

PRICING OPTIONS WITH MATHEMATICAL MODELS

- What we want to accomplish:

Learn the basics of option pricing so you can:

- (i) continue learning on your own, or in more advanced courses;
- (ii) prepare for graduate studies on this topic, or for work in industry, or your own business.

- A rough outline:
 - Basic securities: stocks, bonds
 - Derivative securities, options
 - Deterministic world: pricing fixed cash flows, spot interest rates, forward rates

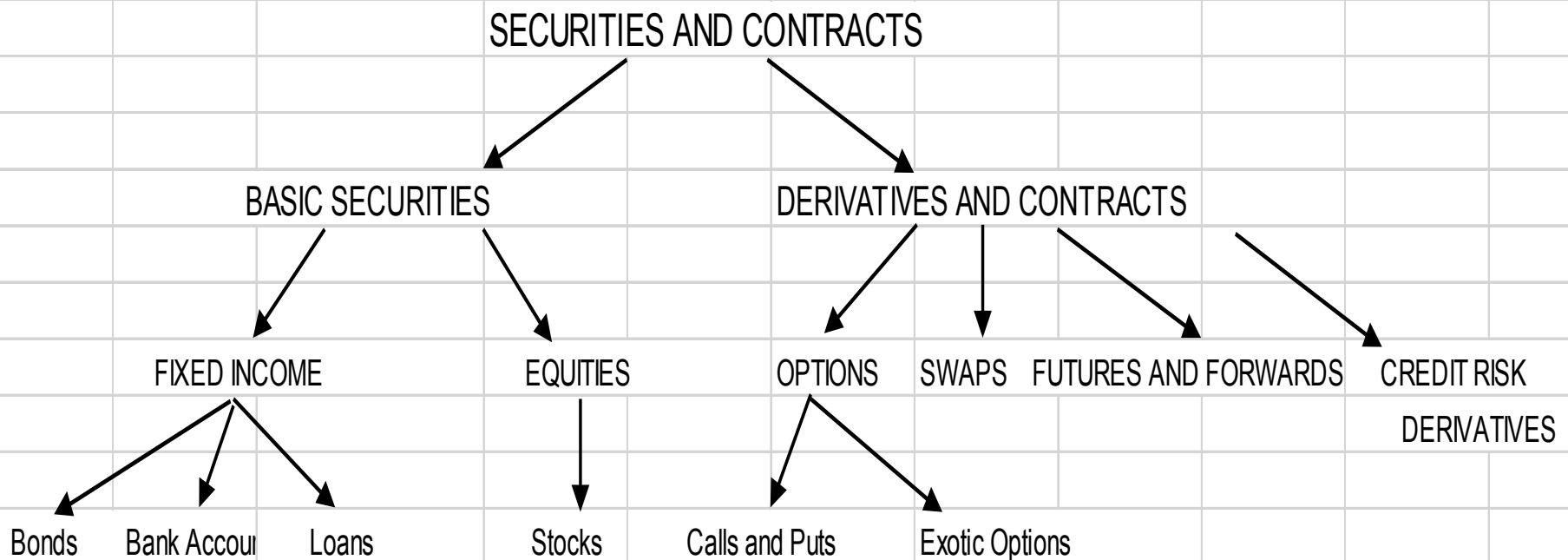
- A rough outline (continued):
 - Stochastic world, pricing options:
 - Pricing by no-arbitrage
 - Binomial trees
 - Stochastic Calculus, Ito's rule, Brownian motion
 - Black-Scholes formula and variations
 - Hedging
 - Fixed income derivatives

Pricing Options with Mathematical Models

2. Stocks, Bonds, Forwards

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

A Classification of Financial Instruments



Stocks

- Issued by firms to finance operations
- Represent ownership of the firm
- Price known today, but not in the future
- May or may not pay dividends

Bonds

- Price known today
- Future payoffs known at fixed dates
- Otherwise, the price movement is random
- Final payoff at maturity: face value/nominal value/principal
- Intermediate payoffs: coupons
- Exposed to default/credit risk

Derivatives

- Sell for a **price/value/premium** today.
- Future value **derived** from the value of the underlying securities (as a function of those).
- Traded at exchanges – standardized contracts, no credit risk;
- or, over-the-counter (OTC) – a network of dealers and institutions, can be non-standard, some credit risk.

Why derivatives?

- To hedge risk
- To speculate
- To attain “arbitrage” profit
- To exchange one type of payoff for another
- To circumvent regulations

Forward Contract

- An agreement to **buy (long)** or **sell (short)** a given **underlying** asset S:
 - At a predetermined future date T (**maturity**).
 - At a predetermined price F (**forward price**).
- F is chosen so that the contract has **zero value today**.
- Delivery takes place at maturity T:
 - Payoff at maturity: $S(T) - F$ or $F - S(T)$
Long Payoff ↓ *Short Payoff ↓*
 - Price F set when the contract is established.
 - $S(T)$ = **spot (market) price** at maturity.

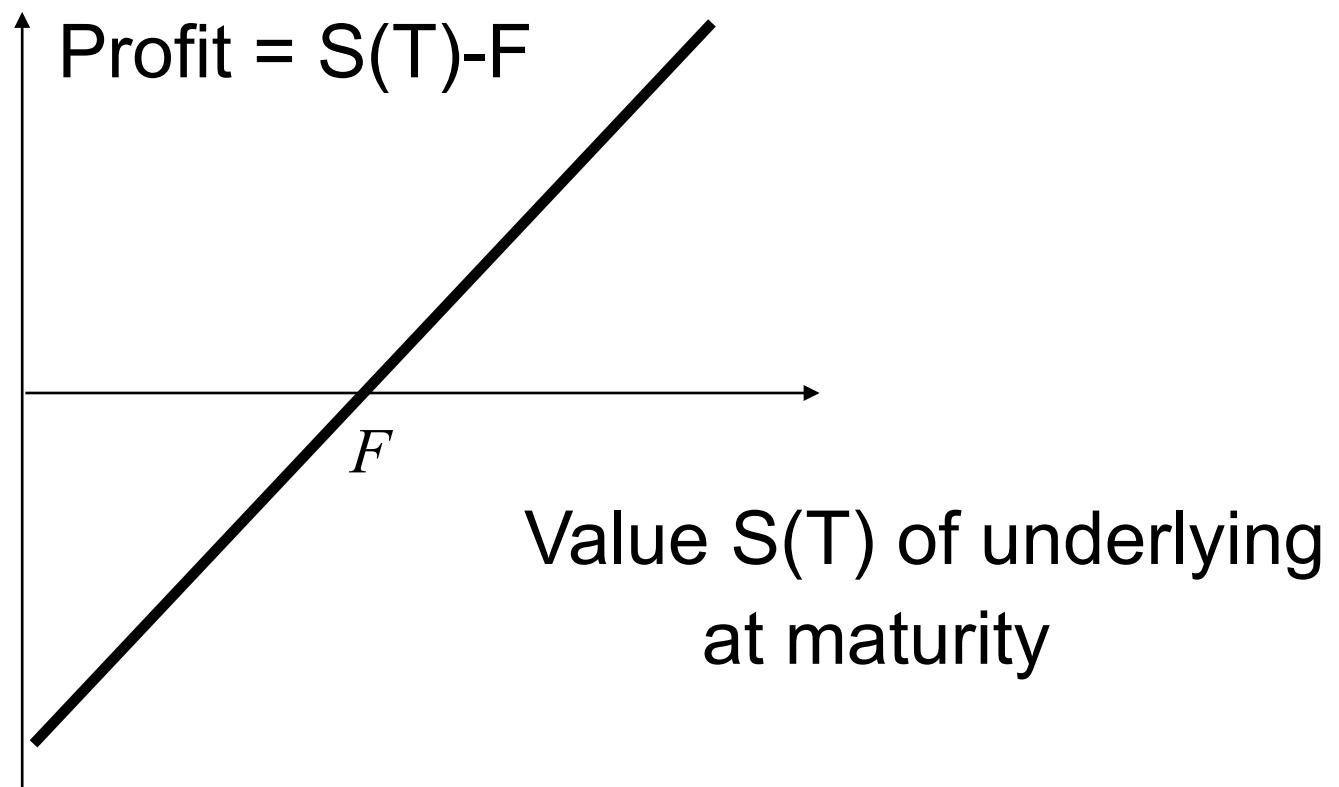
Forward Contract (continued)

- Long position: obligation to buy
- Short position: obligation to sell
- Differences with options:
 - Delivery has to take place.
 - Zero value today.

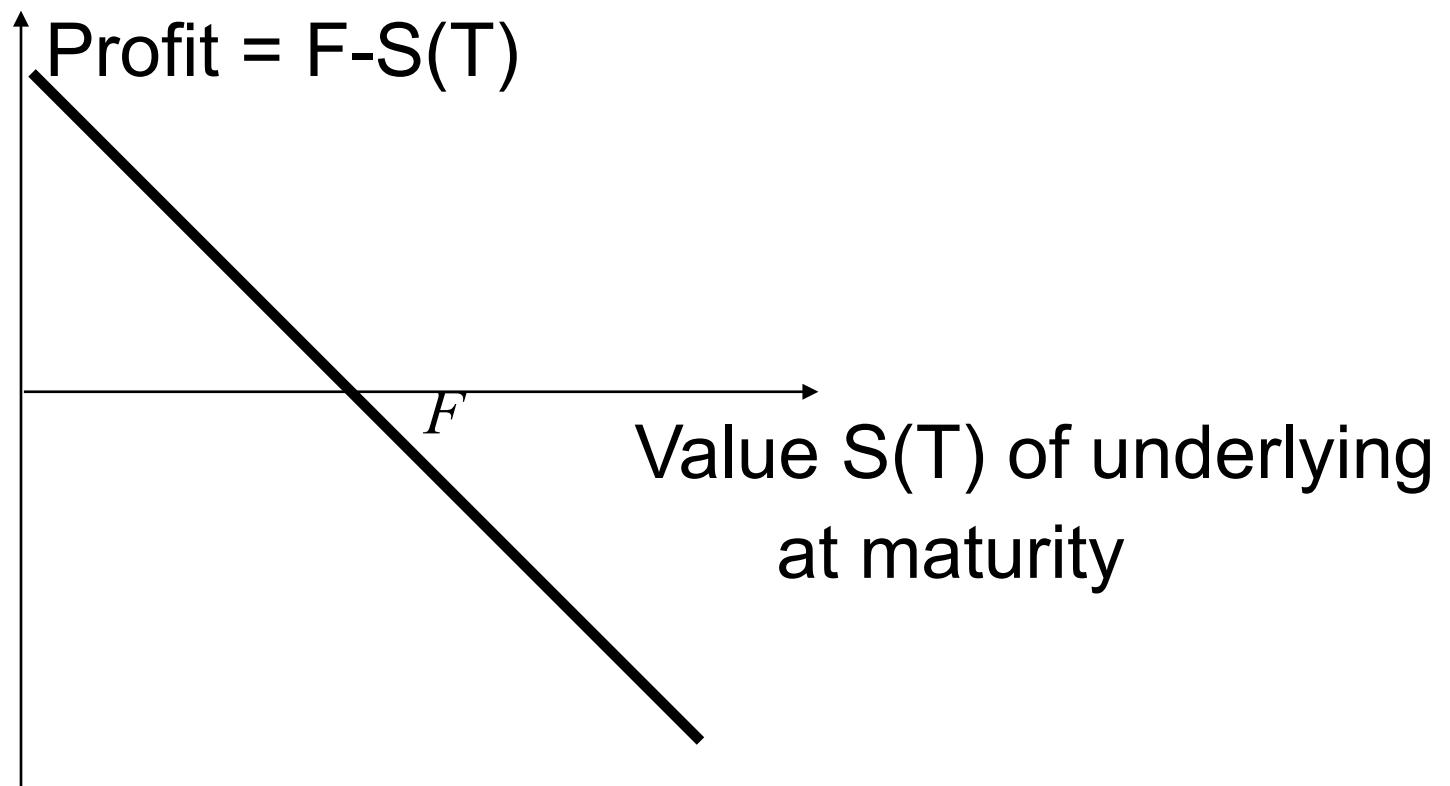
Example

- On May 13, a firm enters into a long forward contract to buy one million euros in six months at an exchange rate of 1.3
- On November 13, the firm pays $F=\$1,300,000$ and receives $S(T)=$ one million euros.
- How does the payoff look like at time T as a function of the dollar value of $S(T)$ spot exchange rate?

Profit from a long forward position



Profit from a short forward position



Pricing Options with Mathematical Models

3. Swaps

- This is a series of forward type contracts.

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Swaps

- Agreement between two parties to exchange two series of payments.
- Classic **interest rate swap**:
 - One party pays **fixed** interest rate payments on a notional amount.
 - Counterparty pays **floating** (random) interest rate payments on the same notional amount.
- Floating rate is often linked to LIBOR (London Interbank Offer Rate), reset at every payment date.

Motivation

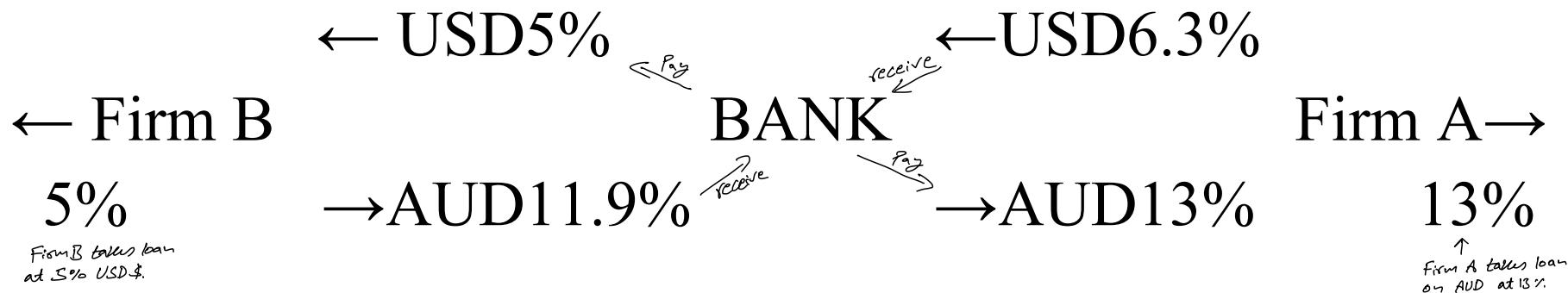
- The two parties may be exposed to different interest rates in different markets, or to different institutional restrictions, or to different regulations.

A Swap Example

- New pension regulations require higher investment in fixed income securities by pension funds, creating a problem: liabilities are long-term while new holdings of fixed income securities may be short-term.
- Instead of selling assets such as stocks, a pension fund can enter a swap, exchanging returns from stocks for fixed income returns.
- Or, if it wants to have an option not to exchange, it can buy **swaptions** instead.

Swap Comparative Advantage

- US firm B wants to borrow AUD, Australian firm A wants to borrow USD
- Firm B can borrow at 5% in USD, 12.6% AUD
- Firm A can borrow at 7% USD, 13% AUD
- Expected gain = $(7-5) - (13-12.6) = 1.6\%$
- Swap: *2 swaps.*



- Bank gains 1.3% on USD, loses 1.1% on AUD, gain=0.2%
- Firm B gains $(12.6-11.9) = 0.7\%$
- Firm A gains $(7-6.3) = 0.7\%$
- Part of the reason for the gain is credit risk involved

A Swap Example: Diversifying

- Charitable foundation CF receives 50mil in stock X from a privately owned firm.
- CF does not want to sell the stock, to keep the firm owners happy
- Equity swap: pays returns on 50mil in stock X, receives return on 50mil worth of S&P500 index.
- A bad scenario: S&P goes down, X goes up; a potential cash flow problem.

Swap Example: Diversifying II

- An executive receives 500mil of stock of her company as compensation.
- She is not allowed to sell.
- Swap (if allowed): pays returns on a certain amount of the stock, receives returns on a certain amounts of a stock index.
- Potential problems: less favorable tax treatment; shareholders might not like it.

Pricing Options with Mathematical Models

4. Call and Put Options

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Vanilla Options

- **Call** option: a right to buy the underlying
- **Put** option: a right to sell the underlying
- **European** option: the right can be **exercised** only at **maturity**
- **American** option: can be exercised at any time before maturity

Various underlying variables

- Stock options
- Index options
- Futures options
- Foreign currency options
- Interest rate options
- Credit risk derivatives
- Energy derivatives
- Mortgage based securities
- Natural events derivatives ...

Exotic options

- **Asian options:** the payoff depends on the average underlying asset price
- **Lookback options:** the payoff depends on the maximum or minimum of the underlying asset price
- **Barrier options:** the payoff depends on whether the underlying crossed a barrier or not
- **Basket options:** the payoff depends on the value of several underlying assets.

Terminology

- *Writing an option*: selling the option
- *Premium*: price or value of an option
- Option **in/at/out of the money**:
 - *At*: strike price equal to underlying price
 - *In*: immediate exercise would be profitable
 - Out*: immediate exercise would not be profitable

Long Call

Outcome at maturity

	$S(T) \leq K$	$S(T) > K$
Payoff:	0	$S(T) - K$
Profit:	$-C(t, K, T)$	$S(T) - K - C(t, K, T)$

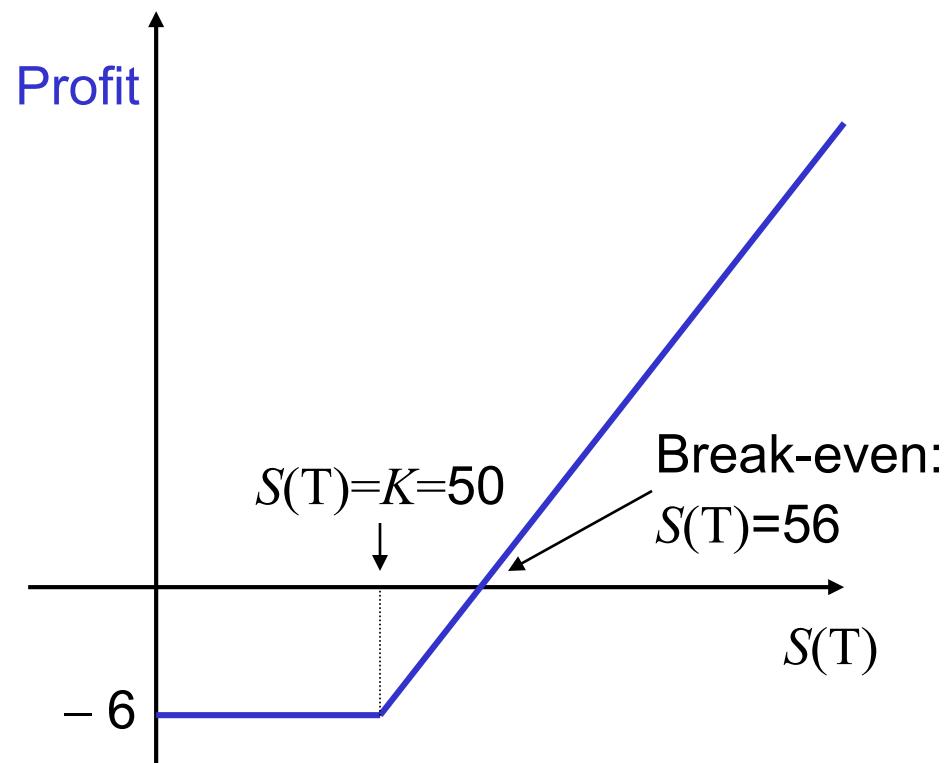
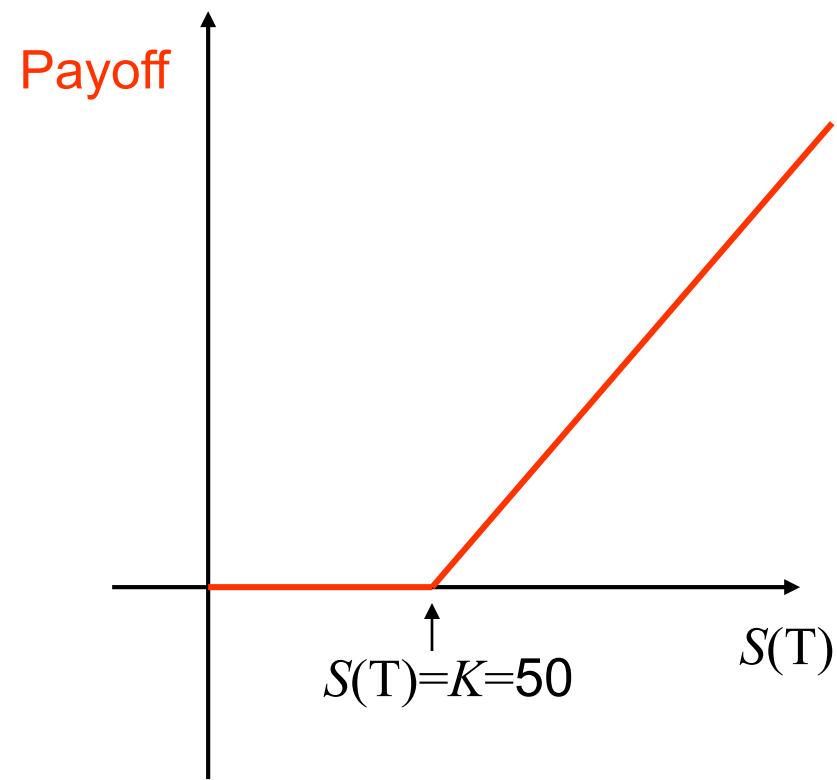
A more compact notation:

Payoff: $\max [S(T) - K, 0] = (S(T) - K)^+$

Profit: $\max [S(T) - K, 0] - C(t, K, T)$

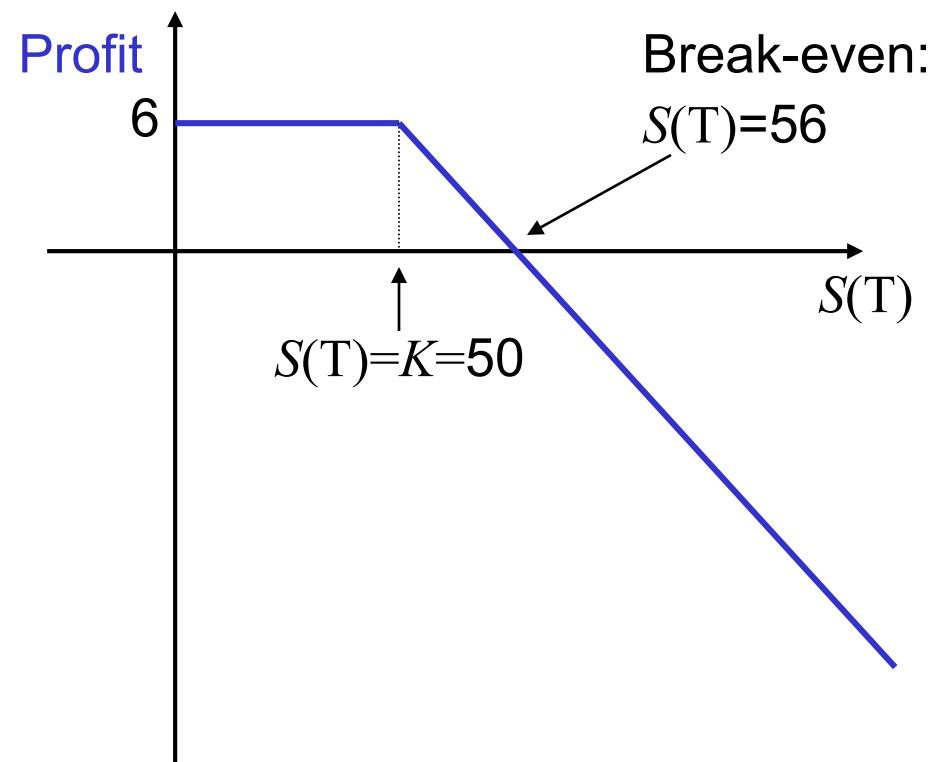
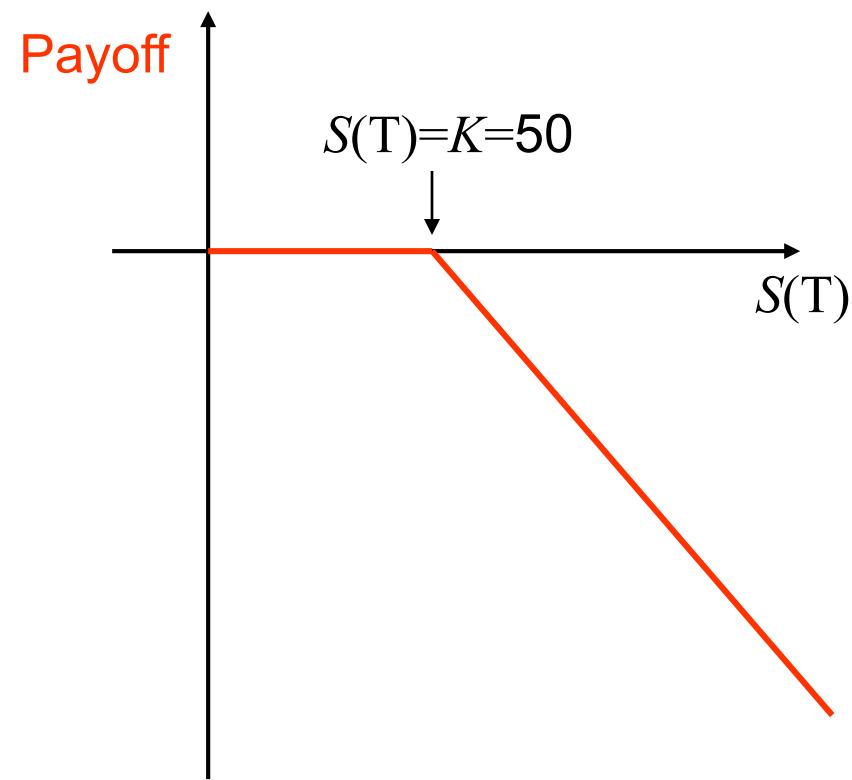
Long Call Position

- Assume $K = \$50$, $C(t, K, T) = \$6$ \leftarrow call price
- Payoff: $\max [S(T) - 50, 0]$
- Profit: $\max [S(T) - 50, 0] - 6$



Short Call Position

- $K = \$50, C(t,K,T) = \6
- Payoff: $- \max [S(T) - 50, 0]$
- Profit: $6 - \max [S(T) - 50, 0]$



Long Put

Outcome at maturity

	$S(T) \leq K$	$S(T) > K$
Payoff:	$K - S(T)$	0
Profit:	$K - S(T) - P(t, K, T)$	$-P(t, K, T)$

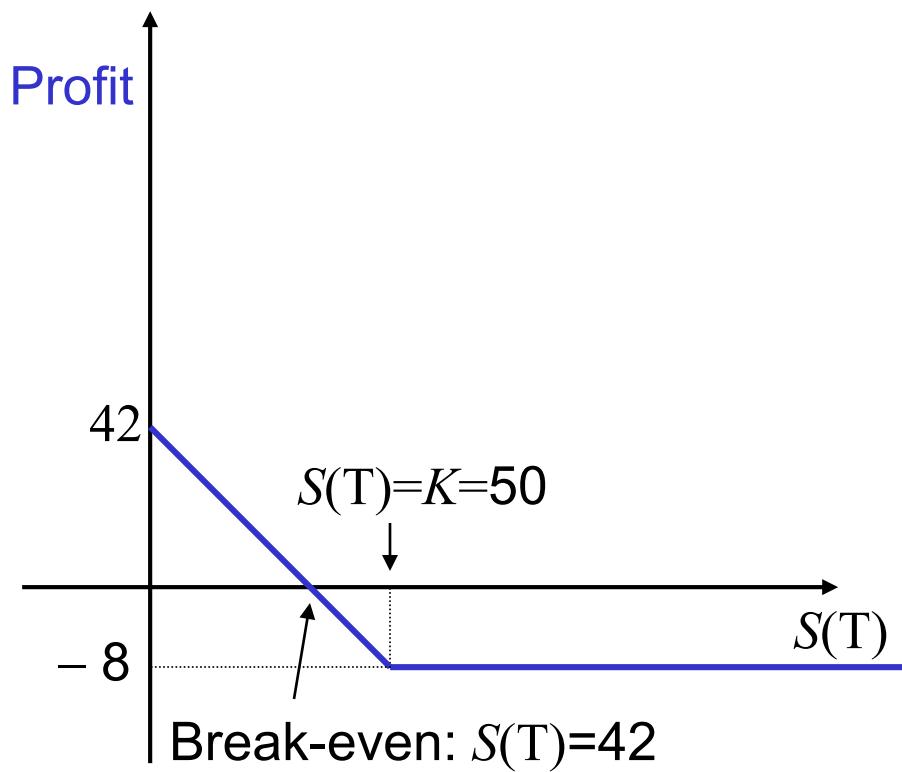
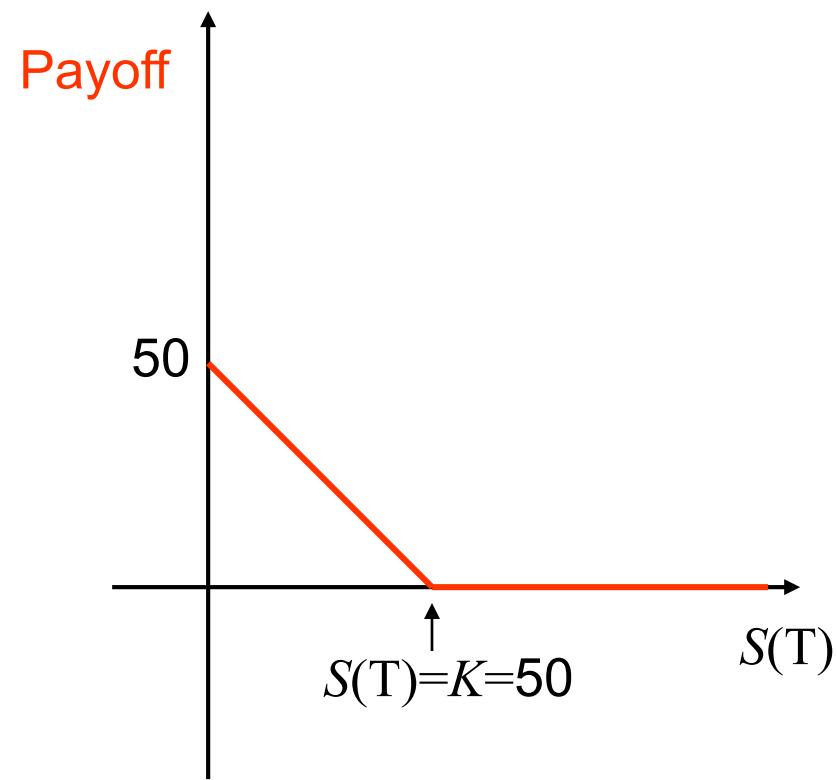
A more compact notation:

Payoff: $\max [K - S(T), 0] = (K - S(T))_+$

Profit: $\max [K - S(T), 0] - P(t, K, T)$

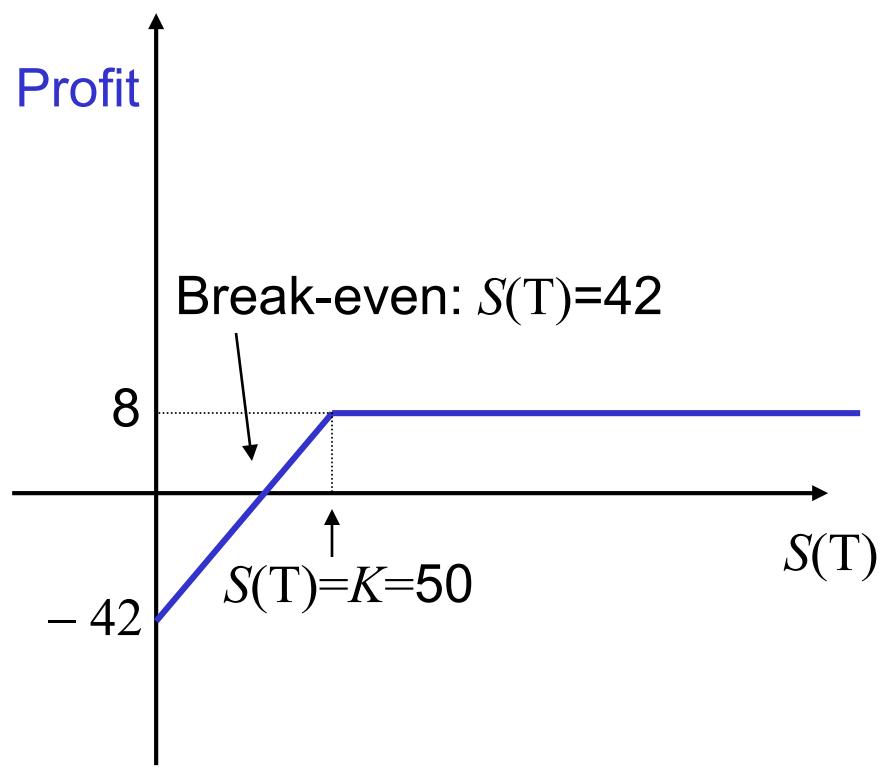
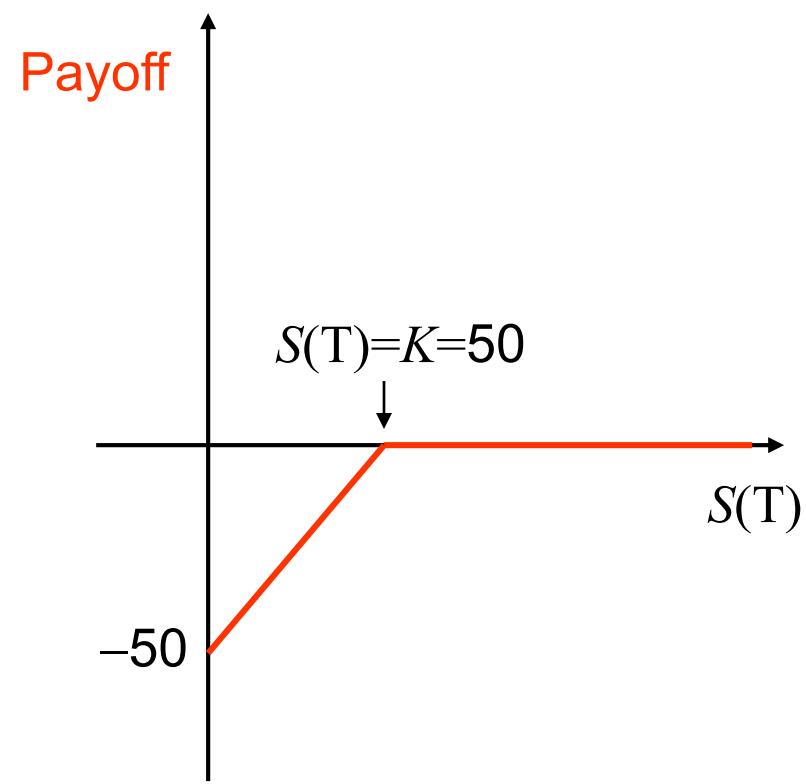
Long Put Position

- Assume $K = \$50$, $P(t, K, T) = \$8$
- Payoff: $\max [50 - S(T), 0]$
- Profit: $\max [50 - S(T), 0] - 8$



Short Put Position

- $K = \$50, P(t,K,T) = \8
- Payoff: $- \max [50 - S(T), 0]$
- Profit: $8 - \max [50 - S(T), 0]$



Implicit Leverage: Example

Borrowing money in your portfolio
↑

- Consider two securities
 - Stock with price $S(0) = \$100$
 - Call option with price $C(0) = \$2.5$ ($K = \$100$)
- Consider three possible outcomes at $t=T$:
 - Good: $S(T) = \$105$
 - Intermediate: $S(T) = \$101$
 - Bad: $S(T) = \$98$

Implicit Leverage: Example

(continued)

Suppose we plan to invest \$100 *i.e. 1 stock or 40 options.*

Invest in:	Stocks	Options
Units	1	40
Return in:	Profit/Loss	Profit/Loss
Good State	5%	100%
Mid State	1%	-60%
Bad State	-2%	-100%

EQUITY LINKED BANK DEPOSIT

- Investment = 10,000
- Return = 10,000 if an index below the current value of 1,300 after 5.5 years
- Return = $10,000 \times (1 + 70\% \text{ of the percentage return on index})$ if index above 1300 after 5.5 yrs.
- Example: Index=1,500. Return =
 $= 10,000 \cdot (1 + (1,500/1,300 - 1) \cdot 70\%) = 11,077$
- Payoff = Bond + call option on index

HEDGING EXAMPLE

Your bonus compensation: 100 shares
of the company, each worth \$150.

