# Financial Markets

# Aims of the module



- Introduction to the objects and problems studied by Financial Mathematics,
- Practical motivation of the problems,
- Precise (i.e. mathematical) descriptions of some common financial contracts,
- Relevance of the other FM modules,
- If time permits, examples on mathematical modelling of financial problems.

# Outline



- Overview of financial markets and institutions,
  - Financial assets: definition, purpose, characteristics, examples
  - Financial markets: purpose, classifications, implementation
  - Financial institutions: different functions, examples, traded products
  - Trading protocols: auctions, clearing procedures,
  - Regulators, . . .
- Overview of certain sectors of financial markets
  - Equities,
  - Money markets, (Interest Rate, short term lending) < 1year maturity
  - Bond markets, (Long term lending) > 1year maturity
  - Mortgages,
  - Foreign exchange,
  - Insurance,
  - Derivatives, . . . (Call & put option, swaps etc)

Financial Agents → Financial institutions like Banks, brokers, investment companies, individuals or Insurance firms.

# Recommended reading

- F. Fabozzi, F. Modigliani and F. Jones, *Fundamentals of Financial Markets and Institutions*, 4th Edition, Pearson, 2009. (**FMJ** for short) → Used in USA undergraduate program.
- P. Howells and K. Bain, *Financial Markets and Institutions*, 5th Edition, Pearson, 2007. (**HB** for short) → UK.
- J. Hull, *Options Future and Other Derivatives*, 8th Edition, Pearson, 2012. → More Mathematical (Risk Neutral)
- L. Harris, *Trading and Exchanges: Market Microstructure for Practitioners*, Oxford University Press, 2003.
- M. Levinson, *Guide to Financial Markets*, The Economist, 2010.

# Introduction

This course is Model free in the sense, we will not use any Probabilistic Model for the financial instruments. You will only need basic Analysis from undergraduate level.
The probabilistic Model are always subjective & not fundamental to financial Markets.

# Financial assets

- Tangible Assets
  - Value is based on physical properties
  - Examples: buildings, land, machinery, food, ...
- Intangible Assets
  - Legal claims to future benefits
  - Examples: **Financial Assets**
- Financial assets give legal claims to **money** in the future.
- Financial and tangible assets are connected:
  - Cash payments of a financial asset (e.g. a pension plan) can be used to buy tangible assets in the future.
  - Buying tangible assets today can be financed by selling financial assets. (This is why governments and corporations sell bonds, i.e. take loans).

# Financial assets



**Key characteristics** of a financial asset are the amount of payments

- at different points in **time**,
- in different **scenarios** of the uncertain future.

**Uncertainty** about the future is

- what makes the analysis of financial assets challenging/ interesting (the subject of financial economics).
- the reason for the existence of most financial assets such as insurance and derivative contracts.
- subjective (knowledge and uncertainty is always subjective).

In **Financial Mathematics**, uncertainty is modelled by **probability spaces**; see FM01 and FM04 for general properties and FM05 for specific instances.

# Financial assets



Examples of financial assets: Withdraw money (when & how much) by Account holder. In financial jargon it means this asset has optionality i.e. payout depends on Account holder.

- **Bank deposit**: payouts are decided by the account owner.
- **Government bond**: payments are known ("fixed income") provided the issuer does not default (fail to pay as promised).
- **Corporate bond**: payments may be reduced and/or postponed if the issuer (borrower) defaults. (Higher interest than Government Bond)
- **Inflation linked bond**: payments depend on the development of the consumer price index. (Government issued – Payout are uncertain as it depends on CPI)
- **Common stock**: dividend payments depend on the earnings of the company and on strategy of the board of directors.
- **Insurance contract**: payments depend on the insured event.
- **Options, futures and other derivatives**: payments depend on a reference variable (price of an underlying asset, weather, . . . )
- **Credit derivatives**: payments depend on a "default event" (a financial institution fails to fulfil its financial obligations).

Ownership in a Company Future payout depends on market price of stock when you sell stock + yearly dividend.

Insurance contract against Bond issuer going default or bankruptcy.

# Financial assets

- By buying or selling financial assets, one can modify the income receivable at different points in **time** and in different future **scenarios**.

Examples (to be discussed in more detail later):

- A bond **issuer** (seller) gets more income today but less in the future when delivering the coupon payments and principal.
- Owners of a company can exchange uncertain future earnings for income today by selling shares of the company.
- Buying insurance, one exchanges today's cash for future cash in scenarios where expenses are high.
- Selling (buying) a futures contract, a producer (consumer) can reduce her uncertainty about future price of the product and thus, of income.

# Financial assets



Selling a financial asset creates a **financial liability** to deliver the future cash-flows of the asset to the buyer.
Examples:

- Bank deposits are a liability to a bank (it has to deliver cash when deposits are withdrawn by customers).
- Insurance contracts are a liability to an insurance company (it has to deliver the insurance claims if the insured loss takes place).
- Issuing bonds (borrowing money) creates the liability to deliver coupon and principal payments in the future.
- Pension annuities are a liability to a pension fund.
- Selling an option creates a liability to the seller.

Buy this to get higher pension (every month) income after retirement. Typically sold by insurance companies. Annuity is bought with some or all of your pension pot (money saved in your pension) s.t. it pays a regular retirement income for life or set period of time. Can only buy annuity after retirement.

# Financial assets



**Fundamental problem in finance**: what assets to buy/sell so that your net cash-flows at different scenarios and points in time are as "nice" as possible? This problem has many names: **portfolio optimization, optimal investment, asset-liability management (ALM), optimal hedging**, ...

- Optimality of a **portfolio** of assets depends on an agent's
  1. views concerning the uncertain future,
  2. risk preferences, (Some people are willing to take higher risk than others)
  3. financial position.

  Described in Financial model as Probabilistic model

  These factors are **subjective**!
- FM studies mathematical modelling of 1,2,3 as well as mathematical and computational techniques for ALM.
- Pricing principles (and, in particular, the Black–Scholes model) are based on asset-liability management. model.

# Financial markets

- Financial markets are where financial assets are traded.
- By trading, market participants can improve their financial position (reduce risks, increase expected returns, . . . ). For example,
  - **diversification** among many assets,
  - buying insurance contracts, forwards or other derivatives.
- Well functioning financial markets reduce **transaction costs** and provide trading opportunities when needed (**liquidity**).
- Financial markets facilitate efficient allocation of capital: supply and demand moves money where it is needed most.
- Because of the connections between financial and tangible assets, financial markets are important for the whole economy, not just to those who trade in financial markets.

# Financial markets



Possible ways to classify financial markets include:

- **Type of assets**: bonds, equity, insurance, derivatives, ...
- **Maturity**: short term investments ("money market") vs. long term investments ("capital market")
  - Maturity is the time to last payment of the asset.
- **Primary** vs. **secondary market**
  - Primary market is where financial assets are first sold (issued). eg: when Gov. issues bond, Stock IPO etc.
  - Secondary market is where one can trade assets after their original issue. eg: Investor selling Gov. Bond issued in past, Stock Exchanges selling stocks etc.
- **Centralised auction** vs. **over-the-counter** (bilateral trade).

Stock Exchange

bilateral (between seller & buyer)

# Financial markets

- Well-known examples of centralised auctions include the stock exchanges in London and New York. (LSE, NASDAQ)
- Most markets (not just financial) are based on an **auction mechanism** where most generous buying and selling offers lead to trades.
- Market prices are determined by **supply and demand** (In particular, market prices are not determined by mathematical models although models may be used by traders to determine their buyin/selling offers).

Keep in mind that the words “price” and “value” have **several different meanings** (accounting values, bid-price, ask-price, transacted price,. . . ). They are routinely misused in practice and academia (also in the books cited on page 4).

# Financial markets



Market participants include

- Individuals,
- Companies (financial and nonfinancial),
- Federal, state, and local governments,
- Central banks,
- Supranationals (International Monetary Fund, European Central Bank, . . . ),

# Financial institutions



Financial institutions provide services related to financial assets:

- providing payment mechanisms (credit cards, bank transfers, checks, . . . ) and thus participating in the **creation of money**,
- investment advice,
- trading financial assets on behalf of customers,
- assisting in the creation of financial assets: initial public offerings, securitizations, . . .
  like Mortgage Backed securities where big Pool of Mortgages are sliced & sold to investors who will be paid the interest of mortgage loan by homeowners.
- **producing** new financial assets by combining existing ones
  e.g: European Call option cash flows can be obtained by combining cash & underlying stock.
- . . .

We will focus on the last (and the second last) task which is where **financial mathematics** is most useful.

# Financial institutions



Spread in finance means difference between two quantities eg prices or rates etc.

- Financial institutions **produce** new (hopefully useful) financial assets by combining existing ones.
- They make profit by selling their products at a price higher than the production costs (like in any other company).
- For example,
  - Retail banks (e.g. HSBC, LloydsTSB, Royal Bank of Scotland, Barclays, HBOS) produce long-term loans (mortgage, car, . . . ) by combining individual deposits. They charge higher interest on the loans than what they give to depositors. They are in the "spread business". (difference between lending rate & borrowing rate)
  - Insurance companies sell insurance products whose payouts are covered by investment returns on insurance premiums.
  - The Black–Scholes formula is based on producing European options out of cash and stocks; see FM02.
- Financial companies are thus not that different from companies that produce tangible products.

# Financial institutions



Financial institutions act as **Financial Intermediaries** by creating assets for buyers and liabilities for sellers which are more attractive to each than would be the case if the parties had to deal with each other directly; HB Sec 1.1.2.

Examples:

- A retail bank creates long-maturity loans (e.g. mortgages or car loans) by combining savings of depositors (**maturity transformation**).
- An insurance company creates insurance products by investing the premiums collected from its customers.
- An investment bank sells customised financial products whose payouts are (approximately) covered by its assets.

These are examples of (approximate) **hedging strategies**; much like "delta-hedging" in the Black–Scholes model; FM02.

# Financial institutions



Diversification can be done in both liability-side & asset-side. In asset-side we don't put all eggs in one basket analogy.

Financial institutions benefit from large size ("**economies of scale**").

Examples:

- Retail banking works when the bank has enough customers whose deposits cover the loans on average.
- Having sold a large number of insurance contracts, an insurance company's liabilities become more predictable on average and easier to cover by investments e.g. in fixed income markets.

These are examples of **diversification** on the liability-side (mathematically, the law of large numbers, see FM01).

**Hedging** and **diversification** are fundamental **risk management** techniques.

# Financial institutions



Financial institutions face the problem of **asset-liability management** (portfolio optimization) of constructing portfolios of assets and liabilities that yield good profits with a low risk.

Examples:

- A retail bank tries to optimise the
  - Asset Side → portfolio of assets to invest the customers' deposits in.
  - Liability - Side → liabilities by offering higher interest rates on **term deposits**. (Fixed Saving account)
- An insurance company tries to optimise the combination of
  - Liability - Side → insurance products to sell (this determines the company's liabilities)
  - Asset Side → portfolio of assets to invest the insurance premiums and shareholder capital in. eg Invest in bond, Corporate Bonds, inflatation linked Bonds etc.

# Financial institutions



- **Financial regulation** enforces rules on market participants in order to maintain well functioning financial markets that benefit the economy as a whole.
- More specific goals include
  - Promoting competitive markets by increasing transparency of the market. This includes the move towards centrally cleared markets and preventing issuers of new securities from concealing relevant information as e.g. in the sub-prime crisis in 2007 (more on this later).
  - Promoting stability of the financial system by requiring/encouraging prudential asset-liability management in financial institutions.

# Financial institutions

Different regulatory frameworks for different market sectors

- Banking: Basel II/III
- Insurance: Solvency II
- US derivatives: Commodity Futures Trading Commission
- ...

# Financial institutions



Three "pillars" in regulatory frameworks:

- **Basel II/III**. Recommendations of the international Basel Committee on Banking Supervision,
  1. Quantitative requirements on how much capital a bank has to hold to cover its liabilities at an acceptable level of risk
  2. Requirements on banks' risk assessment and inclusion of supervisory review process.
  3. Disclosure and transparency requirements
- **Solvency II**. EU-directive on regulation of insurance companies
  1. Quantitative requirements on how much capital an insurer has to hold to cover its liabilities at an acceptable level of risk
  2. Requirements on the governance and risk management of insurers
  3. Disclosure and transparency requirements

     ↳ Publish Financial statement every year that includes their Assets & liabilities.

# Financial institutions



For a very liquid instrument/asset like stocks it's easy to value it by looking at Market price but for Non-liquid assets it's valuation creates challenging problem.

A major problem in any regulatory framework as well as in accounting standards is the **valuation** financial liabilities.

- The value of liabilities appears on the **balance sheet** of every (financial) institution.
- The value of liabilities largely determines the amount of capital a company must hold to cover its liabilities.
- Financial institutions often have liabilities that are not traded in secondary markets (especially if they are in the business of creating new financial products).
- Valuation of such liabilities is often a challenging problem (a central topic in **financial mathematics**).
- The famous result of **Black–Scholes** solves the valuation problem for a European option in **complete markets** only.

# Financial institutions

# Depository institutions



Financial institutions are often classified either as **depository** or **non-depository institutions** depending on whether they take deposits or not.

- Examples of depository institutions:
  - Retail banks
  - Savings and Loan associations (US)
  - Building societies (UK) → Take deposits & produce loan (mortgage) for home buyers.
  - Central banks (deposits of commercial banks)
- Deposits are assets to the depositors but liabilities to a depository institution (deposits have to be paid back).
- Depository institutions are important for the payment system ("creation of money") and the functioning of the whole economy; see FMJ Ch 4-5 or HB Ch 2-3.

# Depository institutions



**Asset-liability management** in a depository institution: invest deposits in financial assets that

- provide better returns than what is paid on the deposits.
- are liquid enough to cover sudden withdrawals.

Relevant risks include:

- **Liquidity risk** (assets have to be liquid enough to pay sudden withdrawals),
- **Credit risk** (risk that borrowers won't pay back as agreed)
- **Interest rate risk** (uncertain future market rates may affect asset returns as well as the interest paid on deposits), Interest rate bank receives by giving loans to businesses are tied to Libor or Euribor (mentioned in contract) which is uncertain in future & creates uncertainty of future payouts.
- **Regulatory risk** (banking regulation may change to the disadvantage of the bank).

# Depository institutions



£85000 is guaranteed by the Government (UK Gov) in UK's case.

Ways to manage liquidity risk:

- Invest in liquid assets (not all loans can be sold, i.e. they are **illiquid** assets). eg mortgages, loans to individuals/businesses. (One way to mitigate this is to give loans & mortgages for shorter time. Bank these days give mortgages for 2 years and at end of 2 yrs, you have to reapply.)
- Use currently owned assets as collateral for loans from other institutions or the **central bank** ("lender of last resort")
- Attract additional deposits by offering higher interest rates to depositors.
- Attract more stable deposits: **Time deposits** (certificates of deposit) have a fixed maturity date and pay higher interest than **savings deposits** which can be withdrawn by customers at any time.
- Help from authorities: **deposit insurance** covers individual deposits up to GBP85000 in order to avoid "bank runs" (depositors rushing to withdraw their money when they fear their bank might go bankrupt).

# Depository institutions



Ways to manage credit risk:

- diversification: give loans to several companies/individuals (and hope that they will not default[1] simultaneously).
- credit derivatives: a **credit default swap** (CDS) is an insurance contract against the default of a money borrower (issuer of a bond). CDSs will be discussed in more detail later.
- Analyzing the financial health of a potential lender: retail banks survey the finances of individuals and small businesses before giving loans. **Credit rating agencies** (Moody's, S&P, Fitch, ...) survey larger companies.

Insurance against default of borrower

[1] An agent is said to **default** if it fails to pay back the money as agreed.

# Depository institutions

Ways to manage interest rate risk:

- **Interest rate derivatives**: swaps, FRAs etc. provide cash-flows that depend on market rates (more details will be given later). Appropriate derivative portfolios may offset some of the interest rate risk.
- Invest in assets whose returns are high when the interest paid on deposits is high (**duration** and **convexity matching** are well-known heuristics for doing that approximately. This will be discussed in more detail in connection with bond markets).

The above are examples of hedging.

Like Insurance Contract against unpleasant movement of interest rate in future.

