\gamma\sin(\alpha+\gamma) + \sin^2\gamma - \sin^2\alpha - \sin^2(\alpha+\gamma)}{\sin^2(\alpha+\gamma)} \\ &= 0, \end{aligned}$$

where the last equality follows by expanding out $\sin(\alpha+\gamma)$, thus proving the claim.

8. IMO 2015 Problem 4

Triangle ABC has circumcircle Ω and circumcentre O. A circle Γ with centre A intersects the segment BC at points D and E, such that B, D, E and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle $ACGE$ and the segmenet CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.



Solution: Suppose $A = (0, 0)$ and Γ is the circle $x^2 + y^2 = r^2$. Let BC be the line $x = 1$. Draw $AM \perp BC$ with M on BC. We suppose that C lies above the x-axis and B lies below. Let $\angle MAD = \alpha$, $\angle MAB = \beta_1$ and $\angle MAF = \gamma_1$. Similarly, let $\angle MAE = \alpha$, $\angle MAC = \beta_2$ and $\angle MAG = \gamma_2$. Suppose AF meets BC at Y and AK meets BC at Z. So, we have $F = (r\cos\gamma_1, -r\sin\gamma_1)$ and $G = (r\cos\gamma_2, r\sin\gamma_2)$. (The negative sign in F is because F is below the x-axis.)

Thus, the equation of the circle Ω is $x^2 + y^2 - hx - ky = 0$, where:

$$h = r\frac{\sin\gamma_1 + \sin\gamma_2}{\sin(\gamma_1 + \gamma_2)}, k = r\frac{\cos\gamma_1 - \cos\gamma_2}{\sin(\gamma_1 + \gamma_2)}.$$

The coordinates of the center O of Ω are $(\frac{h}{2}, \frac{k}{2})$.

In isosceles $\triangle AFD$, we have $\angle AFD = \angle ADF = \frac{\pi}{2} + \frac{\gamma_1 - \alpha}{2}$. Next, $\angle ADM = \frac{\pi}{2} - \alpha$. So, $\angle FDY = \frac{\alpha + \gamma_1}{2}$. Next, as B, F, K and D are concyclic, $\angle KFD = \angle KBD = \frac{\pi}{2} - \beta_1$. Thus,

$\angle KYD = \angle KFD - \angle FDY = \frac{\pi}{2} - \left(\beta_1 + \frac{\alpha+\gamma_1}{2}\right) =: -\theta_1$. So, any point on the line FK can be written as:

$$F + a(\cos\left(\frac{\pi}{2} - \theta_1\right), \sin\left(\frac{\pi}{2} - \theta_1\right)) = (r\cos\gamma_1 + a\sin\theta_1, r\sin\gamma_1 + a\cos\theta_1),$$

where a is a real parameter.

By similar reasoning, any point on the line GL can be written as:

$$G + b(\cos\left(\frac{\pi}{2} - \theta_2\right), \sin\left(\frac{\pi}{2} - \theta_2\right)) = (r\cos\gamma_2 + b\sin\theta_2, r\sin\gamma_2 + b\cos\theta_2),$$

where b is a real parameter and $\theta_2 = \frac{\pi}{2} - \left(\beta_2 + \frac{\alpha+\gamma_2}{2}\right)$. Thus, to find the intersection of FK and GL, we need to solve the following system of equations for a and b:

$$(r\cos\gamma_1 + a\sin\theta_1, r\sin\gamma_1 + a\cos\theta_1) = (r\cos\gamma_2 + b\sin\theta_2, r\sin\gamma_2 + b\cos\theta_2).$$

$$\implies a = r\frac{(\cos\gamma_2 - \cos\gamma_1)\cos\theta_2 - (\sin\gamma_2 - \sin\gamma_1)\sin\theta_2}{\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2} = r\frac{\cos(\gamma_2+\theta_2) - \cos(\gamma_1+\theta_2)}{\sin(\theta_1-\theta_2)}.$$