Your hedging strategy: buy 50 put
options with strike $K = 150$

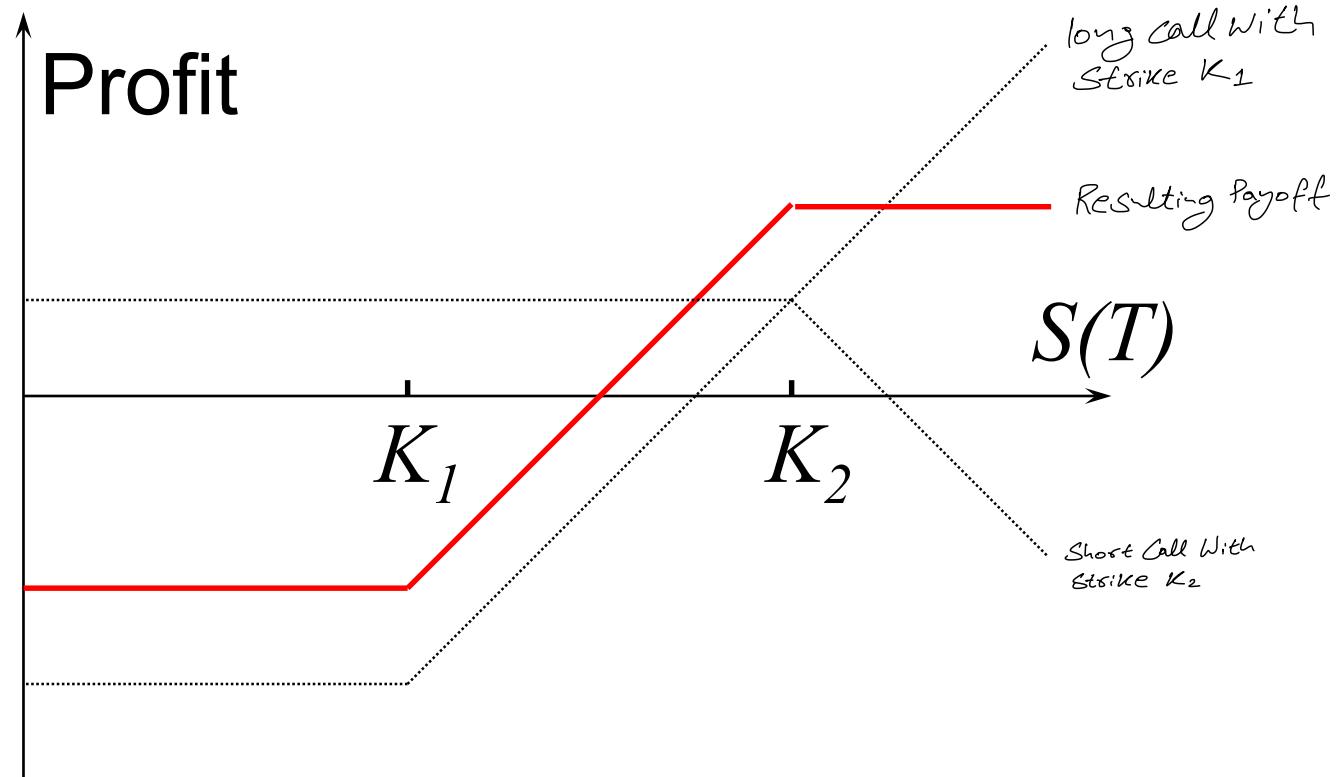
If share value falls to \$100: you lose
\$5,000 in stock, win \$2,500 minus
premium in options

Pricing Options with Mathematical Models

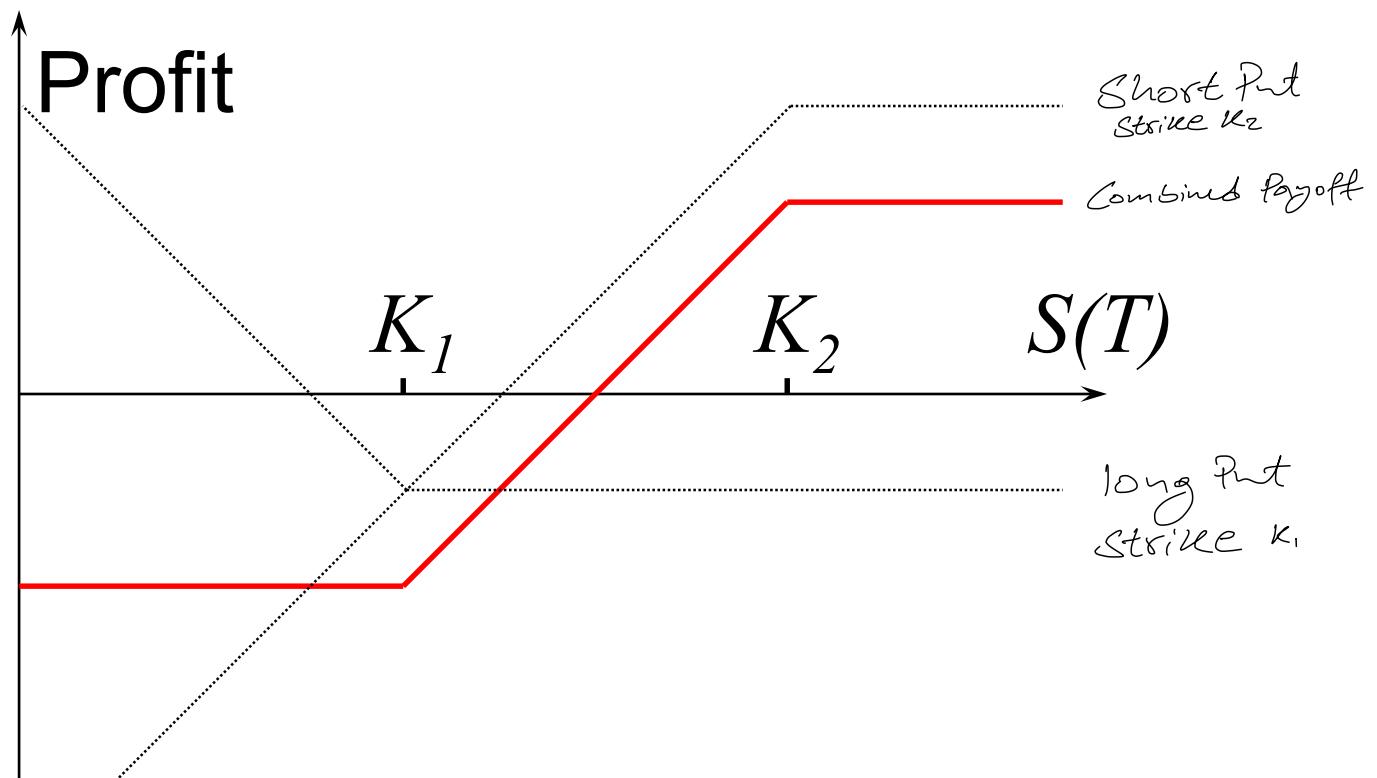
5. Options Combinations

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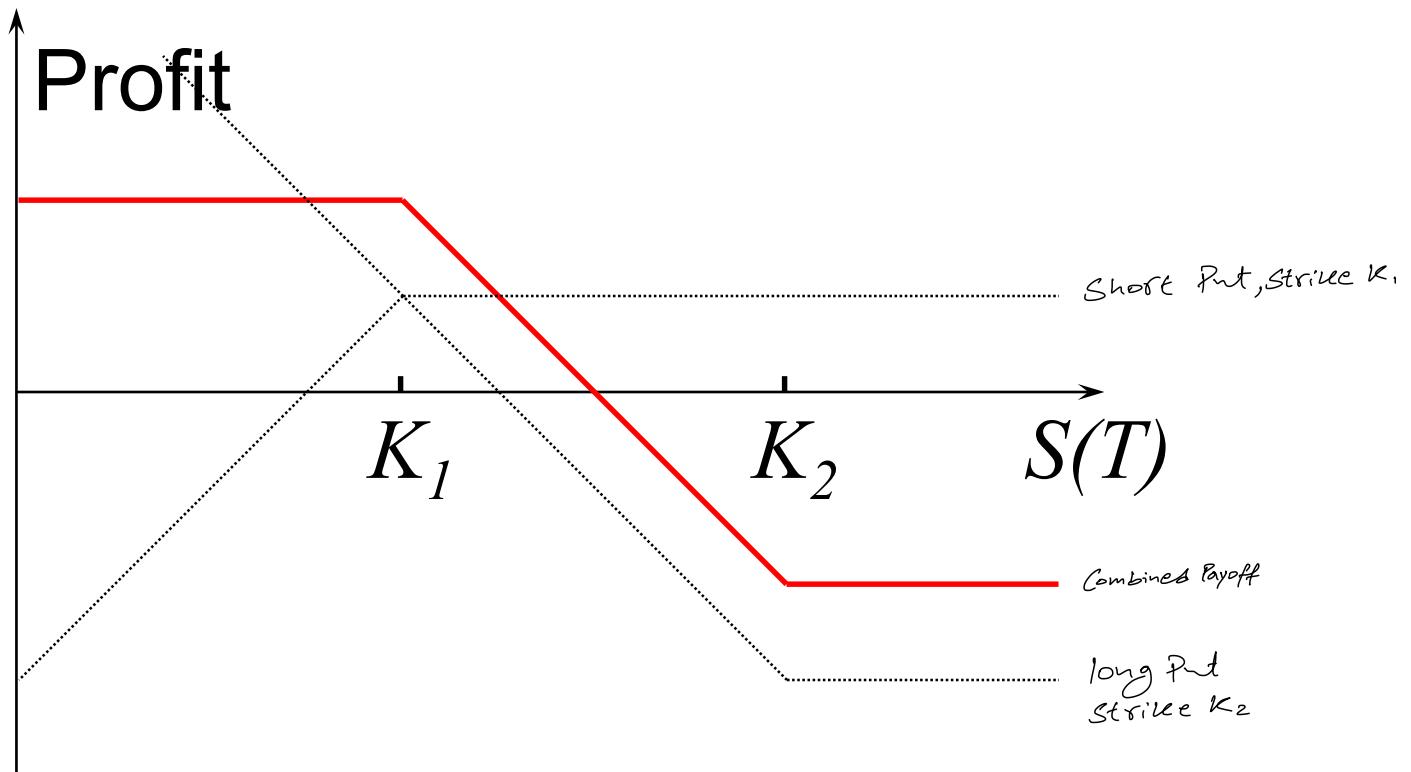
Bull Spread Using Calls



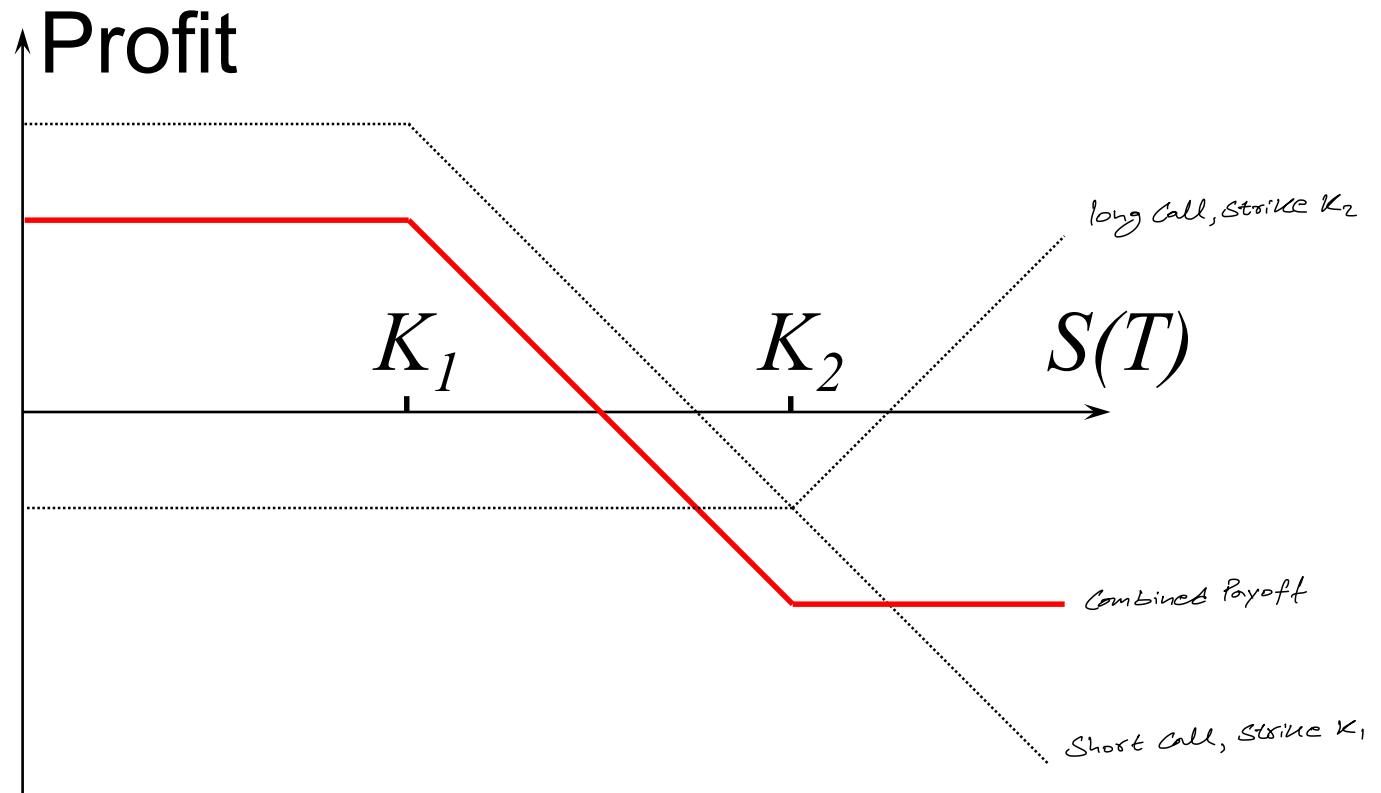
Bull Spread Using Puts



Bear Spread Using Puts

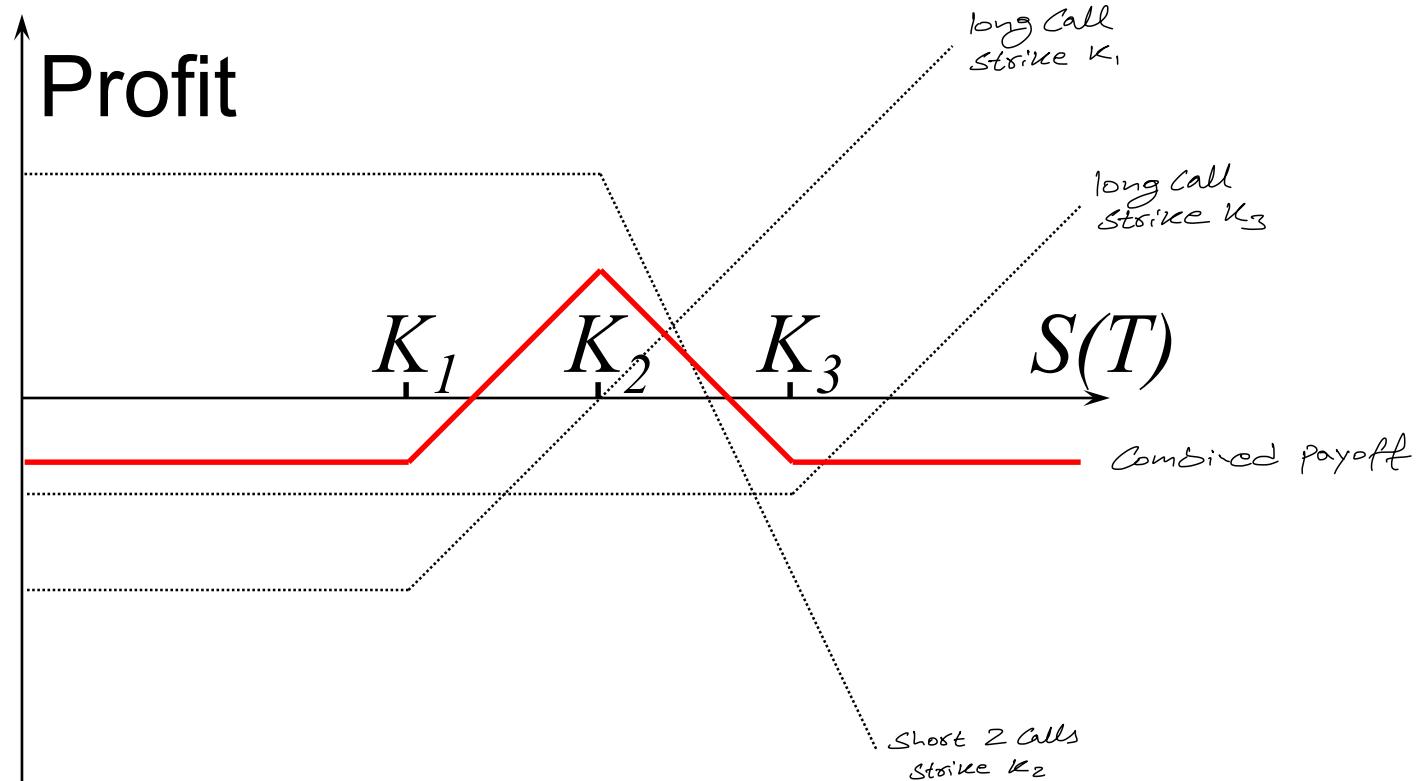


Bear Spread Using Calls

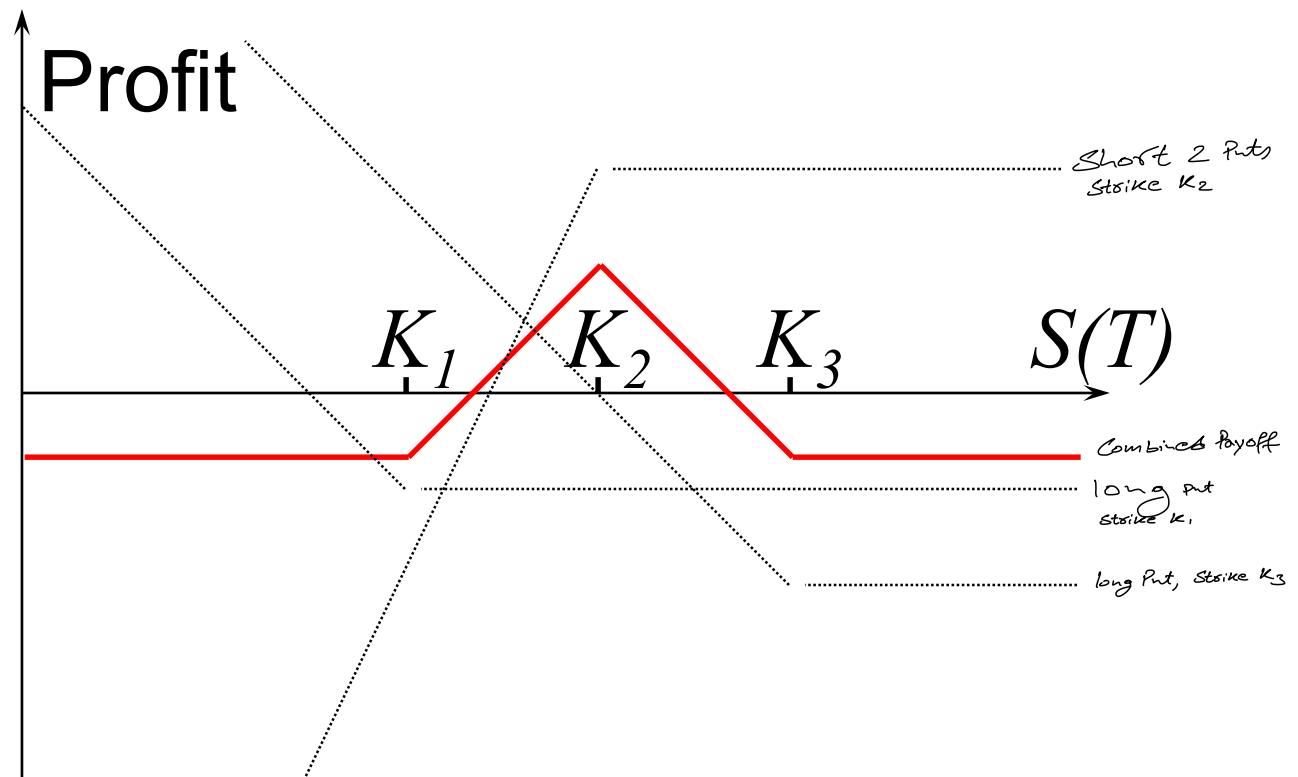


Butterfly Spread Using Calls

Idea: When you believe stock doesn't move too much from position K_2 .

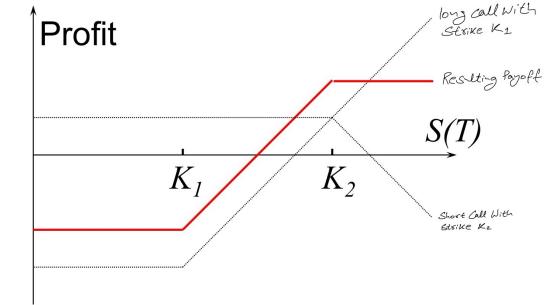


Butterfly Spread Using Puts



Bull Spread (Calls)

- Two strike prices: K_1, K_2 with $K_1 < K_2$
- Short-hand notation: $C(K_1), C(K_2)$



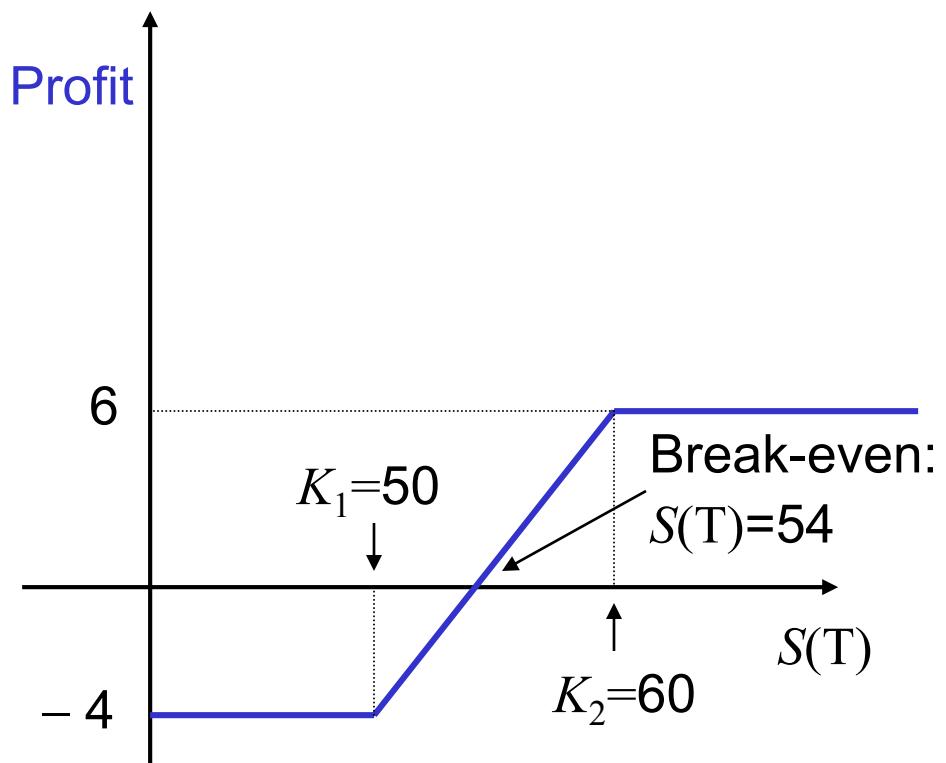
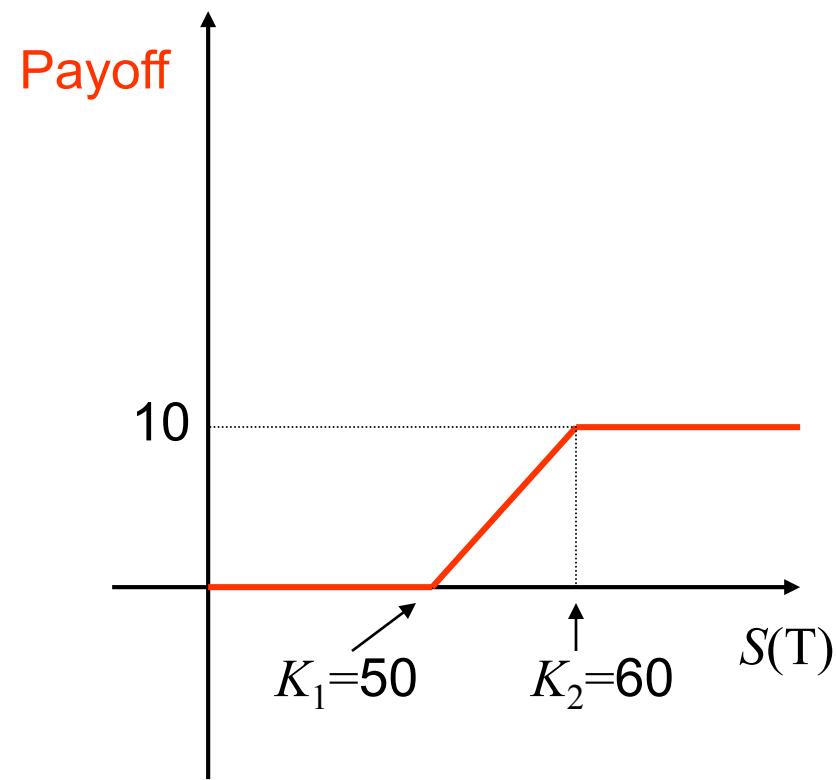
Outcome at Expiration

$$S(T) \leq K_1 \quad K_1 < S(T) \leq K_2 \quad S(T) > K_2$$

Payoff:	0	$S(T) - K_1$	$S(T) - K_1 - (S(T) - K_2) = K_2 - K_1$
Profit:	$C(K_2) - C(K_1)$	$C(K_2) - C(K_1) + S(T) - K_1$	$C(K_2) - C(K_1) + K_2 - K_1$

Bull Spread (Calls)

- Assume $K_1 = \$50$, $K_2 = \$60$, $C(K_1) = \$10$, $C(K_2) = \$6$
- Payoff: $\max [S(T) - 50, 0] - \max [S(T) - 60, 0]$
- Profit: $(6 - 10) + \max [S(T) - 50, 0] - \max [S(T) - 60, 0]$



Bear Spread (Puts)

- Again two strikes: K_1, K_2 with $K_1 < K_2$
- Short-hand notation: $P(K_1), P(K_2)$

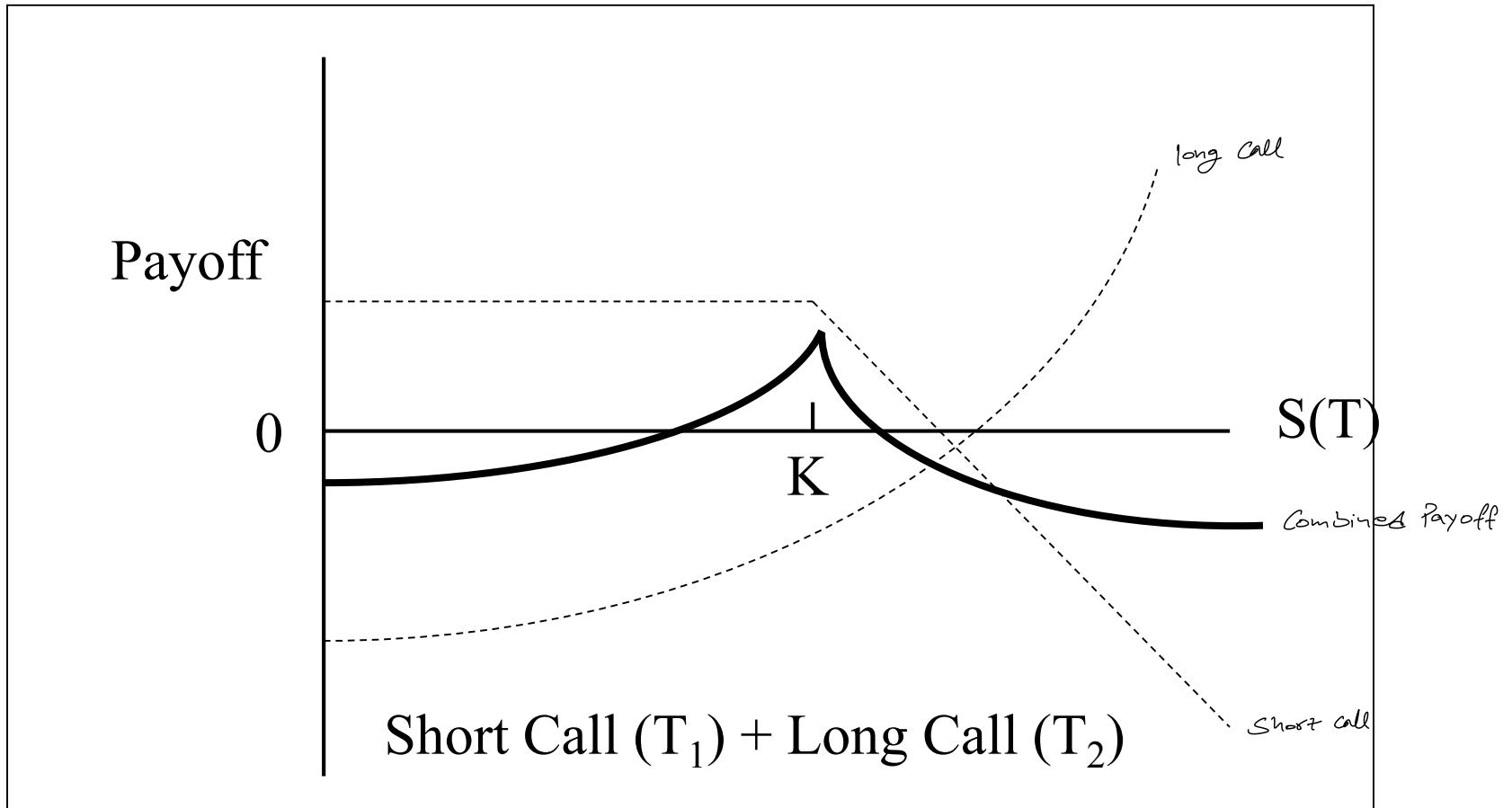
Outcome at Expiration

	$S(T) \leq K_1$	$K_1 < S(T) \leq K_2$	$S(T) > K_2$
Payoff:	$K_2 - S(T) - (K_1 - S(T)) = \\ = K_2 - K_1$	$K_2 - S(T)$	0
Profit:	$P(K_1) - P(K_2) + K_2 - K_1$	$P(K_1) - P(K_2) + \\ + K_2 - S(T)$	$P(K_1) - P(K_2)$

