ALMOST COMMUTING SCHEME OF SYMPLECTIC MATRICES AND QUANTUM HAMILTONIAN REDUCTION

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Abstract. Losev introduced the scheme X of almost commuting matrices in $\mathfrak{g} = \mathfrak{sp}(V)$ for a symplectic vector space V and discussed its algebro-geometric properties. We construct a Lagrangian subscheme X^{nil} of X and show that it is a complete intersection of dimension $\dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$ and compute its irreducible components.

We also study the quantum Hamiltonian reduction of the algebra $\mathcal{D}(\mathfrak{g})$ of differential operators on the Lie algebra $\mathfrak{g}$ tensored with the Weyl algebra, with respect to the action of the symplectic group, and show that it is isomorphic to the spherical subalgebra of a certain rational Cherednik algebra of Type C.

1. Introduction

1.1. Let $V := \mathbb{C}^{2n}$ be a symplectic vector space and let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{sp}(V) = \mathfrak{sp}_{2n}$. The almost commuting scheme X of $\mathfrak{g}$ was defined by Losev in [Los21] as the closed subscheme of $\mathfrak{g} \times \mathfrak{g} \times V$ defined by the ideal I generated by the matrix entries of $[x, y] + i^2$, i.e., by all functions of the form $(x, y, i) \mapsto \lambda([x, y] + i^2)$ for $\lambda \in \mathfrak{g}^*$. Here, we use the fact that $\mathrm{Sym}^2(V)$ can be identified with $\mathfrak{sp}(V)$ to view i^2 as an element of $\mathfrak{sp}(V)$.

The geometrical properties of X were studied by Losev who showed that:

Theorem 1.1.1 ([Los21])**.** *The scheme X is reduced, irreducible and a complete intersection of dimension $2n^2 + 3n$.*

In this paper, we consider the reduced subscheme X^{nil} of X defined as:

$$X^{nil} := \{(x, y, i) \in \mathfrak{g} \times \mathfrak{g} \times V : [x, y] + i^2 = 0 \text{ and } y \text{ is nilpotent}\}.$$

This definition is motivated by Lusztig's notion of *character sheaves* (see [Lus86; Lus91]). By definition, a character sheaf on a reductive algebraic group K is an $Ad(K)$-equivariant perverse sheaf M on K such that the corresponding characteristic variety lies in $K \times \mathcal{N} \subseteq K \times \mathfrak{k}^*$, where $\mathfrak{k} = Lie(K)$ and $\mathcal{N} \subseteq \mathfrak{k}^* \simeq \mathfrak{k}$ is the nilpotent cone. Constructions analogous to the one above were done in Type A in [GG06; FG10a; FG10b] to provide *'mirabolic'* analogs of Lusztig's sheaves.

We describe some notation. It is known (for example, see [CM93, Theorem 5.1.3]) that nilpotent conjugacy classes in $\mathfrak{g}$ are parametrized by the partitions λ of $2n$ in which every odd part appears an even number of times. Let P_n be the set of all such partitions and let $\mathcal{P}_n$ denote the subset of those partitions in P_n in which all the parts are even. For each $\lambda \in P_n$, let $\mathcal{N}_\lambda$ denote the corresponding nilpotent conjugacy class in $\mathfrak{g}$. Define for each $\lambda \in P_n$:

$$X_\lambda := \{(x, y, i) \in X^{nil} : y \in \mathcal{N}_\lambda\}.$$

Note that we can identify $\mathfrak{g} \times \mathfrak{g} \times V$ with $T^*(\mathfrak{g} \times L)$ using the trace form on $\mathfrak{g}$, where L is a Lagrangian subspace of V. This gives $\mathfrak{g} \times \mathfrak{g} \times V$ a natural symplectic structure. Our first main result reads:

Theorem 1.1.2. *(a) The scheme X^{nil} is a complete intersection in $\mathfrak{g} \times \mathfrak{g} \times V$ of dimension $2n^2 + 2n$. The irreducible components of X^{nil} are exactly given by the $\overline{X_\lambda}$ for $\lambda \in \mathcal{P}_n$.*
(b) With the standard symplectic structure, X^{nil} is a Lagrangian subscheme of $\mathfrak{g} \times \mathfrak{g} \times V$.

Such a Lagrangian subscheme was constructed analogously in [GG06] in the context of the almost commuting scheme of the Lie algebra $\mathfrak{gl}_n$. Using this theorem, we can provide an independent proof of Theorem 1.1.1 in the style of [GG06].

In the second half of the paper, we discuss some Hamiltonian reduction problems arising in the context of the scheme X and some other related schemes. For this, we define the following subschemes of $\mathfrak{g} \times \mathfrak{g} = \mathrm{Spec}(\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]) = \mathrm{Spec}(\mathbb{C}[x, y])$. We consider the commuting scheme C which is the (not necessarily reduced) subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined as $\mathrm{Spec}(\mathbb{C}[x, y]/I_1)$, where I_1 is the ideal generated by the matrix entries of the commutator $[x, y]$. Next, we define the scheme A to be the (not necessarily reduced) subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined as $\mathrm{Spec}(\mathbb{C}[x, y]/J)$, where J is the ideal generated by all the 2×2 minors of $[x, y]$.

Note that the set of $\mathbb{C}$-points of the underlying reduced subscheme of C consists of pairs of elements of $\mathfrak{g}$ that commute with each other, whereas that of A consists of pairs of elements of $\mathfrak{g}$ whose commutator has rank lesser than or equal to one. The commuting scheme C is of wide interest, and its geometrical properties (most notably, its reducedness) are largely unknown. It is known that C is irreducible (see [Ric79]).

The schemes X, C and A have an action of the group $G = \mathrm{Sp}(V)$ obtained by the adjoint action on $\mathfrak{g}$ and the natural action on V. Hence, we can consider the respective categorical quotients of these schemes by the action of G. While it isn't known that C is reduced, it is shown in [Los21] that there's an isomorphism:

$$C//G \longrightarrow X//G,$$

which implies that $C//G$ is reduced. (That $C//G$ is reduced was deduced independently in [CC21] slightly earlier, by proving a version of the Chevalley restriction theorem for the commuting scheme of $\mathfrak{g}$.) We extend this isomorphism to show that:

Theorem 1.1.3. *We have an isomorphism of schemes:*

$$X//G \longrightarrow A//G.$$

In particular, the scheme $A//G$ is reduced.

An analog of this isomorphism for the Lie algebra $\mathfrak{gl}_n$ was proved in [GG06].

The above categorical quotients can (and will) all be viewed as classical Hamiltonian reductions of certain schemes under the action of the group G:

- The scheme $X//G$ is the reduction of the scheme $\mathfrak{g} \times \mathfrak{g} \times V$ with respect to G at 0.
- The scheme $A//G$ is the reduction of the scheme $\mathfrak{g} \times \mathfrak{g}$ with respect to G at the closure of the orbit of rank 1 matrices in $\mathfrak{g} \simeq \mathfrak{g}^*$.
- The scheme $C//G$ is the reduction of the scheme $\mathfrak{g} \times \mathfrak{g}$ with respect to G at 0.

So, we can try to study the non-commutative or quantum analogs of these reduction problems. For this, let $\mathcal{U}\mathfrak{g}$ denote the universal enveloping algebra of $\mathfrak{g}$, let $\mathcal{D}(\mathfrak{g})$ denote the algebra of polynomial differential operators on $\mathfrak{g}$ and let W_{2n} denote the Weyl algebra on $2n$ variables, which is the algebra of polynomial differential operators on the affine n-space. (This affine space will be the Lagrangian subspace L in our case.) Then, $\mathcal{D}(\mathfrak{g})$ is a quantization of $\mathbb{C}[\mathfrak{g} \times \mathfrak{g}] \simeq \mathbb{C}[\mathfrak{g} \times \mathfrak{g}^*]$, whereas W_{2n} is a quantization of $\mathbb{C}[V]$. Both the algebras $\mathcal{D}(\mathfrak{g})$ and W_{2n} have a natural $\mathfrak{g}$-action (and thus, so does their tensor product.) So, we get quantum co-moment maps:

$$\Theta_1 : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{g})$$

$$\Theta_2 : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{g}) \otimes W_{2n}.$$

(These maps are elaborated upon in §3.3.) Then, we can consider the following non-commutative algebras:

- The reduction $\Big(\mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g}) \cdot \Theta_1(\mathfrak{g}))\Big)^{\mathfrak{g}}$ of $\mathcal{D}(\mathfrak{g})$ at the augmentation ideal of $\mathcal{U}\mathfrak{g}$.
- The reduction $\Big(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}/(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}$ of $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ at the augmentation ideal of $\mathcal{U}\mathfrak{g}$.
- The reduction $\Big(\mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g}) \cdot \Theta_1(\mathcal{K})\Big)^{\mathfrak{g}}$ at the unique primitive ideal $\mathcal{K} \subseteq \mathcal{U}\mathfrak{g}$ such that $\mathrm{gr}(\mathcal{K}) \subseteq \mathbb{C}[\mathfrak{g}^*]$ is the defining ideal of the orbit of rank 1 matrices.

(Each of these algebras is discussed in detail in §3.3.)

The first of these, the algebra $\Big(\mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g})\cdot\Theta_1(\mathfrak{g}))\Big)^{\mathfrak{g}}$, has been studied classically by Harish-Chandra (see [Har64]) who constructed a surjective algebra homorphismm called the "radial parts" homomorphism:

$$\Phi : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h})^W,$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and W is the Weyl group. The kernel of this homomorphism was shown to be precisely $(\mathcal{D}(\mathfrak{g})\cdot\Theta_1(\mathfrak{g})))^{\mathfrak{g}}$ in the works of Wallach [Wal93] and Lavasseur-Stafford [LS95; LS96], implying that the above algebra is isomorphic to $\mathcal{D}(\mathfrak{h})^W$.

In this paper, we'll discuss the other two quantum Hamiltonian reduction problems. For this, we recall the rational Cherednik algebra H_c of Type C, first defined in [EG00]. Here, the parameter $c = (c_{long}, c_{short})$ lies in $\mathbb{C}^2$. Let $e = \frac{1}{|W|}\sum_{w\in W} w$ be the averaging idempotent of the Weyl group W and consider the spherical subalgebra $eH_ce \subseteq H_c$ of the Cherednik algebra. (The notation is elaborated on in §3.2.)

We prove the following theorem about these algebras:

Theorem 1.1.4. *We have algebra isomorphisms:*

$$\left(\frac{\mathcal{D}(\mathfrak{g})\otimes W_{2n}}{(\mathcal{D}(\mathfrak{g})\otimes W_{2n})\cdot\Theta_2(\mathfrak{g})}\right)^{\mathfrak{g}} \simeq \left(\frac{\mathcal{D}(\mathfrak{g})}{\mathcal{D}(\mathfrak{g})\cdot\Theta_1(\mathcal{K})}\right)^{\mathfrak{g}} \simeq eH_ce,$$

for the parameter $c = (-1/4, -1/2)$.

Analogs of this theorem were proved in [EG00] and [GG06] in the $\mathfrak{gl}_n$-setting.

The first part of the isomorphism involves a generalization of the "radial parts" construction defined in [EG00]. The second part of the isomorphism is proved in a slightly more general setting: Consider any affine variety Y with an action of $G = Sp(V)$ such that the center acts trivially. Then, the quantum Hamiltonian reduction of $\mathcal{D}(Y)$ at the primitive ideal $\mathcal{K} \subseteq \mathcal{U}\mathfrak{g}$ is isomorphic to the quantum Hamiltonian reduction of $\mathcal{D}(Y)\otimes W_{2n}$ at the augmentation ideal of $\mathcal{U}\mathfrak{g}$ (see Theorem 3.5.7.)

Using this theorem and the formalism in [GG06, §7], we describe the construction of a quantum Hamiltonian reduction functor from a certain category $\mathcal{C}$ of holonomic $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-modules supported on X^{nil} to $\mathcal{O}(eH_ce)$, the category $\mathcal{O}$ of the spherical Cherednik algebra eH_ce.

1.2. **Organization.** Here, we give more details about the contents of the paper.

In §2, we prove Theorem 1.1.2. The proof of the fact that X^{nil} is Lagrangian is by embedding it into the Lagrangian subscheme defined in [GG06]. The proof of the rest of the theorem will be seen to be a consequence of this fact and some elementary $\mathfrak{sl}_2$-theory.

In §3.1, we prove Theorem 1.1.3 in a slightly more general setting. In §3.2, we recall some definitions and results about rational Cherednik algebras that we'll be using. In §3.3, we define the algebras and construct algebra homomorphisms between the algebras alluded to in Theorem 1.1.4. In §3.4 and §3.5, we prove that these homomorphisms are indeed isomorphisms. Finally, in §3.6, we note some results about the category $\mathcal{C}$ and provide the construction of an exact functor from $\mathcal{C}$ to $\mathcal{O}(eH_ce)$.

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2. The nilpotent subscheme

In this section, we consider the reduced subscheme of X defined via:

$$X^{nil} := \{(x, y, i) \in \mathfrak{g}\times\mathfrak{g}\times V : [x, y] + i^2 = 0 \text{ and } y \text{ is nilpotent}\}.$$

In §2.1, we'll prove that this is a Lagrangian subscheme of $\mathfrak{g} \times \mathfrak{g} \times V$. We then use this result to provide a new proof of Theorem 1.1.1. Next, in §2.2, we compute the irreducible components of X^{nil}.

