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Accelerated Evaluation of Autonomous Vehicles in the Lane Change Scenario Based on Subset Simulation Technique

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***Abstract*—The testing of autonomous vehicle (AV) safety is receiving a lot of attention due to several high-profile crashes caused by AVs. Popular AV testing techniques include test matrix and Naturalistic Field Operational Test (N-FOT) which are either not comprehensive or too costly and time-consuming. In this paper, a new method based on Subset Simulations (SS), is proposed as an alternative method for evaluating the safety of AV. The lane change scenario is used to demonstrate the performance of SS, and a new Gaussian Mixture Model with 8 parameters describing the stochastic nature of lane change scenario is developed. The new testing procedure achieves a similar level of accelerated rate compared with IS, and SS can easily handle black-box control systems.**

***Index Terms*—Autonomous vehicle, accelerated evaluation, subset simulation**

I. Introduction

Many automakers have announced plans for SAE Level 4 highly automated vehicles (AVs), with some plan for production as early as 2019. These AVs promise improved safety, mobility, energy efficiency, and convenience. Moreover, an IEEE publication [1] estimated that by 2040, up to 75% of vehicles running on road will be autonomous. One substantial barrier to the deployment of AVs is the absence of fast and reliable testing and verification methods.

A common approach used by AV developers is the Naturalistic Field Operational Test (N-FOT) [2], in which AVs are tested on public roads and safety-critical incidents are documented and studied. While N-FOTs are helpful for characterizing real-world AV safety and performance, such studies are time-consuming and costly. An alternative to N-FOTs is the test matrix approach commonly used for advanced driver assistance systems [3]. Test matrices are unsuitable for the evaluation of Level 4+ AVs because pre-defined tests can be "gamed" and thus good scores in these tests do not necessarily imply safety in other scenarios. Another approach is the worst case evaluation method [4], however, by definition worst-case is dependent on the response from the control system, so it is not possible to identify the worst cases for AVs if the control algorithm is a black box.

Variation reduction techniques have been proposed as a solution to the limitations of N-FOTs and test matrices. In general, Monte Carlo simulations have been found to be very inefficient since much of real-world driving consists of non-safety-critical interactions between the host vehicle and its surrounding vehicles. To address this limitation, [5] applied importance sampling (IS) to estimate the probability of rare events significantly faster than conventional Monte Carlo approaches. Instead of relying upon the initial distribution of naturalistic driving data, the IS estimator samples according to the Importance Sampling Distribution (ISD). This distribution is the result of a bijective transformation and therefore testing under this framework can focus on safety-critical scenarios while retaining the probability of such scenarios. Consequently, results under IS can be interpreted in the context of the original distribution, and thus the probability of rare events can be accurately estimated much more quickly. This technique has been used for testing Automated Emergency Braking (AEB) in lane changing/cut-in scenarios in [6] and was found to accelerate testing by around 50 times compared with the Crude Monte Carlo method.

Determining the ISD requires knowledge of the AV algorithm under test - therefore, the traditional IS approach presumes that *failure modes of the vehicle are known a priori*. Special techniques are needed when the tested system is a black box, which is called Adaptive Importance Sampling (AIS). In AIS, testing results are needed to update ISD parameters [7]. Moreover, while IS is efficient for low-dimensional stochastic models, it has limited utility for models of high dimensions because of the difficulty of determining an appropriate ISD as mention in [8].

In this paper, we study another sampling method, Subset Simulation (SS), to address the black-box and high-dimensional model problems. SS is an adaptive stochastic simulation procedure for estimating the low failure probability of a system based on Markov Chain Monte Carlo

This research is founded by Ford company.

method [9]–[11]. It has been used to estimate the structural reliability of civil, aerospace and nuclear systems. Moreover, SS has proven useful in other applications such as sensitivity analysis, design optimization, and uncertainty quantification [10].