# Depository institutions



Ways to manage regulatory risk:

- Lobbying: try to convince regulators to pose rules that do not hurt your business too much.
- Improving risk management: complying with the current and anticipated future regulation. Sensible risk management is good for the business itself. (Regulation, however, does not always encourage sensible risk management.)

# Depository institutions



Bankruptcy happens when Value of Asset < Value of liabilities

Because of their importance to the financial markets and economy as a whole (see FMJ Ch 4, 5 and HB Ch 2, 3), depository institutions are subject to tight **regulation**. (Banks borrow a lot from lenders & invest in asset in hope of getting higher return than the interest they have to pay to lenders.)

- Commercial banks are typically highly **leveraged**: the ratio of **equity capital** (= value of assets − "value" of liabilities) to the value of assets is often less than $8\%$. (Value of assets is only 8% higher than value of liabilities for typical Commercial Banks)
- The owners of a bank have **limited liability**: they only loose the equity capital if the bank goes bankrupt. (In event of Bankruptcy the Banks Shareholders only lose their share & don't have to pay the liabilities to depositors)
- Higher equity capital increases the incentive of a bank to manage its liabilities safely.
- Minimum requirements on equity capital are set according to the recommendations of the international Basel Committee on Banking Supervision.
- For purposes of liquidity, banks must deposit a minimum percentage of assets at a central bank (**reserve requirement**).
  → Cash

(Most liquid Asset is cash.)

# Investment companies



They have large sums of money to invest meaning easier to trade in securities which may be too illiquid for smaller investors. They Generate well diversified Portfolio for their clients.

Investment companies are financial intermediaries that

- sell shares to investors
- invest the investor's money in financial assets.

Benefits to investors:

- **diversification**: an investment company diversifies investments simultaneously for all its customers,
- **reduced costs** of portfolio management: small fees from all the investors add up to cover the cost of professional portfolio management. (The fees can still be high.)

(small fee for the high expertise)

- **liquidity**: an investor can liquidate a well-diversified portfolio simply by selling the shares of the investment company (instead of selling each security of the portfolio separately).

# Investment companies



leveraging is Special class of shorting with cash.

Investment companies may be classified, for example, according to

- **market sector** they invest in: stocks, bonds, money market, emerging markets, commodities, derivatives, ...
- **investment strategy**:
  - passive (try to follow the market), eg: try to follow SNP500 index which is value of portfolio of 500 shares of companies.
  - active (try to beat the market),
  - hedge funds (less regulated, shorting, leverage, ...),
    borrow asset & sell it & use that money to buy other asset.
    borrow money to invest in assets in hope of getting higher return than interest paymt.
  - cashflow driven investment: trying to invest so as to cover the future cashflows (of e.g. pension expenditure) with investment returns (compare with Black–Scholes).
    delta hedging is a cashflow driven investment
- whether they are
  - **open-end**,
  - **closed-end**.

- In Open-end funds, they have underlying assets which they used previous customers money to invest. If you invest in this fund, they use your money to invest in the underlying asset, thereby increasing portfolio size.
- If you are a new investor & want to buy $x$ amount of shares of this fund then you pay ($x \times$ NAV) amount. When this happens the fund will buy more underlying assets with your money thereby increasing wealth of fund.
- $\frac{X}{NAV} = \frac{\text{Amount of Money}}{\text{unit price of share}}$ = No. of shares for price of $X$ you get.

## Open-end funds

- The total number of shares increases (decreases) if there are more (less) new investments than withdrawals during a day.
- The **net asset value per share** (price per share)

There will be liabilities if they did shorting or borrowed money to invest in the fund.

$$\text{NAV} = \frac{\text{Value of assets} - \text{Value of liabilities}}{\text{Total number of shares}}. = \frac{\text{Net Value of Fund}}{\text{Total no. of shares}}$$

is not affected by new investments/withdrawals:

- Assume that current net asset value is $V$ and the number of shares is $N$ (so $\text{NAV} = V/N$). If there are new investments of $X$ (negative $X$ means withdrawals), then $X/\text{NAV}$ new shares are created for the new investors. NAV remains the same:

$$\text{NAV}_{new} = \frac{V+X}{N + X/\text{NAV}_{old}} = \text{NAV}_{old}\,\frac{V+X}{\text{NAV}_{old}\,N + X} = \text{NAV}_{old}\,.$$

i.e. the value of underlying asset in Market change over time.

- NAV changes only when values of the investments change.

# Investment companies



(Closed to New investments)

## Closed-end funds

- Funded once by an **Initial Public Offering** (IPO).
- Do not accept additional investments.
- To invest in a closed end fund, one must buy shares either at the IPO or from investors who already own shares.
- To buy a share, one may have to pay more/less than the $\mathrm{NAV}$.
- The share price is determined by **supply and demand**.
    - The price may be higher than $\mathrm{NAV}$ e.g. if the fund managers are deemed better than others or if the fund has better access to certain assets than other investors.
    - The price may be lower than $\mathrm{NAV}$ e.g. if investors do not like the current asset mix in the portfolio.
- Closed-end funds are often traded on exchanges. Very much like stocks.

# Investment companies



bid price = Price at which you can sell ETF at Exchange

ask price = Price at which you can buy ETF at Exchange

## Exchange-traded funds (ETF), a.k.a. trackers

- Can be seen as a combination of open-end funds and closed-end funds: ETF shares are traded in exchanges while new shares can be created/liquidated by **"authorized participants"** (AP) → Creator of fund.
- An ETF share gives owenership to a fraction of a portfolio of assets.
- If the **bid-price** of an ETF share is higher than the **cost of buying** the corresponding assets, an AP can profit by buying the assets and creating new ETF shares to sell in the market.
- If the **ask-price** of an ETF share is lower than the **liquidation value** of the underlying assets, an AP can profit by buying ETF shares from the market and selling the corresponding assets.
- As a result, the bid-price [ask-price] of an ETF should be lower [higher] than the buying cost [liquidation value] of the underlying assets. In practice, ETF prices may deviate from these bounds to some extent, due to illiquidity etc.
- An **exchange-traded note** ETN is like an ETF where the AP does not necessarily own the underlying shares. ETN's thus have a **credit risk** resulting from the possible default of the AP.

Price at which you can sell your ETF

- If AP goes bust in ETF, you still own the the underlying assets. Not the case for ETN.

# Investment companies

**Asset-liability management** in an investment company:

- An investment company aims to increase its $\mathrm{NAV}$ (in order to attract more business).
- The liabilities of an investment company may be associated with **short selling** or **leveraging**; see below.
- This is typical in “hedge funds”.
- If $\mathrm{NAV}$ is negative, the company is bankrupt (asset values don’t cover liabilities); see LTCM for a famous example. (Board of directors of LTCM included Myron Scholes and Robert Merton who got the Nobel price in economics in 1997. LTCM collapsed in 1998).
- The development of $\mathrm{NAV}$ depends on the investment strategy and the (uncertain) investment returns.
- The goal is to “maximise” the random variable NAV by choosing a good portfolio (problem of **optimal investment**).

# Insurance companies



Types of insurance contracts/companies:

- **Life insurance**: the claims depend on the time of death of an individual (Pension insurance & life insurance are of same category)
- **Health insurance**: covers medical expenses in the event of illness or injury
- **Property and casualty insurance**: automobile, house, catastrophe (more on this in a guest lecture)
- **Reinsurance**: covers a part of the claims of an **insurance portfolio**.
- ...

• Financial contracts are characterised by their cash flows.

Examples of life insurance products:

- **Annuity** (typical in pension insurance)
  - claims: regular payments from some specified date until death
  - premiums[2]: lump sum payment at the start of contract.
- **Whole life coverage**
  - claims: payment at the time of death.
  - premiums: regular payments over the life of the contract
- **Term insurance** (compare with a credit default swap)
  - claims: payment at death if it occurs within a specified period
  - premiums: regular payments over the life of the contract
- **Unit-linked insurance**
  - claims: payment at death depends on the NAV of a portfolio
  - premiums: regular payments over the life of the contract.

cash flows to buyer

cash flows to seller

Insurer takes your money & invest in a portfolio & if the value of amts in portfolio increases, then you will be paid monthly claims accordingly.

[2] The price of an insurance product is often called the **premium**.

# Insurance companies



The above life insurance contracts are examples of a

- **Swap contract**: an agreement to exchange two sequences of payments, the **premium leg** and the **protection leg**.
- If the premium leg consists of constant payments, it is sometimes called the **fixed leg**.
- In an **interest rate swap** (these will be discussed later), the protection leg is often called the **floating leg**.
- The level of premium payments is negotiated at the beginning of the contract. The payment/(unit of time) in the fixed-leg is called the **swap rate**.
- Both legs of a swap can be uncertain (how about in the above life insurance contracts?)
- Most financial contracts can be viewed as swap contracts.

A **credit default swap** (CDS)

- Protection leg: payment at the time of **default** of a financial institution if it occurs within a given period.
- Premium leg: fixed payments at regular intervals over the life of the contract (end of period or time of default, which ever occurs first).

The payment given at time of default is sometimes called the **loss given default** and the swap rate the **premium rate**.

# Insurance companies



- A CDS provides partial protection e.g. to a bond investor: if the bond issuer defaults, the **protection leg** of a CDS covers part of the default loss (the total amount of which is uncertain).
- CDSs are similar to **term insurance** discussed in the context of life insurance except that the triggering event is the default of a company instead of death of a person.
- Unlike (life) insurance contracts, CDS contracts can be traded even if one is not exposed to the insured risk.
- Heavily traded by American International Group (AIG) (one of the largest insurance companies in the world) which got in trouble in 2008 and was bailed out by the United States Federal Reserve Bank.

# Insurance companies



**Pension funds/pension insurance companies** provide retirement benefits for individuals. Main types

Insurer carries risk

- **Defined benefit** (BD) plan  eg: your Employer sets Pension Scheme for you.
  - claims: an annuity (regular payments until death) upon retirement. The level of annuity is often **indexed**: payments are tied e.g. to the development of the **consumer price index**.
  - premiums: fixed percentage (contribution rate, swap rate) of monthly salary before retirement.

Insurance company invest money from premiums you paid into portfolio of amts. Customers carries all risk.

- **Defined contribution** (DC) plan
  - claims: a single payment at retirement. The amount depends on the $\mathrm{NAV}$ of a portfolio.
  - premiums: fixed percentage of monthly salary before retirement.

Who carries the investment risk in the above contracts?

# Insurance companies



**Asset-liability management** in a life insurance company

- Invest premiums so that their payouts cover the claim payments (this is **hedging**), aka. **cashflow driven investment**.
- Sell contracts whose claims are easy to hedge.
  - In a DB contract, the insurer carries the risk that the investments do not cover the annuity claims (**market risk**, **longevity risk** and possibly **inflation risk**).
  - In a DC contract, the insurer is **completely hedged**. The customer carries all risks. (This isn't real insurance.)
- Insurance companies benefit from large size: uncertainty about the average claims is reduced when the number of insurance contracts increases (mathematically: law of large numbers).

# Insurance companies



Other challenges of insurance companies include

- **Asymmetric information**: customers have more information on the future cash-flows of the insurance contract.
- This may lead to **adverse selection** where those who pose the highest risks are the ones most likely to buy insurance.
- **Moral hazard** refers to a situation where an individual behaves more irresponsibly when protected by an insurance contract.

**Regulation** of insurance companies

- A major difficulty in financial supervision of insurance companies, particularly of DB companies, comes from the fact that their liability cash-flows are highly uncertain and difficult to hedge.
- This complicates the valuation of the liabilities and, consequently, the setting of capital requirements.
- In regulatory frameworks such as the Solvency II directive, valuations are often based on discounting expected future claims. This can be dangerous since
  - such values have little to do with actual hedging costs (and asset-liability management).
  - discounted values of long maturity life insurance liabilities are very sensitive to the discount rates used.

# Trading protocols

# Primary markets



Financial markets can be classified either as

- **primary markets**: where a financial asset is sold for the first time (**issued**)
- **secondary markets**: where one can trade assets after their original issue.

Examples of primary markets:

- In a **bond issue** a government or a corporation sells bonds (borrows cash from investors).
- **Initial public offer** (IPO): equities of a company are sold to investors for the first time (Facebook, Alibaba,. . . ).
- **Privatisation**: IPO of a government-owned company (e.g. British Rail, Royal Mail,. . . ).
- **Securitization** is the transformation of private assets into publicly traded assets (aka **securities**). Examples: mortgage-backed securities (more on this later), asset-backed securities, longevity bonds,. . . .
- In **Lloyd's of London**, new insurance products are created and sold.
  (Market place for Insurers)

# Primary markets

**Investment banks** are often involved when new securities are issued. Possible roles of investment banks include

- advise on the issuing procedure.
- advertisement and distribution of the newly created securities to investors.
- buying the securities from the issuer and selling them to investors. The investment bank is then said to be the **underwriter** of the securities. The investment bank makes a profit (loss) if it manages to sell the securities for a higher (lower) price.

eg: IPO of Company.

# Primary markets

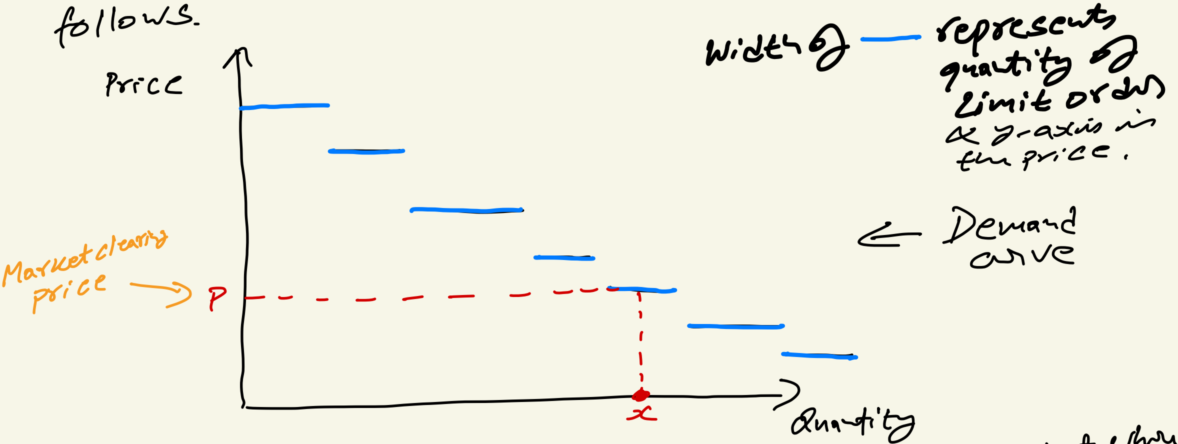
One sided auction

Securities are usually issued through an **auction** where the price is determined by **supply and demand**.

- The issuer may invite offers for the entire issue and then sell **everything** to the investor who offered the highest **price**
- Alternatively, the issuer may ask for offers for buying **fractions** of the issue. In this case,
  - the investors submit buying offers with limits on **price** and **quantity**.
  - the issuer sells fractions of the issue to bidders who offered highest prices.

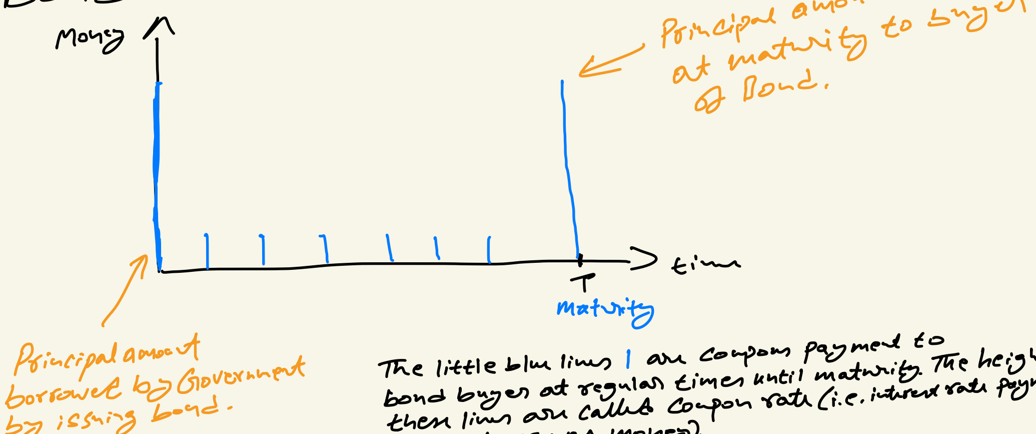
  Common way to issue e.g. government bonds. Successful bidders may be asked to pay
  - what they offered to pay
  - the lowest successful price (**market clearing price**).

