# Risk Neutral Valuation Lecture Notes

# 1 Discrete Time Models

## 1.1 No Arbitrage

Loosely speaking, an arbitrage is an opportunity to make a risk-free profit. Although arbitrages may exists for short periods of time, if markets are efficient, arbitrage opportunities will not last for long. In real markets they may last only a fraction of a second. In this course we shall assume, unless otherwise specified, that markets are arbitrage free. The no arbitrage assumption will allow us to develop a consistent pricing and hedging theory. Furthermore, knowing the arbitrage free price of a contract and the corresponding hedging policy will allow us to recognise an arbitrage and will give us a strategy to take advantage of it.

More formally, assume that the market consists of $M$ risky assets (e.g. stocks, options, forward contracts, etc) whose price we will denote by $A_t^i$ with $i \in \{1, \ldots, M\}$ and one risk-free (deterministic) asset $A_t^0 \equiv B_t = e^{rt}$ which we will call the money market account. We shall assume that the risk-free interest rate $r$ is deterministic and market participants will have $r$ as reference rate for both lending and borrowing transactions.

A **portfolio** or **trading strategy** is a collection of units invested in tradable assets over the relevant time horizon. More precisely, let $\psi_t$ be the amount (positive or negative) invested in the risky asset $A^i$ at time $t$. If we indicate by $\tau$ the set of all possible trading periods, an hedging strategy will be the sequence of vectors:

$$\left(\psi_t^0, \ldots, \psi_t^M\right)_{\{t \in \tau\}}. \tag{1}$$

The **portfolio value** at a given time $t$, say $\pi_t$, will be given by

$$\pi_t \equiv \sum_{i=0}^{M} \psi_t^i A_t^i. \tag{2}$$

Price of asset

units of asset

An arbitrage exists, if it is possible to find a portfolio (the **arbitrage portfolio**) $\left(\psi_t^1, \ldots, \psi_t^M\right)_{\{t \in \tau\}}$ and a time $t$ such that

1. $\pi_0 \leq 0$
2. $\pi_t \geq 0$ a.s.

3. $P(\pi_t > 0) > 0$,

that is, a strategy with zero or negative initial value (in the latter case, somebody is "paying" us to implement the strategy) which is non negative at time $t$ and has a strictly positive probability to have a positive value at time $t$. For example a combination of one stock, one option and the money market account which costs zero at $t = 0$ and has a positive value at time $t = T$ in say, one market scenario out of 10, would constitute an arbitrage.

A portfolio is said to be **self-financing** if

$$\pi_t \equiv \sum_{i=0}^{M} \psi_t^i A_t^i = \sum_{i=0}^{M} \psi_{t-1}^i A_t^i, \tag{3}$$

for all $t \geq 1$ or equivalently

$$\pi_t = \pi_{t-1} + \sum_{s=1}^{t} \psi_{t-1} \cdot (A_t - A_{t-1}). \tag{4}$$

where we have used the "dot" product between vectors $\psi_{t-1}$ and $A_t - A_{t-1}$. All portfolios considered in this course will be self-financing unless otherwise specified.

## 1.2 Forward Contracts

A **Forward Contract** is an obligation to buy or sell a given asset, e.g. a stock $S$, at a pre-determined price (strike) $K$ at a future date $T$. The pay-off of the contract, seen from the point of view of the buyer, at expiry $T$ is given by

$$V_T = (S_T - K). \tag{5}$$

Since forwards contracts are regularly traded before the expiry, two questions arise:

1. What is the value $V_t$ of such a contract for $t \leq T$?
2. How can you hedge the contract by investing in a simple (static) portfolio comprising the asset $S$ and the money market account $B$?

In the following sections we shall assume that $S$ is a non dividend paying asset and that interest rates are constants and equal to $r$.

### 1.2.1 Hedging Approach

We shall show that the $t$-value $V_t$ of the forward contract is equal to

$$V_t = S_t - Ke^{-r(T-t)}. \tag{6}$$

We shall prove the claim for $t = 0$. It is straightforward to extend the result to a generic $t \leq T$.

Assume first that the market value of a forward contract with maturity $T$ and strike $K$ is equal to

$$\tilde{V}_0 = V_0 + x > S_0 - Ke^{-rT}. \tag{7}$$

It is possible to realise a risk-less profit by executing the sequence of operations below.

**At time zero**:

1. Sell the forward contract (i.e. agree to sell $S$ at time $T$ for $K$) and receive $\tilde{V}_0$ in exchange.
2. Borrow an amount of cash equal to $Ke^{-rT}$.
3. Buy the asset and pay for it $S_0$ using part of the cash raised with the first two transactions.
4. Deposit the excess cash $x$ into the money market account.

$\tilde{V}_0 = V_0 + x$

$Ke^{-rT}$

$-S_0$

$-x$

$0$

The net cash flows of the transactions above will be equal to zero.

**At expiry $T$**:

1. Deliver the asset to the counter-party of the forward contract and receive $K$ in exchange.

$+K$

$-K$

$+xe^{rT}$

$xe^{rT}$

2. Repay the debt plus interest, i.e. $K$.
3. Withdraw an amount equal to $xe^{rT}$ from the money market account.

The net cash flow of the transactions above at time $T$ is equal to $xe^{rT}$. We started with a zero cost portfolio and ended up with a strictly positive profit. It is clear that the price $\tilde{V}_0$ allows for arbitrage.

More formally, let $\psi_t$ be the unit invested in the forward contract $\tilde{V}_t$, $\Delta_t$ be the holdings of $S_t$ and $\phi_t$ the amount invested (i.e. cash deposited or borrowed) in the money market account $B_t$ at time $t$. Consider the portfolio $\psi_0 = -1$, $\Delta_0 = 1$ and $\phi_0 = -Ke^{-rT} + x$. The value of the portfolio $\pi_0$ at time zero is equal to

$$\pi_0 = \psi_0 \tilde{V}_0 + \Delta_0 S_0 + \phi_0 B_0 = 0. \tag{8}$$

Using the self-financing property of the portfolio we have that at time $T$, the value of the portfolio is equal to

$$\begin{aligned} \pi_T &= \psi_0 \tilde{V}_T + \Delta_0 S_T + \phi_0 B_T & (9) \\ &= -(S_T - K) + S_T - K + xe^{rT} & (10) \\ &= xe^{rT}. & (11) \end{aligned}$$

Similarly, an arbitrage opportunity arises if

$$\tilde{V}_0 \equiv V_0 - x < S_0 - Ke^{-rT}. \tag{12}$$

The following strategy will allow us to realise a risk-less profit.

**At time zero**:

1. Buy the forward contract (i.e. buy $S$ forward) and pay $\tilde{V}_0$ in exchange.
2. Borrow the stock from the market and sell it for $S_0$.
3. Invest $Ke^{-rT} + x$ in the money market account

The net cash flows of the transactions above will be equal to zero.

**At expiry $T$:**

1. Withdraw $K + xe^{rT}$ from the money market account.
2. Receive the asset $S$ from the counter-party of the forward contract and pay $K$ in exchange.
3. Give back the borrowed stock.

Again, we started with a zero cost portfolio and ended up with a profit. In terms of our arbitrage portfolio, let $\psi_0 = 1$, $\Delta_0 = -1$ and $\phi_0 = Ke^{-rT} + x$. The value of the portfolio $\pi_0$ at time zero is given by

$$\pi_0 = \psi_0 \tilde{V}_0 + \Delta_0 S_0 + \phi_0 B_0 = 0. \tag{13}$$

The value of the portfolio at time $t = T$ was equal to

$$\begin{aligned} \pi_T &= \psi_0 \tilde{V}_T + \Delta_0 S_T + \phi_0 B_T & (14) \\ &= (S_T - K) - S_T + K + xe^{rT} & (15) \\ &= xe^{rT}, & (16) \end{aligned}$$

which is again strictly positve.

If we assume that the market is arbitrage free, then we must conclude that the value at time zero of a forward contract must be equal to

$$V_0 = S_0 - Ke^{-rT}. \tag{17}$$

### 1.2.2 Risk Neutral Pricing Approach

In the previous section we showed how to calculate and hedge a simple contingent claim (a forward contract) by constructing a replicating portfolio which is arbitrage free. The price of such a claim can be calculated using a probabilistic approach. In fact, it is often easier to derive the price of a claim using the probabilistic approach.

Once the no arbitrage price is known, it will be also easier to find the hedging strategy for the claim.

In order to do that we need to introduce the fundamental theorem of asset pricing. In the simple one period framework, the theorem reduces to:

**Theorem 1** *There is no arbitrage if and only if there exists a measure* $\mathbf{Q}$*, which we shall refer to as the* ***Risk Neutral Measure****, such that*

$$E^{\mathbf{Q}}[V_T B_T^{-1}] = V_0, \tag{18}$$

*for any (admissible) asset* $V_T$.

For example we must have that

$$E^{\mathbf{Q}}[S_T B_T^{-1}] = S_0. \tag{19}$$

The price of a forward contract with $V_T = (S_T - K)$ can be derived as follows

$$\begin{aligned} V_0 &= E^{\mathbf{Q}}[V_T B_T^{-1}] \\ &= E^{\mathbf{Q}}[(S_T - K) B_T^{-1}] \\ &= E^{\mathbf{Q}}[S_T B_T^{-1}] - K B_T^{-1} \\ &= S_0 - K e^{-rT}. \end{aligned}$$

In particular, the **Forward Price** $F(0,T)$ at time zero for a forward contract expiring at time $T$, is the strike K such that $V_0 = 0$. From the result above, it is straightforward to see that the forward price is equal to

$$F(0,T) = S_0 e^{rT}. \tag{20}$$

## 1.3 The One-Period Binomial Model

In this section we shall show how to price a simple option using the one period binomial model.

