OLYMPIAD GEOMETRY

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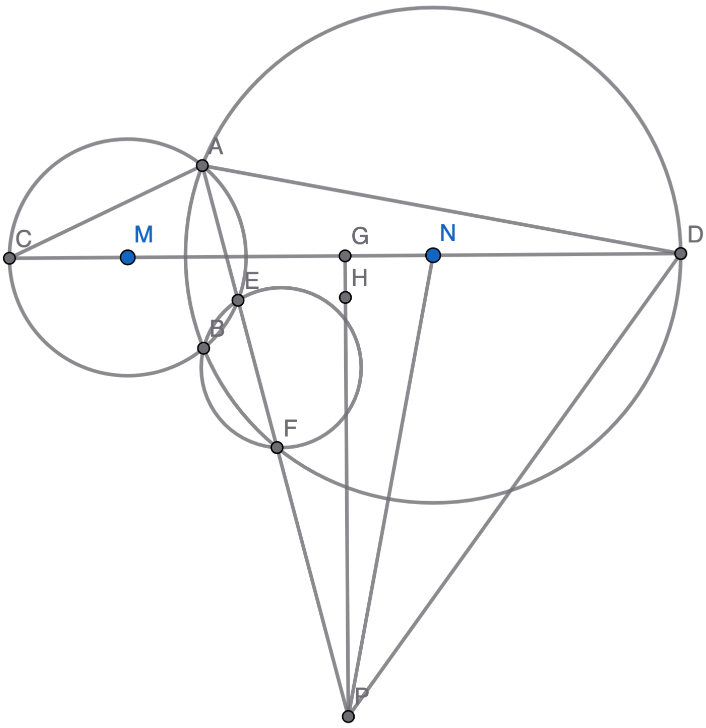
ABSTRACT. This document is a compilation of my attempts at bashing IMO geometry problems using algebraic tools.

CONTENTS

1. IMO 2025 Problem 2

Let Ω and Γ be circles with centres M and N, respectively, such that the radius of Ω is less than the radius of Γ. Suppose circles Ω and Γ intersect at two distinct points A and B. Line MN intersects Ω at C and Γ at D, such that points C, M, N and D lie on the line in that order. Let P be the circumcentre of triangle ACD. Line AP intersects Ω again at $E \neq A$. Line AP intersects Γ again at $F \neq A$. Let H be the orthocentre of triangle PMN. Prove that the line through H parallel to AP is tangent to the circumcircle of triangle BEF.



Solution: Suppose $AB = x, CD = y, \angle ACD = C$ and $\angle ADC = D$. Since $AB \perp CD$, it is clear that we have the following relation between these quantities:

$$y = \frac{x}{2}(\cot C + \cot D) = \frac{x \sin(C + D)}{2 \sin C \sin D}.$$

We start by computing the circumradius R of the $\triangle BEF$. To this end, we note that $\angle AEB = 180° - 2C$, which implies that $\angle BEF = 2C$. Similarly, we have $\angle BFE = 2D$. Next, as P is the circumcenter of $\triangle CAD$, we have $\angle APD = 2C$. So, $\angle PAD = 90° - C$. This implies that:

$$\angle BAE = \angle BAD - \angle EAD = (90° - D) - (90° - C) = C - D.$$

In $\triangle ABE$, by the sine rule, we have:

$$BE = AB \frac{\sin(\angle BAE)}{\sin(\angle AEB)} = x \frac{\sin(C - D)}{\sin(180° - 2C)} = x \frac{\sin(C - D)}{\sin 2C}.$$

Then, the circumradius R of $\triangle BEF$ is given by:

$$R = \frac{BE}{2\sin(\angle BFE)} = x\frac{\sin(C-D)}{2\sin 2D\sin 2C}.$$

Next, we compute the distance of the line EF from the center of the circle BEF. Again by the sine rule, we have $EF = 2R\sin(\angle EBF) = 2R\sin(180^\circ - 2C - 2D) = 2R\sin(2C+2D)$. Hence, the distance of EF from the center is given by:

$$\sqrt{R^2 - R^2\sin^2(2C+2D)} = R|\cos(2C+2D)|.$$

Then, the distance of a tangent line to the circle BEF that is parallel to EF from the line EF is given by:

$$R \pm R|\cos(2C+2D)| = 2R\cos^2(C+D) \text{ or } 2R\sin^2(C+D).$$

In order to show that the parallel to EF that passes through H is parallel to the circle BEF, it suffices to compute the distance of this parallel line from EF and to verify that it is equal to one of the 2 quantities above. The distance of this parallel line from EF is equal to $HP\sin(\angle HPF)$. So, to prove the required claim, it suffices to prove the equality:

$$HP\sin(\angle HPF) = 2R\cos^2(C+D).$$

Note that $\angle HPF = \angle BAE = C - D$ since $AB \parallel PG$, since both are perpendicular to CD.

Next, note that $\angle PCD = \angle PCA - C = 90^\circ - D - C$. Also, PG bisects CD and so $CG = \frac{y}{2}$ Therefore, $PG = CG\tan(\angle PCD) = \frac{y}{2}\cot(C+D)$.

Also, by the sine rule for $\triangle ABC$, we have that the radius CM of the circle ABC is equal to: $CM = \frac{AB}{2\sin(\angle ACB)} = \frac{x}{2\sin 2C}$. Hence,

$$\begin{aligned} MG &= CG - CM \\ &= \frac{y}{2} - \frac{x}{2\sin 2C} \\ &= \frac{x\sin(C+D)}{4\sin C\sin D} - \frac{x}{2\sin 2C} \\ &= \frac{x}{4}\Big(\frac{\sin(C+D)\cos C - \sin D}{\sin C\cos C\sin D}\Big) \\ &= \frac{x}{4}\Big(\frac{\cos(C+D)\sin C}{\sin C\cos C\sin D}\Big) \\ &= \frac{x}{4}\Big(\frac{\cos(C+D)}{\cos C\sin D}\Big). \end{aligned}$$

Similarly, we have $NG = \frac{x}{4}\big(\frac{\cos(C+D)}{\cos D\sin C}\big)$.

