Probability Distribution Functions

Discrete → Probability Mass Function (PMF)

Continuous → Probability density Function (PDF)

→ CDF

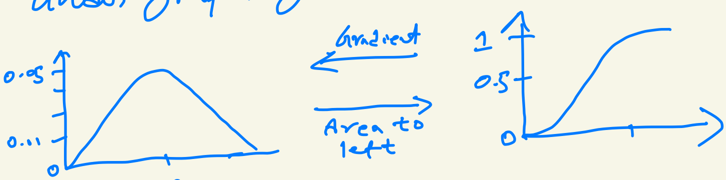
- <u>Discrete</u>: PMF gives prob. of random variable $P(X=x)$. The cdf is $P(X \leq x)$.

- <u>CTS</u>: For CTS dist. $P(X=x)=0$. $f_X(x)$ is the PDF & $F_X(x)$ is the CDF.

  $F_X(x) = P(X \leq x)$; $\int_{-\infty}^{x} f_X(u)\,du = F(x)$

  $F_X'(x) = f_X(x)$

  i.e. Gradient of CDF gives PDF. While area under graph of PDF gives CDF.



PDF

FM02: Applied Probability Revision lecture

Welcome to FM02. The **non-live video lectures** for the course are on **KEATs** (you should watch the first three videos there this week, i.e. all three segments of Lecture 1), which includes the welcome to the course and course admin details (Username and password for course website is FM01Students and Pointbreak), so I won't be repeating that here. In this live lecture we will review some basic but important notions in Applied Probability. Most of these concepts will appear many times on this course and on the other modules for the MSc, and will also be quite important for your summer project next year and in real-life finance jobs. We will not be doing **measure theory** in this lecture (this is covered by the FM01 module). In subsequent lectures I will probably be going over past or mock exam questions.

- A **continuous random variable** X is a random quantity which has the property that for any set A, the probability that the realized value of X lies in the set A is given by

  pdf

$$\mathbb{P}(X \in A) \;=\; \int_A f_X(u)du \tag{1}$$

  where $f_X(x)$ is the **density** of X. Note that X is a random variable here and u is just a dummy variable of integration, and we are assuming that X cannot be $+\infty$ or $-\infty$. In particular, if we set $A = (-\infty, x]$, the **distribution function** of X is defined as

  cdf $\longrightarrow$

$$F_X(x) \;=\; \mathbb{P}(X \in (-\infty, x]) \;=\; \mathbb{P}(X \le x) \;=\; \int_{-\infty}^{x} f_X(u)du\,. \tag{2}$$

  Then differentiating and using the **fundamental theorem of calculus** we see that

$$F'_X(x) \;=\; \frac{d}{dx}F_X(x) \;=\; f_X(x)\,.$$

  Setting $x = \infty$ in (2) we see that

$$\mathbb{P}(X \le \infty) \;=\; \mathbb{P}(X < \infty) \;=\; 1 \;=\; \int_{-\infty}^{\infty} f_X(x)dx$$

  i.e. the density of any continuous random variable has to **integrate to** 1. If we set A equal to the event $\{X \le x\}$, then $A^c = \{X > x\}$, and $\mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c) = 1$, since A and A^c are two **disjoint** events such that $\mathbb{P}(A \cup A^c) = 1$ (see **FM01** for more on this). The **complementary cdf** of X is defined as $\mathbb{P}(X > x)$, which is equal to $1 - \mathbb{P}(A) = 1 - \mathbb{P}(X \le x)$, and differentiating this expression with respect to x we see that

$$\frac{d}{dx}\mathbb{P}(X > x) \;=\; \frac{d}{dx}(1 - \mathbb{P}(X \le x)) \;=\; -f_X(x)$$

  which will be used many times on the course. Note also that

$$\mathbb{P}(X = x) \;=\; \int_x^x f_X(u)du \;=\; 0$$

  i.e. the probability that a continuous random variable X takes a particular value x is zero.