A **Call Option** gives the buyer the right (but not the obligation) to buy a given asset, e.g. a stock $S$, at a pre-determined (strike) price $K$ at a future date $T$. The pay-off of the contract at expiry $T$ is given by

$$C_T \equiv (S_T - K)^+. \tag{21}$$

Let's assume that there asset $S$ follows the binomial dynamics:

$$\begin{tikzcd}
 & S_T^u \\
S_0 \arrow[ur] \arrow[dr] & \\
 & S_T^d
\end{tikzcd}$$

The option dynamics will be then given by

$$\begin{tikzcd}
 & C_T^u \\
C_0 =? \arrow[ur] \arrow[dr] & \\
 & C_T^d
\end{tikzcd}$$

where $C_T^u = (S_T^u - K)^+$ and $C_T^d = (S_T^d - K)^+$.

### 1.3.1 Hedging Approach

The basic idea is to find a (non-trivial) portfolio consisting of an investment in the option $C$, the risky asset $S$ and the money market account $B$ with zero value at time $t = 0$ and expiry $T$. Equivalently, we would like to find a portfolio consisting of an investment in $S$ and $B$ such that its value it is equal to the pay-off of the option $C$ for all possible values of $S$ at time $T$. Without loss of generality we shall set the expiry $T = 1$.

In the simple set up we are analyzing, the hedging problem can be reduced to solving a simple system of two equations in two unknowns, namely $\Delta_0$ i.e. the investment in the risky asset and $\phi_0$ i.e. the investment in the money market account at inception. More precisely, we need to solve the system of equations

$$\begin{cases} C_1^u = \Delta_0 S_1^u + \phi_0 B_1 \\ C_1^d = \Delta_0 S_1^d + \phi_0 B_1, \end{cases} \tag{22}$$

where we have used the self-financing property (3). It is straightforward to see that the system above admits the solution

$$\begin{cases} \Delta_0 = \frac{C_1^u - C_1^d}{S_1^u - S_1^d}, \\ \phi_0 = (C_1^d - \Delta_0 S_1^d) B_1^{-1}. \end{cases} \tag{23}$$

Imagine you are trader and you have sold a call option to an investor. If you buy the portfolio $(\Delta_0, \phi_0)$ constructed above, you will be perfectly hedged at $t = 1$. In other words, any money you may have to pay to investors if the option expires in the money (i.e. has a positive pay-out) will be perfectly off-set by a gain in the hedging portfolio $(\Delta_0, \phi_0)$. Note also that the $\Delta_0$ is reminiscent of the first derivative of the option price with respect to the price of the underlying asset. We shall encounter this quantity again in the continuous time setting.

If we rule out the possibility of an arbitrage, then it follows the value at time $t = 0$ of the hedging portfolio, say $p_0$, must be equal to the initial price of the option $C_0$. Hence, if we know the hedging strategy $(\Delta_0, \phi_0)$ we can calculate the initial price of the option as follows

$$C_0 = \Delta_0 S_0 + \phi_0. \tag{24}$$

### 1.3.2 Risk Neutral Pricing Approach

As for forward contracts, we can calculate the price of a contingent claim using the fundamental theorem of asset pricing. The first step is to derive the risk neutral probabilities explicitly. This can be done by solving the equation in $q$:

$$S_0 = E^{\mathbf{Q}}[S_1 B_1^{-1}] = [q S_1^u + (1-q) S_1^d] B_1^{-1}. \tag{25}$$

It is straightforward to see that the risk neutral probabilities are given by

$$
\begin{aligned}
q &= \frac{S_0 B_1 - S^d}{S^u - S^d} && (26)\\
1 - q &= \frac{S_1^u - S_0 B_1}{S^u - S^d}. && (27)
\end{aligned}
$$

In order to ensure that probabilities above are non negative, the nodes of the tree must satisfy $S_1^d < S_0 B_1 < S_1^u$.

Once the risk neutral probabilities have been calculated, the price of the option follows immediately from a simple application of the fundamental theorem of asset pricing

$$
C_0 = E^{\mathbf{Q}}[C_1 B_1^{-1}] = [qC_1^u + (1-q)C_1^d]B_1^{-1}. \tag{28}
$$

A natural question to ask is whether the price we obtained using (28) is consistent with the price we obtained under the hedging approach (24). To show that this is indeed the case, consider

$$
\begin{aligned}
C_0 &= \Delta_0 S_0 + \phi_0 \\
&= \Delta_0 S_0 + (C_1^d - \Delta_0 S_1^d) B_1^{-1} \\
&= \left[\Delta_0 (S_0 B_1 - S_1^d) + C_1^d\right] B_1^{-1} \\
&= \left[C_1^u \frac{S_0 B_1 - S_1^d}{S_1^u - S_1^d} + C_1^d \left(1 - \frac{S_0 B_1 - S_1^d}{S_1^u - S_1^d}\right)\right] B_1^{-1} \\
&= [qC_1^u + (1-q)C_1^d]B_1^{-1},
\end{aligned}
$$

which proves that the two approaches are consistent and lead to the same result under no-arbitrage.

**Last but not least** note that as far as derivatives, and in general, asset pricing is concerned, real world probabilities are irrelevant. Even if we knew with certainty the true probability distribution of the asset, it wouldn't affect asset pricing. The key for understanding this apparently counter-intuitive concept is to look at the hedging approach. In order to hedge the simple securities presented so far in the course, all we had to do was to borrow or lend money via the money market account and have a long or short investment in the stock. Moreover, we were able to perfectly replicate contingent claims regardless of the final value of the underlying asset. It is natural thus to expect that the “real world” distribution of the asset will not come to play in our pricing formulae. All we have to know is the current price of the asset and the interest rate (both market observable) [1].

## 1.4 Multi-Period Binomial Model

The model presented in the previous section is clearly too simplistic to be used in practical applications. Fortunately the binomial model can be easily extended to a multi-period set up. In what follows we shall discuss a special case of the binomial approach called Cox Ross and Rubinstein (CRR) model. The main features of the model are the following

1. The node of the tree are recombining (lattice model).
2. “Up” and “Down” probabilities are constant across the various nodes of the tree.
3. All time steps between nodes are equal to $\Delta t$.

The first property makes sure that the branches of the tree do not “explode”, making pricing and risk management difficult. The possible outcomes in a non recombining $N$-steps tree are equal to $2^N$, whereas they are equal to $N+1$ in a recombining tree. The second and third properties allow us to derive very simple formulae for the pricing of derivatives contracts. More importantly, we shall show that for a specific choice of the "up" and "down" values of the asset in the tree, the CRR (discrete) asset price process converges, as the number of steps increases and time interval decreases, to a geometric Brownian motion (GBM), which is the foundation of continuous time pricing and hedging.

Consider a fixed time horizon $T$ and subdivide $T$ in $N$ steps of equal size $\Delta t$, i.e. let $\Delta t = T/N$. Let $u$ and $d$ be two real numbers satisfying the constraint

$$d < e^{r\Delta t} < u. \tag{29}$$

[1] The future values of the asset can be chosen arbitrarily as long as $S_1^d < S_0 B_1 < S_1^u$

The asset price $S_{n,i}$ at time $t_n \equiv n\Delta t$ is equal to

$$S_{n,i} = S_0 u^{n-i} d^i. \tag{30}$$

The graph below shows the binomial model for $N = 2$.

$$\begin{tikzcd}
& & S_0u^2 \\
& S_0d \arrow[ur] \arrow[dr] & \\
S_0 \arrow[ur] \arrow[dr] & & S_0ud = S_0du \\
& S_0u \arrow[ur] \arrow[dr] & \\
& & S_0d^2
\end{tikzcd}$$

It is easy to see that the risk neutral probabilities in this model are equal to

$$\begin{aligned}
q &= \frac{B_{\Delta t} - d}{u - d}, \qquad (31) \\
1 - q &= \frac{u - B_{\Delta t}}{u - d}, \qquad (32) \\
& \qquad (33)
\end{aligned}$$

for all nodes of the tree. The distribution of the asset $S$ at a given time step $n$ will be binomial and can be calculated explicitly using the formula

$$\mathbf{Q}\left(S_n = S_0 u^{n-i} d^i\right) = \binom{n}{i} q^{n-i} (1-q)^i \tag{34}$$

where $\mathbf{Q}\left(S_n = x\right)$ indicates the risk neutral probability of $S_n$ taking value $x$.

### 1.4.1 Pricing Options Using the CRR Model

The CRR model is a powerful tool for pricing options. As we shall see, it provides a methodology to price not only call and put options, but a much larger class of derivatives claims depending on the underlying assets. The CRR is particularly useful when no closed formula is available for a given option in the corresponding continuous time GBM set up.

