# OLYMPIAD GEOMETRY

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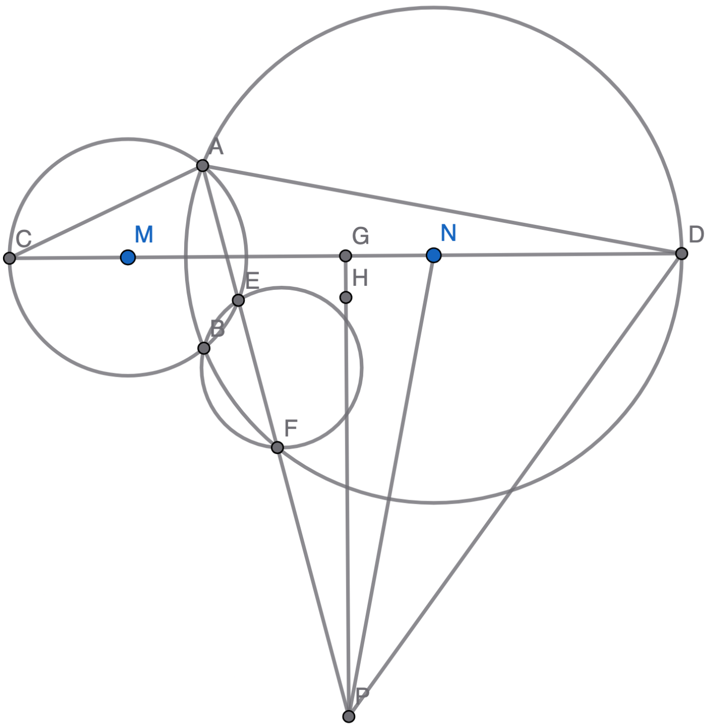
Abstract. This document is a compilation of my attempts at bashing IMO geometry problems using algebraic tools.

## Contents

# 1. IMO 2025 Problem 2

Let $\Omega$ and $\Gamma$ be circles with centres $M$ and $N$, respectively, such that the radius of $\Omega$ is less than the radius of $\Gamma$. Suppose circles $\Omega$ and $\Gamma$ intersect at two distinct points $A$ and $B$. Line $MN$ intersects $\Omega$ at $C$ and $\Gamma$ at $D$, such that points $C, M, N$ and $D$ lie on the line in that order. Let $P$ be the circumcentre of triangle $ACD$. Line $AP$ intersects $\Omega$ again at $E \neq A$. Line $AP$ intersects $\Gamma$ again at $F \neq A$. Let $H$ be the orthocentre of triangle $PMN$. Prove that the line through $H$ parallel to $AP$ is tangent to the circumcircle of triangle $BEF$.




Solution: Suppose $AB = x, CD = y, \angle ACD = C$ and $\angle ADC = D$. Since $AB \perp CD$, it is clear that we have the following relation between these quantities:

$$y = \frac{x}{2}(\cot C + \cot D) = \frac{x \sin(C+D)}{2 \sin C \sin D}.$$

We start by computing the circumradius $R$ of the $\triangle BEF$. To this end, we note that $\angle AEB = 180^\circ - 2C$, which implies that $\angle BEF = 2C$. Similarly, we have $\angle BFE = 2D$. Next, as $P$ is the circumcenter of $\triangle CAD$, we have $\angle APD = 2C$. So, $\angle PAD = 90^\circ - C$. This implies that:

$$\angle BAE = \angle BAD - \angle EAD = (90^\circ - D) - (90^\circ - C) = C - D.$$

In $\triangle ABE$, by the sine rule, we have:

$$BE = AB \frac{\sin(\angle BAE)}{\sin(\angle AEB)} = x \frac{\sin(C-D)}{\sin(180^\circ - 2C)} = x \frac{\sin(C-D)}{\sin 2C}.$$

Then, the circumradius $R$ of $\triangle BEF$ is given by:

$$R = \frac{BE}{2\sin(\angle BFE)} = x\frac{\sin(C-D)}{2\sin 2D \sin 2C}.$$

Next, we compute the distance of the line $EF$ from the center of the circle $BEF$. Again by the sine rule, we have $EF = 2R\sin(\angle EBF) = 2R\sin(180^\circ - 2C - 2D) = 2R\sin(2C+2D)$. Hence, the distance of $EF$ from the center is given by:

$$\sqrt{R^2 - R^2\sin^2(2C+2D)} = R|\cos(2C+2D)|.$$

Then, the distance of a tangent line to the circle $BEF$ that is parallel to $EF$ from the line $EF$ is given by:

$$R \pm R|\cos(2C+2D)| = 2R\cos^2(C+D) \text{ or } 2R\sin^2(C+D).$$

In order to show that the parallel to $EF$ that passes through $H$ is parallel to the circle $BEF$, it suffices to compute the distance of this parallel line from $EF$ and to verify that it is equal to one of the 2 quantities above. The distance of this parallel line from $EF$ is equal to $HP\sin(\angle HPF)$. So, to prove the required claim, it suffices to prove the equality:

$$HP\sin(\angle HPF) = 2R\cos^2(C+D).$$

Note that $\angle HPF = \angle BAE = C - D$ since $AB \parallel PG$, since both are perpendicular to $CD$.

Next, note that $\angle PCD = \angle PCA - C = 90^\circ - D - C$. Also, $PG$ bisects $CD$ and so $CG = \frac{y}{2}$ Therefore, $PG = CG\tan(\angle PCD) = \frac{y}{2}\cot(C+D)$.

Also, by the sine rule for $\triangle ABC$, we have that the radius $CM$ of the circle $ABC$ is equal to: $CM = \frac{AB}{2\sin(\angle ACB)} = \frac{x}{2\sin 2C}$. Hence,

$$\begin{aligned}
MG &= CG - CM \\
&= \frac{y}{2} - \frac{x}{2\sin 2C} \\
&= \frac{x\sin(C+D)}{4\sin C\sin D} - \frac{x}{2\sin 2C} \\
&= \frac{x}{4}\left(\frac{\sin(C+D)\cos C - \sin D}{\sin C\cos C\sin D}\right) \\
&= \frac{x}{4}\left(\frac{\cos(C+D)\sin C}{\sin C\cos C\sin D}\right) \\
&= \frac{x}{4}\left(\frac{\cos(C+D)}{\cos C\sin D}\right).
\end{aligned}$$

Similarly, we have $NG = \frac{x}{4}\left(\frac{\cos(C+D)}{\cos D\sin C}\right)$.

Hence,

$$\tan(\angle PMG) = \frac{PG}{MG} = \frac{\frac{y}{2}\cot(C+D)}{\frac{x}{4}\left(\frac{\cos(C+D)}{\cos C\sin D}\right)} = \cot C.$$