2.1. **Lagrangian subscheme.** To state the precise result, we first describe a symplectic structure on $\mathcal{X} = \mathfrak{g} \times \mathfrak{g} \times V$, which we define as $\omega = \omega_1 + \omega_2$. Here, ω_1 is the symplectic form on $\mathfrak{g} \times \mathfrak{g}$ obtained by identifying it with $T^*(\mathfrak{g})$ using the trace form on $\mathfrak{g}$ and ω_2 is the form on the symplectic vector space V. Next, on the scheme $\mathcal{M} = \mathfrak{gl}(V) \times \mathfrak{gl}(V) \times V \times V^*$, we have a symplectic form ω' obtained by identifying it with $T^*(\mathfrak{gl}(V) \times V)$ using the trace form on $\mathfrak{gl}(V)$.

Theorem 2.1.1. *The scheme X^{nil} is a Lagrangian subscheme of $\mathcal{X}$.*

Recall that a (possibly singular) subscheme Z of $\mathcal{X}$ is said to be Lagrangian if at any smooth point p of Z, the tangent space T_pZ is a Lagrangian subspace of $T_p\mathcal{X}$.

Proof. The scheme X^{nil} is a closed subscheme of $\mathcal{X}$ defined using $\dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$ equations, and so, $\dim(X^{nil}) \geq \dim(\mathcal{X}) - (\dim(\mathfrak{g}) + \frac{1}{2}\dim(V)) = \dim(\mathfrak{g}) + \frac{1}{2}\dim(V) = \frac{1}{2}\dim(\mathcal{X})$. Therefore, to prove the theorem, it suffices to show that X^{nil} is isotropic. For this, consider the embedding:

$$\Phi : \mathcal{X} = \mathfrak{g} \times \mathfrak{g} \times V \longrightarrow \mathfrak{gl}(V) \times \mathfrak{gl}(V) \times V \times V^* = \mathcal{M}$$

$$(x, y, i) \mapsto (x, y, i_1, i_2),$$

where $i_1 = i/2$ and i_2 is the symplectic dual of i in V. (That is, i_2 is the image of i in V^* under the identification $V \simeq V^*$ using the symplectic form.)

Recall from [GG06] the scheme of almost commuting matrices $M \subseteq \mathcal{M}$:

$$M := \{(x, y, i, j) \in \mathfrak{gl}(V) \times \mathfrak{gl}(V) \times V \times V^* : [x, y] + ij = 0\}.$$

Also defined in [GG06] was the closed subscheme M^{nil} of $\mathcal{M}$ obtained by stipulating y to be nilpotent. Then, under the map Φ, we have $\Phi(X) \subseteq M$ and $\Phi(X^{nil}) \subseteq M^{nil}$.

We claim that with the symplectic forms ω and ω' defined above, the map Φ turns out to be a symplectic embedding. To see this, we first observe that we can express the form ω' as a sum $\omega'_1 + \omega'_2$, where ω'_1 is the symplectic form on $\mathfrak{gl}(V) \times \mathfrak{gl}(V)$ identified with $T^*(\mathfrak{gl}(V))$ and ω'_2 is the symplectic form on $V \times V^*$ identified with $T^*(V)$.

Then, it is clear that $\omega_1 = \Phi^*(\omega'_1)$. Next, if i, j are two vectors in V, then we have the computation:

$$\omega'_2((i_1, j_1), (i_2, j_2)) = j_1(i_2) - j_2(i_1) = \frac{1}{2}\omega_2(i, j) - \frac{1}{2}\omega_2(j, i) = \omega_2(i, j),$$

which shows that Φ preserves the symplectic structure.

By [GG06, Theorem 1.1.4], we know that that M^{nil} is a Lagrangian subscheme of $\mathcal{M}$. In particular, it is isotropic. Therefore, by [CG97, Theorem 1.3.30], we get that $\Phi(X^{nil})$ is an isotropic subscheme of $\mathcal{M}$, proving that X^{nil} is an isotropic subscheme of $\mathcal{X}$. $\square$

Corollary 2.1.2. *The scheme X is a complete intersection of dimension* $\dim(\mathfrak{g}) + \dim(V)$.

Proof. The scheme X^{nil} is obtained from X by imposing exactly $n = \frac{1}{2}\dim(V)$ equations. Hence, as $\dim(X^{nil}) = \frac{1}{2}\dim(\mathcal{X}) = \dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$, we must have that $\dim(X) \leq \dim(X^{nil}) + \frac{1}{2}\dim(V) = \dim(\mathfrak{g}) + \dim(V)$. But, the scheme X is obtained from $\mathcal{X}$ by imposing $\dim(\mathfrak{g})$ equations, and so, $\dim(X) \geq \dim(\mathcal{X}) - \dim(\mathfrak{g}) = \dim(\mathfrak{g}) + \dim(V)$. Therefore, X is a complete intersection of dimension $\dim(\mathfrak{g}) + \dim(V)$. $\square$

In fact, we can generalize Theorem 2.1.1 as follows. Let $\mathfrak{h} \subseteq \mathfrak{g}$ denote a Cartan subalgebra of $\mathfrak{g}$ and let W be the Weyl group. Consider the composition map $\phi : \mathfrak{g} \to \mathfrak{g}//G \to \mathfrak{h}//W$, where the first map is the quotient map and the second one is the Chevalley restriction isomorphism. Then, we can consider the morphism:

$$\pi : X \longrightarrow \mathfrak{h}//W,$$

that sends a triple (x, y, i) to $\phi(y)$. It is clear that $X^{nil} = \pi^{-1}(0)$.

Proposition 2.1.3. *All the fibers of the map π are Lagrangian subschemes of $\mathcal{X}$ and have dimension* $\dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$.

Proof. For any $x \in \mathfrak{h}//W$, we have the dimension inequality:

$$\dim(\pi^{-1}(x)) \geq \dim(X) - \dim(\mathfrak{h}//W) = \dim(\mathfrak{g}) + \frac{1}{2}\dim(V) = \frac{1}{2}\dim(\mathcal{X}).$$

Therefore, to show that $\pi^{-1}(x)$ is Lagrangian, it suffices to prove that it is isotropic.

We consider the symplectic embedding $\Phi : \mathcal{X} \to \mathcal{M}$ defined in the proof of Theorem 2.1.1. By Corollary 2.3.4 of [GG06], the image of $\pi^{-1}(x)$ in $\mathcal{M}$ lies inside a Lagrangian subscheme of $\mathcal{M}$. As a result, we conclude that the image of $\pi^{-1}(x)$ must be an isotropic subscheme of $\mathcal{M}$, showing that $\pi^{-1}(x)$ must itself be isotropic. □

Remark 2.1.4. *In fact, by adapting the proof of Proposition 2.3.2 of [GG06], we can prove the following statment: Consider the map:*

$$\mu \times \pi : \mathcal{X} = \mathfrak{g} \times \mathfrak{g} \times V \longrightarrow \mathfrak{g} \times \mathfrak{h}//W,$$

that maps a triple (x, y, i) to the pair $([x, y] + i^2, \phi(y))$. Here, μ is the moment map for the Hamiltonian G-action on $\mathcal{X}$. Then, the map $\mu \times \pi$ is a flat morphism. As a corollary of this fact, we also get that the map $\mu : \mathcal{X} \to \mathfrak{g}$ if flat.

With this, we are ready to prove Losev's theorem. Define the scheme:

$$X^{reg} = \{(x, y, i) \in X : y \text{ is regular, semisimple.}\}$$

In other words, $X^{reg} = \pi^{-1}(\mathfrak{h}^{reg})$, where $\mathfrak{h}^{reg}$ is the regular semisimple locus of $\mathfrak{h}$. By Lemma 2.9 of [Los21], X^{reg} is an irreducible scheme.

Theorem 2.1.5. *(1) We have $\overline{X^{reg}} = X$. In particular, the scheme X is irreducible.*
(2) X is a reduced, complete intersection of dimension $\dim(\mathfrak{g}) + \dim(V)$.

Proof. By Corollary 2.1.2, we already know that X is a complete intersection of $\dim(\mathfrak{g}) + \dim(V)$. Consider the big diagonal $\Delta = (\mathfrak{h} \setminus \mathfrak{h}^{reg})//W$, which is a closed subscheme of $\mathfrak{h}//W$ of codimension 1. Then, we have the equality:

$$X = \overline{X^{reg}} \cup \pi^{-1}(\Delta).$$

Since $\dim(\Delta) = \dim(\mathfrak{h}//W) - 1 = \frac{1}{2}\dim(V) - 1$, by Proposition 2.1.3, we have $\dim(\pi^{-1}(\Delta)) \leq \dim(\mathfrak{g}) + \dim(V) - 1$. However, as X is a complete intersection, any irreducible component must have dimension exactly $\dim(\mathfrak{g}) + \dim(V)$. Therefore, we conclude that $\overline{X^{reg}}$ must be on the only irreducible component, proving that $X = \overline{X^{reg}}$.

Finally, we note that the action of the group G is free on X^{reg}, and thus, X is generically reduced. Therefore, as X is a complete intersection, it must be Cohen-Macaulay, and thus, we conclude that X is reduced. □

2.2. **Irreducible components of X^{nil}.** Let P_n denote the set of all partitions of $2n$ where every odd part occurs an even number of times. Let $\mathcal{P}_n \subseteq P_n$ be the subset of those partitions where each part is even. For any $\lambda \in P_n$, let $\mathcal{N}_\lambda$ denote the corresponding nilpotent conjugacy class of $\mathfrak{g}$ and define:

$$X_\lambda := \{(x, y, i) \in X^{nil} : y \in \mathcal{N}_\lambda\}.$$

Then, it is clear that we have the following disjoint union:

$$X^{nil} = \bigcup_{\lambda \in P_n} X_\lambda.$$

As X^{nil} has been shown to be a complete intersection, every irreducible component has dimension exactly $\dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$. So, in order to complete the proof of Theorem 1.1.2(a) which asserts that the irreducible components of X^{nil} are given exactly by the closures of those X_λ for which $\lambda \in \mathcal{P}_n$, we need to prove the following proposition:

Theorem 2.2.1. *(1) For each $\lambda \in \mathcal{P}_n$, we have* $\dim(X_\lambda) = \dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$.

(2) For each $\lambda \in P_n \setminus \mathcal{P}_n$, we have $\dim(X_\lambda) < \dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$.

Proof. Let y be any nilpotent element in $\mathfrak{g}$. Let $\mathbb{O}$ denote the minimal orbit consisting of rank one elements in $\mathfrak{g}$. Define the reduced schemes: (Also defined in [Los21])

$$X_y = \{(x,i) \in \mathfrak{g} \times V : [x,y] + i^2 = 0\},$$
$$\underline{X_y} = \{(x,z) \in \mathfrak{g} \times \overline{\mathbb{O}} : [x,y] + z = 0\},$$
$$Y_y = \overline{\mathbb{O}} \cap \{[x,y] : x \in \mathfrak{g}\}.$$

We have maps:

$$\rho_1 : X_y \longrightarrow \underline{X_y}$$
$$(x,i) \mapsto (x,i^2),$$

and

$$\rho_2 : \underline{X_y} \longrightarrow Y_y$$
$$(x,z) \mapsto z.$$

The map ρ_1 is finite, with either one or two points in the fiber at any point depending on whether i^2 is zero or not, respectively. The map ρ_2 is an affine bundle map which has fibers of dimension equal to $\dim(\mathfrak{z}_\mathfrak{g}(y))$. Thus, we get that $\dim(X_y) = \dim(\mathfrak{z}_\mathfrak{g}(y)) + \dim(Y_y)$. Further, if $y \in \mathcal{N}_\lambda$ for some $\lambda \in P_n$, we have that $\dim(X_\lambda) = \dim(X_y) + \dim(\mathcal{N}_\lambda) = \dim(\mathfrak{g}) + \dim(Y_y)$.

So, in order to prove the theorem, we are required to show that $\dim(Y_y) = \frac{1}{2}\dim(V)$ for $\lambda \in \mathcal{P}_n$ and $\dim(Y_y) < \frac{1}{2}\dim(V)$ for $\lambda \notin \mathcal{P}_n$. For a fixed nilpotent y, by Jacobson-Morozov theorem, we can find an $\mathfrak{sl}_2$-triple (e,f,h) in $\mathfrak{g}$ with $e = y$. Identifying the subspace $\langle e,f,h\rangle \subseteq \mathfrak{g}$ with $\mathfrak{sl}_2$, we get an $\mathfrak{sl}_2$-action on the vector space V. By $\mathfrak{sl}_2$-theory, the element h acts semisimply on V with integer eigenvalues. Let $V = V_- \oplus V_0 \oplus V_+$ be the decomposition of V, such that V_-, V_0, V_+ denote the span of negative, zero and positive eigenspaces of h respectively.