Thus, the coordinates of the point X are:

$$\begin{aligned}
&r(\cos\gamma_1 + a\sin\theta_1, r\sin\gamma_1 + a\cos\theta_1)\\
&= r(\cos\gamma_1 + \frac{\cos(\gamma_2+\theta_2) - \cos(\gamma_1+\theta_2)}{\sin(\theta_1-\theta_2)}\sin\theta_1, \sin\gamma_1 + \frac{\cos(\gamma_2+\theta_2) - \cos(\gamma_1+\theta_2)}{\sin(\theta_1-\theta_2)}\cos\theta_1)\\
&= \frac{r}{\sin(\theta_1-\theta_2)}(\cos\gamma_1\sin(\theta_1-\theta_2) + \sin\theta_1\cos(\gamma_2+\theta_2) - \sin\theta_1\cos(\gamma_1+\theta_2),\\
&\qquad \sin\gamma_1\sin(\theta_1-\theta_2) + \cos\theta_1\cos(\gamma_2+\theta_2) - \cos\theta_1\cos(\gamma_1+\theta_2))\\
&= \frac{r}{\sin(\theta_1-\theta_2)}(\sin\theta_1\cos(\gamma_2+\theta_2) - \sin\theta_2\cos(\gamma_1+\theta_1), \cos\theta_1\cos(\gamma_2+\theta_2) - \cos\theta_2\cos(\gamma_1+\theta_1)),
\end{aligned}$$

where the last step follows by combining the first and third terms of both coordinates.

To the prove the claim, we need to show that the ratio of the coordinates of X is the same as the ratio of the coordinates of O, that is:

$$\frac{\sin\theta_1\cos(\gamma_2+\theta_2) - \sin\theta_2\cos(\gamma_1+\theta_1)}{\cos\theta_1\cos(\gamma_2+\theta_2) - \cos\theta_2\cos(\gamma_1+\theta_1)} = \frac{h}{k} = \frac{\sin\gamma_1 - \sin\gamma_2}{\cos\gamma_2 - \cos\gamma_1}$$

$$\iff (\cos\gamma_2 - \cos\gamma_1)(\sin\theta_1\cos(\gamma_2+\theta_2) - \sin\theta_2\cos(\gamma_1+\theta_1)) = (\sin\gamma_1 - \sin\gamma_2)(\cos\theta_1\cos(\gamma_2+\theta_2) - \cos\theta_2\cos(\gamma_1+\theta_1)).$$

Computing the LHS of the above expression:

$$\begin{aligned}
&2 \times LHS\\
&= 2(\cos\gamma_2 - \cos\gamma_1)(\sin\theta_1\cos(\gamma_2+\theta_2) - \sin\theta_2\cos(\gamma_1+\theta_1))\\
&= 2\cos\gamma_2\sin\theta_1\cos(\gamma_2+\theta_2) - 2\cos\gamma_1\sin\theta_1\cos(\gamma_2+\theta_2) - 2\cos\gamma_2\sin\theta_2\cos(\gamma_1+\theta_1) + 2\cos\gamma_1\sin\theta_2\cos(\gamma_1+\theta_1)\\
&= \sin(\theta_1+\gamma_2)\cos(\gamma_2+\theta_2) + \sin(\theta_1-\gamma_2)\cos(\gamma_2+\theta_2) - \sin(\theta_1+\gamma_1)\cos(\gamma_2+\theta_2) - \sin(\theta_1-\gamma_1)\cos(\gamma_2+\theta_2)\\
&- \sin(\theta_2+\gamma_2)\cos(\gamma_1+\theta_1) - \sin(\theta_2-\gamma_2)\cos(\gamma_1+\theta_1) + \sin(\theta_2+\gamma_1)\cos(\gamma_1+\theta_1) + \sin(\theta_2-\gamma_1)\cos(\gamma_1+\theta_1).
\end{aligned}$$

Similarly, computing the RHS:

$$\begin{aligned}
&2 \times RHS\\
&= 2(\sin\gamma_1 - \sin\gamma_2)(\cos\theta_1\cos(\gamma_2+\theta_2) - \cos\theta_2\cos(\gamma_1+\theta_1))\\
&= 2\sin\gamma_1\cos\theta_1\cos(\gamma_2+\theta_2) - 2\sin\gamma_2\cos\theta_1\cos(\gamma_2+\theta_2) - 2\sin\gamma_1\cos\theta_2\cos(\gamma_1+\theta_1) + 2\sin\gamma_2\cos\theta_2\cos(\gamma_1+\theta_1)\\
&= \sin(\theta_1+\gamma_1)\cos(\gamma_2+\theta_2) - \sin(\theta_1-\gamma_1)\cos(\gamma_2+\theta_2) - \sin(\theta_1+\gamma_2)\cos(\gamma_2+\theta_2) + \sin(\theta_1-\gamma_2)\cos(\gamma_2+\theta_2)\\
&- \sin(\theta_2+\gamma_1)\cos(\gamma_1+\theta_1) + \sin(\theta_2-\gamma_1)\cos(\gamma_1+\theta_1) + \sin(\theta_2+\gamma_2)\cos(\gamma_1+\theta_1) - \sin(\theta_2-\gamma_2)\cos(\gamma_1+\theta_1).
\end{aligned}$$