Calendar Spread

- Buying & Selling Calls with different Maturity resulting in Non-linear payoff. Same Strike price.

$$T_1 < T_2$$



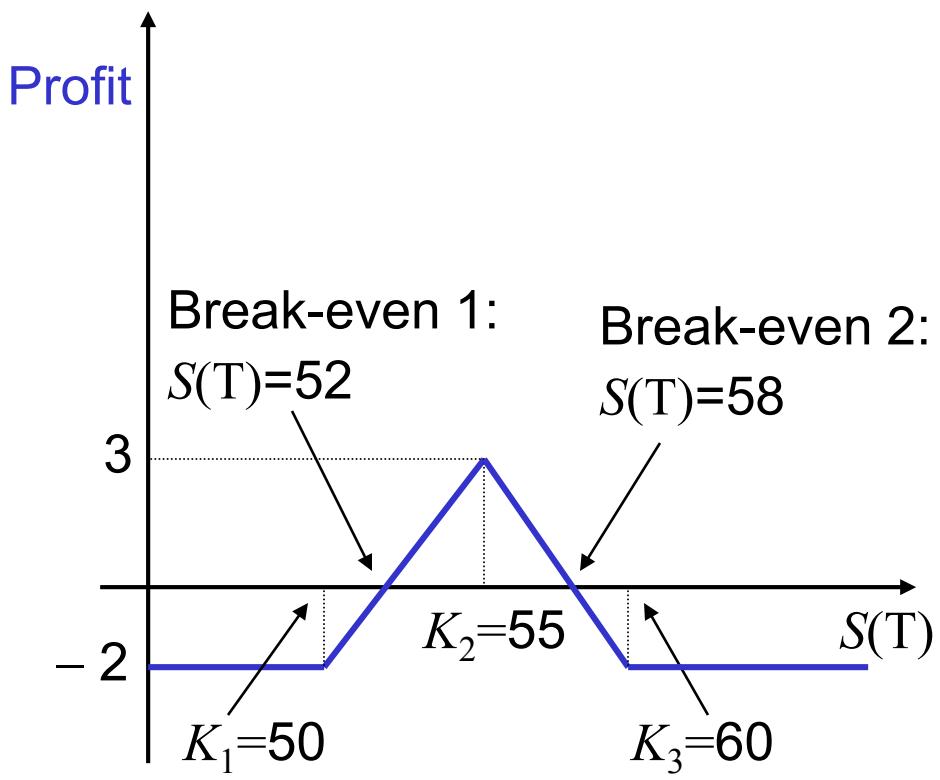
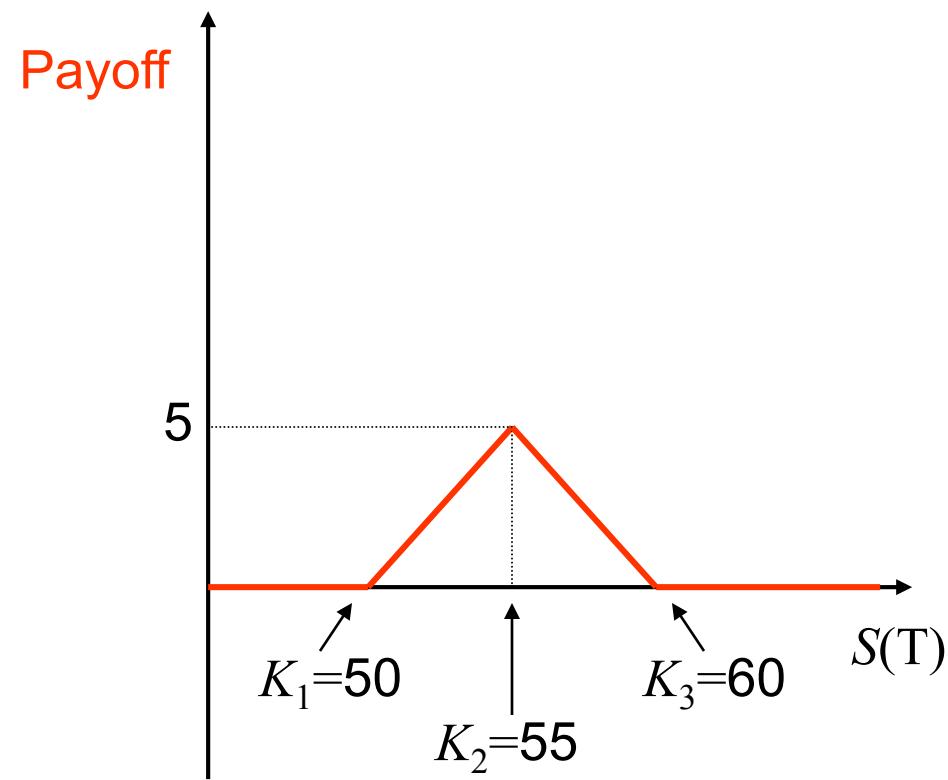
Butterfly Spread

- Positions in **three** options of the same class, with same maturities but different strikes K_1, K_2, K_3
 - Long butterfly spreads: buy one option each with strikes K_1, K_3 , sell two with strike K_2
- $K_2 = (K_1 + K_3) / 2$



Long Butterfly Spread (Puts)

- $K_1 = \$50, K_2 = \$55, K_3 = \$60$
- $P(K_1) = \$4, P(K_2) = \$6, P(K_3) = \$10$

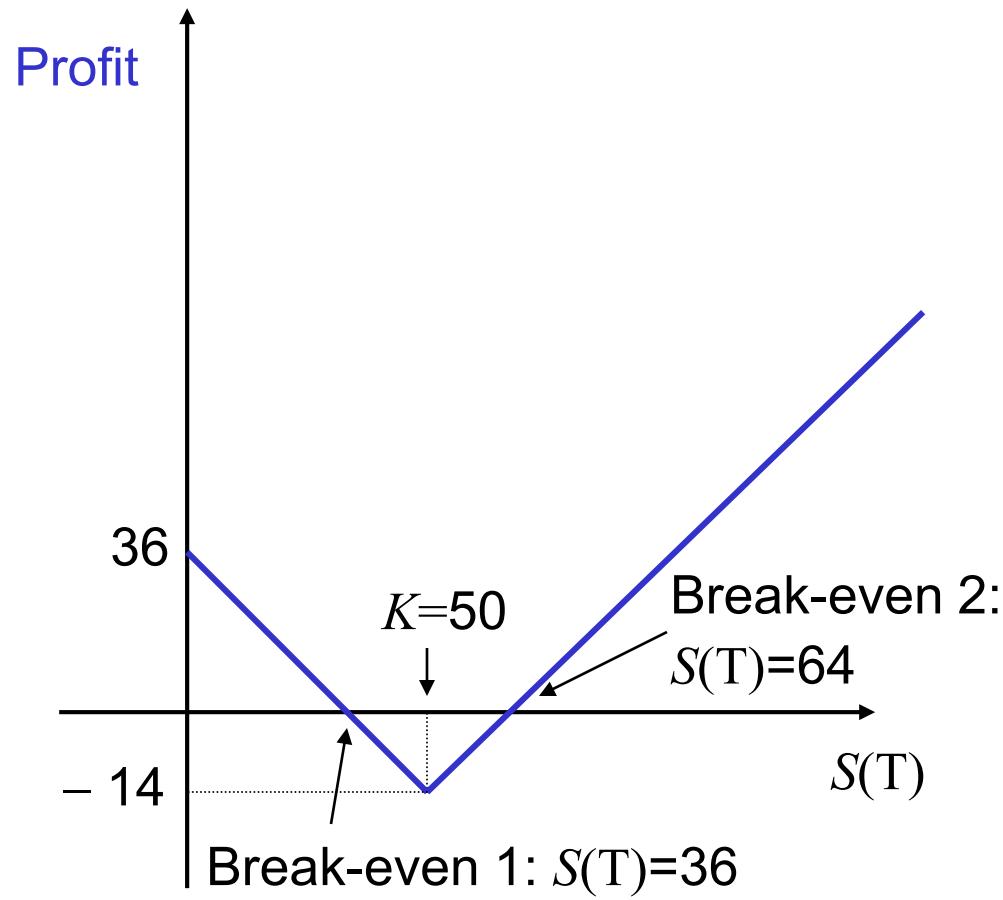
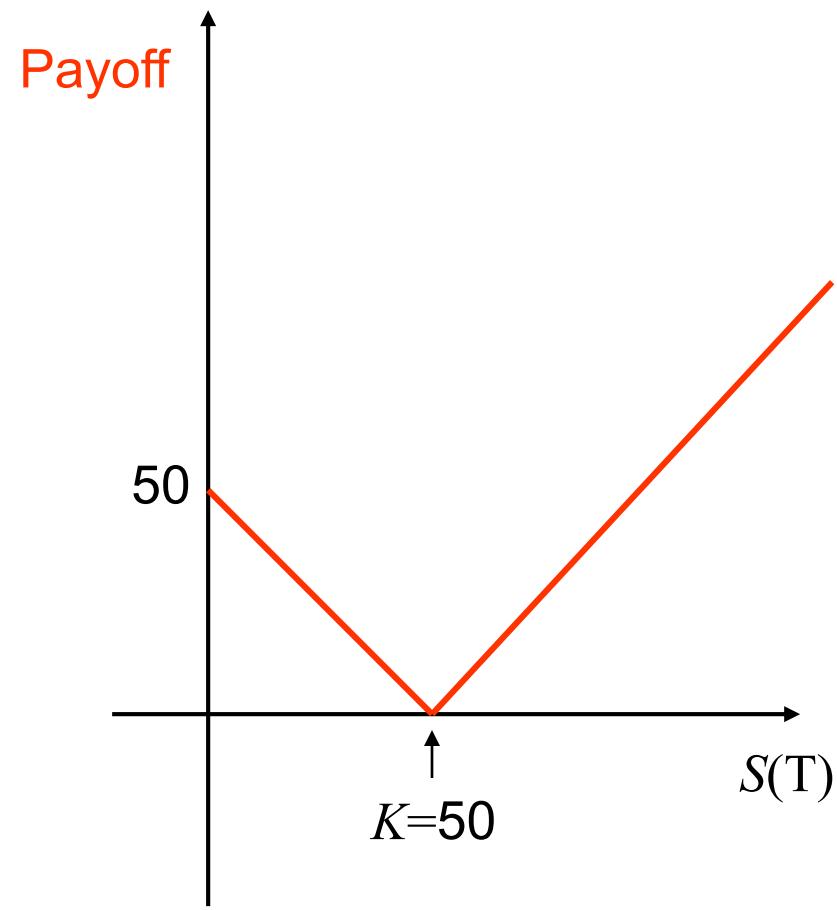


- Betting on high Volatility.

*long Call + long Put
on same strike K*

Bottom Straddle

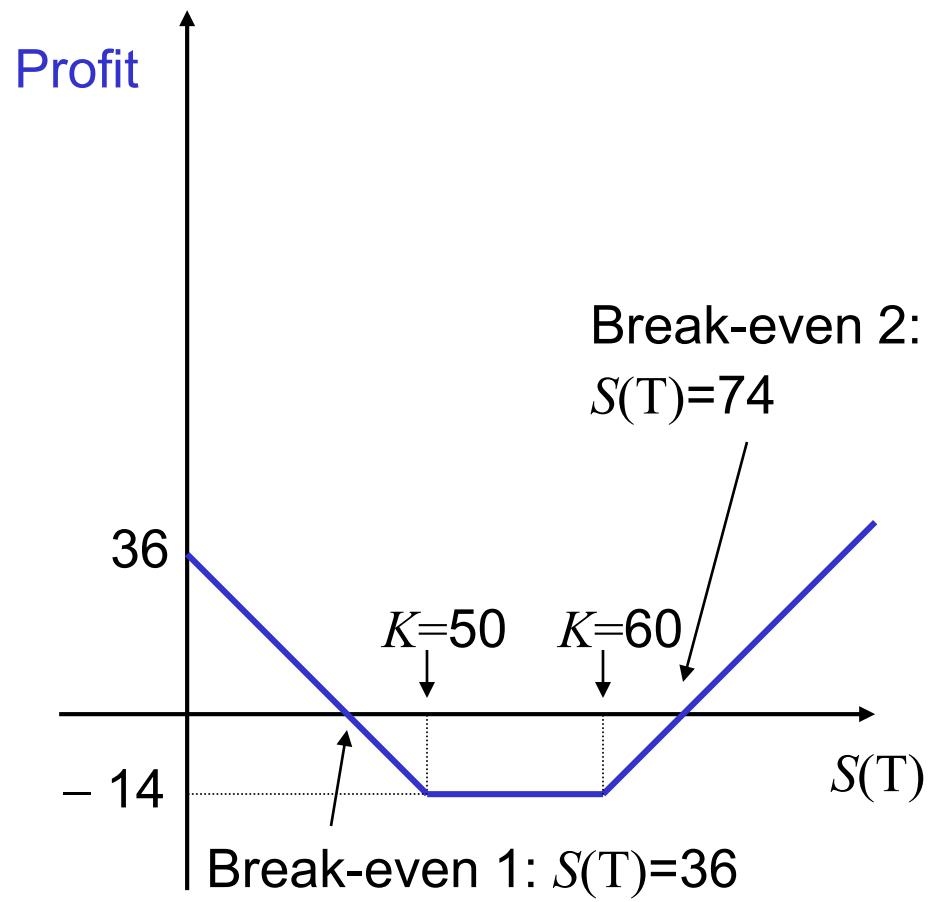
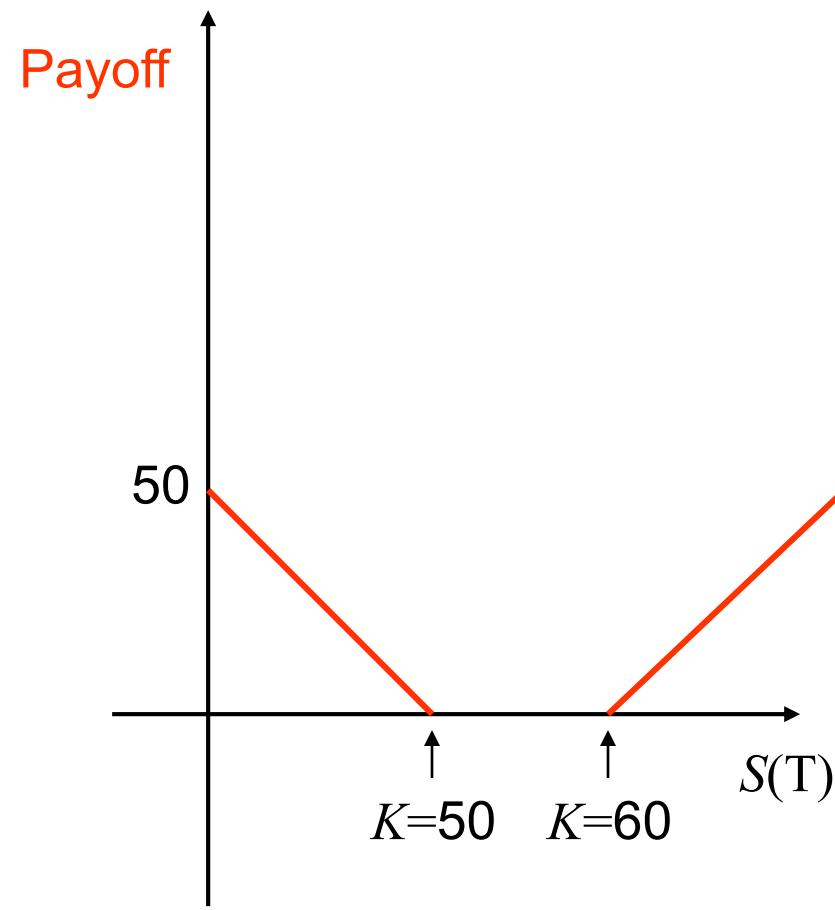
Assume $K = \$50$, $P(K) = \$8$, $C(K) = \$6$



Bottom Strangle

*long Call + long Put
on different strikes K_1 & K_2 .*

Assume $K_1 = \$50$, $K_2 = \$60$, $P(K_1) = \$8$, $C(K_2) = \$6$



Arbitrary payoff shape

- Suppose we want to have a payoff of the form $f(S(T))$ for some function $f(\cdot)$. Assume that call options written on $S(T)$ are traded for all possible strike values K .
- CLAIM:** If $f(\cdot)$ is smooth and $f'(\infty) \cdot 0 = 0$, then

$$f(s) = f(0) + f'(0)s + \int_0^\infty f''(K) \max(S - K, 0) dK$$

Cash

No. of shares to buy/sell.

*Sum of Positions in Call Options with different strikes K .
 $f''(K)$ represents different no. of call options.*

Proof sketch

$$\int_0^{\infty} f''(K) \max(s - K, 0) dK$$

Since if $K > s$, then integral is 0.

= (integration by parts) =

$$= f'(\cancel{s}) \cdot 0 - f'(0) \cdot s - \int_0^{\cancel{s}} f'(K) d[\max(s - K, 0)]$$

$$= -f'(0) \cdot s + \int_0^s f'(K) dK$$

$$= -f'(0) \cdot s + f(s) - f(0).$$

Pricing Options with Mathematical Models

6. Pricing deterministic payoffs

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Future value with constant interest rate

- Suppose you can lend money at annual interest rate r , so that

Present Value (PV)=\$1.00 → Future Value (FV) after 1 year=\$ $(1 + r)$

- Different conventions:
 - Simple interest: after T years, $FV = 1 + T \times r$
 - Interest compounded once a year: after T years, $FV = (1 + r)^T$
 - Interest compounded n times a year: after m compounding periods,

$$FV = (1 + r/n)^m$$

- **Effective annual interest rate r' :**

$$(1 + r/n)^n = 1 + r'$$

EXAMPLE: Quarterly compounding at **nominal** annual rate $r = 8\%$,

$$(1 + 0.08/4)^4 = 1.0824 = 1 + 0.0824$$

Thus, the effective annual interest rate is 8.24%.

- **Continuous compounding:**

- after one year, $FV = \lim_{n \rightarrow \infty} (1 + r/n)^n = e^r$, $e = 2.718 \dots$
- after T years, $FV = \lim_{n \rightarrow \infty} (1 + r/n)^{nT} = e^{rT}$

EXAMPLE: $r = 8\%$, $e^r = 1.0833$, $r' = 8.33\%$

$$\begin{aligned}(1 + \frac{r}{n})^n &= 1 + r' \\ e^r &= 1 + r' \\ r' &= e^{0.08} - 1 \\ &= 0.0833 \\ &= 8.33\%\end{aligned}$$

Price as Present Value

- **LAW OF ONE PRICE:** If two cash flows deliver the same payments in the future, they have the same price (value) today.
- **PRICE DEFINITION:** If one can guarantee having $\$X(T)$ at time T by investing $\$X(0)$ today, then, today's price of $X(T)$ is $X(0)$.
- For deterministic $X(T)$, $X(0)$ is called the **present value**, $PV(X(T))$.
- Thus, if one can invest at compounded rate of r/n , and $T=m$ periods,

$$X(0) = PV(X(T)) = X(T)/(1 + r/n)^m$$

because this is equivalent to $X(0)(1 + r/n)^m = X(T)$.

- **Discount factor:** $1/(1 + r/n)^m$, e^{-rT}


PV of cash flows

- Cash flow $X(0), X(1), X(2), \dots, X(m)$, paid at compounding intervals:

$$PV = X(0) + \frac{X(1)}{\left(1+\frac{r}{n}\right)} + \frac{X(2)}{\left(1+\frac{r}{n}\right)^2} + \dots + \frac{X(m)}{\left(1+\frac{r}{n}\right)^m}$$

- When $X(0)=0$ and $X(i) = X$, we have a geometric series:

$$PV = X \cdot \left(\frac{1}{\left(1+\frac{r}{n}\right)} + \frac{1}{\left(1+\frac{r}{n}\right)^2} + \dots + \frac{1}{\left(1+\frac{r}{n}\right)^m} \right)$$

n = Compounded n times a year
 m = Total periods ($n \times T$)

$$= X \cdot \frac{1}{r/n} \cdot \left(1 - \frac{1}{\left(1+\frac{r}{n}\right)^m} \right)$$

Example

- We want to estimate the value of leasing a gold mine for 10 years. It is estimated that the mine will produce 10,000 ounces of gold per year, at a cost of \$200 per ounce, and that the gold will sell for \$400 per ounce. We also estimate that, if not invested in the mine, we could invest elsewhere at $r=10\%$ return per year.

Annual profit = 10,000 (400-200) = 2 mil

$$PV = \sum_{k=1}^{10} \frac{2 \text{ mil}}{(1+0.1)^k} = 2 \cdot 10 \cdot \left(1 - \frac{1}{(1+0.1)^{10}} \right) = 12.29 \text{ mil}$$

↓
Using formula
from last slide

Loan payments

- Suppose you take a loan with value $PV = \$V$, and the loan is supposed to be paid off (amortized) in equal amounts X over m periods at interest rate $\frac{r}{n}$.
- Inverting the PV formula, we get

$$X = V \cdot \frac{\frac{r}{n} \left(1 + \frac{r}{n}\right)^m}{\left(1 + \frac{r}{n}\right)^m - 1}$$

EXAMPLE 1. You take a 30-year loan on \$400,000, at annual rate of 8%, compounded monthly. What is the amount X of your monthly payments?

With 12 months in a year, the number of periods is $m=30 \cdot 12=360$. The rate per period is $0.08/12=0.0067$. The value of the loan is $V=400,000$. We compute $X= \$2,946$ in monthly payments, approximately.

The loan balance is computed as follows:

- Before the end of the first month, balance = $400,000+0.0067 \cdot 400,000=402,680$
- After the first installment of \$2,946 is paid, balance = $402,680-2,946=399,734$.
- Before the end of the second month, balance = $399,734(1+0.0067)$, and so on.

The future value corresponding to these payments thirty years from now is

$$400,000 \cdot (1 + 0.0067)^{360} = 4,426,747 .$$

EXAMPLE 2 (Loan fees). For mortgage products, there are usually two rates listed: the mortgage interest rate and the APR, or annual percentage rate. The latter rate includes the fees added to the loan amount and also paid through the monthly installments. Consider the previous example with the rate of 7.8% compounded monthly, and the APR of 8.00%. As computed above, the monthly payment at this APR is 2,946. Now, we use this monthly payment of 2,946 and the rate of $7.8/12=0.65\%$ in the formula for V, to find that the total balance actually being paid is 407,851.10. This means that the total fees equal

$$407,851.10 - 400,000 = 7,851.10 .$$

Perpetual annuity

- Pays amount X at the end of each period, for ever.
If the interest rate per period is r ,

$$PV = \sum_{k=1}^{\infty} \frac{X}{(1+r)^k} = \frac{X}{r}$$

Internal rate of return

- The rate r for which

$$0 = X(0) + \frac{X(1)}{\left(1+\frac{r}{n}\right)} + \frac{X(2)}{\left(1+\frac{r}{n}\right)^2} + \dots + \frac{X(m)}{\left(1+\frac{r}{n}\right)^m}$$

*Typically this is -ve
as it will be initial investment.*

Pricing Options with Mathematical Models

7. Bonds

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Bond yield

- **Yield to maturity (YTM)** of a bond is the internal rate of return of the bond, or the rate that makes the bond price equal to the present value of its future payments.
- Suppose the bond pays a **face value** V at maturity $T = m$ periods and n identical coupons a year in the amount of C/n , and its price today is P . Then, the bond's annualized yield corresponding to compounding n times a year is the value y that satisfies

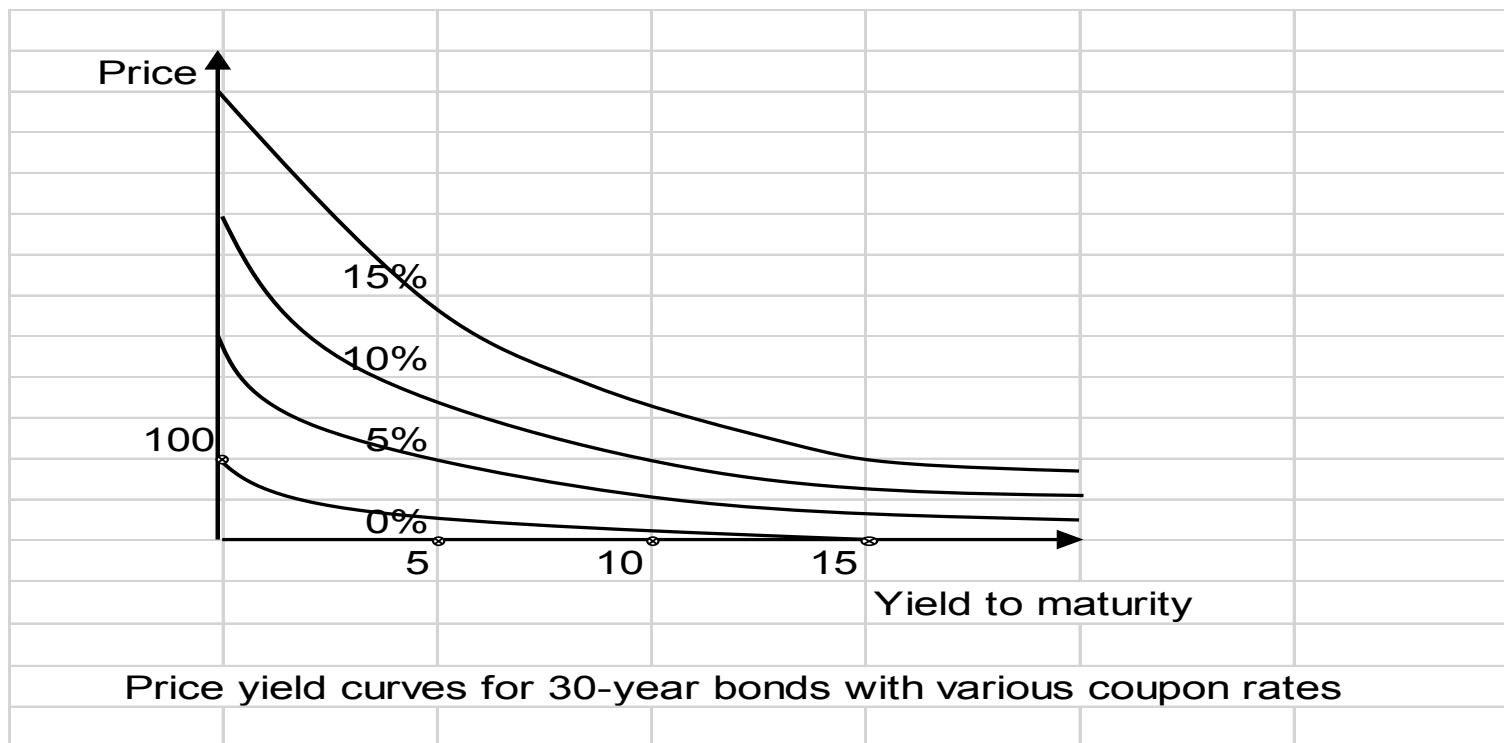
$$P = \frac{V}{\left(1+\frac{y}{n}\right)^m} + \sum_{k=1}^m \frac{C/n}{\left(1+\frac{y}{n}\right)^k} = \frac{V}{\left(1+\frac{y}{n}\right)^m} + \frac{C}{y} \cdot \left(1 - \frac{1}{\left(1+\frac{y}{n}\right)^m} \right)$$

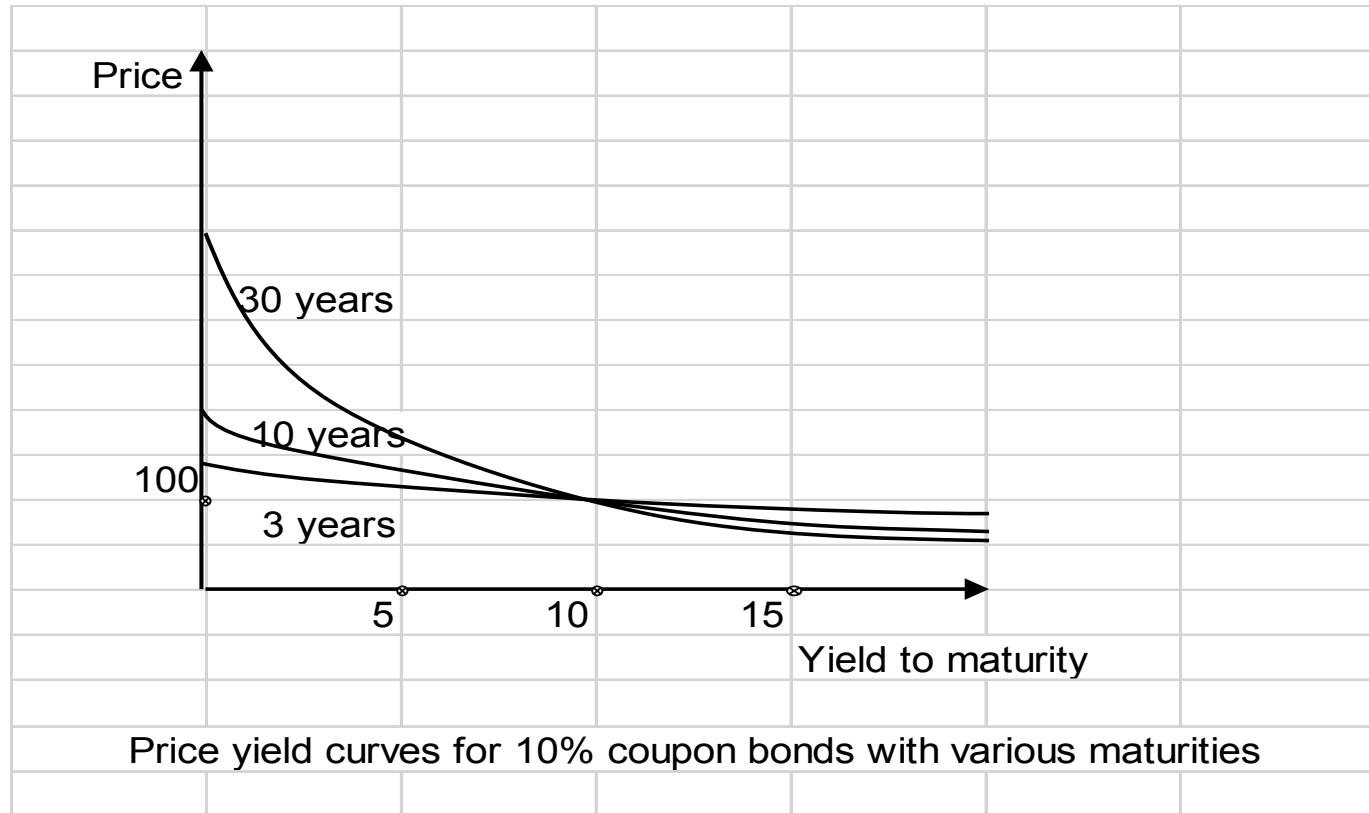
The price of Bond P is observed from Market Q we can calculate the yield y by solving the above equation.