In SS, the probability p_ϵ of the rare event ϵ is represented as a product of conditional probabilities of more frequent events [10]. Moreover, SS does not require any information about the system under evaluation; instead, iterative searches are conducted to find failure zones of the system. The searching method used here is a modified Markov Chain Monte Carlo, or modified Metropolis-Hastings algorithm (MMA). As demonstrated in [11]–[14], MMA has long been used to sample high-dimensional stochastic models. SS is well suited for accelerated testing of autonomous vehicles.

The remainder of this paper is organized as follows: Section II introduces the concept and simulation procedure of SS. Section III considers the AEB example from [5] as a benchmark and reviews the stochastic model used for this study. Section IV discusses simulation results using SS on the AEB case study. Finally, Section V concludes the paper.

II. Subset Simulation Method

A. Subset Simulation Concept

In this section, the subset simulation (SS) method is reviewed. We begin with a well-known stochastic simulation algorithm for estimating expectation: the Crude Monte Carlo simulation (CMC). In CMC, the p_ϵ is estimated by the sample mean:

$$p_\epsilon \approx \hat{p_\epsilon}^{CMC} = \frac{1}{N}\sum_{i=1}^{N} I_\epsilon(x^{(i)}), \tag{1}$$

where $x^{(i)}$ is the i^{th} independent and identically distributed (i.i.d.) sample from the original distribution $\pi(\cdot)$ and $I_\epsilon(\cdot)$ is the indicator function. The CMC is easy to implement, and moreover, it can operate even in a high-dimensional parameter space. The coefficient of variance (c.o.v) of CMC, which is the measure of accuracy, is given by [14]:

$$\delta(\hat{p_\epsilon}^{CMC}) = \sqrt{\frac{1-p_\epsilon}{Np_\epsilon}}. \tag{2}$$

From equation (2), it can be seen that N is inversely proportional to p_ϵ, and thus when p_ϵ is very small, the number of samples N needed is very large. Therefore, a more efficient stochastic simulation method is desired.

To address this problem, we consider the SS technique. The basic idea of SS is to represent a very small probability p_ϵ as a product of larger probabilities: $p_\epsilon = \prod_{m=1}^{M} p_m$; here p_m is estimated sequentially. First let us consider a potentially non-explicit function $Y(x)$ for expectation and let $F = \{x : Y(x) < b\}$ denote the set of failure events. In SS, we seek to express F as the intersection of M intermediate sets: $F = \bigcap_{m=1}^{M} F_m$, where the intermediate sets are nested, i.e. $F_1 \supset F_2 \supset \cdots \supset F_M = F$, and $F_m = \{x : Y(x) < b_m\}$. Therefore, the failure probability p_ϵ is equal to a product of conditional probabilities:

$$p_\epsilon = \Pr(\bigcap_{m=1}^{M} F_m) = \prod_{m=1}^{M} \Pr(F_m|F_{m-1}) = \prod_{m=1}^{M} p_m, \tag{3}$$

where $p_m = \Pr(F_m|F_{m-1})$ is the conditional probability at the m^{th} iteration and $\Pr(F_1|F_0) = \Pr(F_0)$. In this way, the original problem of estimating a small failure probability p_ϵ becomes M intermediate problems corresponding to evaluating larger conditional probabilities. And for each conditional probability is

$$\Pr(F_m|F_{m-1}) = \pi(\cdot|F_{m-1}) = \frac{\pi(\cdot)I_{F_{m-1}}(\cdot)}{\Pr(F_{m-1})}, \tag{4}$$

where $I_{F_{m-1}}$ is the indicator function of subset F_{m-1}. The next problems are (i) the decomposition of the rare event into a sequence of less rare events, and (ii) the evaluation of these conditional probabilities.

In SS, the decomposition is done adaptively and we call the simulation in each subset a new "level" of simulation. The process for each simulation level (m) is described as follows.