Thus, we get that:

$LHS - RHS$

$$
\begin{aligned}
&= \sin(\theta_1+\gamma_2)\cos(\gamma_2+\theta_2) - \sin(\theta_1+\gamma_1)\cos(\gamma_2+\theta_2) - \sin(\theta_2+\gamma_2)\cos(\gamma_1+\theta_1) + \sin(\theta_2+\gamma_1)\cos(\gamma_1+\theta_1) \\
&= 2\sin\left(\frac{\gamma_2-\gamma_1}{2}\right)(\cos\left(\theta_1+\frac{\gamma_1+\gamma_2}{2}\right)\cos(\gamma_2+\theta_2) - \cos\left(\theta_2+\frac{\gamma_1+\gamma_2}{2}\right)\cos(\gamma_1+\theta_1)) \\
&= 2\sin\left(\frac{\gamma_2-\gamma_1}{2}\right)(\cos(\theta_1+\gamma_1)\cos(\gamma_2+\theta_2)\cos\left(\frac{\gamma_2-\gamma_1}{2}\right) - \sin(\theta_1+\gamma_1)\cos(\gamma_2+\theta_2)\sin\left(\frac{\gamma_2-\gamma_1}{2}\right) \\
&\qquad\qquad - \cos(\theta_2+\gamma_2)\cos(\gamma_1+\theta_1)\cos\left(\frac{\gamma_1-\gamma_2}{2}\right) + \sin(\theta_2+\gamma_2)\cos(\gamma_1+\theta_1)\sin\left(\frac{\gamma_1-\gamma_2}{2}\right)) \\
&= -2\sin^2\left(\frac{\gamma_2-\gamma_1}{2}\right)(\sin(\theta_1+\gamma_1)\cos(\gamma_2+\theta_2) + \sin(\theta_2+\gamma_2)\cos(\gamma_1+\theta_1)) \\
&= -2\sin^2\left(\frac{\gamma_2-\gamma_1}{2}\right)\sin(\theta_1+\theta_2+\gamma_1+\gamma_2).
\end{aligned}
$$

Thus, to prove the claim, it suffices to show that $\sin(\theta_1+\theta_2+\gamma_1+\gamma_2) = 0$. This is equivalent to:

$$
\begin{aligned}
&\theta_1+\theta_2+\gamma_1+\gamma_2 = 0 \\
\iff\ &\beta_1-\beta_2 = \frac{1}{2}(\gamma_1-\gamma_2) \\
\iff\ &\sin(\beta_1-\beta_2) = \sin\left(\frac{\gamma_1-\gamma_2}{2}\right).
\end{aligned}
$$

To check this, we compute the coordinates of points B and C. Since the equation of BC is $x=1$, the y-coordinates are given by substituting $x=1$ in the equation of Ω:

$$
\begin{aligned}
&y^2 - ky + 1 - h = 0 \\
\implies\ &y = \frac{k \pm \sqrt{k^2+4h-4}}{2}.
\end{aligned}
$$