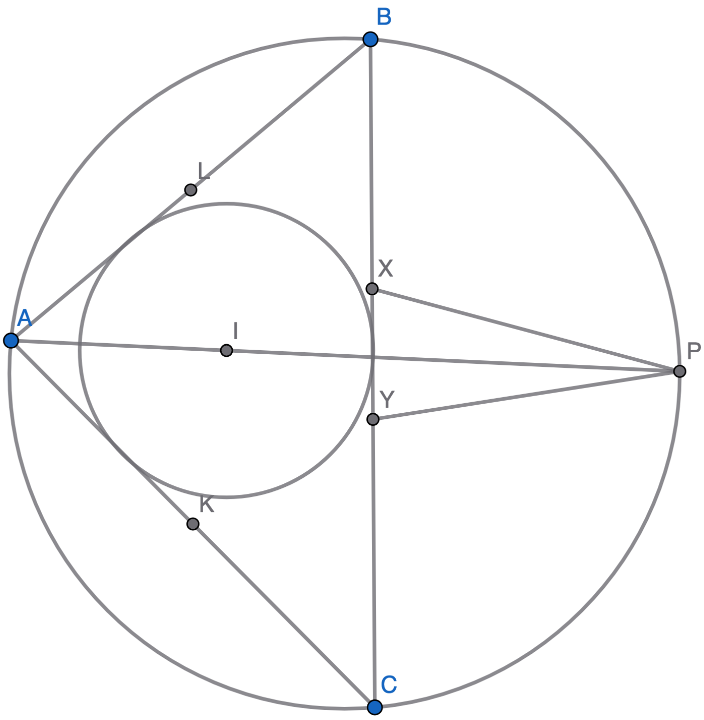
Furthermore, $\angle NHG = 180^\circ - \angle PHG = 180^\circ - (180^\circ - \angle PMG) = \angle PMG$. Therefore, $HG = NG\cot(\angle NHG) = \frac{x}{4}\left(\frac{\cos(C+D)}{\cos D\cos C}\right)$. Hence, we have:

$$\begin{aligned}
HP &= PG - HG \\
&= \frac{x\cos(C+D)}{4\sin D\sin C} - \frac{x\cos(C+D)}{4\cos D\cos C} \\
&= \frac{x\cos^2(C+D)}{4\sin C\sin D\cos C\cos D} \\
&= \frac{x\cos^2(C+D)}{\sin 2C\sin 2D}.
\end{aligned}$$

Thus, $HP\sin(\angle HPF) = \frac{x\cos^2(C+D)}{\sin 2C\sin 2D}\sin(C-D) = 2R\cos^2(C+D)$, completing the proof.

## 2. IMO 2024 Problem 4

Let $ABC$ be a triangle with $AB < AC < BC$. Let the incentre and incircle of triangle ABC be $I$ and $\omega$, respectively. Let $X$ be the point on line $BC$ different from $C$ such that the line through $X$ parallel to $AC$ is tangent to $\omega$. Similarly, let $Y$ be the point on line $BC$ different from $B$ such that the line through $Y$ parallel to $AB$ is tangent to $\omega$. Let $AI$ intersect the circumcircle of triangle $ABC$ again at $P \neq A$. Let $K$ and $L$ be the midpoints of $AC$ and $AB$, respectively. Prove that $\angle KIL + \angle YPX = 180°$.




Solution: Let $I = (0, 0)$ and the radius of the circle $\omega_1 = 1$. Let $B = (1, b)$ and $C = (1, c)$ for some $b > 0$ and $c < 0$. In order to show that $\angle KIL + \angle YPX = 180°$, it suffices to show that $\tan(\angle KIL) = -\tan(\angle YPX)$.

Let the slope of the line $AB$ be $m$. Then the equation of $AB$ is:

$$\frac{y-b}{x-1} = m.$$

As $AB$ is tangent to $\omega_1$, its distance from $I$ should be 1. Thus,

$$\frac{|m-b|}{\sqrt{1+m^2}} = 1 \implies m = \frac{b^2-1}{2b}.$$

Hence, the equation of $AB$ is:

$$\frac{y-b}{x-1} = \frac{b^2-1}{2b}.$$

Similarly, the equation of $AC$ is:

$$\frac{y-c}{x-1}=\frac{c^2-1}{2c}.$$

The intersection of these lines gives the coordinates of point $A$:

$$A=\Big(\frac{1-bc}{1+bc},\frac{b+c}{1+bc}\Big).$$

Then, we compute:

$$\begin{aligned}K&=\frac{A+C}{2}=\Big(\frac{1}{1+bc},\frac{bc^2+2c+b}{2(1+bc)}\Big),\\ L&=\frac{A+B}{2}=\Big(\frac{1}{1+bc},\frac{b^cc+2b+c}{2(1+bc)}\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\text{Slope of } IK&=\frac{bc^2+2c+b}{2},\\ \text{Slope of } IL&=\frac{b^2c+2b+c}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\tan(\angle KIL)&=\frac{\frac{bc^2+2c+b}{2}-\frac{b^2c+2b+c}{2}}{1+\big(\frac{bc^2+2c+b}{2}\big)\big(\frac{b^2c+2b+c}{2}\big)}\\ &=\frac{2(c-b)(bc+1)}{4+(bc^2+2c+b)(b^2c+2b+c)}.\end{aligned}$$

Next, we compute the coordinates of $P$. As $AI$ bisects $\angle BAC$, we have that $PB=PC$. Thus, $P=(r,\frac{b+c}{2})$ for some $r$. Also, since $A,I$ and $P$ are colinear, we have:

$$\text{Slope of } AI=\text{Slope of } PI \implies \frac{1-bc}{b+c}=\frac{2r}{b+c}.$$

This gives that $r=\frac{1-bc}{2}$ implying that $P=(\frac{1-bc}{2},\frac{b+c}{2})$.

Finally, we find coordinates of $X$ and $Y$. The slope of the line through $X$ tangent to $\omega$ is the same as the slope of $AC$, which is equal to $\frac{c^2-1}{2c}$. Hence, the equation of the tangent line is

$$y-\frac{c^2-1}{2c}x=\alpha$$

for some $\alpha$. For this to be tangent to $\omega_1$, its distance from $I$ should be 1. Therefore,

$$\frac{|\alpha|}{\sqrt{1+(\frac{c^2-1}{2c})^2}}=1\implies \alpha=\pm\frac{c^2+1}{2c}.$$

Thus, the equation of the tangent to $\omega$ through the point $X$ is:

$$y-\frac{c^2-1}{2c}x=-\frac{c^2+1}{2c}.$$

(We choose the negative sign, since the positive sign corresponds to the line $AC$.) Intersecting this tangent line with the line $BC$, which is given by $x=1$, we get that $X=(1,-\frac{1}{c})$. Similarly, we get that $Y=(1,-\frac{1}{b})$.

So, we can compute:

$$\begin{aligned}\text{Slope of } PX&=\frac{\frac{b+c}{2}+\frac{1}{c}}{\frac{1-bc}{2}-1}=-\frac{bc+c^2+2}{c(1+bc)},\\ \text{Slope of } PY&=\frac{\frac{b+c}{2}+\frac{1}{b}}{\frac{1-bc}{2}-1}=-\frac{bc+b^2+2}{b(1+bc)}.\end{aligned}$$

Hence,

$$\begin{aligned}\tan(\angle YPX) &= \frac{-\frac{bc+b^2+2}{b(1+bc)} + \frac{bc+c^2+2}{c(1+bc)}}{1+\left(\frac{bc+b^2+2}{b(1+bc)}\right)\left(\frac{bc+c^2+2}{c(1+bc)}\right)} \\ &= \frac{-bc(b+c)(1+bc) - 2c(1+bc) + bc(b+c)(1+bc) + 2b(1+bc)}{bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2)} \\ &= \frac{2(b-c)(1+bc)}{bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2)}.\end{aligned}$$