1) The first level $(m = 1)$ of simulation is conducted by CMC, which directly draw N samples $x_1^{(1)}, ..., x_N^{(1)}$ from the original distribution $\pi(\cdot)$.
2) The function $Y(\cdot)$ is used to characterize failures; $Y(x_i^{(1)})$ is evaluated for each $x_i^{(1)}$, which is the i^{th} sample from the CMC; all $Y(x_i^{(1)})$ are sorted in ascending order to get the list $\{y_1^{(1)} \le ... \le y_N^{(1)}\}$.
3) Set the p_m percentile of the list $\{y_1^{(1)} \le ... \le y_N^{(1)}\}$, denoted as $b^{(1)}$, to be the threshold of the 2nd subset level. This means that $F_1 = \{x : Y(x) < b^{(1)}\}$; those samples inside F_1 are "seeds" for the 2nd level. We denote seeds using $\{\theta_j^{(1)}\}$.
4) In the 2nd level, we sample from the conditional distribution $\pi(\cdot|F_1)$; in general, determining this distribution is not trivial. It is inefficient to use CMC, thus we choose MCMC sampling method for this task.
5) After collecting samples $x_i^{(2)}$, repeat step (2) and obtain $b^{(2)}$ and $F_2 = \{x : Y(x) < b^{(2)}\}$. Once again, those samples inside F_2 are seeds for the 3rd level and denoted as $\{\theta_j^{(2)}\}$. Repeat step (3) for samples of next level.
6) End this process when $b^{(m)} \le b$ or $m + 1 > M$.

The details of Markov Chain Monte Carlo using the Subset Simulation technique will be elaborated in section II-B.

B. Modified Metropolis Algorithm

In the procedure outlined in section II, the key problem is the efficient sampling at each level from the conditional probability distribution $\pi(\cdot|F_{m-1})$. To approach this issue, we consider Markov Chain Monte Carlo (MCMC), which is a class of sampling methods for distributions that cannot be directly sampled efficiently. The basic idea of this method is to construct a Markov Chain whose stationary distribution is the one of interest. By drawing from the Markov Chain, the samples will, in the end, be distributed with the conditional probability $\pi(\cdot|F_{m-1})$. The MCMC used in SS is the Modified Metropolis Algorithm (MMA) which is specifically designed for sampling from conditional distributions.

This approach uses the seeds as described before, and evolves according to the proposal distribution $q_k(\cdot|\cdot)$, $k \in \{1,\ldots,K\}$, which corresponds to k^{th} dimension of the original distribution $\pi(\cdot)$. For level m, we have $N \times p_{m-1}$ seeds, where N is the number of samples from the previous level, and p_{m-1} is the level probability. The MMA process for AV evaluation is described in Algorithm 1.

Algorithm 1: Modified Metropolis Algorithm for each seed $\theta_j^{(m-1)}$ in level m

Input: Initial state: $\theta_j^{(m-1)}$, denoted as x_0 here;
Total # of states of Markov Chain: N_c;
Original distribution: $\pi_k(\cdot)$ for dimension k;
Proposal distribution: $q_k(\cdot|\cdot)$ for dimension k;

for $i = 1, .., N_c - 1$ **do**

Generate candidate state ξ:

for $k = 1, ..., K$ **do**

Sample $\hat{\xi}_k \sim q_k\left(\cdot|x_{i-1}^{(k)}\right)$

Calculate the acceptance ratio:

$$r = \frac{\pi_k(\hat{\xi}_k) q_k\left(x_{i-1}^{(k)}|\hat{\xi}_k\right)}{\pi_k\left(x_{i-1}^{(k)}\right) q_k\left(\hat{\xi}_k|x_{i-1}^{(k)}\right)} \tag{5}$$

Accept or reject $\hat{\xi}_k$:

$$\xi_k = \begin{cases} \hat{\xi}_k, & \text{with probability } \min(r,1) \\ x_{i-1}^{(k)}, & \text{with probability } 1-\min(r,1) \end{cases} \tag{6}$$

Obtain $\xi = [\xi_1, ..., \xi_K]^T$

Check whether $\xi \in F_{m-1}$ by testing. Accept or reject ξ:

$$x_i = \begin{cases} \xi, & \text{if } \xi \in F_{m-1} \\ x_{i-1}, & \text{with probability } \xi \notin F_{m-1} \end{cases} \tag{7}$$

Output: $x_0, ..., x_{N_c-1}$, in total N_c states of a Markov Chain for each seed $\theta_j^{(m-1)}$ for level m