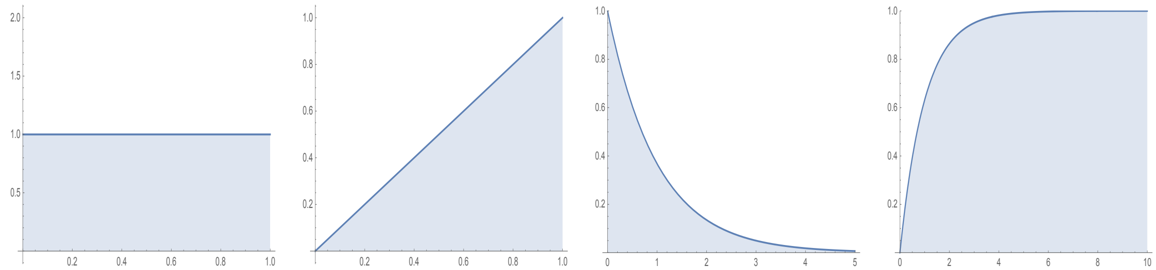


Figure 1: On the left we see the density of a standard $U[0,1]$ uniform random variable, and in the second from left panel we have plotted its distribution function $F_U(x) = x$. In the third panel we have plotted the density of an exponential random variable with parameter $\lambda = 1$, and on the right we have plotted its distribution function.

- **Example**: a standard **uniform random variable** U on $[0,1]$ has density $f_X(x) = 1$ for $x \in [0,1]$ and zero otherwise. The distribution function of U is obtained by integrating this density from 0 to x:

$$\mathbb{P}(U \le x) \quad = \quad \int_0^x 1\, du \quad = \quad u|_{u=x} - u|_{u=0} \quad = \quad x \tag{3}$$

  for $x \in [0,1]$. Note that the lower limit of integration is 0 not $-\infty$ here since the density of U is zero outside $[0,1]$.

- **Example**: an **exponential random variable** X has density $f_X(x) = \lambda e^{-\lambda x}$ for $x \in (0,\infty)$ and zero otherwise, for some parameter $\lambda > 0$ (we say that X is an $\mathrm{Exp}(\lambda)$ random variable). The distribution function of X is again obtained by integrating this density from 0 to x as follows:

$$\mathbb{P}(X \le x) \quad = \quad \int_0^x \lambda e^{-\lambda u} du \quad = \quad -e^{-\lambda u}|_{u=x} \; - \; (-e^{-\lambda u}|_{u=0}) \quad = \quad 1 - e^{-\lambda x}\,.$$

  As a sanity check, we see that $\lim_{x\to 0}\mathbb{P}(X \le x) = \mathbb{P}(X \le 0) = 0$, and $\lim_{x\to+\infty}\mathbb{P}(X \le x) = \mathbb{P}(X < \infty) = 1$, as we would expect since $X \ge 0$ and $X < \infty$.

- Two random variables X and Y are **independent** if $\mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x \cap Y \le y) = \mathbb{P}(X \le x)\mathbb{P}(Y \le y)$. If X and Y are independent, and have density $f_X(x)$ and $f_Y(y)$ respectively, then

$$\mathbb{E}(g(X)h(Y)) \quad = \quad \int_{-\infty}^{\infty} g(x) f_X(x) dx \cdot \int_{-\infty}^{\infty} h(y) f_Y(y) dy. \tag{4}$$

- The **expectation** of a continuous random variable X with density $f_X(x)$ is

$$\mathbb{E}(X) \quad = \quad \int_{-\infty}^{\infty} x f_X(x) dx\,.$$

  The expectation of $g(X)$ for some function g is given by

$$\mathbb{E}(g(X)) \quad = \quad \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{5}$$

  so in particular for any constant $a \in \mathbb{R}$, setting $g(x) = ax$, we have

$$\mathbb{E}(aX) \quad = \quad \int_{-\infty}^{\infty} a x f_X(x) dx \quad = \quad a\mathbb{E}(X)\,. \tag{6}$$