- Higher price corresponds to lower yield.

Price-yield curve

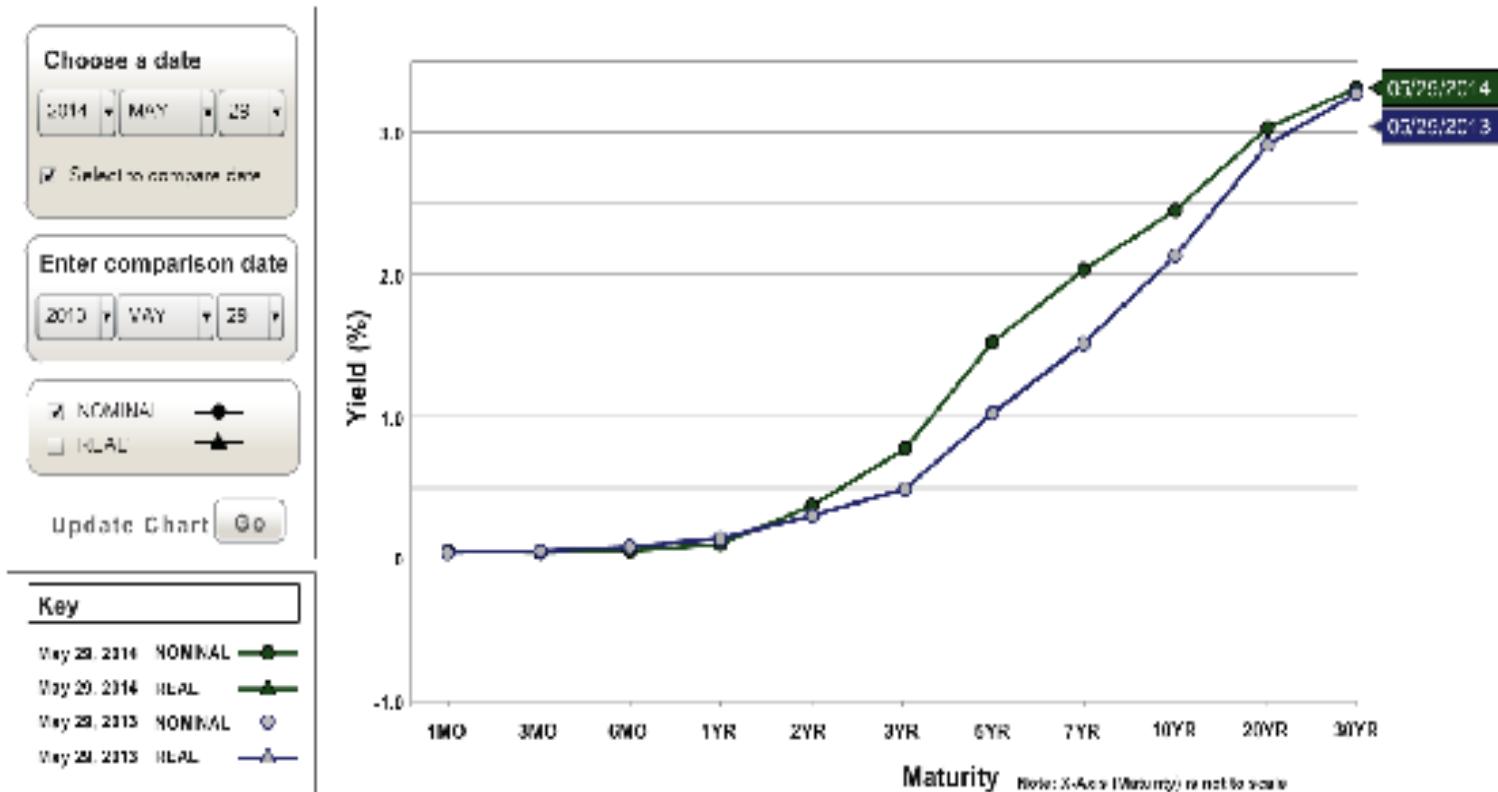
- Terminology: ‘a 10% five-year bond’ is a bond that pays 10% of its face value per year, for five years, plus the face value at maturity.





- Why do they all intersect at the point (10, 100) ? *Price for 10% yield is \$100 today for all 3 bonds. Interest paid through coupons for \$100 face value.*
- Hint: Set $C = y \cdot V$ in the formula for P . We say ‘the bond trades at par’ *→ if Bond trades at face value today.*

Yield curve (term structure of interest rates)



Spot rates and arbitrage

- **Spot rate** = yield of a **zero-coupon bond** (pure discount bond)
- **Arbitrage** (of strong kind) = making positive sure profit with zero investment
- EXAMPLE: A 6-month zero-coupon bond with face value 100 trades at 98.00. A coupon bond that pays 3.00 in 6 months and 103 in 12 months trades at 101.505. What should be the yield of the 1-year zero coupon bond with face value 100?
- REPLICATION: Find a combination of the traded bonds to replicate exactly the payoff of the 1-year bond. (ZCB)
- BUY: one coupon bond; SELL (short): 0.03 units of the 6-month bond
- COST = $101.505 - 0.03 \cdot 98 = 98.565$
- In 6 months: pay 3.00 for the short bond, receive 3.00 as a coupon
- In 12 months: receive 103.00
- $98.565 = \frac{103}{1+r}$, $r = 4.4996\%$; Otherwise arbitrage!

✓
Cancels payments at 6 month

Alternative computation

- First, compute the 6-month spot rate:

$$98 = \frac{100}{(1+y)^{1/2}} \rightarrow y = 4.1233\%$$

- Then, compute the one-year rate from

$$101.505 = \frac{3}{(1+0.041233)^{1/2}} + \frac{103}{1+r} \rightarrow r = 4.4996\%$$

Arbitrage if mispriced

- Suppose the 1-year bond price is 95.00 instead of

$$\frac{100}{1+0.044996} = 95.6942$$

- BUY CHEAP, SELL EXPENSIVE:
 - buy the 1-year bond
 - go short $100/103$ of the portfolio that replicates it:
 - sell short $100/103$ units of the coupon bond;
 - buy $0.03 \cdot 100/103$ units of the 6-month bond.
 - this results in initial profit of 95.6942
- After 6 months: have to pay $3 \cdot 100/103$, and receive the same amount
- After 1 year: have to pay 100 and receive 100
- Total profit: $95.6942 - 95.00 \rightarrow$ arbitrage!

$\frac{100}{103}$ is the scaling since we are getting 100 at the end instead of 103 like in previous slide.

Forward rates

- r_k = annualized spot rate for k periods from now
- Annualized forward rate between the i-th and j-th period, compounding n times a year:

$$(1 + r_j/n)^j = (1 + r_i/n)^i \left(1 + f_{i,j}/n\right)^{j-i}$$

↑
forward rate

- EXAMPLE: The 1-year zero c. bond trades at 95, and the 2-year z.c. bond trades at 89, compounding done once a year.

$$95 \cdot (1 + r_1) = 100 \rightarrow r_1 = 5.2632\%$$

$$89 \cdot (1 + r_2)^2 = 100 \rightarrow r_2 = 5.9998\%$$

$$1.052632 \cdot (1 + f_{1,2}) = (1.059998)^2 \rightarrow f_{1,2} = 6.7416\%$$

- Suppose you believe $f_{1,2}$ is too high:
 - buy one 2-year bond and sell short $89/95 \cdot 100 = 93.6842$ units of the 1-year bond, at zero cost.
- After 1 year: have to pay $89/95 \cdot 100 = 93.6842$
- After 2 years: receive 100 for the second year return of 6.7416%
- If, after 1 year, the 1-year spot rate is, indeed, less than 6.7416%, sell the 1-year bond and receive more than 93.6842, and make *some* profit.

Pricing Options with Mathematical Models

8. Model independent pricing relations: forwards, futures and swaps

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Pricing forward contracts

- Consider a forward contract on asset S , starting at t , with payoff at T equal to
$$S(T) - F(t)$$
- Here, $F(t)$ is the **forward price**, decided at t and paid at T .
- QUESTION: What is the value of $F(t)$ that makes time t value of the contract equal to zero?
- Suppose \$1.00 invested/borrowed at risk-free rate at time t results in payoff $\$B(t,T)$ at time T .

eg: $B(t,T) = e^{r(T-t)}$ or $(1+r)^T$

- **CLAIM:** There is no arbitrage if and only if

$$F(t) = B(t, T)S(t),$$

that is, if the forward price is equal to the time T value of one share worth invested at the risk-free rate at time t .

- Suppose first $F(t) > B(t, T)S(t)$:
 - At t : borrow $S(t)$ to buy one share, and go short in the forward contract;
 - At T : deliver the share, receive $F(t)$, which is more than enough to cover the debt of $B(t, T)S(t)$. Arbitrage!
- Suppose now $F(t) < B(t, T)S(t)$:
 - At t : sell short one share, invest $S(t)$ risk-free, long the forward contract.
 - At T : have more than enough in savings to pay for $F(t)$ for one share and close the short position.

- Suppose S pays deterministic dividends between t and T , with present value $D(t)$. Then,

$$F(t) = B(t, T)(S(t) - D(t)) .$$

- Suppose dividends will be paid between t and T , continuously at a constant rate q . Then,

$$F(t) = B(t, T) e^{-q(T-t)} S(t) .$$



Index like S&P 500 where
dividends are coming quite frequently.

EXAMPLE: $S(t)=100$, a dividend of 5.65 is paid in 6 months. The 1-year continuous interest rate is 10%, and the 6-month continuous annualized rate is 7.41%. The price of the 1-year forward contract should be

$$\text{Using } F(t) = B(t, T)(S(t) - D(t))$$

$$F(t) = e^{0.1} (100 - e^{-0.0741/2} \cdot 5.65) = 104.5$$

Suppose that the price is instead $F(t) = 104$.

- At t : long the forward, sell one share, buy the 6-month bond in the amount of $e^{-0.0741/2} \cdot 5.65 = 5.4445$; invest the remaining balance, $100 - 5.4445 = 94.5555$, in the 1-year bond.
- At 6 months from t : receive 5.65 from the 6-month bond and pay the dividend of 5.65.
- At 1 year from t : receive $e^{0.1} \cdot 94.5555 = 104.5$ from the 1-year bond; pay $F(t) = 104$ for one share, and deliver the share to cover the short position; keep $104.5 - 104 = 0.5$, as profit. Arbitrage!

EXAMPLE: Forward contract on foreign currency. Let $S(t)$ denote the current price in dollars of one unit of the foreign currency. We denote by r_f (r) the foreign (domestic) risk-free rate, with continuous compounding. The foreign interest is equivalent to continuously paid dividends, so we guess that

$$F(t) = e^{(r - r_f)(T-t)} S(t). \quad \leftarrow \text{Using this} \quad F(t) = \underbrace{B(t, T)}_{C^r(T-t)} e^{-q(T-t)} S(t) \quad \downarrow r_f$$

If, for example, $F(t) < e^{(r - r_f)(T-t)} S(t)$:

- At time t : long the forward, borrow $e^{-r_f(T-t)}$ units of foreign currency and invest its value in dollars $e^{-r_f(T-t)} S(t)$ at rate r .
- At time T : use part of the amount $e^{(r - r_f)(T-t)} S(t) > F(t)$ from the domestic risk-free investment to pay $F(t)$ for one unit of foreign currency in the forward contract, and deliver that unit to cover the foreign debt. There is still extra money left. Arbitrage! Similarly if $e^{(r - r_f)(T-t)} S(t) < F(t)$.

Futures *(Forwards done by Exchanges) eg CBOE*

Profit/loss revalued everyday
↑

- Main difference relative to forwards: **marked to market** daily.
- The daily profit/loss is deposited to/taken out of the **margin account**:

Total profit/loss for a contract starting at t , ignoring the margin interest rate, using $F(T)=S(T)$,

→ Futures Contract that immediately matures i.e like buying asset at spot price.

$$= [F(t+1)-F(t)] + \dots + [F(T)-F(T-1)] = S(T)-F(t)$$

- CLAIM: If the interest rate is deterministic, futures price $F(t)$ is equal to the corresponding forward price.
- REPLICATION: At $t=0$, go long $e^{-r(T-1)}$ futures; at $t=1$, increase to $e^{-r(T-2)}$ futures, ..., at $t=T-1$, increase to 1 future contract.

Profit/loss in period $(k, k+1) = [F(k+1) - F(k)]e^{-r(T-(k+1))}$,
the time T value of which is $[F(k+1) - F(k)]$. Thus, time T profit/loss is $S(T)-F(0)$, the same as for a forward contract.

Swaps pricing

- The payoff of the party receiving the floating rate L and paying the fixed rate R at time T_i is

$$C_i := \Delta T [L(T_{i-1}, T_i) - R]$$

long swap = Receive floating L
& pay fixed R

$$\Delta T = T_i - T_{i-1}$$

- LIBOR rate is, by definition,

$$(1 + \angle \Delta T)^P = 1$$

\swarrow ECB

$$L(T_{i-1}, T_i) := \frac{1 - P(T_{i-1}, T_i)}{\Delta T P(T_{i-1}, T_i)},$$

$$P(T_i, T_i) = 1$$

implying

$$C_i = \frac{1}{P(T_{i-1}, T_i)} - (1 + R\Delta T)$$

Price of 1 leg
of the Swap

- The value at time $t < T_0$ of the constant amount $(1 + R\Delta T)$ paid at time $T_i > t$ is $(1 + R\Delta T)P(t, T_i)$. (Why?)

Swaps pricing (continued)

- As for the first term, we claim that the value at time $t < T_0$ of the payoff $1/P(T_{i-1}, T_i)$ paid at time T_i is equal to $P(t, T_{i-1})$. Indeed, if we invest $P(t, T_{i-1})$ at time t in buying a bond with maturity T_{i-1} , we get 1 dollar at time T_{i-1} , with which we can buy exactly $1/P(T_{i-1}, T_i)$ of bonds with maturity T_i , and hence collect $1/P(T_{i-1}, T_i)$ at time T_i . Altogether, the price at time t is

$$C_i(t) = P(t, T_{i-1}) - (1 + R\Delta T)P(t, T_i)$$

Therefore, the price $S(t)$ of the swap at time t is

$$S(t) = \sum_{i=1}^n C_i(t) = \sum_{i=1}^n [P(t, T_{i-1}) - (1 + R\Delta T)P(t, T_i)]$$

$$S(t) = P(t, T_0) - R\Delta T \sum_{i=1}^n P(t, T_i) - P(t, T_n)$$

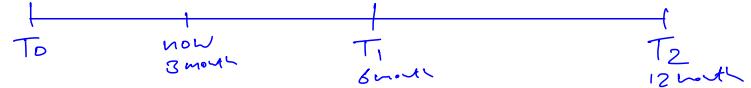
Swaps pricing (continued)

$$S(t) = P(t, T_0) - R\Delta T \sum_{i=1}^n P(t, T_i) - P(t, T_n) .$$

- The **swap rate** R is the fixed rate such that the cost of entering the swap at the initial time 0 is equal to zero. We get

$$R = \frac{P(0, T_0) - P(0, T_n)}{\Delta T \sum_{i=1}^n P(0, T_i)}$$

Example



- Consider a swap investor who receives a fixed rate of 10% in semiannually paid coupons and pays the six-month LIBOR. The swap still has 9 months left to maturity. At the last resetting date, the six-month LIBOR was 6%. The continuous three-month and nine-month rates are 5% and 7%, respectively. Nominal principal is 10,000. In the above notation, T_0 is 3 months in the past, T_1 is 3 months from now, and T_2 is 9 months from now, $\Delta T = 0.5$. We want to find the value of the investor's position today.
- The payoff 3 months from now, on one unit of the notional principal, is

$$C_1 = \frac{1}{P(T_0, T_1)} - (1 + R\Delta T)$$

and 9 months from now it is

$$C_2 = \frac{1}{P(T_1, T_2)} - (1 + R\Delta T)$$

Example (continued)

- The first payoff's price is obtained by multiplying it by the price of the 3-month bond, and is equal to

$$C_1(t) = \left[\frac{1}{P(T_0, T_1)} - (1 + R\Delta T) \right] e^{-0.05 \cdot 0.25}$$

- The bond price $P = P(T_0, T_1)$ can be found from $P(1 + \Delta TL) = 1$, where $L = 0.06$ is the LIBOR rate. We get $P = 1/1.03$, and $C_1(t) = -0.0198$.
- The price of C_2 is found to be

$$C_2(t) = P(t, T_1) - (1 + R\Delta T)P(t, T_2) = e^{-0.05 \cdot 0.25} - (1 + 0.1 \cdot 0.5)e^{-0.07 \cdot 0.75} = -0.0087$$

Example (continued)

- Altogether, the value of the swap for the short position is

$$C(t) = C_1(t) + C_2(t) = -0.0285$$

- Since our investor is long the swap, his value is

$$10,000 \cdot 0.0285 = 285$$

Pricing Options with Mathematical Models

9. Model independent pricing relations: options

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

- Notation:
 - European call and put prices at time t , $c(t), p(t)$
 - American call and put prices at time t , $C(t), P(t)$

- RELATION 1: $c(t) \leq C(t) \leq S(t)$ Since, $\max(S(t)-K, 0) \leq S(t)$
- RELATION 2: $p(t) \leq P(t) \leq K$ Since, $\max(K-S(t), 0) \leq K$
- RELATION 3: $p(t) \leq Ke^{-r(T-t)}$
- RELATION 4: $c(t) \geq S(t) - Ke^{-r(T-t)}$, if S pays no dividends
- Suppose not: $c(t) + Ke^{-r(T-t)} < S(t)$; sell short one share and have more than enough money to buy one call and invest $Ke^{-r(T-t)}$ at rate r .
 - At T: If $S(T) > K$, exercise the option by buying $S(T)$ for K ; & close short position.
 - If $S(T) \leq K$, buy stock from your invested cash. & close short position.
- RELATION 5: $p(t) \geq Ke^{-r(T-t)} - S(t)$

- RELATION 6: $c(t) = C(t)$, if S pays no dividends.

American = European Call
if No dividend.

- Suppose not: $c(t) < C(t)$

1. At t : Sell $C(t)$ and have more than enough to buy $c(t)$;
2. If C exercised at τ : have to pay $S(\tau) - K$, which is possible by selling c , because $c(t) \geq S(t) - Ke^{-r(T-t)} \geq S(t) - K$;
3. If C never exercised, there is no obligation to cover.

Arbitrage!

- COROLLARY: An American call on an asset that pays no dividends should not be exercised early.
 - Indeed, it is better to sell it than to exercise it: $C(t) \geq S(t) - K$.
 - What if there are dividends?
 - What about the American put option? Not equal to European Put.

- **RELATION 7, Put-Call Parity:**

$$c(t) + Ke^{-r(T-t)} = p(t) + S(t)$$

1. Portfolio A: buy $c(t)$ and invest discounted K at risk-free rate;
2. Portfolio B: buy put and one share.

- If $S(T) > K$, both portfolios worth $S(T)$ at time T . $S(T) - K + K = 0 + S(T)$
- If $S(T) \leq K$, both portfolios worth K at time T . $0 + K = K - S(T) + S(T)$

- **RELATION 8:**

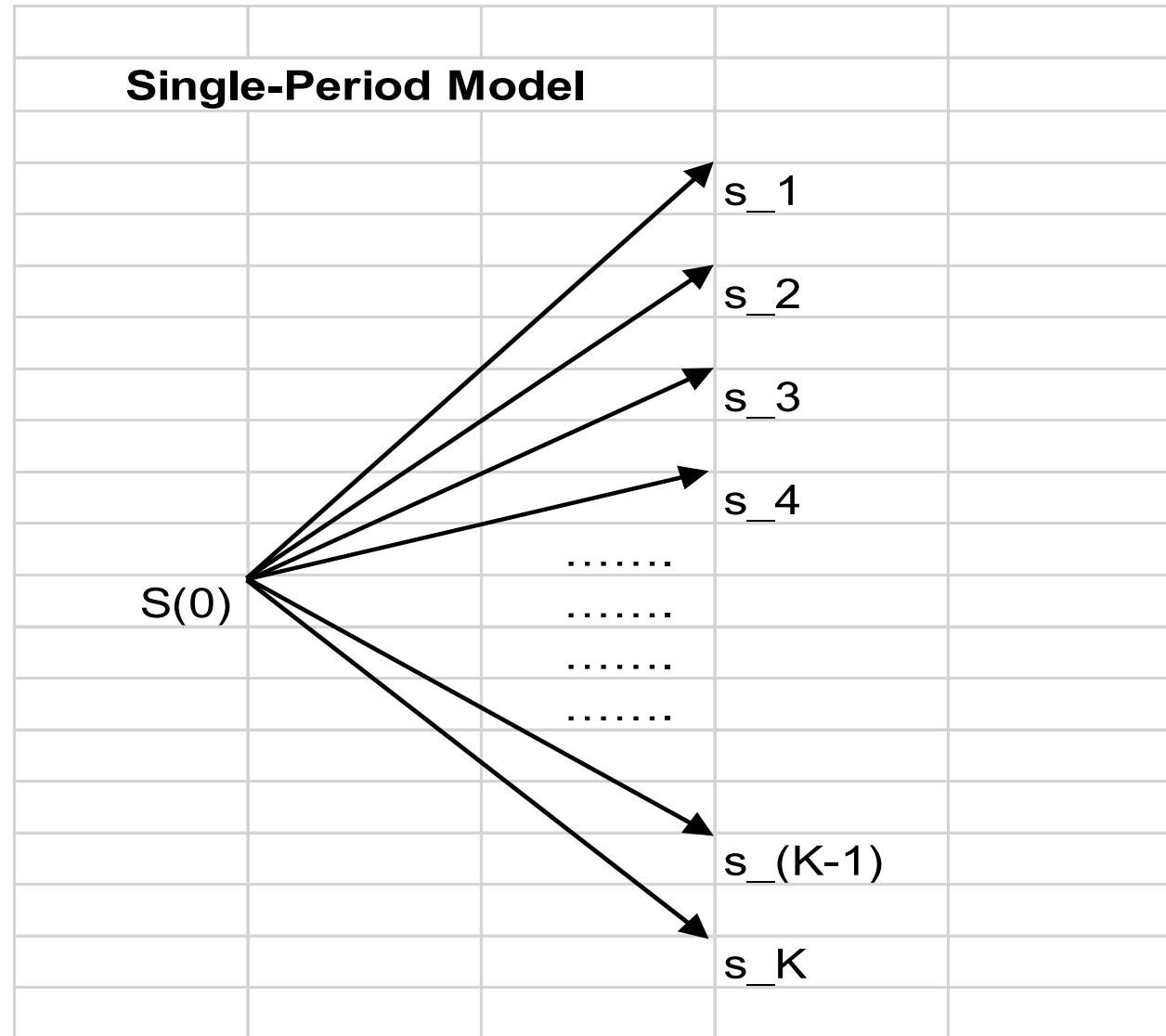
$$S(t) - K \leq C(t) - P(t) \leq S(t) - Ke^{-r(T-t)}$$

- The RH side follows from put-call parity and $P(t) \geq p(t), C(t) = c(t)$.
- For the LHS, suppose not: $S(t) + P(t) > C(t) + K$.
 1. At t : Sell the LHS and have more than enough to buy the RHS;
 2. If P exercised at τ : use the invested cash to pay K for $S(\tau)$;
 3. If P never exercised, exercise C at maturity.

Pricing Options with Mathematical Models

10. Discrete-time models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.



$$P(S(T) = s_i) = p_i$$

Assume:

- Risk-free asset, bank account:

$$B(0) = 1, B(1) = (1 + r)$$

- Initial wealth:

$$X(0) = x \rightarrow \text{initial amount of Money}$$

- Number of shares in asset i : δ_i

- End-of-period wealth:

$$X(1) = \delta_0 B(1) + \delta_1 S_1(1) + \cdots + \delta_N S_N(1)$$

- Budget constraint, self-financing condition:

$$X(0) = \delta_0 B(0) + \delta_1 S_1(0) + \cdots + \delta_N S_N(0)$$

- **Profit/loss, P&L, or the gains of a portfolio strategy:**

$$G(1) = X(1) - X(0)$$

- Discounted version of process Y: $\bar{Y}(t) = Y(t)/B(t)$
- Change in price: $\Delta S_i(1) = S_i(1) - S_i(0)$
- We have

$$G(1) = \delta_0 r + \delta_1 \Delta S_1(1) + \cdots + \delta_N \Delta S_N(1)$$

$$X(1) = X(0) + G(1)$$

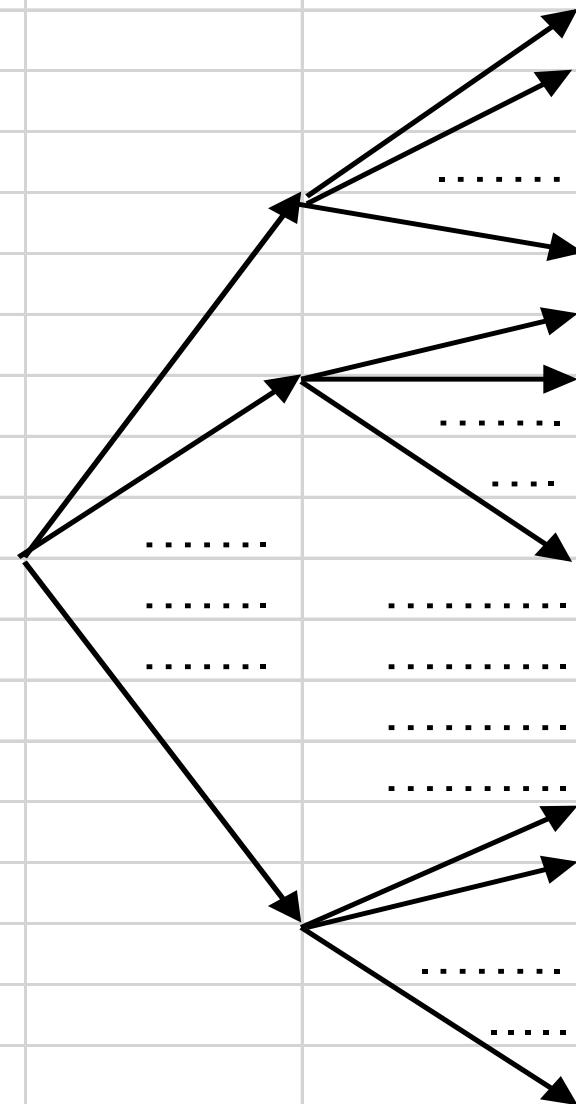
Denoting

$$\Delta \bar{S}_i(1) = \bar{S}_i(1) - S_i(0), \quad \bar{G}(1) = \delta_1 \Delta \bar{S}_1(1) + \cdots + \delta_N \Delta \bar{S}_N(1)$$

one can verify that

$$\bar{X}(1) = X(0) + \bar{G}(1)$$

Multi-Period Model



- Risk-free asset, bank account:

$$B(0) = 1, B(t) = (1 + r(t))B(t - 1)$$

- Number of shares in asset i during the period $[t - 1, t]$: $\delta_i(t)$

- Wealth process:

$$X(t) = \delta_0(t)B(t) + \delta_1(t)S_1(t) + \cdots + \delta_N(t)S_N(t)$$

- Self-financing condition:

$$\begin{aligned} X(t) \\ = \delta_0(t+1)B(t) + \delta_1(t+1)S_1(t) + \cdots + \delta_N(t+1)S_N(t) \end{aligned}$$

$X(t)$ must equal $X(t)$ after changing no. of shares (Position) as no extra money comes in.
 i.e. Money before changing position must equal money after changing position.

- Change in price: $\Delta S_i(t) = S_i(t) - S_i(t - 1)$
- $G(t) = \sum_{s=1}^t \delta_0(s) \Delta B(s) + \sum_{s=1}^t \delta_1(s) \Delta S_1(s) + \cdots + \sum_{s=1}^t \delta_N(s) \Delta S_N(s)$
- It can be checked that $X(t) = X(0) + G(t)$

Denoting

$$\Delta \bar{S}_i(t) = \bar{S}_i(t) - \bar{S}_i(t - 1),$$

$$\bar{G}(t) = \sum_{s=1}^t \delta_1(s) \Delta \bar{S}_1(s) + \cdots + \sum_{s=1}^t \delta_N(s) \Delta \bar{S}_N(s)$$

one can verify that

$$\bar{X}(t) = X(0) + \bar{G}(t)$$

For example, with one risky asset and two periods:

- Change in price: $\Delta S_i(t) = S_i(t) - S_i(t - 1)$
- $G(2) = \delta_0(1)(B(1) - B(0)) + \delta_0(2)(B(2) - B(1))$
 $+ \delta_1(1)(S(1) - S(0)) + \delta_1(2)(S(2) - S(1))$
- Using self-financing

$$\delta_0(1)B(1) + \delta_1(1)S(1) = \delta_0(2)B(1) + \delta_1(2)S(1)$$

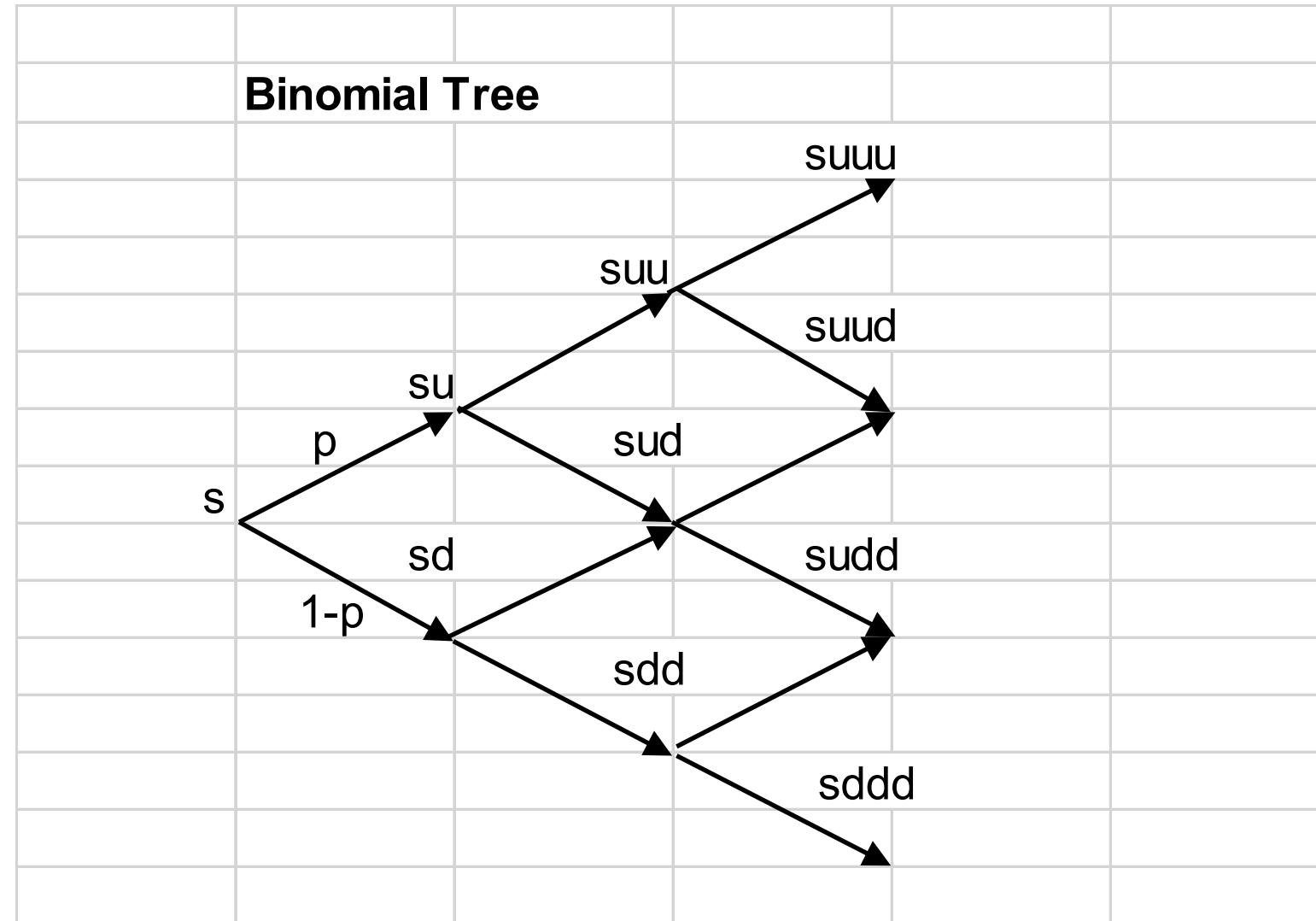
we get

- $G(2) = \underbrace{\delta_0(2)B(2) + \delta_1(2)S(2)}_{X(2)} - \underbrace{\delta_0(1)B(0) - \delta_1(1)S(0)}_{X(0)}$
- This is the same as
$$G(2) = X(2) - X(0)$$

Binomial Tree (Cox-Ross-Rubinstein) model

- $p = P(S(t+1) = u S(t))$, $1-p = P(S(t+1) = d S(t))$
- $u > 1+r > d$

No-Arbitrage Condition



Pricing Options with Mathematical Models

11. Risk-neutral pricing

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Martingale property

- Insurance pricing:

$$C(t) = E_t [e^{-r(T-t)} C(T)]$$

where E_t is expectation given the information up to time t .