Hence,

$$\tan(\angle PMG) = \frac{PG}{MG} = \frac{\frac{y}{2}\cot(C+D)}{\frac{x}{4}\big(\frac{\cos(C+D)}{\cos C\sin D}\big)} = \cot C.$$

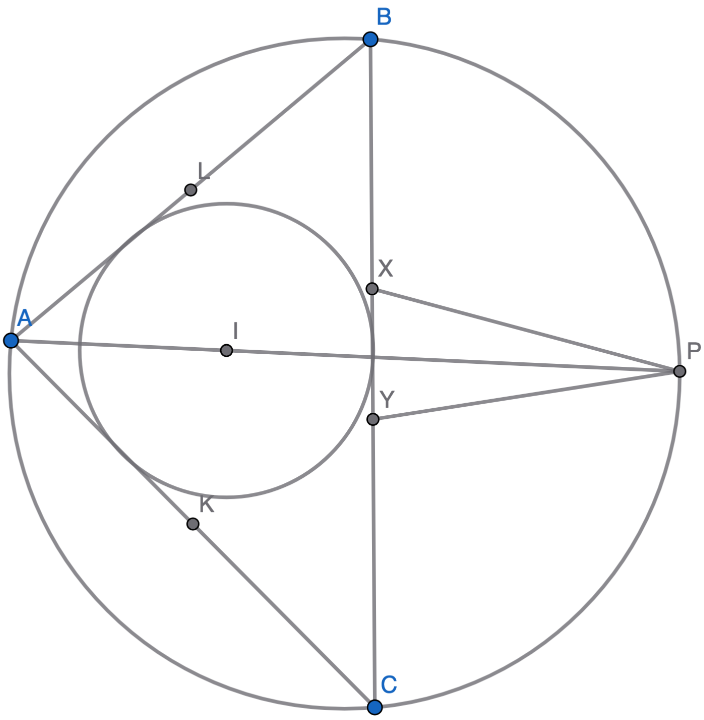
Furthermore, $\angle NHG = 180^\circ - \angle PHG = 180^\circ - (180^\circ - \angle PMG) = \angle PMG$. Therefore, $HG = NG\cot(\angle NHG) = \frac{x}{4}\big(\frac{\cos(C+D)}{\cos D\cos C}\big)$. Hence, we have:

$$\begin{aligned} HP &= PG - HG \\ &= \frac{x\cos(C+D)}{4\sin D\sin C} - \frac{x\cos(C+D)}{4\cos D\cos C} \\ &= \frac{x\cos^2(C+D)}{4\sin C\sin D\cos C\cos D} \\ &= \frac{x\cos^2(C+D)}{\sin 2C\sin 2D}. \end{aligned}$$

Thus, $HP\sin(\angle HPF) = \frac{x\cos^2(C+D)}{\sin 2C\sin 2D}\sin(C-D) = 2R\cos^2(C+D)$, completing the proof.

2. IMO 2024 Problem 4

Let ABC be a triangle with $AB < AC < BC$. Let the incentre and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively. Prove that $\angle KIL + \angle YPX = 180°$.



Solution: Let $I = (0, 0)$ and the radius of the circle $\omega_1 = 1$. Let $B = (1, b)$ and $C = (1, c)$ for some $b > 0$ and $c < 0$. In order to show that $\angle KIL + \angle YPX = 180°$, it suffices to show that $\tan(\angle KIL) = -\tan(\angle YPX)$.

Let the slope of the line AB be m. Then the equation of AB is:

$$\frac{y-b}{x-1} = m.$$

As AB is tangent to ω_1, its distance from I should be 1. Thus,

$$\frac{|m-b|}{\sqrt{1+m^2}} = 1 \implies m = \frac{b^2-1}{2b}.$$

Hence, the equation of AB is:

$$\frac{y-b}{x-1} = \frac{b^2-1}{2b}.$$

Similarly, the equation of AC is:

$$\frac{y-c}{x-1} = \frac{c^2-1}{2c}.$$

The intersection of these lines gives the coordinates of point A:

$$A = \Big(\frac{1-bc}{1+bc}, \frac{b+c}{1+bc}\Big).$$

Then, we compute:

$$K = \frac{A+C}{2} = \Big(\frac{1}{1+bc}, \frac{bc^2+2c+b}{2(1+bc)}\Big),$$
$$L = \frac{A+B}{2} = \Big(\frac{1}{1+bc}, \frac{b^c c+2b+c}{2(1+bc)}\Big).$$

Hence,

$$\text{Slope of } IK = \frac{bc^2+2c+b}{2},$$
$$\text{Slope of } IL = \frac{b^2c+2b+c}{2}.$$

Therefore,

$$\tan(\angle KIL) = \frac{\frac{bc^2+2c+b}{2} - \frac{b^2c+2b+c}{2}}{1+\big(\frac{bc^2+2c+b}{2}\big)\big(\frac{b^2c+2b+c}{2}\big)}$$
$$= \frac{2(c-b)(bc+1)}{4+(bc^2+2c+b)(b^2c+2b+c)}.$$

Next, we compute the coordinates of P. As AI bisects $\angle BAC$, we have that $PB = PC$. Thus, $P = (r, \frac{b+c}{2})$ for some r. Also, since A, I and P are colinear, we have:

$$\text{Slope of } AI = \text{Slope of } PI \implies \frac{1-bc}{b+c} = \frac{2r}{b+c}.$$

This gives that $r = \frac{1-bc}{2}$ implying that $P = (\frac{1-bc}{2}, \frac{b+c}{2})$.

Finally, we find coordinates of X and Y. The slope of the line through X tangent to ω is the same as the slope of AC, which is equal to $\frac{c^2-1}{2c}$. Hence, the equation of the tangent line is

$$y - \frac{c^2-1}{2c}x = \alpha$$

for some α. For this to be tangent to ω_1, its distance from I should be 1. Therefore,

$$\frac{|\alpha|}{\sqrt{1+(\frac{c^2-1}{2c})^2}} = 1 \implies \alpha = \pm\frac{c^2+1}{2c}.$$

Thus, the equation of the tangent to ω through the point X is:

$$y - \frac{c^2-1}{2c}x = -\frac{c^2+1}{2c}.$$

(We choose the negative sign, since the positive sign corresponds to the line AC.) Intersecting this tangent line with the line BC, which is given by $x = 1$, we get that $X = (1, -\frac{1}{c})$. Similarly, we get that $Y = (1, -\frac{1}{b})$.