# If a Government wants to sell Bond of nominal value ₤1 billion (i.e. borrow ₤1 billion) from public, then they invite <u>limit orders</u> from investors (i.e. offers with limits on price & quantity). This is plotted in a <u>Demand Curve</u>. The auction mechanism works as follows.




Officer at Government office look at the cutting point where they will raise ₤1 billion, i.e. x. The price corresponding to x i.e. P is the lowest successful price (<u>market clearing price</u>) for which the bond will be priced & sold. Note at point x, they will raise ₤1 billion. All investors will get the market clearing price in the auction.

# Bond




The little blue lines | are coupon payment to bond buyer at regular times until maturity. The height of these lines are called coupon rate (i.e. interest rate payment on the borrowed money).

# Another Auction Mechanism

This time the offers investors submit to Government are also pair of numbers but this time it will be quantity & Interest rate they demand on loan (coupon rate).

Note, the investor offering the lowest rate will be the most attractive investor to the Government.

This time the plot looks like below:

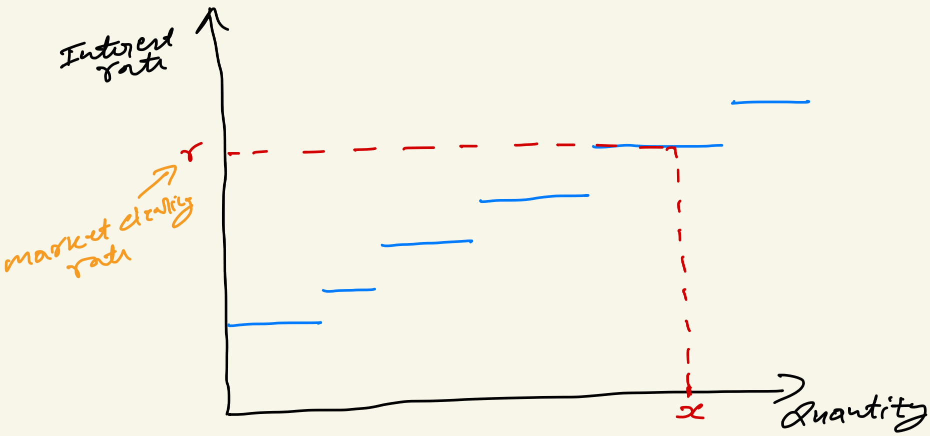



Fig: Supply Curve

This is a increasing curve. Again we look at cut off point when the Government will raise £1 billion denoted by $x$. The rate corresponding to $x$ i.e. $r$ is the market clearing rate. This is how the auction works & all investors will get the same market clearing rate.

# Secondary markets (stock Exchanges)



A **secondary market** is where already issued financial securities are traded.

- The possibility to sell a security in a secondary market
  - makes the security more valuable to its owner.
  - is likely to increase the prices offered for the security already in the primary market. This benefits the original issuer.
- Secondary markets can be **exchanges** or **over the counter** (OTC) markets.
  - In an exchange, all available buying and selling offers are considered simultaneously and the price is set according to supply and demand. A well-known example: NYSE. LSE etc.
  - In an OTC market, traders seek counterparties in a less organized manner. A well-known example: NASDAQ.
  - Standardized products (stocks, European options, futures contracts, . . . ) which are interesting to many investors are often traded in exchanges. Customized products are traded OTC.

# Secondary markets



(you have to big fairly big player to have direct access to stock exchange)

Some terminology:

individual investor

- **Broker** is a middleman who connects sellers and buyers. To trade exchange traded products, smaller agents usually have to go through a broker. (If you have a trading account, it is most likely provided by a broker)
- **Market maker** offers buying and selling prices to other market participants. Exchanges sometimes employ market makers to increase liquidity.
- **Specialist** (e.g. in NYSE) matches buying and selling offers according to an auction mechanism but may also trade on his/her own account.

Market Makers are employed by Exchanges to increase liquidity. They are individuals or institutions who offers bid & ask price & post them for investors.

People you see in NYSE.

The above roles are becoming redundant as markets move towards more transparent (and fairer) market mechanisms and automatic execution of trades.

# Double auctions Mechanism (used by LSE)

- Most Important Market Mechanism
- Secondary Market: Plenty of Buyers & sellers.



- Most exchanges are **double auctions** where both buyers and sellers bid for securities and the most generous offers are selected to participate in trades.
- Exchanges can be **call auctions** or **continuous auctions**.
  - In a **call auction**, the market is cleared once at the end of a bidding period by matching the maximum amount of buying and selling offers.
  - In a **continuous auction**, a new offer is matched immediately (within a fraction of a second) with existing offers if possible. Otherwise, the new offer is added to the **limit order book**.

# Double Auction Mechanism in Call Auction

Call Auction happen at the start & end of trading day during which no other trades happen. There is certain time period (20-30 mins) during which offers from buyers & sellers are invited & at the end of this time period the market is cleared, trades happen & only then continuous trade begins. The reason to do call Auction at the end of trading day is to set the closing price of the day so that it is difficult for individual traders to manipulate the closing price.

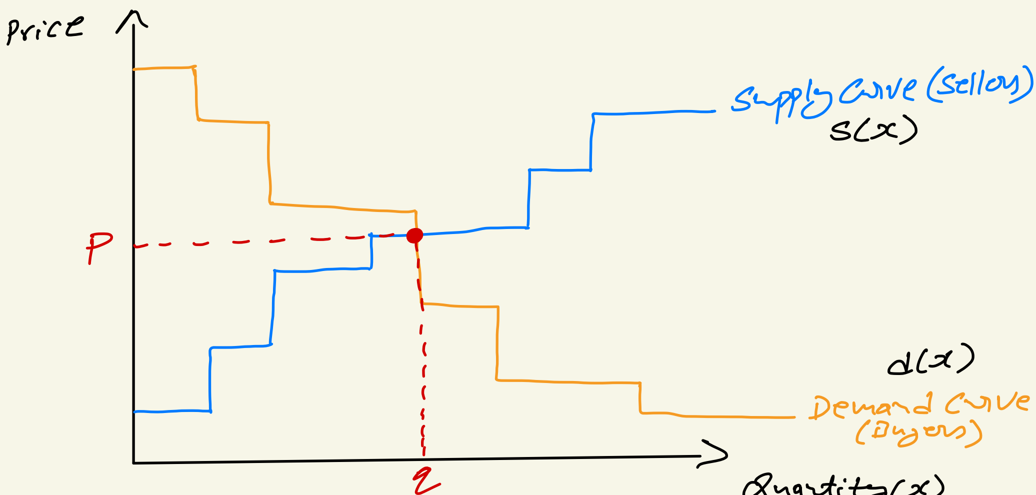



Fig: Call Auction

During the Call option period (20-30 mins) no trades happen & all offers from buyers and sellers are submitted & plotted as Demand & Supply Curve respectively. At the end of Call option period, the Exchange clears the Market. The Market clearing price is where Demand & Supply Curve intersect.

$p$ = Market clearing Price

$q$ = Quantity traded

# Double auctions



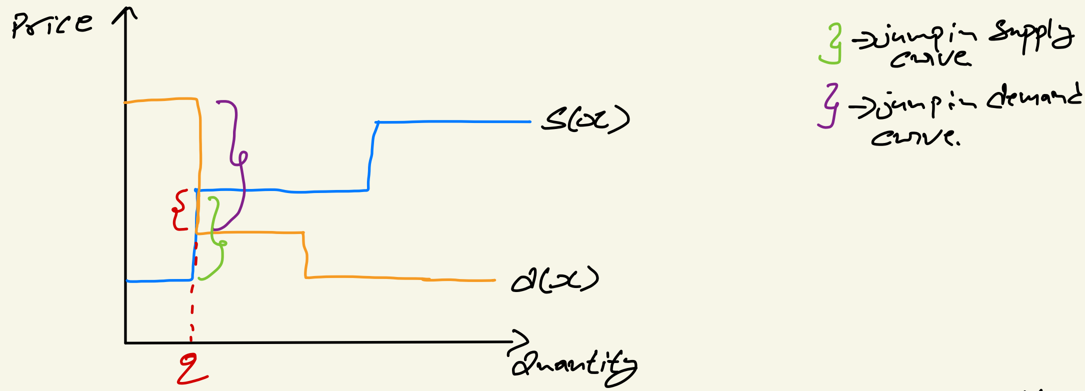
$\bar{x}$ is the largest quantity s.t. supply curve $s(x)$ is less than or equal to demand curve $d(x)$.

## Call auction

- Market participants submit buying or selling offers with limits on quantity and unit price (**limit orders**).
- Arranging selling offers in an increasing order according to the offered price, we get a piecewise constant nondecreasing function $x \mapsto s(x)$, the **supply curve**, that gives the **marginal price** when buying $x$ units of the security.
- Similarly, arranging buying offers in a decreasing order according to the offered price, we get a piecewise constant nonincreasing function $x \mapsto d(x)$, the **demand curve**, that gives the **marginal price** when selling $x$ units of the security
- The market is **cleared** by matching maximum number of trades: $\bar{x} = \sup\{x \mid s(x) \leq d(x)\}$.
- $[\lim_{x \nearrow \bar{x}} d(x), \lim_{x \searrow \bar{x}} d(x)] \cap [\lim_{x \nearrow \bar{x}} s(x), \lim_{x \searrow \bar{x}} s(x)]$ is the interval of **market clearing prices**.

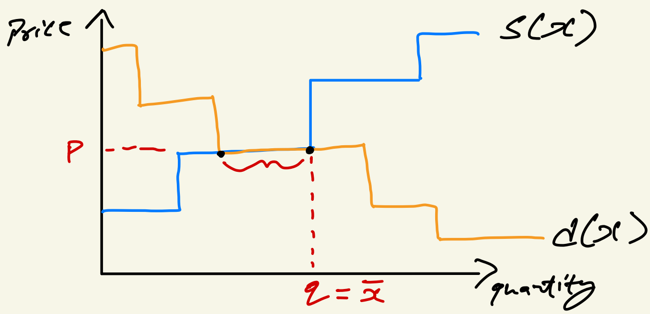
- $[\lim_{x \nearrow \bar{x}} d(x), \lim_{x \searrow \bar{x}} d(x)] \cap [\lim_{x \nearrow \bar{x}} s(x), \lim_{x \searrow \bar{x}} s(x)]$ is the interval of **market clearing prices**.

This says the market clearing price may not be unique. Lets see an illustration:




In this case at quantity q, it is not clear what the market clearing price is.
The left of ∩ in above formula is the jump in the demand curve & right of ∩ is the jump in supply curve i.e. left&right limit as quantity approaches q.
So, the Market clearing prices is anything in the intersection of } and } i.e. the jumps.

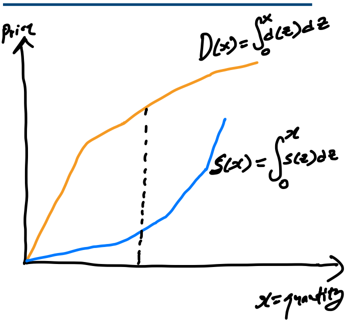
# Market clearing Quantity is Unique




$\bar{x}$ is unique even though it has overlapping region since $\bar{x} = \sup\{x \mid s(x) \le d(x)\}$ & we choose the maximum which is the rightmost point.

# Double auctions






Market clearing can be interpreted as finding the "social optimum":

(Area under the graph of supply&demand curve from 0 to Market Clearing quantity)

- Consider the functions

$$S(x) = \int_0^x s(z)dz \quad \text{and} \quad D(x) = \int_0^x d(z)dz.$$

- $S(x)$ is the cost of buying $x$ units of the security from willing sellers and $D(x)$ is the revenue of selling $x$ units to willing buyers.
- Market is cleared by minimising the difference $S(x) - D(x)$ or, equivalently, maximizing the "consumer surplus" $D(x) - S(x)$. (profit)
- The functions $S$ and $D$ are not differentiable so the justification requires a bit of **convex analysis**.

# Digression to convex analysis

A real-valued function on an interval $I \subseteq \mathbb{R}$ is **convex** iff

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

← line above the f'n.

whenever $x_1, x_2 \in I$ and $\alpha_1, \alpha_2 > 0$ are such that $\alpha_1 + \alpha_2 = 1$.

- Classical analysis is mostly concerned with smooth functions on smooth manifolds.
- In practice, however, optimization problem often involve nonsmooth functions and inequality constraints.
- In many applications, functions involved are **convex**, a property far more useful than smoothness when one needs to optimise.
- Convexity is a "robust" property: it is preserved in sums, compositions, infima, ...
- Classical reference: R.T. Rockafellar, Convex Analysis, 1970.
- Below, we only review differential theory in one dimension.

# Convex function

If we take any two points in a convex graph & draw a straight line connecting them then the line will always be above the graph for convex f'n.

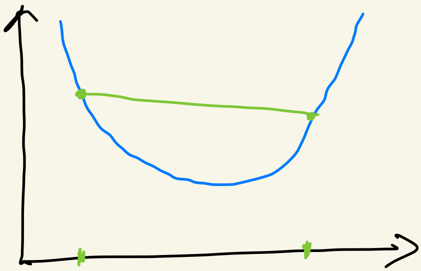


Fig: Convex f'n

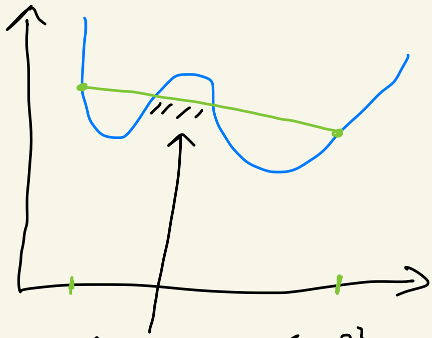



Fig: Non-Convex f'n

Convexity is very important property in Optimization problem.

# Digression to convex analysis



**Theorem 1.** *For a real-valued function $f$ on an interval $I$, the following are equivalent*

*(a) $f$ is convex,*

*(b) There is a nondecreasing function $\phi : \operatorname{int} I \to \mathbb{R}$ such that*

$$f(x) = f(\bar{x}) + \int_{\bar{x}}^{x} \phi(z)dz$$

*for all $\bar{x}, x \in \operatorname{int} I$,*

*(c) $f$ is differentiable on $\operatorname{int} I$ except perhaps on a countable set, its derivative $f'$ is nondecreasing and (b) holds with $\phi = f'$.*

**Corollary 2.** *If $f \in C^2$ and $f'' \geq 0$, then $f$ is convex.*

Theorem 1 may be viewed as the "fundamental theorem of calculus" in the convex case. Convexity allows for a simple proof.

# Digression to convex analysis



*Proof of* $(b) \implies (a)$. Let $x_i \in I$ such that $x_1 < x_2$ and $\alpha_i > 0$ such that $\alpha_1 + \alpha_2 = 1$. With $x = \alpha_1 x_2 + \alpha_2 x_2$, we have

$$f(x) - f(x_1) = \int_{x_1}^{x} \phi(z)dz \leq \phi(x)(x - x_1)$$

and

$$f(x_2) - f(x) = \int_{x}^{x_2} \phi(z)dz \geq \phi(x)(x_2 - x).$$

Multiplying the inequalities by $\alpha_1$ and $-\alpha_2$ and adding up gives

$$\begin{aligned} f(x) &\leq \alpha_1[f(x_1) + \phi(x)(x - x_1)] + \alpha_2[f(x_2) - \phi(x)(x_2 - x)] \\ &= \alpha_1 f(x_1) + \alpha_2 f(x_2). \end{aligned}$$

Thus, $f$ is convex.