- For a stock, this would mean:

$$e^{-rt} S(t) = E_t [e^{-rT} S(T)]$$

→ i.e. discounted stock price today equals the conditional expectation of the discounted stock price in the future.

- If so, we say that $M(t) = e^{-rt} S(t)$ is a **martingale process**:

$$M(t) = E_t [M(T)]$$

Martingale represents fair game. Under Real probability measure discounted stock price is Not Martingale. But under Risk-Neutral Measure Q it becomes Martingale.

Martingale probabilities (measures)

- Typically, the stock price process will not be a martingale under the actual (physical) probabilities, but it may be a martingale under some other probabilities.
- Those are called **martingale**, or **risk-neutral**, or **pricing probabilities**.
- Such probabilities are typically denoted Q, q_i , sometimes P^*, p_i^* .
- We write:

$$e^{-rt} S(t) = E_t^Q [e^{-rT} S(T)],$$

or
$$e^{-rt} S(t) = E_t^* [e^{-rT} S(T)]$$

Risk-neutral pricing formula

- Thus, we expect to have, for some risk-neutral probability Q

Price of claim today

= expected value, under Q , of the claim's discounted future payoff

or

$$C(t) = E_t^Q [e^{-r(T-t)} C(T)]$$

if $C(T)$ is paid at T , and the continuously compounded risk-free rate r is constant.

- How to justify this formula?
- Which Q ? Are there any? How many?

Example: A Single Period Binomial model

- $r=0.005$, $S(0)=100$, $s^u = 101$, $s^d = 99$, that is, $u=1.01$, $d=0.99$.
- The payoff is an European Call Option, with payoff

$$\max\{S(1) - 100, 0\}$$

- It will be \$1 if the stock goes up and \$0 if the stock goes down. Looking for the **replicating portfolio**, we solve

$$\begin{aligned}\delta_0(1 + 0.005) + \delta_1 101 &= 1 ; \\ \delta_0(1 + 0.005) + \delta_1 99 &= 0.\end{aligned}$$

We get

$$\delta_0 = -49.254, \delta_1 = 0.5$$

i.e. buy 0.5 of stock (\$50) & borrow 49.254
so the difference ($50 - 49.254$) is the cost
of replicating portfolio & this should be the price of option.

Example continued

- $\delta_0 = -49.254$, $\delta_1 = 0.5$
- This means borrow 49.254, and buy one share of the stock. This costs

$$C(0) = 0.5 \times 100 - 49.254 = 0.746$$

This is the no arbitrage price:

- 1) Suppose the price is higher, say 1.00. Sell the option for 1.00, invest 1-0.746 at the risk-free rate; use 0.746 to set up the replicating strategy; have 1 if stock goes up, and 0 if it goes down, exactly what you need. Arbitrage.
- 2) Suppose the price is lower, say 0.50. Buy the option for 0.50, sell short half a share for 50, invest 49.254 at the risk-free rate; This leaves you with extra 0.246 today. If stock goes up you make 1.00 from the option; together with 49.254×1.005 , this covers 101/2 to close your short position. If stock goes down, use 49.254×1.005 to cover 99/2 when closing your short position. Arbitrage.

Martingale pricing

If discounted stock price is Martingale
then discounted wealth process is also
Martingale.

- Suppose the discounted wealth process \bar{X} is a martingale under Q , and suppose it replicates $C(T)$, so that $X(T)=C(T)$. By the martingale property,

$$\bar{X}(t) = E_t^Q \bar{X}(T) = E_t^Q \bar{C}(T)$$

For example, if discounting is continuous at a constant rate r , this gives

$$X(t) = E_t^Q [e^{-r(T-t)} C(T)]$$

This is the cost of replication at time t , therefore, for any such probability Q ,

the price/value of the claim at time t is equal to the expectation, under Q , of the discounted future payoff of the claim.

Single Period Binomial model

- The future wealth value is

$$X(1) = \delta_0(1 + r) + \delta_1 S(1)$$

thus, when discounted,

$$\bar{X}(1) = \delta_0 + \delta_1 \bar{S}(1)$$

Therefore, if the discounted (non-dividend paying) stock is a martingale, so is the discounted wealth. For the stock to be a Q -martingale, we need to have

$$S(0) = E^Q \frac{S(1)}{1 + r} = \frac{1}{1 + r} (q \times s^u + (1 - q) \times s^d)$$

Solving for q , we get, with $s^u = S(0)u, s^d = S(0)d$,

$$q = \frac{(1 + r) - d}{u - d}, 1 - q = \frac{u - (1 + r)}{u - d}$$

Example (the same as above)

$$s^u = 100 \times 1.01, s^d = 100 \times 0.99,$$

$$q = \frac{(1+r) - d}{u - d} = \frac{1.005 - 0.99}{1.01 - 0.99} = 0.75$$

Thus, the price of the call option is

$$C(0) = E^Q \frac{C(1)}{1+r} = \frac{1}{1+r} (q \times C^u + (1-q) \times C^d)$$

$$= \frac{1}{1+0.005} (0.75 \times (101 - 100) + (1 - 0.75) \times 0)$$

$$= 0.746 \quad (\text{the same as } \text{before, Using Replication & law of one price}).$$

Forwards (Using Martingale Pricing)

- Let D denote the process used for discounting, for example

$$D(t) = e^{-rt}$$

- We want the forward price $F(t)$ to be such that the value of the forward contract zero at the initial time t :

$$0 = E_t^Q \left[\{S(T) - F(t)\} \frac{D(T)}{D(t)} \right]$$

$F(t)$ & $D(t)$ known at t so can be taken out of E .

Since DS is a Q -martingale, we have $E_t^Q[D(T)S(T)] = D(t)S(t)$, and we get

$$F(t) = S(t) \frac{D(t)}{E_t^Q[D(T)]}$$

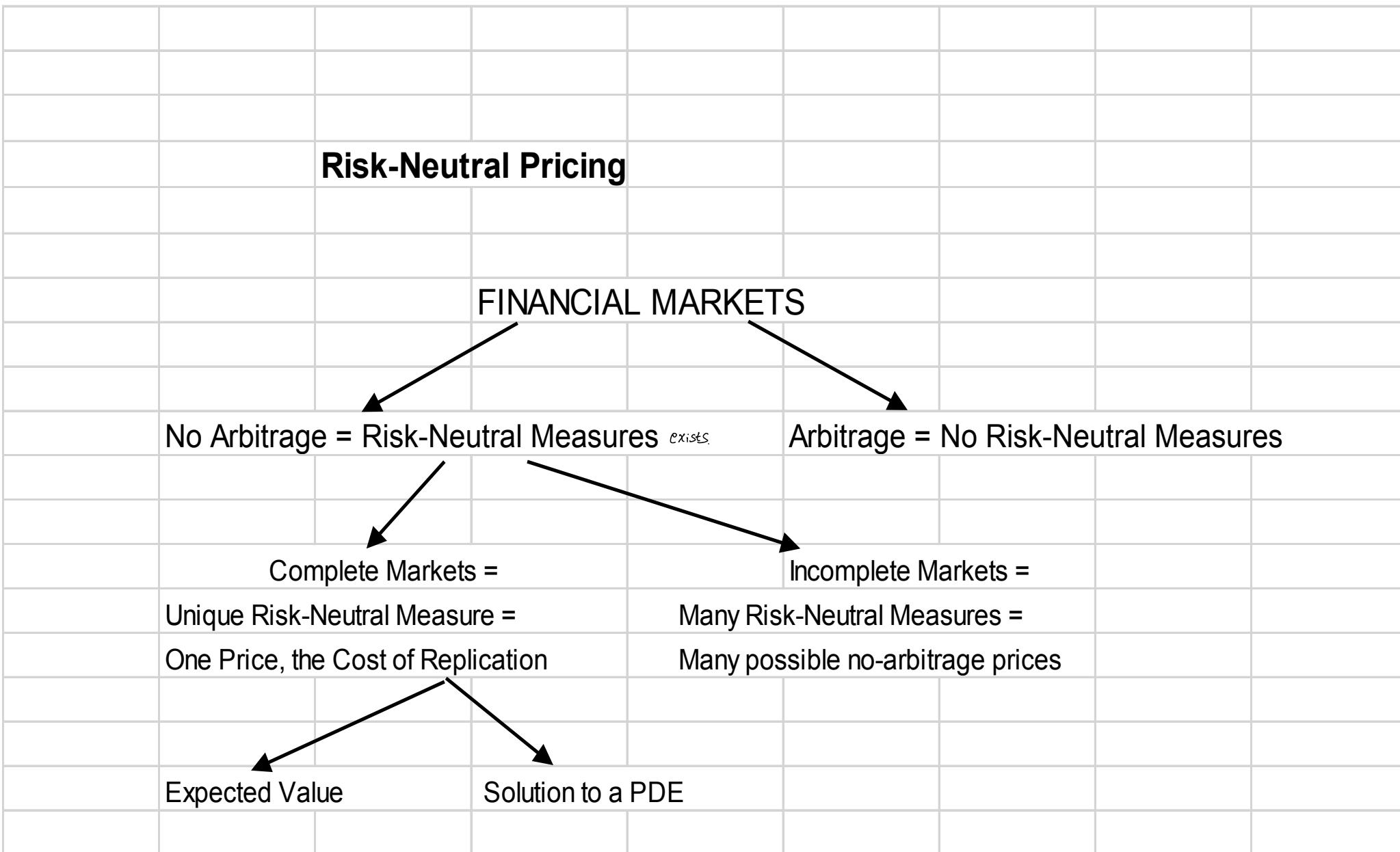
which, for the above $D(t)$, is the same as

$$F(t) = S(t)e^{r(T-t)} = S(t)B(t, T)$$

Pricing Options with Mathematical Models

12. Fundamental theorems of asset pricing

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.



Equivalent martingale measures (EMM's)

Recall

$$q = \frac{(1+r) - d}{u - d}, 1 - q = \frac{u - (1+r)}{u - d}$$

Thus, q and $1 - q$ are strictly between zero and one if and only if

$$d < 1 + r < u$$

Then, the events of non-zero P probability also have non-zero Q probability, and vice-versa. We say that P and Q are **equivalent probability measures**, and Q is called an **equivalent martingale measure (EMM)**. Note also that Q is the only EMM.

$Q = (q, 1-q)$ for Binomial Model.

- First fundamental theorem of asset pricing

No arbitrage = existence of at least one EMM

Definition of arbitrage: there exists a strategy such that, for some T ,

$X(0) = 0, X(T) \geq 0$ with probability one, and

$$P(X(T) > 0) > 0$$

One direction: suppose there exists an EMM Q , and a strategy with $X(T)$ as above. Then, *by Martingale Property*

$$X(0) = E^Q \bar{X}(T) > 0, \text{ so, no arbitrage.}$$

Since $X(0)=0$ for Arbitrage defn.

- Second fundamental theorem of asset pricing
 - **Definition of completeness:** a market (model) is complete if every claim can be replicated by trading in the market.

*Completeness and no arbitrage
= existence of exactly one EMM*

- In a complete market, every claim has a unique price, equal to the cost of replication, also equal to the expectation under the unique EMM.
- Even in an incomplete market, one assumes that there is one EMM Q (among many), that the market chooses to price all the claims.
- How to compute it? *Calibration*

$$\min_{\alpha} I^{\alpha} \bar{Z}(T) \leq C(s) \leq \max_{\alpha} I^{\alpha} \bar{Z}(T)$$

↓
No-Arbitrage Price in Incomplete Markets is any Price between Min & Max.

Example: Binomial tree model is arbitrage free and complete if $d < 1 + r < u$

Pricing Options with Mathematical Models

13. Binomial tree pricing

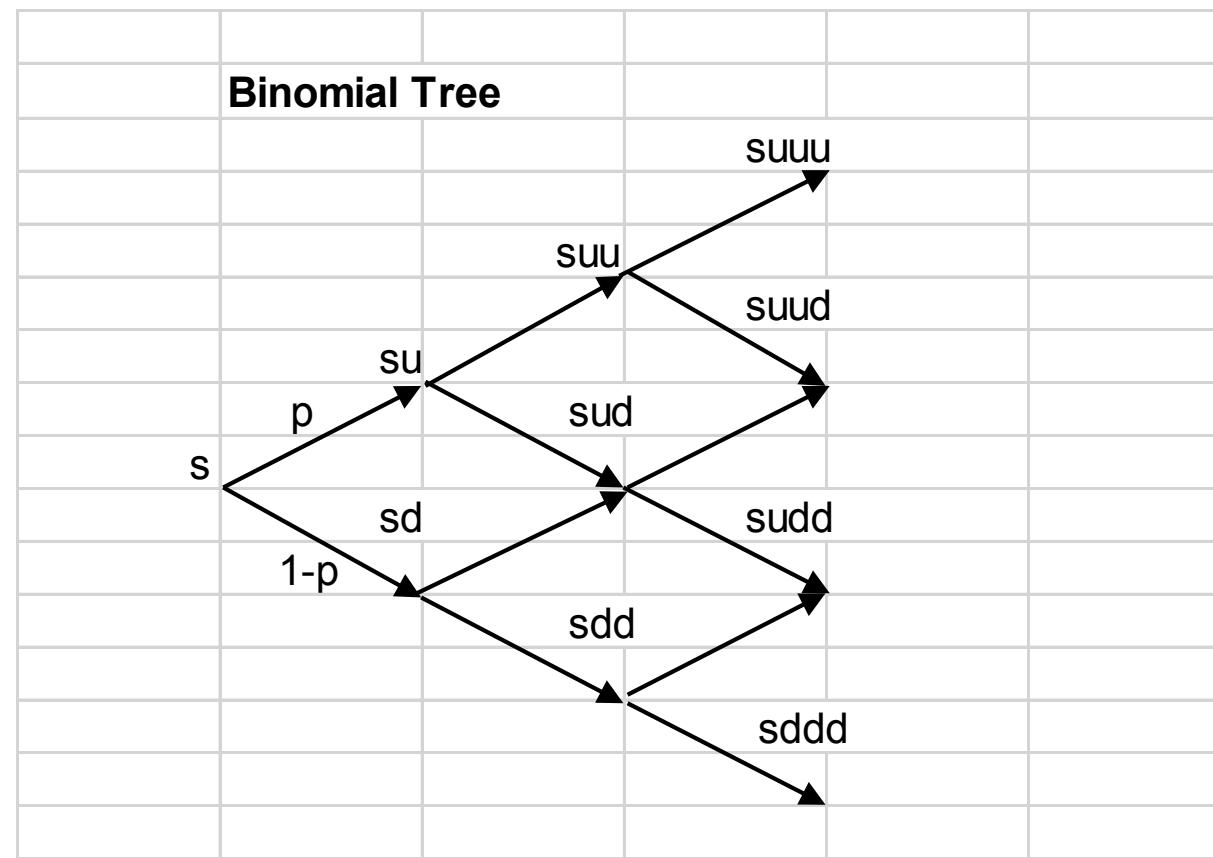
Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Δt can be chosen to be 1 day. Good enough Approximation to B-S formula.

Binomial Tree (Cox-Ross-Rubinstein) model

- $p = P(S(t+1) = u S(t))$,
- $1 - p = P(S(t+1) = d S(t))$
- $u > e^{r\Delta t} > d$

Δt is in terms of a year.



Expectation formula

- CLAIM: Given a random variable X whose value will be known at time T , process $M(t) = E_t[X]$ is a martingale. Indeed, for $s < t, \leq T$

$$E_s[M(t)] = E_s E_t [X] = E_s[X] = M(s)$$

where the middle equality is the so- called ***law of iterated expectations***.

- Since

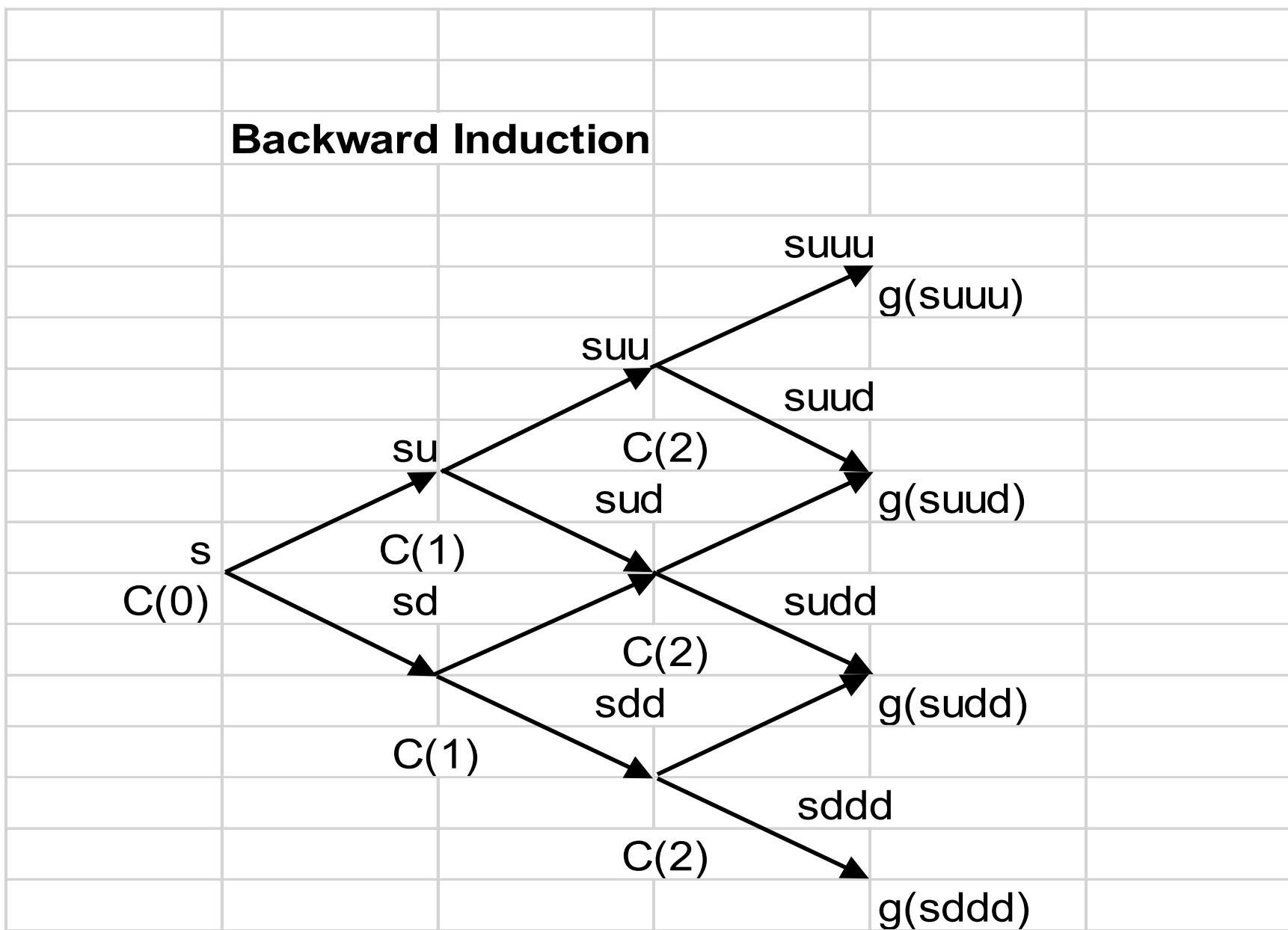
$$e^{-rt} C(t) = E_t^Q [e^{-rT} C(T)]$$

we conclude that $e^{-rt} C(t)$ is a Q -martingale. Therefore,

$e^{-rt} C(t) = E_t^Q [e^{-r(t+\Delta t)} C(t + \Delta t)]$, and we get the **expectation formula**

$$\begin{aligned} C(t) &= E_t^Q [e^{-r\Delta t} C(t + \Delta t)] \\ &= e^{-r\Delta t} [q \times C^u(t + \Delta t) + (1 - q) \times C^d(t + \Delta t)] \end{aligned}$$

Pricing path-independent payoff $g(S(T))$



Example: a call option

		European Option Price	
S(0)	100.0000	121.0000	21.0000
u	1.1000	110.0000	15.1088
d	0.9000	99.0000	0.0000
K	100.0000	100.0000	10.8703
r	0.0500	90.0000	0.0000
p*	0.7564	81.0000	0.0000
Delta t	1.0000		

$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.7564$$

$$15.1088 = e^{-r\Delta t} [q \times 21 + (1 - q) \times 0]$$

$$10.87 = e^{-r\Delta t} [q \times 15.1088 + (1 - q) \times 0]$$

American options

- Consider an American option that pays $g(\tau)$ dollars if exercised at time $\tau \leq T$.
- It can be shown that, in a complete market with discrete time and time intervals of length Δt , its no-arbitrage price $A(t)$ is given by

$$A(t) = \max_{t \leq \tau \leq T} E_t^Q [e^{-r(\tau-t)} g(\tau)]$$

- The expectation formula is given by the dynamic programming principle:

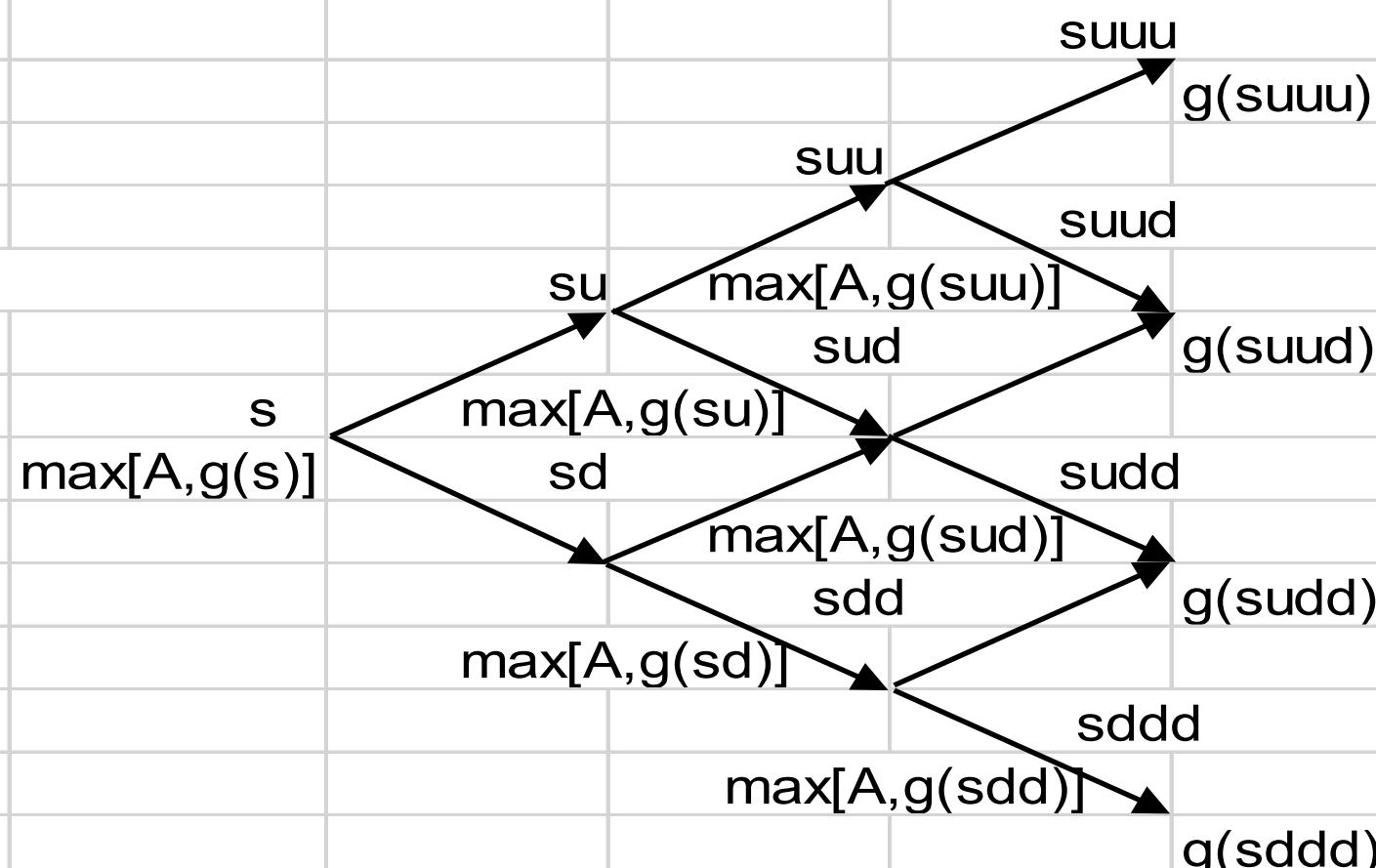
$$A(t) = \max \{ g(t), E_t^Q [e^{-r\Delta t} g(t + \Delta t)] \}$$

or, in the binomial model,

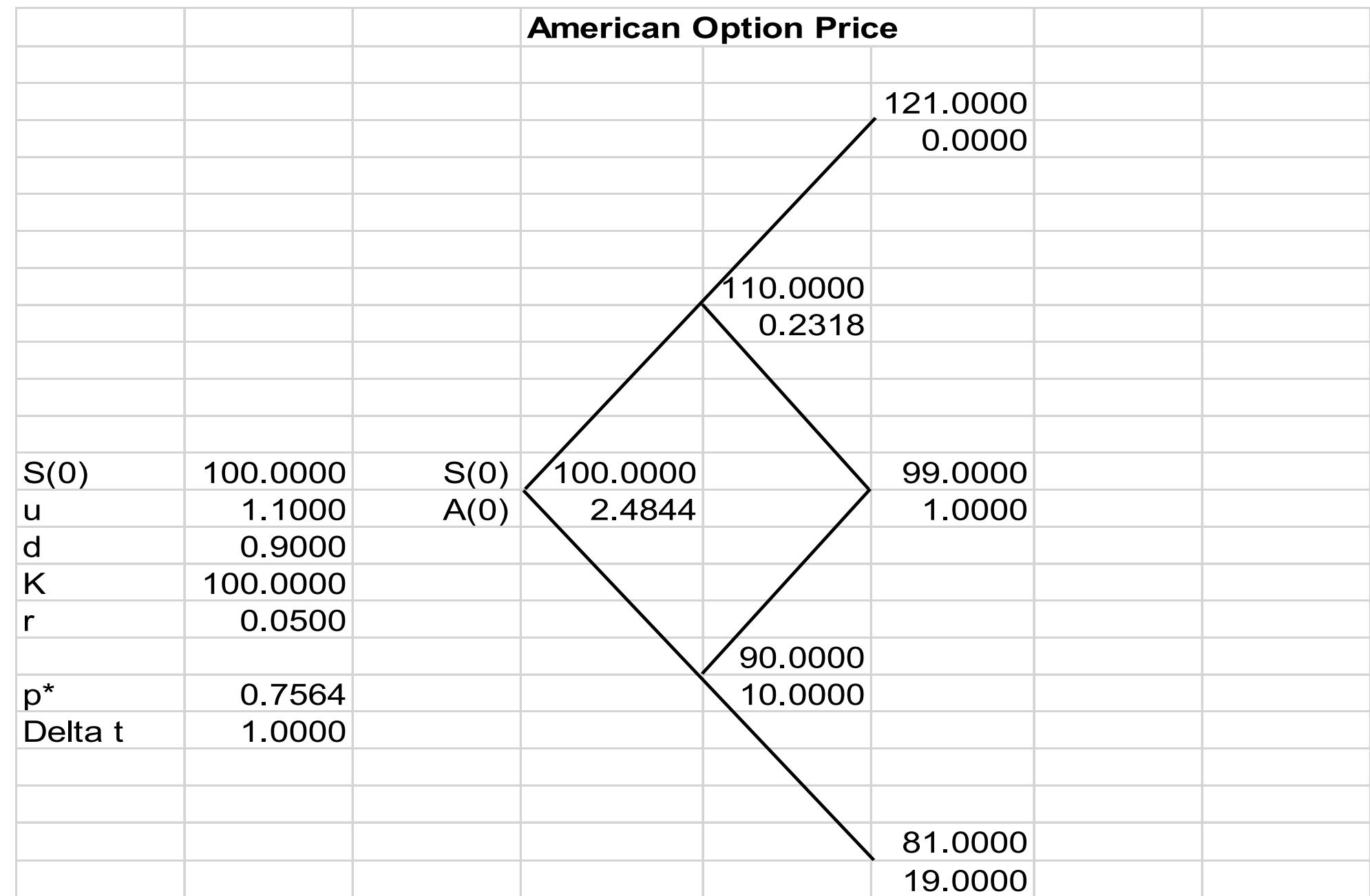
$$A(t) = \max[g(t), e^{-r\Delta t} \{ q \times A^u(t + \Delta t) + (1 - q) \times A^d(t + \Delta t) \}]$$


Option Exercise Value
at time t

Backward Induction for American Options



Example: a put option



$$U = \frac{1}{d} , \quad U = e^{\sigma \sqrt{\Delta t}}$$

where σ = Vol. of Stock

$\Delta t = 1 \text{ day}$
 $r = \text{Risk-free rate}$

$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.7564$$

$$\begin{aligned} 10 &= \max\{10, e^{-r\Delta t} [q \times 1 + (1 - q) \times 19] \} \\ &= \max\{10, 5.1229\} \end{aligned}$$

$$0.2318 = \max\{0, e^{-r\Delta t} [q \times 0 + (1 - q) \times 1] \}$$

$$2.4844 = \max\{0, e^{-r\Delta t} [q \times 0.2318 + (1 - q) \times 10] \}$$

Pricing Options with Mathematical Models

14. Brownian motion process

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

History

- Brown, 1800's
- Bachelier, 1900
- Einstein 1905, 1906
- Wiener, Levy, 1920's, 30's
- Ito, 1940's
- Samuelson, 1960's
- Merton, Black, Scholes, 1970's

A short introduction to the Merton-Black-Scholes model

- Risk-free asset

$$B(t) = e^{rt}$$

- Stock has a **lognormal distribution**:

$$\log S(t) = \log S(0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}z(t)$$

where $z(t)$ is a standard normal random variable. Thus,

$$S(t) = S(0) e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}z(t)}$$

and it can be shown that

$$ES(t) = S(0) e^{\mu t}, \quad \frac{1}{t} Var \left[\log \frac{S(t)}{S(0)} \right] = \sigma^2 \quad \xrightarrow{\text{Variance of log stock return per unit time.}}$$

Discretized Brownian motion

- $W(0) = 0$
- $W(t_{k+1}) = W(t_k) + \sqrt{\Delta t} z(t_k)$

$$\Delta t = t_{k+1} - t_k$$

where $z(t_k)$ are independent standard normal random variables.