So, we can compute:

$$\text{Slope of } PX = \frac{\frac{b+c}{2}+\frac{1}{c}}{\frac{1-bc}{2}-1} = -\frac{bc+c^2+2}{c(1+bc)},$$
$$\text{Slope of } PY = \frac{\frac{b+c}{2}+\frac{1}{b}}{\frac{1-bc}{2}-1} = -\frac{bc+b^2+2}{b(1+bc)}.$$

Hence,

$$\begin{aligned}\tan(\angle YPX) &= \frac{-\frac{bc+b^2+2}{b(1+bc)} + \frac{bc+c^2+2}{c(1+bc)}}{1+\left(\frac{bc+b^2+2}{b(1+bc)}\right)\left(\frac{bc+c^2+2}{c(1+bc)}\right)} \\ &= \frac{-bc(b+c)(1+bc) - 2c(1+bc) + bc(b+c)(1+bc) + 2b(1+bc)}{bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2)} \\ &= \frac{2(b-c)(1+bc)}{bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2)}.\end{aligned}$$

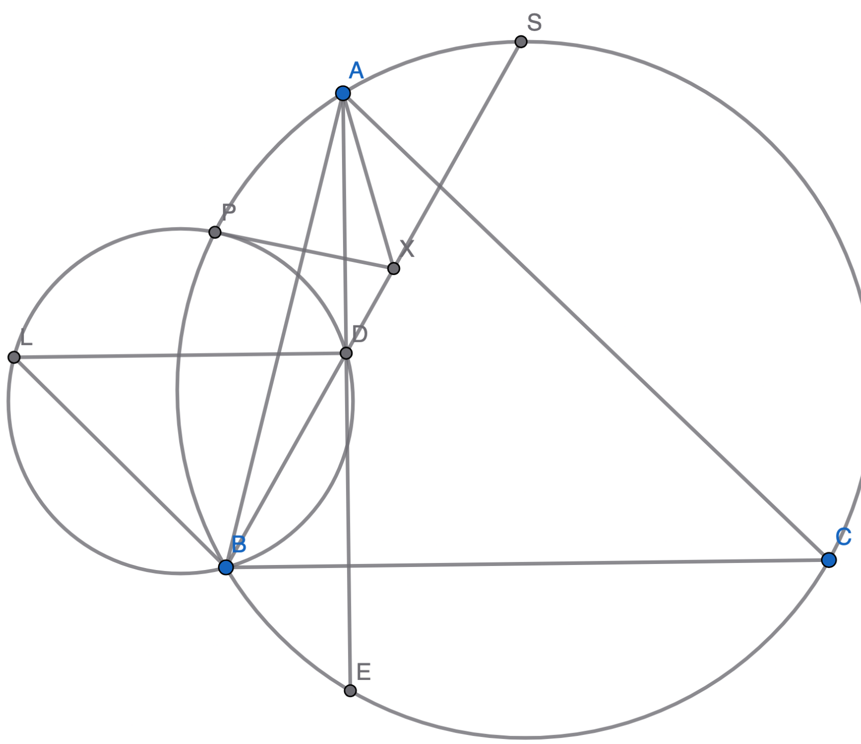
Therefore, from our expressions for $\tan(\angle KIL)$ and $\tan(\angle YPX)$, it follows that the equality $\tan(\angle KIL) = -\tan(\angle YPX)$ is equivalent to the following algebraic identity:

$$4 + (bc^2 + 2c + b)(b^2c + 2b + c) = bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2),$$

which is easily verified.

3. IMO 2023 Problem 2

Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$. Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.



Solution: We have $\angle SBC = \angle SCB = 90^\circ - \frac{A}{2}$. Also, $\angle CBE = \angle CAE = 90^\circ - C$. Thus, $\angle LBD = 180^\circ - \angle SBE = 180^\circ - (\angle SBC + \angle CBE) = 90^\circ + \frac{C-B}{2}$.

Let $\angle PBS = t$. In ω, sine rule gives:

$$\begin{aligned}\frac{PD}{\sin t} &= \frac{LD}{\sin(\angle LBD)} \\ &= \frac{LD}{\sin(90^\circ + \frac{C-B}{2})} \\ &= \frac{LD}{\cos(\frac{B-C}{2})}.\end{aligned}$$

Next, we have $(\angle PAD + \angle DAC) + \angle PBC = 180^\circ$. Thus,

$$\begin{aligned}\angle PAD &= 180^\circ - \angle PBC - \angle DAC \\ &= 180^\circ - (t + \frac{B+C}{2}) - (90^\circ - C)\end{aligned}$$

$$= 90^\circ - t + \frac{C-B}{2}.$$

Similarly, we have $(\angle DPA + \angle DPB) + \angle ACB = 180^\circ$, Thus,

$$\begin{aligned}\angle DPA &= 180^\circ - \angle DPB - \angle ACB \\ &= 180^\circ - \angle DLB - C \\ &= 180^\circ - (90^\circ - \angle LED) - C \\ &= 180^\circ - (90^\circ - C) - C \\ &= 90^\circ.\end{aligned}$$

Hence, we have that $PD = AD\sin(\angle PAD) = AD\sin(90^\circ - t + \frac{C-B}{2}) = AD\cos(\frac{C-B}{2} - t)$. Inserting this into the above sine rule equation, we get:

$$\begin{aligned}\frac{AD\cos(\frac{C-B}{2} - t)}{\sin t} &= \frac{LD}{\cos(\frac{B-C}{2})} \\ \implies AD\cos\left(\frac{C-B}{2}\right)\cot t + AD\sin\left(\frac{C-B}{2}\right) &= \frac{LD}{\cos(\frac{B-C}{2})} \\ \implies \cot t &= \frac{LD}{AD\cos^2(\frac{B-C}{2})} + \tan\left(\frac{B-C}{2}\right).\end{aligned}$$

(1) First, suppose PX is tangent to the circle ω. Then, $\angle DPX = \angle DBP = t$. So, by the sine rule in $\triangle PDX$ we get:

$$\begin{aligned}\frac{DX}{\sin t} &= \frac{PX}{\sin PDX} \\ &= \frac{PX}{\sin PLB}.\end{aligned}$$

Now, $\angle PLB = \angle PLD + \angle BLD = \angle PBD + \angle BLD = 90^\circ + t - C$. Hence,

$$\frac{PX}{DX} = \frac{\sin(90^\circ + t - C)}{\sin t} = \frac{\cos(t - C)}{\sin t} = \cot t\cos C + \sin C.$$

Computing the power of the point X with respect to ω, we get $PX^2 = BX \cdot DX$. Hence,

$$\frac{BX}{DX} = \frac{BX}{PX}\cdot\frac{PX}{DX} = \left(\frac{PX}{DX}\right)^2 = (\cot t\cos C + \sin C)^2.$$