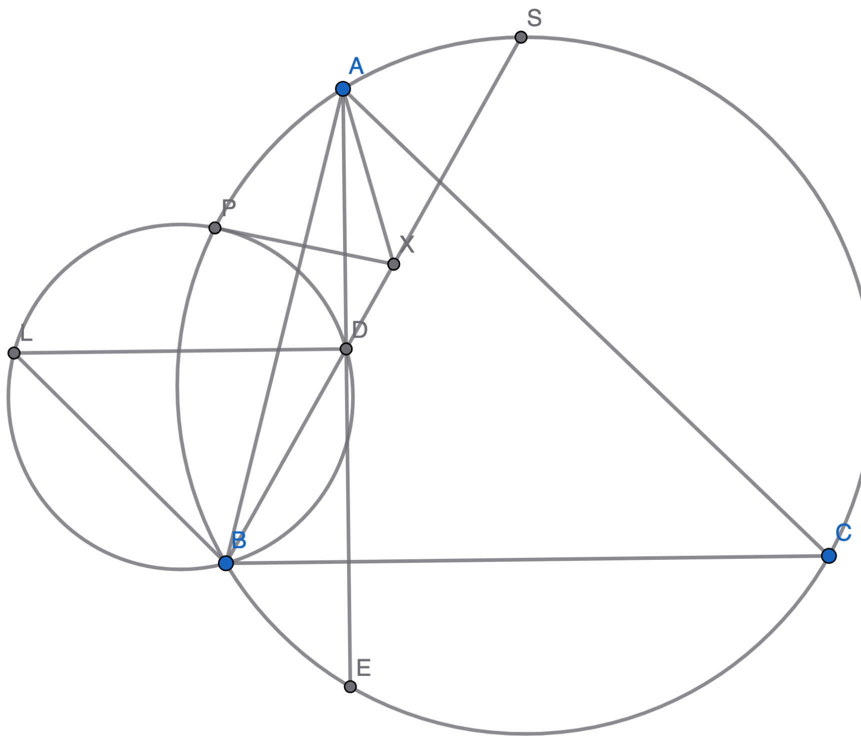
Therefore, from our expressions for $\tan(\angle KIL)$ and $\tan(\angle YPX)$, it follows that the equality $\tan(\angle KIL) = -\tan(\angle YPX)$ is equivalent to the following algebraic identity:

$$4 + (bc^2 + 2c + b)(b^2c + 2b + c) = bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2),$$

which is easily verified.

# 3. IMO 2023 Problem 2

Let $ABC$ be an acute-angled triangle with $AB < AC$. Let $\Omega$ be the circumcircle of $ABC$. Let $S$ be the midpoint of the arc $CB$ of $\Omega$ containing $A$. The perpendicular from $A$ to $BC$ meets $BS$ at $D$ and meets $\Omega$ again at $E \neq A$. The line through $D$ parallel to $BC$ meets line $BE$ at $L$. Denote the circumcircle of triangle BDL by $\omega$. Let $\omega$ meet $\Omega$ again at $P \neq B$. Prove that the line tangent to $\omega$ at $P$ meets line $BS$ on the internal angle bisector of $\angle BAC$.




Solution: We have $\angle SBC = \angle SCB = 90° - \frac{A}{2}$. Also, $\angle CBE = \angle CAE = 90° - C$. Thus, $\angle LBD = 180° - \angle SBE = 180° - (\angle SBC + \angle CBE) = 90° + \frac{C-B}{2}$.

Let $\angle PBS = t$. In $\omega$, sine rule gives:

$$\begin{aligned}\frac{PD}{\sin t} &= \frac{LD}{\sin(\angle LBD)} \\ &= \frac{LD}{\sin(90° + \frac{C-B}{2})} \\ &= \frac{LD}{\cos(\frac{B-C}{2})}.\end{aligned}$$

Next, we have $(\angle PAD + \angle DAC) + \angle PBC = 180°$. Thus,

$$\begin{aligned}\angle PAD &= 180° - \angle PBC - \angle DAC \\ &= 180° - (t + \frac{B+C}{2}) - (90° - C)\end{aligned}$$

$$= 90^\circ - t + \frac{C-B}{2}.$$

Similarly, we have $(\angle DPA + \angle DPB) + \angle ACB = 180^\circ$, Thus,

$$\begin{aligned}
\angle DPA &= 180^\circ - \angle DPB - \angle ACB \\
&= 180^\circ - \angle DLB - C \\
&= 180^\circ - (90^\circ - \angle LED) - C \\
&= 180^\circ - (90^\circ - C) - C \\
&= 90^\circ.
\end{aligned}$$

Hence, we have that $PD = AD\sin(\angle PAD) = AD\sin(90^\circ - t + \frac{C-B}{2}) = AD\cos(\frac{C-B}{2} - t)$. Inserting this into the above sine rule equation, we get:

$$\begin{aligned}
\frac{AD\cos(\frac{C-B}{2} - t)}{\sin t} &= \frac{LD}{\cos(\frac{B-C}{2})} \\
\implies AD\cos\left(\frac{C-B}{2}\right)\cot t + AD\sin\left(\frac{C-B}{2}\right) &= \frac{LD}{\cos(\frac{B-C}{2})} \\
\implies \cot t &= \frac{LD}{AD\cos^2(\frac{B-C}{2})} + \tan\left(\frac{B-C}{2}\right).
\end{aligned}$$

(1) First, suppose $PX$ is tangent to the circle $\omega$. Then, $\angle DPX = \angle DBP = t$. So, by the sine rule in $\triangle PDX$ we get:

$$\begin{aligned}
\frac{DX}{\sin t} &= \frac{PX}{\sin PDX} \\
&= \frac{PX}{\sin PLB}.
\end{aligned}$$

Now, $\angle PLB = \angle PLD + \angle BLD = \angle PBD + \angle BLD = 90^\circ + t - C$. Hence,

$$\frac{PX}{DX} = \frac{\sin(90^\circ + t - C)}{\sin t} = \frac{\cos(t - C)}{\sin t} = \cot t\cos C + \sin C.$$

Computing the power of the point $X$ with respect to $\omega$, we get $PX^2 = BX \cdot DX$. Hence,

$$\frac{BX}{DX} = \frac{BX}{PX}\cdot\frac{PX}{DX} = \left(\frac{PX}{DX}\right)^2 = (\cot t\cos C + \sin C)^2.$$

(2) Now, suppose $AX$ bisects $\angle BAC$. Then, $\angle AXB = 180^\circ - \angle XAB - \angle XBA = 180^\circ - \frac{A}{2} - (B - \frac{B+C}{2}) = 90^\circ + C$. Also, $\angle DAX = \angle DAC - \angle XAC = 90^\circ - C - \frac{A}{2}$. So, applying the sine rule in $\triangle ADX$, we get:

$$\begin{aligned}
\frac{DX}{\sin(\angle DAX)} &= \frac{AD}{\sin(\angle AXD)} \\
\implies DX = AD\frac{\sin(90^\circ - C - \frac{A}{2})}{\sin(90^\circ + C)} &= AD\frac{\cos(C + \frac{A}{2})}{\cos C}.
\end{aligned}$$

Next, applying the sine rule in $\triangle BAX$, we get:

$$\begin{aligned}
\frac{BX}{\sin(\angle BAX)} &= \frac{BA}{\sin(\angle AXB)} \\
\implies BX = AB\frac{\sin(\frac{A}{2})}{\sin(90^\circ + C)} &= \frac{c\sin(\frac{A}{2})}{\cos C},
\end{aligned}$$

where we suppose $AB = c$. Combining the two expressions above, we get that:

$$\frac{BX}{DX} = \frac{c\sin(\frac{A}{2})}{AD\cos(C + \frac{A}{2})}.$$

We have obtained expressions for $\frac{BX}{DX}$ in both of the above cases. Thus, to prove the required claim, it suffices to prove the equality:

$$(\cot t \cos C + \sin C)^2 = \frac{c \sin(\frac{A}{2})}{AD \cos(C + \frac{A}{2})}.$$

To that end, we first compute AD. Note that $\angle ADB = 180^\circ - \angle ABD - \angle BAD = 180^\circ - (B - \frac{B+C}{2}) - (90^\circ - B) = 180^\circ - \frac{A}{2}$. Then, using the sine rule in $\triangle BAD$, we get:

$$AD = AB \frac{\sin(\angle ABD)}{\sin(\angle ADB)} = \frac{c \sin(\frac{B-C}{2})}{\sin(180^\circ - \frac{A}{2})} = \frac{c \sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}.$$

