Representation theory of mirabolic quantum $\mathfrak{sl}_n$

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Notation

- $\mathbb{F}_q$ = Finite field with q elements
- $v = q^{1/2}$
- n, d = Positive integers
- $V = \mathbb{F}_q^d$
- $G = GL(V)$
- $B \subset G$ Borel subgroup of upper triangular matrices
- G/B = Variety of complete flags in V
- $\mathcal{F}(n, d)$ = Variety of n-step partial flags in V
- $U_v(\mathfrak{sl}_n)$ = Quantized enveloping algebra of $\mathfrak{sl}_n(\mathbb{C})$

Geometrical construction of $U_v(\mathfrak{sl}_n)$

The space of functions $\mathbb{C}[\mathcal{F}(n, d) \times \mathcal{F}(n, d)]$ has a convolution product given by:

$$f \star g(F, F') := \sum_{H \in \mathcal{F}(n,d)} f(F, H)g(H, F').$$

This restricts to a product on the invariant space $\mathbb{C}[\mathcal{F}(n, d) \times \mathcal{F}(n, d)]^G$.

> **Theorem [Beilinson-Lusztig-MacPherson]**
>
> The space $\mathbb{C}[\mathcal{F}(n, d) \times \mathcal{F}(n, d)]^G$ with the convolution product is isomorphic to the *quantum Schur algebra* $U_v(n, d)$.

By a stabilization procedure as $d \to \infty$, we obtain the quantum group $U_v(\mathfrak{sl}_n)$.

Mirabolic subgroup of G

The mirabolic subgroup $P \subset G$ is the stabilizer of a non-zero vector $v \in V$. For any G-variety X, there is a bijection:

$$\{G\text{-diagonal orbits on } X \times (V \setminus \{0\})\} \leftrightarrow \{P\text{-orbits on } X\}.$$

The data of an extra vector on the left is often referred to as the 'mirabolic' setting.

Mirabolic quantum group

Instead of pairs of flags, we can consider triples consisting of two partial flags in $\mathcal{F}(n, d)$ and a vector in V. Rosso defined a convolution product on the space $\mathbb{C}[\mathcal{F}(n, d) \times \mathcal{F}(n, d) \times V]$:

$$f \star g(F, F', v) := \sum_{H \in \mathcal{F}(n,d), u \in V} f(F, H, u)g(H, F', v - u). \tag{1}$$

The resulting product on the space $\mathbb{C}[\mathcal{F}(n, d) \times \mathcal{F}(n, d) \times V]^G$ gives rise to the *mirabolic quantum Schur algebra* $MU(n, d)$.

> **Definition**
>
> The *mirabolic quantum group*, denoted by $MU(n)$, is defined as the $\mathbb{C}$-algebra with generators ℓ, and e_i, f_i, k_i, k_i^{-1} for $1 \leq i \leq n-1$, subject to the usual relations of the quantum group $U_v(\mathfrak{sl}_n)$, plus the following additional ones involving ℓ:
>
> $$\ell^2 = \ell, \quad k_i \ell = \ell k_i$$
> $$\ell e_i = e_i \ell, \quad \ell f_i = f_i \ell \quad \text{if } i \geq 2$$
> $$\ell e_1 \ell = \ell e_1, \quad \ell f_1 \ell = f_1 \ell$$
> $$(v + v^{-1}) e_1 \ell e_1 = v^{-1} e_1^2 \ell + v \ell e_1^2$$
> $$(v + v^{-1}) f_1 \ell f_1 = v^{-1} \ell f_1^2 + v f_1^2 \ell.$$

> **Theorem [Rosso, Fan-Zhang-Ma]**
>
> The algebra $MU(n, d)$ is a finite-dimensional quotient of the algebra $MU(n)$.