- Recall that if X is a random variable with expectation $\mathbb{E}(X) = \mu$, the **variance** of X is

$$\mathrm{Var}(X) \quad = \quad \mathbb{E}[(X-\mu)^2] \quad = \quad \mathbb{E}[X^2 - 2X\mu + \mu^2] \quad = \quad \mathbb{E}(X^2) - 2\mathbb{E}(X)\mu + \mu^2 \quad = \quad \mathbb{E}(X^2) \; - \; \mu^2\,.$$

  From this we see that for any constant $a \in \mathbb{R}$

$$\mathrm{Var}(aX) \quad = \quad \mathbb{E}((aX)^2) \; - \; (\mathbb{E}(aX))^2 \quad = \quad a^2\mathbb{E}(X^2) \; - \; a^2(\mathbb{E}(X))^2 \quad = \quad a^2\mathrm{Var}(X)$$

  where I am using (6) to obtain the middle equality here.

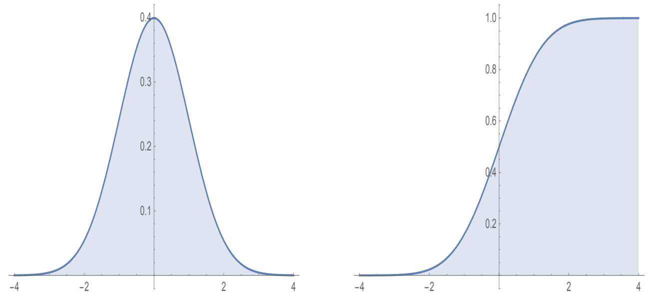


Figure 2: Here we have plotted the density $n(x)$ (left) and the distribution function $\Phi(x)$ (right) for the standard Normal distribution.

- For two random variables X and Y we have

$$\mathrm{Var}(X+Y) \;=\; \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$$

where $\mathrm{Cov}(X,Y) := \mathbb{E}((X-\mu_X)(Y-\mu_Y))$ is known as the **covariance** of X and Y, where $\mu_X := \mathbb{E}(X)$ and $\mu_Y := \mathbb{E}(Y)$.

- If X and Y are **independent** random variables, then $\mathbb{E}((X-\mu_X)(Y-\mu_Y)) = \mathbb{E}(X-\mu_X)\mathbb{E}(Y-\mu_Y) = 0 \times 0 = 0$, so

$$\mathrm{Var}(X+Y) \;=\; \mathrm{Var}(X) + \mathrm{Var}(Y)\,.$$

- If Z has a standard $N(0,1)$ **Normal distribution**, then the density of Z is

$$n(z) \;=\; \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

for $z \in \mathbb{R}$, and the distribution function of Z is

$$\mathbb{P}(Z \le x) \;=\; \Phi(x) \;=\; \int_{-\infty}^{x} n(z)dz$$

and hence

$$\Phi'(x) \;=\; n(x)$$

(see third and fourth plots in Figure 2 above).

- Proving that $\int_{-\infty}^{\infty} n(z)dz = 1$ is not trivial, and requires working in **polar coordinates** with 2 i.i.d. (independent and identically distributed) standard Normal random variables. $\Phi(x)$ cannot be computed exactly, but there are useful **asymptotic formula** for $\Phi(x)$ when x is small or x is large, or we can look up Φ in **tables**.

- For a general Normal random variable $X \sim N(\mu, \sigma^2)$ with mean μ and variance σ^2, it can be shown that $Z = (X-\mu)/\sigma$ is a standard $N(0,1)$ random variable. Then

$$\mathbb{P}(X > x) \;=\; \mathbb{P}(\frac{X-\mu}{\sigma} > \frac{x-\mu}{\sigma}) \;=\; \mathbb{P}(Z > \frac{x-\mu}{\sigma}) \;=\; \Phi^c(z)$$

where $z = \frac{x-\mu}{\sigma}$ and $\Phi^c(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 1 - \Phi(z)$. We can then look up $\Phi^c(z)$ in a Normal table or compute it in e.g. Matlab, Excel, Python or Mathematica.

