Padé Approximation of Delays in Cooperative ACC Based on String Stability Requirements

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Abstract—Cooperative adaptive cruise control (CACC) improves road throughput by employing intervehicle wireless communications. The inherent communication time delay and vehicle actuator delay significantly limit the minimum intervehicle distance in view of string stability requirements. Hence, controller design needs to consider both delays, which result in a nonrational transfer function representation of the CACC-controlled string. Padé approximations can be applied to arrive at a finite-dimensional model, which allows for many standard control methods. Our objective is to provide a method to decide for the lowest possible order of the Padé approximation, which is sufficiently accurate in view of CACC (string) stability analysis. The constant time gap strategy and a one-vehicle look-ahead topology are adopted to develop a CACC stable string. First, based on the stable controller parameter region, a suitable order of Padé approximations of the vehicle actuator delay can been carried out in view of individual vehicle stability. Then, the minimum string-stable time gaps for a CACC system with both exact and approximated delays have been compared. The procedure with a Proportional-derivative controller to choose the approximation order of delays has been given, followed by the time-domain simulation validation.

Index Terms—Cooperative adaptive cruise control (CACC), Padé approximation, string stability, wireless communication delay.

I. INTRODUCTION

ADVANCED driver assistance systems have significantly developed in the last decades. Adaptive cruise control (ACC) systems are penetrating into the market. Equipped with radar or camera to detect the preceding vehicle, ACC systems relieve the drivers' task by automatically keeping a desired intervehicle distance [1]. To realize a shorter intervehicle distance and, consequently, to improve highway capacity, cooperative adaptive cruise control (CACC) systems, which use wireless intervehicle communications, have been developed [2], [3]. String stability, which refers to the attenuation of the effects of disturbances in upstream direction of the string of vehicles, is a primary requirement for a CACC string to prevent traffic jams while increasing highway throughput. Furthermore, string stability is

beneficial for safety and for fuel consumption particularly for commercial vehicles [4].

However, imperfections induced by the wireless communication, such as delays, sampling intervals, packet loss, and communication constraints, can affect string stability [5]. Thus, it is critical to take communication delays into account to design and analyze a CACC string.

The influence of communication delay on string stability has been well-known in the literature. In [6], for instance, the effects of communication delays from lead and preceding vehicles were analyzed through a partial fraction expansion approach. Naus *et al.* [2] derived a necessary and sufficient frequency-domain condition for string stability, considering communication delay. Ploeg *et al.* [7] presented theoretical analysis and experimental results, indicating that the existence of communication delay compromises string stability. In the Grand Cooperative Driving Challenge 2011 [8]–[11] the information of vehicles was communicated at 10 Hz. In the GCDC 2016 [12], vehicle motion data update frequency was required to be 25 Hz.

In a CACC string, besides wireless communication delays, there is also the vehicle actuator delay, which not only affects individual vehicle stability [13], but also has effects on string stability. From recent experiments on vehicle model validation [7], [14], there is an observable vehicle actuator delay value, which cannot be ignored. In [11], the vehicle actuator delay and actuator response lag were taken into account when designing CACC systems. Xiao *et al.* [15] presented a requirement for minimum time gap to guarantee string stability, considering parasitic vehicle delays and phase lags. Based on the Lyapunov–Razumikhin theorem, Huang and Ren [16] constructed a conservative upper bound for the vehicle actuator delay to guarantee individual vehicle stability, and derived a condition for string stability.

Hence, both the vehicle actuator delay and the wireless communication delay are of utmost importance in view of string stability in CACC systems.

However, considering time delays leads to a nonrational transfer function, which poses a problem for many controller design methods. To deal with the nonrational representation of time delays, some studies have paid attention to the applications of approximations for time delays. As an example, Morbidi *et al.* [17] analytically studied sufficient conditions for string stability with communication and actuator delays, while the delays were approximated with Maclaurin series expansions under the assumption that the product of the delay and the frequency was sufficiently small. However, Padé approximation is more frequently used for systems with time delays, since it often gives

a better approximation of the delay system than the Maclaurin series of the same order, and it may still result in a suitable representation, whereas the Maclaurin series of the same order does not converge [18].

Padé approximations of time delays have been applied in CACC systems in some publications. In [11], a linear controller was designed, where vehicle actuator delay was approximated by a 2nd-order Padé approximation in a CACC string. In [19], a 4th-order Padé approximation was adopted to permit a finite-dimensional model of the vehicle dynamics in view of the vehicle response delay. In [20], both the vehicle and communication delays were described by a 3rd-order Padé approximation. Although a higher order Padé approximation leads to a more accurate representation, the resulting system is more complex. Thus, it is necessary to explicitly motivate a suitable order of the Padé approximation used in CACC systems. In addition, a too high-order Padé approximation may cause numerical problems, e.g., in [20], increasing the Padé approximation order for both delays by n leads to an increment of the controller order by $2n$, which will require an additional order-reduction step in the $\mathcal{H}_\infty$ controller synthesis to make the controller applicable in practice.

This paper focusses on a procedure to find a suitable (low) order for the approximation for the delays in view of both individual vehicle stability and string stability. To this end, first, to preserve a similar (individual) stability property of both the CACC system with approximated delays and the one with exact delays, both systems should have approximately the same range of controller parameters that lead to stability. Next, in view of string stability, the minimum time gap for the approximated CACC system with selected order should be close to the one of the original system, where a variety of vehicle model parameters and communication delays have to be considered.

Based on a particular Proportional-derivative (PD) controller, the selected order of the Padé approximation of the delays is possibly also suitable when designing other controllers for CACC systems, which has been illustrated with a comparison of the PD controller as applied in this paper and the $\mathcal{H}_\infty$ controller from [20]. Thus, in CACC systems, the method proposed in this paper can be used to decide on the order of Padé approximations, with many other linear controllers including robust controllers with regards to model uncertainties.

The outline of this paper is as follows. Section II introduces a model of CACC-controlled vehicle string with both the vehicle actuator delay and the wireless communication delay. Section III presents a method to pursue the appropriate order of the Padé approximation in view of string stability, in the frequency-domain. In Section IV, the proposed Padé approximation order is validated by means of time-domain simulations. Section V summarizes the conclusions.