This implies that:

$$\frac{c \sin(\frac{A}{2})}{AD \cos(C + \frac{A}{2})} = \frac{\sin^2(\frac{A}{2})}{\cos(C + \frac{A}{2}) \sin(\frac{B-C}{2})} = \frac{\sin^2(\frac{A}{2})}{\sin^2(\frac{B-C}{2})}$$

since $\frac{B-C}{2} = 90^\circ - (C + \frac{A}{2})$. Thus, we are reduced to proving the equality:

$$\cot t \cos C + \sin C = \frac{\sin(\frac{A}{2})}{\sin(\frac{B-C}{2})}.$$

Recall that $\cot t = \frac{LD}{AD \cos^2(\frac{B-C}{2})} + \tan\left(\frac{B-C}{2}\right)$. To compute this, we need to find $LD$. Note that $LD = DE \tan(\angle LED) = DE \tan C = (AE - AD) \tan C$. Applying the sine rule in circle $\Omega$, we get:

$$AE = AB \frac{\sin(\angle ABE)}{\sin(\angle ACB)} = \frac{c \sin(90 + B - C)}{\sin C} = \frac{c \cos(B - C)}{\sin C} = c \cot C \cos B + c \sin B.$$

Therefore,

$$\begin{aligned} LD &= \tan C(AE - AD) \\ &= c \tan C\Big(\cot C \cos B + \sin B - \frac{\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}\Big) \\ &= c \cos B + c \tan C \sin B - c\frac{\tan C \sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{LD}{AD \cos^2(\frac{B-C}{2})} &= \frac{\cos B + \tan C \sin B - \frac{\tan C \sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}}{\frac{\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})} \cos^2(\frac{B-C}{2})} \\ &= \frac{\sin(\frac{A}{2})(\cos B + \tan C \sin B) - \tan C \sin(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})}. \end{aligned}$$

This implies that:

$$\begin{aligned} \cot t &= \frac{LD}{AD \cos^2(\frac{B-C}{2})} + \tan\left(\frac{B-C}{2}\right) \\ &= \frac{\sin(\frac{A}{2})(\cos B + \tan C \sin B) - \tan C \sin(\frac{B-C}{2}) + \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})}. \end{aligned}$$

Next,

$$\begin{aligned} \cos C \cot t &= \frac{\sin(\frac{A}{2})(\cos B \cos C + \sin C \sin B) - \sin C \sin(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\ &= \frac{\sin(\frac{A}{2}) \cos(B - C) - \sin C \sin(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})}. \end{aligned}$$

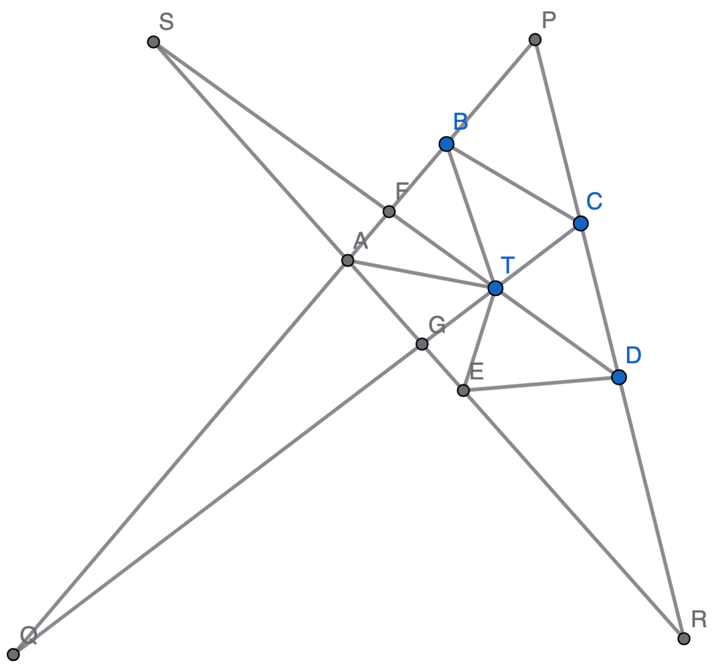
Finally,

$$
\begin{aligned}
\sin C + \cos C \cot t &= \frac{\sin C \sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2}) + \sin(\frac{A}{2}) \cos(B-C) - \sin C \sin(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) - \sin C \sin^3(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin^2(\frac{B-C}{2}) \cos(C + \frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin^2(\frac{B-C}{2}) \cos(\frac{B+C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin(\frac{A}{2}) \sin^2(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2})(\cos(B-C) + \sin^2(\frac{B-C}{2}))}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2})}{\sin(\frac{B-C}{2})},
\end{aligned}
$$

completing the proof.

# 4. **IMO 2022 Problem 4**

Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point $T$ inside $ABCDE$ with $TB = TD, TC = TE$ and $\angle ABT = \angle TEA$. Let line $AB$ intersect lines $CD$ and $CT$ at points $P$ and $Q$, respectively. Assume that the points $P, B, A, Q$ occur on their line in that order. Let line $AE$ intersect lines $CD$ and $DT$ at points $R$ and $S$, respectively. Assume that the points $R, E, A, S$ occur on their line in that order. Prove that the points $P, S, Q, R$ lie on a circle.




Solution: Suppose $TB = TD = s$ and $TC = TE = r$. We normalize $BC = DE = 1$. As we have $\triangle BTC \cong \triangle DTE$, we have the following equalities of angles:

$$\alpha := \angle BTC = \angle DTE$$

$$\beta := \angle TCB = \angle TED$$

$$\gamma := \angle CBT = \angle EDT.$$

Let $\angle DTC = \phi$ and $\angle ABT = \angle TEA = \theta$. Next, let $\angle TDC = m$ and $\angle TCD = n$. Finally, let $\angle FAT = f$ and $\angle GAT = g$. Note that $\angle BTE = 360° - (\angle BTC + \angle DTE + \angle DTC) = 360° - 2\alpha - 2\phi$. Thus, we have the equality:

$$f + g = 2\alpha + 2\phi - 2\theta.$$

As the angles at vertex $B$ add to $180°$, we have $\angle CBP = 180° - \gamma - \theta$. Similar consideration at vertex $C$ gives that $\angle BCP = 180° - \beta - n$. Thus, $\angle P = 180° - (\angle CBP + \angle BCP) = \theta + n - \alpha$.

Similarly, $\angle R = \theta + m - \alpha$. Hence, by the sine rule in $\triangle APR$, we get:

$$\frac{AP}{AR} = \frac{\sin(\theta + m - \alpha)}{\sin(\theta + n - \alpha)}.$$

Next, $\angle Q = 180^\circ - (\angle P + \angle QCP) = \alpha - \theta$.

By exactly the same argument, we get that $\angle S = \alpha - \theta$. By the sine rule in $\triangle QAT$ and $\triangle SAT$, we have:

$$AQ = \frac{\sin(\angle ATQ)}{\angle Q} AT$$
$$AS = \frac{\sin(\angle ATS)}{\angle S} AT,$$

and so, $\frac{AQ}{AS} = \frac{\sin(\angle ATQ)}{\sin(\angle ATS)}$. Note that $\angle ATQ = \angle TAF - \angle Q = f - \alpha + \theta$. Similarly, $\angle ATS = g - \alpha + \theta$. So,

$$\frac{AQ}{AS} = \frac{\sin(f - \alpha + \theta)}{\sin(g - \alpha + \theta)}.$$

In order to show that $P, S, Q$ and $R$ are concylic, it suffices to show that:

$$\begin{aligned}
&AP \cdot AQ = AR \cdot AS \\
\iff\ &\frac{AP}{AR} = \frac{AS}{AQ} \\
\iff\ &\frac{\sin(\theta + m - \alpha)}{\sin(\theta + n - \alpha)} = \frac{\sin(g - \alpha + \theta)}{\sin(f - \alpha + \theta)} \\
\iff\ &\frac{\sin(\theta + m - \alpha) + \sin(\theta + n - \alpha)}{\sin(\theta + m - \alpha) - \sin(\theta + n - \alpha)} = \frac{\sin(g - \alpha + \theta) + \sin(f - \alpha + \theta)}{\sin(g - \alpha + \theta) - \sin(f - \alpha + \theta)} \\
\iff\ &\frac{\tan(\theta - \alpha + \frac{m+n}{2})}{\tan(\frac{m-n}{2})} = \frac{\tan(\theta - \alpha + \frac{f+g}{2})}{\tan(\frac{g-f}{2})} \\
\iff\ &\frac{\tan(\theta - \alpha + \frac{180^\circ - 2\phi}{2})}{\tan(\frac{m-n}{2})} = \frac{\tan(\theta - \alpha + \frac{2\alpha + 2\phi - 2\theta}{2})}{\tan(\frac{g-f}{2})} \\
\iff\ &\frac{\cot(\phi - \theta + \alpha)}{\tan(\frac{m-n}{2})} = \frac{\tan(\phi)}{\tan(\frac{g-f}{2})} \\
\iff\ &\tan\left(\frac{g-f}{2}\right)\cot(\phi - \theta + \alpha) = \tan\left(\frac{m-n}{2}\right)\tan(\phi) \\
\iff\ &\tan\left(\frac{g-f}{2}\right)\cot(\frac{g+f}{2}\Big) = \tan\left(\frac{m-n}{2}\right)\cot(\frac{m+n}{2}\Big) \\
\iff\ &\frac{\sin(g) - \sin(f)}{\sin(g) + \sin(f)} = \frac{\sin(m) - \sin(n)}{\sin(m) + \sin(n)} \\
\iff\ &\frac{\sin(g)}{\sin(f)} = \frac{\sin(m)}{\sin(n)}.
\end{aligned}$$

Note that in $\triangle CDT$ by the sine rule:

$$\frac{\sin(m)}{\sin(n)} = \frac{TC}{TD} = \frac{r}{s}.$$

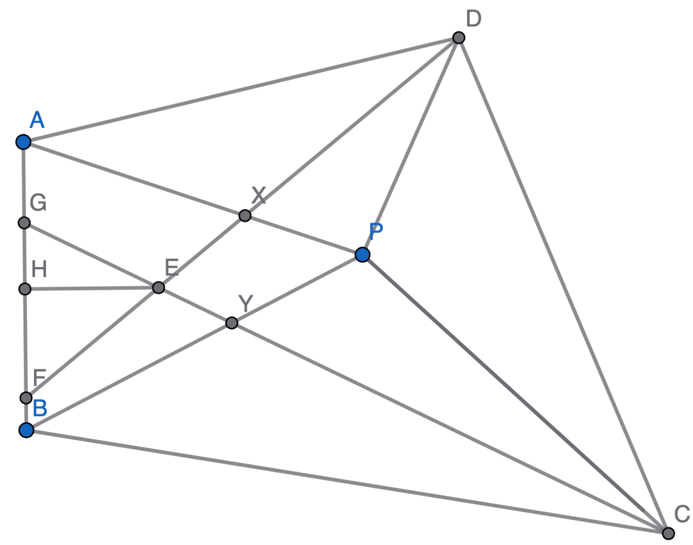
On the other hand, by the sine rule in $\triangle BAT$ and $\triangle EAT$, we have:

$$\frac{\sin(g)}{\sin(f)} = \frac{\frac{TE}{TA}\sin(\angle TEA)}{\frac{TB}{TA}\sin(\angle TBA)} = \frac{TE\sin(\theta)}{TB\sin(\theta)} = \frac{r}{s},$$

completing the proof.

# 5. IMO 2020 Problem 1

Consider the convex quadrilateral $ABCD$. The point $P$ is in the interior of $ABCD$. The following ratio equalities hold: $\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC$. Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment $AB$.




Solution: Let $DF$ and $CG$ be the bisectors of $\angle ADP$ and $\angle PCB$ respectively. Let $BP = a$, $AP = b$ and $R$ be the circumradius of $\triangle PAB$. Let $\angle PBA = 2x$ and $\angle PAB = 2y$. Draw $EH \perp AB$. It suffices to show that $H$ is the mid-point of $AB$.

Since $AD$ bisects $\angle ADP$, we have by the sine rule in $\triangle ADP$:

$$\frac{AX}{XP} = \frac{AD}{DP} = \frac{\sin(\angle APD)}{\sin(\angle PAD)} = \frac{\sin 3x}{\sin x}.$$

Thus, $AX = b\frac{\sin 3x}{\sin x + \sin 3x} = b\frac{\sin 3x}{2\sin 2x \cos x} = R\frac{\sin 3x}{\cos x}$, by the sine rule in $\triangle APB$.

Next, $\angle AXF = \angle XAD + \angle XDA = \angle XAD + \frac{1}{2}\angle PDA = x + 90^\circ - 2x = 90^\circ - x$. This implies that $\angle XFA = 180^\circ - \angle FAX - \angle AXF = 90^\circ + x - 2y$. Then, by the sine rule in $\triangle AFX$, we get:

$$\begin{aligned}
AF &= AX\frac{\sin(\angle AXF)}{\sin(\angle XFA)} \\
&= R\frac{\sin 3x}{\cos x}\frac{\cos x}{\cos(2y - x)} \\
&= R\frac{\sin 3x}{\cos(2y - x)}
\end{aligned}$$

$$= R\frac{\sin((2x+2y)-(2y-x))}{\cos(2y-x)}$$
$$= R(\sin(2x+2y) - \cos(2x+2y)\tan(2y-x)).$$

Similarly, we have $BG = R(\sin(2x+2y) - \cos(2x+2y)\tan(2x-y))$. Finally, $AB = 2R\sin(\angle APB)) = 2R\sin(180^\circ - 2x - 2y) = 2R\sin(2x+2y)$. Thus, we have:

$$GF = AF + BG - AB = -R\cos(2x+2y)(\tan(2y-x) + \tan(2x-y)) = -R\cos(2x+2y)\frac{\sin(x+y)}{\cos(2y-x)\cos(2x-y)}.$$

Note that $\angle GFE = 90^\circ + x - 2y$ and $\angle FGE = 90^\circ + y - 2x$. Thus, we have $\angle GEF = x + y$. Hence, by the sine rule in $\triangle GFE$, we get:

$$EF = GF\frac{\sin(\angle FGE)}{\sin(\angle GEF)} = -R\cos(2x+2y)\frac{\sin(x+y)}{\cos(2y-x)\cos(2x-y)}\frac{\cos(2x-y)}{\sin(x+y)} = -R\frac{\cos(2x+2y)}{\cos(2y-x)}.$$

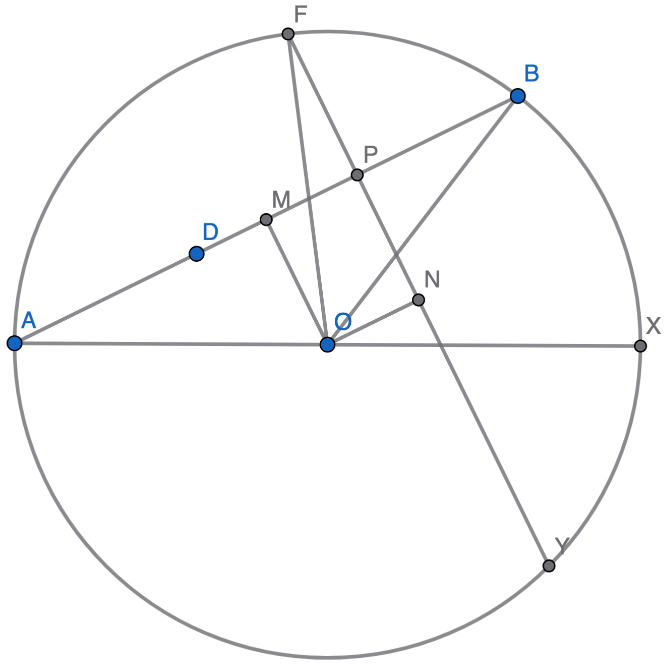
Next,

$$FH = EF\cos(\angle HFE) = -R\frac{\cos(2x+2y)}{\cos(2y-x)}\sin(2y-x) = -R\cos(2x+2y)\tan(2y-x).$$

Finally, $AH = AF - FH = (R(\sin(2x+2y) - \cos(2x+2y)\tan(2y-x))) - (-R\cos(2x+2y)\tan(2y-x)) = R\sin(2x+2y) = \frac{1}{2}AB$, completing the proof.