Now, we consider the identification of $\mathfrak{sl}_2$-representations $\mathfrak{g} = \mathfrak{sp}(V) = \mathrm{Sym}^2(V)$. By Lemma 2.2.2, for any $v \in V$, we have:

$$v \in V_+ \iff v^2 = \mathrm{ad}_e(x) = \mathrm{ad}_y(x) = [y,x] \text{ for some } x \in \mathfrak{g}.$$

Hence, we conclude that $\dim(Y_y) = \dim(V_+)$. As there is a one-to-one correspondence between positive and negative eigenvectors of h, we have that $\dim(V_-) = \dim(V_+)$, and so, $\dim(Y_y) \leq \frac{1}{2}\dim(V)$. The equality holds exactly when $V_0 = 0$, that is, when h has no zero eigenvalues. Zero eigenvalues for h occur only in irreducible representations having odd dimension. Hence, the dimension inequality becomes an equality exactly when the space V decomposes into a sum of irreducibles each having even dimension. However, irreducible components of V correspond exactly to the Jordan blocks of e. Therefore, the dimension equality holds exactly when each Jordan block of y having even size, that is $y \in \mathcal{N}_\lambda$ for some $\lambda \in \mathcal{P}_n$, thus completing the proof. □

Lemma 2.2.2. *Let V be a representation of $\mathfrak{sl}_2(\mathbb{C}) = \langle e,f,h\rangle$. Let $W = \mathrm{Sym}^2(V)$ with the action of $\mathfrak{sl}_2$ induced from the one on V. Then, for any $v \in V$, we have that $v^2 = e \cdot w$ for some $w \in W$ if and only if v lies in the span of the positive eigenspaces of h in V.*

Proof. Suppose v lies in the span of the positive eigenspaces of h in V. Then, we must have that v^2 lies in the span of the positive eigenspaces of h in W, and thus, by $\mathfrak{sl}_2$-theory, we must have $v^2 = e \cdot w$ for some $w \in W$.

Now, we prove the converse. Henceforth, instead of working with $\mathrm{Sym}^2(V)$, we will work with $W = V \otimes V$, for notational convenience. As $V \otimes V$ contains $\mathrm{Sym}^2(V)$ as a proper subrepresentation, if $v^2 = e \cdot w$ for some $w \in \mathrm{Sym}^2(V)$, we must have that $v \otimes v = e \cdot \tilde{w}$ for some $\tilde{w} \in V \otimes V$.

Let $V = \oplus_i V_i$ be the decomposition of V into a direct sum of $\mathfrak{sl}_2$-representations. Then, we get a corresponding decomposition $v = \sum_i v_i$ such that $v_i \in V_i$ for all i. Then, as $v \otimes v = e \cdot w$, it is clear that there exist $w_i \in V_i \otimes V_i$ such that $v_i \otimes v_i = e \cdot w_i$ for each i. Therefore, without loss of generality, we can assume that V is irreducible.

Suppose $\dim(V) = n+1$ for some $n \in \mathbb{Z}_{\geq 0}$. Then, there exists an h-eigenbasis $\{x_0, x_1, \dots, x_n\}$ of V such that the e and f actions are given by:

$$e \cdot x_i = x_{i+1}, f \cdot x_j = x_{j-1},$$

for all $0 \leq i \leq n-1$ and $1 \leq j \leq n$.

Write $v = \sum_i c_i x_i$ for $c_i \in \mathbb{C}$. Then, we have $v \otimes v = \sum_{i,j} c_i c_j x_i \otimes x_j$. Pick the smallest k such that $c_k \neq 0$. Then, $c_k^2 x_k \otimes x_k$ is the summand in $v \otimes v$ having strictly the smallest eigenvalue for the h-action. Therefore, as we have that $v \otimes v = e \cdot w$ for some $w \in V \otimes V$, there must exist $w' \in V \otimes V$ such that $x_k \otimes x_k = e \cdot w'$. Hence, without loss of generality, we can assume that $v = x_k$ for some k.

So, given that $x_k \otimes x_k = e \cdot w$ for some $w \in V \otimes V$, we want to show that h acts on x_k with a positive eigenvalue. For the sake of a contradiction, suppose h acts on x_k with a non-positive eigenvalue. Therefore, we must have that h actis on $x_k \otimes x_k$ with a non-positive eigenvalue. As $x_k \otimes x_k = e \cdot w$, we must have that h acts on w with a strictly negative eigenvalue. Let $V' \subseteq V$ be the subspace spanned by $\{x_0, x_1, \dots, x_{n-1}\}$ (that is, all but the highest weight vector). Then, we must have that $w \in V' \otimes V' \subseteq V \otimes V$.

Define the linear function:

$$\begin{aligned} f : V \otimes V &\longrightarrow \mathbb{C} \\ x_i \otimes x_j &\mapsto (-1)^i. \end{aligned}$$

Then, we claim that $f(e \cdot x) = 0$ for all $x \in V' \otimes V'$. To see this, we note that $V' \otimes V'$ is spanned by vectors for the form $x_i \otimes x_j$ for $0 \leq i, j \leq n-1$. Hence, for such i, j, we have:

$$e \cdot (x_i \otimes x_j) = x_{i+1} \otimes x_j + x_i \otimes x_{j+i},$$

which makes it clear that $f(e \cdot (x_i \otimes x_j)) = 0$.

In particular, as $x_k \otimes x_k = e \cdot w$, we must have $f(x_k \otimes x_k) = 0$. However, it follows from the definition of f that $f(x_k \otimes x_k) = (-1)^k \neq 0$. This gives a contradiction, and so, the eigenvalue corresponding to x_k must have been positive, completing the proof. $\square$

3. Hamiltonian reduction

3.1. **Classical setting.** In this section, we prove the isomorphism $X//G \simeq A//G$. We will actually prove a slightly more general statement, and will first define some notation to formulate the precise statement.

Let ω denote the symplectic form on the vector space V. Owing to the natural action of G on the vector space V that preserves ω, we get a moment map:

$$\mu_0 : V \longrightarrow \mathfrak{g}^*,$$

that maps the element $v \in V$ to $v^2 \in \mathfrak{g} \simeq \mathfrak{g}^*$. Here, we use the identification $\mathrm{Sym}^2(V) = \mathfrak{g}$. We can dualize this map to get a co-moment map:

$$\theta_0 : \mathfrak{g} \longrightarrow \mathbb{C}[V].$$

By [CG97, Proposition 1.4.6], we have the following formula for this co-moment map: For any $x \in \mathfrak{g}$, we have the following polynomial function on V:

$$v \mapsto \frac{1}{2}\omega(x \cdot v, v)$$

for all $v \in V$. In particular, the image of θ_0 in $\mathbb{C}[V]$ is exactly the vector space of polynomial functions on V having degree 2. We can extend the above map multiplicatively to get a map $\mathbb{C}[\mathfrak{g}] \simeq \mathrm{Sym}(\mathfrak{g}) \to \mathbb{C}[V]$, which we also call θ_0, whose image is exactly the subalgebra $\mathbb{C}[V]_{even}$ of polynomials have even total degree. Define $K := \ker(\theta_0) \subseteq \mathbb{C}[\mathfrak{g}]$. We give a more explicit description of this ideal K below.

Lemma 3.1.1. *Let $\overline{\mathbb{O}}$ denote the closure of the orbit $\mathbb{O}$ of rank one matrices in $\mathfrak{g}$, such that the scheme structure on $\overline{\mathbb{O}}$ is given by the reduced structure on it. Then, the radical ideal in $\mathbb{C}[\mathfrak{g}]$ that defines the scheme $\overline{\mathbb{O}}$ is generated by the 2×2 minors.*

Proof. Consider the map:

$$\mu_0 : V \longrightarrow \mathfrak{g}^* \simeq \mathfrak{g}$$
$$v \mapsto v^2.$$

The image of μ_0 is exactly $\overline{\mathbb{O}}$. Also, the pre-image of any point in $\mathbb{O}$ consists of exactly two vectors in V that are negatives of each other, whereas the pre-image of zero is exactly the zero vector. Therefore, we get an induced map from the categorical quotient:

$$\overline{\mu_0} : V//\{\pm 1\} \longrightarrow \overline{\mathbb{O}}.$$

By the above discussion, $\overline{\mu_0}$ is a closed embedding that is a bijection on $\mathbb{C}$-points. Therefore, as the scheme $\overline{\mathbb{O}}$ is reduced, the map $\overline{\mu_0}$ is an isomorphism.

Hence, the coordinate ring $\mathbb{C}[\overline{\mathbb{O}}]$ of $\overline{\mathbb{O}}$ is isomorphic to the invariant ring $\mathbb{C}[V]^{\{\pm 1\}}$. Choosing coordinates $p_1, p_2, \ldots, p_{2n}$ in V, this invariant ring is equal to $\mathbb{C}[p_1, p_2, \ldots, p_{2n}]^{\{\pm 1\}}$. By Weyl's first fundamental theorem for the orthogonal group $O(1) = \{\pm 1\}$ ([Wey46, Thereom 2.9A]), the invariant ring is generated by the polynomials $q_{ij} = p_i p_j$ for $1 \leq i, j \leq 2n$. Also, by the second fundamental theorem ([Wey46, Theorem 2.17A]), the relations between these generators are exactly given by $R_{ijkl} = q_{ij}q_{kl} - q_{il}q_{kj}$ for $1 \leq i, j, k, l \leq 2n$. Since the pullbacks of these relations R_{ijkl}'s to the coordinate ring of $\mathfrak{g}$ are exactly given by the 2×2 minors, this shows that the defining ideal is exactly generated by these elements. □

Corollary 3.1.2. *The map θ_0 induces an isomorphism of algebras:*

$$\theta_0 : \mathbb{C}[\mathfrak{g}]/K \longrightarrow \mathbb{C}[V]^{\{\pm 1\}} = \mathbb{C}[V]_{even}.$$

The ideal K is the defining ideal of $\overline{\mathbb{O}}$ in $\mathbb{C}[\mathfrak{g}]$ and is generated by the 2×2 minors.

Next, let Y be an affine algebraic variety with an action of the group $G = Sp(V)$ such that the center $\{\pm 1\}$ of G acts on it trivially. (We are interested in the case when $Y = \mathfrak{g}$, and G acts on it by the adjoint action.) The G-action on Y lifts to a Hamiltonian action on the symplectic variety T^*Y. Then, we get a moment map:

$$\mu_1 : T^*Y \longrightarrow \mathfrak{g}^*,$$

which we can dualize to get a co-moment map $\theta_1 : \mathrm{Sym}(\mathfrak{g}) \to \mathbb{C}[T^*Y]$.

Finally, we have a diagonal G-action on the space $T^*Y \times V$. The moment map $\mu_2 : T^*Y \times V \to \mathfrak{g}^*$ for this action is equal to $\mu_1 + \mu_0$, whereas the co-moment map $\theta_2 : \mathrm{Sym}(\mathfrak{g}) \to \mathbb{C}[T^*Y \times V] \simeq \mathbb{C}[T^*Y] \otimes \mathbb{C}[V]$ is defined via $\theta_2(x) = \theta_1(x) \otimes 1 + 1 \otimes \theta_0(x)$ for all $x \in \mathfrak{g}$.

We define the schemes that we'll be dealing with:

- We define the scheme $X(Y) \subseteq T^*Y \times V$ as the zero fiber of the moment map μ_2. More specifically, the defining ideal I of $X(Y)$ in $\mathbb{C}[T^*Y \times V]$ is the one generated by $\theta_2(\mathfrak{g})$ in $\mathbb{C}[T^*Y \times V]$.
- We define the scheme $A(Y) \subseteq T^*Y$ as the pre-image of $\overline{\mathbb{O}}$ under the moment map μ_1. That is, the defining ideal J of $A(Y)$ in $\mathbb{C}[T^*Y]$ is the one generated by $\theta_1(K)$.

There is a projection morphism $\phi : X(Y) \to A(Y)$ that maps a pair $(x, v) \in X(Y)$ to the element $x \in A(Y)$. We are going to show that:

Theorem 3.1.3. *The induced morphism on the categorical quotients:*

$$\phi : X(Y)//G \longrightarrow A(Y)//G,$$

is an isomorphism.