# Digression to convex analysis



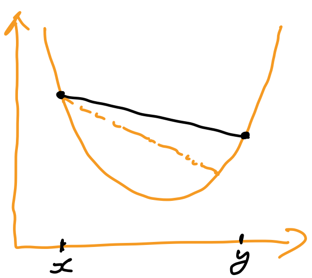



Fig: Right directional derivative $f'_+(x)$

*Proof of* $(a) \implies (c)$. Convexity of $f$ implies that the difference quotient $(f(y)-f(x))/(y-x)$ is nondecreasing both in $y$ and $x$, so the **directional derivatives**

left derivative (directional)

Right derivative (directional)

$$f'_-(x) := \lim_{y \nearrow x} \frac{f(y)-f(x)}{y-x} \quad \text{and} \quad f'_+(x) := \lim_{y \searrow x} \frac{f(y)-f(x)}{y-x}$$

y approaches x from right

exist, are nondecreasing, $f'_-(x) \le f'_+(x)$ and

$$f'_+(x) \le \frac{f(y)-f(x)}{y-x} = \frac{f(x)-f(y)}{x-y} \le f'_-(y) \quad \forall y > x.$$

We thus have $f'_-(x) \le f'_+(x) \le \lim_{y \searrow x} f'_-(y)$, where all the functions are nondecreasing and thus, continuous and equal to each other except perhaps at countably many points.

# Digression to convex analysis



*Proof of* $(a) \implies (c)$ *continued.* Since the directional derivatives are nondecreasing, $f'_+(x) \le \lim_{y \searrow x} f'_+(y)$. On the other hand, convexity implies that $f$ is continuous on $\operatorname{int} I$, so that

$$\frac{f(y)-f(x)}{y-x} = \lim_{z \searrow x} \frac{f(y)-f(z)}{y-z} \ge \lim_{z \searrow x} f'_+(z)$$

and thus, $f'_+(x) \ge \lim_{z \searrow x} f'_+(z)$. Combining this with the inequalities on the previous page, we see that $f'_+$ is a right-continuous function and

$$f'_+(x) = \lim_{y \searrow x} f'_-(y) = \lim_{y \searrow x} f'_+(y).$$

By a similar argument, $f'_-$ is left-continuous and

$$f'_-(x) = \lim_{y \nearrow x} f'_-(y) = \lim_{y \nearrow x} f'_+(y).$$

# Digression to convex analysis

*Proof of* $(a) \implies (c)$ *continued.* Let $\phi$ be such that $f'_- \le \phi \le f'_+$ and

$$h(x) = f(\bar{x}) + \int_{\bar{x}}^{x} \phi(z)dz.$$

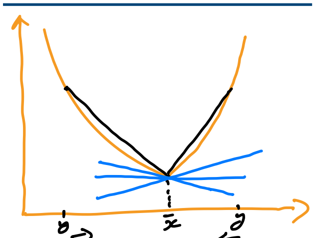
Since $\phi$ is nondecreasing,

$$\phi_+(x) \le \frac{\int_x^y \phi(z)dz}{y-x} \le \phi(y),$$

for all $y > x$. In other words,

$$\phi_+(x) \le \frac{h(y)-h(x)}{y-x} \le \phi(y),$$

so $h'_+ = \phi_+$. Similarly, $h'_- = \phi_-$. Since $\phi_+ = f'_+$ and $\phi_- = f'_-$ (see the previous page), the directional derivatives of the function $f - h$ vanish everywhere so it must be a constant function. $\square$




If we look at non-differential point $\bar{x}$, a family of linear f'ns which touches the f'n at point $\bar{x}$ & is always below the f'n denoted by blue lines. The slope of these blue lines is called subgradient of f at $\bar{x}$.

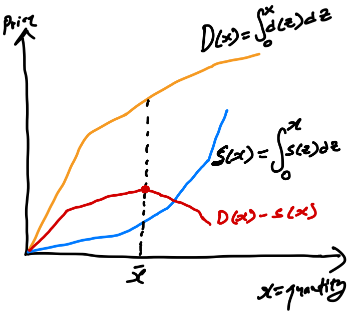
- Note that $\phi$ in (b) of Theorem 1 is not unique: any $\phi$ with $f'_- \le \phi \le f'_+$ will do; see the proof of $(c) \implies (b)$.
- A $v \in \mathbb{R}$ is a **subgradient** of $f$ at $\bar{x}$ if $v \in [f'_-(\bar{x}), f'_+(\bar{x})]$, or equivalently,

$$f(x) \ge f(\bar{x}) + v \cdot (x - \bar{x}) \quad \forall x \in I.$$

- The set of subgradients of $f$ at $\bar{x}$ is known as the **subdifferential** of $f$ at $\bar{x}$ and it is denoted by $\partial f(\bar{x})$.
- A convex function $f$ is differentiable at $\bar{x}$ if and only if $\partial f(\bar{x})$ is a singleton and then, $\partial f(\bar{x}) = \{f'(x)\}$.
- Note that, an $\bar{x} \in I$ minimizes $f$ over $I$ if and only if $0 \in \partial f(\bar{x})$. (i.e. 0 is subgradient of f)

# Double auctions






• is the maximum of $D(x)-S(x)$ when 0 belongs to subdifferential i.e. $0 \in \partial(S-D)(\bar{x})$

Back to market clearing:

- Since the functions $s$ and $-d$ are nondecreasing, Theorem 1 implies that the functions $S$ and $-D$ are convex.
- Thus, an $\bar{x}$ minimizes $S(x) - D(x)$ iff $0 \in \underbrace{\partial(S - D)(\bar{x})}_{\text{collection of all subgradients of the function.}}$.
- We have

$$S(x) - D(x) = \int_0^x (s(z) - d(z))dz$$

so $0 \in \partial(S - D)(\bar{x})$ means that

$\bar{x}$ satisfies this condition

$$\overbrace{0 \in [\underbrace{s_-(\bar{x}) - d_-(\bar{x})}_{\text{left limit of difference}}, \underbrace{s_+(\bar{x}) - d_+(\bar{x})}_{\text{right limit of difference } S-d}].}$$

- Note that $\bar{x} := \sup\{x \,|\, s(x) \leq d(x)\}$ is the largest of the minimizers of $S - D$.

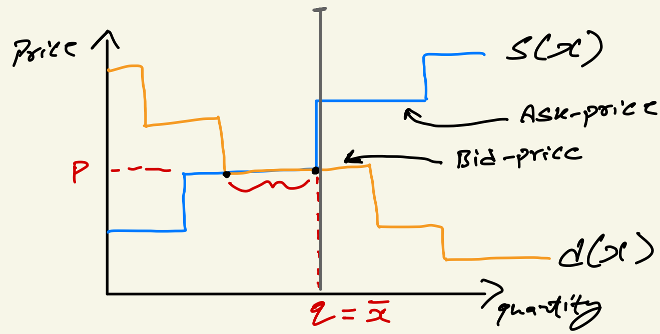
# Double auctions



- After market clearing, prices of the remaining selling offers are all strictly higher than prices of the remaining buying offers.
- The offers remaining after market clearing are recorded in the so called **limit order book** (LOB).
- The LOB gives the marginal prices for buying or selling a given quantity at the best available prices.
- Interpreting negative purchases as sales, the marginal prices can be incorporated into a single function $x \mapsto s(x)$ giving the marginal price for buying $x$ units of the security.
- If $x$ is negative, buying $x$ shares is interpreted as selling $-x$ units of the security.
- Flatter the curve $s$, more **liquid** the market.
- Traditional math finance (e.g. Black–Scholes) assumes perfectly liquid market which corresponds to a constant $s$.

# What happens after Call auction i.e. after the Market is cleared?

⇒




At the end of call option everything to left of T line is taken away (those offers disappear). What's left goes into record which is public to the traders.

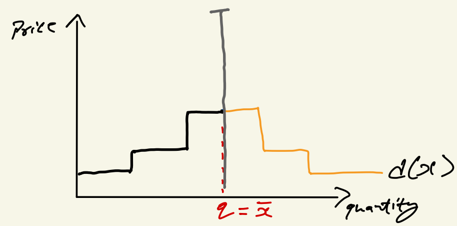
After Market clearing, the best available price at which you can buy the share is called ask price & the best available price to sell the share is called bid price.

These ask & bid price are exactly what you see in the Bloomberg terminals. What you don't see is the 2nd or 3rd best prices in the terminals but they are there in the exchange.

So After call auction, these ask & bid prices that are recorded are published to the traders. This is called the Limit order Book which is the record of all existing buying & selling offers that has not been matched yet by other traders.

- Selling 2 shares means buying −2 shares & selling −2 shares means buying 2 shares.

  eg: buy $x$ amount means selling $-x$ i.e you flip the demand curve at gray line.




This increasing curve gives you the available offer for buying or selling the shares.

Now the Limit order book is described by single piecewise increasing curve.

# Double auctions

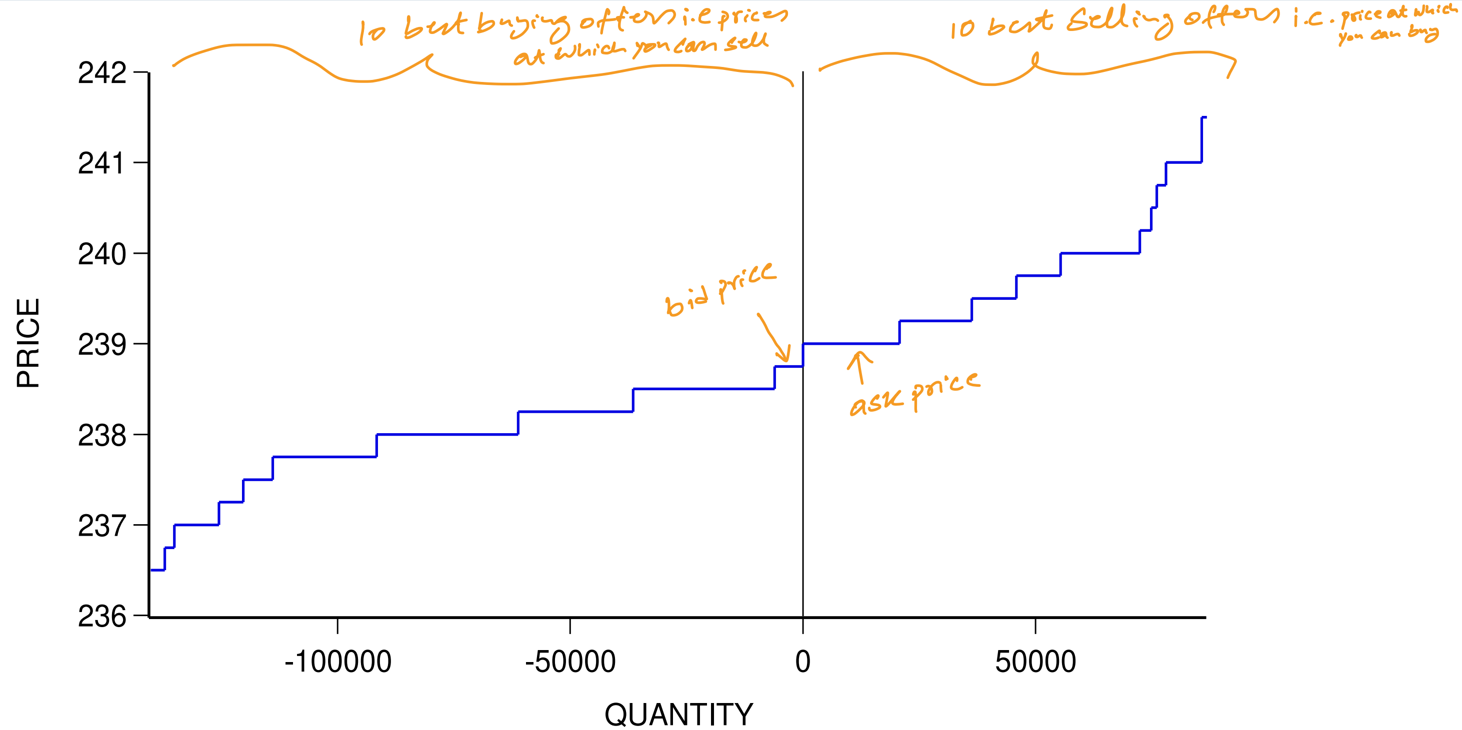



Figure 1: Marginal price curve for shares of the Danish telecom company TDC A/S observed in Copenhagen Stock Exchange on January 12, 2005 at 13:58:19.43. The horizontal axis gives the cumulative depth of the book measured in the number of shares. The prices are in Danish krone. The data was provided by OMX market research.

# Double auctions



In Continuous Trading, whenever a new order comes in, it's either executed immediately or if the limit price is not generous enough it will not be executed but instead will be added to the Limit order Book & the increasing piecewise limit curve will be updated.

## Continuous auction

- Many stock exchanges operate in "continuous" time: market is cleared very frequently (e.g. every $10^{-6}$ seconds).
- The limit order book (first created with a call auction at the beginning of a trading day) is updated almost immediately when a new order arrives.
- Limit sell orders above the best ask-price and limit buy orders below the best bid-price **increase liquidity**.
- A **market order** is an order to buy/sell a given amount at the best available prices. Market orders **reduce liquidity**.
- A trading day ends with another call auction: the last clearing period lasts several minutes instead of $10^{-6}$ seconds.
- The purpose of the closing call auction is to prevent the manipulation of **closing prices** which are often used in accounting and regulation.

# Double auctions



- Liquidity is the constancy of supply curve & illiquidity is the Non-constancy of supply curve.

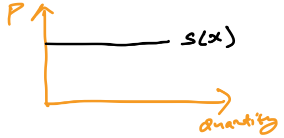



Figure showing perfect liquid Market as s(x) is constant.

The steeper the s(x) curve the more illiquid as you pay more to buy large quantities.

- Given a LOB with marginal price curve $s$, the cost of a market order of $x$ (sell order if $x < 0$) shares is given by

$$S(x) = \int_0^x s(z)dz.$$

- By Theorem 1, $S$ is a **convex** function of $x$.
- If $s$ is nonnegative, $S$ is nondecreasing.
- Why would $s$ be negative?
- The state of a limit order market is described by $s$ or, equivalently, by $S$.
- In order to model limit order markets, one needs to model the future evolution of $S$. This is much more demanding than modelling a single price in liquid markets (modelling a function instead of a real number).

# Double auctions

- Limit order books are often described by the "supply function"

$$\tilde{S}(x) = S(x)/x.$$

  This gives the price per share when buying $x$ shares.
- The supply function is always nondecreasing: if $x_1 \leq x_2$, convexity gives

$$S(x_1) = S\left(\frac{x_1}{x_2}x_2 + (1 - \frac{x_1}{x_2})0\right) \leq \frac{x_1}{x_2}S(x_2) + (1 - \frac{x_1}{x_2})S(0),$$

  where $S(0) = 0$ so that $\tilde{S}(x_1) \leq \tilde{S}(x_2)$.
- However, there are nondecreasing functions which do not correspond to nondecreasing marginal price functions. An example:

$$\tilde{S}(x) = \begin{cases} x & \text{if } x \leq \bar{x}, \\ \bar{x} & \text{if } x \geq \bar{x}. \end{cases}$$

  What goes wrong here? Hint: Theorem 1.

# Double auctions



- The above discussion about auctions was concerned with offers on **price** and quantity. This applies to products like stocks and options for which the full price is paid when a trade is agreed.
- However, most of what was said applies just as well to auctions concerning **swap contracts**, where payments are made in the future in terms of a **premium leg**.
- In case of a credit default swap (CDS), for example, a buy limit order specifies the number of contracts and the **premium rate** one is willing to pay over the life of the contract.

↳ In this auction you offer long or short positions with rate & no. of contracts.

# Double auctions



Another example:

- **Betting auctions** are double auction markets for financial products that pay $1$ pound if certain event occurs and $0$ otherwise ("digital options"). Typical events are
  - Team X wins a football game,
  - Candidate X wins the presidential election,
  - ...
- Typically, a limit order to sell $q$ units of such a product for unit price $s$ is quoted as offering a multiplier of $1/s$ for a cash investment up to $qs$ if the event occurs.
- Examples of betting exchanges include Betdaq and Betfair.

# Double auctions



- **High-frequency trading** and **algorithmic trading** are new areas where one tries to anticipate the market behaviour by studying the dynamics of the limit order book and **order flow** (the arrival of orders).
- Predictable phenomena in the limit order book are used to device automated trading strategies.
- High-frequency trading has been criticised for giving an unfair advantage to traders who can afford to buy high-speed connections to a trading system.