Let's consider a generic pay-off $V_T = \Xi(S_T)$. This is an European style pay-off because it depends solely on the terminal value of $S$. The initial price $V_0$ of the claim will be equal to

$$V_0 = E^{\mathbf{Q}}[\Xi(S_T)B_T^{-1}] = \sum_{i=0}^{N} \Xi(S_0 u^{N-i} d^i) \binom{N}{i} q^{N-i}(1-q)^i. \tag{35}$$

If $\Xi(S_T) = (S_T - K)^+$ for example, $V_0$ will be the price of a call option. It is instructive to simplify formula (35) in the case a call option (a similar result can be obtained for put option). We shall see that the formula will look surprisingly similar to Black and Scholes formula. Formally let $C_0$ represent the price at time zero of a call option and let $a \equiv \sup\{i : S_0 u^{N-i} d^i > K\}$, then

$$\begin{aligned}
C_0 &= e^{-rT} \sum_{i=0}^{N} (S_0 u^{N-i} d^i - K)^+ \binom{N}{i} q^{N-i}(1-q)^i \\
&= e^{-rN\Delta t} \sum_{i=a}^{N} (S_0 u^{N-i} d^i - K) \binom{N}{i} q^{N-i}(1-q)^i \\
&= S_0 \sum_{i=a}^{N} \binom{N}{i} (uqe^{-r\Delta t})^{N-i} (de^{-r\Delta t}(1-q))^i - Ke^{-r\Delta t} \sum_{i=a}^{N} \binom{N}{i} q^{N-i}(1-q))^i \\
&= S_0 \sum_{i=a}^{N} \binom{N}{i} \bar{q}^{N-i}(1-\bar{q})^i - Ke^{-rT} \sum_{i=a}^{N} \binom{N}{i} q^{N-i}(1-q)^i \\
&= S_0 \mathbf{Q}_1 - Ke^{-rT} \mathbf{Q}_2,
\end{aligned}$$

where we have defined $\bar{q} \equiv uqe^{-r\Delta t}$,

$$\mathbf{Q}_1 \equiv \sum_{i=a}^{N} \binom{N}{i} \bar{q}^{N-i}(1-\bar{q})^i, \tag{36}$$

and

$$\mathbf{Q}_2 \equiv \sum_{i=a}^{N} \binom{N}{i} q^{N-i}(1-q)^i. \tag{37}$$

$\mathbf{Q}_2$ is the (binomial) probability of the underlying asset being greater than the strike at maturity. $\mathbf{Q}_1$ is a similar measure but with slightly modified risk neutral probabilities. We shall see that the Black and Scholes formula derived in a continuous time setting resemble closely the formula above. Also, we shall show that as $N \to \infty$ and $\Delta t \to 0$, the distribution of the CRR asset process $S$ converges to the distribution of a geometric Brownian motion. Moreover, option prices under the binomial model converge to the corresponding GBM prices.

So far we have taken the shape of the CRR tree as given. Is there an intuitive way to choose $u$ and $d$? A convenient choice is the following

$$\begin{aligned} u &= e^{\mu\Delta t+\sigma\sqrt{\Delta t}} \\ d &= e^{\mu\Delta t-\sigma\sqrt{\Delta t}}, \end{aligned}$$

where $\mu \in \mathbf{R}$ and $\sigma \in \mathbf{R}^+$ are constants representing the drift and volatility of the asset respectively.

We would like to derive the limit distribution of the process $S$ as the time step decreases, i.e. for $N \to \infty$ (or equivalently $\Delta t \to 0$). In order to do so, define the random variable $X_N$ to be the number of up moves in the first N steps. The asset process can be thus be written as

$$
\begin{aligned}
S_T^{(N)} &= S_0 u^{X_N} d^{N-X_N} \\
&= S_0 \exp\left(X_N(\mu\Delta t + \sigma\sqrt{\Delta t}) + (N - X_N)(\mu\Delta t - \sigma\sqrt{\Delta t})\right) \\
&= S_0 \exp\left(\mu T + \sqrt{\Delta t}\sigma(2X_N - N)\right) \\
&= S_0 \exp\left(\mu T + \sqrt{T}\sigma\frac{X_N - 1/2N}{\frac{1}{2}\sqrt{N}}\right) \\
&= S_0 \exp\left(\mu T + \sqrt{T}\sigma Z_N\right)
\end{aligned}
$$

where we have defined

$$
Z_N \equiv \frac{X_N - 1/2N}{\frac{1}{2}\sqrt{N}}. \tag{38}
$$

$X_N$ can be represented as a sum of i.i.d. random variables. In particular

$$
X_N = \sum_{i=1}^{N} \epsilon_i, \tag{39}
$$

where

$$
\epsilon_i = \begin{cases} 1 \text{ if } S_i/S_{i-1} = u & \text{with prob} \quad q \\ 0 \text{ if } S_i/S_{i-1} = d & \text{with prob} \quad 1-q. \end{cases} \tag{40}
$$

Each $\epsilon_i$ has a Bernoulli distribution with mean and variance

$$
E^{\mathbf{Q}}[\epsilon_i] = q \tag{41}
$$

$$
Var^{\mathbf{Q}}[\epsilon_i] = q(1-q), \tag{42}
$$

respectively. It follows that $X_N$ has the following mean

$$\begin{aligned} E^{\mathbf{Q}}[X_N] &= E^{\mathbf{Q}}[\sum_{i=1}^{N} \epsilon_i] && (43) \\ &= \sum_{i=1}^{N} E^{\mathbf{Q}}[\epsilon_i] && (44) \\ &= Nq, && (45) \end{aligned}$$

and variance

$$\begin{aligned} Var^{\mathbf{Q}}[X_N] &= Var^{\mathbf{Q}}[\sum_{i=1}^{N} \epsilon_i] && (46) \\ &= \sum_{i=1}^{N} Var^{\mathbf{Q}}[\epsilon_i] && (47) \\ &= Nq(1-q). && (48) \end{aligned}$$

In order to make further progresses, we need to find a simple approximation for the risk neutral probability $q$. This will allow us to derive a simple explicit expression for the mean and variance of $X_N$ and ultimately $Z_N$. Consider the Taylor expansion of $u$, $d$ and $B_{\Delta t}$ in their respective exponents up to terms of order $\Delta t$

$$\begin{aligned} u &= e^{\mu\Delta t + \sigma\sqrt{\Delta t}} \\ &\approx 1 + \mu\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t, \end{aligned}$$

$$\begin{aligned} d &= e^{\mu\Delta t - \sigma\sqrt{\Delta t}} \\ &\approx 1 + \mu\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t, \end{aligned}$$

and

$$\begin{aligned} B_{\Delta t} &= e^{r\Delta t} \\ &\approx 1 + r\Delta t. \end{aligned}$$

Substituting the expansions of $u$, $d$ and $B_{\Delta t}$ into (31) and (32) , we obtain

$$\begin{aligned} q &\approx \frac{1}{2} + \frac{1}{2}\sqrt{\Delta t}\frac{(r-\mu) - 1/2\sigma^2}{\sigma} & (49)\\ 1-q &\approx \frac{1}{2} - \frac{1}{2}\sqrt{\Delta t}\frac{(r-\mu) - 1/2\sigma^2}{\sigma}. & (50) \end{aligned}$$

Using the results above, the mean of $X_N$ can be approximated as follows

$$\begin{aligned} E^{\mathbf{Q}}[X_N] &= Nq & (51)\\ &= \frac{1}{2}N + \frac{1}{2}\sqrt{N}\sqrt{N\Delta t}\frac{(r-\mu) - 1/2\sigma^2}{\sigma} & (52)\\ &= \frac{1}{2}N + \frac{1}{2}\sqrt{N}\sqrt{T}\frac{(r-\mu) - 1/2\sigma^2}{\sigma}. & (53) \end{aligned}$$

It is then straightforward to see that the mean of $Z_N$ is equal to

$$E^{\mathbf{Q}}[Z_N] = \frac{E^{\mathbf{Q}}[X_N] - 1/2N}{1/2\sqrt{N}} = \sqrt{T}\frac{(r-\mu) - 1/2\sigma^2}{\sigma}. \qquad (54)$$

Similarly the the variance of $X_N$ is given by

$$\begin{aligned} Var^{\mathbf{Q}}[X_N] &= Nq(1-q)\\ &= \frac{1}{4}N - \frac{1}{4}N\Delta t\left(\frac{(r-\mu) - 1/2\sigma^2}{\sigma}\right)^2 \end{aligned}$$

and

$$\begin{aligned} Var^{\mathbf{Q}}[Z_N] &= \frac{V^{\mathbf{Q}}[X_N]}{1/4N}\\ &= 1 - \Delta t\left(\frac{(r-\mu) - 1/2\sigma^2}{\sigma}\right)^2. \end{aligned}$$

Note that the mean of $Z_N$ does not depend on $N$. Moreover that variance of $Z_N \to 1$ as $\Delta t \to 0$ (or equivalently $N \to \infty$). Using the Central Limit Theorem ($Z_N$ is a rescaled, mean adjusted sum of the i.i.d. random variables $\epsilon_i$), we can prove that the distribution of $Z_N$ converges to the distribution of normal random variable with mean $m$ and standard deviation $v$ given by

$$
\begin{aligned}
m &= \sqrt{T}\frac{(r-\mu) - 1/2\sigma^2}{\sigma} && (55) \\
v &= 1. && (56)
\end{aligned}
$$

Let $Z = \lim_{N\to\infty} Z_N$ and $Y$ be $N(0,1)$. Then $Z$ admits the following representation

$$Z = m + Y. \tag{57}$$

We have now all the ingredients to derive the limiting distribution of the asset process $S_T^{(N)}$

$$
\begin{aligned}
S_T &\equiv \lim_{N\to\infty} S_T^{(N)} \\
&= S_0 \exp\left(\mu T + \sqrt{T}\sigma Z\right) \\
&= S_0 \exp\left(\mu T + \sqrt{T}\sigma(m+Y)\right) \\
&= S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma W_T\right)
\end{aligned}
$$

where we have defined $W_T \equiv Y\sqrt{T}$. We have just shown that as $N \to \infty$, the dynamics of the asset $S$ under the CRR model converge to a GBM which form the basis of continuous time pricing and the Black and Scholes framework.