In particular, this proves Theorem 1.1.3, by taking $Y = \mathfrak{g}$. To prove the above theorem, we need to show that the induced map on the coordinate rings $\phi^* : \mathbb{C}[A(Y)]^G \to \mathbb{C}[X(G)]^G$ is an isomorphism. In terms of the ideals defined above, this is the map:

$$\phi^* : \Big(\mathbb{C}[T^*Y]/J\Big)^G \to \Big(\mathbb{C}[T^*Y \times V]/I\Big)^G.$$

We prove that this is an isomorphism by constructing an inverse $\phi' : \Big(\mathbb{C}[T^*Y \times V]/I\Big)^G \to \Big(\mathbb{C}[T^*Y]/J\Big)^G$. We need the following lemma:

Lemma 3.1.4. *Given the G-action on the space $T^*Y \times V$, any G-invariant polynomial in $\mathbb{C}[T^*Y \times V] = \mathbb{C}[T^*Y] \otimes \mathbb{C}[V]$ can be written as a sum of monomials, each having even total degree in the V-variables. In particular, we have an equality of algebras:*

$$(\mathbb{C}[T^*Y \times V]/I)^G = (\mathbb{C}[T^*Y] \otimes \mathbb{C}[V]_{even}/I)^G$$

Proof. This follows by observing that the element $-1 \in G$ sends a pair (x, i) to $(x, -i)$ for any $(x, i) \in T^*Y \times V$. ☐

Remark 3.1.5. *This lemma is the only point in the whole proof where we use the fact that the center $\{\pm 1\}$ of G acts trivially on Y.*

Let $s : \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{g}]$ be the algebra isomorphism that is induced by sending $x \mapsto -x$ for all $x \in \mathfrak{g} \simeq \mathfrak{g}^*$. Next, by Corollary 3.1.2, we have an algebra isomorphism $\theta_0 : \mathbb{C}[\mathfrak{g}]/K \to \mathbb{C}[V]_{even}$. We consider the inverse of this map:

$$\theta_0^{-1} : \mathbb{C}[V]_{even} \longrightarrow \mathbb{C}[\mathfrak{g}]/K.$$

Next, given the map $\theta_1 : \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[T^*Y]$, we can quotient by K to get a map:

$$\theta_1 : \mathbb{C}[\mathfrak{g}]/K \to \mathbb{C}[T^*Y]/\theta_1(K).$$

Hence, we can consider the composition map:

$$\theta_1 \circ s \circ \theta_0^{-1} : \mathbb{C}[V]_{even} \to \mathbb{C}[T^*Y]/\theta_1(K) = \mathbb{C}[T^*Y]/J.$$

Tensoring both sides by $\mathbb{C}[T^*Y]$, we get an algebra homomorphism which we call ϕ':

$$\phi' : \mathbb{C}[T^*Y] \otimes \mathbb{C}[V]_{even} \to \mathbb{C}[T^*Y]/J.$$

Proposition 3.1.6. *The ideal I lies in the kernel of ϕ'. Furthermore, the restriction of ϕ' to the space of G-invariants gives an inverse to ϕ^*.*

Proof. The ideal I is generated by elements of the form $\theta_2(x) = \theta_1(x) \otimes 1 + 1 \otimes \theta_0(x)$ for $x \in \mathfrak{g}$. Then, the first part of the claim follows from the computation:

$$\phi'(1 \otimes \theta_0(x)) = \theta_1 \circ s \circ \theta_0^{-1}(\theta_0(x)) = -\theta_1(x) = -\phi'(\theta_1(x) \otimes 1).$$

The same computation also implies that these maps are inverses to each other, completing the proof. ☐

Proof of Theorem 3.1.3. The inverse to the map ϕ^* is the one induced by ϕ'. This follows from Lemma 3.1.4 and Proposition 3.1.6. ☐

We now restrict to the case $Y = \mathfrak{g}$. We have morphisms:

$$C//G \xrightarrow{\psi} A//G \xleftarrow{\phi} X//G.$$

Set-theoretically, these morphisms are as follows: The map ψ sends a pair (x, y) of commuting matrices to the almost commuting pair (x, y). The map ϕ sends a triple (x, y, i) to the pair (x, y). We have shown that ϕ is an isomorphism. The morphism ψ is an isomorphism too, because it is proven in [Los21] that we have an isomorphism $C//G \to X//G$, and that morphism composed with ϕ gives ψ.

Remark 3.1.7. *The fact that ψ is an isomorphism can also be proven independently by mimicking the proof of Theorem 12.1 of [EG00], making use of Weyl's fundamental theorem of invariant theory for $\mathfrak{g} = \mathfrak{sp}(V)$.*

Combining Theorem 1.1.3 with Theorem 1.3 of [Los21], we have the following corollary:

Corollary 3.1.8. *We have an algebra isomorphism:*

$$\mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W = \mathbb{C}[(\mathfrak{h} \times \mathfrak{h})//W] \xrightarrow{\sim} \mathbb{C}[A//G] = \Big(\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]/J\Big)^{\mathfrak{g}}.$$

3.2. **Reminder on rational Cherednik algebras.** Fix a Cartan sublagebra $\mathfrak{h} \subseteq \mathfrak{g} = \mathfrak{sp}(V)$ and a root system $R \subseteq \mathfrak{h}^*$. The space $\mathfrak{h}$ has an action of the Weyl group $W = (\mathbb{Z}/(2))^n \rtimes S_n$. For each $\alpha \in R$, let $s_\alpha \in W$ denote the reflection of $\mathfrak{h}$ relative to the root α. Fix a W-invariant function $c : R \to \mathbb{C}$. Then, we recall from [EG00] the definition of the rational Cherednik algebra H_c of Type C, which is the quotient of the algebra $\mathrm{Sym}(\mathfrak{h}) \otimes \mathrm{Sym}(\mathfrak{h}^*) \otimes \mathbb{C}[W]$ by the ideal generated by the following relations:

$$wxw^{-1} = w(x), wyw^{-1} = w(y),$$
$$[y, x] = \langle x, y\rangle - \frac{1}{2}\sum_{\alpha\in R} c(\alpha)\langle \alpha, y\rangle\langle x, \alpha^\vee\rangle,$$

for all $w \in W$, $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$. Let $e = \frac{1}{|W|}\sum_{w\in W} w$ be the averaging idempotent in $\mathbb{C}[W]$. Then, we can construct the spherical subalgebra $eH_ce \subseteq H_c$, which will be our main object of interest. It is known that the subalgebras $\mathbb{C}[\mathfrak{h}]^W$ and $\mathrm{Sym}(\mathfrak{h})^W$ generate the algebra eH_ce. Another description of the algebra eH_ce given in [EG00] will be useful for us, and we recall that here. Define the rational Calogero-Moser operator L_c, which is a differential operator on the space $\mathfrak{h}^{reg}$, as follows (see [OP83]):

$$L_c := \Delta_{\mathfrak{h}} - \frac{1}{2}\sum_{\alpha\in R}\frac{c(\alpha)(c(\alpha)+1)}{\alpha^2}\cdot(\alpha, \alpha),$$

where Δ_h is the Laplacian operator on the Cartan subalgebra $\mathfrak{h}$. Let $\mathcal{D}(\mathfrak{h}^{reg})_-$ denote the subalgebra of $\mathcal{D}(\mathfrak{h}^{reg})$ spanned by differential operators $D \in \mathcal{D}(\mathfrak{h}^{reg})$ such that $degree(D) + order(D) \leq 0$. Let $\mathcal{C}_c$ denote the centralizer of the operator L_c in $\mathcal{D}(\mathfrak{h}^{reg})_-$. Finally, consider the subalgebra $\mathcal{B}_c$ of $\mathcal{D}(\mathfrak{h}^{reg})$ generated by $\mathcal{C}_c$ and $\mathbb{C}[\mathfrak{h}]^W$, the algebra of W-invariant polynomial functions on $\mathfrak{h}$.

Theorem 3.2.1. *[EG00, Theorem 4.8] We have an embedding (known as the Dunkl embedding)* $\Theta : eH_ce \hookrightarrow \mathcal{D}(\mathfrak{h}^{reg})^W$ *such that* $\Theta(\mathbb{C}[\mathfrak{h}]^W) = \mathbb{C}[\mathfrak{h}]^W$ *and* $\Theta(\mathrm{Sym}(\mathfrak{h})^W) = \mathcal{C}_c$. *Moreover,* $\Theta(\Delta_{\mathfrak{h}}) = L_c$. *Furthermore,* Θ *induces an isomorphism of algebras* $eH_ce \simeq \mathcal{B}_c$.

The algebra H_c has a filtration such that all the elements of W and the generators of $\mathrm{Sym}(\mathfrak{h}^*)$ have degree zero, and the generators of $\mathrm{Sym}(\mathfrak{h})$ have degree one. This induces a filtration on the spherical subalgebra eH_ce. The algebra $\mathcal{D}(\mathfrak{h}^{reg})$ has a filtration given by the degree of differential operators. Then, we have the following PBW theorem for Cherednik algebras:

Proposition 3.2.2. *[EG00, Corollary 4.4] With the above filtrations, we have an isomorphism:*

$$\mathrm{gr}(eH_ce) \simeq \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W.$$

Furthermore, under this isomorphism, the associated graded map $\mathrm{gr}(\Theta) : \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W \simeq \mathrm{gr}(eH_ce) \to \mathrm{gr}(\mathcal{D}(\mathfrak{h}^{reg})) = \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]^W$ *is exactly the algebra homomorhpism induced by the natural embedding* $\mathfrak{h}^{reg} \hookrightarrow \mathfrak{h}$.

Henceforth, we fix the standard embedding $\mathfrak{sp}(V) = \mathfrak{sp}_{2n} \subseteq \mathfrak{gl}_{2n}$ as the subspace consisting of block matrices of the form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = -D^T, B = B^T$ and $C = C^T$. Let $\mathfrak{h}$ be the space of diagonal matrices in $\mathfrak{sp}_{2n}$. Next, there are exactly two W-conjugacy classes in R depending on the length of a root, and so, we can denote the function c by a pair (c_{long}, c_{short}), where c_{long} (resp. c_{short}) is the value of the function c on the long (resp. short) roots. Identifying $\mathfrak{h} \simeq \mathfrak{h}^*$ using the trace form, we fix an orthonormal basis $r_1, r_2, \ldots, r_n$ of $\mathfrak{h}^*$ given by $r_i = (E_{i,i} - E_{i+n,i+n})/\sqrt{2}$, where $E_{i,j} \in \mathfrak{gl}_{2n}$ is the elementary matrix whose only non-zero entry is in the i^{th} row of the j^{th} column and is equal to 1.

Then, we have a choice of a root system R given by:

$$R := \left\{ \pm\frac{(r_i + r_j)}{\sqrt{2}}, \pm\frac{(r_i - r_j)}{\sqrt{2}}, \pm\sqrt{2}r_i : 1 \leq i < j \leq n \right\}.$$

Here, the long roots are given the vectors $\pm\sqrt{2}r_i$, whereas the rest are all short roots. The set of positive roots $R^+ \subseteq R$ is obtained by replacing all '$\pm$' by '$+$' in the above defintion. Let $\{e_\alpha\}_{\alpha\in R}$

denote the set of root vectors in $\mathfrak{h}$ that form a Cartan-Weyl basis of the Lie algebra $\mathfrak{g}$ chosen so that $(e_\alpha, e_{-\alpha}) = 1$, where $(.,.)$ is the trace form. Then, we can express the e_α's explicitly in terms of elementary matrices $E_{i,j}$ as follows:

$$\alpha = \frac{(r_i + r_j)}{\sqrt{2}} \implies e_\alpha = e_{-\alpha}^T = \frac{E_{i,j+n} + E_{j,i+n}}{\sqrt{2}}$$
$$\alpha = \frac{(r_i - r_j)}{\sqrt{2}} \implies e_\alpha = e_{-\alpha}^T = \frac{E_{i,j} - E_{j+n,i+n}}{\sqrt{2}}$$
$$\alpha = \sqrt{2} r_i \implies e_\alpha = e_{-\alpha}^T = E_{i,i+n}.$$

Next, we recall the construction of the 'univeral' Harish-Chandra homomorphism from [EG00]. Let $\mathfrak{g}^{rs}$ denote the subset of regular semisimple elements of $\mathfrak{g}$. Then, we have that $\mathfrak{h}^{reg} = \mathfrak{h} \cap \mathfrak{g}^{rs}$. Next, inside the universal enveloping algebra $\mathcal{U}\mathfrak{g}$, consider the space $(\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})}$ of $\mathrm{ad}(\mathfrak{h})$-invariants. Then, $(\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})} \cdot \mathfrak{h}$ is a two-sided ideal of the algebra $(\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})}$, and so, we can define the quotient algebra $(\mathcal{U}\mathfrak{g})_\mathfrak{h} := (\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})}/((\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})} \cdot \mathfrak{h}$.

By Proposition 6.1 of [EG00], and the paragraph following its proof, there exists a canonical algebra isomorphism:

$$\Psi : \mathcal{D}(\mathfrak{g}^{reg})^\mathfrak{g} \longrightarrow (\mathcal{D}(\mathfrak{h}^{reg}) \otimes (\mathcal{U}\mathfrak{g})_\mathfrak{h})^W.$$

Next, fix a $\mathcal{U}\mathfrak{g}$-module $\mathcal{V}$ and let $\mathcal{V}\langle 0\rangle$ denote its zero weight space. Then, by definition, this space $\mathcal{V}\langle 0\rangle$ is acted upon by $\mathfrak{h}$ by zero, and is thus stable under the action of $(\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})}$. Hence, we have an action of $(\mathcal{U}\mathfrak{g})_\mathfrak{h}$ on $\mathcal{V}\langle 0\rangle$, giving an algebra homomorphism $\chi : (\mathcal{U}\mathfrak{g})_\mathfrak{h} \to \mathrm{End}_\mathbb{C}(\mathcal{V}\langle 0\rangle)$. In particular, if $V\langle 0\rangle$ is one-dimensional, we get a homomorphism $\chi : (\mathcal{U}\mathfrak{g})_\mathfrak{h} \to \mathbb{C}$, and so, we can compose it with the above algebra isomorphism to get:

$$\Psi_\mathcal{V} := \chi \circ \Psi : \mathcal{D}(\mathfrak{g}^{reg})^\mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg})^W.$$

This is the Harish-Chandra homomorphism associated with the representation $\mathcal{V}$.

Let $\mathrm{Ann}(\mathcal{V}) \cdot \mathcal{U}\mathfrak{g}$ be the annihilator of $\mathcal{V}$ and consider the ideal $(\mathcal{D}(\mathfrak{g}) \cdot \mathrm{ad}(\mathrm{Ann}(\mathcal{V})))^\mathfrak{g}$, which is a two-sided ideal inside $\mathcal{D}(\mathfrak{g})$. Then, it follows from definitions and the proof of [EG00, Proposition 6.1] that the kernel of the homomorphism $\Psi_\mathcal{V}$ contains the ideal $(\mathcal{D}(\mathfrak{g}) \cdot \mathrm{ad}(\mathrm{Ann}(\mathcal{V})))^\mathfrak{g}$.