(2) Now, suppose AX bisects $\angle BAC$. Then, $\angle AXB = 180^\circ - \angle XAB - \angle XBA = 180^\circ - \frac{A}{2} - (B - \frac{B+C}{2}) = 90^\circ + C$. Also, $\angle DAX = \angle DAC - \angle XAC = 90^\circ - C - \frac{A}{2}$. So, applying the sine rule in $\triangle ADX$, we get:

$$\begin{aligned}\frac{DX}{\sin(\angle DAX)} &= \frac{AD}{\sin(\angle AXD)} \\ \implies DX = AD\frac{\sin(90^\circ - C - \frac{A}{2})}{\sin(90^\circ + C)} &= AD\frac{\cos(C + \frac{A}{2})}{\cos C}.\end{aligned}$$

Next, applying the sine rule in $\triangle BAX$, we get:

$$\begin{aligned}\frac{BX}{\sin(\angle BAX)} &= \frac{BA}{\sin(\angle AXB)} \\ \implies BX = AB\frac{\sin(\frac{A}{2})}{\sin(90^\circ + C)} &= \frac{c\sin(\frac{A}{2})}{\cos C},\end{aligned}$$

where we suppose $AB = c$. Combining the two expressions above, we get that:

$$\frac{BX}{DX} = \frac{c\sin(\frac{A}{2})}{AD\cos(C + \frac{A}{2})}.$$

We have obtained expressions for $\frac{BX}{DX}$ in both of the above cases. Thus, to prove the required claim, it suffices to prove the equality:

$$(\cot t\cos C+\sin C)^2=\frac{c\sin(\frac{A}{2})}{AD\cos(C+\frac{A}{2})}.$$

To that end, we first compute AD. Note that $\angle ADB=180^\circ-\angle ABD-\angle BAD=180^\circ-(B-\frac{B+C}{2})-(90^\circ-B)=180^\circ-\frac{A}{2}$. Then, using the sine rule in $\triangle BAD$, we get:

$$AD=AB\frac{\sin(\angle ABD)}{\sin(\angle ADB)}=\frac{c\sin(\frac{B-C}{2})}{\sin(180^\circ-\frac{A}{2})}=\frac{c\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}.$$

This implies that:

$$\frac{c\sin(\frac{A}{2})}{AD\cos(C+\frac{A}{2})}=\frac{\sin^2(\frac{A}{2})}{\cos(C+\frac{A}{2})\sin(\frac{B-C}{2})}=\frac{\sin^2(\frac{A}{2})}{\sin^2(\frac{B-C}{2})}$$

since $\frac{B-C}{2}=90^\circ-(C+\frac{A}{2})$. Thus, we are reduced to proving the equality:

$$\cot t\cos C+\sin C=\frac{\sin(\frac{A}{2})}{\sin(\frac{B-C}{2})}.$$

Recall that $\cot t=\frac{LD}{AD\cos^2(\frac{B-C}{2})}+\tan\left(\frac{B-C}{2}\right)$. To compute this, we need to find LD. Note that $LD=DE\tan(\angle LED)=DE\tan C=(AE-AD)\tan C$. Applying the sine rule in circle Ω, we get:

$$AE=AB\frac{\sin(\angle ABE)}{\sin(\angle ACB)}=\frac{c\sin(90+B-C)}{\sin C}=\frac{c\cos(B-C)}{\sin C}=c\cot C\cos B+c\sin B.$$

Therefore,

$$\begin{aligned}LD&=\tan C(AE-AD)\\&=c\tan C\Big(\cot C\cos B+\sin B-\frac{\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}\Big)\\&=c\cos B+c\tan C\sin B-c\frac{\tan C\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{LD}{AD\cos^2(\frac{B-C}{2})}&=\frac{\cos B+\tan C\sin B-\frac{\tan C\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}}{\frac{\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}\cos^2(\frac{B-C}{2})}\\&=\frac{\sin(\frac{A}{2})(\cos B+\tan C\sin B)-\tan C\sin(\frac{B-C}{2})}{\sin(\frac{B-C}{2})\cos^2(\frac{B-C}{2})}.\end{aligned}$$

This implies that:

$$\begin{aligned}\cot t&=\frac{LD}{AD\cos^2(\frac{B-C}{2})}+\tan\Big(\frac{B-C}{2}\Big)\\&=\frac{\sin(\frac{A}{2})(\cos B+\tan C\sin B)-\tan C\sin(\frac{B-C}{2})+\sin^2(\frac{B-C}{2})\cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2})\cos^2(\frac{B-C}{2})}.\end{aligned}$$

Next,

$$\begin{aligned}\cos C\cot t&=\frac{\sin(\frac{A}{2})(\cos B\cos C+\sin C\sin B)-\sin C\sin(\frac{B-C}{2})+\cos C\sin^2(\frac{B-C}{2})\cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2})\cos^2(\frac{B-C}{2})}\\&=\frac{\sin(\frac{A}{2})\cos(B-C)-\sin C\sin(\frac{B-C}{2})+\cos C\sin^2(\frac{B-C}{2})\cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2})\cos^2(\frac{B-C}{2})}.\end{aligned}$$

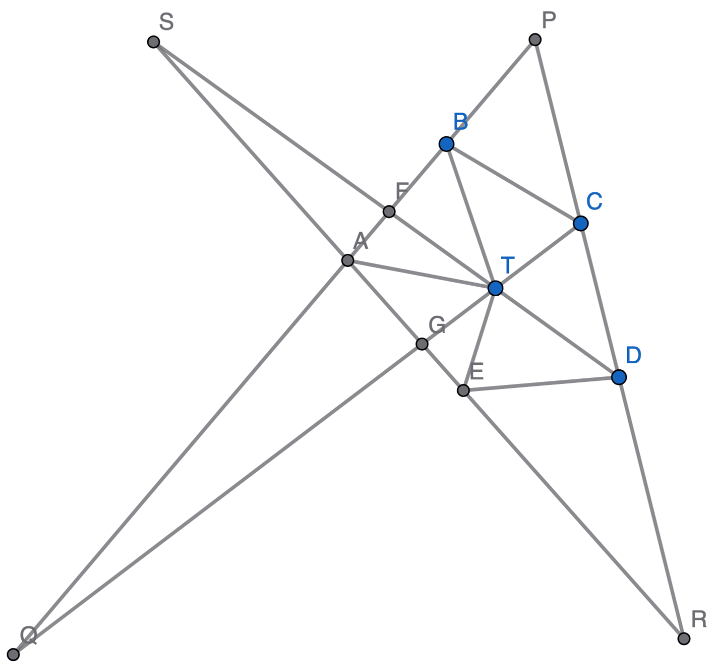
Finally,

$$\begin{aligned}
\sin C + \cos C \cot t &= \frac{\sin C \sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2}) + \sin(\frac{A}{2}) \cos(B-C) - \sin C \sin(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) - \sin C \sin^3(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin^2(\frac{B-C}{2}) \cos(C + \frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin^2(\frac{B-C}{2}) \cos(\frac{B+C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin(\frac{A}{2}) \sin^2(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2})(\cos(B-C) + \sin^2(\frac{B-C}{2}))}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2})}{\sin(\frac{B-C}{2})},
\end{aligned}$$

completing the proof.