### 1.4.2 Martingales and The Fundamental Theorem of Asset Pricing

In a multi-period model, the version of the fundamental theorem of asset pricing introduced in section 1.2.2 needs to be generalised slightly.

**Theorem 2** *There is no arbitrage if and only if there exists a measure* $\mathbf{Q}$*, the* ***Risk Neutral Measure****, with respect to which discounted asset prices are martingales.*

Remember that a process $M$ is called a **Martingale** (relative to ($\mathbf{Q}$, $\mathcal{F}_n$)) if

1. $M$ is adapted,
2. $E^{\mathbf{Q}}[| M_n |] < \infty$, for all $n$,
3. $E^{\mathbf{Q}}[M_n \mid \mathcal{F}_{n-1}] = M_{n-1}$ a.s. $(n \geq 1)$.

Is the CRR model free of arbitrage then? The answer is of course yes. . In order to prove the claim, it is sufficient to show that the process $M_n \equiv S_n B_n^{-1}$ is a $(\mathbf{Q}, \mathcal{F}_n)$ martingale. For simplicity, we shall assume that $\{\mathcal{F}\}_{n \geq 0}$ is the filtration generated by the process $S$.

Consider a generic time $1 \leq n \leq N$ and let $\epsilon_i$ be the series of i.i.d. random variables defined in (40). Observe that

$$
\begin{aligned}
E^{\mathbf{Q}}[M_n \mid \mathcal{F}_{n-1}] &= E^{\mathbf{Q}}[S_n B_n^{-1} \mid \mathcal{F}_{n-1}] \\
&= E^{\mathbf{Q}}[S_{n-1} B_{n-1}^{-1} u^{\epsilon_n} d^{1-\epsilon_n} e^{-r\Delta t} \mid \mathcal{F}_{n-1}] \\
&= S_{n-1} B_{n-1}^{-1} E^{\mathbf{Q}}[u^{\epsilon_n} d^{1-\epsilon_n} e^{-r\Delta t} \mid \mathcal{F}_{n-1}] \\
&= S_{n-1} B_{n-1}^{-1} E^{\mathbf{Q}}[u^{\epsilon_n} d^{1-\epsilon_n}] e^{-r\Delta t} \\
&= S_{n-1} B_{n-1}^{-1} [qu + (1-q)d] e^{-r\Delta t} \\
&= S_{n-1} B_{n-1}^{-1} [q(u-d) + d] e^{-r\Delta t} \\
&= S_{n-1} B_{n-1}^{-1} [e^{r\Delta t} - d + d] e^{-r\Delta t} \\
&= S_{n-1} B_{n-1}^{-1} \\
&= M_{n-1}.
\end{aligned}
$$

Furthermore note that for each $n$, the process $S$ is positive and bounded, which implies that for all $n$

$$E^{\mathbf{Q}}[| S_n |] < \infty. \tag{58}$$

It follows that CRR model is arbitrage free.

### 1.4.3 Hedging in the CRR Model

Hedging in the CRR model is no more difficult than hedging in the simple one period binomial model. The CRR model can in fact be decomposed in a series of one period binomial models.

Start from the penultimate time step and for each node consider the only two possible outcomes. More precisely, consider node $i$ and time $t = N-1$ where $i \in \{0, \ldots, N-1\}$

$$\begin{tikzcd} & S_{N,i} \\ S_{N-1,i} \arrow[ur] \arrow[dr] & \\ & S_{N,i+1} \end{tikzcd}$$

and the corresponding option prices

$$\begin{tikzcd} & C_{N,i} \\ C_{N-1,i} =? \arrow[ur] \arrow[dr] & \\ & C_{N,i+1} \end{tikzcd}$$

Solving a system of equations similar to the one encountered in the one period model, we can derive the hedging ratios as follows

$$\begin{aligned} \Delta_{N-1,i} &= \frac{C_{N,i} - C_{N,i+1}}{S_{N,i} - S_{N,i+1}} \\ \phi_{N-1,i} &= [C_{N,i+1} - \Delta_{N-1,i} S_{N,i+1}] B_{\Delta t}^{-1} \end{aligned}$$

By construction the hedging portfolio above will be equal to the option pay-off $C_N$ no matter whether the stock price goes up or down. Next, we need to derive the option price $C_{N-1,i}$ which can be done using the martingale property of discounted asset (and thus option) prices, i.e.

$$
\begin{aligned}
C_{N-1,i} &= B_{\Delta t}^{-1} E^{\mathbf{Q}}[C_N \mid S_{N-1} = S_{N-1,i}] \\
&= e^{-r\Delta t}[qC_{N,i} + (1-q)C_{N,i+1}].
\end{aligned}
$$

The steps above need to be carried out for all $i \in \{0, \ldots, N-1\}$. Next, move back on time step and repeat the exact same calculations. Summing up, we need to run the following algorithm:

For $k = 1$ to $N$

For $i = 0$ to $N - k$

$$
\begin{aligned}
\Delta_{N-k,i} &= \frac{C_{N-k+1,i} - C_{N-k+1,i+1}}{S_{N-k+1,i} - S_{N-k+1,i+1}} \\
\phi_{N-k,i} &= [C_{N-k+1,i+1} - \Delta_{N-k,i} S_{N-k+1,i+1}] B_{\Delta t}^{-1} \\
C_{N-k,i} &= e^{-r\Delta t}[qC_{N-k+1,i} + (1-q)C_{N-k+1,i+1}].
\end{aligned}
$$

end

end

# 2 Continuous Time Models

## 2.1 Brownian Motion

Brownian motion is one of the fundamental concepts in stochastic calculus and mathematical finance. It is a stochastic process $W$ satisfying:

- $W_0 = 0$.
- $W_t$ is continous in $t$.
- For all $0 = t_0 < t_1 < \cdots < t_m$, the increments $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$, $\ldots$, $W_{t_m} - W_{t_{m-1}}$ are independent.
- Given any positive $t$ and $s$ the random variable $W_{t+s} - W_t$ is normally distributed with mean zero and variance $s$.

In what follows we shall analyse a few properties of the Brownian motion which will be key in the construnction of stochastic integrals and the derivation of Ito's formula.

Partition the time interval $[0, t]$ as follows $\Pi \equiv \{0 = t_0 < t_1, \ldots t_n = t\}$ and let

$$\mid \Pi \mid \equiv \max_{0 \leq i \leq n-1} t_{i+1} - t_i,$$

be the largest interval in the partition.

The quadratic variation of a process $X$ over a fixed interval, say $[0, t]$, is defined as

$$[X, X]_t = \lim_{|\Pi| \downarrow 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2. \tag{59}$$

Note that the quadratic variation is defined path-wise and will in general be a random variable for fixed $t$ or a stochastic process if seen as a function of $t$. We will show however, that in the case of Brownian motion, the quadratic variation $[W, W]_t$ is deterministic and equal to $t$.

**Lemma 3** *The sampled quadratic variation of the Brownian motion $[W, W]_t^n$ over the interval $[0, t]$ converges to $t$ in $\mathcal{L}^2$, more precisely*

$$\lim_{|\Pi|\downarrow 0} E\left([W,W]_t^n - t\right)^2 = 0.$$

*Here we have defined the sampled quadratic variation, i.e. the quadratic variation corresponding to a given partition, as*

$$[W,W]_t^n \equiv \sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2$$

**Proof.** Consider first the expected sampled quadratic variation

$$\begin{aligned}
E[W,W]_t^n &= E\left[\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2\right] \\
&= \sum_{i=0}^{n-1} E[(W_{t_{i+1}} - W_{t_i})^2] \\
&= \sum_{i=0}^{n-1}(t_{i+1} - t_i) \\
&= t.
\end{aligned}$$

The expected sampled quadratic variation is thus independent of partition under consideration. What about the variance?

$$
\begin{aligned}
Var([W,W]_t^n) &= E\left([W,W]_t^n - E[W,W]_t^n\right)^2 \\
&= E\left([W,W]_t^n - t\right)^2 \\
&= E\left(\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 - t\right)^2 \\
&= E\left(\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)\right)^2 \\
&= \sum_{i=0}^{n-1} E\left[(W_{t_{i+1}} - W_{t_i})^4 - 2(t_{i+1} - t_i)(W_{t_{i+1}} - W_{t_i})^2 + (t_{i+1} - t_i)^2\right] \\
&= \sum_{i=0}^{n-1} E\left[3(t_{i+1} - t_i)^2 - 2(t_{i+1} - t_i)^2 + (t_{i+1} - t_i)^2\right] \\
&= 2\sum_{i=0}^{n-1}(t_{i+1} - t_i)^2 \\
&\leq 2 \mid \Pi \mid \sum_{i=0}^{n-1}(t_{i+1} - t_i), \\
&= 2 \mid \Pi \mid t
\end{aligned}
$$

where we have used the fact that increments of the Brownian motion on non overlapping intervals are independent. Now, letting $\mid \Pi \mid$ go to zero (or equivalently letting $n \uparrow \infty$) it is easy to see that $Var([W,W]_t^n) \downarrow 0$. Note also that the second line of the derivation above implies $\mathcal{L}^2$ convergence. ∎

We shall summarise the result in differential form as follows

$$d[W,W]_t = (dW_t)^2 = dt.$$

The quadratic variation of the Brownian motion will play a key role in the construction of Ito's integral. Another important result which we shall need in the derivation of Ito's formula is the following:

**Lemma 4** *The quantity* $[W,T]_t^n$ *defined as*

$$[W,T]_t^n \equiv \sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i),$$

*converges to* 0 *in* $\mathcal{L}^1$, *i.e.*

$$\lim_{|\Pi|\downarrow 0} \mid [W,T]_t^n - 0 \mid = 0.$$

**Proof.** Consider the following

$$\begin{aligned}
\mid [W,T]_t^n \mid &\leq \sum_{i=0}^{n-1} \mid (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i) \mid \\
&= \sum_{i=0}^{n-1} \mid (W_{t_{i+1}} - W_{t_i}) \mid (t_{i+1} - t_i) \\
&\leq \max_{0 \leq i \leq n-1} \mid (W_{t_{i+1}} - W_{t_i}) \mid \sum_{i=0}^{n-1} (t_{i+1} - t_i) \\
&\leq \max_{0 \leq i \leq n-1} \mid (W_{t_{i+1}} - W_{t_i}) \mid t
\end{aligned}$$

which tends to 0 as $\mid \Pi \mid \downarrow 0$ because Brownian paths are continuous. ∎

We shall summarize this result by writing

$$d[W,T]_t = dW_t dt = 0.$$

The last important result we shall need in the derivation of Ito's formula is the following

**Lemma 5**

$$[T,T]_t \equiv \lim_{|\Pi|\downarrow 0} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 \equiv \lim_{|\Pi|\downarrow 0} [T,T]_t^n$$

**Proof.**

$$\begin{aligned}
[T,T]_t^n &= \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 && (60) \\
&\leq \mid \Pi \mid \sum_{i=0}^{n-1} (t_{i+1} - t_i) && (61) \\
&= \mid \Pi \mid t. && (62)
\end{aligned}$$

Letting $\mid \Pi \mid \downarrow 0$ we obtain the desired result. ■

In differential form, we shall write

$$d[T,T]_t = (dt)^2 = 0.$$

## 2.2 Stochastic Integrals

Let $\Delta_t$ be an $\mathcal{F}_t$ adapted process such that

$$E[\int_0^t \Delta_u^2 du] < \infty \tag{63}$$

for all $t$. We would like to make sense of integrals of the type

$$I_t \equiv \int_0^t \Delta_u dW_u \tag{64}$$

where $W_t$ is a Brownian motion adapted to $\mathcal{F}_t$. The stochastic integral above is often called Ito's integral and it is a stochastic process if seen as a function of $t$. Does such an integral exists and in what sense? The integral does not exists in the Rienman-Stieltjes sense because the path of the Brownian motion are nowhere differentiable. However, the stochastic integral can be defined as the limit (in $\mathcal{L}^2$) of simpler integrals.

Changing the notation slightly to emphasise that $I_t$ is a stochastic process, let $0 \leq t \leq T$ and consider the partition $\Pi \equiv \{0 = t_0 < t_1 < \cdots < t_n = T\}$. A **simple process** $\Delta_t^n$ is a piece-wiese constant stochastic process. In particular, let $\Delta_t^n$ be an $\mathcal{F}_t$ adapted process which is constant for all $t \in [t_j, t_{j+1})$ and $j = 0, \ldots, n-1$. For $t \leq T$, let's define the stochastic integral with respect to the simple process above as

$$I_t^n \equiv \int_0^t \Delta_u^n dW_u = \sum_{j=0}^{k-1} \Delta_{t_j}(W_{t_{j+1}} - W_{t_j}) + \Delta_{t_k}(W_t - W_{t_k}). \tag{65}$$

**Lemma 6** *The process $I_t^n$ is a Martingale.*

**Proof.** It is straightforward to see that $E[\mid I_t^n \mid] < \infty$. Moreover $I_t^n$ is adapted because both $W_t$ and $I_t^n$ are $\mathcal{F}_t$ adapted. It remains to prove that

$$E[I_t^n \mid \mathcal{F}_s] = I_s^n, \tag{66}$$

for all $0 \leq s < t$. For simplicity of exposition, we shall limit our attention to the set $s \in \{0 = t_0, \ldots, t_{n-1}\}$.

For $i < k$ consider

$$\begin{aligned} E[I_t^n \mid \mathcal{F}_{t_i}] &= E\left[\sum_{j=0}^{k-1} \Delta_{t_j}(W_{t_{j+1}} - W_{t_j}) + \Delta_{t_k}(W_t - W_{t_k}) \mid \mathcal{F}_{t_i}\right] \\ &= \sum_{j=0}^{k-1} E[\Delta_{t_j}(W_{t_{j+1}} - W_{t_j}) \mid \mathcal{F}_{t_i}] + E[\Delta_{t_k}(W_t - W_{t_k}) \mid \mathcal{F}_{t_i}]. \end{aligned}$$

For $j \geq i$ we have that

$$\begin{aligned} E[\Delta_{t_j}(W_{t_{j+1}} - W_{t_j}) \mid \mathcal{F}_{t_i}] &= E[\Delta_{t_j} E[(W_{t_{j+1}} - W_{t_j}) \mid \mathcal{F}_{t_j}] \mid \mathcal{F}_{t_i}] \qquad (67) \\ &= E[\Delta_{t_j}(E[W_{t_{j+1}} \mid \mathcal{F}_{t_j}] - W_{t_j}) \mid \mathcal{F}_{t_i}] = 0 \qquad (68) \end{aligned}$$

where we have used the tower property of conditional expectations, the fact that the Brownian is a martingale with respect to $\mathcal{F}_t$ and the adaptness of $\Delta_t$. Similarly we have that

$$E[\Delta_{t_k}(W_{t_t} - W_{t_k}) \mid \mathcal{F}_{t_i}] = E[\Delta_{t_k}(E[W_t \mid \mathcal{F}_{t_k}] - W_{t_k}) \mid \mathcal{F}_{t_i}] = 0. \tag{69}$$

The calculations above imply that

$$E[I_t^n \mid \mathcal{F}_{t_i}] = E\left[\sum_{j=0}^{i-1} \Delta_{t_j}(W_{t_{j+1}} - W_{t_j}) \mid \mathcal{F}_{t_i}\right] = I_{t_i}^n. \tag{70}$$

The proof for a general time $s$ is similar. Hence, the stochastic integral for simple integrand processes is a martingale. ∎

Another fundamental property of stochastic integrals is the so called Ito's isometry. This property is very important because it will be used to prove the existence of stochastic integrals for (more) general integrands.

**Lemma 7** *Let $I^n_t$ be defined as in (64); the following equality holds*

$$E[(I^n_t)^2] = E\left[\left(\int_0^t \Delta^n_u dW_u\right)^2\right] = E\left[\int_0^t (\Delta^n_u)^2 du\right]. \tag{71}$$

**Proof.** In order to simplify the notation let

$$\begin{aligned} D_j &= W_{t_{j+1}} - W_{t_j} \qquad j = 0, \ldots, k-1 \\ D_k &= W_t - W_{t_k}. \end{aligned}$$

From elementary algebra it follows that

$$(I^n_t)^2 = \sum_0^k (\Delta^n_{t_j})^2 D_j^2 + 2 \sum_{0 \le i < j \le k} \Delta^n_{t_i} \Delta^n_{t_j} D_i D_j.$$

Note that for $i < j$, using the tower property of conditional expectations and the fact that $\Delta^n$ and $W$ are adapted, we have

$$\begin{aligned} E[\Delta^n_{t_i} \Delta^n_{t_j} D_i D_j] &= E[E[\Delta^n_{t_i} D_i \Delta^n_{t_j} D_j \mid \mathcal{F}_j]] \\ &= E[\Delta^n_{t_i} D_i \Delta^n_{t_j} E[D_j \mid \mathcal{F}_j]] \\ &= E[\Delta^n_{t_i} D_i \Delta^n_{t_j}] E[D_j] \\ &= 0 \end{aligned}$$

where we have used the fact that

$$\begin{aligned} E[D_j \mid \mathcal{F}_j]] &= E[D_j] && (72) \\ &= E[W_{t_{j+1}} - W_{t_j}] && (73) \\ &= E[W_{t_{j+1}}] - E[W_{t_j}] = 0 + 0. && (74) \end{aligned}$$

From the calculations above and the linearity of the expectation operator, it follows that

$$E[(I^n_t)^2] = \sum_{j=0}^k E[(\Delta^n_{t_j})^2 D_j^2]. \tag{75}$$

Using again the tower property, the terms on the right hands side of the summation can be simplified as follows

$$\begin{aligned} E[(\Delta^n_{t_j})^2 D_j^2] &= E[E[(\Delta^n_{t_j})^2 D_j^2 \mid \mathcal{F}_j]] \\ &= E[(\Delta^n_{t_j})^2 E[D_j^2 \mid \mathcal{F}_j]] \\ &= E[(\Delta^n_{t_j})^2 E[(W_{t_{j+1}} - W_{t_j})^2 \mid \mathcal{F}_j]] \\ &= E[(\Delta^n_{t_j})^2 E[(W_{t_{j+1}} - W_{t_j})^2]] \\ &= E[(\Delta^n_{t_j})^2] E[(W_{t_{j+1}} - W_{t_j})^2] \\ &= E[(\Delta^n_{t_j})^2](t_{j+1} - t_j) \end{aligned}$$

Finally, substituting the result above into (75) and exchanging the oder of summation and expectation, we obtain the desired result, i.e.

$$E[(I^n_t)^2] = \sum_0^k E[(\Delta^n_{t_j})^2](t_{j+1} - t_j) = E\left[\int_0^t (\Delta^n_u)^2 du\right].$$

■

We have thus succedeed in proving that Ito's isometry holds when the integrand is a simple process. What about the quadratic variation of such stochastic integrals?