3.3. **Definitions and statement of the main theorem.** We set up some notation. Let ω denote the symplectic form on the vector space V. Let L be a fixed Lagrangian subspace of V. Then, we define the Weyl algebra W_{2n} as the algebra of polynomial differential operators on L, also denoted by $\mathcal{D}(L)$. Explicitly, the Weyl algebra is an associative $\mathbb{C}$-algebra generated by the variables $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ that satisfy the following relations:

$$[x_i, x_j] = 0, [y_i, y_j] = 0, [y_i, x_j] = \delta_{i,j}, 1 \le i, j \le n.$$

Here, the x_i's correspond to a choice of coordinates on the vector space L^* and and the y_i's represent the partial derivatives ∂_{x_i}'s with respect to x_i's. We have a direct sum decomposition $W_{2n} = W_{2n,even} \oplus W_{2n,odd}$ as vector spaces, where $W_{2n,even}$ is the space spanned by all the monomials in the x_i's and y_i's having even total degree and $W_{2n,odd}$ is the space spanned by the monomials having odd total degree. It is clear that $W_{2n,even}$ is a subalgebra of W_{2n} which is generated by all monomials of the form $x_i x_j, y_i y_j$ and $x_i y_j$ for $1 \le i, j \le n$.

Recall from §3.1 that corresponding to the G action on V, we have a co-moment map:

$$\theta_0 : \mathfrak{g} \longrightarrow \mathbb{C}[V],$$

such that the image of ϕ in $\mathbb{C}[V]$ is exactly the vector space of polynomial functions on V having degree 2.

Next, we note that the algebra W_{2n} is a quantization of $\mathbb{C}[V]$. To make this statement precise, we define the symmetrization map, which is the following map of vector spaces:

$$\mathrm{Sym} : \mathbb{C}[V] \longrightarrow W_{2n}$$
$$\lambda_1 \lambda_2 \cdots \lambda_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \lambda_{\sigma(1)} \lambda_{\sigma(2)} \cdots \lambda_{\sigma(k)},$$

where each λ_i is a linear function on V. This map is a vector space isomorphism. Note that both of these spaces have Lie algebra structures, where the Lie bracket on $\mathbb{C}[V]$ is given by the Poisson bracket and the bracket on W_{2n} is the one induced by the commutator of the associative product. The following lemma follows by a straightforward computation:

Lemma 3.3.1. *The restriction of the map* Sym *to the subspace* $\mathbb{C}[V]_{\leq 2} \subseteq \mathbb{C}[V]$ *of polynomials of degree lesser than or equal to two is a Lie algebra homomorphism .*

Now, we define the composition:

$$\Theta_0 := \mathrm{Sym} \circ \theta_0 : \mathfrak{g} \longrightarrow W_{2n}.$$

Viewing the Lie algebra $\mathfrak{sp}(V) = \mathfrak{sp}_{2n}$ as a subspace of $\mathfrak{gl}_{2n}$, using the embedding defined in the previous section, the map Θ_0 can be written in terms of coordinates via the formula:

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \mapsto \frac{1}{2}\left(\sum_{i,j=1}^{n} (2a_{i,j}x_iy_j - b_{i,j}x_ix_j + c_{i,j}y_iy_j) + \mathrm{Tr}(A) \right),$$

where $A = (a_{i,j}), B = (b_{i,j})$ and $C = (c_{i,j})$.

Then, Θ_0 is a Lie algebra homomorphism, and so it induces an algebra homomorphism $\mathcal{U}\mathfrak{g} \to W_{2n}$, which we also denote by Θ_0. Define $\mathcal{K} := \ker(\Theta_0) \subseteq \mathcal{U}\mathfrak{g}$.

Next, we define the action of $\mathfrak{g}$ on the space $\mathcal{D}(\mathfrak{g})$ of polynomial differential operators on $\mathfrak{g}$. For this, fix $x \in \mathfrak{g}$ and consider the linear map:

$$\mathrm{ad}_x : \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$y \mapsto [x, y].$$

Using the identification $\mathfrak{g} \simeq \mathfrak{g}^*$, the assignment $x \mapsto \mathrm{ad}_x$ gives a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$, where $\mathrm{End}(\mathfrak{g}^*)$ denotes the space of vector space endomorphisms of $\mathfrak{g}^*$. Any element in $\mathrm{End}(\mathfrak{g}^*)$ can be uniquely extended to a derivation on $\mathrm{Sym}(\mathfrak{g}^*) \simeq \mathbb{C}[\mathfrak{g}]$ via Leibniz rule. Since $\mathrm{Der}(\mathbb{C}[\mathfrak{g}]) \subseteq \mathcal{D}(\mathfrak{g})$, we have constructed a map:

$$\Theta_1 : \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{g}),$$

which is a Lie algebra homomorphism. Hence, this induces an algebra homomorphism $\mathcal{U}\mathfrak{g} \to \mathcal{D}(\mathfrak{g})$, which we also denote by Θ_1. Then, given $x \in \mathfrak{g}$ and $d \in \mathcal{D}(\mathfrak{g})$, the action of $\mathfrak{g}$ on $\mathcal{D}(\mathfrak{g})$ is defined via:

$$x \cdot d := [\Theta_1(x), d].$$

We now define the algebras we'll be working with:

- Consider the algebra $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$. We have the diagonal $\mathfrak{g}$-action on this algebra: Given $d \otimes w \in \mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ and $x \in \mathfrak{g}$, we define the action via:

  $$x \cdot (d \otimes w) := [\Theta_1(x), d] \otimes w + d \otimes [\Theta_0(x), w].$$

  Let $\Theta_2 : \mathfrak{g} \to \mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ be the Lie algebra homomorphism defined via $\Theta_2 = \Theta_1 \otimes 1 + 1 \otimes \Theta_0$. This can be extended to get an algebra homomorphism $\Theta_2 : \mathcal{U}\mathfrak{g} \to \mathcal{D}(\mathfrak{g}) \otimes W_{2n}$, which is the co-moment map for the above action. Then, we have the quantum Hamiltonian reduction $\Big(\mathcal{D}(\mathfrak{g}) \otimes W_{2n} / (\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}$ of $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ at the augmentation ideal $\mathcal{U}\mathfrak{g}^+$ of $\mathcal{U}\mathfrak{g}$.
- Next, define $\mathfrak{J} \subseteq \mathcal{D}(\mathfrak{g})$ to be the left ideal generated by the image of $\mathcal{K} \subseteq \mathcal{U}\mathfrak{g}$ under the co-moment map Θ_1, that is, $\mathfrak{J} := \mathcal{D}(\mathfrak{g}) \cdot (\Theta_1(\ker(\Theta_0)))$. Then, we have the algebra $\Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}}$ which is the quantum Hamiltonian reduction of the algebra $\mathcal{D}(\mathfrak{g})$ at the ideal $\mathfrak{J}$.
- Finally, we consider the spherical subalgebra eH_ce of the rational Cherednik algebra H_c with the paramater c given by $c = (c_{long}, c_{short}) = (-1/4, -1/2)$.

The goal now is to prove Theorem 1.1.4 by constructing algebra isomorphisms:

$$\Psi : \Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}} \longrightarrow eH_ce,$$

$$\Phi : \Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}} \longrightarrow \Big(\mathcal{D}(\mathfrak{g}) \otimes W_{2n} / (\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}.$$

These isomorphisms are established in Theorems 3.4.3 and 3.5.7 respectively.

Before we move further, we define some filtrations on our algebras. On the algebra $\mathcal{D}(\mathfrak{g})$, we have an increasing filtration given by the degree of the differential operators. The associated graded with respect to this filtration is given by $\mathrm{gr}(\mathcal{D}(\mathfrak{g})) \simeq \mathbb{C}[T^*\mathfrak{g}] \simeq \mathbb{C}[\mathfrak{g} \times \mathfrak{g}]$, identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ using the trace pairing. Similarly, we have a filtration on the Weyl algebra $W_{2n} = \mathcal{D}(L)$ defined by the degree of the differential operators, the associated graded with respect to which is given by $\mathrm{gr}(W_{2n}) \simeq \mathbb{C}[V]$. Using these two filtrations, we also get a tensor product filtration on $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ such that the associated graded is $\mathrm{gr}(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) = \mathbb{C}[\mathfrak{g} \times \mathfrak{g} \times V]$. Next, on the algebra $\mathcal{U}\mathfrak{g}$, we have the PBW filtration, and so by PBW theorem, we have the associated graded with respect to this filtration $\mathrm{gr}(\mathcal{U}\mathfrak{g}) \simeq \mathrm{Sym}\,\mathfrak{g}$.

Remark 3.3.2. *Note that the $\mathfrak{g}$-action on each of these algebras is filtration preserving, and so, as $\mathfrak{g}$ is reductive, taking $\mathfrak{g}$-invariants commutes with computing associated graded.*

Finally, the Cherednik algebra H_c has a filtration such that all the elements of W and the generators of $\mathrm{Sym}(\mathfrak{h}^*)$ have degree zero, and the generators of $\mathrm{Sym}(\mathfrak{h})$ have degree one. This also induces a filtration on the spherical subalgebra eH_ce.

Lemma 3.3.3. *We have an inclusion of ideals $J \subseteq \mathrm{gr}(\mathfrak{J})$, where $J \subseteq \mathbb{C}[\mathfrak{g} \times \mathfrak{g}]$ is the defining ideal of pairs of matrices whose commutator has rank lesser than or equal to 1.*

Recall from the definitions given in §3.1 that the ideal J is generated by all 2×2 minors of the commutator $[x, y]$ for $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

Proof. As Θ_0 and Θ_1 are filtration preserving maps between filtered algebras, they induce maps between the respective associated graded algebras. We describe these associated graded maps in order to compute $\mathrm{gr}(\mathfrak{J})$.

The map $\Theta_0 : \mathcal{U}\mathfrak{g} \to W_{2n}$ was defined on $\mathfrak{g} \subseteq \mathcal{U}\mathfrak{g}$ as the composition $Sym \circ \theta_0$. Then, we have:

$$\theta_0 = \mathrm{gr}(\Theta_0) : \mathrm{Sym}\,\mathfrak{g} = \mathrm{gr}(\mathcal{U}\mathfrak{g}) \longrightarrow \mathrm{gr}(W_{2n}) = \mathbb{C}[V].$$

This induces a map $V = \mathrm{Spec}(\mathbb{C}[V]) \to \mathrm{Spec}(\mathrm{Sym}\,\mathfrak{g})) = \mathfrak{g}^* \simeq \mathfrak{g}$ that sends a vector v to the rank one endomorphism $v^2 \in \mathfrak{g}$. By Lemma 3.1.1, we know that the ideal $\ker(\theta_0)$ is generated by all 2×2 minors in the entries of $\mathfrak{g}$.

Next, we consider the map $\Theta_1 : \mathcal{U}\mathfrak{g} \to \mathcal{D}(\mathfrak{g})$. In this case, we have the associated graded map:

$$\theta_1 = \mathrm{gr}(\Theta_1) : \mathrm{Sym}\,\mathfrak{g} = \mathrm{gr}(\mathcal{U}\mathfrak{g}) \longrightarrow \mathrm{gr}(\mathcal{D}(\mathfrak{g})) = \mathbb{C}[\mathfrak{g} \times \mathfrak{g}].$$

This is the co-moment map for the diagonal adjoint action of G on $\mathfrak{g} \times \mathfrak{g}$. This morphism θ_1 induces a map $\mathfrak{g} \times \mathfrak{g} = \mathrm{Spec}(\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]) \to \mathrm{Spec}(\mathrm{Sym}\,\mathfrak{g}) = \mathfrak{g}^* \simeq \mathfrak{g}$ sending a pair (x, y) to the commutator $[x, y]$.

Hence, if we consider the ideal generated by the image of $\ker(\theta_0)$ under the map θ_1, then this is exactly generated by the 2×2 minors of the commutator $[x, y]$ for $(x, y) \in \mathfrak{g} \times \mathfrak{g}$. But these are exactly the generators of the ideal J, and so $J = \mathbb{C}[\mathfrak{g} \times \mathfrak{g}] \cdot \theta_1(\ker(\theta_0))$. Also, by definition, $\mathfrak{J} = \mathcal{D}(\mathfrak{g}) \cdot \Theta_1(\ker(\Theta_0))$. Hence, to prove that $J \subseteq \mathrm{gr}(\mathfrak{J})$, it suffices to show that $\mathrm{gr}(\ker(\Theta_0)) = \ker(\theta_0)$. It is clear that $\mathrm{gr}(\ker(\Theta_0)) \subseteq \ker(\theta_0)$.