# 6. IMO 2018 Problem 1

Let $\Gamma$ be the circumcircle of acute-angled triangle $ABC$. Points $D$ and $E$ lie on segments $AB$ and $AC$, respectively, such that $AD = AE$. The perpendicular bisectors of $BD$ and $CE$ intersect the minor arcs $AB$ and $AC$ of $\Gamma$ at points $F$ and $G$, respectively. Prove that the lines $DE$ and $FG$ are parallel (or are the same line).




Solution: Let $O$ be center of $\Gamma$ and $AO$ meets $\Gamma$ at X. Let $FY$ be the perpendicular bisector of $BD$ and draw $OM \perp AB$ and $ON \perp FY$. Let $P = FY \cap AB$. Let $\angle BOX = 2\beta$, $AD = 2d$ and $R$ be the radius of $\Gamma$. We assume that $0° < 2\beta < 180°$.

We start by observing that $\angle BAO = \angle ABO = \beta$. Since $FY$ is perpendicular to both $AB$ and $ON$, we have $ON \parallel AB$. Therefore, $\angle NOX = \angle BOA = \beta$.

Next, since $FY$ and $OM$ are both perpendicular to $AB$, we have that $ONMP$ is a rectangle. Therefore, $ON = MP = MB - PB = \frac{1}{2}AB - \frac{1}{2}DB = \frac{1}{2}AD = d$. Thus, $\angle FON = \cos^{-1}\left(\frac{ON}{OF}\right) = \cos^{-1}\left(\frac{d}{R}\right)$. Hence, $\angle FOX = \beta + \cos^{-1}\left(\frac{d}{R}\right)$.

Similarly, suppose the point $C$ is chosen on $\Gamma$ such that $\angle COX = 2\gamma$ and $-180° < 2\gamma < 0$. Then, we will have that $\angle GOX = \gamma - \cos^{-1}\left(\frac{d}{R}\right)$.

Thus, $\angle FOG = \beta - \gamma + 2\cos^{-1}\left(\frac{d}{R}\right)$. This implies that $\angle OFG = \angle OGF = 90° - \frac{\beta-\gamma}{2} - \cos^{-1}\left(\frac{d}{R}\right)$. Hence, the angle that the line $FG$ makes with $AX$ is equal to:

$$\angle OFG + \angle FOX = \left(90° - \frac{\beta - \gamma}{2} - \cos^{-1}\left(\frac{d}{R}\right)\right) + \left(\beta + \cos^{-1}\left(\frac{d}{R}\right)\right) = 90° + \frac{\beta + \gamma}{2}.$$

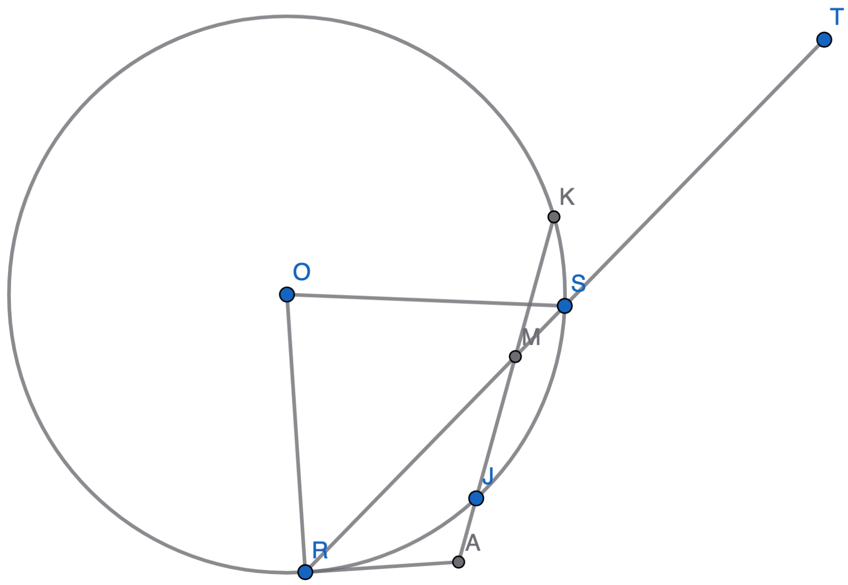
But then, exactly by a similar argument, we have $\angle DOE = \beta - \gamma$ and $\angle ADE = \angle AED = 90^\circ - \frac{\beta-\gamma}{2}$. Thus, the angle that the line $DE$ makes with AX is equal to:

$$\angle ADE + \angle DAX = 90^\circ - \frac{\beta - \gamma}{2} + \beta = 90^\circ + \frac{\beta + \gamma}{2}.$$

As the two angles above are equal, the claim stands proven.

# 7. IMO 2017 Problem 4

Let $R$ and $S$ be different points on a circle $\Omega$ such that $RS$ is not a diameter. Let $\ell$ be the tangent line to $\Omega$ at $R$. Point $T$ is such that $S$ is the midpoint of the line segment $RT$. Point $J$ is chosen on the shorter arc $RS$ of $\Omega$ so that the circumcircle $\Gamma$ of triangle $JST$ intersects $\ell$ at two distinct points. Let $A$ be the common point of $\Gamma$ and $\ell$ that is closer to $R$. Line $AJ$ meets $\Omega$ again at $K$. Prove that the line $KT$ is tangent to $\Gamma$.




Solution: Let $M = RT \cap AK$ and let $\angle ROS = 2\alpha$ and $\angle RMA = \gamma$. Finally, let $RM = x$ and $RS = d$. To prove the required claim, we need to show that $KT^2 = KJ \cdot KA$.

To that end, we compute the power of the point $M$ with respect to $\Omega$ and $\Gamma$. First, note that $MJ \cdot MA = MS \cdot MT$. Thus, $MJ = \frac{(d-x)(2d-x)}{MA}$. Next, $MJ \cdot MK = MR \cdot MS$ and so $MK = \frac{x(d-x)}{MJ} = \frac{x}{2d-x}MA$. This implies that $KJ = MK + MJ = \frac{x}{2d-x}MA + \frac{(d-x)(2d-x)}{MA}$. Finally, $KA = MK + MA = \frac{x}{2d-x}MA + MA = \frac{2d}{2d-x}MA$.