- The **moment generating function** (mgf) of a general random variable X is defined as

$$M(p) \;:=\; \mathbb{E}(e^{pX}) \;=\; \int_{-\infty}^{\infty} e^{px} f_X(x)dx$$

dominated Convergence Thm.

for $p \in \mathbb{R}$, where I am using (5) for the final equality. Then (assuming we can interchange differentiation and expectation, which can be rigourously justified but we will not do this here) we see that $M'(p) = \mathbb{E}(Xe^{pX})$, so in particular $M'(0) = \mathbb{E}(X)$. Similarly $M^{(n)}(p) := \frac{d^n}{dp^n}\mathbb{E}(e^{pX}) = \mathbb{E}(X^n e^{pX})$, so $M^{(n)}(0) = \mathbb{E}(X^n)$, which is the nth moment of X. This is why M is called the moment generating function.

If $X_1, ..., X_n$ is a sequence of i.i.d. random variables, then

$$\mathbb{E}(e^{p(X_1+...+X_n)}) \;=\; \mathbb{E}(e^{pX_1})...\mathbb{E}(e^{pX_n}) \;=\; \mathbb{E}(e^{pX_1})^n\,.$$

Completing the Sq +

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-p\sigma)^2} = 1$$

- If $X \sim N(\mu, \sigma^2)$, then using (5) we can show that the mgf of X is given by

$$\mathbb{E}(e^{pX}) \;=\; \mathbb{E}(e^{p(\sigma Z+\mu)}) \;=\; \int_{-\infty}^{\infty} e^{p(\sigma z+\mu)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \;=\; \ldots \;=\; e^{\mu p + \frac{1}{2}\sigma^2 p^2}$$

where $Z \sim N(0,1)$, and we have used that $X \sim \sigma Z + \mu$, since we already know that $(X-\mu)/\sigma = Z$, and I have skipped some tedious details to get to the final expression.

Revision questions

1. For a **continuous random variable** X with density $f_X(x)$, prove that $\mathbb{E}(1_A(X)) = \mathbb{P}(X \in A)$ for any set A, where $1_A(X) = 1$ if $X \in A$ and zero otherwise. $1_A(X)$ is known as the **indicator function**, and will be used many times in the course, and we sometimes abbreviate this to just 1_A.

Solution.

$$\mathbb{E}(1_A(X)) \;=\; \int_{-\infty}^{\infty} 1_A(x) f_X(x) dx \;=\; \int_A f_X(x) dx \;=\; \mathbb{P}(X \in A)\,.$$

2. Let W_t be a random function such that $W_t \sim N(0,t)$ (**Brownian motion** has this nice property as we shall see in the second half of the course). Compute $\mathbb{P}(W_t > x)$.

Solution.

$$\mathbb{P}(W_t > x) \;=\; \mathbb{P}\Big(\frac{W_t - 0}{\sqrt{t}} > \frac{x-0}{\sqrt{t}}\Big) \;=\; \mathbb{P}\Big(Z > \frac{x}{\sqrt{t}}\Big) \;=\; \Phi^c\Big(\frac{x}{\sqrt{t}}\Big)$$

since $Z = (W_t - 0)/\sqrt{t}$ is a standard $N(0,1)$ random variable. The general rule here is: **do to one side what you do to the other side**.

by doing $F_X^{-1}(U)$ where U is Uniform dist. in computer.

3. **Simulating random variables with a given distribution**. Let X be a random variable with a continuous strictly increasing distribution function $F_X(x)$. What is the distribution of $F_X^{-1}(U)$, where U is a standard Uniform random variable on $[0,1]$?