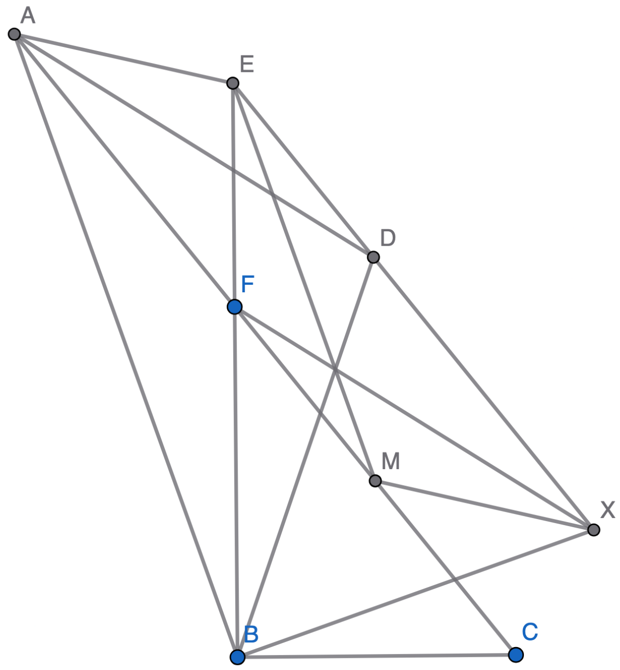
Thus, $B = (1, \frac{k-\sqrt{k^2+4h-4}}{2})$ and $C = (1, \frac{k+\sqrt{k^2+4h-4}}{2})$. Note that $M = (1,0)$. Thus, $AM = 1, BM = \frac{\sqrt{k^2+4h-4}-k}{2}$ and $CM = \frac{k+\sqrt{k^2+4h-4}}{2}$. Therefore, we can compute:

$$
\begin{aligned}
\sin(\beta_1-\beta_2) &= \sin\beta_1\cos\beta_2 - \cos\beta_1\sin\beta_2 \\
&= \frac{BM}{AB}\cdot\frac{AM}{AC} - \frac{AM}{AB}\cdot\frac{CM}{AC} \\
&= \frac{BM-CM}{AB\cdot AC} \\
&= \frac{-k}{\sqrt{1+(\frac{k-\sqrt{k^2+4h-4}}{2})^2}\sqrt{1+(\frac{k+\sqrt{k^2+4h-4}}{2})^2}} \\
&= \frac{-4k}{\sqrt{2k^2+4h-2k\sqrt{k^2+4h-4}}\sqrt{2k^2+4h+2k\sqrt{k^2+4h-4}}} \\
&= \frac{-2k}{\sqrt{(k^2+2h)^2-k^2(k^2+4h-4)}} \\
&= \frac{-k}{\sqrt{k^2+h^2}} \\
&= \frac{-r\frac{\sin(\frac{\gamma_2-\gamma_1}{2})}{\cos(\frac{\gamma_1+\gamma_2}{2})}}{\frac{r}{\cos(\frac{\gamma_1+\gamma_2}{2})}} \\
&= \sin\left(\frac{\gamma_1-\gamma_2}{2}\right),
\end{aligned}
$$

proving the claim.

9. IMO 2016 Problem 1

$\triangle BCF$ has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen such that $DA = DC$ and AC is the bisector $\angle DAB$. Point E is chosen such that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that BD, FX and ME are concurrent.



Solution: Let $\angle BFC = y$. Also, let $BF = c$ and $FC = b$. Since $AF = FB$, we have $\angle FBA = \angle BAF = \angle FAD = \angle DAE = \frac{y}{2}$.

We start by showing that the point D lies on the EX. To prove this, it suffices to show that:

$$\frac{AD}{\sin(\angle AEX)} = \frac{DE}{\sin(\angle DAE)}.$$

Now, $AD = \frac{1}{2}AC\sec(DAF) = \frac{b+c}{2}\sec(\frac{y}{2})$. Next, $\angle AEX = \pi - \angle FAE = \pi - y$. Finally, $DE = AE = \frac{1}{2}AD\sec(\angle DAE) = \frac{b+c}{4}\sec^2(\frac{y}{2}) = \frac{b+c}{4\cos^2(\frac{y}{2})} = \frac{b+c}{2(\cos y+1)} = \frac{b}{2}$. Then, we see that:

$$\frac{AD}{\sin(\angle AEX)} = \frac{\frac{b+c}{2}\sec(\frac{y}{2})}{\sin(y)} = \frac{b+c}{4\sin(\frac{y}{2})\cos^2(\frac{y}{2})} = \frac{DE}{\sin(\frac{y}{2})} = \frac{DE}{\sin(\angle DAE)},$$