# Short positions

(Owe -ve unit of Asset)

(Sell Security)



One is said to have a **short position** in an asset if one **owes the asset** ("owing" is the opposite of "owning").

- Borrowing cash creates a short position in cash.
- Buying assets with borrowed cash is known as **leveraging** (or **buying on margin**).
- Leveraging is common in banking business.
- Leveraging amplifies both profits and losses.

Margin Account is the bank Account with Cash with your broker. When you borrow money from Margin Account & buy asset, your Margin Account will be -ve. This is called buying on Margin.

# Short positions

- Selling a borrowed asset is known as **short selling**.
- Short selling allows for the possibility of "selling high and buying low" (in that order).
- However, if the price rises after short sales, one may have to buy higher than the selling price.
- When buying an asset, one faces the risk of losing the invested money but in short sales, the **loss may be unbounded**.
- An example: "Market neutral" investment strategies take long positions in "relatively under-priced" securities and short positions in "relatively over priced" securities. Aim at being immune to general market movements.

# Short positions



- Short positions create financial **liabilities**:
  - Borrowed cash has to be paid back (with interest) in the future.
  - When short selling stocks, one has to deliver the dividend payments and the stock price when closing the position.
  - When short selling bonds, a government has to deliver the coupon payments and the principal.
  - When selling insurance, and insurance company has to deliver the claim payments.
- Just like one can get out of the short position in a stock, by buying stocks from the market, an insurance company can buy **reinsurance** on its insurance portfolio (this is known as a **buy-in**).

Buy-Out ⇒ When insurance company gives the whole portfolio to the reinsurance company to handle.

# Short positions

- Regulation poses restrictions on short selling.
- **Margin requirements** specify a minimum ratio for the "market values" of an investors net assets and liabilities.
- Margin requirements are often defined by posing minimum allowed values for either

$$\frac{A-L}{A} \quad \text{or} \quad \frac{A-L}{L},$$

  where $A$ is the "market value" of assets (long positions) and $L$ is the "market value" of liabilities (short positions).
- Note that both amount to requiring that $cA \geq L$ for some constant $c < 1$.
- Again, for illiquid assets/liabilities, the values $A$ and $L$ may be difficult to estimate.

# Market sectors

- **Primitive assets**: equities, money markets, bonds, mortgages, currencies, insurance, commodities, real estate,
- **Derivative assets**: options, futures, swaps, American options, exotic options,. . .

# Stock market

- **Common stock** represents ownership in a company.
- Stock investments are characterized by **dividends** received while owning the stock and the **price** received when selling.
- Dividend payments are based on the company's yearly earnings and the amount is determined by the **board of directors** that represents the owners.
- In case of **bankruptcy**, the owners of common stock own what is left after the company has paid back all its **debt**. Debt with highest **seniority** is paid first, **subordinated** debt next.
- **Capital structure of a firm** refers to the mix of a company's debt and **equity** (the asset value − liability value).
- **Market capitalisation** (value) of a company is the "market price" of a share multiplied by the total number of shares.
- Stocks are heavily traded by institutional investors such as pension funds and life insurance companies.

# Stock market



- **Private equity** refers to common stock of companies that are **not publicly traded in exchanges**.
- Private equity is less liquid than equity of publicly traded companies.
- Most institutional investors invest in private equity through **private equity funds** the shares of which may be much more liquid than direct private equity.
- Private equity investment is risky but it can result in high returns in the long run if the company becomes publicly traded or gets bought by another company.
- For example, Facebook was valued at 104 billion USD in its IPO in February 2012. Facebook paid 19 billion USD for WhatsApp in February 2014.

# Stock market

- Most **publicly traded stocks** are traded in **electronic limit order markets** where limit orders are automatically matched by market orders or recorded in limit order books.
- Market prices of stocks (like the prices of all assets) are based on supply and demand.
- Professional traders often base their buying and selling offers on "fundamental analysis" which studies
  - financial statements of companies
  - supply and demand for the company's products
  - macroeconomic factors (that may affect a company's earnings as well as supply and demand of its shares)
  - managers of companies
  - ...

# Stock market



**Mathematics** used in stock market analysis includes

- Statistical analysis of market prices of stocks (see FM05).
  - Dynamic properties of market prices.
  - Distributional properties of prices at a given point in time (skewness, heavy tails, . . . ).
  - Combinations of the above
  - . . .
- Portfolio optimization. This involves
  - statistical models of stock prices
  - risk preferences of investors
  - numerical optimization algorithms.

Mathematics finds more uses in market sectors where the cash-flows of traded products have more structure (fixed income, derivatives, . . . )

# Money market



(Inter Bank lending)

- **Money market** refers to short-term (less than a year) lending/borrowing of money.
- Lending can be seen as a purchase of "zero coupon bonds" (bonds without coupon payments) from the borrower.
- A **zero-coupon bond** (or a "discount bond") with maturity $T$ is a financial instrument that pays one pound at time $T$.
- Examples:
  - **Treasury bills** are issued by governments. (ZCB issued by Government)
  - **Interbank deposits** are unsecured (without collateral) loans issued by banks (overnight, LIBOR, EURIBOR: AA-rated or higher).
  - **Repurchase agreement** is secured lending backed by assets.

    ↳ with collateral

# Money market

There are many different ways to define interest rate.



- Price is whats quoted in the market. The interest rate is a mathematical definition based on the price. There are many different mathematical ways to define interest rate.

- Prices of bonds are often quoted in terms of **“interest rates”**.
- Rates associated with zero-coupon bonds are called **zero rates**.
- There are **many ways to define zero rates**.
- The zero rate with maturity $T$ and **compounding frequency** $m$ is defined as the unique solution $Y_T$ of the equation

(Interest rate obtained by Discrete Compounding with finite frequency.)

$$P_T \left(1 + \frac{Y}{m}\right)^{Tm} = 1 \iff P_T = \left(1 + \frac{Y}{m}\right)^{-Tm},$$

where $P_T$ is the “price” of a zero coupon bond with maturity $T$.
- **Interpretation**: Investing $P_T$ pounds with annual rate of interest $Y_T$ paid regularly $m$ times per year results in one pound in $T$ years.
- Terminology: Annual compounding: $m = 1$, semi-annual: $m = 2$, quarterly: $m = 4$, monthly: $m = 12$, etc
- Note that

$$\lim_{m\to\infty} \left(1 + \frac{Y}{m}\right)^{-Tm} = e^{-YT}.$$

# Money market

- The continuously compounded **zero rate** $Y_T$ of maturity $T$ is defined by

$$P_T = e^{-Y_T T} \iff \boxed{Y_T = -\frac{\ln P_T}{T}.}$$

CTS Compounded Zero Rate

- Continuously compounded **forward rates** $f_t$ are defined in terms of $P_T$ by

$$P_T = \underbrace{e^{-\int_0^T f_t dt}}_{\text{discounted by forward rate}} \iff \boxed{f_T = -\frac{d \ln P_T}{dT}}$$

This only makes sense if Price is the f'n of CTS variable T. But in practice there are only finite numbers of Bonds & maturity dates for which you can find prices. But this CTS assumption makes modelling simple.

(for this to make sense, one would need prices of zero coupon bonds for all $T$).

- Thus

$$P_s = P_t e^{\int_s^t f_u du} \quad \text{for } s < t.$$

- The **zero curve** $s \mapsto P_s$ is decreasing iff $f_t > 0$ for all $t$.

Bond prices as f'n of maturity.

# Money market

Libor zero rates

mainly used as underlying risk factor for Interest Rate derivatives



- The **LIBOR rate $Y_T$** of maturity $T$ is defined by

T in years.

$$P_T = \frac{1}{1 + Y_T T} \iff Y_T = \frac{1}{T}\left(\frac{1}{P_T} - 1\right).$$

- Note that $\ln P_T = -\ln(1 + Y_T T) \approx -Y_T T$, so the LIBOR rate can be seen as an approximation of the continuously compounded rate.
- **LIBOR forward rates $F_{s,t}$** for $s < t$ are defined by

Here we look at ZCB prices of two different maturity s & t.

$$P_t = P_s \frac{1}{1 + F_{s,t}(t - s)} \iff F_{s,t} = \frac{1}{t - s}\left(\frac{P_s}{P_t} - 1\right).$$

- LIBOR rates are used to define the payouts of interest rate derivatives (to be studied later).

# Money market



- Debt instruments are subject to **credit risk** i.e. the risk that the lender does not make the payments as agreed.
- Higher the credit risk, lower the demand and market price of a debt instrument.
- Similarly, a lender with high credit risk has to offer cheap zero coupon bonds to borrow money.
- Since the price is a decreasing function of the zero rate, higher credit risk results in higher zero rates.
- **Credit ratings** (e.g. AAA, AA, . . . ) reflect trustworthiness of a lender as judged by private companies such as Standard & Poor's, Moody's Investor Service and Fitch Ratings.
- Differences between zero rates of poorly and highly rated companies are known as **credit spreads**.

# Money market



ZCB Price as f'n of maturity.

- The function $T \mapsto P_T$ is known as the **zero curve**.

zero Rate as f'n of maturity.

- The function $T \mapsto Y_T$ is known as the **yield curve** (recall that there are different definitions of $Y_T$).
- Note that $Y_T > 0$ if and only if $P_T < 1$ (a pound today is worth more than a pound tomorrow).
- More generally, $F_{s,t} > 0$ if and only if $P_t < P_s$ when $s < t$ (a pound earlier is worth more than a pound later).
- Traditional models of financial mathematics describe forward rates as positive processes.
- Recently, however, we have observed negative rates in the market.

# Money market

Banks fill out questionair on ask/bid rate price. Not based on actual traded instrument



- Note first that LIBOR/EURIBOR rates are not "real": they are not rates at which money is lent/borrowed:
  - "Panel banks provide daily quotes of the rate, rounded to two decimal places, that each panel bank **believes** one prime bank is quoting to another prime bank for interbank term deposits within the euro zone"; see the Euribor Code of Conduct.
  - If a bank's assets/liabilities depend on LIBOR rates, they may not want to report what they really believe; see the "LIBOR fixing scandal"
- The problems with LIBOR have motivated the **"LIBOR transition"**: LIBOR rates cease to exist after 2021 `https://www.fca.org.uk/markets/libor`.
- EONIA/SONIA (Euro/Sterling OverNight Index Average) are average overnight rates of unsecured lending **transactions** (actual trades).
- EONIA/SONIA is harder to manipulate than LIBOR.
- More info on EURIBOR and EONIA can be fund at `http://www.emmi-benchmarks.eu`

# Money market



Think of -ve interest rate of eonia i.e. overnight rate as <u>storage</u> <u>cost</u> of putting banks money.

- Recently, EONIA has been **negative**:
  `http://www.euribor-rates.eu/eonia.asp`
- Why would a bank lend money to another bank at a negative rate rather than hold money itself?
- What is "money"? (government "backed" money, M0).
- **Holding money** means putting bills and coins in the vault or making a deposit at the central bank.
- The central bank overnight deposit rate (see **Deposit facility**) is lower than EONIA:
  `www.ecb.europa.eu/mopo/implement/sf/html/index.en.html`
- Many banks prefer lending money to another bank if they get higher rate than what central bank offers even if this rate is still negative.
- Negative rates have invalidated many widely used interest rate models and pricing formulas.

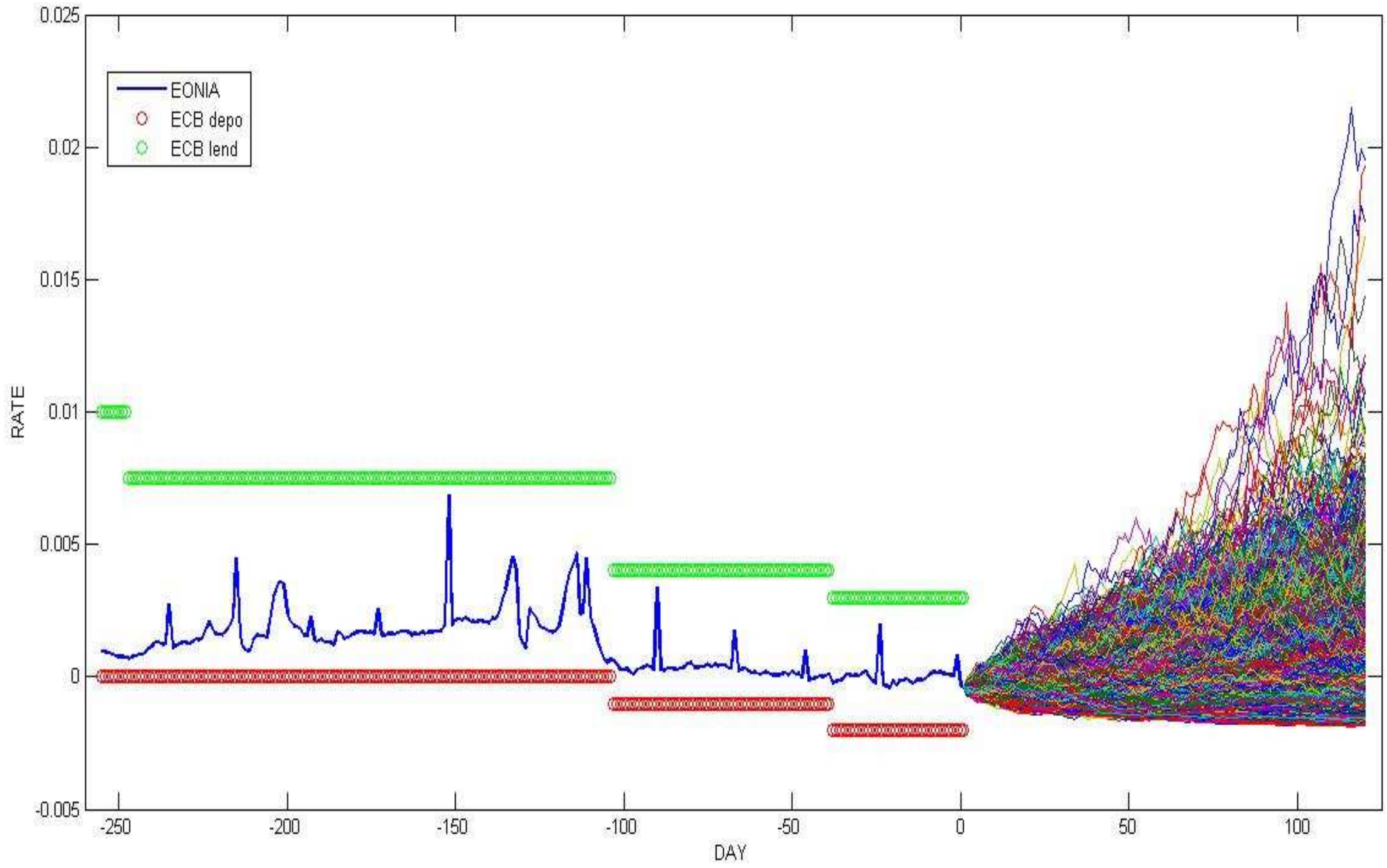
# Money market (Negative Interest Rate Model)

- Example: **Black-Karasinski model** describes the logarithmic rate $y := \ln Y$ as a mean reverting (Ornstein-Uhlenbeck) process:

$$dy = -a(y - \bar{y})dt + \sigma dW,$$

  where $a, \sigma > 0$ and $\bar{y}$ (the **mean reversion level**) are constant parameters.
- Future rates are thus modeled by $Y = \exp y$ which is a **strictly positive** process.
- Knowing that $Y > Y^{ECB}$, suggests modeling the **credit spread** $S := Y - Y^{ECB}$ with Black-Karasinski instead.
- Long-term modeling of $Y^{ECB}$ would require more knowledge of central banking and **macro economics**.