4. IMO 2022 Problem 4

Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD, TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.



Solution: Suppose $TB = TD = s$ and $TC = TE = r$. We normalize $BC = DE = 1$. As we have $\triangle BTC \cong \triangle DTE$, we have the following equalities of angles:

$$\alpha := \angle BTC = \angle DTE$$

$$\beta := \angle TCB = \angle TED$$

$$\gamma := \angle CBT = \angle EDT.$$

Let $\angle DTC = \phi$ and $\angle ABT = \angle TEA = \theta$. Next, let $\angle TDC = m$ and $\angle TCD = n$. Finally, let $\angle FAT = f$ and $\angle GAT = g$. Note that $\angle BTE = 360° - (\angle BTC + \angle DTE + \angle DTC) = 360° - 2\alpha - 2\phi$. Thus, we have the equality:

$$f + g = 2\alpha + 2\phi - 2\theta.$$

As the angles at vertex B add to $180°$, we have $\angle CBP = 180° - \gamma - \theta$. Similar consideration at vertex C gives that $\angle BCP = 180° - \beta - n$. Thus, $\angle P = 180° - (\angle CBP + \angle BCP) = \theta + n - \alpha$.

Similarly, $\angle R = \theta + m - \alpha$. Hence, by the sine rule in $\triangle APR$, we get:

$$\frac{AP}{AR} = \frac{\sin(\theta + m - \alpha)}{\sin(\theta + n - \alpha)}.$$

Next, $\angle Q = 180° - (\angle P + \angle QCP) = \alpha - \theta$.

By exactly the same argument, we get that $\angle S = \alpha - \theta$. By the sine rule in $\triangle QAT$ and $\triangle SAT$, we have:

$$AQ = \frac{\sin(\angle ATQ)}{\angle Q} AT$$
$$AS = \frac{\sin(\angle ATS)}{\angle S} AT,$$

and so, $\frac{AQ}{AS} = \frac{\sin(\angle ATQ)}{\sin(\angle ATS)}$. Note that $\angle ATQ = \angle TAF - \angle Q = f - \alpha + \theta$. Similarly, $\angle ATS = g - \alpha + \theta$. So,

$$\frac{AQ}{AS} = \frac{\sin(f - \alpha + \theta)}{\sin(g - \alpha + \theta)}.$$

In order to show that P, S, Q and R are concylic, it suffices to show that:

$$\begin{aligned}
&AP \cdot AQ = AR \cdot AS \\
&\iff \frac{AP}{AR} = \frac{AS}{AQ} \\
&\iff \frac{\sin(\theta + m - \alpha)}{\sin(\theta + n - \alpha)} = \frac{\sin(g - \alpha + \theta)}{\sin(f - \alpha + \theta)} \\
&\iff \frac{\sin(\theta + m - \alpha) + \sin(\theta + n - \alpha)}{\sin(\theta + m - \alpha) - \sin(\theta + n - \alpha)} = \frac{\sin(g - \alpha + \theta) + \sin(f - \alpha + \theta)}{\sin(g - \alpha + \theta) - \sin(f - \alpha + \theta)} \\
&\iff \frac{\tan(\theta - \alpha + \frac{m+n}{2})}{\tan(\frac{m-n}{2})} = \frac{\tan(\theta - \alpha + \frac{f+g}{2})}{\tan(\frac{g-f}{2})} \\
&\iff \frac{\tan(\theta - \alpha + \frac{180° - 2\phi}{2})}{\tan(\frac{m-n}{2})} = \frac{\tan(\theta - \alpha + \frac{2\alpha + 2\phi - 2\theta}{2})}{\tan(\frac{g-f}{2})} \\
&\iff \frac{\cot(\phi - \theta + \alpha)}{\tan(\frac{m-n}{2})} = \frac{\tan(\phi)}{\tan(\frac{g-f}{2})} \\
&\iff \tan\left(\frac{g-f}{2}\right)\cot(\phi - \theta + \alpha) = \tan\left(\frac{m-n}{2}\right)\tan(\phi) \\
&\iff \tan\left(\frac{g-f}{2}\right)\cot(\frac{g+f}{2}\Big) = \tan\left(\frac{m-n}{2}\right)\cot(\frac{m+n}{2}\Big) \\
&\iff \frac{\sin(g) - \sin(f)}{\sin(g) + \sin(f)} = \frac{\sin(m) - \sin(n)}{\sin(m) + \sin(n)} \\
&\iff \frac{\sin(g)}{\sin(f)} = \frac{\sin(m)}{\sin(n)}.
\end{aligned}$$

Note that in $\triangle CDT$ by the sine rule:

$$\frac{\sin(m)}{\sin(n)} = \frac{TC}{TD} = \frac{r}{s}.$$

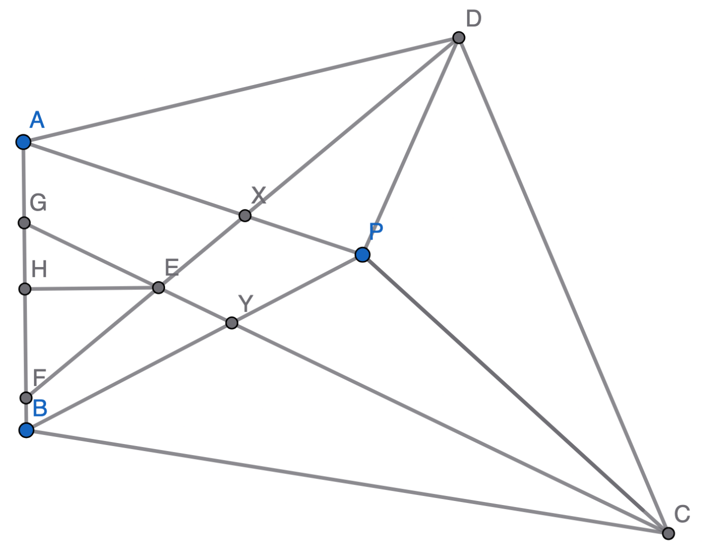
On the other hand, by the sine rule in $\triangle BAT$ and $\triangle EAT$, we have:

$$\frac{\sin(g)}{\sin(f)} = \frac{\frac{TE}{TA}\sin(\angle TEA)}{\frac{TB}{TA}\sin(\angle TBA)} = \frac{TE\sin(\theta)}{TB\sin(\theta)} = \frac{r}{s},$$

completing the proof.