II. CACC WITH DELAYS

A. Delays in a CACC String

A homogeneous CACC string is considered in this paper, as shown in Fig. 1, where l, q_i, v_i, and u_i are the length, position, velocity, and desired acceleration of vehicle i, respectively; $d_{r,i}$ represents the desired distance between vehicle i and its pre-
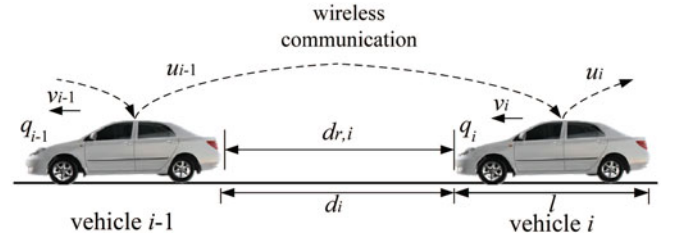

Fig. 1.   CACC-equipped string of vehicles.

ceding vehicle $i-1$, and d_i is the actual intervehicle distance. Focussing on the analysis of delays in a CACC system, a one-vehicle look-ahead topology is adopted. It has been shown in [20] that a one-vehicle look-ahead topology leads to smaller admissible time gaps than a multiple-vehicle look-ahead topology below a certain break-even communication delay. In a homogeneous vehicle string, the wireless communication delay θ_c and the vehicle length l are independent of the vehicle index i.

Due to nonlinear dynamics of engine and drive train, aerodynamic drag, and rolling resistance, feedback linearization is often adopted to arrive at a simplified model for CACC design [21]. However, time delays cannot be omitted. Adopting [7], the acceleration a_i of vehicle i is as follows:

$$\tau \dot{a}_i(t) + a_i = u_i(t - \theta_a) \tag{1}$$

where θ_a represents the vehicle actuator delay, and τ is a time constant, representing the longitudinal vehicle response. Both τ and θ_a are set identical for both acceleration and brake situations, for the sake of simplicity. In addition, considering a model gain k_g with a nominal value of 1, the Laplace transfer function from the desired acceleration U_i to the position Q_i reads

$$\frac{Q_i(s)}{U_i(s)} = D_a(s)G(s) = e^{-\theta_a s} \frac{k_g}{s^2(\tau s + 1)} \tag{2}$$

where $s \in \mathbb{C}$ is the Laplace variable. Here, the representations of vehicle dynamics is as follows:

$$D_a(s) = e^{-\theta_a s} \tag{3a}$$

$$G(s) = \frac{k_g}{s^2(\tau s + 1)} \tag{3b}$$

are adopted to make it clear when replacing the exact delay by its Padé approximations. This model can adequately describe the longitudinal vehicle dynamics in view of CACC systems [8], [11], [20]. Note that, both τ and k_g are uncertain in practice. Thus, a variety of vehicle parameters will be considered in the next of this paper to pursue a suitable order of Padé approximations.

In a CACC system, the wireless intervehicle link is employed for feedforward purposes. In our case, the feedforward input is the desired acceleration of the preceding vehicle, i.e., u_{i-1} (see Section II-B). Although the wireless communication time delay has decreased largely with the recent developments of both software and hardware [12], [20], it still has a significant effect on string stability of CACC systems. Thus, considering the wireless communication θ_c, the actual feedforward input of

vehicle i is

$$u_{i-1,c}(t) = u_{i-1}(t - \theta_c) \tag{4}$$

where θ_c consists of the total time delay due to queueing, contention, transmission, and propagation. Note that other network effects in wireless communication, such as packet loss and sampling intervals, are not taken into account in this paper.

B. CACC Controller

In a CACC string, each vehicle (except the leading one) aims to keep a desired distance $d_{r,i}$ to its preceding vehicle. The constant time gap policy is utilized, which is the most common one to improve string stability, see [2] and the references contained therein. Consequently, the desired intervehicle distance involves a standstill distance and a velocity-dependent part:

$$d_{r,i}(t) = r + hv_i(t) \tag{5}$$

where h is a constant time gap, and r is the standstill distance, which are identical for all vehicles in the string. Pursuing a small value of h can increase traffic throughput. However, inherent time delays in the wireless communication yield a lower bound for the time gap from a string-stability perspective [2], [7].

The actual intervehicle distance d_i reads

$$d_i(t) = q_{i-1}(t) - q_i(t) - l \tag{6}$$

which is measured by the radar sensor. To realize the vehicle-following objective, the intervehicle distance error e_i, defined as

$$e_i(t) = d_i(t) - d_{r,i}(t) \tag{7}$$

should asymptotically converge to zero. To this end, a variety of controllers have been proposed [4], [8], [9], [13], [16], [20], [22]. However, a linear PD controller is most widely adopted, especially in experimental applications [7], [14]. Here, we adopted the algorithm in [7], where the Routh–Hurwitz stability criterion is applied to analyze the stability of PD controlled CACC systems with no vehicle actuator delay. The control structure is briefly introduced as follows:

with $k_g = 1$, the third derivative of (7) reads

$$\dddot{e}_i(t) = -\frac{1}{\tau}\ddot{e}_i(t) - \frac{1}{\tau}\xi_i(t) + \frac{1}{\tau}u_{i-1}(t - \theta_c) \tag{8}$$

where a new input ξ_i is defined as

$$\xi_i(t) := h\dot{u}_i(t) + u_i(t). \tag{9}$$

To stabilize the error dynamics, the control law for ξ_i follows:

$$\xi_i(t) = u_{i-1}(t - \theta_c) + \omega_p e_i(t) + \omega_d \dot{e}_i(t). \tag{10}$$

In the Laplace domain, a standard PD controller reads

$$K(s) = \omega_p + \omega_d s \tag{11}$$

where ω_p and ω_d represent the proportional and derivative parameters, respectively. ω_p is designed to be ω_d^2 for notational and analytic convenience, (see also [2], [19]), which can reduce an overflow of controller parameters in the sequel.
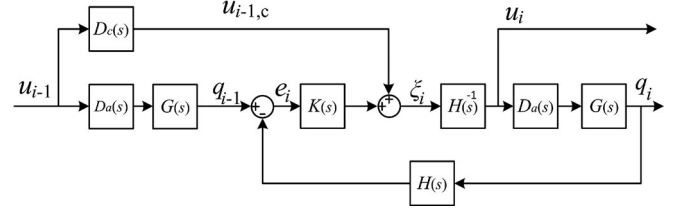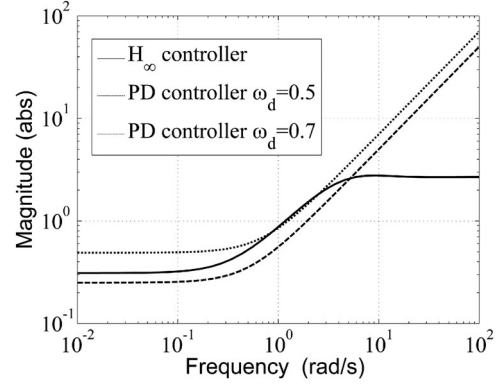

Fig. 2. Block scheme of the CACC system.