By applying MMA algorithm, the resulting stationary distribution will be the conditional distribution $\pi(\cdot|F_{m-1})$. Here we observe that the total number of sample in level m is $N \times p_{(m-1)} \times N_c$. For convenience, if we set $N_c = p_{(m-1)}^{-1}$, then at each level, we will have the same number of total samples. Also in accordance with [10], we select each of the level probabilities to be $p_m = p_0 = 0.1$, $\forall m$. The MCMC method is known to handle high-dimensional stochastic models efficiently, which is an important consideration because the stochastic model for a realistic environment of an AV can be complicated.

Another point requires elaboration is the proposal distribution $q_k(\cdot|\cdot)$. In general, any one-dimensional distribution centered at the seed could suffice. However, different proposed candidate will affect the speed of convergence. The most well-studied candidate distribution is the normal distribution, i.e. $q_k(\cdot|\alpha) = \mathrm{N}(\alpha, \sigma)$, where α is the mean value and σ is the standard deviation of the normal distribution. In [12], the optimal standard deviation is found to be related to the "roughness" of the original distribution, denoted as I. It is given by $\sigma \approx 2.4/\sqrt{dI}$, where d is the dimension of original distribution and $I = E_f[((\log f)')^2] = \int_{-\infty}^{\infty} \frac{(\pi'(x))^2}{\pi(x)} dx$.

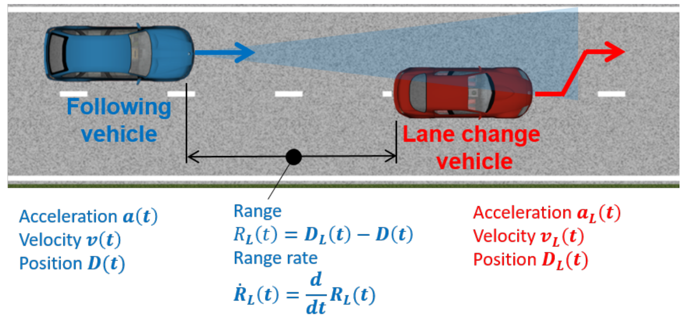


Fig. 1: Lane change scenario

Finally, an additional benefit of using the MMA with SS is that no wasteful burn-in period is needed [14]. Since the seeds in each level start with the stationary distribution of the Markov chain, this MCMC method provides a very effective way to generate conditionally distributed samples.

III. Case Study: The Cut-in Scenario

A. Problem Setting

1) Black Box System: In order to demonstrate the benefits of the SS method for AV safety evaluation, we consider the cut-in scenario previously studied in [5]. In this scenario, the leading vehicle is controlled by a human driver (HV) which attempts to change lane in front of the AV, as shown in Fig. 1. The function under test is the AV's AEB system. The AEB model is taken from [5] with some modification of the inputs. In this case study, the AEB is treated as a black-box and the SS procedure requires no additional information.

2) Original Distribution Model: We simulate the AEB system by samples from two different stochastic models of the environment described in section III-B. The first one is the same as described in [5]. The goal is to compare the result from the SS method with the result from the IS method in terms of acceleration rate. The second one is an 8-dimensional Gaussian Mixture Model which is more realistic. The goal of the second case is to demonstrate that the SS method can deal with high-dimensional stochastic models.

B. Original Distribution Model for Subset Simulation

The database we use to build the model is the Safety Pilot Model Deployment Program (SPMD) [16]. The SPMD dataset is collected from about 2,800 vehicles. Querying from the database, we get around 400,000 cut-in cases. After data cleaning, the data is used to build the stochastic model.

TABLE I: GMM Variables

$X_1 \sim X_3$	$X_4 \sim X_6$	X_7	X_8
Following vehicle's speed	Lead vehicle speed	Duration of lane change	Initial Range
2nd order polynomial parameters: $a_1t^2 + b_1t + c_1$	2nd order polynomial parameters: $a_2t^2 + b_2t + c_2$	$T_{duration}$	R_0

1) Three Independent Random Variables Stochastic Model: The first stochastic model we used is the same as in [5] which is build from SPMD. The variables of this stochastic model are: (1) lead vehicle's initial speed; (2) initial range R_0 and (3) the initial Time-To-Collision (TTC), i.e. $TTC_0 = -R_0/(\dot{R}_0)$.