5. IMO 2020 Problem 1

Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold: $\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC$. Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.



Solution: Let DF and CG be the bisectors of $\angle ADP$ and $\angle PCB$ respectively. Let $BP = a$, $AP = b$ and R be the circumradius of $\triangle PAB$. Let $\angle PBA = 2x$ and $\angle PAB = 2y$. Draw $EH \perp AB$. It suffices to show that H is the mid-point of AB.

Since AD bisects $\angle ADP$, we have by the sine rule in $\triangle ADP$:

$$\frac{AX}{XP} = \frac{AD}{DP} = \frac{\sin(\angle APD)}{\sin(\angle PAD)} = \frac{\sin 3x}{\sin x}.$$

Thus, $AX = b\frac{\sin 3x}{\sin x + \sin 3x} = b\frac{\sin 3x}{2\sin 2x \cos x} = R\frac{\sin 3x}{\cos x}$, by the sine rule in $\triangle APB$.

Next, $\angle AXF = \angle XAD + \angle XDA = \angle XAD + \frac{1}{2}\angle PDA = x + 90^\circ - 2x = 90^\circ - x$. This implies that $\angle XFA = 180^\circ - \angle FAX - \angle AXF = 90^\circ + x - 2y$. Then, by the sine rule in $\triangle AFX$, we get:

$$\begin{aligned} AF &= AX\frac{\sin(\angle AXF)}{\sin(\angle XFA)} \\ &= R\frac{\sin 3x}{\cos x}\frac{\cos x}{\cos(2y - x)} \\ &= R\frac{\sin 3x}{\cos(2y - x)} \end{aligned}$$

$$= R\frac{\sin((2x+2y)-(2y-x))}{\cos(2y-x)}$$
$$= R(\sin(2x+2y) - \cos(2x+2y)\tan(2y-x)).$$

Similarly, we have $BG = R(\sin(2x+2y) - \cos(2x+2y)\tan(2x-y))$. Finally, $AB = 2R\sin(\angle APB)) = 2R\sin(180° - 2x - 2y) = 2R\sin(2x+2y)$. Thus, we have:

$GF = AF + BG - AB = -R\cos(2x+2y)(\tan(2y-x) + \tan(2x-y)) = -R\cos(2x+2y)\dfrac{\sin(x+y)}{\cos(2y-x)\cos(2x-y)}$.

Note that $\angle GFE = 90° + x - 2y$ and $\angle FGE = 90° + y - 2x$. Thus, we have $\angle GEF = x + y$. Hence, by the sine rule in $\triangle GFE$, we get:

$EF = GF\dfrac{\sin(\angle FGE)}{\sin(\angle GEF)} = -R\cos(2x+2y)\dfrac{\sin(x+y)}{\cos(2y-x)\cos(2x-y)}\dfrac{\cos(2x-y)}{\sin(x+y)} = -R\dfrac{\cos(2x+2y)}{\cos(2y-x)}$.

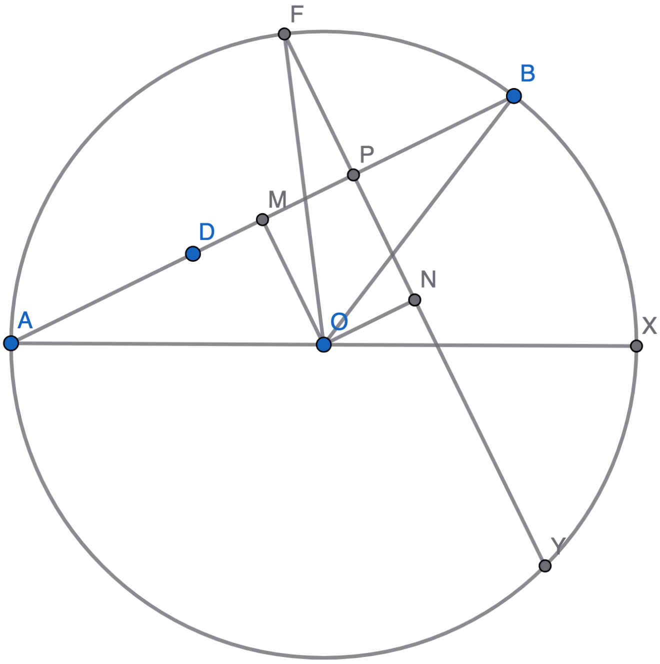
Next,

$$FH = EF\cos(\angle HFE) = -R\frac{\cos(2x+2y)}{\cos(2y-x)}\sin(2y-x) = -R\cos(2x+2y)\tan(2y-x).$$

Finally, $AH = AF - FH = (R(\sin(2x+2y) - \cos(2x+2y)\tan(2y-x))) - (-R\cos(2x+2y)\tan(2y-x)) = R\sin(2x+2y) = \frac{1}{2}AB$, completing the proof.

6. IMO 2018 Problem 1

Let Γ be the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line).



Solution: Let O be center of Γ and AO meets Γ at X. Let FY be the perpendicular bisector of BD and draw $OM \perp AB$ and $ON \perp FY$. Let $P = FY \cap AB$. Let $\angle BOX = 2\beta$, $AD = 2d$ and R be the radius of Γ. We assume that $0° < 2\beta < 180°$.

We start by observing that $\angle BAO = \angle ABO = \beta$. Since FY is perpendicular to both AB and ON, we have $ON \parallel AB$. Therefore, $\angle NOX = \angle BOA = \beta$.

Next, since FY and OM are both perpendicular to AB, we have that $ONMP$ is a rectangle. Therefore, $ON = MP = MB - PB = \frac{1}{2}AB - \frac{1}{2}DB = \frac{1}{2}AD = d$. Thus, $\angle FON = \cos^{-1}\left(\frac{ON}{OF}\right) = \cos^{-1}\left(\frac{d}{R}\right)$. Hence, $\angle FOX = \beta + \cos^{-1}\left(\frac{d}{R}\right)$.

Similarly, suppose the point C is chosen on Γ such that $\angle COX = 2\gamma$ and $-180° < 2\gamma < 0$. Then, we will have that $\angle GOX = \gamma - \cos^{-1}\left(\frac{d}{R}\right)$.