To see that the inclusion is an equality, we first describe the images of the maps θ_0 and Θ_0. We have that $\theta_0(\mathfrak{g})$ is exactly the space of degree 2 polynomials in $\mathbb{C}[V]$, and so, $Im(\theta_0)$ is equal to the subalgebra $\mathbb{C}[V]_{even}$ of all polynomials having even total degree. Applying the symmetrization map, we see that $Im(\Theta_0)$ is exactly $W_{2n,even}$. So, we have a short exact sequence of filtered $\mathcal{U}\mathfrak{g}$-modules:

$$0 \longrightarrow \ker(\Theta_0) \longrightarrow \mathcal{U}\mathfrak{g} \longrightarrow W_{2n,even} \longrightarrow 0.$$

We claim that the filtrations on $\ker(\Theta_0)$ and $W_{2n,even}$ are equal to the ones induced on them by the one on $\mathcal{U}\mathfrak{g}$. For the kernel, this is true by definition. Next, as noted above, the function Θ_0 maps $\mathfrak{g} \subseteq \mathcal{U}\mathfrak{g}$ to exactly the space spanned by the 'degree 2' elements x_ix_j, y_iy_j and $x_iy_j + y_jx_i$. These elements generate the algebra $W_{2n,even}$ and they define the same filtration on $W_{2n,even}$ as the one induced from W_{2n}. Thus, this filtration is the same as the one induced on $W_{2n,even}$ by viewing it as a quotient of $\mathcal{U}\mathfrak{g}$. So, we have that the above is a strict short exact sequence of filtered algebras, and therefore, by [Sjö73, Lemma 1], we can take the associated graded to get an exact sequence of $\mathrm{gr}(\mathcal{U}\mathfrak{g}) = \mathrm{Sym}\,\mathfrak{g}$-modules:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{gr}(\ker(\Theta_0)) \arrow[r] \arrow[d, hook] & \mathrm{gr}(\mathcal{U}\mathfrak{g}) \arrow[r] \arrow[d, equal] & \mathrm{gr}(W_{2n,even}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(\theta_0) \arrow[r] & \mathrm{Sym}\,\mathfrak{g} \arrow[r] & \mathbb{C}[V]_{even} \arrow[r] & 0
\end{tikzcd}.$$

This implies that $\mathrm{gr}(\ker(\Theta_0)) = \ker(\theta_0)$, completing the proof. □

Remark 3.3.4. *The equality* $\mathrm{gr}(\ker(\Theta_0)) = \ker(\theta_0)$ *can also be seen in a more general setting in the works of Joseph, where he constructs minimal realizations of simple Lie algebras in the Weyl algebra (see [Jos74], [Jos76]).*

3.4. **Construction of the isomorphism Ψ.** In this section, we construct the isomorphism:

$$\Psi : \Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}} \longrightarrow eH_c e.$$

For this, we recall the map $\Theta_0 : \mathfrak{g} \to W_{2n}$ defined in the previous section. The Weyl algebra W_{2n} is the space of the polynomial differential operators on the vector space L, and we have fixed coordinates $x_1, x_2, \ldots, x_n$ on L^*. Let $\mathcal{V}$ be the vector space spanned by all expressions of the form $(x_1 x_2 \cdots x_n)^{-1/2} \cdot P$, where P is a Laurent polynomial in $x_1, x_2, \ldots, x_n$. Then, the map Θ_0 gives an action of $\mathfrak{g}$ on $\mathcal{V}$, where any element $x \in \mathfrak{g}$ acts on $\mathcal{V}$ by $\Theta_0(x) \subseteq W_{2n}$ via formal differentiation of Laurent polynomials.

Recall that the Cartan $\mathfrak{h} \subseteq \mathfrak{g}$ is spanned by elements of the form $E_{i,i} - E_{n+i,n+i}$ for $1 \le i \le n$. Then, under Θ_0, the image of this element is the differential operator $(x_i y_i + y_i x_i)/2 = x_i y_i + \frac{1}{2} = x_i \partial_{x_i} + \frac{1}{2}$. Hence, the $\mathfrak{g}$-action on $\mathcal{V}$ has a one-dimensional zero weight space $\mathcal{V}\langle 0\rangle$ spanned by the element $(x_1 x_2 \cdots x_n)^{-1/2} \cdot 1$. Then, using the construction of the radial parts homomorphism in §3.2, we get an algebra homomorphism:

$$\Psi : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg})^W.$$

Let $\Delta_{\mathfrak{g}}$ and $\Delta_{\mathfrak{h}}$ be the Laplacian on $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Recall from §3.2 the Caloger-Moser differential operator L_c for $c = (-1/4, -1/2)$ and let $\mathcal{C}_c$ denote the centralizer of L_c in $\mathcal{D}(\mathfrak{h}^{reg})$. Also, let $\mathrm{Sym}(\mathfrak{g}) \subseteq \mathcal{D}(\mathfrak{g})$ denote the subalgebra of differential operators on $\mathfrak{g}$ with constant coefficients.

Lemma 3.4.1. *(a) We have the equality:*

$$\Psi(\Delta_{\mathfrak{g}}) = L_c.$$

(b) The map Ψ *induces an isomorphism:*

$$\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} C_c.$$

Proof. (a) By Proposition 6.2 of [EG00], we have:

$$\Psi(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{h}} - \sum_{\alpha \in R} \frac{e_\alpha \cdot e_{-\alpha}}{\alpha^2}.$$

We will evaluate $e_\alpha \cdot e_{-\alpha}|_{\mathcal{V}\langle 0\rangle}$ for the long roots and the short roots separately:

(i) Suppose α is a long root. (For such roots, $(\alpha, \alpha) = 2$.) Then, $\alpha = \sqrt{2} r_i$ for some i, where $r_i = (E_{i,i} - E_{i+n,i+n})/\sqrt{2}$. Then,

$$\Theta_0(e_\alpha \cdot e_{-\alpha}) = \frac{(-x_i{}^2) \cdot (\partial^2_{x_i})}{4}.$$

It is straightforward to see that this operator acts by $\frac{-3}{16}\,\mathrm{Id}_{\mathcal{V}\langle 0\rangle} = c_{long}(c_{long}+1)\,\mathrm{Id}_{\mathcal{V}\langle 0\rangle}$ on the space spanned by $(x_1 x_2 \cdots x_n)^{-1/2} \cdot 1$.

(ii) Suppose α is a short root. (For such roots, $(\alpha,\alpha)=1$.) Then, $\alpha=(r_i+r_j)/\sqrt{2}$ or $\alpha=(r_i-r_j)/\sqrt{2}$ for some $i\neq j$. In the first case,

$$\Theta_0(e_\alpha\cdot e_{-\alpha})=\frac{(-x_ix_j)\cdot(\partial_{x_i}\partial_{x_j})}{2},$$

whereas in the second case,

$$\Theta_0(e_\alpha\cdot e_{-\alpha})=\frac{(x_i\partial_{x_j})\cdot(x_j\partial_{x_i})}{2}.$$

Then, in either case, the differential operator acts by $\frac{-1}{8}\operatorname{Id}_{\mathcal{V}\langle 0\rangle}=\frac{1}{2}c_{short}(c_{short}+1)\operatorname{Id}_{\mathcal{V}\langle 0\rangle}$ on the space spanned by $(x_1x_2\cdots x_n)^{-1/2}\cdot 1$.

Hence, we conclude that:

$$\Psi(\Delta_{\mathfrak{g}})=\Delta_{\mathfrak{h}}-\frac{1}{2}\sum_{\alpha\in R}\frac{c(\alpha)(c(\alpha)+1)}{\alpha^2}\cdot(\alpha,\alpha),$$

which is exactly the Calogero-Moser operator L_c of Type C for the parameter c.

(b) This follows from the proof of Proposition 7.2 of [EG00], which works for any reductive Lie algebra $\mathfrak{g}$.

$\square$

By the above lemma, we see that $\mathcal{C}_c\subseteq\operatorname{Im}(\Psi)$. Next, the restriction of Ψ to $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}\subseteq\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ is exactly the Chevalley restriction map $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}\to\mathbb{C}[\mathfrak{h}]^W\subseteq\mathcal{D}(\mathfrak{h}^{reg})$, and so, $\mathbb{C}[\mathfrak{h}]^W\subseteq\operatorname{Im}(\Psi)$. Now, under the Dunkl homomorphism $\Theta: eH_ce\to\mathcal{D}(\mathfrak{h}^{reg})^W$, the image of the spherical subalgebra eH_ce is exactly the subalgebra of $\mathcal{D}(\mathfrak{h}^{reg})^W$ generated by $\mathcal{C}_c$ and $\mathbb{C}[\mathfrak{h}]^W$. Hence, we have the following corollary:

Corollary 3.4.2. *We have the inclusion of algebras* $\Theta(eH_ce)\subseteq\operatorname{Im}(\Psi)$.

Next, let $\operatorname{Ann}(\mathcal{V})\subseteq\mathcal{U}\mathfrak{g}$ denote the annihilator of the representation $\mathcal{V}$. Then, as remarked in §3.2, we have that $(\mathcal{D}(\mathfrak{g})\cdot\operatorname{ad}(\operatorname{Ann}(\mathcal{V})))^{\mathfrak{g}}\subseteq\ker(\Psi)$. Finally, we note that the action of $\mathfrak{g}$ on $\mathcal{V}$ was defined via the map Θ_0, and so, $\ker(\Theta_0)\subseteq\operatorname{Ann}(\mathcal{V})$, which implies that $\mathfrak{J}^{\mathfrak{g}}=(\mathcal{D}(\mathfrak{g})\cdot\Theta_1(\ker(\Theta_0)))^{\mathfrak{g}}\subseteq(\mathcal{D}(\mathfrak{g})\cdot\operatorname{ad}(\operatorname{Ann}(\mathcal{V})))^{\mathfrak{g}}$.

Theorem 3.4.3. *We have an isomorphism of algebras:*

$$\Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}}\simeq eH_ce.$$

Proof. The proof of this theorem is based on a commutative diagram that is very similar to the one present in the proof of Theorem 1.3.1 of [GG06].

As noted above, we have $\mathfrak{J}^{\mathfrak{g}}\subseteq(\mathcal{D}(\mathfrak{g})\cdot\operatorname{ad}(\operatorname{Ann}(\mathcal{V})))^{\mathfrak{g}}\subseteq\ker(\Psi)$, and so, the map Ψ can be factored to induce an algebra hommomorphism (which we also denote by Ψ):

$$\Psi:\Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}}\longrightarrow\mathcal{D}(\mathfrak{h}^{reg}).$$

Identifying $\mathfrak{h}$ with $\mathfrak{h}^*$ using the trace form, we get an algebra isomorphism $\phi:\mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]\to\mathbb{C}[\mathfrak{h}\times\mathfrak{h}]$. Then, we have the following diagram:

$$\begin{tikzcd}
\mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]^W \arrow[r, "\phi", "\sim"'] \arrow[d, "\text{Proposition 3.2.2}"', "\sim"] & \mathbb{C}[\mathfrak{h}\times\mathfrak{h}]^W \arrow[r, "\text{Cor. 3.1.8}", "\sim"'] & \Big(\mathbb{C}[\mathfrak{g}\times\mathfrak{g}]/J\Big)^{\mathfrak{g}} \arrow[d, two heads, "\text{Lemma 3.3.3}"] \\
\operatorname{gr}(eH_ce) \arrow[d, "\operatorname{gr}\Theta"', "\sim"] & & \Big(\operatorname{gr}(\mathcal{D}(\mathfrak{g}))/\operatorname{gr}(\mathfrak{J})\Big)^{\mathfrak{g}} \arrow[d, two heads, "\text{proj}"] \\
\operatorname{gr}\Theta(eH_ce) \arrow[r, hook, "\text{Cor. 3.4.2}"] & \operatorname{gr}\Big(\Psi\Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}}\Big) & \operatorname{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}} \arrow[l, two heads, "\operatorname{gr}\Psi"']
\end{tikzcd}.$$

This diagram commutes, and so, we get that all the arrows must be bijections. In particular, the image of $\operatorname{gr}(\Psi)$ in $\mathcal{D}(\mathfrak{h}^{reg})$ can be identified with the image of $\operatorname{gr}(\Theta)$ and both of these maps are injective. This identification can also be obtained as the associated graded of the embedding

$\Theta(eH_ce) \subseteq \mathrm{Im}(\Psi)$ from Corollary 3.4.2, and so, this embedding must itself be an equality. Hence, we can compose with Θ^{-1} (as Θ is injective) to get an algebra homomorphism:

$$\Theta^{-1} \circ \Psi : \Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}} \longrightarrow eH_ce.$$

It follows from the commutative diagram that the associated graded version of this map gives a bijection between $\mathrm{gr}\Big((\mathcal{D}(\mathfrak{g})/\mathfrak{J})^{\mathfrak{g}}\Big)$ and $\mathrm{gr}(eH_ce)$. Hence, the map $\Theta^{-1} \circ \Psi$ is itself a bijection, which is exactly the claim of the theorem. $\square$

Corollary 3.4.4. *(1) We have an isomorphism of commutative algebras:*

$$\mathrm{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{gr}(eH_ce).$$

(2) All the maps in the above commutative diagram are isomorphisms. In particular, we get that:

(a) We have an isomorphism:

$$\mathbb{C}[A//G] = \Big(\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]/J\Big)^{\mathfrak{g}} \simeq \mathrm{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}}.$$

(b) We have the equality of ideals $\mathrm{gr}(\mathfrak{J})^{\mathfrak{g}} = J^{\mathfrak{g}}$ *in the ring* $\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]^{\mathfrak{g}}$ *(cf. Lemma 3.3.3).*

3.5. **Construction of the isomorphism Φ.** In this section, we construct an isomorphism:

$$\Phi : \Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}} \longrightarrow \Big(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}/(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}.$$

We will work in a slightly more general setting. Let Y be an affine algebraic variety with an action of the group $G = Sp(V)$ such that the center $\{\pm 1\}$ of G acts on it trivially. (We are interested in the case when $Y = \mathfrak{g}$, and G acts on it by the adjoint action.) Then, differentiating the G-action on Y, we get a map from the Lie algebra $\mathfrak{g}$ to vector fields on Y. Thus, viewing these vector fields as differential operators of order one on Y, we get a Lie algebra homomorphism:

$$\Theta_1 : \mathfrak{g} \longrightarrow \mathcal{D}(Y),$$

which can be extended to get an algebra homomorphism $\Theta_1 : \mathcal{U}\mathfrak{g} \to \mathcal{D}(Y)$. Then, we can define the ideal $\mathfrak{J}$ of $\mathcal{D}(Y)$ as $\mathcal{D}(Y) \cdot (\Theta_1(\mathcal{K}))$, where $\mathcal{K}$ is the kernel of the algebra homomorphism $\Theta_0 : \mathcal{U}\mathfrak{g} \to W_{2n}$.