As described in [5], these three variables are independent. Moreover, the lead vehicle's speed is set to be constant during each lane change, and the duration of each lane change is fixed. These assumptions are not realistic. Therefore, we develop a new stochastic model.

2) 8 Dependent Random Variables Gaussian Mixture Model: The new stochastic model we use is the Gaussian Mixture Model (GMM), which is a parametric probability density function represented as a weighted sum of Gaussian component densities. GMM can be described by

$$p(X) = \sum_{i=1}^{N} \omega_i \phi(X|\mu_i, \Sigma_i) \tag{8}$$

where X is K dimension random variable vector, N is the total number of component, ω_i is the mixture weight for the ith component and $\phi(X|\mu_i, \Sigma_i)$ is the ith component Gaussian probability density function with mean vector μ_i and covariance matrix Σ_i. GMM is commonly used as a parametric model of the probability distribution for multivariate data with an arbitrarily complex pdf. The GMM parameters are estimated from training data using the iterative Expectation-Maximization (EM) algorithm. This technique is very mature and is described thoroughly in papers [17] - [18].

To describe the cut-in cases, we choose 8 variables and then use the EM algorithm to estimate the joint distribution using a 10-component GMM (the number of components is chosen according to the AIC test). The 8 variables are described in TABLE I. These variables can fully describe the lane changes. Denote the pdf of this GMM as $p(X_1, ..., X_8)$.

Before using this GMM, preprocessing is needed. When sampling for test cases, we can only sample the initial speed of the AV and have no control of its subsequent speed. Therefore, we need to calculate the marginal distribution of the AV's speed parameters other than the initial speed parameter (X_3). Thus we actually want to sample from

$$p(X_3, ..., X_8) = \int_{X_1, X_2} p(X_1, ..., X_8) dx_1 dx_2. \tag{9}$$

Then we transform these dependent random variables to a space U consisting of independent standard normal random variables by one-to-one mapping $U = T(X)$. Since we have the joint distribution, we define the mapping by Rosenblatt transformation [19] as

$$\begin{aligned} T_1 : X \longmapsto Y &= \begin{bmatrix} F_3(X_3) \\ F_{4|3}(X_4|x_3) \\ \cdots \\ F_{8|7,...,3}(X_8|x_7, ..., x_3) \end{bmatrix} \\ T_2 : Y \longmapsto U &= \begin{bmatrix} \Phi^{-1}(Y_1) \\ \cdots \\ \Phi^{-1}(Y_6) \end{bmatrix} \\ T &= T_1T_2 : X \longmapsto U, \end{aligned} \tag{10}$$

where Φ is the standard normal cdf. Using the Rosenblatt transformation, the SS can explore the space of U and inverse map to X to test the AEB for each $Y(x)$.

TABLE II: Subset Simulation parameters

	Baseline stochastic model	GMM
Original distribution $p(X)$	3 independent distribution of TTC_0, $Range_0$ and v_{L0}	$p(X_3, ..., X_8) \Leftrightarrow$ 6 independent distribution of $U_1, ..., U_6$
Proposal distribution $q_k(\cdot\|\cdot)$	$N(x_i, \sigma)$, $\sigma \approx$ 0.11, $x_i \sim f(TTC_0)$ 0.003, $x_i \sim f(Range_0)$ 1, $x_i \sim f(v_{L0})$	In space U: $N(x_i, \sigma)$, $\sigma \approx 1.07$
Performance function $Y(x)$	min(range) during a lane change	
Level probability p_0	0.1	
Total sample for each level N	5000	
Number of seeds for each level $N \times p_{m-1}$	$0.1 \times 5000 = 500$	
Total state for each Markov Chain N_c	$p_0^{-1} = 10$	
Stop criterion	$m + 1 > 10$ or $b_{m+1} < 0.2$	