Fig. 3. Comparison of PD and $\mathcal{H}_\infty$ controllers.

Consequently, the control structure with both wireless communication delay θ_c and vehicle actuator delay θ_a can be depicted as in Fig. 2, where $H(s)$ and $D_c(s)$ represent the Laplace transforms

$$H(s) = hs + 1 \tag{12a}$$

$$D_c(s) = e^{-\theta_c s}. \tag{12b}$$

C. Comparison of PD and $\mathcal{H}_\infty$ Controllers

In order to determine a suitable approximation order of time delays in CACC systems, we need to consider a variety of vehicle parameters and communication delays. Although controller design is not pursued in this paper, PD controllers can perform similar to other linear controllers in the frequency-range of interest for this particular application, such as an $\mathcal{H}_\infty$ controller [20]. To motivate this statement, Fig. 3 shows the magnitudes of the $\mathcal{H}_\infty$ feedback controller in [20] and the PD controller (11) with $\omega_d = 0.5$ and 0.7. At frequencies higher than 10 rad/s, PD controllers cannot eliminate the signal noise as the $\mathcal{H}_\infty$ controller, which has a higher order. However, the similarity in both frequency responses indicates that the method to choose a suitable order of Padé approximations with one particular PD controller can be used for other linear controllers, although the resulting approximation order may be different.

III. CACC STABILITY BASED ON PADÉ APPROXIMATION

In this section, we examine the suitable order of the Padé approximations of both the actuator and the communication delays in view of individual vehicle stability and string stability.
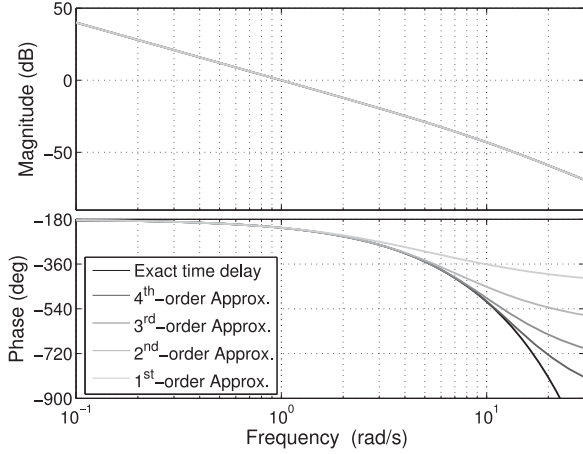
Fig. 4. Frequency-domain responses of vehicle dynamics with exact and approximated delays.

A. Padé Approximation

Padé approximations of a delay yield a finite-dimensional model with a rational transfer function, of which the expressions with the same order of the numerator and the denominator are as follows:

$$e^{-Ts} \cong \frac{\sum_{k=0}^{p} \beta_k (-Ts)^k}{\sum_{k=0}^{p} \beta_k (Ts)^k} \quad (13)$$

where $T \in \{\theta_a, \theta_c\}$ represents the time delay, and p is the order of the Padé approximation. The coefficients β_k are given by

$$\beta_k = \frac{(2p-k)! \, p!}{(2p)! \, k! \, (p-k)!}. \quad (14)$$

Increasing the order can lead to a better approximation, while the resulting model is more complex. The 1st-, 2nd-, 3rd-, and 4th-order Padé approximations of e^{-Ts} are represented as

$$P_1(s) = \frac{1 - \frac{1}{2}Ts}{1 + \frac{1}{2}Ts} \quad (15a)$$

$$P_2(s) = \frac{1 - \frac{1}{2}Ts + \frac{1}{12}(Ts)^2}{1 + \frac{1}{2}Ts + \frac{1}{12}(Ts)^2} \quad (15b)$$

$$P_3(s) = \frac{1 - \frac{1}{2}Ts + \frac{1}{10}(Ts)^2 - \frac{1}{120}(Ts)^3}{1 + \frac{1}{2}Ts + \frac{1}{10}(Ts)^2 + \frac{1}{120}(Ts)^3} \quad (15c)$$

$$P_4(s) = \frac{1 - \frac{1}{2}Ts + \frac{3}{28}(Ts)^2 - \frac{1}{84}(Ts)^3 + \frac{1}{1680}(Ts)^4}{1 + \frac{1}{2}Ts + \frac{3}{28}(Ts)^2 + \frac{1}{84}(Ts)^3 + \frac{1}{1680}(Ts)^4}. \quad (15d)$$

Then, the frequency-domain responses for the exact and approximated transfer functions from the desired acceleration u_i to the position q_i, i.e., $D_a(s)G(s)$ are shown in Fig. 4, with $\tau = 0.1\,\text{s}$ and $\theta_a = 0.5\,\text{s}$ as an example. In the frequency-domain responses, the magnitudes of exact and approximated models have hardly visible differences, which are smaller than $0.02\,\text{dB}$, while the phases differ for the higher frequencies. Since choosing a suitable order of the Padé approximation mainly depends on the system dynamics, it is necessary to apply a large range of parameters, to find an approximation order leading to the sufficiently similar stability properties as the aforementioned CACC system with exact delays.

Next, we will check the individual-vehicle-stable PD controller parameter regions and the minimum string-stable time gaps with the exact and approximated models, respectively.

B. Individual Vehicle Stability

Individual vehicle stability, as mentioned in the previous section, requires that the system should have a stable equilibrium. With the selected order Padé approximation of the delays, the resulting CACC system and the one with exact delays should have approximately the same (individual) stability properties. Thus, the PD controller parameter region for which the approximated system is stable, needs to be close enough to the stable PD controller parameter region for the system with the exact time delay. We start with a 2nd-order Padé approximation, due to the inaccuracy of the 1st-order Padé approximation.

Considering the control structure shown in Fig. 2, the transfer function from the desired acceleration to the intervehicle distance error reads

$$C(s) = \frac{D_a(s)G(s)(1 - D_c(s))}{1 + D_a(s)G(s)K(s)}. \quad (16)$$

It is clear that the communication delay θ_c occurring in the feedforward loop does not influence individual vehicle stability in the presented CACC structure.