**Theorem 8**

$$E([I^n, I^n]_t) = E\left[\int_0^t (\Delta^n_u)^2 du\right] = E\left[(I^n_t)^2\right]. \tag{76}$$

*In other words, the quadratic variation of the Ito's integral is equal to its variance.*

**Proof.** We can sub-divide each interval $[t_j, t_{j+1})$ of the partition $\Pi$ in smaller sub-intervals $[s_{i,j}, s_{i+1,j})$ for $i = 0, \ldots, m-1$ such that $s_{0,j} = t_j$ and $s_{m,j} = t_{j+1}$. For simplicity, but without loss of generality, assume that $t = T = T_n$.

$$
\begin{aligned}
[I^n, I^n]_t &= \lim_{m\uparrow\infty} \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} (I^n_{s_{i+1,j}} - I^n_{s_{i,j}})^2 \\
&= \sum_{j=0}^{n-1} (\Delta^n_{t_j})^2 \lim_{m\uparrow\infty} \sum_{i=0}^{m-1} \left(W_{s_{i,j+1}} - W_{s_{i,j}}\right)^2 \\
&= \sum_{j=0}^{n-1} (\Delta^n_{t_j})^2 (t_{j+1} - t_j) \\
&= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (\Delta^n_{t_j})^2 du \\
&= \int_0^t (\Delta^n_u)^2 du
\end{aligned}
$$

■

Is it possible to generalise the notion of stochastic integral to more general integrands? The answer is of course yes. In what follows, we shall consider the class of adapted integrand $\Delta_t$ satisfying property (63). Note that the process $\Delta$ may in general be discontinuous.

**Lemma 9** *Given a process $\Delta$ satisfying property (63) and a time $t$, there exists a sequence of simple processes $\Delta^n$ converging to $\Delta$ in the sense below*

$$
\lim_{n\uparrow\infty} E\left[\int_0^t (\Delta^n_u - \Delta_u)^2 du\right] = 0. \tag{77}
$$

*The Ito's integral for such a process will be defined as*

$$
I_t \equiv \int_0^t \Delta_u dW_u = \lim_{n\to\infty} \int_0^t \Delta^n_u dW_u.
$$

*where the limits holds in $\mathcal{L}^2$.*

**Proof.** A full proof of the lemma above is beyond the scope of this notes and we shall present only a sketch of the main steps. In particular, we shall take for granted that given $\Delta_u$, an approximating sequence satististisfying (77) actually exists.

We need to show that if we construct a series of stochastic integrals $I_t^n$ starting from the simple processes $\Delta^n$, this sequence converges to something. We shall define this 'something' to be the stochastic integral for $\Delta$. Note that the integrals

$$I_t^n \equiv \int_0^t \Delta_u^n dW_u,$$

form a Cauchy sequence in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbf{P})$. Using Ito's isometry we can show that for any $\epsilon > 0$, there exists an integer $N$, such that for all $n, m \geq N$,

$$\begin{aligned} E\left[(I_t^n - I_t^m)^2\right] &= E\left(\int_0^t \Delta_u^n dW_u - \int_0^t \Delta_u^m dW_u\right)^2 \\ &= E\left(\int_0^t (\Delta_u^n - \Delta_u^m) dW_u\right)^2 \\ &= E\left[\int_0^t (\Delta_u^n - \Delta_u^m)^2 du\right] < \epsilon. \end{aligned}$$

The calculations above show that the sequence $I_t^n$ is Cauchy and converges in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbf{P})$.
■

Note also that the proof above and (76) imply that

$$E\left[I_t^2\right] = E\left(\int_0^t \Delta_u dW_u\right)^2 = E\left(\int_0^t \Delta_u^2 du\right),$$

i.e., Ito's isometry holds for general integrands. Last but not least, the process $I_t$ is a martingale.

**Summary of Important Properties**

- The Ito's integral $I_t$ for general adapted, square integrable processes (i.e. processes satisfying condition (63) )is well defined.
- $I_t$ is a martingale.
- Ito's isometry holds:
$$E[I_t^2] = E\left[\int_0^t \Delta_u^2 du\right].$$
- The expected quadratic variation of $I_t$ is equal to its variance, i.e.
$$E([I, I]_t) = E\left[\int_0^t (\Delta_u)^2 du\right] = E[I_t^2].$$

## 2.3 Ito's Formula

Ito's formula allows us to study the behaviour of functions of the Brownian motions and other more general processes. It is a very powerful tool in stochastic calculus and derivatives pricing. We shall make use of Ito's formula extensively in our analysis of the celebrated Black and Scholes model.

**Lemma 10** *Let $f(t,w)$ be a real-valued function with continuous first and second derivative and let $W_t$ be standard Brownian motion. The following holds:*

$$f(T,W_T) = f(t,W_t)+\int_t^T f_t(u,W_u)du+\int_t^T f_w(u,W_u)dW_u+\frac{1}{2}\int_t^T f_{ww}(u,W_u)dt. \quad (78)$$

*where $0 \leq t \leq T$ and $f_t$, $f_w$, $f_{ww}$ indicate the partial derivative of $f(t,w)$ with respect to $t$ and $w$.*

It is convenient, when doing calculations, to write formula above in differential form. The following expression does not have a mathematical meaning of its own and should only be consider a short-cut.

$$df(t,W_t) = f_t(t,W_t)dt + f_w(t,W_t)dW_t + \frac{1}{2}f_{ww}(t,W_t)dt. \quad (79)$$

**Proof.** We shall present a heuristic proof of Ito's formula. Consider the stochastic Taylor expansion of $f(t,W_t)$

$$\begin{aligned} df(t,W_t) = f_t(t,W_t)dt + f_w(t,W_t)dW_t &+ \frac{1}{2}f_{ww}(t,W_t)(dW_t)^2 + \frac{1}{2}f_{tt}(t,W_t)(dt)^2 \\ &+ f_{tw}(t,W_t)dtdW_t + \mathcal{O}(3,3) \end{aligned}$$

where $\mathcal{O}(3,3)$ indicate terms of oder 3 and above in $t$ and $W_t$. Using the equalities

$$\begin{aligned} d[W,W]_t &= (dW_t)^2 = dt \\ d[T,T]_t &= (dt)^2 = 0 \\ d[W,T]_t &= dtdW_t = 0. \end{aligned}$$

it is easy to see that terms $\mathcal{O}(3,3)$ must all be equal to zero and equation (80) simplifies to (79). $\blacksquare$

A simple extension of ito's formula will allow us to deal with functions $f(t, X_t)$ of general Ito's processes.

**Lemma 11** *Let $f(t,w)$ be a real-valued function with continuous first and second derivative and let $X_t$ an Ito's process, i.e. a stochastic process of the form*

$$X_t = X_0 + \int_0^t \Delta_u dW_u + \int_0^t \phi_u dB_u, \tag{80}$$

*where $B_t$ is an adapted process of finite variation. Here $\Delta_t$ and $B_t$ satisfy*

$$E\left[\int_0^T \Delta_t^2 dt + \int_0^T \mid B_t \mid dt\right] < \infty.$$

*The following holds*

$$f(T, X_T) = f(t, X_t) + \int_t^T f_t(u, X_u)du + \int_t^T f_w(u, X_u)dX_u + \frac{1}{2}\int_t^T \Delta_u^2 f_{ww}(u, X_u)du, \tag{81}$$

*where as before $0 \leq t \leq T$ and $f_t$, $f_w$, $f_{ww}$ indicate the partial derivative of $f(t,w)$ with respect to $t$ and $w$ respectively.*

**Proof.** The proof of Ito's formula for general Ito processes is similar to the one encountered for the Brownian motion. The first step is to perform a Taylor expansion of $f(t, X_t)$

$$\begin{aligned} df(t, X_t) = f_t(t, X_t)dt + f_w(t, X_t)dX_t &+ \frac{1}{2}f_{ww}(t, X_t)(dX_t)^2 + \frac{1}{2}f_{tt}(t, X_t)(dt)^2 \\ &+ f_{tw}(t, X_t)dtdX_t + \mathcal{O}(3,3). \end{aligned}$$

Note that $X_t$ can be express in diffential form as

$$dX_t = \Delta_t dW_t + \phi_t dB_t. \tag{82}$$

Now, the quadratic variation $d[X,X]_t$ of the process $X_t$ is equal to

$$\begin{aligned} d[X,X]_t = (dX_t)^2 &= (\Delta_t dW_t + \phi_t dB_t)^2 \\ &= \Delta_t (dW_t)^2 + \phi_t^2 (dB_t)^2 + \Delta_t \phi_t dB_t dW_t \\ &= \Delta_t^2 dt \end{aligned}$$

where we have used the fact that the quadratic variation $d[B,B]_t = (dB_t)^2$ of a finite variation process is equal to zero and the covariation $d[B,W]_t = dB_t dW_t$ of the Brownian motion with a finite variation process is also equal to zero

$$\begin{aligned} dB_t &= B_t^{'} dt \\ (dB_t)^2 &= (B_t^{'})^2 (dt)^2 = 0 \\ dB_t dW_t &= B_t^{'} (dt dW_t) = 0. \end{aligned}$$

Similar calculations show that the term $dtdX_t$ also vanishes. Substituting the results above into the Taylor expansion of $f(t, X_t)$ we obtain the desired result. ■

Another powerful tool is the stochastic version of the integration by parts (IBP) formula which we shall report without proof.