Thus, $\angle FOG = \beta - \gamma + 2\cos^{-1}\left(\frac{d}{R}\right)$. This implies that $\angle OFG = \angle OGF = 90° - \frac{\beta-\gamma}{2} - \cos^{-1}\left(\frac{d}{R}\right)$. Hence, the angle that the line FG makes with AX is equal to:

$$\angle OFG + \angle FOX = \left(90° - \frac{\beta - \gamma}{2} - \cos^{-1}\left(\frac{d}{R}\right)\right) + \left(\beta + \cos^{-1}\left(\frac{d}{R}\right)\right) = 90° + \frac{\beta + \gamma}{2}.$$

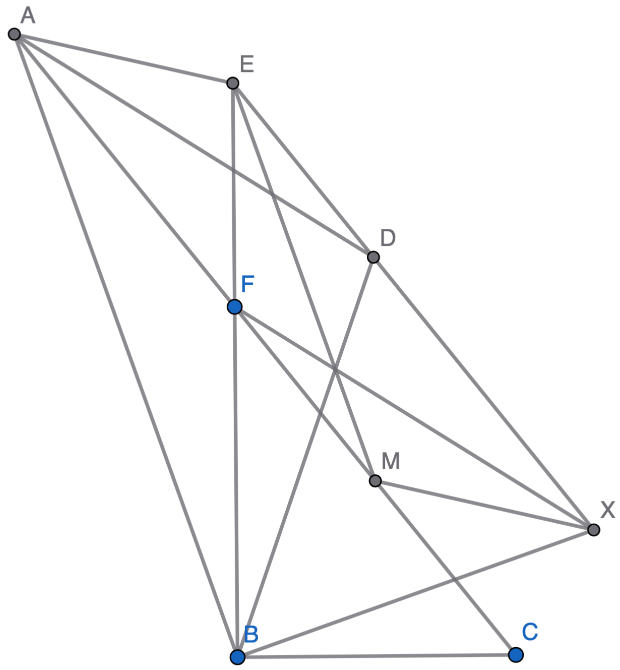
But then, exactly by a similar argument, we have $\angle DOE = \beta - \gamma$ and $\angle ADE = \angle AED = 90^\circ - \frac{\beta-\gamma}{2}$. Thus, the angle that the line DE makes with AX is equal to:

$$\angle ADE + \angle DAX = 90^\circ - \frac{\beta - \gamma}{2} + \beta = 90^\circ + \frac{\beta + \gamma}{2}.$$

As the two angles above are equal, the claim stands proven.

7. IMO 2016 Problem 1

$\triangle BCF$ has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen such that $DA = DC$ and AC is the bisector $\angle DAB$. Point E is chosen such that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that BD, FX and ME are concurrent.



Solution: Let $\angle BFC = y$. Also, let $BF = c$ and $FC = b$. Since $AF = FB$, we have $\angle FBA = \angle BAF = \angle FAD = \angle DAE = \frac{y}{2}$.

We start by showing that the point D lies on the EX. To prove this, it suffices to show that:

$$\frac{AD}{\sin(\angle AEX)} = \frac{DE}{\sin(\angle DAE)}.$$

Now, $AD = \frac{1}{2}AC\sec(DAF) = \frac{b+c}{2}\sec(\frac{y}{2})$. Next, $\angle AEX = 180° - \angle FAE = 180° - y$. Finally, $DE = AE = \frac{1}{2}AD\sec(\angle DAE) = \frac{b+c}{4}\sec^2(\frac{y}{2}) = \frac{b+c}{4\cos^2(\frac{y}{2})} = \frac{b+c}{2(\cos y+1)} = \frac{b}{2}$. Then, we see that:

$$\frac{AD}{\sin(\angle AEX)} = \frac{\frac{b+c}{2}\sec(\frac{y}{2})}{\sin(y)} = \frac{b+c}{4\sin(\frac{y}{2})\cos^2(\frac{y}{2})} = \frac{DE}{\sin(\frac{y}{2})} = \frac{DE}{\sin(\angle DAE)},$$

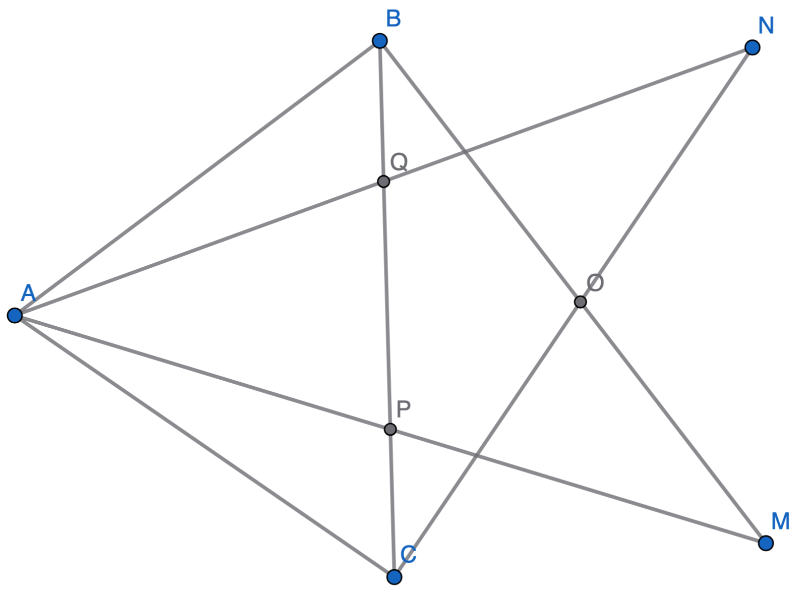
proving that D lies on EX.

Next, we show that F lies on BE. For this, it suffices to show that $y = \angle BFC = \angle AFE$. Since we already know that $\angle FAE = y$, this is equivalent to showing that $AE = EF$. For this, we use the cosine rule in $\triangle FAE$ to observe that $FE^2 = AE^2 + AF^2 - 2AE \cdot AF\cos(\angle FAE)$. So, it suffices to show that $AF = 2AE\cos(\angle FAE)$. Note that $AF = c$ whereas $2AE\cos(\angle FAE) = b\cos y = c$, completing the proof.

Next, we note that $EX = AM = AF + FM = c + \frac{b}{2}$. On the other hand, $EB = EF + FB = \frac{b}{2} + c$, showing that $EB = EX$. Since we already know that $EF = EA = ED$, this also implies that $DX = FB$. Hence, applying Ceva's theorem to $\triangle EBX$, the the lines EM, BD and FX will be concurrent if any only if EM bisects the side BX, which is equivalent to EM bisecting $\angle BEX$, since the triangle is isosceles. To that end, we observe that $MB = \frac{b}{2} = AE - MX$. Hence, $\triangle EMB \cong \triangle EMX$, which proves that EM is the angle bisector, thus proving the claim.