Next, as before, we can define the map $\Theta_2 : \mathcal{U}\mathfrak{g} \to \mathcal{D}(Y) \otimes W_{2n}$ defined on $\mathfrak{g}$ via $x \mapsto \Theta_1(x) \otimes 1 + 1 \otimes \Theta_0(x)$, which gives a diagonal action of $\mathfrak{g}$ on $\mathcal{D}(Y) \otimes W_{2n}$. We will show that there exists an algebra isomorphism:

$$\Phi : \Big(\mathcal{D}(Y)/\mathfrak{J}\Big)^{\mathfrak{g}} \longrightarrow \Big(\mathcal{D}(Y) \otimes W_{2n}/(\mathcal{D}(Y) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}.$$

The proof of this fact will follow a refined version of the argument given in §3.1.

We define this map via $\Phi(D) = D \otimes 1$ for all $D \in \mathcal{D}(Y)$. To see that this map is well defined, we need the following proposition.

Proposition 3.5.1. *For all* $D \in \mathfrak{J}$*, we have* $D \otimes 1 \in (\mathcal{D}(Y) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})$.

Before we prove this proposition, we need to prove a technical result about the structure of the ideal $\mathfrak{J}$. Owing to the Hopf algebra structure on the universal enveloping algebra $\mathcal{U}\mathfrak{g}$, we have an antipode $S : \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g}$ which is an algebra anti-homomorphism defined via $S(x) = -x$ for all $x \in \mathfrak{g} \subseteq \mathcal{U}\mathfrak{g}$. Similarly, we have an algebra anti-homomorphism $S' : W_{2n} \to W_{2n}$ defined by sending the generators $x_1, \ldots, x_n, y_1, \ldots, y_n$ of W_{2n} to $ix_1, \ldots ix_n, iy_1, \ldots, iy_n$, respectively, where $i = \sqrt{-1}$. Then, it follows that the map Θ_0 is an intertwiner, that is, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U}\mathfrak{g} & \xrightarrow{\Theta_0} & W_{2n} \\ {\scriptstyle S}\big\downarrow & & \big\downarrow{\scriptstyle S'} \\ \mathcal{U}\mathfrak{g} & \xrightarrow{\Theta_0} & W_{2n} \end{array}.$$

One way to see this is to use the explicit formula for Θ_0 described in §3.3. In particular, we get the following corollary:

Corollary 3.5.2. *The ideal $\mathcal{K} \subseteq \mathcal{U}\mathfrak{g}$ is stable under the action of the map S.*

Lemma 3.5.3. *For any $f \in \mathcal{U}\mathfrak{g}$, we have in the algebra $\mathcal{D}(Y) \otimes W_{2n}$, the following equality:*

$$\Theta_1(f) \otimes 1 = 1 \otimes \Theta_0(S(f)) \ mod \ ((\mathcal{D}(Y) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})).$$

Proof. Without loss of generality, suppose $f = x_1 x_2 \cdots x_k$ where $x_i \in \mathfrak{g} \subseteq \mathcal{U}\mathfrak{g}$ for all i. The proof will be by induction on n. Suppose the claim is true for all monomials f which are a product of fewer than n elements of $\mathfrak{g}$.

Recall that for all $x \in \mathfrak{g}$, $\Theta_2(x) = \Theta_1(x) \otimes 1 + 1 \otimes \Theta_0(x)$ and so the claim of the lemma is true for $f \in \mathfrak{g} \subseteq \mathcal{U}\mathfrak{g}$. We then compute as follows:

$$\begin{aligned}
\Theta_1(f) \otimes 1 &= \Theta_1(x_1 x_2 \cdots x_n) \otimes 1 \\
&= (\Theta_1(x_1 x_2 \cdots x_{n-1}) \otimes 1) \cdot (\Theta_1(x_n) \otimes 1) \\
&= (\Theta_1(x_1 x_2 \cdots x_{n-1}) \otimes 1) \cdot (1 \otimes (\Theta_0(S(x_n))) \text{ mod } ((\mathcal{D}(Y) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})) \\
&= (1 \otimes \Theta_0(S(x_n)) \cdot (\Theta_1(x_1 x_2 \cdots x_{n-1}) \otimes 1) \\
&= (1 \otimes \Theta_0(S(x_n)) \cdot (1 \otimes \Theta_0(S(x_1 x_2 \cdots x_{n-1})) \text{ mod } ((\mathcal{D}(Y) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})) \\
&\text{(by induction assumption)} \\
&= 1 \otimes \Theta_0(S(x_1 x_2 \cdots x_n)) \\
&\text{(as } S \text{ is an algebra anti-homomorphism)} \\
&= 1 \otimes \Theta_0(S(f)),
\end{aligned}$$

completing the proof. □

Proof of Proposition 3.5.1. Any element $D \in \mathfrak{J}$ is of the form $D = \sum_i D_i \cdot \Theta_1(f_i)$ where $D_i \in \mathcal{D}(\mathfrak{g})$ and $f_i \in \mathcal{K}$ for all i. Then, using the preceding lemma:

$$D \otimes 1 = \sum_i D_i \cdot \Theta_1(f_i) \otimes 1 = \sum_i D_i \otimes \Theta_0(S(f_i)) \text{ mod } ((\mathcal{D}(Y) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})).$$

By Corollary 3.5.2, $\Theta_0(S(f_i)) = 0$ for all $f_i \in \mathcal{K}$, and so, the above expression is zero proving that $D \in (\mathcal{D}(Y) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})$. □

Hence, we have a well-defined algebra homomorphism:

$$\Phi : \Big(\mathcal{D}(Y)/\mathfrak{J}\Big)^{\mathfrak{g}} \longrightarrow \Big(\mathcal{D}(Y) \otimes W_{2n}/(\mathcal{D}(Y) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}.$$

In order to show that the map Φ is an isomorphism, we'll describe the construction of an inverse Φ'. For this, we state a lemma which describes the structure of the algebra $(\mathcal{D}(Y) \otimes W_{2n})^{\mathfrak{g}}$.

Lemma 3.5.4. *We have the inclusion of algebras:*

$$(\mathcal{D}(Y) \otimes W_{2n})^{\mathfrak{g}} \subseteq \mathcal{D}(Y) \otimes W_{2n,even}.$$

Hence, we have the equality of algebras:

$$\Big(\mathcal{D}(Y) \otimes W_{2n}/(\mathcal{D}(Y) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}} = \Big(\mathcal{D}(Y) \otimes W_{2n,even}/(\mathcal{D}(Y) \otimes W_{2n,even}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}.$$

Proof. We start by writing $\mathcal{D}(Y) \otimes W_{2n} = (\mathcal{D}(Y) \otimes W_{2n,even}) \oplus (\mathcal{D}(Y) \otimes W_{2n,odd})$. Then, the $\mathfrak{g}$-action on $\mathcal{D}(Y) \otimes W_{2n}$ preserves this direct sum decomposition. Hence, to prove the claim, we need to show that $(\mathcal{D}(Y) \otimes W_{2n,odd})^{\mathfrak{g}} = 0$. Now, we observe that the associated graded of this space $\mathrm{gr}(\mathcal{D}(Y) \otimes W_{2n,odd})^{\mathfrak{g}} \subseteq \mathbb{C}[T^*(Y) \times V]^{\mathfrak{g}}$ consists exactly of those polynomials, which are sums of monomials having odd total degree in the V-variables. Thus, by Lemma 3.1.4, the principal symbol of any element in $(\mathcal{D}(Y) \otimes W_{2n,odd})^{\mathfrak{g}}$ is 0, showing that the space itself must be zero, proving the claim. □

Proposition 3.5.5. *There exists a map*

$$\Phi' : \Big(\mathcal{D}(Y) \otimes W_{2n,even}/(\mathcal{D}(Y) \otimes W_{2n,even}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}} \longrightarrow \Big(\mathcal{D}(Y)/\mathfrak{J}\Big)^{\mathfrak{g}},$$

such that Φ' is the inverse to Φ.

Proof. Recall that we have a $\mathfrak{g}$-equivariant map $\Theta_0 : \mathcal{U}\mathfrak{g} \to W_{2n}$, such that the image of Θ_0 is exactly $W_{2n,even}$. Thus, we can construct an inverse map:

$$\Theta_0^{-1} : W_{2n,even} \longrightarrow \mathcal{U}\mathfrak{g}/\ker(\Theta_0).$$

Next, we have the antipode map $S : \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g}$ which preserves the ideal $\ker(\Theta_0)$. Also, we have constructed an algebra homomorphism $\Theta_1 : \mathcal{U}\mathfrak{g} \to \mathcal{D}(Y)$. We can quotient by the ideal $\ker(\Theta_0) \subseteq \mathcal{U}\mathfrak{g}$ and the left ideal generated by its image in $\mathcal{D}(Y)$ to get a map of vector spaces:

$$\Theta_1 : \mathcal{U}\mathfrak{g}/\ker(\Theta_0) \longrightarrow \mathcal{D}(Y)/\mathcal{D}(Y) \cdot \Theta_1(\ker(\Theta_0)).$$

Therefore, we can construct the composition map:

$$\Theta_1 \circ S \circ \Theta_0^{-1} : W_{2n,even} \longrightarrow \mathcal{D}(Y)/\mathcal{D}(Y) \cdot \Theta_1(\ker(\Theta_0)) = \mathcal{D}(Y)/\mathfrak{J},$$

which is a map of vector spaces. Then, we tensor both sides (over $\mathbb{C}$) by $\mathcal{D}(Y)$ to get a map:

$$\mathcal{D}(Y) \otimes W_{2n,even} \longrightarrow \mathcal{D}(Y) \otimes \mathcal{D}(Y)/\mathfrak{J}.$$

Next, we compose by the quotient $\mathcal{D}(Y) \otimes \mathcal{D}(Y)/\mathfrak{J} \to \mathcal{D}(Y)/\mathfrak{J}$ to get a map of $\mathcal{D}(Y)$-modules:

$$\Phi' : \mathcal{D}(Y) \otimes W_{2n,even} \longrightarrow \mathcal{D}(Y)/\mathfrak{J}.$$

We will show that the left ideal $(\mathcal{D}(Y) \otimes W_{2n,even}) \cdot \Theta_2(\mathfrak{g})$ lies in the kernel of Φ'. Then, the inverse to the map Φ is obtained by restricting the above map Φ' to the space of $\mathfrak{g}$-invariants. That these maps are inverse to each other follows directly from definitions using Lemma 3.5.3.

We observe the following equality of left ideals in the algebra $\mathcal{D}(Y) \otimes W_{2n,even}$:

$$(\mathcal{D}(Y) \otimes W_{2n,even}) \cdot \Theta_2(\mathfrak{g}) = (\mathcal{D}(Y) \otimes 1) \cdot \Theta_2(\mathcal{U}\mathfrak{g}^+),$$

where $\mathcal{U}\mathfrak{g}^+$ denotes the augmentation ideal of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$. This follows by noting that the algebra $W_{2n,even}$ is generated by the monomials x_ix_j, x_iy_j and y_iy_j and each of these monomials can expressed in the form $\Theta_2(x) - \Theta_1(x) \otimes 1$ for a suitable choice of $x \in \mathfrak{g} \subseteq \mathcal{U}\mathfrak{g}$. Hence, as Φ' is a map of $\mathcal{D}(Y)$-modules, in order to show that $(\mathcal{D}(Y) \otimes W_{2n,even}) \cdot \Theta_2(\mathfrak{g}) \subseteq \ker(\Phi')$, it suffices to show that $\Theta_2(\mathcal{U}\mathfrak{g}^+) \subseteq \ker(\Phi')$. For this, pick any monomial $f = x_1x_2 \cdots x_n \in \mathcal{U}\mathfrak{g}$ where $x_i \in \mathfrak{g}$ for all i. Then,

$$\Theta_2(f) = \prod_{i=1}^{n} \Theta_2(x_i) = \prod_{i=1}^{n} (\Theta_1(x_i) \otimes 1 + 1 \otimes \Theta_0(x_i)).$$