Co-module structure

The algebra $MU(n)$ is a co-module algebra over $U_v(\mathfrak{sl}_n)$ via the map:

$$\rho : MU(n) \to U_v(\mathfrak{sl}_n) \otimes_{\mathbb{C}} MU(n) \tag{2}$$
$$\rho(e_i) = 1 \otimes e_i + e_i \otimes k_i, \qquad \rho(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1,$$
$$\rho(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \qquad \rho(\ell) = 1 \otimes \ell.$$

Representation theory of $MU(n)$

$W = \mathbb{C} - \text{span}\langle w_1, w_2, \ldots, w_n \rangle :=$ Defining representation of $U_v(\mathfrak{sl}_n)$
$W_k := \wedge^k V$ is naturally a $U_v(\mathfrak{sl}_n)$-representation which has a basis given by

$$w_I := w_{i_1} \wedge w_{i_2} \wedge \cdots \wedge w_{i_k}$$

for $I = \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$.
W_k can be turned into an $MU(n)$-representation by defining:

$$\ell \cdot w_I = \begin{cases} 0 & \text{if } 1 \in I \\ w_I & \text{if } 1 \notin I \end{cases}.$$

Given any $U_v(\mathfrak{sl}_n)$-representation M, the space $M \otimes_{\mathbb{C}} W_k$ is naturally an $MU(n)$-representation via the co-module map ρ defined in Equation (2).

Theorem [G.-Rosso]

- The category of finite dimensional $MU(n)$-representations is semisimple.
- A finite dimensional $MU(n)$-representation is uniquely determined by its simultaneous (k_i, ℓ)-weight spaces.
- Every simple finite dimensional $MU(n)$-representation is of the form $L_{\lambda,k}^\pi := L_\lambda^\pi \otimes_{\mathbb{C}} W_k$, where L_λ^π is a simple $U_v(\mathfrak{sl}_n)$-representation and $0 \leq k \leq n$.

Example

The following diagrams represent $MU(2)$-representations, where the dots are one dimensional spaces labeled by their (k_1, ℓ)-weights and the arrows represent the action of f_1.
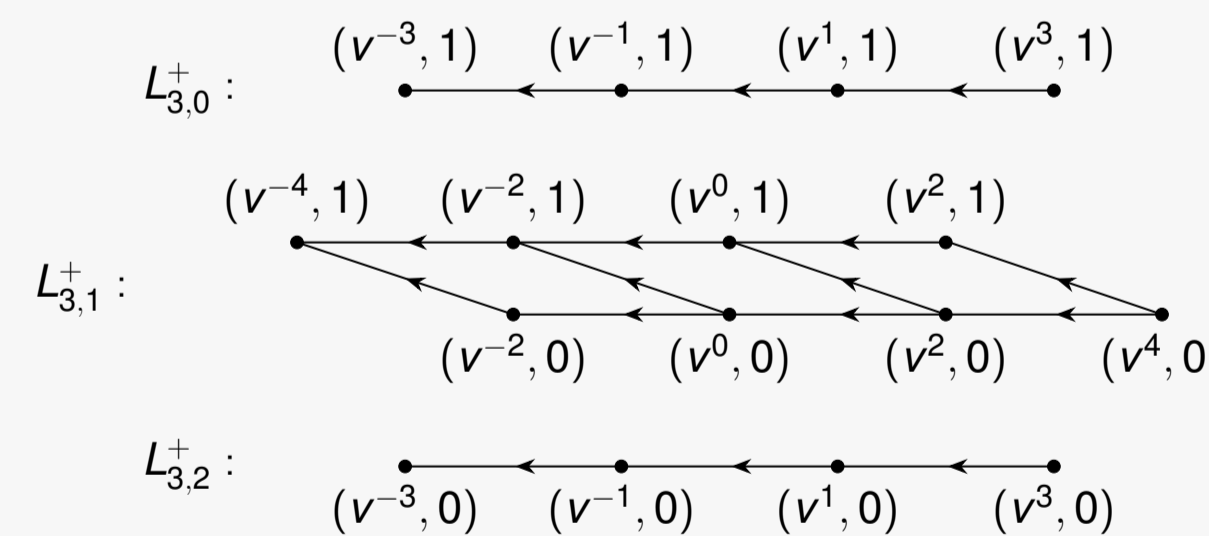
Mirabolic Hecke algebra

Just like the quantum group, there is a mirabolic version of the Iwahori-Hecke algebra of Type A.
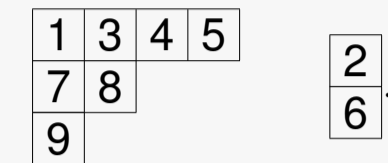
> **Definition**
>
> The *mirabolic Hecke algebra*, denoted by MH_d, is defined as the space $\mathbb{C}[G/B \times G/B \times V]^G$ with a convolution product given by the same formula as in (1).