First, we consider the stable PD controller region in case of the CACC model with exact delays. Making use of the Nyquist stability criterion, the PD controller parameter region to guarantee individual vehicle stability can be graphically determined. In Fig. 2, the open loop transfer function reads

$$L(s) = D_a(s)G(s)K(s) \quad (17)$$

which should not clock-wise encircle the point $(-1, 0)$ since the system is stable.

Next, considering the CACC system with the Padé approximated time delays, the Routh–Hurwitz stability criterion can be applied to calculate the stable PD controller region. In case of ideal vehicle dynamics, where no vehicle actuator delay exists, i.e., $\theta_a = 0$, it follows that the CACC string can be stabilized for $0 < \omega_d < \frac{1}{\tau}$ [23]. Now, consider $\theta_a > 0$. Substituting (3), (11), and (13) in (16), we obtain finite-dimensional transfer functions, such that the Routh–Hurwitz stability criterion can be applied. The roots of the denominator of (16), i.e.,

$$1 + D_a(s)G(s)K(s)$$
$$= 1 + \frac{k_g(\omega_p + \omega_d s)\sum_{k=0}^{p} \beta_k(-\theta_a s)^k}{s^2(\tau s + 1)\sum_{k=0}^{p} \beta_k(\theta_a s)^k} \quad (18)$$

should be in the left half of the complex plane to achieve stable systems. Thus, it requires a positive ω_d limited to a maximum value $\omega_{d,\max}$ to guarantee individual vehicle stability, i.e., leading to the stable PD controller parameter region

$$\omega_d \in (0, \omega_{d,\max}(\theta_a, \tau)). \quad (19)$$

TABLE I
STABLE PD CONTROLLER GAINS $\omega_{d,\text{MAX}}(\theta_a, \tau)$ WITHOUT THE VEHICLE ACTUATOR DELAY; AND WITH THE VEHICLE ACTUATOR DELAY, AND ITS 2ND-, 3RD-, AND 4TH-ORDER PADÉ APPROXIMATIONS

case	$\omega_{d,\text{max}}(\theta_a, \tau)$		
	$\tau = 0.1$	$\tau = 0.3$	$\tau = 0.5$
$\theta_a = 0$	10	3.333333	2
exact delay $\theta_a = 0.1$	3.7732	2.0830	1.4577
2nd-order Padé of $\theta_a = 0.1$	3.776279	2.083767	1.458203
3rd-order Padé of $\theta_a = 0.1$	3.771906	2.083407	1.458121
4th-order Padé of $\theta_a = 0.1$	3.776158	2.083763	1.458203
exact delay $\theta_a = 0.3$	1.7980	1.2577	0.9840
2nd-order Padé of $\theta_a = 0.3$	1.800136	1.258760	0.984279
3rd-order Padé of $\theta_a = 0.3$	1.793552	1.257302	0.983764
4th-order Padé of $\theta_a = 0.3$	1.799742	1.258719	0.984271
exact delay $\theta_a = 0.5$	1.1909	0.9157	0.7546
2nd-order Padé of $\theta_a = 0.5$	1.191522	0.916885	0.755256
3rd-order Padé of $\theta_a = 0.5$	1.185621	0.914896	0.754381
4th-order Padé of $\theta_a = 0.5$	1.191091	0.916803	0.755232

This stability region thus includes the case of $\theta_a = 0$, with $\omega_{d,\text{max}}(0, \tau) = \frac{1}{\tau}$. Applying 2nd-, 3rd-, and 4th-order Padé approximations of $e^{-\theta_a s}$, respectively, the corresponding values of $\omega_{d,\text{max}}(\theta_a, \tau)$ can be obtained.

As a result, Table I shows the maximum PD controller gain $\omega_{d,\text{max}}(\theta_a, \tau)$ in view of individual vehicle stability with parameters $\theta_a \in \{0.1, 0.3, 0.5\}$ s and $\tau \in \{0.1, 0.3, 0.5\}$ s. Here, the model gain is assumed with the nominal value $k_g = 1$. With different $k_g = \bar{k}_g$, the values of $\omega_{d,\text{max}}(\theta_a, \tau)$ with exact and approximated delays change with the same proportion, which has no effect on the accuracy of Padé approximations. Note that adopting the Padé approximations does not guarantee that the resulting $\omega_{d,\text{max}}(\theta_a, \tau)$ is smaller than the one obtained from the Nyquist mapping of the model with exact delays. Table I indicates that the values of $\omega_{d,\text{max}}(\theta_a, \tau)$ for CACC systems with 2nd- and 4th-order Padé approximations are slightly larger than these for the system with the exact delay, whereas representing the CACC system with 3rd-order Padé approximations results in smaller values of $\omega_{d,\text{max}}(\theta_a, \tau)$ except the cases of $\theta_a = 0.1$ s and $\tau \in \{0.3, 0.5\}$ s. However, in spite of the inaccuracy of graphical Nqyuist methods and the calculations from the Routh-Hurwitz criterion, the values of $\omega_{d,\text{max}}(\theta_a, \tau)$ for 2nd-, 3rd-, and 4th-order approximated CACC systems are all sufficiently close to the one for the original system. Since a smaller value of $\omega_{d,\text{max}}(\theta_a, \tau)$ will never compromise stability of the original system, it is reasonable to apply the 3rd-order Padé approximation on the vehicle actuator delay in a CACC system.

In the next section, we check the suitable order Padé approximation from a string stability perspective.

C. String Stability

Since the constant time gap policy is introduced to achieve string stability, we choose the minimum string-stable time gap as the string stability property to arrive at the lowest possible Padé approximation order. In other words, the minimum string-stable time gaps for the CACC system with approximated delays

and with exact delays should be sufficiently close. To this end, a range of values for the parameters of the vehicle actuator delays, the communication delays, and the PD controller gain are considered. Because of the results of the previous section, we start with the 3rd-order Padé approximation to find the suitable order in view of string stability.

In the widely applied performance-oriented approach, string stability is characterized by the amplification in upstream direction of the signal of interest [2], [7]. Define the string stability transfer function as $S(s)$, which describes the relation between a relevant (scalar) signal of vehicle i and the corresponding signal of its preceding vehicle $i - 1$. In CACC systems, the signals of interest generally involve the intervehicle distance error, the acceleration, the velocity, and the position. Then the system of CACC-controlled vehicles is string stable if

$$\sup_{\omega} |S(j\omega)| \leq 1 \tag{20}$$

with the frequency $\omega \in \mathbb{R}^+$.