C. Evaluate AV's Algorithm using Subset Simulation

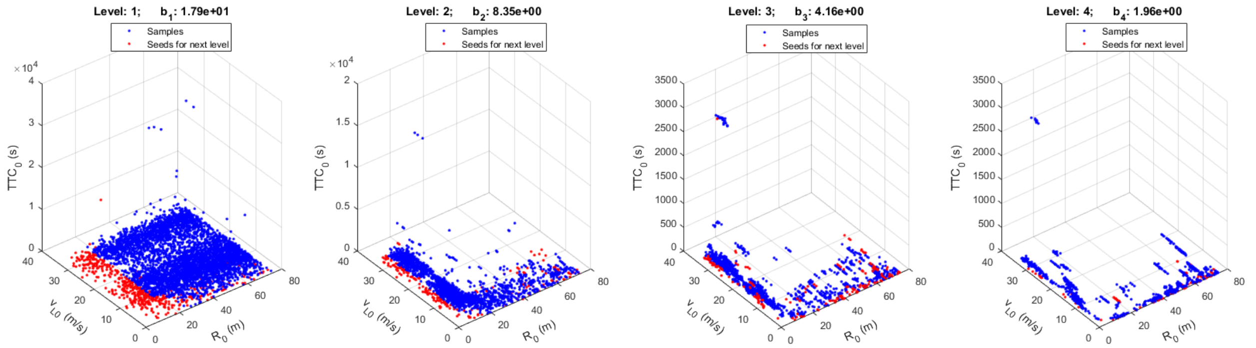
1) Three Independent Variables Model: The simulation of AV is done using MATLAB/Simulink with the input of leading vehicle's initial speed (v_{L0}), initial Time-To-Collision (TTC_0) and initial range ($Range_0$).The SS can be conducted directly since the variables are independent. As mentioned in section II-B, the proposal distribution is the normal distribution centered at each state with $\sigma \approx 2.4/\sqrt{dI}$. The IS result is compared with the SS result in section IV. The parameters used in SS is shown in TABLE II.

2) 8 Variables GMM: Sampling from the 8-dependent variables GMM model, the Rosenblatt transformation is needed. At each subset level, the distribution is first mapped to the standard normal random variable space U and then expand according to the proposal distribution. During testing, the proposed states are mapped back to the original random variable space X and then tested. The parameters used in this SS is shown in TABLE II.

IV. Simulation Results

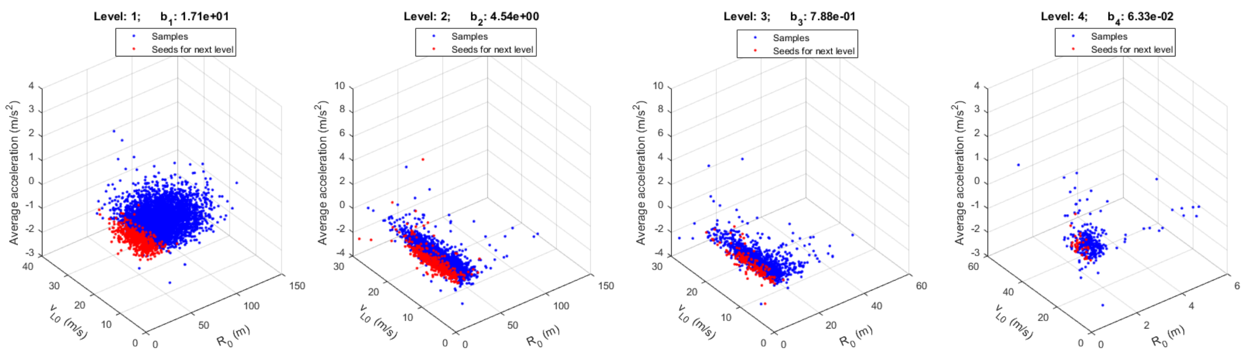
A. Using Baseline Stochastic Model

In this section, the AEB of the following vehicle is tested by the initial condition sampled from baseline stochastic