**Lemma 12** *Let $X_t$ and $Y_t$ be two Ito processes. The following is true*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_u dY_u + \int_0^t Y_u dX_u + [X,Y]_t \tag{83}$$

*which in differential form reads*

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t, \tag{84}$$

*where as usual we have used the short-cut $d[X,Y]_t = dX_t dY_t$.*

## 2.4 Applications of Ito's lemma and IBP formula

### 2.4.1 Black and Scholes Stock Dynamics

Consider the following stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{85}$$

for some real $\mu$ and positive $\sigma$. The process above is often used to model stock dynamics. Note that (85) is a stochastic differential equation because $S_t$ appears on both sides of the equal sign. We can use Ito's formula to find a solution to the SDE. Consider the function

$$f(w) = \ln w.$$

Using Ito's formula to calculate $df(S_t)$ and noting that $f_t(w) = 0$, $f_w(w) = 1/w$ and $f_{ww}(w) = -1/w^2$, we have that

$$\begin{aligned}
df(S_t) = d\ln S_t &= \frac{1}{S_t}dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2 \\
&= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2}(\mu S_t dt + \sigma S_t dW_t)^2 \\
&= \mu dt + \sigma dW_t - \frac{1}{2}\frac{1}{S_t^2}(\mu S_t^2 (dt)^2 + 2\sigma\mu dt dW_t + \sigma^2 S_t^2 dW_t) \\
&= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t,
\end{aligned}$$

which in integral form can be written as

$$\begin{aligned}
\ln S_t &= \ln S_0 + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) du + \int_0^t \sigma dW_u \\
&= \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.
\end{aligned}$$

Finally exponentiating both side of equality

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right]. \tag{86}$$

### 2.4.2 Vasicek Model

Vasicek model, often use in interest rate theory modelling, provides us with an interesting application of the integration by parts formula. Consider the SDE

$$dX_t = (\alpha - \beta X_t)dt + \sigma dW_t.$$

The standard way to solve the SDE above is to apply the IBP formula to the product $e^{\beta t}X_t$

$$\begin{aligned} d(e^{\beta t}X_t) &= e^{\beta t}\left((\alpha - \beta X_t)dt + \sigma dW_t\right) + \beta e^{\beta t}X_t + de^{\beta t}dX_t & (87) \\ &= e^{\beta t}\left(\alpha dt + \sigma dW_t\right). & (88) \end{aligned}$$

Multiplying both side of the equality $e^{\beta t}$ and integrating after some re-arrangment we obtain

$$X_t = X_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma \int_0^t e^{-\beta(t-u)}dW_u.$$

## 2.5 The Black and Scholes Model

Despite its limitations, Black and Scholes is still fundamental to understand the pricing and hedging of financial derivatives in continuous time. Assume that the dynamics of the underlying asset $S$ satisfy the SDE

$$dS_t = \mu S_t dt + \sigma dW_t,$$

under the *physical measure* $\mathcal{P}$. As usual $\mu$ is a real number, $\sigma$ is a positive real number and $W_t$ is a standard Brownian motion.

### 2.5.1 Hedging Approach

Assume we have bought a claim which pays $C_T = \Phi(S_T)$ at maturity, for some pay-off function $\Phi(x)$. How do we go about hedging the claim? What is the value of such a claim at time before expiry?

Similarly to the discrete time framework, we need to form a portfolio consisting of $\Delta_t$ units of the asset and $\phi_t$ units of the money market account with value equal to the option price (say $C_t$) for any $t$ before expiry. In order words, we shall impose that

$$\pi_t \equiv \Delta_t S_t + \phi_t B_t = C_t, \tag{89}$$

for all $0 \leq t < T$. Again we shall restrict our attention to self-financing portfolios. In continous time a portfolio is self-financing if it satisfies

$$\pi_t = \pi_0 + \int_0^t \Delta_u dS_u + \int_0^t \phi_u dB_u, \tag{90}$$

for all $0 \leq t < T$. Equations (89) and (90) imply that the initial price of the option will be given by

$$C_0 = \pi_0 = \Delta_0 S_0 + \phi_0, \tag{91}$$

and

$$C_T = \pi_0 + \int_0^T \Delta_u dS_u + \int_0^T \phi_u dB_u.$$

Pricing and hedging the option amounts to finding the hedging ratios $\Delta_t$ and $\phi_t$ for all $0 \leq t < T$. If we assume for now that we have found a portfolio ($\Delta_0$, $\phi_0$) such that (91) holds, in oder for $\pi_t$ to equal $C_t$ or equivalently for $X_t \equiv C_t - pi_t$ to hold for all $t$, we must have

$$dX_t = dC_t - d\pi_t = 0.$$

Assuming that $C_t = C(t, S_t)$ and $C(t, S) \in \mathcal{C}^2$ i.e. the price of the option is a function of the current time and asset price with twice differentiable, continuous derivatives, we can apply Ito's formula to $dX_t$

$$
\begin{aligned}
dX_t &= dC(t,S_t) - \Delta_t dS_t - \phi_t dB_t \\
&= \left( C_t(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{SS}(t,S_t) \right) dt + C_S(t,S_t)dS_t - \Delta_t dS_t - r\phi_t B_t dt.
\end{aligned}
$$

By choosing

$$\Delta_t = C_S(t,S_t), \tag{92}$$

we can eliminate the asset risk impact from the portfolio $X_t$. Moreover from (89) and the equation above, it follows that

$$\phi_t = (C(t,S_t) - C_S(t,S_t)S_t)B_t^{-1}, \tag{93}$$

which in turns implies

$$dX_t = \left( C_t(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{SS}(t,S_t) - rC(t,S_t) + rS_t C_S(t,S_t) \right) dt \tag{94}$$

In oder for $dX_t = 0$ and $X_T = 0$, the following PDE needs to be satisfied

$$\begin{cases} C_t(t,S) + \frac{1}{2}\sigma^2 S^2 C_{SS}(t,S) + rSC_S(t,S) - rC(t,S) = 0 \\ C(T,S) = \Phi(S) \end{cases} \tag{95}$$

for all $S$ and $0 \leq t < T$.

The price of the option $C(t,S)$ can be calculated by solving the PDE above analytically (if possible) or numerically. Once the price of the option has been calculated, the hedging policy can be obtained formulae (92) and (93).

### 2.5.2 Feynman-Kac Theorem

In the case of call and put options it is possible to solve the Black and Scholes PDE above explicitly by reducing it, via some variable transformation, to the heat equation. However, it is easier as well as instructive to solve the pricing problem via a probabilistic approach and show that the solution is also the solution of the Black and Scholes PDE.

Feynman-Kac theorem establishes a connection between the PDE and the probabilistic representation of the option price.

**Theorem 13** *Let $F(t,x)$ solve the following PDE*

$$\begin{cases} F_t(t,x) + \frac{1}{2}\sigma^2 x^2 F_{xx}(t,x) + kxF_x(t,x) - kF(t,x) = 0 \\ C(T,x) = \Phi(x) \end{cases}$$

*and let*

$$dX_t = kX_t dt + \sigma X_t dW_t$$

*where $W_t$ is a standard Brownian motion, $k, \sigma \in \mathbf{R}$. If*

$$E\left[\int_0^T \left(e^{-kt}\sigma X_t F_x(t,X_t)\right)^2 dt\right] < \infty, \tag{96}$$

*then*

$$F(t,X_t) = E\left[e^{-k(T-t)}\Phi(X_T) \mid \mathcal{F}_t\right].$$

**Proof.** In order to prove the claim, let's apply the IBP formula and Ito's formula to the product $e^{-kt}F(t,X_t)$

$$\begin{aligned} d(e^{-kt}F(t,X_t)) &= -ke^{-kt}F(t,X_t)dt + e^{-kt}dF(t,X_t) \\ &= -ke^{-kt}F(t,X_t)dt + e^{-kt}\left(F_t(t,X_t) + \frac{1}{2}\sigma^2 X_t^2 F_{xx}(t,X_t) + kX_tF_x(t,x)\right)dt \\ &+ e^{-kt}\sigma X_t F_x(t,x)dW_t \\ &= e^{-kt}\left(F_t(t,X_t) + \frac{1}{2}\sigma^2 X_t^2 F_{xx}(t,X_t) + kX_tF_x(t,x) - kF(t,X_t)\right)dt \\ &+ e^{-kt}\sigma X_t F_x(t,x)dW_t. \end{aligned}$$

The $dt$ terms in the expression above is equal to 0 by assumption. Integrating and taking expectations on both side of the equality, we obtain

$$E\left[e^{-kT}F(T,X_T)\mid\mathcal{F}_t\right]=e^{-kt}F(t,X_t)+E\left[\int_t^T\sigma e^{-ku}X_uF_x(u,x)dW_u\mid\mathcal{F}_t\right].$$

The last term on the RHS of the equality is equal to zero because the stochastic integral

$$I_t=\int_0^t\sigma e^{-ku}X_uF_x(u,x)dW_u$$

is a Martingale (note condition (96)). Rearranging and using the terminal condition, we finally obtain

$$F(t,X_t)=E\left[e^{-k(T-t)}\Phi(X_T)\mid\mathcal{F}_t\right]$$

■

We can now make use of Feynman-Kac theorem to solve the Black and Scholes PDE. Setting $k=r$ and amending the notation slightly, we have that

$$C(t,S_t)=E^{\mathcal{Q}}\left[e^{-r(T-t)}\Phi(S_T)\mid\mathcal{F}_t\right],\tag{97}$$

where

$$dS_t=rS_tdt+\sigma S_td\tilde{W}_t,\tag{98}$$

and $\tilde{W}_t$ is a $\mathcal{Q}$ Brownian motion.