Let $E_i(s)$, $U_i(s)$, $V_i(s)$, and $Q_i(s)$ represent the Laplace transforms of the intervehicle distance error, the vehicle input (desired acceleration), the velocity, and the position of vehicle i, respectively. The velocity is chosen in view of traffic analysis [7]. The position has to be considered for heterogeneous traffic [2], whereas the acceleration is concerned for vehicle throttle/brake constraints, and the intervehicle error presents the vehicle following accuracy. Under the assumption of homogeneous traffic, the string stability transfer function $S(s)$ does not depend on the choice of the signal, yielding

$$S(s) = \frac{E_i(s)}{E_{i-1}(s)} = \frac{U_i(s)}{U_{i-1}(s)} = \frac{V_i(s)}{V_{i-1}(s)} = \frac{Q_i(s)}{Q_{i-1}(s)}$$
$$= \frac{1}{H(s)} \frac{D_c(s) + D_a(s)G(s)K(s)}{1 + D_a(s)G(s)K(s)}. \tag{21}$$

Substituting (2), (11), (12), and (21) in (20) yields

$$\left| \frac{1}{hj\omega + 1} \right| \cdot$$
$$\left| \frac{e^{-\theta_c j\omega} + \frac{k_g}{(j\omega)^2(\tau j\omega + 1)} e^{-\theta_a j\omega}(\omega_p + \omega_d j\omega)}{1 + \frac{k_g}{(j\omega)^2(\tau j\omega + 1)} e^{-\theta_a j\omega}(\omega_p + \omega_d j\omega)} \right| \leq 1 \tag{22}$$

to realize string stability.

In the case without communication delay, (20) reduces to

$$\sup_{\omega} \left| \frac{1}{hj\omega + 1} \right| \leq 1 \tag{23}$$

which is fulfilled for any nonnegative time gap, i.e., $h \geq 0$ s. However, as previously stated, in reality a communication delay exists, which plays a significant role in designing the time gap for string stability.

When $\theta_c > 0$ s, the magnitude of the string stability transfer function can be expressed as

$$|S(j\omega)| = \frac{1}{\sqrt{(h\omega)^2 + 1}} \left| \frac{M(j\omega)}{N(j\omega)} \right| \tag{24}$$
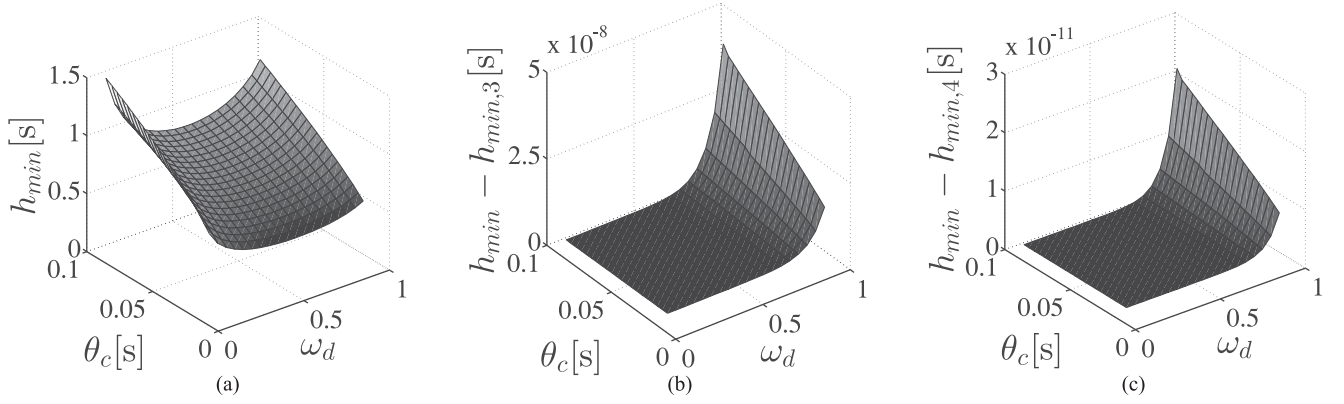
Fig. 5. (a) Minimum string-stable time gap $h_{\min}$ for CACC with exact delays; difference $h_{\min} - h_{\min,p}$ of (b) the 3rd-order Padé approximation and of (c) the 4th-order Padé approximation, with $\tau = 0.3$ s, $\theta_a = 0.3$ s, $\theta_c \in [0.02, 0.1]$ s, $k_g = 1$, and $\omega_d \in [0.1, 1.0]$.

where

$$M(j\omega) = e^{-\theta_c j\omega}$$
$$+ \frac{k_g}{(j\omega)^2(\tau j\omega + 1)} e^{-\theta_a j\omega}(\omega_p + \omega_d j\omega) \quad (25a)$$

$$N(j\omega) = 1 + \frac{k_g}{(j\omega)^2(\tau j\omega + 1)} e^{-\theta_a j\omega}(\omega_p + \omega_d j\omega) \quad (25b)$$

from which it follows that string stability can be guaranteed for $h \geq h_{\min}$, where

$$h_{\min} = \sup_\omega \left(\frac{\sqrt{\left|\frac{M(j\omega)}{N(j\omega)}\right|^2 - 1}}{\omega} \right). \quad (26)$$

Substituting (15) and (25) in (26), results in an approximated minimum time gap $h_{\min,p}$, where p represents the order of Padé approximation.

To judge the quality of Padé approximations for the vehicle actuator delay and the communication delay, we adopt the minimum string-stable time gap as the criterion in the sense that the system with approximated delays should yield a minimum time gap $h_{\min,p}$ which is close to $h_{\min}$. Note that $h_{\min,p}$ is not guaranteed to be larger than $h_{\min}$. The values of vehicle actuator lag $\tau \in [0.1, 0.5]$ s and model gain $k_g \in [0.5, 1.5]$ s are adopted, so that different vehicle dynamics are considered. It is necessary to compare the values of $h_{\min}$ and $h_{\min,p}$ with a range of vehicle actuator delays and communication delays. Thus, $\theta_a \in [0.1, 0.5]$ s is considered. In view of the communication delay, the vehicle motion data update frequency varies from 10 Hz in the GCDC 2011 [8]–[11] to 25 Hz in [20], with the development of both hardware and software. In the GCDC 2016 [12], the same update rate as in [20] was used. Thus, the communication delay indeed is $\theta_c \cong 0.02$ s. In this paper, we took the range $\theta_c \in [0.02, 0.1]$ s. The upper bound $\theta_c = 0.1$ s, also mentioned as a communication requirement in [12], which tolerates more calculation time due to complex controllers or more information communicated.