Next, in $\triangle MRA$, we have $\angle RMA = \gamma$ and $\angle MRA = \alpha$. So, $\angle RAM = 180° - \alpha - \gamma$. Hence, by the sine rule, $MA = x\frac{\sin\alpha}{\sin(\alpha+\gamma)}$ and $RA = x\frac{\sin\gamma}{\sin(\alpha+\gamma)}$. Next, since $AR$ is tangent to $\Omega$, we have $AR^2 = AJ \cdot AM$. Thus,

$$AR^2 = (AM - MJ) \cdot AK$$

$$AR^2 = \left(MA - \frac{(d-x)(2d-x)}{MA}\right)\frac{2d}{2d-x}MA$$

$$AR^2 = \frac{2d}{2d-x}MA^2 - 2d(d-x)$$

$$x^2\frac{\sin^2\gamma}{\sin^2(\alpha+\gamma)} = x^2\frac{2d}{2d-x}\frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)} - 2d(d-x)$$

$$-2d(d-x) = x^2\frac{\sin^2\gamma}{\sin^2(\alpha+\gamma)} - x^2\frac{2d}{2d-x}\frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)}.$$

Finally, we can compute the required expressions.

$$\begin{aligned}
KJ\cdot KA &= \Big(\frac{x}{2d-x}MA + \frac{(d-x)(2d-x)}{MA}\Big)\frac{2d}{2d-x}MA \\
&= \frac{2dx}{(2d-x)^2}MA^2 + 2d(d-x) \\
&= \frac{2dx^3}{(2d-x)^2}\frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)} + 2d(d-x).
\end{aligned}$$

Next, by the cosine rule in $\triangle MKT$, we get:

$$\begin{aligned}
KT^2 &= MK^2 + MT^2 - 2MK\cdot MT\cos(\angle KMT) \\
&= \frac{x^4}{(2d-x)^2}\frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)} + (2d-x)^2 - 2x^2\frac{\sin\alpha\cos\gamma}{\sin(\alpha+\gamma)}.
\end{aligned}$$

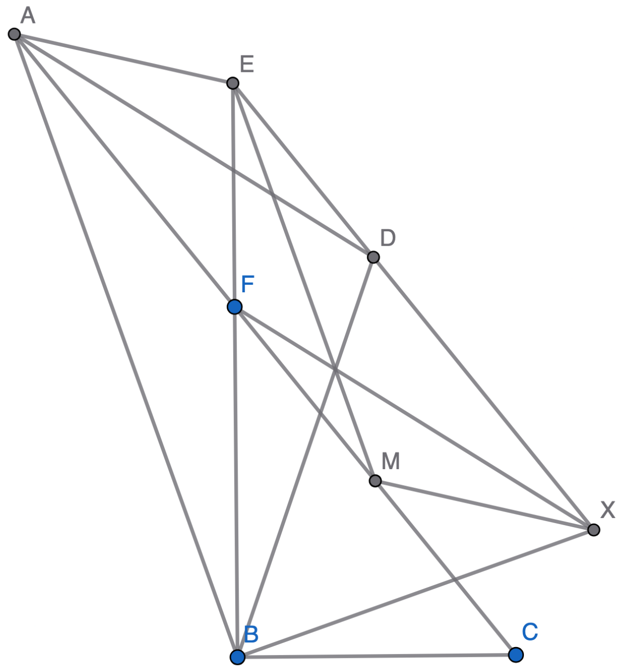
Therefore,

$$\begin{aligned}
KJ\cdot KA - KT^2 &= \frac{x^3\sin^2\alpha}{(2d-x)\sin^2(\alpha+\gamma)} + 2x^2\frac{\sin\alpha\cos\gamma}{\sin(\alpha+\gamma)} - 2d(d-x) - x^2 \\
&= \frac{x^3\sin^2\alpha}{(2d-x)\sin^2(\alpha+\gamma)} + 2x^2\frac{\sin\alpha\cos\gamma}{\sin(\alpha+\gamma)} + x^2\frac{\sin^2\gamma}{\sin^2(\alpha+\gamma)} - x^2\frac{2d}{2d-x}\frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)} - x^2 \\
&= 2x^2\frac{\sin\alpha\cos\gamma}{\sin(\alpha+\gamma)} + x^2\frac{\sin^2\gamma}{\sin^2(\alpha+\gamma)} - x^2\frac{\sin^2\alpha}{\sin^2(\alpha+\gamma)} - x^2 \\
&= x^2\frac{2\sin\alpha\cos\gamma\sin(\alpha+\gamma) + \sin^2\gamma - \sin^2\alpha - \sin^2(\alpha+\gamma)}{\sin^2(\alpha+\gamma)} \\
&= 0,
\end{aligned}$$

where the last equality follows by expanding out $\sin(\alpha+\gamma)$, thus proving the claim.

## 8. IMO 2016 Problem 1

$\triangle BCF$ has a right angle at $B$. Let $A$ be the point on line $CF$ such that $FA = FB$ and $F$ lies between $A$ and $C$. Point $D$ is chosen such that $DA = DC$ and $AC$ is the bisector $\angle DAB$. Point $E$ is chosen such that $EA = ED$ and $AD$ is the bisector of $\angle EAC$. Let $M$ be the midpoint of $CF$. Let $X$ be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that $BD$, $FX$ and $ME$ are concurrent.




Solution: Let $\angle BFC = y$. Also, let $BF = c$ and $FC = b$. Since $AF = FB$, we have $\angle FBA = \angle BAF = \angle FAD = \angle DAE = \frac{y}{2}$.

We start by showing that the point $D$ lies on the $EX$. To prove this, it suffices to show that:

$$\frac{AD}{\sin(\angle AEX)} = \frac{DE}{\sin(\angle DAE)}.$$

Now, $AD = \frac{1}{2}AC\sec(DAF) = \frac{b+c}{2}\sec(\frac{y}{2})$. Next, $\angle AEX = 180^\circ - \angle FAE = 180^\circ - y$. Finally, $DE = AE = \frac{1}{2}AD\sec(\angle DAE) = \frac{b+c}{4}\sec^2(\frac{y}{2}) = \frac{b+c}{4\cos^2(\frac{y}{2})} = \frac{b+c}{2(\cos y+1)} = \frac{b}{2}$. Then, we see that:

$$\frac{AD}{\sin(\angle AEX)} = \frac{\frac{b+c}{2}\sec(\frac{y}{2})}{\sin(y)} = \frac{b+c}{4\sin(\frac{y}{2})\cos^2(\frac{y}{2})} = \frac{DE}{\sin(\frac{y}{2})} = \frac{DE}{\sin(\angle DAE)},$$

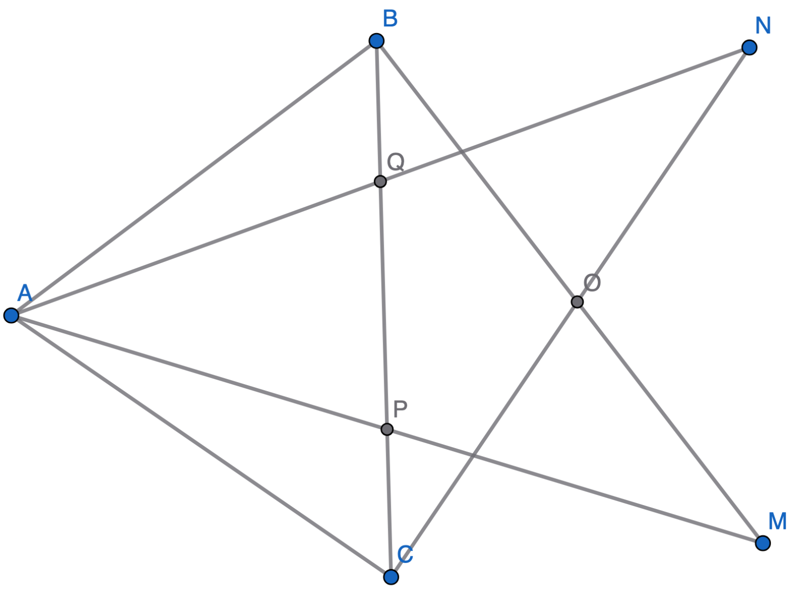
proving that $D$ lies on $EX$.

Next, we show that $F$ lies on $BE$. For this, it suffices to show that $y = \angle BFC = \angle AFE$. Since we already know that $\angle FAE = y$, this is equivalent to showing that $AE = EF$. For this, we use the cosine rule in $\triangle FAE$ to observe that $FE^2 = AE^2 + AF^2 - 2AE \cdot AF\cos(\angle FAE)$. So, it suffices to show that $AF = 2AE\cos(\angle FAE)$. Note that $AF = c$ whereas $2AE\cos(\angle FAE) = b\cos y = c$, completing the proof.