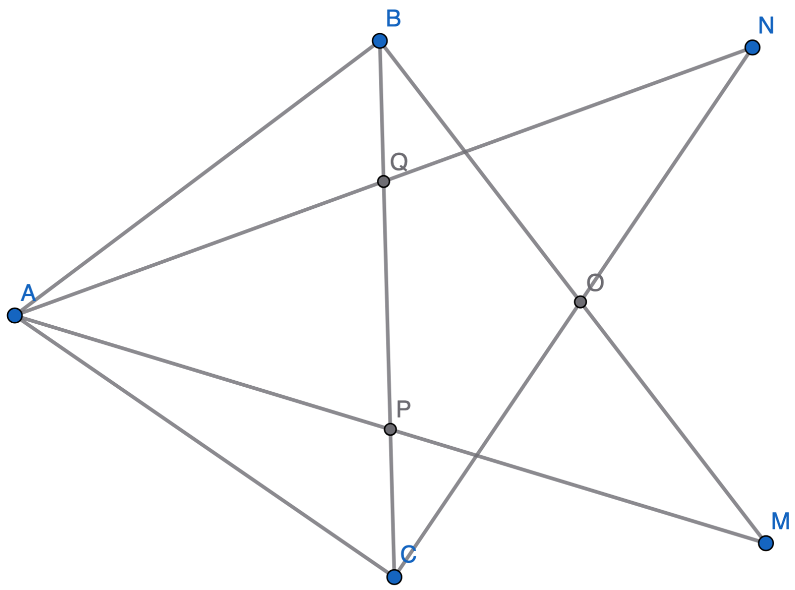
proving that D lies on EX.

Next, we show that F lies on BE. For this, it suffices to show that $y = \angle BFC = \angle AFE$. Since we already know that $\angle FAE = y$, this is equivalent to showing that $AE = EF$. For this, we use the cosine rule in $\triangle FAE$ to observe that $FE^2 = AE^2 + AF^2 - 2AE \cdot AF\cos(\angle FAE)$. So, it suffices to show that $AF = 2AE\cos(\angle FAE)$. Note that $AF = c$ whereas $2AE\cos(\angle FAE) = b\cos y = c$, completing the proof.

Next, we note that $EX = AM = AF+FM = c+\frac{b}{2}$. On the other hand, $EB = EF+FB = \frac{b}{2}+c$, showing that $EB = EX$. Since we already know that $EF = EA = ED$, this also implies that $DX = FB$. Hence, applying Ceva's theorem to $\triangle EBX$, the the lines EM, BD and FX will be concurrent if any only if EM bisects the side BX, which is equivalent to EM bisecting $\angle BEX$, since the triangle is isosceles. To that end, we observe that $MB = \frac{b}{2} = AE - MX$. Hence, $\triangle EMB \cong \triangle EMX$, which proves that EM is the angle bisector, thus proving the claim.

10. IMO 2014 Problem 4

Points P and Q lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.



Solution: Note that $\angle AQP = \angle APQ = \angle CAB$. Thus $AQ = AP$. We establish a coordinate system such that $A = (0,0), B = (1,b), C = (1,c), Q = (1,q)$ and $P = (1,-q)$ with $b, q > 0$ and $c < 0$. Since $\angle CAQ = \angle ABC$, we must have:

$$\frac{q-c}{1+qc} = \frac{1}{b} \implies bq - bc - qc = 1 \implies q = \frac{1+bc}{b-c}.$$

The equation of the circumcircle ABC is given by:

$$x^2 + y^2 - (1-bc)x - (b+c)y = 0.$$

Next, we have that $N = 2Q = (2, 2q)$ and $M = 2P = (2, -2q)$. Then we have the equation of BM:

$$\frac{y-b}{x-1} = \frac{b+2q}{-1} \implies y + (b+2q)x = 2b + 2q.$$

Similarly, the equation of CN is:

$$y + (c-2q)x = 2c - 2q.$$

The angle between these two lines is the arctan of:

$$\frac{(b+2q)-(c-2q)}{1+(b+2q)(c-2q)} = \frac{b-c+4q}{1+bc+2cq-2bq-4q^2} = \frac{b-c+4q}{-1-bc-4q^2}.$$