Thus,

- $W(t_l) - W(t_k) = \sqrt{\Delta t} \sum_{i=k}^{l-1} z(t_i)$

is normally distributed, with zero mean and variance $(l - k)\Delta t = t_l - t_k$

Brownian motion definition

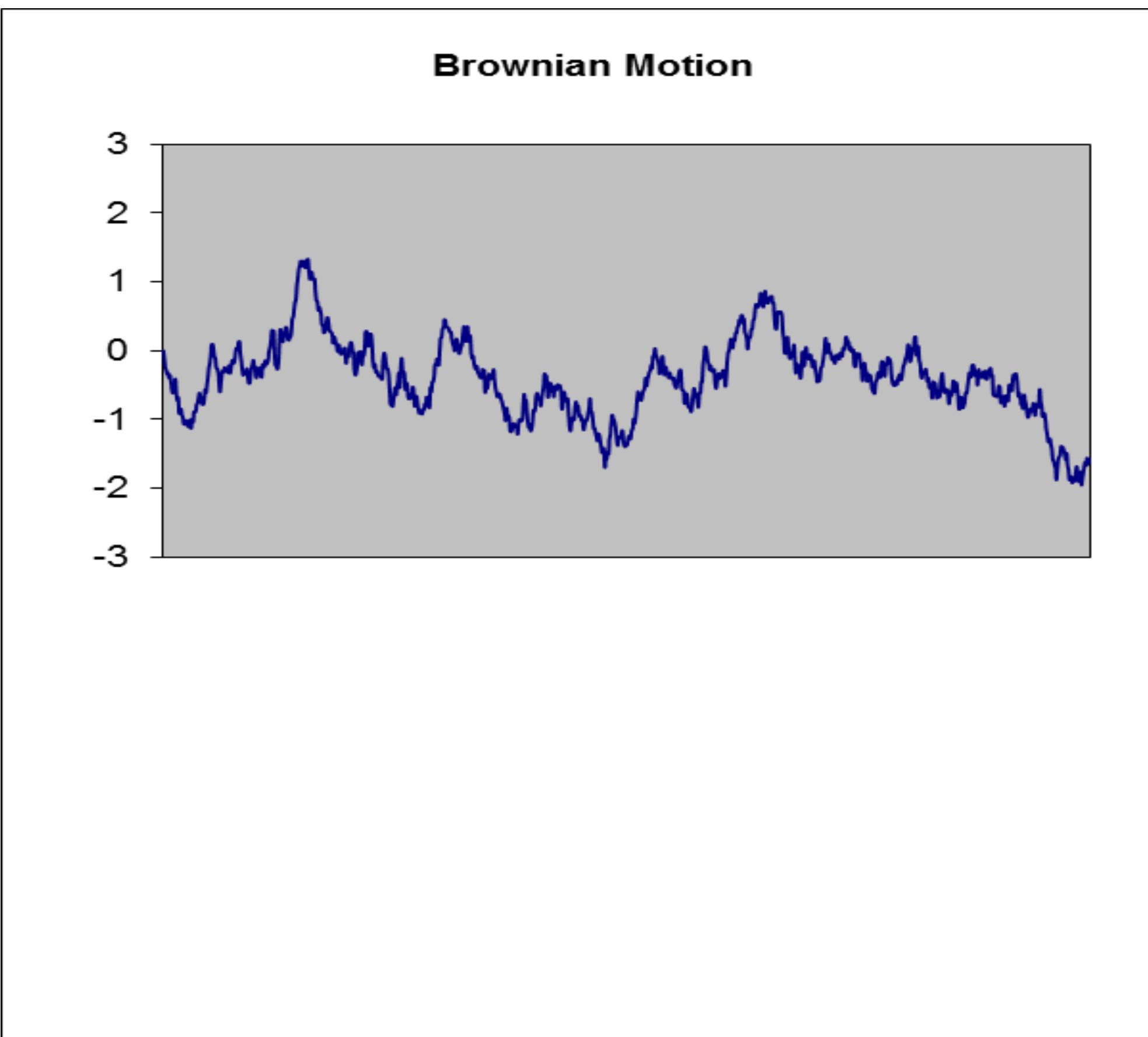
- (i) $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$, for $s < t$.
- (ii) The process W has independent increments: for any set of times $0 \leq t_1 < t_2 < \dots < t_n$, the random variables

$$W(t_2) - W(t_1), W(t_3) - W(t_2), \dots, W(t_n) - W(t_{n-1})$$

are independent.

- (iii) $W(0) = 0$.
- (iv) The sample paths $\{W(t); t \geq 0\}$ are continuous functions of t .

A simulated path of Brownian motion



Brownian motion properties

- Not differentiable: $E \frac{[W(t) - W(s)]^2}{(t-s)^2} = \frac{1}{t-s} \rightarrow \infty$ as $(t-s) \rightarrow 0$.
- A Markovian process: the distribution of the future value $W(t)$ given information up to time $s < t$ depends only on $W(s)$ and not on the past values.
- Martingale property:

$$E_s W(t) = W(s), t > s$$

because

$$E_s W(t) = E[W(t)|W(s)] = E[W(t) - W(s)|W(s)] + E[W(s)|W(s)]$$

(conditioning point independent $W(s) - W(0)$)

$$= E[W(t) - W(s)] + W(s)$$

$$= W(s)$$

\mathbb{E} of increment of Brownian motion is 0.

Pricing Options with Mathematical Models

15. Stochastic integral

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Stochastic Differential Equations

- Modeling a process in time with an Ordinary Differential Equation:

$$\frac{dX(t)}{dt} = \mu(t, X(t))$$

which may be informally written as

$$dX(t) = \mu(t, X(t))dt$$

- We would like to have a Stochastic Differential Equation (SDE):

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t)$$

- We will define it in the integral form:

$$X(t) = X(0) + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s)$$

Stochastic integral, Ito integral

- Fix a process Y adapted to the information given by W , such that

$$E \left[\int_0^t Y^2(u) du \right] < \infty$$

- **Construction:** split interval $[0, t]$ into n subintervals $[t_i, t_{i+1}]$ and consider

$$I_n(t) := \sum_i Y(t_i) [W(t_{i+1}) - W(t_i)]$$

Ito showed that there is a limit $I(t)$, *(L^2 limit)*

$$E[(I_n(t) - I(t))^2] \rightarrow 0$$

and called the limit the **stochastic integral**:

$$I(t) = \int_0^t Y(s) dW(s)$$

Stochastic integral properties

- Process $I(t) = \int_0^t Y(u)dW(u)$ is a martingale with mean zero, or

$$E \left[\int_0^t Y(u)dW(u) \right] = 0$$

$$E_s \left[\int_0^t Y(u)dW(u) \right] = \int_0^s Y(u)dW(u)$$

and the variance is

$$E \left[\left(\int_0^t Y(u)dW(u) \right)^2 \right] = E \left[\int_0^t Y^2(u)du \right]$$

Reasons why the martingale property

- We have

$$\begin{aligned} E_s \int_0^t Y(u) dW(u) &= E_s \int_0^s Y(u) dW(u) + E_s \int_s^t Y(u) dW(u) \\ &= \int_0^s Y(u) dW(u) + E_s \int_s^t Y(u) dW(u) \end{aligned}$$

- We claim that

$$E_s \int_s^t Y(u) dW(u) = 0$$

For example, for $t_{j+1}, t_j > s$,

$$\begin{aligned} E_s[Y(t_j)(W(t_{j+1}) - W(t_j))] &= E_s E_{t_j}[Y(t_j)(W(t_{j+1}) - W(t_j))] \\ &= E_s \{Y(t_j) E_{t_j}[(W(t_{j+1}) - W(t_j))]\} \\ &= 0 \end{aligned}$$

Reasons why the variance

- We have, for example

$$\begin{aligned} E[Y^2(t_1)(W(t_2) - W(t_1))^2] &= E[E_{t_1}\{Y^2(t_1)(W(t_2) - W(t_1))^2\}] \\ &= E[Y^2(t_1)E\{(W(t_2) - W(t_1))^2\}] \\ &= E[Y^2(t_1)\underbrace{(t_2 - t_1)}_{\Delta t}] = \mathbb{E} \sum_i Y^2(t_i) \Delta t = \mathbb{E} \left(\int_0^t Y^2(u) du \right) \end{aligned}$$

Here, we used the fact that

$$E_{t_1}\{(W(t_2) - W(t_1))^2\} = E\{(W(t_2) - W(t_1))^2\}$$

because $(W(t_2) - W(t_1))$ is independent of the information available up to time t_1 .

Pricing Options with Mathematical Models

16. Ito's rule, Ito's lemma

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Ito's rule

Think of $f(t)$ as option price & $x(t)$ as stock price.

- Standard calculus:

$$\frac{d}{dt} f(t, x(t)) = \frac{\partial}{\partial t} f(t, x(t)) + \frac{\partial}{\partial x} f(t, x(t)) \frac{d}{dt} x(t)$$

- or, denoting partial derivatives with subscripts,

$$df(t, x(t)) = f_t(t, x(t))dt + f_x(t, x(t))dx(t)$$

- In stochastic calculus, for

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2}\sigma^2 f_{xx}(t, X(t))dt$$

or

$$df = \left[f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx} \right] dt + \sigma f_x dW$$

Reason why – quadratic variation

- Split interval $[0, t]$ into pieces of length Δt .
- Consider the sum of absolute increments to the p -th power

$$Q_p(t, W) := \sum_i |W(t_{i+1}) - W(t_i)|^p$$

- For $p = 2$, its limit is called **quadratic variation** and for Brownian motion we have

$$Q_2(t, W) \rightarrow t , \text{ as } \Delta t \rightarrow 0$$

while $Q_1(t, W) \rightarrow \infty$.

- For a differentiable function f ,

$$Q_1(t, f) \rightarrow \int_0^t |f'(s)| ds , \text{ and } Q_2(t, f) \rightarrow 0$$

“Proof” of Itô’s rule

- Taylor expansion:

$$f(t + \Delta t, X(t + \Delta t)) - f(t, X(t)) = f_t \Delta t + f_x \Delta X$$

$$+ \frac{1}{2} f_{xx} (\Delta X)^2 + \underbrace{\text{higher order terms}}_{\substack{\text{for smooth fns this term goes to zero but not for Brownian motion.} \\ \text{Need to show higher order term goes to zero for Brownian Motion.}}}$$

- The second order term does not disappear:

$$(\Delta X)^2 = (\mu \Delta t + \sigma \Delta W)^2 = \mu^2 (\Delta t)^2 + 2\mu\sigma \Delta W \Delta t + \sigma^2 (\Delta W)^2$$

In the limit when $\Delta t \rightarrow 0$ this gives

$$(dX)^2 = \sigma^2 dt$$

- We get Itô’s rule: $df = f_t dt + f_x dX + \frac{1}{2} f_{xx} \sigma^2 dt$

More on Ito's rule

- We can write

$$df = f_t dt + f_x dX + \frac{1}{2} f_{xx} dX \cdot dX$$

using the following informal rules:

$$dt \cdot dt = 0, \quad dt \cdot dW = 0, \quad dW \cdot dW = dt$$

- This gives

$$dX \cdot dX = (\mu dt + \sigma dW) \cdot (\mu dt + \sigma dW) = \sigma^2 dt$$

Example: $W^2(t)$

Standard calculus:

$$\int_0^t s ds = \frac{t^2}{2}$$

$$\int_0^t W(s) dW(s) = ? \quad \frac{W^2(t)}{2} - \frac{t}{2}$$

Consider function $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$. Since

$$dW = 0 \times dt + 1 \times dW$$

$\mu=0$ $\sigma=1$

we have, by Ito's rule,

$$dW^2(t) = 2W(t)dW(t) + \frac{1}{2} \times 2dt$$

which can be written as *(Integrating between 0 & t)*

$$W^2(t) - W^2(0) = \int_0^t 2W(s)dW(s) + \int_0^t ds$$

which gives

$$2 \int_0^t W(s)dW(s) = W^2(t) - t$$

Exponential of Brownian motion

- The process

$$Y(t) = e^{aW(t)+bt}$$

is a function $Y(t) = f(t, W(t))$ with

$$f(t, x) = e^{ax+bt}, \quad f_t(t, x) = bf(t, x), \quad f_x(t, x) = af(t, x), \quad f_{xx}(t, x) = a^2 f(t, x)$$

- Applying Itô's rule we get

$$dY = \left[b + \frac{1}{2}a^2 \right] Y dt + aY dW$$

- If $b = -\frac{1}{2}a^2$, so that $Y(t) = e^{aW(t)-\frac{1}{2}a^2t}$ we have a **martingale**:

$dY = aY dW$, and from $E_s[Y(t)] = Y(s)$ we get

$$\begin{aligned} E_s \left[e^{aW(t)-\frac{1}{2}a^2t} \right] &= e^{aW(s)-\frac{1}{2}a^2s} \\ e^{-\frac{1}{2}a^2t} E_s \left[e^{aW(t)} \right] &= e^{aW(s)-\frac{1}{2}a^2s} \\ E_s \left[e^{aW(t)} \right] &= e^{aW(s)+\frac{1}{2}a^2t-\frac{1}{2}a^2s} \end{aligned}$$

$$E_s[e^{aW(t)}] = e^{aW(s)+\frac{1}{2}a^2(t-s)}$$

→ This formula gives us a way to compute conditional IE of exponential fun of Brownian Motion.

Two-dimensional Ito's rule

Two Random factors eg: $X \& Y$ being 2 stocks.

- Correlated Brownian motions:

$$E[W_X(t)W_Y(t)] = \rho t, \quad dW_X dW_Y = \rho dt$$

- Consider a model with two processes

$$dX = \mu_X dt + \sigma_X dW_X(t), \quad dY = \mu_Y dt + \sigma_Y dW_Y(t)$$

- Ito's rule:

$$df(X(t), Y(t)) = f_x dX + f_y dY + \frac{1}{2} f_{xx} (dX)^2 + \frac{1}{2} f_{yy} (dY)^2 + f_{xy} dXdY$$

$$= \underbrace{f_x dX + f_y dY}_{\text{drift terms}} + \left[\frac{1}{2} f_{xx} \sigma_X^2 + \frac{1}{2} f_{yy} \sigma_Y^2 + \underbrace{f_{xy} \rho \sigma_X \sigma_Y}_{\text{covariance term}} \right] dt$$

- Product Rule: $d(XY) = \overbrace{XdY + YdX} + \overbrace{\rho \sigma_X \sigma_Y dt}$

Pricing Options with Mathematical Models

17. Black-Scholes-Merton pricing

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

The model

- The risk-free asset satisfies the ODE

$$dB(t) = rB(t)dt, \quad B(0) = 1$$

implying $B(t) = e^{rt}$.

- The stock satisfies the SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

(
GBM
(linear
SDE)

Using Ito's rule we can check that

$$S(u) = S(t)e^{(\mu - \frac{1}{2}\sigma^2)(u-t) + \sigma[W(u) - W(t)]}$$

$t = \text{initial time}$
 $u = \text{future time}$

Black-Scholes-Merton pricing: PDE approach

- We want to find the price of a European path-independent claim with payoff $C(T) = g(S(T))$.
- It is reasonable to guess that the price will be a function $C(t, S(t))$ of the current time and price of the underlying. If so, from Ito's rule,

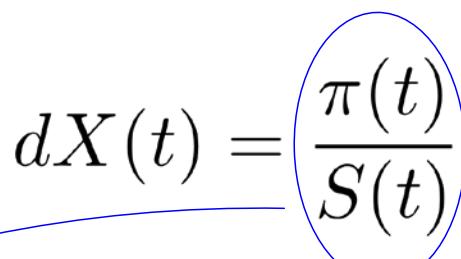
$$dC = \left[C_t + \frac{1}{2} \sigma^2 S^2 C_{ss} + \mu S C_s \right] dt + \sigma S C_s dW$$

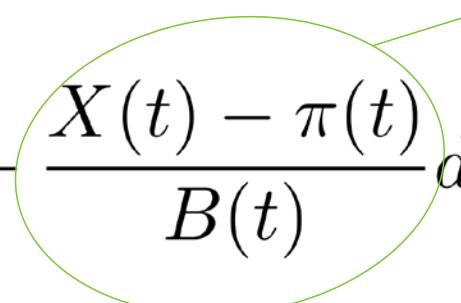
- On the other hand, with $\pi(t)$ = amount invested in stock at time t , a self-financing wealth process satisfies:

$$dX(t) = \frac{\pi(t)}{S(t)} dS(t) + \frac{X(t) - \pi(t)}{B(t)} dB(t)$$

No. of Shares in Bank Account

) Plugging dS & dB

No. of Shares (Delta) 

$dX = [rX + (\mu - r)\pi]dt + \sigma\pi dW$ 

Replication produces a PDE

- If we want replication, $C(t) = X(t)$, we need the dt terms to be equal, and the dW terms to be equal.
- Comparing the dW terms we get that the number of shares needs to be equal to the so-called **delta of the option**

$$\frac{\pi(t)}{S(t)} = C_s(t, S(t))$$

- Using this and comparing the dt terms we get the Black-Scholes PDE:

sub $\pi = SC_s$ & $X = C$:

$$C_t + \frac{1}{2}\sigma^2 s^2 C_{ss} + r(sC_s - C) = 0 \quad \forall t, s \geq 0 \\ t \leq T$$

- subject to the boundary condition,

Black-Scholes PDE

$$C(T, s) = g(s)$$

The bottom line

- If the PDE has a unique solution $C(t, s)$, it means we can replicate the option by holding delta shares at each time. The option price at time t when the stock price is equal to s is given by $C(t, s)$, and the option delta is the derivative of the option price with respect to the underlying.
- The PDE and the option price do not depend on the mean return rate μ of the underlying!

Black-Scholes formula

- For an European call option $g(s) = (s - K)^+$, the solution of the PDE is given by the Black-Scholes formula:

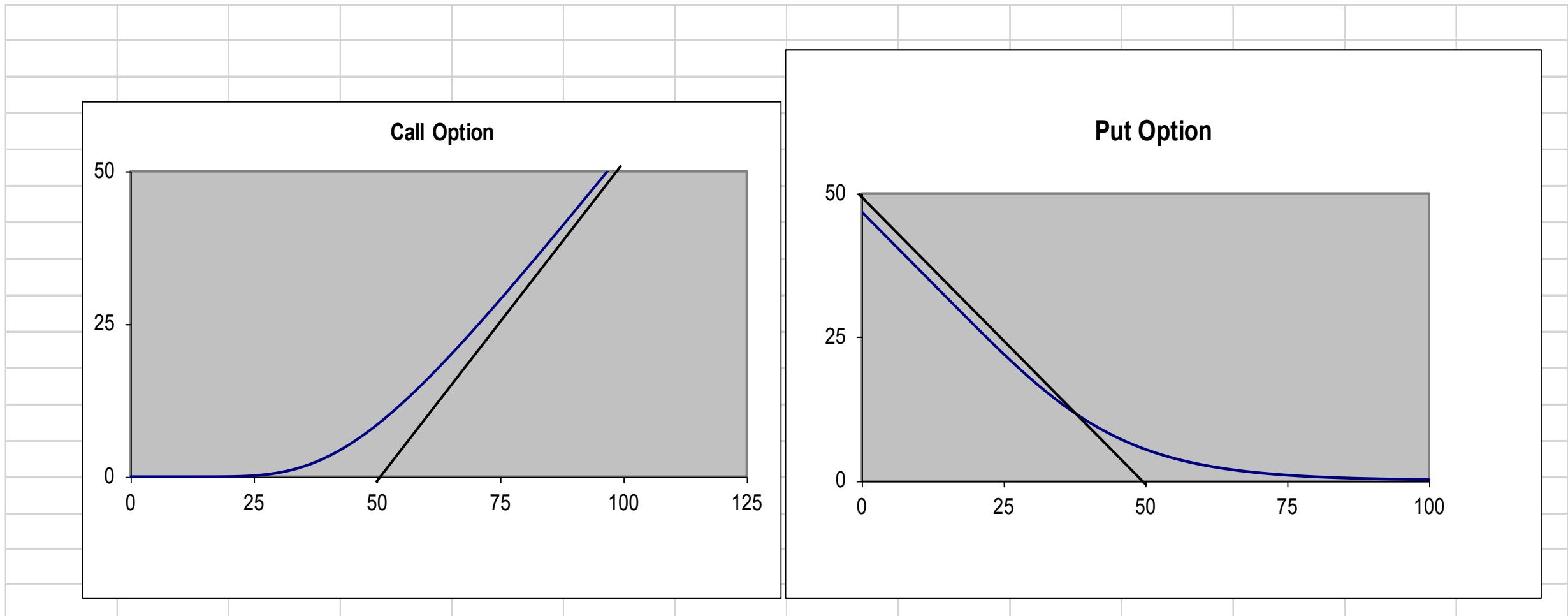
$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$N(x) := P[Z \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}} [\log(S(t)/K) + (r + \sigma^2/2)(T-t)] \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}} [\log(S(t)/K) + (r - \sigma^2/2)(T-t)] \\ &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

Graphs of the PDE solutions



Pricing Options with Mathematical Models

18. Risk-neutral pricing: Black-Scholes-Merton model

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Risk-neutral probability in B-S-M model

$$\text{Price} = \mathbb{E}^Q [e^{-rt} g(S(T))]$$

- Let us find the dynamics of S under the risk-neutral probability Q .
- Denote by W^Q the Brownian motion under Q .
- We claim that if we replace μ by r , that is, if the stock satisfies

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW^Q(t)$$

then the discounted stock price is a Q -martingale.

- Indeed, this is because Ito's rule then gives *(Product Rule)*

$$\begin{aligned} d(e^{-rt} S(t)) &= e^{-rt} dS(t) + S(t) d(e^{-rt}) \\ &= e^{-rt} [rS(t)dt + \sigma S(t)dW^Q(t)] - S(t)re^{-rt}dt = 0 \times dt + \sigma \bar{S}(t)dW^Q(t) \end{aligned}$$

↑
discounted Stock Price
 $e^{-rt} S(t)$

Girsanov theorem

$$W^Q(t) = W(t) + \frac{\text{some constant}}{\text{constant}} \times t \rightarrow \text{is}$$

a Brownian Motion.

$$dW^Q = dW + \text{constant } dt$$

- In order to have the above dynamics and also

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t),$$

we need to have

$$W^Q(t) = W(t) + \frac{\mu - r}{\sigma} t$$

$dW^Q = dW + \frac{\mu - r}{\sigma} dt$

- The famous **Girsanov theorem** tells us that this is possible: there exists a unique probability Q under which so-defined W^Q is a Brownian motion.
- The discounted wealth process is also a Q -martingale:

$$d(e^{-rt} X(t)) = e^{-rt} (\mu - r) \pi(t) dt + e^{-rt} \sigma \pi(t) dW(t)$$

$$= e^{-rt} \sigma \pi(t) dW^Q(t)$$

(Full workout on next page)

$$dX = [rX + (\mu - r)\pi]dt + \sigma\pi dW$$

Using Ito's Product rule,

$$\begin{aligned}
 d(e^{-rt} X(t)) &= e^{-rt} dX(t) - X(t) r e^{-rt} dt \\
 &= e^{-rt} (rX + (\mu - r)\pi) dt + e^{-rt} \sigma \pi dW - X r e^{-rt} dt \\
 &= \cancel{e^{-rt} rX dt} + e^{-rt} (\mu - r)\pi dt + e^{-rt} \sigma \pi dW - \cancel{X r e^{-rt} dt} \\
 &= e^{-rt} \mu \pi dt - e^{-rt} r \pi dt + e^{-rt} \sigma \pi dW \\
 dW^Q &= dW + \frac{\mu - r}{\sigma} dt \\
 &= e^{-rt} \mu \pi dt - e^{-rt} r \pi dt + e^{-rt} \sigma \pi (dW^Q - \frac{\mu - r}{\sigma} dt) \\
 &= e^{-rt} \mu \pi dt - e^{-rt} r \pi dt + e^{-rt} \sigma \pi dW^Q - e^{-rt} \pi (\mu - r) dt \\
 &= e^{-rt} \pi (\mu - r) dt + e^{-rt} \sigma \pi dW^Q - \cancel{e^{-rt} \pi (\mu - r) dt} \\
 &= e^{-rt} \sigma \pi dW^Q \\
 &= 0 \cdot dt + e^{-rt} \sigma \pi dW^Q.
 \end{aligned}$$

Black-Scholes formula as an expected value

- Option prices can then be computed taking expected values under Q .
- To do that, we note that we can write

$$S(T) = S(0)e^{\sigma W^Q(T) + (r - \frac{1}{2}\sigma^2)T}$$

- We have to compute

$$\begin{aligned} & E^Q[e^{-rT}(S(T) - K)^+] \\ &= E^Q[e^{-rT}S(T)\mathbf{1}_{\{S(T)>K\}}] - Ke^{-rT}E^Q[\mathbf{1}_{\{S(T)>K\}}] \end{aligned}$$

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } A \text{ happens} \\ 0 & \text{if Not.} \end{cases}$$

Computing the expected values

- For the second term we compute the price of a Digital (Binary) option:

$$E^Q e^{-rT} \mathbf{1}_{\{S(T)>K\}} = e^{-rT} Q(S(T) > K)$$

$$\begin{aligned} Q(S(T) > K) &= Q\left(S(0)e^{(r-\sigma^2/2)T+\sigma W^Q(T)} > K\right) \\ &= Q\left(\frac{W^Q(T)}{\sqrt{T}} > -d_2\right) \\ &= N(d_2) \end{aligned}$$

where the middle equality follows by taking logs and re-arranging.

- The first term is computed using the formula

$$E\left[g\left(\frac{W^Q(T)}{\sqrt{T}}\right)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-x^2/2} dx$$

↳ divided by \sqrt{T} s.t. we have Std. Normal R.V $\sim N(0,1)$.

Reminder: Black-Scholes formula

$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$N(x) := P[Z \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}} [\log(S(t)/K) + (r + \sigma^2/2)(T-t)] \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}} [\log(S(t)/K) + (r - \sigma^2/2)(T-t)] \\ &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

Another way to get the PDE

(Easy Method of Getting PDE)

- Under risk-neutral probability Q , by Ito's rule

$$dC(t, S(t)) = [C_t + rS(t)C_s + \frac{1}{2}\sigma^2 S^2(t)C_{ss}]dt + \sigma C_s S(t)dW^Q(t)$$

This term has to be Martingale i.e. zero when I discount C .

Then, discounting,

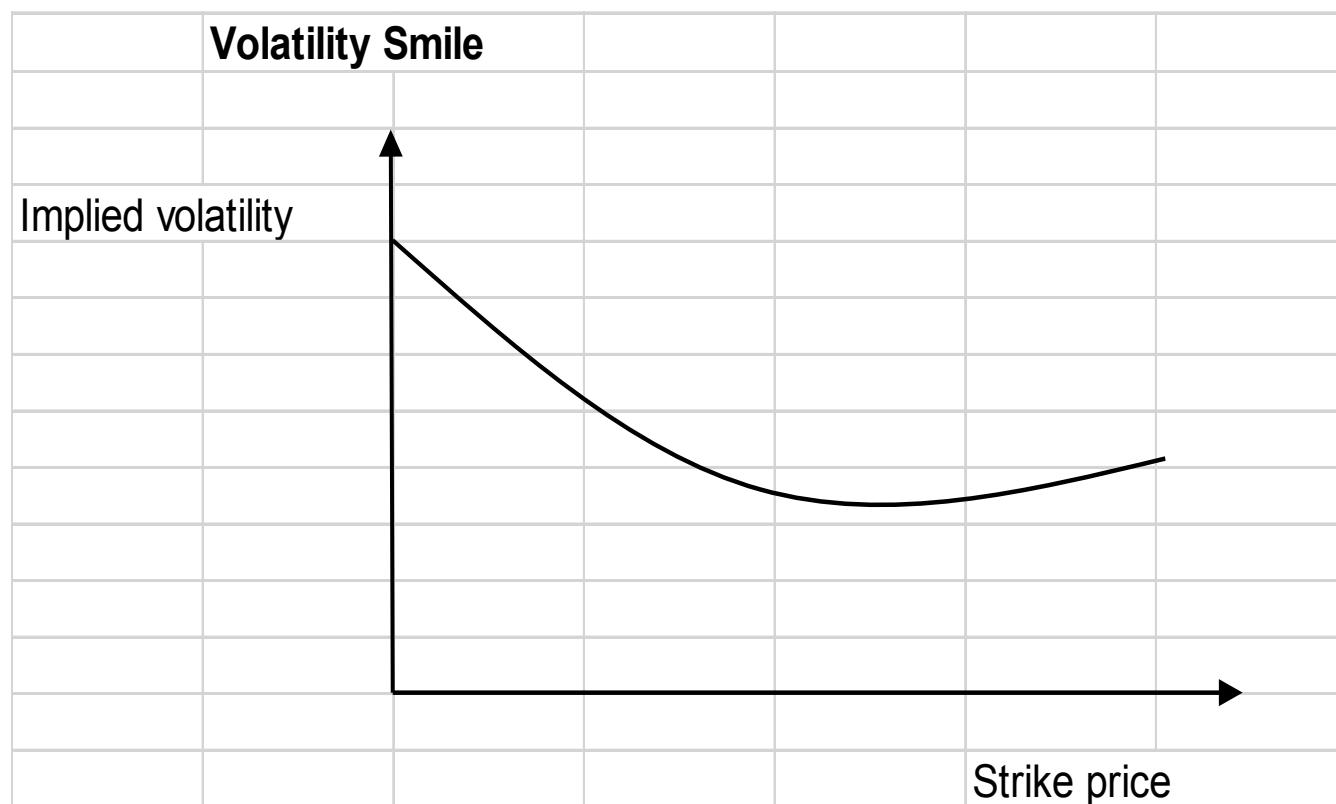
$$\begin{aligned} & d(e^{-rt}C(t, S(t))) \quad (\text{Applying Ito's Productrule}) \\ &= e^{-rt}[(C_t + rS(t)C_s + \frac{1}{2}\sigma^2 S^2(t)C_{ss} - rC)]dt \\ & \quad + e^{-rt}\sigma C_s S(t)dW^Q \end{aligned}$$

- This has to be a Q martingale, which means that the dt term has to be zero, resulting in the Black-Scholes PDE.

$$C_t + rS(t)C_s + \frac{1}{2}\sigma^2 S^2(t)C_{ss} - rC = 0$$

Implied volatility

- It is the value of σ that matches the theoretical Black-Scholes price of the option with the observed market price of the option



- In the Black - Scholes model, volatility is the same for all options on the same underlying
- However, this is not the case for implied volatilities: **volatility smile**

Pricing Options with Mathematical Models

19. Variations on Black-Scholes-Merton

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Dividends paid continuously

Discounted stock price will not be Martingale in this case. Hence, we have to make discounted wealth process Martingale.

- Assume the stock pays a dividend at a continuous rate q . Total value of holding one share of stock is

$$G(t) := S(t) + \int_0^t qS(u)du \quad \begin{array}{l} \text{Total gains} \\ \text{---} \\ dG = dS + \varepsilon S dt \end{array}$$

- Therefore, the wealth process of investing in this stock and the bank account is

$$dX = (X - \pi)dB/B + \pi dG/S$$

sub. dB & dG

- $dX(t) = [rX(t) + \pi(t)(\mu + q - r)]dt + \pi(t)\sigma dW(t)$

- To get the discounted wealth process to be a martingale, that is,

- $dX(t) = rX(t)dt + \pi(t)\sigma dW^Q(t)$

we need to have

$$W^Q(t) = W(t) + t(\mu + q - r)/\sigma$$

Equating the two $dX(t)$ from previous slide (•):

$$[rx + \pi(\mu+q-r)]dt + \pi\sigma dw = rxdt + \pi\sigma dw^Q$$

~~$$rxdt + \cancel{\pi(\mu+q-r)dt} + \cancel{\pi\sigma dw} = rxdt + \cancel{\pi\sigma dw^Q}$$~~

$$(\mu+q-r)dt + \sigma dw = \sigma dw^Q$$

$$dw^Q = \frac{(\mu+q-r)dt}{\sigma} + dw$$

Integrate to get:

$$w^Q = \frac{\mu+q-r}{\sigma} t + w$$

- This makes the stock dynamics

$$\begin{aligned} \text{Sub. } dW &= dW^Q - \frac{(r-q-\sigma^2)}{\sigma} dt \text{ into GBM} \\ \text{i.e. } dS &= S(rdt + \sigma dW) \end{aligned}$$

↙

$$dS(t) = S(t)[(r - q)dt + \sigma dW^Q(t)]$$

and the pricing PDE is

$$C_t + \frac{1}{2}\sigma^2 s^2 C_{ss} + (r - q)sC_s - rC = 0$$

- The solution, for the European call option, is obtained by replacing the underlying price s with $se^{-q(T-t)}$:

$$C(t, s) = se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}[\log(s/K) + (r - q + \sigma^2/2)(T - t)] \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}}[\log(s/K) + (r - q - \sigma^2/2)(T - t)] \end{aligned}$$

Dividends paid discretely

at known times with known values.

- Assume the stock pays deterministic dividends, and denote the process of discounted dividends by $\bar{D}(t)$.
- Assume that the process

$$S_G(t) = S(t) - \bar{D}(t)$$

satisfies

$$dS_G = S_G[\mu dt + \sigma dW(t)]$$

Then, the option price is obtained by replacing $s = S(t)$ by $S(t) - \bar{D}(t)$.

in B-S formula.

- Options on futures *(Here the underlying is Futures Contract)*

- Since $F(t) = e^{r(T-t)}S(t)$, Using Ito's product rule:

$$dF = F(\mu - r)dt + F\sigma dW$$

- With $W^Q(t) = W(t) + t(\mu - r)/\sigma$, we get

$$dF = F\sigma dW^Q$$

So, futures price without discounting is a Martingale under Q.

- Thus, the PDE for path independent options is

$$C_t + \frac{1}{2}\sigma^2 f^2 C_{ff} - rC = 0$$

- The solution for the call option is

$$\begin{aligned}
 C(t, f) &= e^{-r(T-t)} [fN(d_1) - KN(d_2)] \\
 d_1 &= \frac{1}{\sigma_F \sqrt{T-t}} [\log(f/K) + (\sigma_F^2/2)(T-t)] \\
 d_2 &= \frac{1}{\sigma_F \sqrt{T-t}} [\log(f/K) - (\sigma_F^2/2)(T-t)]
 \end{aligned}$$

Pricing Options with Mathematical Models

20. Currency options

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Currency options in the B-S-M model

- Consider the payoff, evaluated in the domestic currency, equal to

$$(R(T) - K)^+$$

*R(T) is a Random process
in the dollar value of 1 unit of foreign
currency.*

where $R(T)$ is the exchange rate, the time T domestic value of one unit of foreign currency.