First, with a constant vehicle actuator delay $\theta_a = 0.3$ s, we choose the vehicle parameters $\tau = 0.3$ s and $k_g = 1$, and a range of communication delay $\theta_c \in [0.02, 0.1]$ s, in order to judge the

quality of Padé approximations with different PD controller parameters. Table I indicates that $\omega_{d,\max}(\theta_a, \tau) = 1.2577$ for the system with an exact time delay. Thus, selecting $\omega_d \in [0.1, 1.0]$ for the sake of individual vehicle stability, Fig. 5 shows the minimum time gap $h_{\min}$ with the nominal value of 1 s, and the minimum string-stable time gap difference $h_{\min} - h_{\min,p}$. Time gaps above the surface in Fig. 5(a) will guarantee string stability of the CACC string. Fig. 5(b) and (c) indicate that the difference $h_{\min} - h_{\min,p}$ is less than 5.0×10^{-8} s and 3.0×10^{-11} s for 3rd- and 4th-order Padé approximations, respectively, which are sufficiently small. Hence, there is no necessity to consider a higher order Padé approximation.

Then, choosing the maximum communication delay $\theta_c = 0.1$ s, we study how the vehicle parameters τ and k_g influence the accuracy of the Padé approximations. The PD controller parameter $\omega_d = 0.6$ is selected, considering the controller comparison in Section II-C and the individual vehicle stability requirement in Table I. Fig. 6 shows $h_{\min}$ and $h_{\min} - h_{\min,p}$ as a function of $\tau \in [0.1, 0.5]$ s and $\theta_a \in [0.1, 0.5]$ s, while $k_g = 1$ is adopted. With $k_g \in [0.5, 1.5]$ and $\tau = 0.3$ s, $h_{\min}$ and $h_{\min} - h_{\min,p}$ are presented in Fig. 7. Both Figs. 6 and 7 indicate that vehicle parameters indeed affect the quality of Padé approximations, i.e., $h_{\min} - h_{\min,p}$ increases with increasing k_g and τ. This may seem counter-intuitive, but realize that, although $h_{\min} - h_{\min,p}$ would increase with smaller τ when only accounting for the same frequency, different values of τ lead to the corresponding $h_{\min}$ at different frequencies. Thus, the frequency also influences the values of $h_{\min} - h_{\min,p}$.

Finally, both time delays are considered with a range of values, i.e., $\theta_c \in [0.02, 0.1]$ s and $\theta_a \in [0.1, 0.5]$ s. According to Figs. 6 and 7, $k_g = 1.5$ and $\tau = 0.5$ s are adopted to check the difference $h_{\min} - h_{\min,p}$. $\omega_d = 0.6$ can still keep individual vehicle stability ($\omega_{d,\max}(\theta_a, \tau) \cong 0.6596$, with $\theta_a = 0.5$ s, $\tau = 0.1$ s and $k_g = 1.5$), while the stable margin gets smaller than that in Table I with $k_g = 1$. The numerical results are shown in Fig. 8. In Fig. 8(a), with exact delays, the minimum time gap $h_{\min}$ increases with both the vehicle actuator delay and the wireless communication delay. In Fig. 8(b) and (c), the differences of the minimum time gap for the 3rd- and 4th-order Padé approximated delays are less than 1.0×10^{-6} s and 1.0×10^{-9} s,
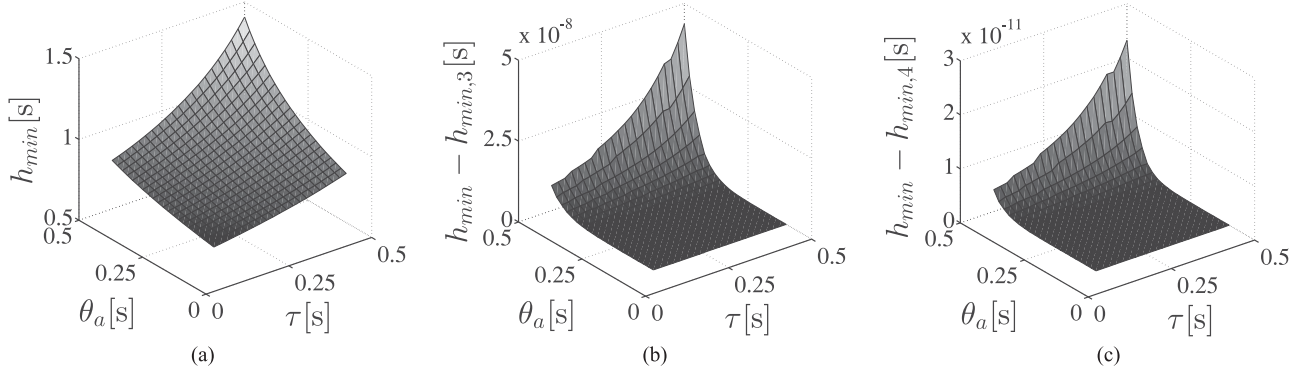
Fig. 6. (a) Minimum string-stable time gap h_{min} for CACC with exact delays; difference $h_{min} - h_{min,p}$ of (b) the 3rd-order Padé approximation and of (c) the 4th-order Padé approximation, with $\tau \in [0.1, 0.5]$ s, $\theta_a \in [0.1, 0.5]$ s, $\theta_c = 0.1$ s, $k_g = 1$, and $\omega_d = 0.6$.
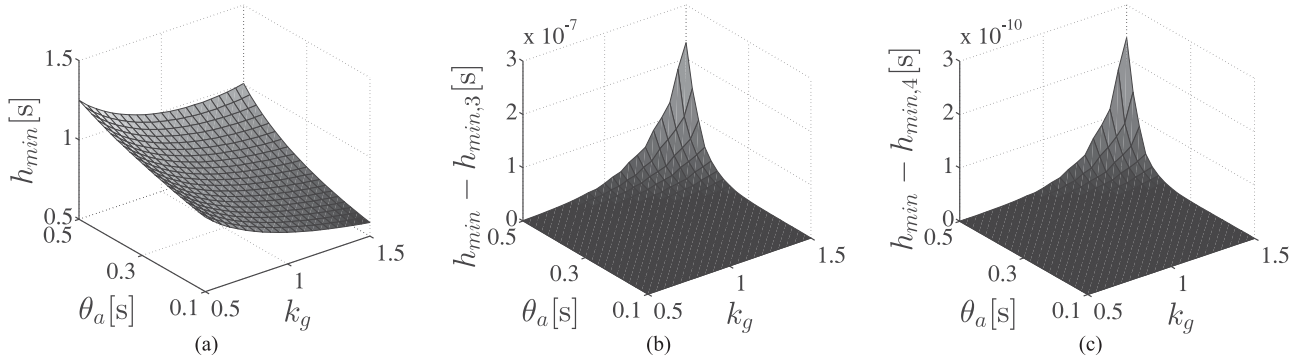