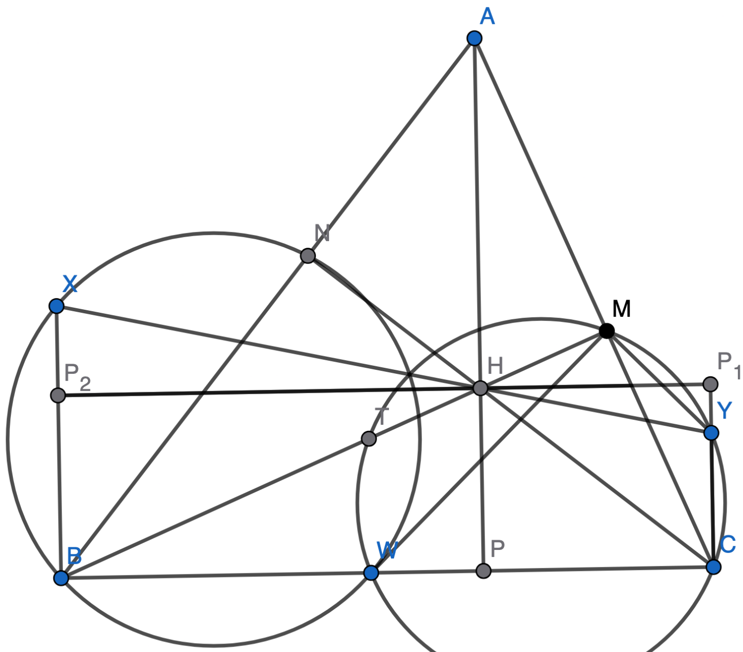
To show that A, B, O and C are concyclic, it suffices to show that this is equal to $-\tan(\angle BAC)$, that is:

$$\begin{aligned} \frac{b-c+4q}{-1-bc-4q^2} &= -\frac{b-c}{1+bc} \\ \iff 4(1+bc)q &= 4q^2(b-c) \\ \iff q &= \frac{1+bc}{b-c}, \end{aligned}$$

which was already shown, thus completing the proof.

11. IMO 2013 Problem 4

Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.



Solution: Let $a = BC, b = CA$ and $c = AB$. Let R be the circumradius of $\triangle ABC$. Construct P_1P_2 parallel to BC through the point H. To prove the required claim, it suffices to show that $\angle XHP_2 = \angle YHP_1$.

We have $\angle HMC = \angle WMY = \frac{\pi}{2}$. Therefore, $\angle TMW = \angle CMY$, which implies that $TW = CY$. Consider the power of the point B with respect to the circle ω_2:

$$BT \cdot BM = BW \cdot BC.$$

$$\begin{aligned} BT &= \frac{BW \cdot BC}{BM} \\ &= \frac{aBW}{a \sin C} = \frac{BW}{\sin C}. \end{aligned}$$

Next, note that $\angle TBW = \frac{\pi}{2} - C$. Applying cosine law to $\triangle BWT$:

$$\begin{aligned} TW^2 &= BT^2 + BW^2 - 2BT \cdot BW \cos(TBW) \\ &= BT^2 + BW^2 - 2BT \cdot BW \sin C \end{aligned}$$

$$
\begin{aligned}
&= \frac{BW^2}{\sin^2 C} + BW^2 - 2\frac{BW}{\sin C} \cdot BW \sin C \\
&= BW^2 \csc^2 C - BW^2 \\
&= BW^2 \cot^2 C.
\end{aligned}
$$

Thus, $TW = BW \cot C$. This implies that $CY = BW \cot C$. Similarly, we get that $BX = CW \cot B$.

As $XB \perp BC$, we must have $XB \perp P_1P_2$ since $P_1P_2 \parallel BC$. Similarly, $YC \perp P_1P_2$. Therefore,

$$
\begin{aligned}
\tan(\angle XHP_2) &= \frac{XP_2}{HP_2} \\
&= \frac{BX - BP_2}{HP_2} \\
&= \frac{BX - HP}{BP} \\
&= \frac{CW \cot B - 2R \cos B \cos C}{2R \sin C \cos B} \\
&= \frac{CW}{2R \sin B \sin C} - \cot C.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\tan(\angle YHP_1) &= \frac{YP_1}{HP_1} \\
&= \cot B - \frac{BW}{2R \sin B \sin C}.
\end{aligned}
$$

Then, we see that:

$$
\begin{aligned}
\tan(\angle XHP_2) - \tan(\angle YHP_1) &= \frac{BW + CW}{2R \sin B \sin C} - \cot B - \cot C \\
&= \frac{a}{2R \sin B \sin C} - \frac{\sin(B + C)}{\sin B \sin C} \\
&= \frac{\sin A}{\sin B \sin C} - \frac{\sin A}{\sin B \sin C} \\
&= 0,
\end{aligned}
$$

which shows that $\angle XHP_2 = \angle YHP_1$, completing the proof.