## Shifted Black-Karasinski model of EONIA:



# Bonds



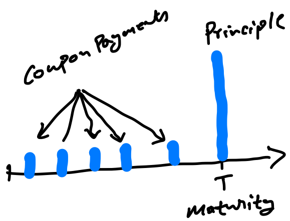



Fig: Fixed Coupon Bond

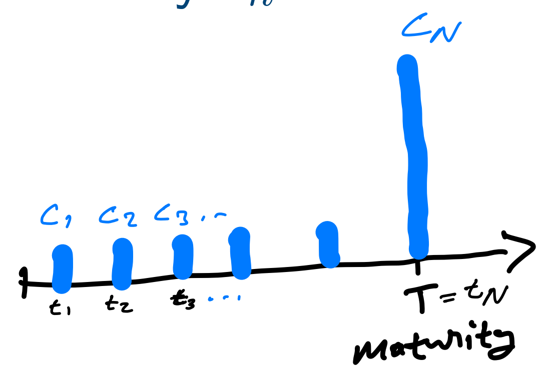
- **Bonds** are long term debt instruments that give **"coupon payments"** at regular time intervals and **"principal payment"** at maturity. If the agreed payments are deterministic, one speaks of **fixed-rate** bonds.
- Examples:
  - **Government bonds** (UK: gilts, US: Treasury bonds).
  - **Corporate bonds** are issued by companies or other entities with limited liability. (subject to Credit Risk)
  - **Index-linked bonds** have coupons and principal payments that are multiples of an index (typically consumer price index). (Future payments are uncertain as it is linked to index) → both principle & coupons uncertain.
- Together with money market instruments, bonds are often referred to as **fixed income instruments** although the payments they provide are not always fixed (index-linkage, credit risk).

# Bonds

$C_n$ = Outstanding payment of Bond either Coupons or Principle.
$t_n$ = time $C_n$ is paid

- If there is no credit risk, a fixed-rate bond (or a bond portfolio) with payments $c_n$ at times $t_n$, $n = 1, \ldots, N$ can be **replicated** by a portfolio of zero-coupon bonds: buy $c_n$ units of zero-coupon bond with maturity $t_n$.
- The **replication cost** is

$$\sum_{n=1}^{N} c_n P^a_{t_n},$$




where $P^a_{t_n}$ denotes the **ask-price** of a zero coupon bond with maturity $t_n$.

- How is the replication argument affected by credit risk/index-linkage?

This equation provides link between YTM of Bond portfolio & the Yield Curve.

# Bonds

YTM is a characteristic of a given Bond Portfolio while Yield curve is the characteristic of the Market constructed from prices of ZCB at different maturities observed from market.

Given a **coupon bond** or a **bond portfolio** with a market price $B_t$ and payments $c_n$ at $t_n$, $n = 1, \ldots, N$, its **yield to maturity** (YTM) $Y_t$ at time $t < t_1$ is defined as a solution to

$$B_t = \sum_{n=1}^{N} c_n e^{-Y(t_n - t)}.$$

current time

Think of $Y_t$ as discount rates that equates discounted future payments with Market price of Bond $B_t$.

- Since $x \mapsto e^x$ is a strictly increasing continuous function with range $(0, \infty)$, the equation has a unique solution and the YTM is a **decreasing** function of $B_t$.
- Note that the YTM is a **characteristic of a bond** while a **yield curve** $s \mapsto Y_{t,s}$ is a **characteristic of the market**. In perfectly liquid markets, the two are related by

$$\sum_{n=1}^{N} c_n e^{-Y_t(t_n - t)} = \sum_{n=1}^{N} c_n P_{t_n} = \sum_{n=1}^{N} c_n e^{-Y_{t,t_n}(t_n - t)}.$$

Price of ZCB with CTS compounding

$Y_{t,t_n}$ = zero rate at time $t$ for a ZCB with maturity $t_n$

# Bonds

YTM like interest rates are mathematical construction whereas price B is the real data.



- Bond prices are often quoted in terms of YTM (much like option prices are quoted in terms of "implied volatility").
- YTM is reported for many publicly available **bond indices**; see e.g. Bloomberg terminals.
- The behaviour of bond prices can be conveniently described in terms of YTM. Indeed, we have $B_t = B(t, Y_t)$, where $B$ is the function defined by

$$B(t, Y) = \sum_{n=1}^{N} c_n e^{-Y(t_n - t)}.$$

- We will make a Taylor approximation of the logarithm of $B$. Taking the exponential of that, we will obtain a **positive approximation** of the price.

# Bonds

$[t, s] \to$ holding period

present time $\uparrow t$, $s \uparrow$ future time



**First order approximation** of $\Delta \ln B := \ln B_s - \ln B_t$ over a fixed holding period $[t, s]$ can be written

$$\Delta \ln B \approx \frac{\partial \ln B}{\partial t}(t, Y_t)\Delta t + \frac{\partial \ln B}{\partial Y}(t, Y_t)\Delta Y$$

1st order Taylor Approx.

where $\Delta t = s - t$ and $\Delta Y = Y_s - Y_t$. We have

$$\frac{\partial \ln B}{\partial t}(t, Y) = Y \quad \text{and} \quad \frac{\partial \ln B}{\partial Y}(t, Y) = -D,$$

$$\frac{\partial \ln B}{\partial t} = \frac{1}{B}\frac{\partial B}{\partial t} = \frac{1}{B}\frac{\partial}{\partial t}\left[\sum c_n e^{-Y(t_n - t)}\right] = \frac{1}{B}\sum Y \underbrace{c_n e^{-Y(t_n-t)}}_{B} = \frac{Y}{B}\cdot B = Y$$

where the **duration** $D$ is given by

$$D := -\frac{1}{B}\frac{\partial B}{\partial Y}(t, Y) = \frac{1}{B}\sum_{n=1}^{N}(t_n - t)e^{-Y(t_n - t)}c_n.$$

(a weighted average of the times $(t_n - t)$ to payments).

# Bonds



$$\frac{\partial^2 \ln B}{\partial Y^2} = \frac{\partial}{\partial Y}\left[\frac{1}{B}\frac{\partial B}{\partial Y}\right]$$
$$= -\frac{1}{B^2}\frac{\partial B}{\partial Y}\cdot\frac{\partial B}{\partial Y} + \frac{1}{B}\frac{\partial^2 B}{\partial Y^2}$$
$$= \frac{1}{B}\frac{\partial^2 B}{\partial Y^2} - \left(\frac{1}{B}\frac{\partial B}{\partial Y}\right)^2$$
$$= C - D^2$$

Second order derivatives of $\ln B$ can be written as

$$\frac{\partial^2 \ln B}{\partial t^2}(t, Y) = 0, \ \frac{\partial^2 \ln B}{\partial t \partial Y}(t, Y) = 1, \ \frac{\partial^2 \ln B}{\partial Y^2}(t, Y) = C - D^2,$$

where

$$C := \frac{1}{B}\frac{\partial^2 B}{\partial Y^2}(t, Y) = \frac{1}{B}\sum_{n=1}^{N}(t_n - t)^2 e^{-Y(t_n - t)} c_n,$$

is known as the **convexity** of the bond (portfolio). The **second-order approximation** becomes

$$\Delta \ln B \approx Y_s \Delta t - D_t \Delta Y + \frac{1}{2}(C_t - D_t^2)(\Delta Y)^2.$$

$$\boxed{\Delta \ln B \approx \frac{\partial \ln B}{\partial t}\Delta t + \frac{\partial \ln B}{\partial Y}\Delta Y + \frac{1}{2}\frac{\partial^2 \ln B}{\partial t^2}(\Delta t)^2 + \frac{1}{2}\frac{\partial^2 \ln B}{\partial Y^2}(\Delta Y)^2 + \frac{\partial^2 \ln B}{\partial t \partial Y}\Delta t \Delta Y}$$

$2^{nd}$ order Taylor Approx

# Bonds

The bond (portfolio) **returns** can now be approximated by

$$\frac{B_s}{B_t} = \exp(\Delta \ln B) \approx \exp\left(Y_s \Delta t - D_t \Delta Y + \frac{1}{2}(C_t - D_t^2)(\Delta Y)^2\right)$$

- Recall that, in general,

$$B_t = \sum_{n=1}^{N} c_n e^{-Y_{t,t_n}(t_n - t)}$$

  which depends on the **whole yield curve** and the **payment structure** $(c_n, t_n)$, $n = 1, \ldots, N$.
- Our approximation gives the return in terms of a **single risk factor**, the YTM, while the payment structure is summarised by just **two numbers**: $D_t$ and $C_t$.

# Bonds



- We perform statistical analysis of the above return formula using historical values of total return indices, yields and durations.
- Monthly observations from December 1996 to October 2013 of Bloomberg EFFAS market data for France, Germany, Italy, Spain, United Kingdom and United States.
- We fit the following two models to the data

1st order Taylor Approx + linear Regression (For $\Delta t$ term we took average of YTM of holding period)

$$\begin{aligned} \text{Model 1:} \quad & \Delta \ln B \approx c + \tfrac{Y_s + Y_t}{2} \Delta t - D_s \Delta Y, \\ \text{Model 2:} \quad & \Delta \ln B \approx c + \tfrac{Y_s + Y_t}{2} \Delta t - D_s \Delta Y + \gamma (\Delta Y)^2. \end{aligned}$$

- The parameters $c$ and $\gamma$ are estimated by ordinary least squares. (linear regression)

2nd order Approx + linear regression

# Bonds



Table 1: Regression statistics for fixed-rate government bonds

| | FRA | GER | IT | UK | US | EURO |
|---|---|---|---|---|---|---|
| **Model 1** | | | | | | |
| $100 * c$ | 0.0147 | 0.0083 | 0.0084 | -0.0195 | -0.0202 | 0.015 |
| | (3.5842) | (1.7215) | (1.9339) | (-4.8403) | (-3.0658) | (4.1467) |
| $R^2$ | 99.76% | 99.64% | 99.68% | 99.84% | 99.71% | 99.79% |
| **Model 2** | | | | | | |
| $100 * c$ | 0.007 | 0.0007 | -0.0001 | -0.0256 | -0.0277 | 0.01 |
| | (1.4255) | (0.1186) | (-0.0178) | (-5.7509) | (-3.6689) | (2.2687) |
| $\gamma$ | 24.4889 | 24.1545 | 32.5155 | 4.8396 | 11.1262 | 17.031 |
| | (2.7949) | (2.2625) | (2.5933) | (3.0699) | (1.9703) | (1.9131) |
| $R^2$ | 99.77% | 99.65% | 99.70% | 99.85% | 99.72% | 99.79% |
| Partial-$R^2$ | 5.11% | 3.41% | 4.43% | 2.63% | 2.84% | 2.46% |
| Data start | 1997-12 | 1997-12 | 1997-12 | 1980-12 | 1998-12 | 1997-12 |
| Data end | 2010-03 | 2010-03 | 2010-03 | 2010-03 | 2010-03 | 2010-03 |

Model 1: $\Delta \ln B \approx c + Y_s \Delta t - D_t \Delta Y$,
Model 2: $\Delta \ln B \approx c + Y_s \Delta t - D_t \Delta Y + \gamma(\Delta Y)^2$,
Numbers in parentheses are the $t$-statistics for the estimated coefficients.
Partial-$R^2$ is defined as the $R^2$-statistic obtained by regressing the residual of Model 1 with the quadratic term.

# Bonds



The above analysis can be extended to **index-linked bonds** where the payment at time $t_n$ is $c_n I_{t_n}$ for a given index $I$.

- In **inflation-linked bonds**, $I$ is the **consumer price index** (or retail price index).
- The index $I$ can also be used to model **default losses** in **corporate bonds**. In this case, $I$ gives the "cumulative recovery rate" in the portfolio over time.

Defining the YTM by

$I_t$ = Index value at time of calculation of YTM (current time)

$$B_t = \sum_{n=1}^{N} c_n I_t e^{-Y(t_n - t)},$$

the second order Taylor approximation wrt $(t, Y, I)$ becomes

$$\frac{B_s}{B_t} \approx \exp\left(Y_s \Delta t - D_t \Delta Y + \frac{1}{2}(C_t - D_t^2)(\Delta Y)^2 + \Delta \ln I\right).$$

# Bonds

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## Table 2: Regression statistics for inflation-linked bonds

| | CAN | FRA | SA | SWE | UK | US |
|---|---|---|---|---|---|---|
| **Model 1** | | | | | | |
| 100*c | 0.1642 | 0.1328 | 0.4935 | 0.1114 | 0.2073 | 0.1966 |
| t-stat | (6.201) | (5.544) | (9.367) | (3.678) | (7.611) | (5.888) |
| $R^2$ | 97.48% | 95.46% | 78.6% | 91.29% | 96.06% | 94.41% |
| **Model 2** | | | | | | |
| 100*c | 0.0046 | -0.0098 | -0.0069 | 0.0087 | -0.0151 | -0.0004 |
| t-stat | (0.628) | (-0.839) | (-0.389) | (1.465) | (-1.918) | (-0.066) |
| $R^2$ | 99.80% | 98.91% | 97.58% | 99.66% | 99.67% | 99.82% |
| Partial-$R^2$ | 92.22% | 76.02% | 88.69% | 96.14% | 91.43% | 96.78% |
| Data start | 1996-12 | 1998-09 | 2000-03 | 1996-12 | 1996-01 | 1997-03 |
| Data end | 2010-03 | 2010-03 | 2010-03 | 2010-03 | 2010-03 | 2010-03 |

1st order Taylor ignoring Index I

1st order Taylor with Index value I

The table contains estimation results for the models:
Model 1: $\Delta \ln B \approx c + Y_s \Delta t - D_t \Delta Y$,
Model 2: $\Delta \ln B \approx c + Y_s \Delta t - D_t \Delta Y + \Delta \ln I$,
Numbers in parentheses are the $t$-statistics for the estimated coefficients.

## Table 3: Regression statistics for corporate bonds

| | Model 1 | | | | Model 2 | | | |
|---|---|---|---|---|---|---|---|---|
| | 1-3 Y | 3-5 Y | 5-7 Y | 7-10 Y | 1-3 Y | 3-5 Y | 5-7 Y | 7-10 Y |
| $100 * c$ | -0.022 | -0.038 | -0.064 | -0.094 | 0.013 | 0.007 | 0.028 | 0.012 |
| | (-3.431) | (-5.014) | (-4.357) | (-5.754) | (1.811) | (0.818) | (1.753) | (0.675) |
| $\alpha$ | | | | | 0.509 | 0.633 | 1.312 | 1.502 |
| | | | | | (7.424) | (8.246) | (8.921) | (9.426) |
| $R^2$ | 96.38% | 98.63% | 97.5% | 98.21% | 97.33% | 99.05% | 98.35% | 98.87% |
| Partial-$R^2$ | | | | | 26.35% | 30.63% | 34.07% | 36.58% |
| Data start | 1997-1 | 1997-1 | 1997-1 | 1997-1 | 1997-1 | 1997-1 | 1997-1 | 1997-1 |
| Data end | 2010-1 | 2010-1 | 2010-1 | 2010-1 | 2010-1 | 2010-1 | 2010-1 | 2010-1 |

$$\text{Model 1}: \quad \Delta \ln B \approx c + Y_s \Delta t - D_t \Delta Y$$
$$\text{Model 2}: \quad \Delta \ln B \approx c + Y_s \Delta t - D_t \Delta Y - \alpha S_s \Delta t.$$

Corporate YTM
Government YTM

In Model 2, we have used the approximation $\Delta I \approx -\alpha S_t \Delta t$, where $S_t = Y_t - Y_t^G$ is the **credit spread** ($Y^G$ denotes the yield on government bonds).

# Mortgages → Bond issued by Home buyer to the Bank.



- **Mortgage** is a loan to buy real estate (e.g. a home).
- Borrower (seller of the mortgage) is a home buyer and the lender (buyer of the mortgage) is a financial institution, such as a bank, credit union or building society.
- A home buyer receives money at the beginning in exchange for making regular (e.g. monthly) payments to the investor over the life of the contract (e.g. 20 years).
- The regular payments consist of **capital** and **interest**.
  - Capital payments reduce the loan amount so that at the end of the contract the loan has been paid back in full.
  - Interest payments depend on the remaining loan amount and the interest rate specified in the contract.
- In a **variable rate** mortgage, the interest rate is the sum of a **reference rate** (e.g. 3m EURIBOR) and a **margin**.

# Mortgages



- A mortgage loan is backed by the property that is bought with the loan: if the borrower cannot make the payments, the lender has the right to repossess the property to cover the loan.
- The interest rate margin is negotiated and depends on the **credit worthiness** of the lender. Credit indicators include **loan-to-value ratio** and **payment-to-income ratio**.
- Typically, the borrower has the option to pay back the loan ahead of schedule. This is referred to as **prepayment**.
- From the point of view of the investor (lender) a mortgage is subject to
  - **credit risk** (the lender may default on the payments),
  - **interest rate** risk (the reference rate may fluctuate),
  - **prepayment risk** (the lender may pay the loan back earlier than expected thus changing the cash-flows the lender was expecting to receive).