Fig. 7. (a) Minimum string-stable time gap h_{min} for CACC with exact delays; difference $h_{min} - h_{min,p}$ of (b) the 3rd-order Padé approximation and of (c) the 4th-order Padé approximation, with $\tau = 0.3$ s, $k_g \in [0.5, 1.5]$, $\theta_a \in [0.1, 0.5]$ s, $\theta_c = 0.1$ s, and $\omega_d = 0.6$.
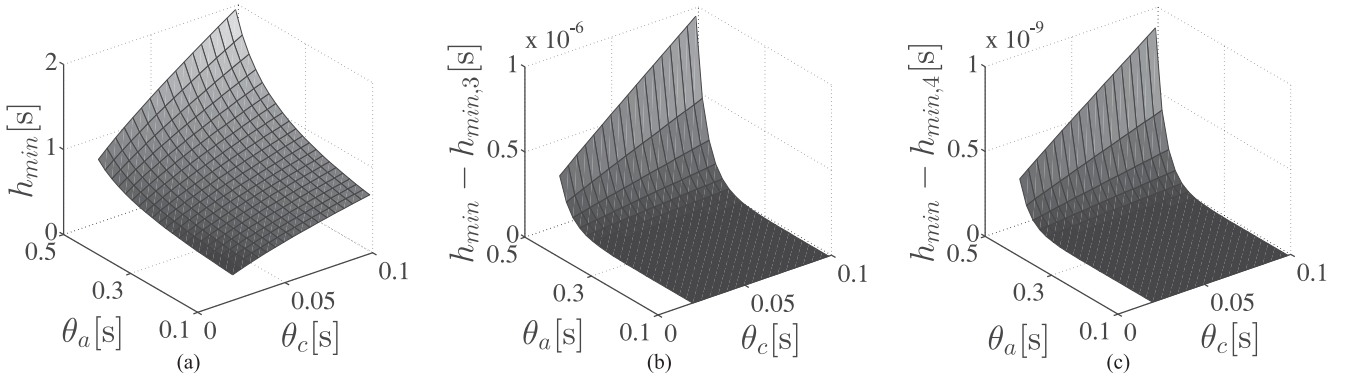

Fig. 8. (a) Minimum string-stable time gap h_{min} for CACC with exact delays; difference $h_{min} - h_{min,p}$ of (b) the 3rd-order Padé approximation and of (c) the 4th-order Padé approximation, with $\tau = 0.5$ s, $k_g = 1.5$, $\theta_a \in [0.1, 0.5]$ s, $\theta_c \in [0.02, 0.1]$ s, and $\omega_d = 0.6$.

respectively. Note that different PD controller gains can be used to pursue the driving comfort or to decrease the intervehicle distance error, whereas the resulting difference $h_{min} - h_{min,p}$ has the similar accuracy level. Therefore, adopting the 3rd- and 4th-order Padé approximations in this CACC setting is sufficient in view of string stability.

If independent proportional and derivative parameters are considered, numerical analyses of the quality of Padé approximations involve one more parameter, and rather complex simulation study is required. However, the procedure proposed in this paper can also be followed.

In the given setting of the CACC system, taking a 3rd-order Padé approximation for both the vehicle actuator delays and the communication delays is suitable from the perspective of string stability properties.

IV. SIMULATION IN TIME-DOMAIN

To validate the accuracy of the selected order Padé approximation by the proposed method, time-domain simulations have been carried out for a CACC string with four vehicles. In this particular case, the 3rd-order Padé approximation is chosen
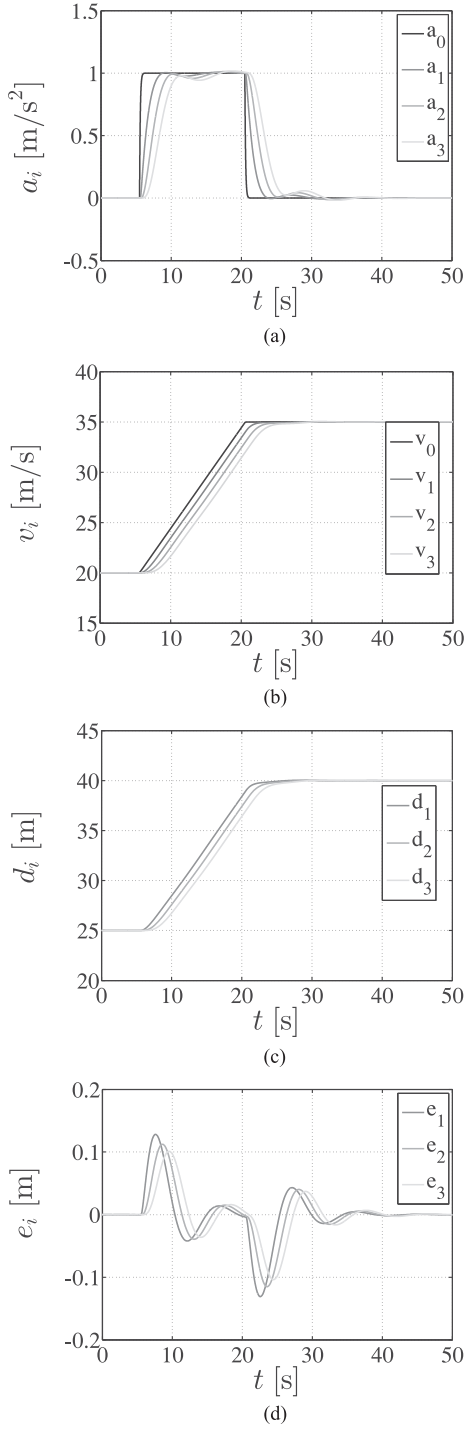
Fig. 9. Time responses of (a) the acceleration a_i, (b) the velocity v_i, (c) the intervehicle distance d_i, and (d) the distance error e_i of CACC with exact delays.
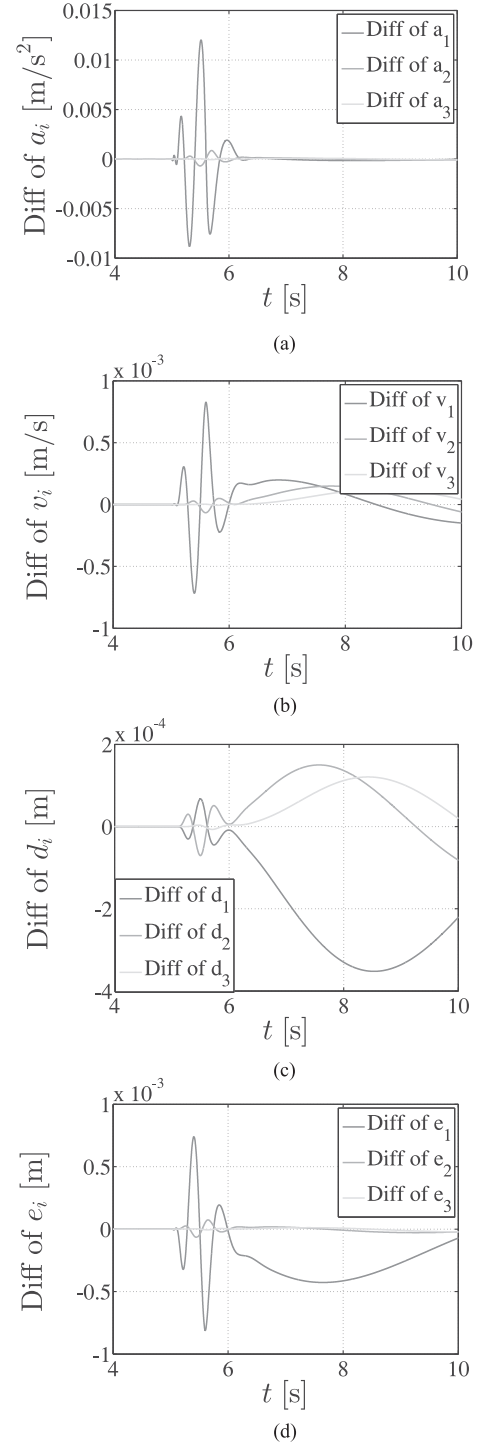