- Assume that the exchange rate process is given by

$$dR(t) = R(t)[\mu_R dt + \sigma_R dW(t)]$$

} → GBM dynamics
although this is not
the best Model for
Exchange rate.

- The pricing formula is the same as in the case of a dividend-paying underlying, but with q replaced by r_f , the foreign risk-free rate.

*Foreign Currency for domestic investor is a Risky Asset like Stock which pays dividend.
The dividend in this case is the foreign Interest rate the foreign Bank pays the Investor.
So we have two Assets to trade-in, the domestic Bank & the foreign Bank.*

Reasons why

Using Ito's product rule:

$$\begin{aligned}
 d(R(t) e^{rt}) &= R(t) r e^{rt} dt + e^{rt} dR(t) \\
 &= R(t) r e^{rt} dt + e^{rt} R(t) M_R dt + e^{rt} R(t) \sigma_R dW(t) \\
 &= R^*(t) [r dt + M_R dt + \sigma_R dW(t)] \\
 &= R^*(t) [(r + M_R) dt + \sigma_R dW(t)]
 \end{aligned}$$

- We trade in the domestic and foreign risk-free accounts.
 - The dollar value of one unit of the foreign account is

$$R^*(t) := R(t)e^{r_f \cdot t} \quad \xrightarrow{\text{Domestic Value of foreign Bank Account.}}$$

Using Ito's Product rule shown top right

- The wealth dynamics (in domestic currency) of a portfolio of π dollars in the foreign account and the rest in the domestic account are

$$dX = \frac{X - \pi}{B} dB + \frac{\pi}{R^*} dR^* = [rX + \pi(\mu_R + r_f - r)]dt + \pi\sigma_R dW$$

In domestic dollar value.

domestic Bank

foreign Bank

Sub. $dB = r_B dt$ & $dR^* =$

- This is exactly the same as for dividends with q replaced by r_f .

$$W^Q(t) = W(t) + t(\mu_R - (r - r_f))/\sigma_R$$

$dW = dW^Q - [\mu_R - (r - r_f)]/\sigma_R dt$

Now, plug dW into dX & using Ito's rule the discounted wealth process will be Martingale.

Call option formula

- The dollar value of the call option is

$$c(t, R) = Re^{-r_f(T-t)}N(d_1) - Ke^{-r(T-t)}[N(d_2)]$$

where

$$d_1 = \frac{1}{\sigma_R \sqrt{T-t}} [\log(R/K) + (r - r_f + \sigma_R^2/2)(T-t)]$$

$$d_2 = \frac{1}{\sigma_R \sqrt{T-t}} [\log(R/K) + (r - r_f - \sigma_R^2/2)(T-t)] = d_1 - \sigma_R \sqrt{T-t}$$

Example: Quanto options

- $S(t)$: a domestic equity index
- Payoff: $S(T) - F$ units of **foreign currency**; quanto forward
 \uparrow
Constant Forward price

E.g.: S&P 500 is 300 & F=100 then the option pays €800 Euros.
- As in the previous slide, we have

$$W^Q(t) = W(t) + t(\mu_R - (r - r_f)) / \sigma_R$$

and thus

$$dR(t) = R(t)[(r - r_f)dt + \sigma_R dW^Q(t)]$$

- Assume Non-dividend Index following GBM:

$$dS(t) = S(t)[rdt + \sigma_S dZ^Q(t)]$$

$$IE^Q(Z^Q(t).W^Q(t)) = S^Q t$$

where BMP Z^Q has instantaneous correlation ρ with W^Q . We have
Using Ito's Product rule:

$$d(S(t)R(t)) = S(t)R(t)[(2r - r_f + \rho\sigma_R\sigma_S)dt + \sigma_R dW^Q(t) + \sigma_S dZ^Q(t)]$$

- $S(T) - F$ units of foreign currency is the same as $(S(T) - F)R(T)$ units of domestic currency. The domestic value is

$$e^{-rT}(E^Q[S(T)R(T)] - FE^Q[R(T)])$$

Need dynamics of
 $R(t)$ & $S(t), R(t)$ from
previous page.

- To make it equal to zero $e^{-rT}(IE^Q[S(T)R(T)] - FE^Q[R(T)]) = 0$

(Since for forward contract
the initial price must be
zero.)

$$F = \frac{E^Q[S(T)R(T)]}{E^Q[R(T)]}$$

- We have

$$E^Q[S(T)R(T)] = S(0)R(0)e^{(2r - r_f + \rho\sigma_S\sigma_R)T}$$

$$E^Q[R(T)] = R(0)e^{(r - r_f)T}$$

} Using formula
on next page

- We get

$$F = S(0)e^{(r + \rho\sigma_S\sigma_R)T}$$

- If

$$dX = aXdt + bXdW + cXdZ$$

Integrate & take IE then 2^{nd} & 8^{th} term on RHS goes to zero.

then

$$EX(t) = X(0)e^{at}$$

- This is because

$$d(EX(t)) = a \times (EX(t))dt$$

and the solution to this ODE, that has initial value $X(0)$, is the one above.

$$\frac{d}{dt} \underbrace{EX(t)}_{f(t)} = a EX(t)$$

$$f'(t) = a f(t)$$

Pricing Options with Mathematical Models

21. Exotic options

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Most popular exotic options

- Barrier options: they pay a call/put payoff only if the underlying price reaches a given level (barrier) before maturity. Thus, they depend on the maximum or the minimum price of the underlying during the life of the option.
- Asian options: a call/put written on the average stock price until maturity. Useful when the price of the underlying may be very volatile. *Eg: Energy Market. No closed formula for Price.*
- Compound options: the underlying is another option. Call on a call:

$$E_0^Q e^{-rT_1} [\text{BS}(T_1) - K_1]^+$$

\downarrow
*Black-Scholes Price of option at T_1 which
expires in T_2 .*

Example: a forward start option (call)

For eg $t_1 = 3$ months from now & $T = 6$ months from now

- A call with the strike price $S(t_1)$, $t_1 > 0$. Note that

$$S(0) \frac{S(T)}{S(t_1)} = S(0) \exp\{\sigma(W^Q(T) - W^Q(t_1)) + (r - \sigma^2/2)(T - t_1)\}$$

- We first compute the value at t_1 :

$$\begin{aligned} E_{t_1}^Q \left[e^{-r(T-t_1)} (S(T) - S(t_1))^+ \right] &= E_{t_1}^Q \left[e^{-r(T-t_1)} \frac{S(t_1)}{S(0)} \left(\frac{S(0)S(T)}{S(t_1)} - S(0) \right)^+ \right] \\ &= \frac{S(t_1)}{S(0)} \text{BS}(T - t_1, S(0)) . \end{aligned}$$

factor this out
 Black-Scholes price of option with strike $S(0)$, starting at $S(0)$
 & maturity $(T - t_1)$.

taken out of IE as
 it's known at time t_1 .

- Today's value

$$E_0^Q \left[e^{-rt_1} \frac{S(t_1)}{S(0)} \text{BS}(T - t_1, S(0)) \right] = \text{BS}(T - t_1, S(0)) E_0^Q \left[e^{-rt_1} \frac{S(t_1)}{S(0)} \right] = \text{BS}(T - t_1, S(0))$$

= 1
 because of Martingale Property
 $E_0^Q \left(e^{rt_1} \frac{S(t_1)}{S(0)} \right) = \frac{1}{S(0)} \cdot E_0^Q \left(e^{rt_1} S(t_1) \right) = \frac{1}{S(0)} e^{-rt_1} S(0) = 1.$

Example: a chooser option

eg: Buy option today & decide 3 months from now whether its a call or put for maturity 6 months from now.

- The holder can decide at time t_1 whether the payoff will be a call or a put, with the same strike price and maturity. Thus, the value at time t_1 is, using put-call parity,

$$\max(c(t_1), p(t_1)) = \max(c(t_1), c(t_1) + Ke^{-r(T-t_1)} - S(t_1))$$

Put call parity

$$= c(t_1) + \max(0, Ke^{-r(T-t_1)} - S(t_1))$$

Put option with strike $Ke^{-r(T-t_1)}$ & maturity t_1 .

- It is a package of a call option with maturity T and strike price K , and a put option with maturity t_1 and strike price $Ke^{-r(T-t_1)}$.

Pricing Options with Mathematical Models

22. Pricing options on more underlyings

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Two risky assets (Two Underlyings)

$$dS_1 = S_1[\mu_1 dt + \sigma_1 dW_1]$$

$$dS_2 = S_2[\mu_2 dt + \sigma_2 dW_2]$$

Equivalently,

$$dS_1 = S_1[\mu_1 dt + \sigma_1 dB_1]$$

$$dS_2 = S_2[\mu_2 dt + \sigma_2 \rho dB_1 + \sigma_2 \sqrt{1 - \rho^2} dB_2]$$

This is because, given two independent Brownian Motions B_1 and B_2 , we can set

$$W_1 = B_1, \quad W_2 = \rho B_1 + \sqrt{1 - \rho^2} B_2$$

$$\text{IE}(W_1(t) W_2(t)) = \cancel{\text{IE}(B_1^2(t))} + \sqrt{1 - \rho^2} \underbrace{\text{IE}(B_1(t) B_2(t))}_{=0} \text{ Since } B_1 \text{ & } B_2 \text{ are independent, the IE of product is product of IE & each IE is 0.}$$

$$= \cancel{\rho t}.$$

Hence, correlation is ρ .

$$\text{Var}(W_2(t)) = \cancel{\rho^2 \text{Var}(B_1(t))} + (1 - \rho^2) \text{Var}(B_2(t))$$

$$= \cancel{\rho^2 t} + t - \cancel{\rho^2 t}$$

$$= t //$$

Wealth process

-

$$dX = \frac{\pi_1}{S_1} dS_1 + \frac{\pi_2}{S_2} dS_2 + \frac{X - (\pi_1 + \pi_2)}{B} dB .$$

Sub. dS_1, dS_2, dB :
 This gives

No. of shares in S_1
 No. of shares in S_2
 No. of Units in Bank Account

$$dX = [rX + \pi_1(\mu_1 - r) + \pi_2(\mu_2 - r)]dt + \pi_1\sigma_1 dW_1 + \pi_2\sigma_2 dW_2 .$$

For the discounted wealth process to be a martingale under the risk-neutral probability Q , we need to have

$$dX = rXdt + \pi_1\sigma_1 dW_1^Q + \pi_2\sigma_2 dW_2^Q$$

for some Q -Brownian Motions W_i^Q with correlation ρ . For that to be the case, we must have

$$W_i^Q(t) = W_i(t) + t(\mu_i - r)/\sigma_i$$

The New Brownian Motion also keeps the same correlation ρ .

Under \mathcal{Q} : $dS_i = \gamma S_i dt + \sigma_i S_i dW_i^Q$

The pricing PDE with two factors

- Suppose $C(T) = g(S_1(T), S_2(T))$. Using the two-dimensional Ito's rule $\frac{\partial C}{\partial t} + \dots$

$$dC = \left[C_t + rS_1C_{s_1} + rS_2C_{s_2} + \frac{1}{2}\sigma_1^2S_1^2C_{s_1s_1} + \frac{1}{2}\sigma_2^2S_2^2C_{s_2s_2} + \rho\sigma_1\sigma_2S_1S_2C_{s_1s_2} \right] dt \\ + \sigma_1S_1C_{s_1}dW_1^Q + \sigma_2S_2C_{s_2}dW_2^Q .$$

Comparing the dt term with the wealth equation, or making the drift of the discounted C equal to zero,

$$C_t + \frac{1}{2}\sigma_1^2s_1^2C_{s_1s_1} + \frac{1}{2}\sigma_2^2s_2^2C_{s_2s_2} + \rho\sigma_1\sigma_2s_1s_2C_{s_1s_2} + r(s_1C_{s_1} + s_2C_{s_2} - C) = 0 .$$

$$C(T, s_1, s_2) = g(s_1, s_2) \xrightarrow{\text{Terminal/Boundary Condition}}$$

$$\frac{\pi_1}{S_1} = C_{s_1}, \quad \frac{\pi_2}{S_2} = C_{s_2}$$

Example: exchange option

Exchange one Asset for another.

The payoff is

$$g(S_1(T), S_2(T)) = (S_2(T) - S_1(T))^+ = \max(S_2(T) - S_1(T), 0)$$

Since we have

$$(s_2 - s_1)^+ = s_1 \left(\frac{s_2}{s_1} - 1 \right)^+$$

it is reasonable to expect that the option price will be of the form

$$C(t, s_1, s_2) = s_1 D(t, z)$$

for some function D and a new variable $z = s_2/s_1$. After some computations, we can show that D has to satisfy

$$D_t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)z^2 D_{zz} = 0, \quad D(T, z) = (z - 1)^+$$

Boundary Condition

Example: exchange option (continued)

This is the Black-Scholes PDE corresponding to the European call option with strike price $K = 1$, interest rate $r = 0$, and volatility

$$\sigma_E = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \quad .$$

Using the Black-Scholes formula for D , and the fact that $C = s_1 D$, we get

$$C(t, s_1, s_2) = s_2 N(d_1) - s_1 N(d_2) \quad ,$$

$$d_1 = \frac{1}{\sigma_E \sqrt{T-t}} [\log(s_2/s_1) + (\sigma_E^2/2)(T-t)]$$

$$d_2 = \frac{1}{\sigma_E \sqrt{T-t}} [\log(s_2/s_1) - (\sigma_E^2/2)(T-t)] = d_1 - \sigma_E \sqrt{T-t} \quad ,$$

Pricing Options with Mathematical Models

23. Stochastic volatility

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Complete markets

$$dS(t) = S(t)[r(t, S(t))dt + \sigma(t, S(t))dW^Q(t)]$$

- We have

$$S(T) = S(t)e^{\int_t^T [r(u, S(u)) - \frac{1}{2}\sigma^2(u, S(u))]du + \int_t^T \sigma(u, S(u))dW^Q(u)}$$

The PDE for the value of the payoff $g(S(T))$ is

$$C_t + \frac{1}{2}\sigma^2(t, s)C_{ss} + r(t, s)[sC_S - C] = 0$$

Constant Elasticity of Variance, CEV model, with $0 < \beta < 1$:

$$\sigma(t, s) = \frac{\sigma}{s^\beta}$$

- Our Model is complete if you have same number of Brownian motion (sources of risk) as the number of Risky assets.

Complete markets (continued)

- If $r(t)$ and $\sigma(t)$ are deterministic functions of time, random variable $\int_t^T \sigma(u)dW(u)$ has normal distribution with zero mean and variance $\int_t^T \sigma^2(u)du$.
- For a payoff $g(S(T))$, the value at time t is obtained by replacing $\sigma^2 \times (T-t)$ with $\int_t^T \sigma^2(u)du$ and replacing $r \times (T-t)$ with $\int_t^T r(u)du$.

Incomplete markets

- Consider two independent BMP's W_1 and W_2 , and

$$\begin{aligned} dS(t) &= S(t)[\mu(t)dt + \sigma_1(t, V(t))dW_1(t) + \sigma_2(t, V(t))dW_2(t)] \\ dV(t) &= \alpha(t)dt + \gamma(t)dW_2(t) \end{aligned}$$

We can show how that there are many possible Martingale Measures. Here is how we construct them:

- Denote by $\kappa(t)$ any (adapted) stochastic process. For each, there is a risk-neutral measure Q_κ . In particular, for any such process κ we can set

$$\begin{aligned} dW_1^{Q_\kappa}(t) &= dW_1(t) + \frac{1}{\sigma_1(t)}[\mu(t) - r(t) - \sigma_2(t)\kappa(t)]dt \\ dW_2^{Q_\kappa}(t) &= dW_2(t) + \kappa(t)dt \quad \Rightarrow W_2^{Q_\kappa}(t) = W_2(t) + \int_0^t \kappa(s)ds \end{aligned}$$

- It can be checked that discounted S is then a Q_κ -martingale and

$$dV(t) = [\alpha(t) - \kappa(t)\gamma(t)]dt + \gamma(t)dW_2^{Q_\kappa}(t)$$

The discounted Second Asset $V(t)$ is Not Martingale.

Incomplete markets (continued)

- For constant κ and constant parameters, the PDE becomes

$$\begin{aligned} C_t + \frac{1}{2}C_{ss}s^2(\sigma_1^2 + \sigma_2^2) + \frac{1}{2}C_{vv}\gamma^2 \\ + sC_{sv}\gamma\sigma_2 + r(sC_S - C) + C_v(\alpha - \kappa\gamma) = 0 \end{aligned}$$

- The parameters are **calibrated** to the market, that is, chosen so that the market prices of liquidly traded options are matched to the model prices as well as possible.

If we introduce extra stochastic factor in our Model, we must include that extra variable in option price $C(t, s, v)$.

Examples

- Heston's model: (Incomplete Model when Heston introduced it as we can't trade-in volatility which is one of the underlying)

$V(t)$ = Volatility (risky asset)
 B = long term mean level
 A = Speed of reversion

$$dS(t) = S(t)[rdt + \sqrt{V(t)}dW^Q(t)]$$

$$dV(t) = A(B - V(t))dt + \gamma\sqrt{V(t)}dZ^Q(t)$$

for some other risk-neutral Brownian motion Z^Q having correlation ρ with W^Q . Price is a function $C(t, s, v)$ satisfying

$$0 = C_t + \frac{1}{2}v[s^2C_{ss} + \gamma^2C_{vv}] + r(sC_s - C) + A(B - v)C_v + \rho\gamma vsC_{sv}$$

- SABR model:

$$dS(t) = S(t)[rdt + \sigma(t)\frac{1}{S^\beta(t)}dW^Q(t)]$$

$$d\sigma(t) = \alpha\sigma(t)dZ^Q(t)$$

In an Incomplete Market, there will be many Risk-Neutral Measures \mathcal{Q} , & hence many prices. Under different \mathcal{Q} we will have different parameters value of A and B in Heston's Model & α in SABR Model.

- Which \mathcal{Q} to use when many \mathcal{Q} gives No-Arbitrage Price?

Assume Market knows well & prices at correct \mathcal{Q} -measure.

We try to estimate from Market data what that \mathcal{Q} is.

For eg for Heston's Model we calibrate to Market to find parameters A, B, S & γ .

Compare

$$\min_{(A,B,S,\gamma)} \sum_{\substack{\text{All liquidly traded} \\ \text{options}}} (\text{Theoretical Price}(A, B, S, \gamma) - \text{Market Price})^2$$

Get your optimized A, B, S & γ & use it to price custom made derivatives your Investment Bank is selling to customers.

Ideally, you want as few parameters as possible that don't change that much from one day to another.

You want tractable Model, otherwise it will be computationally expensive even if you have Super Computers.

Pricing Options with Mathematical Models

24. Jump-diffusion models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Merton's jump-diffusion model

- Suppose the jumps arrive according to a Poisson process, that is, at independent exponentially distributed intervals.
- The number $N(t)$ of jumps up to time t is given by Poisson distribution:

$$P[N(t) = k] = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

$\lambda = \text{freq. of jumps}$

$\& [N(t) = k] \leq$

- The stock price satisfies the following dynamics:

$$dS(t) = S(t)[r - \lambda m]dt + S(t)\sigma dW^Q(t) + dJ(t) ,$$

where m is such that the discounted stock price is a Q -martingale, and dJ is the actual jump size.

Merton's jump-diffusion model (continued)

- More precisely, $dJ(t) = 0$ if there is no jump at time t , and $dJ(t) = S(t)X_i - S(t)$ if the i -th jump occurs at time t , where X_i are iid random variables. Therefore,

$$S(t) = S(0) \cdot X_1 \cdot X_2 \cdot \dots \cdot X_{N(t)} \cdot e^{(r-\sigma^2/2-\lambda m)t + \sigma W^Q(t)}$$

- The price of payoff $g(S(T))$ is

$$C(0) = \sum_{k=0}^{\infty} E^Q \left[e^{-rT} g(S(T)) \mid N(T) = k \right] Q(N(T) = k)$$

which is equal to

$$\sum_{k=0}^{\infty} E^Q \left[e^{-rT} g \left(S(0) X_1 \dots X_k \cdot e^{(r-\sigma^2/2-\lambda m)T + \sigma W^Q(T)} \right) \right] \times e^{-\lambda T} \frac{(\lambda T)^k}{k!}$$

Merton's jump-diffusion model (continued)

- If X_i 's are lognormally distributed, the price of the option can be represented as

$$C(0) = \sum_{k=0}^{\infty} e^{-\tilde{\lambda}T} \frac{(\tilde{\lambda}T)^k}{k!} BS_k$$

for $\tilde{\lambda} = \lambda(1 + m)$ and BS_k is the Black and Scholes formula with appropriately chosen $r = r_k$ and $\sigma = \sigma_k$.

This is a 3 parameter model $\sigma, \tilde{\lambda}, m$, which is more flexible. It can fit data better & still not a lot of parameters.

Pricing Options with Mathematical Models

25. Static hedging with futures

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Perfect hedge with futures

Futures are convenient for hedging as they have liquid Market & bid-ask Spreads are narrow.

- There is a futures contract that trades:
 - exactly the asset we want to hedge.
 - with the exact maturity we want to hedge.
- Otherwise, “asset mismatch” or “maturity mismatch”. Then, the solution is “crosshedging”.

Crosshedging

$$\text{Cov}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\cdot\mathbb{E}[Y]$$

$$\rho(X,Y) = \frac{\text{Cov}[X,Y]}{\sigma_X \cdot \sigma_Y}$$

- Hedging payoff $S_1(T)$ with futures F_2 maturing at $U \geq T$: the unknown quantity X , called **basis** is, with δ the number of futures contracts,

$$X = S_1(T) - \delta F_2(T, U)$$

- The risk is measured as

$$\text{Var}_t[X] = \text{Var}_t[S_1(T)] + \delta^2 \text{Var}_t[F_2(T, U)] - 2\delta \text{Cov}_t[S_1(T), F_2(T, U)]$$

- If we want to minimize the variance, taking the derivative with respect to δ and setting it equal to zero gives us the optimal δ :

$$\delta = \frac{\text{Cov}_t[S_1(T), F_2(T, U)]}{\text{Var}_t[F_2(T, U)]} = \rho \frac{\sigma_S}{\sigma_F},$$

$\frac{\sigma_S \sigma_F}{\sigma_F^2}$
 \uparrow

where ρ is the correlation between $S_1(T)$ and $F_2(T, U)$, and σ_S^2 , σ_F^2 are their variances.

Crosshedging (continued)

$$\begin{aligned} \text{Var}_t[X] &= \text{Var}_t[S_1(T)] + \frac{\text{Cov}_t^2[S_1(T), F_2(T, U)]}{\text{Var}_t[F_2(T, U)]} \text{Var}_t[F_2(T, U)] - 2 \frac{\text{Cov}_t^2[S_1(T), F_2(T, U)]}{\text{Var}_t[F_2(T, U)]} \\ &= \text{Var}_t[S_1(T)] - \frac{\text{Cov}_t^2[S_1(T), F_2(T, U)]}{\text{Var}_t[F_2(T, U)]} \end{aligned}$$

- The minimal variance is *Sub. δ into $\text{Var}_t[X]$ in previous page:*

$$Var_t[X] = Var_t[S_1(T)] - \frac{Cov_t^2[S_1(T), F_2(T, U)]}{Var_t[F_2(T, U)]} .$$

$\hookrightarrow S_2$ is the asset
on which futures contract is written.

- In the case of perfect hedge, $S_1 = S_2$, and $U = T$, we have $S_1(T) = F_1(T) = F_2(T, U)$. Then, $\delta = 1$, and, using the fact that $\text{Cov}[S, S] = \text{Var}[S]$, we see that $Var_t[X] = 0$ and there is no basis risk involved.

$$\begin{aligned} \delta &= \frac{\text{Cov}_t[S_1(T), F_2(T, U)]}{\text{Var}_t[F_2(T, U)]} = \frac{\text{Cov}_t[S_1(T), S_1(T, T)]}{\text{Var}_t[S_1(T, T)]} \\ &= \frac{\text{Var}_t[S_1(T)]}{\text{Var}_t[S_1(T)]} \\ &= 1. \end{aligned}$$

So, To hedge, for each unit of asset you buy/sell 1 unit of Future contract.

Example

- Consider a U.S. firm that will receive 1 million of currency A , six months from now. The firm will hedge by shorting δ units of six-month futures contracts on a highly correlated currency B .
- Suppose that the exchange rates are $Q_A = 0.1$ dollar/ A and $Q_B = 0.2$ dollar/ B . The exchange rate A/B has to be $Q = Q_A/Q_B = 0.5$. However, this does not mean that the company should short 0.5 million of B futures.
- Suppose historical data gives us $\sigma_{Q_A} = 0.03$, and $\sigma_{Q_B} = 0.02$, and that the correlation between the two is 0.9. Thus, the covariance is equal to $0.9 \cdot 0.02 \cdot 0.03 = 0.00054$. We get

$$\delta = \frac{0.00054}{0.0004} = 1.35$$

\uparrow $\frac{\sigma_{Q_A} \sigma_{Q_B}}{\sigma_{Q_B}^2}$

- Therefore, for each unit of A the U.S. company should short an amount of B equivalent to 1.35 of A , thus $1,000,000 \cdot 1.35/2 = 675,000$ of B .
- The minimal variance (per unit of currency) is 0.000171, quite a bit smaller than the variance of the dollar/ A exchange rate, equal to $\sigma_{Q_A}^2 = 0.0009$.

$$\text{Min Var} = \sigma_{Q_A}^2 - \frac{\sigma_{Q_A}^2 \sigma_{Q_B}^2}{\sigma_{Q_B}^2} = \frac{(0.03)^2 - (0.5)^2 \cdot (0.03)^2}{(0.02)^2} = 0.000171.$$

Rolling the hedge forward: the story of Metallgesellschaft

Since they are delivering oil, To hedge this they have to go long on Futures Contract on Oil.

- In early 1990s, MG sold huge volume of long-term fixed price forward-type contracts to deliver oil.
- Hedging: rolling over short-term future contracts to receive oil.
- Oil price went down: good for fixed price contracts.
- Bad for futures: large margin calls.
- All contracts closed out at a huge loss.

Pricing Options with Mathematical Models

26. Static hedging with bonds

Hedging fixed payments where payoff's are mostly driven by Interest rates.

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Duration (*Sensitivity of 1st order*)

- Suppose the bond price is given by

$$P = \sum_{i=1}^T \frac{C_i}{(1+y)^i}$$

Compounding once a year
so $n=1$.

- The **sensitivity** of the price with respect to yield is

$$\frac{\partial P}{\partial y} = - \sum_{i=1}^T i \frac{C_i}{(1+y)^{i+1}} = - \frac{P}{1+y} \sum_{i=1}^T i \frac{1}{P} \frac{C_i}{(1+y)^i}$$

- (Macaulay) **duration** is defined as

$$D = \sum_{i=1}^T i \frac{1}{P} \frac{C_i}{(1+y)^i}$$

Measures sensitivity of Bond Price
w.r.t the yield.

- It is an average of the coupon payment times, weighted by the relative size of the coupons; it is equal to maturity T for the zero-coupon bond.

Bond immunization

Want Both Sensitivity close to zero.

- A second order measure of yield risk, the **convexity**, is defined as

$$C = \frac{1}{P} \frac{\partial^2 P}{\partial y^2}$$

- Duration is a static version of the delta of an option, and convexity is a static version of the gamma of an option.
- **Hedging future cash payments** that have to be delivered at specified times: use a portfolio of bonds with the same duration and the same convexity as the cash payments. This is called **immunization**. It is based on the following approximation:

$$\Delta P \approx -\frac{D}{1+y} P \Delta y + \frac{1}{2} C P (\Delta y)^2$$

Pricing Options with Mathematical Models

27. Perfect hedging - replication

B-S type Complete Models.

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Replication in binomial trees

- Consider a call with $S(0) = 55.5625, \sigma = 0.32, r = 0, K = 55, T = 0.08$,

$$S^u = S(0)e^{\sigma\sqrt{T-t}} = 55.5625e^{0.32\sqrt{0.08}} = 60.826$$

$$S^d = S(0)e^{-\sigma\sqrt{T-t}} = 55.5625e^{-0.32\sqrt{0.08}} = 50.7544$$

- Option payoff is either $60.826 - 55 = 5.826$ or zero. To replicate it, we solve

$$\begin{aligned} \text{Initial Amount in Bank} \\ \downarrow \\ \delta_0 + 60.826\delta_1 &= 5.826 \\ \text{No. of shares} \\ \downarrow \\ \delta_0 + 50.7544\delta_1 &= 0 \end{aligned}$$

The Cost for replicating
this portfolio (option price) is
 $\delta_0 + \delta_1 S(0)$

- Suppose $S(T) = 68.8125$. Then, the final profit/loss is

$$\delta_1(68.8125 - 55.5625) + 55 - 68.8125 = -6.1479$$

Change in Stock Price *Option Exercise*

- In general, the number δ_1 of shares of the underlying is given by

$$\delta_1 = \frac{C^u - C^d}{S^u - S^d} = \frac{\Delta C}{\Delta S} \xrightarrow{\Delta t \rightarrow 0} C'(S)$$

Replication in the B-S-M model

- In the B-S-M model the delta of payoff $C(T) = g(S(T))$ is the derivative of its price with respect to the underlying,

$$\Delta_C := \frac{\partial C(t, s)}{\partial s}$$

- For the European call it is

$$\Delta_C = N(d_1)$$

- For the perfect (theoretical) hedge, re-balancing must take place continuously. This requires that the model and its parameters are exactly correct.

A real data example

- A call option on Microsoft stock 20 consecutive days in year 2000, $T = 20/252 = 0.08$ years.
- Daily data: $Y(i) = \log S(i+1) - \log S(i)$ $\bar{Y} = \text{Sample Average}$
- Sample standard deviation $= \sqrt{\frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y}_i)^2} = 0.019978$
- Annual st dev $\sigma = 0.019978 \times \sqrt{252} = 0.317139$
- $K = 55, S(0) = 55.5625$. We set $r = 0$.

A real data example (continued)

- Using B-S-M, $C(0) = 2.2715$, $\Delta(0) = 0.5629$
- Borrow $\Delta(0)S(0) - C(0)$
- Next day: $S(1) = 55.3750$, $T - 1 = 19/252$
- Portfolio value: $X(0) = C(0)$ and

$$X(1) = X(0) + \Delta(0)[S(1) - S(0)] = 2.166$$

- $\Delta(1) = 0.5483$, sell $\Delta(0) - \Delta(1)$ shares
- $X(2) = X(1) + \Delta(1)[S(2) - S(1)] = 2.2003$

Replication Experiment				
Time	Stock Price	Call Price	Delta	Wealth
0	55.5625	2.2715	0.5629	2.2715
1	55.3750	2.1176	0.5483	2.1660
2	55.4375	2.1009	0.5539	2.2003
3	56.5625	2.7256	0.6481	2.8235
4	59.1250	4.5831	0.8268	4.4841
5	60.3125	5.5702	0.8899	5.4659
6	61.3125	6.4562	0.9313	6.3558
7	60.6250	5.7970	0.9166	5.7155
8	62.6875	7.7346	0.9724	7.6060
9	61.2500	6.3360	0.9507	6.2082
10	63.2500	8.2681	0.9873	8.1095
11	64.1875	9.1933	0.9953	9.0351
12	64.2500	9.2531	0.9972	9.0973
13	65.0000	10.0007	0.9992	9.8452
14	63.0000	8.0025	0.9974	7.8467
15	64.1875	9.1877	0.9997	9.0310
16	65.8125	10.8125	1.0000	10.6556
17	68.2500	13.2500	1.0000	13.0931
18	68.1250	13.1250	1.0000	12.9681
19	68.8125	13.8125	1.0000	13.6556

A real data example (continued)

- Loss in option:

$$S(T) - K = S(T) - 55 = 13.8125$$

- Portfolio value: $X(T) = 13.6556$

- Loss with hedging: $13.8125 - 13.6556 = 0.1569$

- Loss without hedging: $13.8125 - C(0) = 11.5410$

Pricing Options with Mathematical Models

28. Hedging portfolio sensitivities

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Option sensitivities

Keeping Sensitivities close to zero to lower risk of Portfolio.