If we expand out the right hand side, we get a sum of 2^n monomials indexed by disjoint partitions (P, Q) of the set $\{1, 2, \ldots, n\}$, that is, sets P and Q such that $P \cap Q = \phi$ and $P \cup Q = \{1, 2, \ldots, n\}$. Given such a partition (P, Q) we have in the expansion of $\Theta_2(f)$ a monomial of the form:

$$\Theta_1(x_{p_1}x_{p_2} \cdots x_{p_k}) \otimes \Theta_0(w_{q_1}x_{q_2} \cdots x_{q_l}),$$

where $P = \{p_1 < p_2 < \cdots < p_k\}$ and $Q = \{q_1 < q_2 < \cdots < q_l\}$. We can evaluate Φ' on such a monomial as follows:

$$\begin{aligned}\Phi'(\Theta_1(x_{p_1}x_{p_2} \cdots x_{p_k}) \otimes \Theta_0(x_{q_1}x_{q_2} \cdots x_{q_l})) &= \Theta_1(x_{p_1}x_{p_2} \cdots x_{p_k}) \cdot \Theta_1 \circ S \circ \Theta_0^{-1}(\Theta_0(x_{q_1}x_{q_2} \cdots x_{q_l})) \\ &= (-1)^l \Theta_1(x_{p_1}x_{p_2} \cdots x_{p_k}) \cdot \Theta_1(x_{q_l}x_{q_{l-1}} \cdots x_{q_1}) \\ &= \Theta_1((-1)^l x_{p_1}x_{p_2} \cdots x_{p_k}x_{q_l}x_{q_{l-1}} \cdots x_{q_1}).\end{aligned}$$

Hence, in order to show that Φ' maps $\Theta_2(f)$ to zero, we need to show that Θ_1 maps the sum (over all partitions (P, Q) of $\{1, 2, \ldots, n\}$) of the monomials:

$$(-1)^l x_{p_1}x_{p_2} \cdots x_{p_k}x_{q_l}x_{q_{l-1}} \cdots x_{q_1}$$

to zero. This will be established by showing that the sum of these monomials is itself zero, which is done in the following lemma. ☐

Lemma 3.5.6. *Let A be any associative algebra over a field k and let $a_1, a_2, \ldots, a_n$ be arbitrary elements in A. Then, we have the identity:*

$$\sum_{\substack{P=\{p_1<p_2<\cdots<p_k\} \\ Q=\{q_1<q_2<\cdots<q_l\}}} (-1)^l a_{p_1} a_{p_2} \cdots a_{p_k} a_{q_l} a_{q_{l-1}} \cdots a_{q_1} = 0,$$

where the sum is over all sets P, Q such that $P \cap Q = \phi$ and $P \cup Q = \{1, 2, \ldots, n\}$.

Proof. We'll prove the statement when A is the free associative algebra $A = k\langle a_1, a_2, \ldots, a_n\rangle$. For an arbitrary associative algebra A, the statement follows by considering the natural homomorphism from a free associative algebra to A that maps the generators to the elements $a_1, a_2, \ldots, a_n \in A$.

Let $\mathcal{V}$ be a vector space with basis $a_1, a_2, \ldots, a_n$. Then, the algebra $A = k\langle a_1, a_2, \ldots, a_n\rangle = T(V)$ has a Hopf algebra structure with the following data associated with it:

(1) Multiplication $\nabla : A \otimes A \to A$ defined by concatenation
(2) Comultiplication $\Delta : A \to A \otimes A$ defined via $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in \mathcal{V}$
(3) Unit $\eta : k \to A$ defined via $\eta(1) = 1$
(4) Counit $\epsilon : A \to k$ defined via $\epsilon(v) = 0$ for all $v \in \mathcal{V}$
(5) Antipode $S : A \to A$ defined via $S(v) = -v$ for all $v \in \mathcal{V}$.

Using this notation, the above sum can be expressed as:

$$\begin{aligned}
&\sum_{\substack{P=\{p_1<p_2<\cdots<p_k\} \\ Q=\{q_1<q_2<\cdots<q_l\}}} (-1)^l a_{p_1} a_{p_2} \cdots a_{p_k} a_{q_l} a_{q_{l-1}} \cdots a_{q_1} \\
&= \sum_{\substack{P=\{p_1<p_2<\cdots<p_k\} \\ Q=\{q_1<q_2<\cdots<q_l\}}} \nabla(a_{p_1} a_{p_2} \cdots a_{p_k} \otimes (-1)^l a_{q_l} a_{q_{l-1}} \cdots a_{q_1}) \\
&= \sum_{\substack{P=\{p_1<p_2<\cdots<p_k\} \\ Q=\{q_1<q_2<\cdots<q_l\}}} \nabla(a_{p_1} a_{p_2} \cdots a_{p_k} \otimes S(a_{q_1} a_{q_2} \cdots a_{q_l})) \\
&= \nabla \circ (1 \otimes S)\left(\sum_{\substack{P=\{p_1<p_2<\cdots<p_k\} \\ Q=\{q_1<q_2<\cdots<q_l\}}} (a_{p_1} a_{p_2} \cdots a_{p_k} \otimes a_{q_1} a_{q_2} \cdots a_{q_l}) \right) \\
&= \nabla \circ (1 \otimes S)\left(\prod_{i=1}^{n} (a_i \otimes 1 + 1 \otimes a_i) \right) \\
&= \nabla \circ (1 \otimes S) \circ \Delta(a_1 a_2 \cdots a_n).
\end{aligned}$$

So, by using the identity $\nabla \circ (1 \otimes S) \circ \Delta = \eta \circ \epsilon$ for Hopf algebras, we get that the sum is equal to

$$\nabla \circ (1 \otimes S) \circ \Delta(a_1 a_2 \cdots a_n) = \eta \circ \epsilon(a_1 a_2 \cdots a_n) = \eta(0) = 0,$$

which completes the proof. ☐

Theorem 3.5.7. *There is an isomorphism of algebras:*

$$\Big(\mathcal{D}(Y)/\mathfrak{J}\Big)^{\mathfrak{g}} \simeq \Big(\mathcal{D}(Y) \otimes W_{2n}/(\mathcal{D}(Y) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}.$$

In particular, the above isomorphism holds for $Y = \mathfrak{g}$.

Proof. The isomorphism is given by the map Φ. This follows from Lemma 3.5.4 and Proposition 3.5.5. ☐

Corollary 3.5.8. *We have isomorphisms of commutative algebras:*

$$\operatorname{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}} \simeq \operatorname{gr}\Big(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}/(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}} \simeq \Big(\mathbb{C}[\mathfrak{g} \times \mathfrak{g} \times V]/I\Big)^{\mathfrak{g}} = \mathbb{C}[X//G].$$

Proof. Recall from §3.1 that we have an isomorphism of schemes $\phi : X//G \to A//G$. This gives an isomorphism between the coordinate rings:

$$\phi^* : \Big(\mathbb{C}[\mathfrak{g}\times\mathfrak{g}]/J\Big)^{\mathfrak{g}} = \mathbb{C}[A//G] \to \mathbb{C}[X//G] = \Big(\mathbb{C}[\mathfrak{g}\times\mathfrak{g}\times V]/I\Big)^{\mathfrak{g}}.$$

Then, we can consider the commutative diagram:

$$\begin{tikzcd} \Big(\mathbb{C}[\mathfrak{g}\times\mathfrak{g}]/J\Big)^{\mathfrak{g}} \arrow[r, "\phi^*", "\sim"'] \arrow[d, "proj"] & \Big(\mathbb{C}[\mathfrak{g}\times\mathfrak{g}\times V]/I\Big)^{\mathfrak{g}} \arrow[d, "proj"] \\ \mathrm{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}} \arrow[r, "\mathrm{gr}(\Phi)"] & \mathrm{gr}\Big(\mathcal{D}(\mathfrak{g})\otimes W_{2n}/(\mathcal{D}(\mathfrak{g})\otimes W_{2n})\cdot\Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}} \end{tikzcd},$$

In this diagram, the top map and the left one are already known to be bijective. We have now shown that Φ is bijective too. Also, it's clear from their respective definitions that Φ and its inverse Φ' are both filtration preserving maps. Therefore, we conclude that $\mathrm{gr}(\Phi)$ must also be a bijection. This forces the fourth map in the commutative diagram to be a bijection. □

3.6. **Quantum Hamiltonian Reduction functor.** Let $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-*mod* denote the category of finitely generated $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-modules. The algebra $\mathcal{D}(\mathfrak{g})\otimes W_{2n}$ contains a subalgebra $Z = \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}}$ of invariant differential operators on $\mathfrak{g}$ with constant coefficients. Let $Z_+ \subseteq Z$ be the augmentation ideal, consisting of differential operators with zero constant term.

Definition 3.1. Let $\mathcal{C}$ be the full subcategory of $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-*mod* whose objects M are such that the action on M of the subalgebra Z_+ is locally nilpotent.

We can identify $\mathcal{D}(\mathfrak{g})\otimes W_{2n}$ with the ring of differential operators $\mathcal{D}(\mathfrak{g}\times L)$. Then, for any finitely generated $\mathcal{D}(\mathfrak{g})\otimes W_{2n}$-module M, there exists a characteristic variety $\mathrm{Ch}(M) \subseteq T^*(\mathfrak{g}\times L) = \mathfrak{g}\times\mathfrak{g}\times V$.

Proposition 3.6.1. *For any $M \in (\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-mod, $M \in \mathcal{C}$ if and only if* $\mathrm{Ch}(M) \subseteq X^{nil}$.
Furthermore, all the objects in $\mathcal{C}$ are holonomic $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-modules.

Proof. The proof of the fact that $M \in \mathcal{C}$ if and only if $\mathrm{Ch}(M) \subseteq X^{nil}$ follows by essentially repeating the proof of Proposition 5.3.2 of [GG06] replacing $\mathfrak{gl}(V)\times\mathbb{P}$ by $\mathfrak{sp}(V)\times L$ everywhere. The holonomicity of the objects in $\mathcal{C}$ follows from the fact that their singular support lies in X^{nil}, which is a Lagrangian subvariety of $T^*(\mathfrak{g}\times L)$. □

Now we define the quantum Hamiltonian reduction functor. Let Q be the quotient $(\mathcal{D}(\mathfrak{g})\cdot W_{2n})/((\mathcal{D}(\mathfrak{g})\otimes W_{2n})\cdot\Theta_2(\mathfrak{g}))$. We have the quantum Hamiltonian reduction of $\mathcal{D}(\mathfrak{g})\otimes W_{2n}$ with respect to the $\mathfrak{g}$-action, given by $\mathcal{A} := Q^{\mathfrak{g}} = \Big((\mathcal{D}(\mathfrak{g})\otimes W_{2n})/((\mathcal{D}(\mathfrak{g})\otimes W_{2n})\cdot\Theta_2(\mathfrak{g}))\Big)^{\mathfrak{g}}$. By Theorem 1.1.4, we have an isomorphism $\mathcal{A} \simeq eH_ce$, where eH_ce is the spherical Cherednik algebra with parameter $c = (-1/4.-1/2)$. Let eH_ce-*mod* be the category of finitely generated eH_ce-modules. Then, by Proposition 7.2.2 and Corollary 7.2.4 of [GG06], we have:

Proposition 3.6.2. *(1) The algebra Q is an object of $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-mod.*

(2) There is an exact functor $\mathbb{H}$:

$$\mathbb{H} : (\mathcal{D}(\mathfrak{g})\otimes W_{2n})\text{-}mod \longrightarrow eH_ce\text{-}mod$$
$$M \mapsto \mathrm{Hom}_{\mathcal{D}(\mathfrak{g})\otimes W_{2n}}(Q, M) = M^{\mathfrak{g}}.$$

(3) The functor $\mathbb{H}$ has a left adjoint ${}^T\mathbb{H}$:

$${}^T\mathbb{H} : eH_ce\text{-}mod \longrightarrow (\mathcal{D}(\mathfrak{g})\otimes W_{2n})\text{-}mod$$
$$M \mapsto Q\otimes_{\mathcal{A}} M,$$

such that the canonical adjunction morphism $M \to {}^T\mathbb{H}(\mathbb{H}(M))$ is an isomorphism for all $M \in (\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-mod.

(4) The full subcategory $\ker(\mathbb{H})$ *is a Serre subcategory of $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-mod and the functor $\mathbb{H}$ induces an equivalence of categories:*

$$(\mathcal{D}(\mathfrak{g})\otimes W_{2n})\text{-}mod/\ker(\mathbb{H}) \simeq eH_ce\text{-}mod$$

Next, we recall that the algebra eH_ce is generated by the subalgebras $\mathbb{C}[\mathfrak{h}]^W$ and $\mathrm{Sym}(\mathfrak{h})^W$, defined in §3.2. Let $\mathrm{Sym}(\mathfrak{h})^W_+$ denote the augmentation ideal of $\mathrm{Sym}(\mathfrak{h})^W$. Let $\mathcal{O}(eH_ce)$ be the category of $\mathcal{O}$ of the algebra H_c, which is the full subcategory of eH_ce-*mod* whose objects are finitely generated eH_ce-modules with locally nilpotent action of $\mathrm{Sym}(\mathfrak{h})^W_+ \subseteq eH_ce$. Now, under the isomorphism $\mathcal{A} \simeq eH_ce$, the subalgebra Z_+ is mapped exactly to the subalgebra $\mathrm{Sym}(\mathfrak{h})^W_+$ of eH_ce. Therefore, we conclude that the above proposition implies the following:

Proposition 3.6.3. *The functor* $\mathbb{H}$ *restricts to an exact functor* $\mathbb{H} : \mathcal{C} \to \mathcal{O}(eH_ce)$. *This induces an equivelence of categories* $\mathcal{C}/\ker(\mathbb{H}) \simeq \mathcal{O}(eH_ce)$.

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