Fig. 10. Differences of time responses of (a) the acceleration $a_i - a_{i,3}$, (b) the velocity $v_i - v_{i,3}$, (c) the intervehicle distance $d_i - d_{i,3}$, and (d) the distance error $e_i - e_{i,3}$, between the CACC models with exact delays and with 3rd-order Padé approximated delays.

according to the simulation results in Section III. The standstill distance is set as $r = 5$ m and the length of vehicle as $l = 3$ m. Since larger delays can show the difference between models with the exact and with approximated delays more clearly, $\theta_a = 0.5$ s and $\theta_c = 0.1$ s are chosen. The value of the model gain $k_g = 1$ is chosen. Considering a smaller $\tau = 0.1$ s leads to a faster response of the vehicle model. The PD controller gain is chosen

as $\omega_d = 0.6$ and the time gap as $h = 1$ s to guarantee string stability.

In view of string stability, the vehicle acceleration is the most significant disturbance. In addition, the accuracy of approximations of delays is the most sensitive to the step response. Thus, in the time-domain simulations, the desired acceleration for the

leading vehicle is set as

$$u_0 = \begin{cases} 1 \, \text{m/s}^2 & 5 \, \text{s} \leqslant t \leqslant 20 \, \text{s} \\ 0 & \text{other.} \end{cases} \qquad (27)$$

The initial velocity of all vehicles in this CACC string is $v_{\text{initial}} = 20 \, \text{m/s}$. All vehicles start with the desired distance. The acceleration duration is set as $15 \, \text{s}$ in order to create enough time for the followers to reach the desired acceleration.

Fig. 9 shows the time responses of the CACC string with exact delays. It is observable that there is a slight overshoot in the acceleration responses in Fig. 9(a). Note that the time gap from the string stability criterion (20) does not guarantee the absence of overshoot in time-domain [7], [24]. The velocity, intervehicle distance, and the distance error responses in Fig. 9(b), (c), and (d), respectively, clearly show that this system is string stable.

Applying the 3rd-order Padé approximation for both the communication delay and the vehicle actuator delay, results in the approximated responses $a_{i,3}, v_{i,3}, d_{i,3}$. The differences between the CACC string responses with exact delays and the ones with approximated delays are shown in Fig. 10, for vehicle $i = 1$, 2, and 3, which are CACC-controlled. We zoom in around $t = 5 \, \text{s}$, when the acceleration changes and the largest difference occurs. Similar results around $t = 20 \, \text{s}$ are omitted here. In Fig. 10, all differences decrease with the vehicle number i increasing, due to the sting stable performance. Thus, the responses of the first follower vehicle 1 in the string are considered. In Fig. 10(a), the acceleration differences reach $0.015 \, \text{m/s}^2$, which is much smaller than the step input of $1 \, \text{m/s}^2$. Accounting for the velocities in Fig. 10(b), the difference due to the 3rd-order Padé approximation of delays is less than $1 \times 10^{-3} \, \text{m/s}$. The differences of intervehicle distances and the distance errors are smaller than $4 \times 10^{-4} \, \text{m}$ and $1 \times 10^{-3} \, \text{m}$ as shown in Fig. 10(c) and (d), respectively, which are even less than the possible measurement noise of the intervehicle distance.

V. CONCLUSION

In a CACC-controlled vehicle string, the communication delay and vehicle actuator delay inherently exist. Adopting a Padé approximation for the delays leads to a rational transfer function representation of the CACC string, which allows many controller design methods. Choosing a low order of Padé approximation, which is accurate enough, can significantly promote the controller design. In this paper, we proposed a method to arrive at a low possible Padé approximation order, based on the individual vehicle stability and the string stability properties for the system with exact delays and with approximated delays. We used a particular PD controller as an example when judging the quality of Padé approximations for a given CACC string. However, the same method can be used in case of other linear controllers. From the individual vehicle stability perspective, we arrived at a suitable order Padé approximation based on the requirement of similar regions for the system with exact delays and the system with approximated delays. The values of the minimum string-stable time gaps for the CACC model were chosen as the string stability criterion to find the suitable approximation order. Different values of vehicle dynamics pa-

rameters have been considered. With the results of this method, time-domain simulations were conducted to validate that the selected order Padé approximation was suitable for the given CACC string.

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