In other words, the Black and Scholes PDE, implies that the risk neutral drift rate of the underlying process must be given by $r$. We shall prove that this are the risk neutral dynamics, i.e. $\mathcal{Q}$ is the risk neutral measure and under $\mathcal{Q}$ all discounted asset price processes (including $S_t$) are Martingales.

In order to derive the price $C(t,S_t)$ of the option we can either solve the expectation (97) or the Black and Scholes PDE. For general pay-offs, one approach may be better than the other, but in the case of calls and puts they are both relatively simple and lead to close form solutions. Before deriving Black and Scholes formula explicitly however we shall introduce the concept change of measure.

### 2.5.3 Change of Measure and Girsanov Theorem

Two measures $\mathcal{P}$ and $\mathcal{Q}$ are said to be equivalent if they agree on a set of probability zero, i.e. if $A$ is a set such that $\mathcal{P}(A)=0$ then $\mathcal{Q}(A)=0$ and vice-versa. Whenever two measures are equivalent, we can define one as a functional of the other.

**Theorem 14** *Let $\mathcal{P}$ and $\mathcal{Q}$ be equivalent, then there exists a random variable $Z$, such that*

$$\mathcal{P}(\{Z > 0\}) = 1,$$

$$E^{\mathcal{P}}[Z] = 1$$

*and*

$$\mathcal{Q}(A) = E^{\mathcal{P}}[1_{\{A\}}Z]$$
$$\mathcal{P}(A) = E^{\mathcal{P}}\left[1_{\{A\}}\frac{1}{Z}\right]$$

*for all $A \in \mathcal{F}$.*

*The reverse is also true, i.e. if such $Z$ exists, then the measures $\mathcal{P}$ and $\mathcal{Q}$ defined above are equivalent.*

The theorem above, known as Radon-Nykodim theorem, also implies that if $\mathcal{P}$ and $\mathcal{Q}$ are equivalent, then

$$E^{\mathcal{Q}}(X) = E^{\mathcal{P}}[XZ]$$

and

$$E^{\mathcal{P}}(X) = E^{\mathcal{Q}}\left[X\frac{1}{Z}\right]$$

where $X$ is an integrable random variable.

How do the dynamics of the Brownian motion and in general the asset dynamics $S_t$ change when we change the measure? What is the relationship between the physical measure $\mathcal{P}$ and the risk neutral measure $\mathcal{Q}$? Girsanov's theorem will help us answering those questions.

**Theorem 15** *Let $W_t$ be a Brownian motion with respect to $(\mathcal{P}, \mathcal{F}_t)$ and let $\theta_t$ be an $\mathcal{F}_t)$ adapted process such that*

$$E\left[\exp\left(\frac{1}{2}\int_0^T \theta_u^2 du\right)\right] < \infty.$$

*Define the process*

$$L_t \equiv \frac{dQ}{dP}\mid_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\int_0^t \theta_u^2 du - \int_0^t \theta_u dW_u\right),$$

*and*

$$Q(A) = E^{\mathcal{P}}[1_{\{A\}} L_t]$$

*for all $A \in \mathcal{F}$. Note that $L_t$ is a $(\mathcal{P}, \mathcal{F}_t)$ Martingale.*

*Then the process*

$$\tilde{W}_t = W_t + \int_0^t \theta_u du$$

*is a $(\mathcal{Q}, \mathcal{F}_t)$ Brownian motion. Moreover, given a stochastic process $X_t$ and $0 \leq s < t$, the following equality holds*

$$E^{\mathcal{Q}}[X_t \mid \mathcal{F}_s] = \frac{E^{\mathcal{P}}[X_t L_t \mid \mathcal{F}_s]}{L_s} \tag{99}$$

Intuitively, changing measure is equivalent to shifting the mean of Brownian motion from zero to $-\int_0^t \theta_u du$. If $W_t$ is Brownian motion under $\mathcal{P}$, under $\mathcal{Q}$ it will be a Brownian motion (a $\mathcal{Q}$ Brownian motion to be precise) minus a drift term.

The following theorem makes use of Girsanov's theorem to give a characterisation of the notion of no-arbitrage.

**Theorem 16** *If there is no arbitrage, then there exists a probability measure $\mathcal{Q}$ (the Risk Neutral Measure) equivalent to $\mathcal{P}$ (the Physical or Real Word Measure) under which discounted asset prices are Martingales. In particular the process*

$$\tilde{S}_t \equiv S_t e^{-rt},$$

*is a* $(\mathcal{Q}, \mathcal{F}_t)$ *Martingale. Moreover, the change of measure density is given by*

$$L_t \equiv \frac{dQ}{dP} \mid_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\theta^2 t - \theta W_t\right),$$

*with*

$$\theta = \frac{\mu - r}{\sigma}.$$

**Proof.** Under $\mathcal{P}$ a simple application of the IBP formula shows that the dynamics of $\tilde{S}_t$ are given by

$$d\tilde{S}_t = (\mu - r)\,\tilde{S}_t dt + \sigma \tilde{S}_t dW_t.$$

Making use of Girsanov's theorem, we can derive the dynamics of $W_t$ under $\mathcal{Q}$

$$dW_t = d\tilde{W}_t - \theta dt = d\tilde{W}_t - \frac{\mu - r}{\sigma} dt,$$

where $\tilde{W}_t$ is a $\mathcal{Q}$ Brownian motion. The dynamics of $\tilde{S}_t$ under the risk neutral measure will then be

$$d\tilde{S}_t = \sigma \tilde{S}_t d\tilde{W}_t$$

or equivalently

$$\tilde{S}_t = S_0 + \sigma \int_0^t \tilde{S}_u d\tilde{W}_u$$

It follows that $\tilde{S}_t$ is a Martingale under the risk neutral measure $\mathcal{Q}$.

■

From the results above,it is easy to see (the proof is left as an exercise) that the process $S_t$ has the following $\mathcal{Q}$ dynamics

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t,$$

which are the same dynamics we found using the PDE approach in conjunction with Feynman-Kac theorem.

## 2.6 Black and Scholes Formula

We are now ready to calculate the price of a call option (deriving the price of a put option is left as an exercise). We shall solve the problem using the probabilistic approach rather than solving the PDE. As mentioned before however, the two approaches are equivalent and lead to the same result. Remember that the pay-off of a call option with strike $K$ and maturity $T$ is equal to

$$\Phi(S) = (S - K)^+.$$

The price of a call option at any time $t$ prior to expiry will be the given by

$$\begin{aligned}
C(t, S_t) &= E_t^{\mathcal{Q}}\left[(S_T - K)^+\right] e^{-r(T-t)} \\
&= E_t^{\mathcal{Q}}\left[(S_T - K)\, 1_{\{S_T > K\}}\right] e^{-r(T-t)} \\
&= e^{-r(T-t)} E_t^{\mathcal{Q}}[S_T 1_{\{S_T > K\}}] - Ke^{-r(T-t)} \mathcal{Q}_t(\{S_T > K\}) \\
&\equiv P_1 - P_2.
\end{aligned}$$

Using the properties of Brownian motion, the second term of the RHS of the equal sign reduces to

$$\begin{aligned}
P_2 &\equiv Ke^{-r(T-t)} \mathcal{Q}_t(\{S_T > K\}) \\
&= Ke^{-r(T-t)} \mathcal{Q}_t(\{S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T - W_t)} > K\}) \\
&= Ke^{-r(T-t)} \mathcal{Q}_t\left(\{\tilde{W}_T - \tilde{W}_t > \frac{\ln(K/S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma}\}\right) \\
&= Ke^{-r(T-t)} \mathbf{N}\left(\frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\}\right)
\end{aligned}$$

where $\mathbf{N}(\cdot)$ is the cumulative distribution function of a normal-$(0,1)$ random variable. We can solve the first expectation in a similar way.

$$
\begin{aligned}
P_1 &\equiv e^{-r(T-t)}E_t^{\mathcal{Q}}[S_T 1_{\{S_T>K\}}] \\
&= E_t^{\mathcal{Q}}[S_t e^{\frac{1}{2}\sigma^2(T-t)+\sigma(W_T-W_t)} 1_{\{S_T>K\}}]
\end{aligned}
$$

It is convenient and instructive to use a change of measure to calculate the expectation above. Set

$$
\Lambda_t \equiv \frac{d\hat{Q}}{dQ} |_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\sigma^2 t + \sigma \tilde{W}_t\right)
$$

and let

$$
\hat{W}_t = \tilde{W}_t - \sigma t
$$

be a $\hat{\mathcal{Q}}$-Brownian motion. Using formula (99), we can use the change of expectation to simplify $P_1$ as follows

$$
\begin{aligned}
P_1 &= S_t E_t^{\mathcal{Q}}[\frac{\Lambda_T}{\Lambda_t} 1_{\{S_T>K\}}] \\
&= S_t E_t^{\hat{\mathcal{Q}}}[1_{\{S_T>K\}}] \\
&= S_t \hat{\mathcal{Q}}_t(\{S_T > K\}) \\
&= S_t \hat{\mathcal{Q}}_t\left(\{\hat{W}_T - \hat{W}_t > \frac{\ln(K/S_t) - (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma}\}\right) \\
&= S_t \mathbf{N}\left(\frac{\ln(S_t/K) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}
$$

In summary the Black and Scholes price of a call option is given by

$$
C(t, S_t) = S_t \mathbf{N}(d_t^+) - K e^{-r(T-t)} \mathbf{N}(d_t^-)
$$

where

$$d_t^{+/-} = \frac{\ln(S_t/K) + (r +/- \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$