Next, we note that $EX = AM = AF + FM = c + \frac{b}{2}$. On the other hand, $EB = EF + FB = \frac{b}{2} + c$, showing that $EB = EX$. Since we already know that $EF = EA = ED$, this also implies that $DX = FB$. Hence, applying Ceva's theorem to $\triangle EBX$, the the lines $EM, BD$ and $FX$ will be concurrent if any only if $EM$ bisects the side $BX$, which is equivalent to $EM$ bisecting $\angle BEX$, since the triangle is isosceles. To that end, we observe that $MB = \frac{b}{2} = AE - MX$. Hence, $\triangle EMB \cong \triangle EMX$, which proves that $EM$ is the angle bisector, thus proving the claim.

# 9. **IMO 2014 Problem 4**

Points $P$ and $Q$ lie on side $BC$ of acute-angled triangle $ABC$ so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points $M$ and $N$ lie on lines $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$, and $Q$ is the midpoint of $AN$. Prove that lines $BM$ and $CN$ intersect on the circumcircle of triangle $ABC$.




Solution: Note that $\angle AQP = \angle APQ = \angle CAB$. Thus $AQ = AP$. We establish a coordinate system such that $A = (0,0), B = (1,b), C = (1,c), Q = (1,q)$ and $P = (1,-q)$ with $b,q > 0$ and $c < 0$. Since $\angle CAQ = \angle ABC$, we must have:

$$\frac{q-c}{1+qc} = \frac{1}{b} \implies bq - bc - qc = 1 \implies q = \frac{1+bc}{b-c}.$$

The equation of the circumcircle $ABC$ is given by:

$$x^2 + y^2 - (1-bc)x - (b+c)y = 0.$$

Next, we have that $N = 2Q = (2, 2q)$ and $M = 2P = (2, -2q)$. Then we have the equation of $BM$:

$$\frac{y-b}{x-1} = \frac{b+2q}{-1} \implies y + (b+2q)x = 2b + 2q.$$

Similarly, the equation of $CN$ is:

$$y + (c-2q)x = 2c - 2q.$$

The angle between these two lines is the arctan of:

$$\frac{(b+2q)-(c-2q)}{1+(b+2q)(c-2q)} = \frac{b-c+4q}{1+bc+2cq-2bq-4q^2} = \frac{b-c+4q}{-1-bc-4q^2}.$$

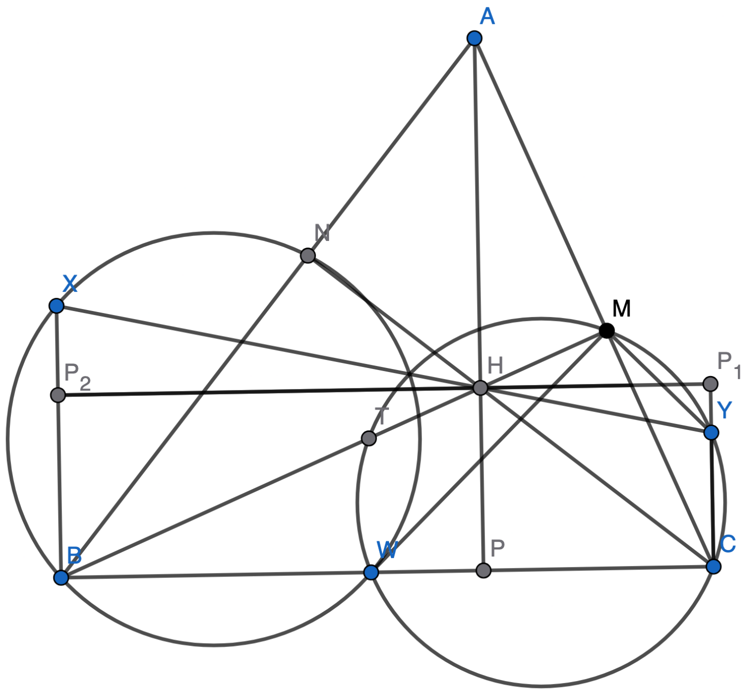
To show that $A, B, O$ and $C$ are concyclic, it suffices to show that this is equal to $-\tan(\angle BAC)$, that is:

$$\begin{aligned} \frac{b-c+4q}{-1-bc-4q^2} &= -\frac{b-c}{1+bc} \\ \iff 4(1+bc)q &= 4q^2(b-c) \\ \iff q &= \frac{1+bc}{b-c}, \end{aligned}$$

which was already shown, thus completing the proof.

## 10. **IMO 2013 Problem 4**

Let $ABC$ be an acute-angled triangle with orthocentre $H$, and let $W$ be a point on the side $BC$, lying strictly between $B$ and $C$. The points $M$ and $N$ are the feet of the altitudes from $B$ and $C$, respectively. Denote by $\omega_1$ the circumcircle of $BWN$, and let $X$ be the point on $\omega_1$ such that $WX$ is a diameter of $\omega_1$. Analogously, denote by $\omega_2$ the circumcircle of $CWM$, and let $Y$ be the point on $\omega_2$ such that $WY$ is a diameter of $\omega_2$. Prove that $X$, $Y$ and $H$ are collinear.




Solution: Let $a = BC, b = CA$ and $c = AB$. Let $R$ be the circumradius of $\triangle ABC$. Construct $P_1P_2$ parallel to $BC$ through the point $H$. To prove the required claim, it suffices to show that $\angle XHP_2 = \angle YHP_1$.

We have $\angle HMC = \angle WMY = 90°$. Therefore, $\angle TMW = \angle CMY$, which implies that $TW = CY$. Consider the power of the point $B$ with respect to the circle $\omega_2$:

$$BT \cdot BM = BW \cdot BC.$$

$$\begin{aligned} BT &= \frac{BW \cdot BC}{BM} \\ &= \frac{aBW}{a \sin C} = \frac{BW}{\sin C}. \end{aligned}$$

Next, note that $\angle TBW = 90° - C$. Applying cosine law to $\triangle BWT$:

$$\begin{aligned} TW^2 &= BT^2 + BW^2 - 2BT \cdot BW \cos(TBW) \\ &= BT^2 + BW^2 - 2BT \cdot BW \sin C \end{aligned}$$

$$
\begin{aligned}
&= \frac{BW^2}{\sin^2 C} + BW^2 - 2\frac{BW}{\sin C} \cdot BW \sin C \\
&= BW^2 \csc^2 C - BW^2 \\
&= BW^2 \cot^2 C.
\end{aligned}
$$

Thus, $TW = BW \cot C$. This implies that $CY = BW \cot C$. Similarly, we get that $BX = CW \cot B$.

As $XB \perp BC$, we must have $XB \perp P_1P_2$ since $P_1P_2 \parallel BC$. Similarly, $YC \perp P_1P_2$. Therefore,

$$
\begin{aligned}
\tan(\angle XHP_2) &= \frac{XP_2}{HP_2} \\
&= \frac{BX - BP_2}{HP_2} \\
&= \frac{BX - HP}{BP} \\
&= \frac{CW \cot B - 2R \cos B \cos C}{2R \sin C \cos B} \\
&= \frac{CW}{2R \sin B \sin C} - \cot C.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\tan(\angle YHP_1) &= \frac{YP_1}{HP_1} \\
&= \cot B - \frac{BW}{2R \sin B \sin C}.
\end{aligned}
$$

Then, we see that:

$$
\begin{aligned}
\tan(\angle XHP_2) - \tan(\angle YHP_1) &= \frac{BW + CW}{2R \sin B \sin C} - \cot B - \cot C \\
&= \frac{a}{2R \sin B \sin C} - \frac{\sin(B + C)}{\sin B \sin C} \\
&= \frac{\sin A}{\sin B \sin C} - \frac{\sin A}{\sin B \sin C} \\
&= 0,
\end{aligned}
$$

which shows that $\angle XHP_2 = \angle YHP_1$, completing the proof.