> **Theorem [Rosso]**
>
> The category of finite dimensional MH_d-representations is semisimple. The simple finite dimensional MH_d-representations $M^{\lambda,k}$ are indexed by pairs (λ, k) of partitions λ of $d - k$, where $0 \leq k \leq d$.

Example

The representation $M^{\lambda,k}$ has a basis given by bitableaux of shape $(\lambda, 1^k)$ on which the (mirabolic) Jucys-Murphy elements $\{L_1, L_2, \ldots, L_d\}$ act via simultaneous eigenvalues. For instance, the following bitableau is an example of a basis element of the representation $M^{(4,2,1),2}$ of MH_9:

On this bitableau, the action of the Jucys-Murphy elements is given by:

$$L_1 = v^0, \ L_3 = v^2, \ L_4 = v^4, \ L_5 = v^6, \qquad L_2 = 0,$$
$$L_7 = v^{-2}, \ L_8 = v^0, \qquad\qquad\qquad\qquad L_6 = 0,$$
$$L_9 = v^{-4}.$$

Mirabolic quantum Schur-Weyl duality

> **Definition**
>
> The *mirabolic tensor space* $MV_{n,d}$ is defined as the $\mathbb{C}$-vector space $\mathbb{C}[\mathcal{F}(n, d) \times G/B \times V]^G$.

By the same formula as in Equation (1), the space $MV_{n,d}$ has a left action of the mirabolic quantum Schur algebra $MU(n, d)$ and a right action of the mirabolic Hecke algebra MH_d.

> **Theorem [Rosso, Fan-Zhang-Ma]**
>
> The actions of $MU(n, d)$ and MH_d on the space $MV_{n,d}$ satisfy the double centralizer property.

The following result is motivated by an analogous formulation of the usual quantum Schur-Weyl duality between the quantum group $U_v(\mathfrak{sl}_n)$ and the Hecke algebra H_d by Lusztig and Grojnowski.

Theorem [G.-Rosso]

As an $(MU(n), MH_d)$-bimodule, the mirabolic tensor space decomposes as follows:

$$MV_{n,d} = \bigoplus_{(\lambda,k) \in M\Lambda_{n,d}} L_{\lambda,k}^+ \otimes_{\mathbb{C}} M^{\lambda,k},$$

where $M\Lambda_{n,d} = \{(\lambda, k) : |\lambda| + k = d, k \leq n \text{ and } \lambda \text{ has } \leq n \text{ parts}\}$.

References

[1] A. A. Beilinson, G. Lusztig, and R. MacPherson.
A geometric setting for the quantum deformation of GL_n.
Duke Math. J., 61(2):655–677, 1990.

[2] Z. Fan, Z. Zhang, and H. Ma.
Geometric approach to mirabolic Schur-Weyl duality of type A.
arXiv:2404.05594, 2024.

[3] P. Goyal and D. Rosso.
Representation theory of mirabolic quantum $\mathfrak{sl}_n$.
arXiv:2510.07469, 2025.

[4] I. Grojnowski and G. Lusztig.
On bases of irreducible representations of quantum GL_n.
In *Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989)*, volume 139 of *Contemp. Math.*, pages 167–174. Amer. Math. Soc., Providence, RI, 1992.

[5] D. Rosso.
The mirabolic Hecke algebra.
J. Algebra, 405:179–212, 2014.

[6] D. Rosso.
Mirabolic quantum $\mathfrak{sl}_2$.
Transform. Groups, 23(1):217–255, 2018.