# Mortgages

- The cash-flow uncertainty can be reduced by **pooling** several mortgages together.
- Large banks may have direct investments in large pools.
- Other investors have access to pooled mortgages through **mortgage backed securities** (securitized mortgages).
- MBSs are created by companies such as Fannie Mae, Freddie Mac and Ginnie Mae who
  - buy large amounts of mortgages from mortgage issuers,
  - sell shares of mortgage pools to other investors.
- Mortgages can be pooled in many different ways:
  - by credit rating: **prime loans** and **subprime loans**,
  - by tranches: first $X\%$ of defaults affects only the first pool,
  - capital only/interest only, . . .
- **The subprime crisis** was largely caused by unrealistic credit ratings assigned by mortgage issuers.

# Currency market



- Also known as **foreign exchange** (forex/FX) market.
- The largest market as measured by turnover.
- Currencies are largely traded in OTC markets.
- Electronic limit order markets include: Spot FX, Hotspot FX.
- Exchange rates are affected by inflation, interest rates and other macroeconomic factors in the different currencies.
- Companies are exposed to FX risk if they have investments or liabilities nominated in foreign currencies.

# Currency market

- Consider a market with a finite set $J$ of currencies.
- Denote the **ask-price** of buying currency $j$ with currency $i$ by $\pi^{ij}$. In other words, $\pi^{ij}$ is the number of units of currency $i$ needed to buy one unit of currency $j$.
- From the point of view of currency $i$, the **bid-price** of currency $j$ is $1/\pi^{ji}$, the number of units of currency $i$ one gets by selling one unit of currency $j$.
- After market clearing, we must have $1/\pi^{ji} < \pi^{ij}$ (why?)
- More generally, arbitrage exists unless

$i_3$ → $i_2$ → $i_1$ → $i_m$ …

Going in a Circle
using $i_2$ to buy $i_1$
" $i_3$ to buy $i_2$
⋮
using $i_1$ to buy $i_m$

cost of buying 1 unit of currency $i_1$ using $i_1$ →

$$\pi^{i_1,i_2}\pi^{i_2,i_3}\cdots\pi^{i_m,i_1} \geq 1$$

for all sequences of currencies $i_1, i_2, \ldots, i_m$.

# Insurance

- Insurance products are financial products
  - whose payouts are determined by harmful events to the buyer of insurance,
  - that have been legally classified as insurance products.
- To buy an insurance contract, you must have **insurable interest**, i.e. exposure to the insured event.
- Insurance products and insurance companies are often regulated differently from financial products and financial companies, mainly because of historical reasons.
- For example, **credit default swaps** have the structure of insurance contracts but they are not classified as insurance contracts. Thus, anyone can buy them. They were heavily traded by speculators before 2008.

# Insurance

- Usually, no secondary market exist for insurance contracts: you cannot sell your car insurance to a third party.
- Insurers, however, can buy **reinsurance** from reinsurance companies (Swiss Re, Munich Re, Berkshire Hathaway,...)
- Examples of reinsurance contracts:
  - Proportional: the reinsurer pays a fixed percentage of the insurance claims.
  - Stop loss: the reinsurer covers claims exceeding an agreed threshold. The payoff is thus of the form

$$c = \max\{X - K, 0\},$$

where $X$ is the total claims and $K$ is the agreed threshold.

# Insurance

Some forms of insurance have been securitized.

- **Catastrophe bonds** (CAT bonds) are bonds issued by an insurance (or a reinsurance) company.
  - Like regular bonds except that a CAT bond defaults if a specified catastrophic "event" happens.
  - The "event" may be defined e.g. in terms of wind speeds or magnitude of an earthquake in a given area.
  - When the event happens, home insurance contracts tend to have high claims while CAT bonds have low claims.
  - The hedge is not perfect. This is known as **basis risk**.
- **Life insurance securitizations**
  - Attempts have been made to securitize pension insurance contracts as well. So far, these have not been successful.

# Derivatives

A **derivative** is a financial contract whose payoff(s) is a **function** of an **underlying** random variable(s).

- Various possible underlyings: future values of
  - market **prices** of traded assets (stocks, stock indices, bonds, foreign currencies, energy, other derivatives, . . . )
  - market **interest rates**
  - **non-tradable underlyings** related to e.g. weather or default events, . . .
- Various possible functions:
  - forwards and futures,
  - put and call options,
  - American options,
  - swaps, . . .

# Forwards and futures

- A **forward contract** is an agreement to buy/sell an asset at a future time $T$ (**maturity**) for price $K$ (**forward price**).
- **Long position** in a forward with **cash-delivery** has payoff $(S_T - K)$, where $S_T$ is the "market price" of the asset at time $T$. **Short position** has payoff $(K - S_T)$.
- The forward price $K$ is negotiated (offered in exchanges) so that nothing is paid at time $0$.
- Forward contracts can be used to reduce the risk associated with the future price $S_T$ if one needs to buy/sell the asset at time $T$.
- For example, a manufacturer has to buy ingredients or a producer has to sell products. Who would take long/short positions here?
- The underlying can be e.g. a liquidly traded financial asset or a physical commodity.

# Forwards and futures

$S_0^a - K P_T^b \geq 0$
$K P_T^a - S_0^b \geq 0$
$\frac{S_0^b}{P_T^a} \leq K \leq \frac{S_0^a}{P_T^b}$

a = ask price
b = bid price



- If the underlying asset is a perfectly liquid **financial asset** with market price $S_0$, then one can **replicate** the payoff $S_T - K$ by buying the asset now and selling $K$ units of a zero-coupon bond with maturity $T$.
- If the zero-coupon bond is perfectly liquid with price $P_T$, then the **replication cost** for the long position is $S_0 - KP_T$.
- Replication of a short position costs $KP_T - S_0$ (buy the bond and short-sell the underlying).
- Thus, in perfectly liquid markets, the unique **arbitrage-free** forward price would be $K = S_0/P_T$ (and forward contracts would be redundant assets).
- In practice, however, there are no perfectly liquid assets.
- How would bid/ask-spread change the replication costs?
- For **non-financial assets**, storage and lending costs may increase the hedging costs even further.

# Forwards and futures



- Some standardised forward contracts are traded in exchanges.
- Such contracts are called **futures contracts** and their forward price $K$ is called the **futures price**.
- Positions in exchange traded products are subject to **margin requirements**: $cA \geq L$ where and $A$ and $L$ are the "values" of your long and short positions (see page 76).
- Assume that today, you enter a long position in a futures contract with futures price $K_0$ (recall that entering the contract doesn't cost anything).
- If tomorrow's futures price is $K_1$, the value of your contract is then $(K_1 - K_0)P_T$.
- Indeed, the payoff $S_T - K_0$ of your contract can be **replicated** by entering a futures contract with payout $S_T - K_1$ and buying $K_1 - K_0$ zero-coupon bonds with maturity $T$. The **replication cost** is $(K_1 - K_0)P_T$.

# Call and put options

Assumed very liquid

- **Call option** on an underlying $S$, with **maturity** $T$ and **strike price** $K \in \mathbb{R}$ is a derivative which pays

$$(S_T - K)^+ := \max\{S_T - K, 0\}.$$

- **Put option** pays

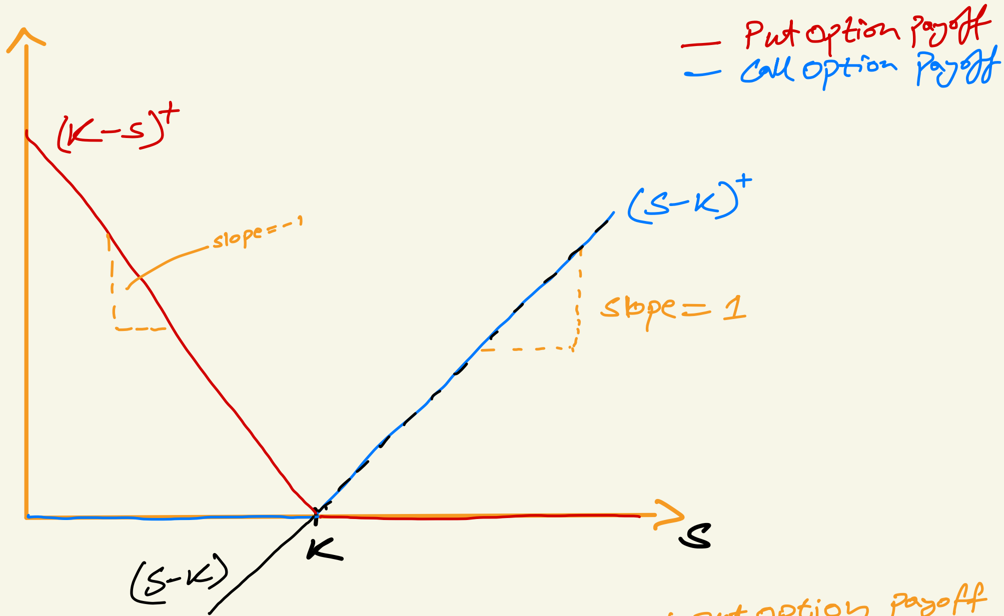
$$(K - S_T)^+ := \max\{K - S_T, 0\}.$$

- We have

$$\overbrace{(S_T - K)^+}^{\text{long in call}} - \overbrace{(K - S_T)^+}^{\text{short in put}} = S_T - K,$$

so the put option can be **replicated** with a portfolio of a call, $K$ units of zero-coupon bond and $-1$ units of the underlying.

# Payoff




The difference between call option and put option payoff gives payoff of Forward Contract $(S-K)$

i.e. $(S_T-K)^+ - (K-S_T)^+ = \underbrace{S_T-K}_{\text{Forward Contract}}$.

- Replicating Put Option

$$(K-S)^+ = K - S + (S-K)^+$$

$K$ units of ZCB with maturity $T$.

$-1$ unit of underlying stock $S$.

$1$ unit of a call option.

$Cost = K P_T^a - S_0^b + C(K)^a$ is cost of Replicating Put option.

- $$\left.\begin{aligned} P(K)^b &\le C(K)^a + K P_T^a - S_0^b \\ P(K)^a &\ge C(K)^b + K P_T^b - S_0^a \end{aligned}\right\} \text{No arbitrage Bounds for Prices}$$

$P(K) = C(K) + K P_T - S_0 \Leftarrow$ Put-call Parity in perfect Market.

# Call and put options



- If the ask-prices of the call option and the zero-coupon bond are $C(K)^a$ and $P_T^a$ and bid-price of the underlying is $S_0^b$, then the put option can be replicated at cost

$$C(K)^a + KP_T^a - S_0^b.$$

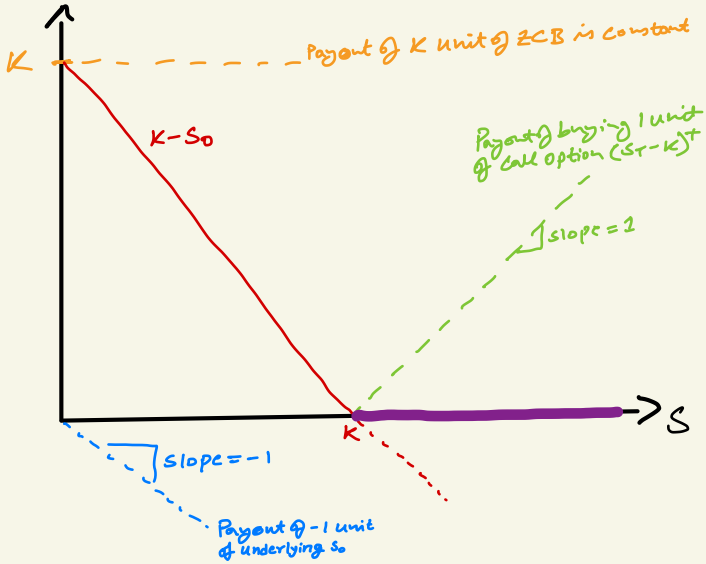
  If the bid-price of the put is higher than this, there is **arbitrage**.
- By a similar argument, there is arbitrage if the ask-price of the put option is lower than

$$C(K)^b + KP_T^b - S_0^a.$$

- If all the assets are perfectly liquid, the unique arbitrage-free put price $P(K)$ is given by the **put-call parity**

$$P(K) = C(K) + KP_T - S_0.$$

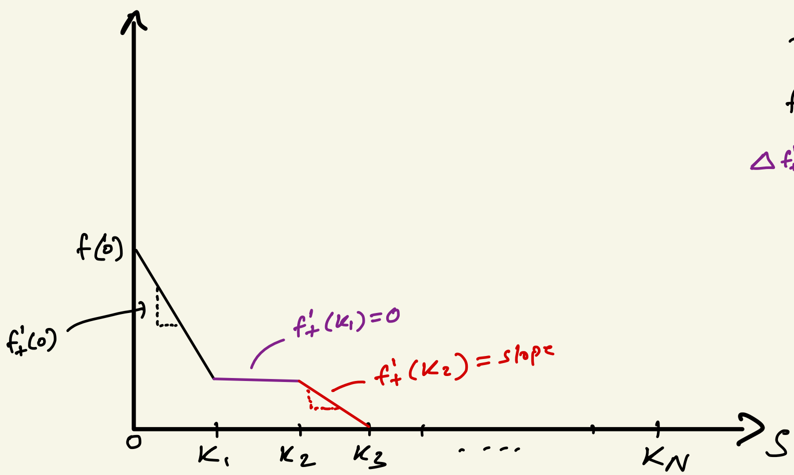
# Put option payoff Replication




Slope = No. of Units of instruments eg Bond, Call stock.

combined: $K - S_0 + (S_T - K)^+ = 0$ ← Put call payout replication

# Replicating Payoff of Complicated functions (look formula in next page)




$K_i$ = strike

$f(x)$ = Payout f^n of the underlying

$f_+'(0)$ = Right derivative of f^n at 0.

$\Delta f_+'(K_1) = f_+'(K_1) - f_+'(0)$ = slope

Replication of above function:

- $f(0)$ units of ZCB.
- $f_+'(0)$ units of the underlying.
- $\Delta f_+'(K_1)$ units of call with strike $K_1$.
- $\Delta f_+'(K_2)$ units of call with strike $K_2$.

# Call and put options (Any CTS piecewise linear f'n of the payoff of underlying)



- More generally, using zero-coupon bonds, the underlying and call options with strike prices $K_1 < K_2 < \cdots < K_N$, we can replicate any continuous payoff function $f$ which is linear on intervals that do not contain strike prices.
- Indeed, such a function $f$ can be expressed as

$$\left[ f(S) = f(0) + f'_+(0)S + \sum_{i=1}^{N} (S - K_i)^+ \Delta f'(K_i), \right]$$

where $\Delta f'(K)$ is the increment in the slope of $f$ at $K$:

$$\Delta f'(K) := f'_+(K) - f'_-(K) = \lim_{L \searrow K} f'(L) - \lim_{L \nearrow K} f'(L).$$

- Thus, the payout $f$ is replicated by a portfolio of $f(0)$ units of a zero coupon bond, $f'_+(0)$ units of the underlying and $\Delta f'(K_i)$ units of a call option with strike $K_i$, for $i = 1, \dots, N$.

# Call and put options

- In perfectly liquid markets (where unit prices of zero-coupon bonds, the underlying and call options are independent of the direction and quantity of trade), the **replication cost** would be

$$\pi(f) = f(0)P_T + f'_+(0)S_0 + \sum_{i=1}^{N} C(K_i)\Delta f'(K_i)$$

- Applying the above to the function $f(S) = (K - S)^+$ gives the put-call parity as a special case.
- What are the replication costs for $f$ and $-f$ when bid and ask prices of zero-coupon bonds, the underlying and call options are different?