8. IMO 2014 Problem 4

Points P and Q lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.



Solution: Note that $\angle AQP = \angle APQ = \angle CAB$. Thus $AQ = AP$. We establish a coordinate system such that $A = (0,0), B = (1,b), C = (1,c), Q = (1,q)$ and $P = (1,-q)$ with $b,q > 0$ and $c < 0$. Since $\angle CAQ = \angle ABC$, we must have:

$$\frac{q-c}{1+qc} = \frac{1}{b} \implies bq - bc - qc = 1 \implies q = \frac{1+bc}{b-c}.$$

The equation of the circumcircle ABC is given by:

$$x^2 + y^2 - (1-bc)x - (b+c)y = 0.$$

Next, we have that $N = 2Q = (2,2q)$ and $M = 2P = (2,-2q)$. Then we have the equation of BM:

$$\frac{y-b}{x-1} = \frac{b+2q}{-1} \implies y + (b+2q)x = 2b + 2q.$$

Similarly, the equation of CN is:

$$y + (c-2q)x = 2c - 2q.$$

The angle between these two lines is the arctan of:

$$\frac{(b+2q)-(c-2q)}{1+(b+2q)(c-2q)} = \frac{b-c+4q}{1+bc+2cq-2bq-4q^2} = \frac{b-c+4q}{-1-bc-4q^2}.$$

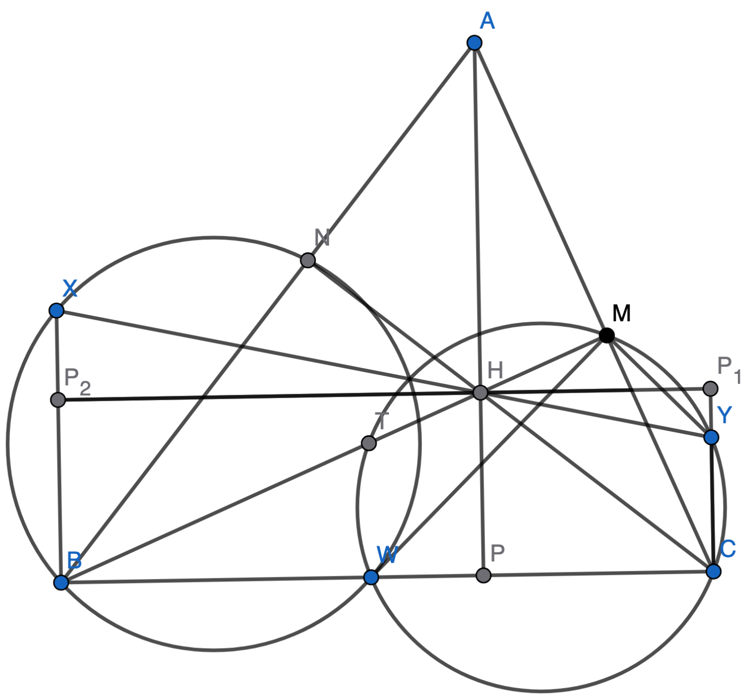
To show that A, B, O and C are concyclic, it suffices to show that this is equal to $-\tan(\angle BAC)$, that is:

$$\begin{aligned} \frac{b-c+4q}{-1-bc-4q^2} &= -\frac{b-c}{1+bc} \\ \iff 4(1+bc)q &= 4q^2(b-c) \\ \iff q &= \frac{1+bc}{b-c}, \end{aligned}$$

which was already shown, thus completing the proof.

9. IMO 2013 Problem 4

Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.



Solution: Let $a = BC, b = CA$ and $c = AB$. Let R be the circumradius of $\triangle ABC$. Construct P_1P_2 parallel to BC through the point H. To prove the required claim, it suffices to show that $\angle XHP_2 = \angle YHP_1$.

We have $\angle HMC = \angle WMY = 90°$. Therefore, $\angle TMW = \angle CMY$, which implies that $TW = CY$. Consider the power of the point B with respect to the circle ω_2:

$$BT \cdot BM = BW \cdot BC.$$

$$\begin{aligned} BT &= \frac{BW \cdot BC}{BM} \\ &= \frac{aBW}{a \sin C} = \frac{BW}{\sin C}. \end{aligned}$$

Next, note that $\angle TBW = 90° - C$. Applying cosine law to $\triangle BWT$:

$$\begin{aligned} TW^2 &= BT^2 + BW^2 - 2BT \cdot BW \cos(TBW) \\ &= BT^2 + BW^2 - 2BT \cdot BW \sin C \end{aligned}$$

$$
\begin{aligned}
&= \frac{BW^2}{\sin^2 C} + BW^2 - 2\frac{BW}{\sin C} \cdot BW \sin C \\
&= BW^2 \csc^2 C - BW^2 \\
&= BW^2 \cot^2 C.
\end{aligned}
$$

Thus, $TW = BW \cot C$. This implies that $CY = BW \cot C$. Similarly, we get that $BX = CW \cot B$.

As $XB \perp BC$, we must have $XB \perp P_1P_2$ since $P_1P_2 \parallel BC$. Similarly, $YC \perp P_1P_2$. Therefore,

$$
\begin{aligned}
\tan(\angle XHP_2) &= \frac{XP_2}{HP_2} \\
&= \frac{BX - BP_2}{HP_2} \\
&= \frac{BX - HP}{BP} \\
&= \frac{CW \cot B - 2R \cos B \cos C}{2R \sin C \cos B} \\
&= \frac{CW}{2R \sin B \sin C} - \cot C.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\tan(\angle YHP_1) &= \frac{YP_1}{HP_1} \\
&= \cot B - \frac{BW}{2R \sin B \sin C}.
\end{aligned}
$$

Then, we see that:

$$
\begin{aligned}
\tan(\angle XHP_2) - \tan(\angle YHP_1) &= \frac{BW + CW}{2R \sin B \sin C} - \cot B - \cot C \\
&= \frac{a}{2R \sin B \sin C} - \frac{\sin(B+C)}{\sin B \sin C} \\
&= \frac{\sin A}{\sin B \sin C} - \frac{\sin A}{\sin B \sin C} \\
&= 0,
\end{aligned}
$$

which shows that $\angle XHP_2 = \angle YHP_1$, completing the proof.