- In general, a portfolio value X is sensitive to changes in values of all parameters.
- Partial derivatives of the portfolio value with respect to parameters are called greeks
 - **Delta:** $\Delta = \frac{\partial}{\partial s} X$
 - **Theta:** $\Theta = \frac{\partial}{\partial t} X$
 - **Gamma:** $\Gamma = \frac{\partial^2}{\partial s^2} X$
 - **Vega:** $\mathcal{V} = \frac{\partial}{\partial \sigma} X$
 - **rho:** $\rho = \frac{\partial}{\partial r} X$

Approximation by Taylor expansion

- From Taylor's expansion,

$$X(t + \Delta t, s + \Delta s) = X(t, s) + \frac{\partial X(t, s)}{\partial s} \Delta S + \frac{\partial X(t, s)}{\partial t} \Delta t \\ + \frac{1}{2} \frac{\partial^2 X(t, s)}{\partial s^2} \Delta S^2 + \frac{1}{2} \frac{\partial^2 X(t, s)}{\partial t^2} \Delta t^2 + \frac{\partial^2 X(t, s)}{\partial s \partial t} \Delta S \Delta t + \dots$$

where

$$\Delta S = S(t + \Delta t) - S(t)$$

- Approximation:

$$X(t + \Delta t, s + \Delta s) \approx X(t, s) + \Delta \cdot \Delta S + \Theta \cdot \Delta t + \frac{1}{2} \Gamma \cdot \Delta S^2$$

Changes in
Portfolio

$$\Delta X \approx \Delta \cdot \Delta S + \Theta \cdot \Delta t + \frac{1}{2} \Gamma \cdot \Delta S^2$$

Approximation by Taylor expansion (continued)

- If the portfolio is delta-neutral, that is, its Δ is zero, then

$$\Delta X \approx \Theta\Delta t + \frac{1}{2}\Gamma\Delta S^2$$

- If Γ is strictly positive, any change in the value of the underlying tends to increase the value of the portfolio.
- If σ and r are stochastic:

$$\begin{aligned} & X(t + \Delta t, s + \Delta s, \sigma + \Delta\sigma, r + \Delta r)) \\ = & X(t, s, \sigma, r) + \frac{\partial X}{\partial s} \Delta s + \frac{\partial X}{\partial \sigma} \Delta\sigma + \frac{\partial X}{\partial r} \Delta r + \frac{\partial X}{\partial t} \Delta t \\ & + \frac{1}{2} \frac{\partial^2 X}{\partial s^2} \Delta S^2 + \frac{1}{2} \frac{\partial^2 X}{\partial \sigma^2} \Delta\sigma^2 + \frac{1}{2} \frac{\partial^2 X}{\partial r^2} \Delta r^2 + \dots \end{aligned}$$

Example

- Denote by Γ the gamma of portfolio X , and by Γ_C the gamma of a contingent claim C . We want to buy/sell n contracts of C in order to make the portfolio gamma neutral, that is,

$$\Gamma + n\Gamma_C = 0$$

- This implies

$$n = -\frac{\Gamma}{\Gamma_C}$$

n is the no. of units of contracts you buy/sell. If n is -ve you sell & if +ve you buy.

- However, taking this additional position in C will change the delta of the portfolio. We then buy/sell some shares of the underlying asset in order to make the portfolio delta-neutral. This does not change the gamma, because the underlying asset S has zero gamma. (Why?).

$$\frac{\partial^2 S}{\partial S^2} = \frac{\partial}{\partial S} 1 = 0$$

Example (continued)

- As an example, let's say a delta-neutral portfolio X has a gamma $\Gamma = -5,000$. A traded option has $\Delta_C = 0.4$ and $\Gamma_C = 2$. We buy $n = 5,000/2 = 2,500$ option contracts, making the portfolio gamma-neutral. This makes the delta of the portfolio equal to

$$\Delta_X = 2,500 \cdot 0.4 = 1,000 .$$

- Thus, we have to sell 1,000 shares of the underlying asset to keep the portfolio delta-neutral. *(since each share has $\Delta=1$)*.

Example (continued)

- If we want to make a portfolio vega-neutral, in addition to delta-neutral and gamma-neutral, then it is necessary to hold two different contingent claims written on the underlying asset. In this case we want to have

$$\Gamma + n_1\Gamma_1 + n_2\Gamma_2 = 0$$

$$\mathcal{V} + n_1\mathcal{V}_1 + n_2\mathcal{V}_2 = 0 \quad ,$$

where $n_i, \Gamma_i, \mathcal{V}_i$ are the number of contracts, the gamma and the vega of claim i .

- This is a system of two equations with two unknowns which can typically be solved. At the end, we still have to adjust the number of shares in the new portfolio in order to make it delta-neutral, similarly as above.

Portfolio insurance

- Idea: A put option with our portfolio as the underlying protects against portfolio losses
- Problem: such options are not traded.
- Solution: a synthetic put - replicating the put payoff by trading.
- However, it led to large losses in the crash of October 1987, due to loss of liquidity.

Hedging Put option: Short Stock.

The Story of Long Term Capital Management

- Merton and Scholes were partners
- Anticipated spreads between various rates to become narrower *cg: Spreads Between Corporate bonds & US Treasury Bonds.*
- “Russian crisis” pushed the spreads even wider
- LTCM was highly leveraged – margin calls forced it to start selling assets, others also did, their prices went even lower, losses huge in 1998
- Bailed out by government effort
- The reasons: high leverage and unprecedented extreme market moves

Pricing Options with Mathematical Models

29. Introduction to interest rate models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Bond price as expected value

From now on all E are under pricing probability (risk-Neutral) P .

- We assume that the price of a security is the expected value of the discounted payoff under some risk-neutral probability, now denoted P , not Q .
- For a pure discount bond paying \$1.00 at maturity, in discrete time, with maturity in n periods,

$$P(t, n) = E_t \left[\frac{1}{\prod_{i=0}^{n-1} (1 + r_i)} \right]$$

r_i is different for each period.

- In continuous time, with maturity T ,

↓
 ZCB price at time t
 with Maturity T .

$$P(t, T) = E_t \left[e^{- \int_t^T r(u) du} \right]$$

We Model Bond prices by modelling short rates $r(t)$ & Computing discounted E instead of directly modelling Bond prices like we did for stocks in B-S model because this guarantees us that discounted Bond prices will be Martingale.

$$e^{- \int_0^t r(u) du} \cdot P(t, T) = E_t \left[e^{- \int_0^T r(u) du} \cdot 1 \right]$$

Martingale since it is a fixed R.V. doesn't depend on t .

$M_t = E_t[X]$ for $s < t$, $E_s[M_t] = E_s[E_t[X]] = E_s[X] = M_s$, Hence Martingale \Rightarrow No Arbitrage Price

In practice we do:

- (short rate)
- ① Have Model for Interest Rate $\delta(t)$ but not sure about parameters.
- ② Use this $P(t, T) = E_t \left[e^{-\int_t^T r(u) du} \right]$ to calculate Theoretical Bond prices.
- ③ Observe $P(t, T)$ bond prices from Market.
- ④ Calibrate to find the parameters that matches Market price.
You want to find the Model where the parameters don't change that much from one day to another.

Price of bond option as expected value

Maturity of Bond at T .

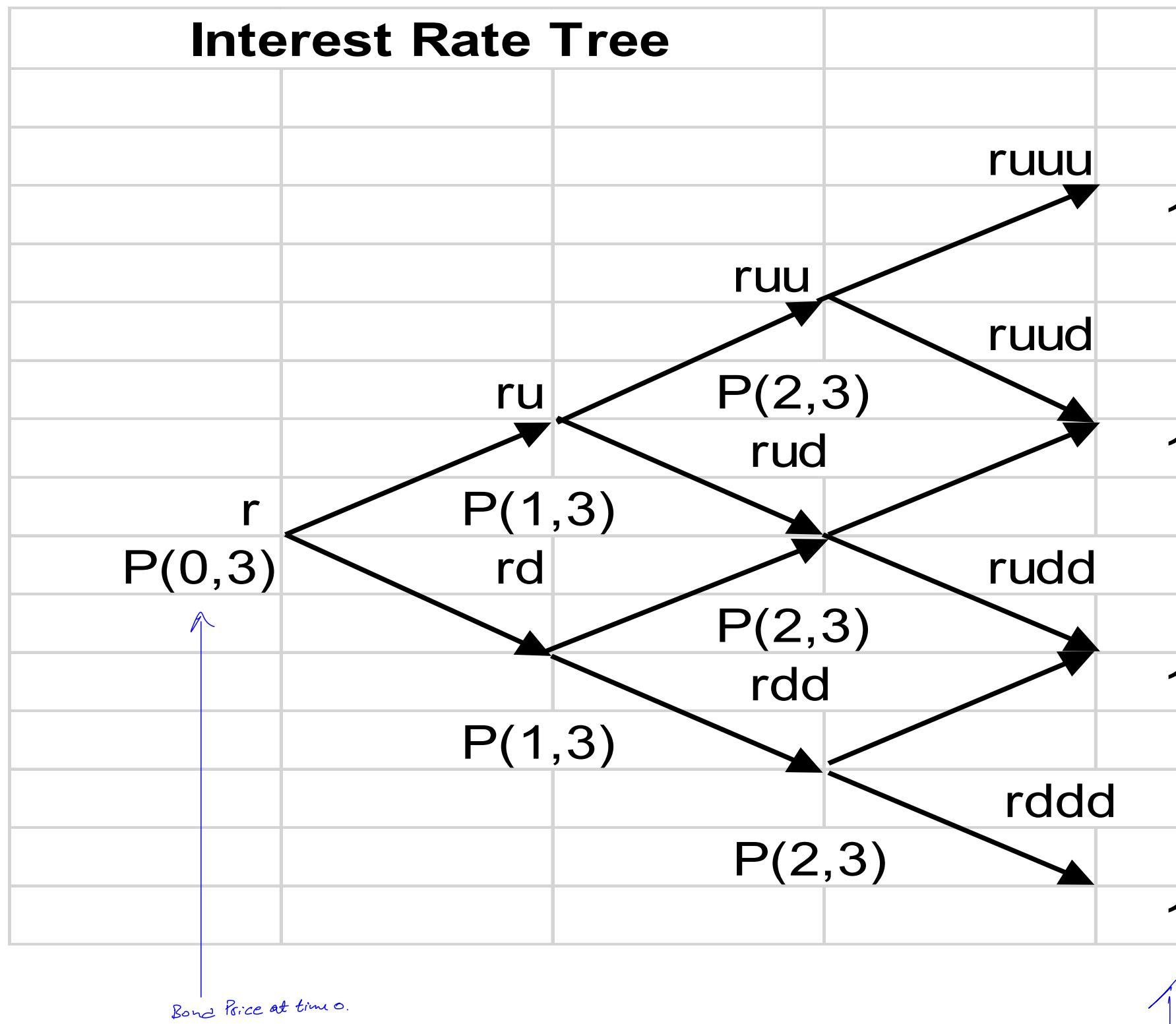
- For a European call on the bond, with call's maturity at τ , and $t < \tau < T$:
- In discrete time, (where τ is equal to m periods)

$$C(t) = E_t \left[\frac{1}{\prod_{i=0}^{m-1} (1+r_i)} \left(E_t \left[\frac{1}{\prod_{j=m}^{n-1} (1+r_j)} \right] - K \right)^+ \right]$$

- In continuous time

$$C(t) = E_t \left[e^{-\int_t^\tau r(u)du} \left(E_\tau \left[e^{-\int_\tau^T r(u)du} \right] - K \right)^+ \right]$$

• Discrete Time Interest Rate Binomial Tree Model.



↑
Bond Price
at Maturity

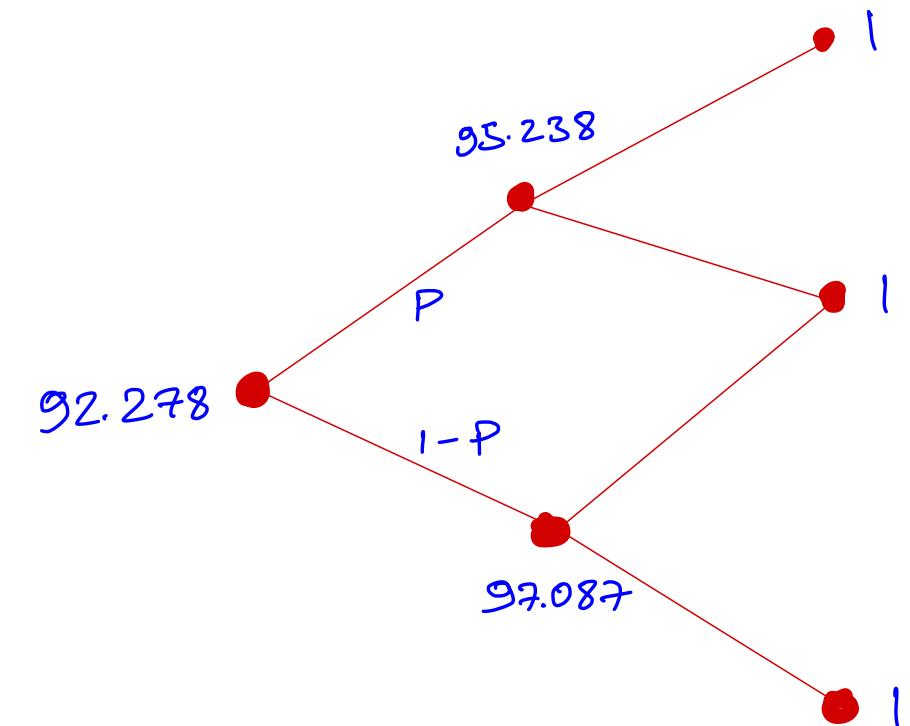
Example

- One-year interest rate today is 4%, one year from now it can go up to 5% or go down to 3%.
- A two-year zero-coupon bond with nominal value \$100 trades at \$92.278.
- We want to price an European call option on the bond, with $\gamma = 1$, $K = 96$.
- One year from now, the price of the bond will be, if $r = 5\%$,

$$\frac{100}{1.05} = 95.238$$

and if $r = 3\%$,

$$\frac{100}{1.03} = 97.087$$



Example (continued)

- By no-arbitrage, if p is the risk-neutral probability that the interest rate will be 5% one year from now, we necessarily have

$$92.278 = \frac{1}{1.04} (p \cdot 95.238 + (1 - p) \cdot 97.087)$$

- Then $p = 0.605$ and we can now price the call option:

$$\begin{aligned} & \frac{1}{1.04} (p(95.238 - 96)^+ + (1 - p)(97.087 - 96)^+) \\ &= \frac{1}{1.04} (1 - 0.605)1.087 = 0.413 \end{aligned}$$

↑
Price of Call option on Bond

Pricing Options with Mathematical Models

30. Continuous-time interest rate models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Modeling the short rate $(s(t))$

- Historically, the first approach was to model the dynamics of r , the **short rate**, using an SDE of the type

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$$

- It is typically modeled directly under the risk-neutral probability, that is, W now denotes W^Q .
- The models of the short rate with just one Brownian motion are called **one-factor models**.

↳ For Bond prices we need Multi-factor Models to fit data well.

Vasicek model

- The interest rate satisfies

$$dr = a(b - r)dt + \sigma dW$$

↑
long term Mean process
↑
Speed of Mean reversion

where a, b and σ are constant.

- This is a well-known Ornstein-Uhlenbeck process in stochastic calculus.
- It is a mean-reverting process.
- Interest rate has normal distribution and can become negative.
- It results in closed-form solutions for bond and option prices on Bond.

$d(r_t e^{at})$ using Ito's rule can be shown to have Normal dist. & so r_t has Normal dist.

Cox-Ingersoll-Ross (CIR) model

- The interest rate satisfies

Sq.root so that we get closed form Soln.
↓

$$dr = a(b - r)dt + \sigma\sqrt{r}dW$$

where a, b and σ are constant.

- The interest rate cannot become negative.
- The volatility of the interest rate is stochastic, as a function of the interest rate level.
- It also results in closed-form solutions for bond prices.

More one-factor models

- Ho-Lee model:

$$dr = b(t)dt + \sigma dW$$

Normal dist.

as many parameters $b(t)$ for as many Bonds (i.e. fitting the yield curve).

- Hull-White model:

$$dr = [b(t) - ar]dt + \sigma dW$$

- Black-Derman-Toy model:

$$dr = r[a(t)dt + \sigma dW]$$

Doesn't work in continuous time
since:
 $E[e^{\int_0^t r_s ds}] = \infty \quad \forall t.$

- The pricing PDE for the option value with payoff $g(r(T))$ is
for $dr = \mu(t, r)dt + \sigma(t, r)dW(t) \quad \& \quad C(t, r)$.

$$C_t + \frac{1}{2}\sigma^2 C_{rr} + \mu C_r - rC = 0, \quad C(T, r) = g(r)$$

Affine models of the term structure

- We wish the price of the bond to be of the form, for deterministic functions A, B ,

$$P(t, T) = e^{A(t, T) - B(t, T)r(t)}$$

- Since $P(T, T) = 1$, we need to have

$$A(T, T) = 0, \quad B(T, T) = 0$$

- The pricing PDE for the bond price is

$$P_t + \frac{1}{2}\sigma^2 P_{rr} + \mu P_r - rP = 0, \quad P(T, T) = 1$$

Need to find forms of $\mu(t, r)$ & $\sigma(t, r)$ for which the sol'n of PDE
is Affine: $P(t, T) = e^{A(t, T) - B(t, T)\sigma(t)}$

Affine models (continued)

- If we assume **affine dynamics** for μ & σ :

$\alpha(t), \beta(t), \gamma(t), \delta(t) \rightarrow$ All deterministic functions of time.

$$\mu(t, r) = \alpha(t)r + \beta(t), \quad \sigma^2(t, r) = \gamma(t)r + \delta(t)$$

the PDE becomes $\text{Sub : } p(t, T) = e^{A(t, T) - B(t, T)\sigma(t)}$ & above $\mu(t, \sigma)$ & $\sigma(t, \sigma)$ in PDE:

$$A_t - \beta B + \frac{1}{2}\delta B^2 - r \left[1 + B_t + \alpha B - \frac{1}{2}\gamma B^2 \right] = 0$$

- Both terms have to be zero, which yields two ODE's

$$B_t + \alpha B - \frac{1}{2}\gamma B^2 = -1, \quad A_t = \beta B - \frac{1}{2}\delta B^2$$

that have explicit solutions in the Vasicek and Cox-Ingersoll-Ross models.

- $p(t, T) = e^{A(t, T) - B(t, T)\sigma(t)}$ Use this to find Bond prices with the obtained A & B .

In AFFINE Models where μ & σ are linear in t ,

$$P(t, T) = e^{A(t, T) - B(t, T)\sigma(t)}$$

where A & B can be computed from the differential equations.

$$\text{Affine Models: } d\sigma = (\alpha \sigma + \beta)dt + (\gamma \sigma + \delta)dW(t)$$

Affine Models include all the Models we have mentioned Vasicek..etc
except Black-Derman-Toy Model.

Pricing Options with Mathematical Models

31. Forward rate models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Forward rates

Modelled as Random processes Moving in time.

What the rate is corresponding to investing \$1 at future time S & have it grow until time T .

S -Maturity ECB



- **Forward rate investment:** At time t , $t < S < T$, sell short one S -bond, buy $P(t, S)/P(t, T)$ T -bonds. At time S pay 1 and get $\underbrace{P(t, S)/P(t, T)}_{\text{many dollars}}$ at time T . Continuous rate R :

$$1 \cdot e^{R(t; S, T)(T-S)} = \frac{P(t, S)}{P(t, T)}$$

Forward Rate at time t
between S & T .



$$R(t; S, T) = -\frac{\log P(t, T) - \log P(t, S)}{T - S}$$

- **Instantaneous forward rate:** AS $S \rightarrow T$,

$$f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}$$

derivative w.r.t. Maturity

Integrating :

$$P(t, T) = e^{-\int_t^T f(t, u) du}$$

$$f(t, t) = r(t)$$

HJM is a family of Models/approach.

Heath-Jarrow-Morton (HJM) model

Modelling forward rates which depends on today's time & Maturity instead of short rate $r(t)$.

- The objective is to model simultaneously the whole term structure, not just the short-term rate, which is useful for calibration purposes.
- The price of a zero-coupon bond is

$$P(t, T) = E_t \left[e^{- \int_t^T r(u) du} \right] = e^{- \int_t^T f(t, u) du}$$

Short rates are not observable in Market. However, we can recover forward rates from Bond price data because we don't have E for forward rates in above formula.

- HJM model is a model for forward rates:

$$df(t, T) = \alpha(t, T) dt + \underbrace{\sigma^r(t, T) dW(t)}_{\sum_i \sigma_i(t, T) dw_i(t)}$$

Under Pricing Probability

- Here, ${}^T r$ denotes “transpose” of a matrix, in case W is multidimensional.

- We can have more than one factor in this Model for different Maturities Whereas Since Short rate doesn't depend on Maturity so by its nature it's one-dimensional/one-factor object.

HJM model (continued)

No freedom to choose α once we choose σ .

- No arbitrage implies that the drift of the bond price is $r(t)P(t)dt$.
- After some algebra, this implies:

$$\alpha(t, T) = \sigma^{Tr}(t, T) \int_t^T \sigma(t, u) du$$

so that the term structure of the drift is uniquely determined by the term structure of volatility.

- In calibration, we use the fact that

$$f(0, T) = -\frac{\partial \log P(0, T)}{\partial T}$$

- Then, a model for volatility needs to be chosen.

HJM example

- Flat term structure:

$$\sigma(t, T) \equiv \sigma \quad \text{Constant Volatility}$$

$$\alpha(t, T) = \sigma \int_t^T \sigma du = \sigma^2(T - t)$$

Therefore,

$$df(t, T) = \sigma^2(T - t)dt + \sigma dW(t)$$

(Integrating) $\rightarrow f(t, T) = f(0, T) + \sigma^2 t(T - t/2) + \sigma W(t)$

$$r(t) = f(t, t) = f(0, t) + \sigma^2 t^2/2 + \sigma W(t)$$

Taking derivative w.r.t 2nd argument of f

$$dr(t) = \left[\underbrace{\frac{\partial}{\partial T} f(0, t)}_{\frac{\partial f(0, T)}{\partial T}} + \sigma^2 t \right] dt + \sigma dW(t)$$

This is actually the Ho-Lee Model:

$$d\sigma(t) = b(t)dt + \sigma dW(t) \quad \text{So, } b(t) = \frac{\partial}{\partial T} f(0, t) + \sigma^2 t$$

BGM market model

- The forward LIBOR rate:

*Investing \$1 at T_{i-1} to T_i at simple rate
Using forward rate def'n:*

$$1 \cdot [1 + \Delta T L(t, T_i)] = \frac{P(t, T_{i-1})}{P(t, T_i)}$$

$$\Delta T = T_i - T_{i-1}$$

$$L(t, T_i) = \frac{P(t, T_{i-1})}{\Delta T P(t, T_i)} - \frac{1}{\Delta T}$$

discounting by T_i -Bond by T_i -Bond

- We want to consider it under probability P^{T_i} , called T_i -forward measure, under which **discounting** by T_i -bond results in a martingale:

Libor
 $L(t, T_i)$

$$dL(t, T_i) = L(t, T_i) \gamma(t, T_i) dW^{T_i}(t) \quad \leftarrow \text{Martingale}$$

Pricing a caplet in the market model

- Caplet payoff:

$$(L(T_{i-1}, T_i) - R_C)^+$$

*Call option on
LIBOR rate*

*↑
fixed rate*

The value is given by

$$P(t, T_i) E_t^{T_i} [(L(T_{i-1}, T_i) - R_C)^+]$$

*↑
Price of discounted Asset
i.e. T_i Bond*

Black caplet formula: volatility determined from

$$\text{Average of Volatility} \rightarrow \left\{ \sigma^2 = \frac{1}{T_{i-1} - t} \int_t^{T_{i-1}} \gamma^2(u, T_i) du \right.$$

Assuming γ is Not-Random, this is like computing call option price (Black-Scholes) with asset γ
which has deterministic Volatility with Interest rate = 0. Hence, use Black-Scholes formula.

(Another Way to Price Caplet)

A general way to price a caplet

Price of T_i maturity Bond at T_{i-1}



$$L(t, T_i) = \frac{P(t, T_{i-1})}{\Delta T P(t, T_i)} - \frac{1}{\Delta T} \quad \begin{matrix} \text{With } t = T_{i-1} \\ P(T_{i-1}, T_{i-1}) = 1 \end{matrix}$$



- Denote by $P = P(T_{i-1}, T_i)$, and recall that $L\Delta T = 1/P - 1$. Thus, the caplet payoff at time T_i is given by

$$C(T_i) = \left(\frac{1}{P} - (1 + R_C \Delta T) \right)^+ = \frac{1 + R_C \Delta T}{P} \left(\frac{1}{1 + R_C \Delta T} - P \right)^+ .$$

- Denoting $K = \frac{1}{1 + R_C \Delta T}$, the payoff is

$$C(T_i) = \frac{1}{KP} (K - P)^+$$

A general way to price a caplet (continued)

- Note that this payoff is paid at time T_i , but it is known at time T_{i-1} .
- The price at moment T_{i-1} of payoff $C(T_i)$ paid at T_i is $P \times C$. Why?
At T_{i-1} I buy C bonds which means I have C dollars at time T_i as each bond pays 1 dollar. So to buy C bonds I need exactly $P \times C$ dollars as each bond costs P .
- This means that the payoff $C(T_i)$ at time T_i is equivalent to the payoff

$$P \times C(T_i) = \frac{1}{K}(K - P)^+$$

at time T_{i-1} , equivalent to $1/K$ put options on the bond maturing at T_i , with option maturity equal to T_{i-1} , and strike price equal to K .

Pricing Options with Mathematical Models

32. Change of numeraire method

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

No special reason to discount by Bank Account other than historical as this was the first Asset.
we can discount by other Assets.

Other pricing probabilities

C = Any Security stocks, bonds, derivative etc.

S = Fixed Asset used for discounting e.g.: Used Bank Account before.

- Suppose C/S is a martingale under probability P^S , where S is a numeraire security, and C is any other security. Then

$$E_t^S[C(T)/S(T)] = C(t)/S(t) \quad \leftarrow \text{Martingale}$$

- This is true for any payoff $C(T)$, and the price is

$$C(t) = S(t)E_t^S[C(T)/S(T)]$$

General Pricing Formula

For eg. if S is Bank Account then $S(t) = e^{rt}$, so

$$\begin{aligned} C(t) &= e^{rt} E_t^{\mathcal{Q}} [e^{-rT} C(T)] \\ &= E_t^{\mathcal{Q}} [e^{-r(T-t)} C(T)] \end{aligned}$$

This Method helps to find
Explicit formula for Price
if it exists. If not we need to
use Numerical Methods.

Example (Black-Scholes Call)

$$C(t) = S(t)E_t^S[C(T)/S(T)]$$

→ General Pricing Formula

- Compute

$$D(t) = E_t^Q[e^{-r(T-t)}S(T)\mathbf{1}_{\{S(T)>K\}}]$$

↙ 1st Term in the Payoff
of call option when exercised.
(look pg 159-160).

- Change to P^S :

(Changing Numeraire to S & using above general pricing formula i.e. Multiply by numeraire $S(t)$ &
Change to new measure S , don't discount by bank account so, e^{-rt} disappears but discount by $S(T)$)

$$D(t) = S(t)E_t^S[\mathbf{1}_{\{S(T)>K\}}] = S(t)P_t^S(S(T) > K) = S(t)P_t^S \left(\frac{e^{r(T-t)}}{S(t)} < \frac{e^{r(T-t)}}{K} \right)$$

↑
Under P^S Measure when we divide asset price by S , we get martingale.

- Since $M(t) := \frac{e^{r(t-T)}}{S(t)}$ is a P^S -martingale, we have using Itô's rule,

$$dM(t) = -\sigma M(t)dW^S(t)$$

$$M(T) = M(t)e^{-\frac{1}{2}\sigma^2(T-t)-\sigma[W^S(T)-W^S(t)]}$$

Normal dist. with $(0, T-t)$

$$\begin{aligned} P_t^S(S(T) > K) &= P_t^S \left(\frac{1}{S(T)} < \frac{1}{K} \right) \\ &= P_t^S \left(\frac{e^{r(T-t)}}{S(t)} < \frac{e^{r(T-t)}}{K} \right) \end{aligned}$$

- It is now easy to compute $P_t^S(M(T) < 1/K)$

$$= N(d_1)$$

Black-Scholes-Merton formula for bond options

- Consider zero-coupon bond prices $P(t, T)$ and an asset price process S such that $F(t) = S(t)/P(t, T)$ has deterministic volatility $\sigma_F(t)$. Then, the European call price is

$$C(0) = S(0)N(d_1) - KP(0, T)N(d_2)$$

where

$$d_1 = \frac{1}{\Sigma_F(T)} \left(\log \frac{S(0)}{KP(0, T)} + \frac{1}{2} \Sigma_F^2(T) \right)$$

$$d_2 = \frac{1}{\Sigma_F(T)} \left(\log \frac{S(0)}{KP(0, T)} - \frac{1}{2} \Sigma_F^2(T) \right)$$

$$\Sigma_F(T) := \sqrt{\int_0^T \sigma_F^2(u) du}$$

option Maturity is T .
 ZCB $P(t, T)$ with same Maturity
 as option used as Numeraire.

Replace e^{-rt}
 with $P(0, T)$ &
 $\sigma\sqrt{T}$ with $\Sigma_F(T)$.

Bond option example

$$\text{Hence, } S(t) = P(t, T_2)$$

- **Call option price in the Vasicek model.** Consider a call option with maturity T_1 , on a T_2 -bond, $T_1 \leq T_2$. The process $F(t) = P(t, T_2)/P(t, T_1)$ is of the form

In Affine Models which includes Vasicek, $P(t, T) = e^{A(t, T) - B(t, T)r(t)}$

$$F(t) = e^{A(t, T_2) - A(t, T_1) - (B(t, T_2) - B(t, T_1))r(t)}$$

for some deterministic functions A and B , and

(For Vasicek Volatility = σ)

$$\sigma_F(t) = -\sigma[B(t, T_2) - B(t, T_1)]$$

Thus, it is deterministic, and we can use the formula, on previous page

with $T = T_1$ & $S(0) = P(0, T_2)$.
 $P(0, T) = P(0, T_1)$.

Pricing Options with Mathematical Models

33. Introduction to credit risk models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Structural models

(Not used as often in Industry)

- Denote by V the “value of the firm”. We assume the B-S-M model for it:

$$dV = (r - \delta)Vdt + \sigma_V VdW$$

δ = Some kind of payments like dividends going out of firm.

This firm issues debt (bond) & equity (stocks). So, $V(t) = E(t) + D(t)$.

→ By this we mean debt is a single Bond (ZCB) with maturity T & final value D .

- Denote by E the value of equity, (by T the maturity of the debt and by D the nominal value of debt.) The **value of equity** is a call option on the **value of the firm**:

$$E(T) = \max(V(T) - D, 0)$$

→ Value of Equity at maturity of debt.

- Bondholders receive *(Modelling Corporate Bond)*

$$\min(V(T), D) = D - \max(D - V(T), 0)$$

Pricing Bond payoff is pricing constant (Present value) minus price of Put-option on value of firm with strike = D .

If Firm goes Bankrupt i.e. $V(T) < D(T)$.

Structural models (continued)

- Firm value $V(0)$ not observable. However,

$$E(0) = BS(V(0), \sigma_V) \quad \begin{matrix} \text{also } \sigma_V \\ \uparrow \\ \text{Black-Scholes underlying } V(0) \\ \& V_0 = \sigma_V \end{matrix}$$
$$\sigma_E E(0) = N(d_1(V_0, \sigma_V))V(0)\sigma_V$$

where the second equality comes from equating the volatilities of E and the call option on V .

$$\begin{aligned} dE &= \dots dt + \alpha_E \sigma_V V dw && \leftarrow \text{Using It\^o's rule} \\ dE &= \dots dt + \sigma_E E dw && \leftarrow \text{Using } E \text{ instead of } S \\ \text{So, } \alpha_E \sigma_V V &= \sigma_E E \\ \sigma_E E &= N(d_1) \sigma_V V \end{aligned}$$

Reduced-form, intensity models

Modelling Default of Single firm.

- The simplest case: the probability distribution of default is exponential, with intensity λ .
- Denote by A the event that default has not occurred by time T .
- Price of a contingent claim with payoff $C(T)$ (independent of default event):

$$\begin{aligned} C(0) &= E^Q \left[e^{-rT} C(T) \mathbf{1}_A \right] = E^Q \left[e^{-rT} C(T) \right] E[\mathbf{1}_A] \quad (\text{By independence assumption}) \\ &= E^Q \left[e^{-rT} C(T) \right] P[A] \\ &= E^Q \left[e^{-rT} C(T) \right] e^{-\lambda T} \end{aligned}$$

Reduced-form, intensity models (continued)

- Thus, the price of a defaultable claim is obtained by discounting at a higher rate $r + \lambda$:

$$C(0) = E^{\mathcal{Q}} \left[e^{-(r+\lambda)T} C(T) \right]$$

- If the interest rate is stochastic and the intensity a function of time,

$$C(0) = E^{\mathcal{Q}} \left[e^{-\int_0^T [r(u)+\lambda(u)] du} C(T) \right]$$

\mathcal{Q} is calibrated using liquid vanilla credit products eg corporate bonds, credit default swaps etc.

- Empirically, **implied default probability** is much larger than the historical default probability.

↓
Which comes from
Calibration of \mathcal{Q} .