(a) Level 1 samples (b) Level 2 samples (c) Level 3 samples (d) Level 4 samples

Fig. 2: Samples expended by SS using baseline model



(a) Level 1 samples (b) Level 2 samples (c) Level 3 samples (d) Level 4 samples

Fig. 3: Samples expended by SS using 8 variables GMM

model described in [5]. The inputs of the AEB are initial range, initial leading vehicle's speed and initial TTC described in [5]. The samples tested using SS are shown in Fig. 2 with x-axis being initial range, y-axis being initial leading vehicle speed and z-axis being the initial TTC. Only first 4 levels are shown for abbreviation. The failure probability $p_\epsilon = 3.1 \times 10^{-7}$ is calculated by equation (3) using only 32,000 test results from SS. Compared with the result using IS technique done in [5], with the same confidence level, around 74,100 test cases are needed via the IS method.

Shown in Fig. 2, the dangerous cases keep shrinking to lower TTC value from level to level. The lower the TTC, the shorter time for the following AEB to react to. The results also show that a few clusters of dangerous cases emerge. As shown in Fig. 2(d), there are three clusters of dangerous cases: (1) low initial range with various initial leading vehicle's speed from 2m/s to 35m/s and very low initial TTC; (2) low initial leading vehicle's speed with range from 2m to 75m and very low initial TTC; (3) high initial TTC (around 2600s) with around 2m initial range and 30m/s of initial leading vehicle's speed. Seeds in level 4 give dangerous cases that would happen with probability 10^{-4} to happen. These cases can be the focus for AV testing and evaluation.

We compare the SS method with IS method by testing the AEB using the baseline stochastic model. It is shown that the SS and IS converge to the same failure probability p_ϵ, while exhibiting similar levels of accelerating performance.

B. Using GMM

In this section, the AEB of the following AV is tested. The samples tested at each level using Subset Simulation is shown in Fig. 3 with x-axis being the initial range, y-axis being the initial lead vehicle speed and z-axis being the average acceleration of the lead vehicle during lane change. The red points are those chosen to be the seeds for the next level. In total, 18,500 cases are tested and in level 4, $b_4 = 0.063 < 0.2$, thus the simulation is terminated. Calculated by equation (3), the failure probability of this AEB is $p_\epsilon = 3.45 \times 10^{-4}$. It is much higher than using the baseline stochastic model which is 3.1×10^{-7}. This is because during the lane change, the 8-variable GMM enable the leading vehicle to decelerate which will endanger the following AV.

As can be seen in Fig 3(d), the AEB performs poorly when the lead vehicle decelerates during a lane change, and when the initial range is short. However, we do not see obvious clusters like in the baseline stochastic model. The reason is that the SS only select top 10% of most dangerous cases at each level and when the leading vehicle is able to decelerate, cases with lower initial range are more dangerous than other cases in the same probability level. Also, when leading vehicle's speed is between 20m/s (45mph) to 40m/s (89mph), it is more likely to decelerate during a lane change.

From these results, it is clear that the SS method can handle higher-dimension stochastic models. The AEB's performance is worse when testing in a more realistic environment model which indicate the needs for more realistic model.

V. Conclusion and Future Works

This paper presents the Subset Simulation as an adaptive sampling method for accelerated evaluation of autonomous vehicles. The Subset Simulation has two main advantages. First, SS is able to deal with black-box systems i.e. no information about the AV control algorithm is needed. Moreover, as described in section II, the Markov Chain Monte Carlo (MCMC) method is used in SS (particularly MMA) to extend inside the variable space. This property enables SS to deal with high dimension stochastic models. These two advantages are very important when assessing AV's safety.

We demonstrated the ability of SS to accelerate the evaluation in section IV. In general, SS is a variance reduction technique aiming at using fewer tests to estimate the performance. As shown in section IV, SS has comparable accelerating performance as IS. In future research, more complicated scenarios will be considered.

In this paper, another stochastic model is developed to describe the vehicle motions during lane changes. This 8-parameter model allows the leading vehicle being able to accelerate or decelerate during lane change with more parameters than the baseline stochastic model.

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