# Call and put options

- Under bid-ask spreads, the replication cost of $f$ becomes

$$\pi(f) = f(0)^+ P_T^a + f'_+(0)^+ S_0^a + \sum_{i=1}^N C(K_i)^a \Delta f'(K_i)^+$$
$$- f(0)^- P_T^b - f'_+(0)^- S_0^b - \sum_{i=1}^N C(K_i)^b \Delta f'(K_i)^-.$$

where $f(0)^+ = \max\{f(0), 0\}$ and $f(0)^- = -\min\{f(0), 0\}$ etc

- Similarly, the replication cost for $-f$ becomes

unlike in liquid Markets, the replicating cost is not simply $-\pi(f)$.

$$\pi(-f) = f(0)^- P_T^a + f'_+(0)^- S_0^a + \sum_{i=1}^N C(K_i)^a \Delta f'(K_i)^-$$
$$- f(0)^+ P_T^b - f'_+(0)^+ S_0^b - \sum_{i=1}^N C(K_i)^b \Delta f'(K_i)^+.$$

# Call and put options



Applying the integration by parts formula for BV functions (Bounded Variation f'ns)

Class of f'ns that do not need to be differentiable.

$$\int_0^b F dG + \int_0^b G dF = F(b)G(b) - F(0)G(0)$$

with $F(K) = f'(K)$ and $G(K) = (S-K)^+$ gives

$$-\int_0^{S\wedge b} f'(K)dK + \int_0^b (S-K)^+ df'(K) = f'(b)(S-b)^+ - f'(0)S$$

so

$$f(S\wedge b) = f(0) + f'(0)S - f'(b)(S-b)^+ + \int_0^b (S-K)^+ df'(K).$$

Letting $b \nearrow \infty$, gives

$$\boxed{f(S) = f(0) + f'(0)S + \int_0^\infty (S-K)^+ df'(K).}$$

# Call and put options



- Thus, if call options were available for all strikes and if markets were perfectly liquid, then the payout $f$ could be replicated at cost

$$\pi[f] = f(0)P_T + f'(0)S_0 + \int_0^\infty C(K)df'(K). \tag{1}$$

- If the bid and ask prices are $C(K)^b < C(K)^a$, then the replication cost becomes

$$\pi[f] = f(0)P_T + f'(0)S_0 + \int_0^\infty C(K)^a df'_+(K) - \int_0^\infty C(K)^b df'_-(K),$$

  where $f'_+$ and $f'_-$ are increasing functions such that $f' = f'_+ - f'_-$.
- The above argument works when $f'$ is of **bounded variation**
- The construction is unrealistic since in real markets, liquidly traded call options exist only for a finite number of strikes.
- Integrating by parts twice (and assuming that all the derivatives exist) gives

$$\int_0^\infty C(K)df'(K) = [C(K)f'(K)]_{K=0}^\infty - [C'(K)f(K)]_{K=0}^\infty + \int_0^T C''(K)f(K)dK.$$

# Call and put options



- Above,

$$[C(K)f'(K)]_{K=0}^{\infty} := C(\infty)f'(\infty) - C(0)f'(0),$$
$$[C'(K)f(K)]_{K=0}^{\infty} := C'(\infty)f(\infty) - C'(0)f(0).$$

- The call option with strike $K = 0$ has payout $S_T$, so $C(0) = S_0$, while a call option with $K = \infty$ has zero payout, so $C(\infty) = 0$.
- Since $C(K) \searrow 0$ as $K \nearrow \infty$, we have $C'(\infty) = 0$.
- Finally,

$$C'(0) = \lim_{K \searrow 0} \frac{C(K) - C(0)}{K},$$

where the difference quotient is the price of a portfolio of $1/K$ units of a call with strike $K$ and $-1/K$ units of a call with strike $0$. The payout of such a portfolio converges to constant $-1$ as $K \searrow 0$, so $C'(0) = -P_T$.

- Combining the above with (1) gives the so called Breeden–Litzenberger formula

$$\pi(f) = \int_0^{\infty} f(K)C''(K)dK.$$

# Optionality



- **Optionality** in a financial contract means that the counter parties have some control over the cash-flows.
- In a European call the optionality is trivial: exercise the option only if $S_T > K$. Similarly for a put.
- An **American option** allows the owner to buy/sell the underlying at price $K$ at any time before $T$. Exercising an American call at time $t$, pays $(S_t - K)$ at time $t$, a put pays $(K - S_t)$. Finding the optimal timing is a nontrivial problem in general. American options are typical for individual stocks while European options are typical for indices.
- A **Bermudan option** is an American option with restrictions on the time of exercise.
- **Callable bond** allows the bond issuer to get out of the short position by returning the principal before maturity.
- **Convertible bond** allows the holder to convert the bond into a specified number of shares of the issuing company.
- Most **life insurance contracts** have optionalities.
- **Mortgages** often allow the homeowner to pay some or all of the loan back early.

**Lemma 3.** *In perfectly liquid markets with a positive interest rate $r$, it is not optimal to exercise an American call option before maturity.*

*Proof.* Let $t < T$. If $S_t \leq K$, it clearly does not make sense to exercise at time $t$. If $S_t > K$, exercise would result in $(S_t - K)e^{(T-t)r}$ at maturity. If instead, we exercise at $T$, short one unit of the underlying and invest the proceeds in the bank account, we would get

$$\begin{aligned}(S_T - K)^+ + S_t e^{(T-t)r} - S_T &= (S_T - K)^+ + (K - S_T) \\ &\quad + (S_t - K)e^{(T-t)r} + (e^{(T-t)r} - 1)K \\ &= (S_t - K)e^{(T-t)r} \\ &\quad + (K - S_T)^+ + (e^{(T-t)r} - 1)K,\end{aligned}$$

where the last two terms are nonnegative. □

Why does the above fail in practice?

# Exotic derivatives



- Many European and American options are exchange traded
- Various more exotic derivatives are traded in OTC markets
- **Binary (digital) options** pay one pound if a specified event happens (underlying hits a given level, sports betting, ...)
- **Path-dependent options**: American/Bermudan, Asian options (functions of the average), look-back (functions of min or max), barrier options (knock-in/out), variance options (functions of variation of the path).
- **Rainbow options**: the payout depends on several underlyings.
- ...

# Interest rate derivatives



Eg:
Cap: Bought by homeowner if they have mortgage with variable interest.

Floor: Bought by Bank to ensure they get same interest payment in the future.

For Both Cap & Floor the Interest is on the Nominal Specified in the contract.

- **Interest rate derivatives** have payouts depending on an interest rate $Y$ such as LIBOR, EURIBOR or EONIA
- Their payouts often depend on the value of $Y$ at several points in time $t_0, \ldots, t_N$ (path dependence). (Risk factor prevailing is I.R.)
- A **cap** with strike $\bar{Y}$ pays $(Y_{t_{n-1}} - \bar{Y})^+$ at each $t_n$.
- A **floor** with strike $\bar{Y}$ pays $(\bar{Y} - Y_{t_{n-1}})^+$ at each $t_n$.
- In an **interest rate swap** with swap rate $\bar{Y}$,
  - the **floating leg** pays $Y_{t_{n-1}}(t_n - t_{n-1})$,
  - the **fixed leg** pays $\bar{Y}(t_n - t_{n-1})$

  at each $t_n$, $n = 1, \ldots, N$. The floating leg receiver is said to have a long position in the swap.

# Interest rate derivatives

- Assuming that markets are perfectly liquid and that there is no credit risk, the payouts of a LIBOR interest rate swap can be replicated as follows.
- Recall that at each time $t_n$ for $n = 1, \ldots, N$,
  - the **floating leg** pays $Y_{t_{n-1}}(t_n - t_{n-1})$,
  - the **fixed leg** pays $\bar{Y}(t_n - t_{n-1})$.
- The fixed-leg can be **replicated** with zero-coupon bonds. In perfectly liquid markets, the replication cost would be

$$\sum_{n=1}^{N} \underbrace{\bar{Y}(t_n - t_{n-1})}_{\text{Units of ZCB}} \underbrace{P_{t_n}}_{\text{Price of ZCB}},$$

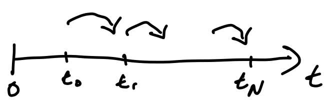
ZCB maturity dates must match payments of fixed leg.

where $P_{t_n}$ is today's price of a zero-coupon bond with maturity $t_n$.

# Interest rate derivatives

This replication is self-financing.






We can **replicate** the floating leg as follows.

- Buy a zero-coupon bond with maturity $t_0$ and short a zero-coupon bond with maturity $t_N$.
- Invest the pound received at $t_0$ into zero-coupon bonds maturing at $t_1$. This yields $1/P_{t_0,t_1}$ units of cash at $t_1$.
- By definition of LIBOR rates, $1/P_{t_0,t_1} = 1 + Y_{t_0}(t_1 - t_0)$.
- Deliver $Y_{t_0}(t_1 - t_0)$ to the floating leg receiver and invest the remaining one pound in zero-coupon bonds maturing at $t_2$. This yields $1 + Y_{t_1}(t_2 - t_1)$ at time $t_2$.
- Continue this way until time $t_N$ when the remaining pound is used to cover the short position in the zero-coupon bond.

In perfectly liquid markets, the cost of implementing the above strategy is $P_{t_0} - P_{t_N}$.

# Interest rate derivatives



In summary, if zero-coupon bonds are perfectly liquid and default free with prices $P_t$ today, the net cash-flows of floating leg receiver can be replicated at cost

$$P^{swap}(\bar{Y}) := \underbrace{P_{t_0} - P_{t_N}}_{\text{floating}} - \overbrace{\bar{Y} \sum_{n=1}^{N} (t_n - t_{n-1}) P_{t_n}}^{\text{fixed}}.$$

long position in swap

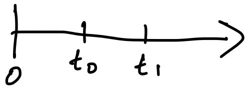
- If someone is willing to take the floating side of the swap with a rate $\bar{Y}$ such that $P^{swap}(\bar{Y}) < 0$, we could earn $P^{swap}(\bar{Y})$ pounds by entering the fixed side of swap and buying the replicating portfolio.
- In perfectly liquid markets, where both sides of the swap are offered at the same swap rate $\bar{Y}$, an **arbitrage opportunity** exists unless $P^{swap}(\bar{Y}) = 0$.

No arbitrage condition

# Interest rate derivatives

Contract lives from $t_0$ to $t_1$ & only one payment date at $t_1$






A **forward rate agreement** (FRA) is an interest rate swap with $N = 1$. Thus, in perfectly liquid markets, the value of a replicating portfolio for the floating leg receiver would be

$$\begin{aligned} P^{FRA} &= P_{t_0} - P_{t_1} - P_{t_1}\bar{Y}(t_1 - t_0) \\ &= P_{t_1}\left[\frac{P_{t_0}}{P_{t_1}} - 1 - \bar{Y}(t_1 - t_0)\right] \end{aligned}$$

Using the definition of the LIBOR forward rate

$$F_{t_0,t_1} = \frac{1}{t_1 - t_0}\left(\frac{P_{t_0}}{P_{t_1}} - 1\right),$$

Substituting $\left(\frac{P_{t_0}}{P_{t_1}} - 1\right) = F_{t_0,t_1}(t_1 - t_0)$

we can write the replication cost as

$$P^{FRA} = P_{t_1}[F_{t_0,t_1} - \bar{Y}](t_1 - t_0).$$

swap rate

In perfectly liquid arbitrage free markets, $\bar{Y} = F_{t_0,t_1}$.

Libor Forward Rate

# Interest rate derivatives

(Constructing ZCB prices that do not exist in the Market.)



How to construct the **yield curve** $T \mapsto Y_T$ from observable data? Since $Y_T$ and $P_T$ are in one-to-one correspondence, it is enough to construct the **zero-curve** $T \mapsto P_T$.

- For maturities up to one year, we can observe the prices $P_T$ directly in LIBOR markets.
- For maturities longer than one year, we observe **market prices** $P^i$ of government bonds and **market swap rates** $\bar{Y}^j$ of interest rate swaps
- This gives a system of equations

$$\sum_{n=1}^{N} c_n^i P_{t_n} = B^i \quad i = 1, \ldots, I,$$

$$P_{t_0^j} - P_{t_N^j} - \sum_{n=1}^{N} \bar{Y}^j (t_n^j - t_{n-1}^j) P_{t_n^j} = 0 \quad j = 1, \ldots, J.$$

Using Market prices of Swaps & Gov. Bond. We can find ZCB prices for different maturities by solving this equation → This gives us Swap curve.

- One then finds the values of $P_t$ that give the "best fit". This can be solved with numerical optimization techniques.


# Credit derivatives

- Payouts of **credit derivatives** depend on defaults of one or more bond issuer.
- **Credit default swap** (CDS) pays a fixed amount when a given issuer defaults within a given time period. The owner of a CDS contract pays a fixed periodic premium until maturity or default.
- Payouts of **Index CDS** depend on defaults of $M$ issuers. If there are $K_t$ defaults in $(t-1, t]$, a long position gives $K_t/M$ pounds at time $t$. The premium paid at time $t$ is a multiple of the number $M_t = M - \sum_{s=1}^{t} K_s$ of survivors.
- **Credit default obligation** (CDO) is specified in terms of two numbers $\underline{M} < \overline{M}$ (a "tranche"). The protection leg pays $K_t$ if $M_t \in (\overline{M}, \underline{M})$ and the premium leg pays a multiple of $\min\{M_t - \underline{M}, \overline{M} - \underline{M}\}$ while $M_t > \underline{M}$.

# Derivatives



- Many pricing formulas are based on the assumptions that
  - markets are perfectly liquid,
  - there is no credit risk.
- Both assumptions fail in practice:
  - **There are no perfectly liquid** assets: unit prices depend both on the sign and quantity traded.
  - **Long positions are subject to credit risk**: payouts may be less than agreed.
- As a result, exact replication is rarely possible in practice.
- This invalidates many classical pricing formulas.
- For example, the put-call parity and replication of interest rate swaps should be only taken as approximations. In other words puts and swaps are **nonredundant** assets.

# Derivatives



- Practitioners are well aware of the limitations of replication arguments.
- Instead of correcting the pricing models, however, the current practice is to adjust the incorrect valuations.
- **Credit valuation adjustment** (CVA) tries to correct for credit risk that was ignored by a pricing model.
- **Funding valuation adjustment** (FVA) tries to correct for incorrect lending/borrowing rates.
- Such adjustments are under active research and debate.
- However, the adjustments often lack economic justification (practitioners and even academics are often confused by the multiple meanings of the words "price" and "valuation").

# Mathematical finance

# Probability and statistics



**Mathematical analysis** of optimal investment and valuation of financial contracts starts with **probabilistic modeling** of the uncertain asset returns and liabilities:

- Identification of
  - the relevant **asset classes**,
  - the **risk factors** that affect the asset returns and the cash-flows associated with the liabilities.
- Expressing the asset returns and cash-flows as **functions** of the risk factors (recall e.g. futures, options, bonds, ...).
- Modeling future values of risk factors as **random variables**.

The last step is important and challenging. The probabilistic model should incorporate the investor's **views** of the uncertain future.

# Optimal investment



- Given a probabilistic description of assets and liabilities and an **investment strategy**, one gets a probabilistic description of the **profit and loss** (P&L) and thus, the future net wealth of the investor
- The problem is then to find an investment strategy that yields a P&L as "nice" as possible.
- To formulate the problem mathematically, one needs a measure of "niceness" of a **random** P&L. Such a description depends on the agents **risk preferences** which is an elusive and highly **subjective** notion.
- A **quantitative representation of risk preferences** is a real-valued function on the space of **random variables** (the probabilistic descriptions of P&Ls).

# Valuation of contingent claims



Can't replicate everything

- In **incomplete markets**, the hedging argument for valuation of contingent claims has two natural generalizations:
  - **accounting value**: How much cash do we need to cover our liabilities at an acceptable level of risk?
  - **indifference price**: What is the least price we can sell a financial product for without increasing our risk?
- The former is important in accounting, financial reporting and supervision (SII, IFRS) and in the BS-model.
- The latter is more relevant in trading.
- Classical math finance makes no distinction between the two.

# Valuation of contingent claims



- In **incomplete markets**, the hedging argument for valuation of contingent claims has two natural generalizations:
  - **accounting value**: How much cash do we need to cover our liabilities at an acceptable level of risk?
  - **indifference price**: What is the least price we can sell a financial product for without increasing our risk?
- In general, such values depend on our **views**, **risk preferences** and **financial position**.
- Subjectivity is the driving force behind trading.
- In complete markets, the two notions coincide and they are independent of the subjective factors
- Optimal investment and contingent claim valuation will be studied in more detail in FM12, Incomplete markets.