Solution.

$$\mathbb{P}(F_X^{-1}(U) \leq x) \;=\; \mathbb{P}(U \leq F_X(x)) \;=\; F_X(x) \tag{7}$$

since $F_X(F_X^{-1}(x)) = x$, i.e. $F_X^{-1}(U) \sim X$, where we have used that $\mathbb{P}(U \leq x) = x$ from (3), i.e. $F_X^{-1}(U)$ has the same distribution as X. This is how we typically generate a random variable with a given distribution in practice on a computer.

Similarly, we can compute the distribution function of $F_X(X)$ as

$$\mathbb{P}(F_X(X) \leq x) \;=\; \mathbb{P}(X \leq F_X^{-1}(x)) \;=\; F_X(F_X^{-1}(x)) \;=\; x \tag{8}$$

so we see that $F_X(X)$ has the same distribution function as a $U[0,1]$ random variable (see Eq (3) above), so $F_X(X) \sim U[0,1]$. These facts will be used for the Gaussian copula below.

More advanced concepts

- If X and Y are independent random variables on $\mathbb{R}$ with density $f_X(x)$ and $f_Y(y)$ respectively, then the density of $Z = X + Y$ is

$$f_Z(x) \;=\; \int_{-\infty}^{\infty} f_X(u) f_Y(x-u) du\,. \tag{9}$$

Here we are conditioning on $X = u$ and $Y = x - u$ so $X + Y = x$. This is known as the **convolution** of two random variables. **Exercise for reader**: Compute the density of the sum of two independent $U[0,1]$ random variables or two independent Exponential random variables. Since these random variables cannot be negative, we have to make a minor modification to (9) ...

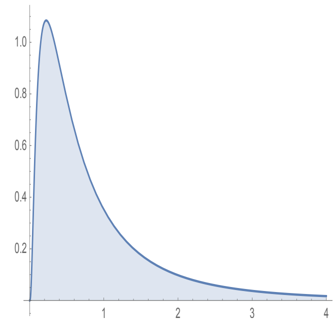


Figure 3: Plot of lognormal density for $\sigma = 1$ and $\mu = \frac{1}{2}$.

- **Strong law of large numbers** (SLLN). Let $X_1, X_2, ...$ denote an infinite sequence of independent identically distributed (i.i.d) random variables with $\mathbb{E}(X_i) = \mu$ and $\mathbb{E}(X_i^4) < \infty$ for all $i = 1, 2, ...$ Let $S_n = \sum_{i=1}^n X_i$. Then the SLLN says that

$$\mathbb{P}(\frac{S_n}{n} \to \mu) \;=\; 1$$

i.e. the **sample average** $\frac{S_n}{n}$ tends to the true expectation μ of X_i as $n \to \infty$, as we might expect. This result is the cornerstone of **Monte Carlo simulation**, namely that if μ is unknown, we can estimate μ with $\hat{\mu}_n = \frac{S_n}{n}$ for n large (see e.g. chapter 7 of David Williams book "Probability with Martingales" for proof). Take the FM06 module next semester to learn more about numerical simulation methods which will be required for your summer project.

- **Transformations of a 1-d random variable**. Let X be a continuous random variable with density $f_X(x)$, and let $Y = g(X)$, where g is differentiable and strictly increasing, which implies that $g'(x) > 0$ for all $x \in \mathbb{R}$, which further implies that g has a unique inverse g^{-1} such that $g^{-1}(g(x)) = x$. Then the distribution function of Y is

$$\mathbb{P}(Y \leq y) \;=\; \mathbb{P}(X \leq g^{-1}(y)) \;=\; F_X(g^{-1}(y))\,.$$

Differentiating with respect to y, we obtain the density of Y:

$$f_Y(y) \;=\; \frac{d}{dy}\mathbb{P}(Y \leq y)\,.$$

- **Important Financial Example - the Lognormal distribution**. Assume $X \sim N(\mu, \sigma^2)$. Compute the distribution function and density of $Y = e^X$.

Solution. Set $g(x) = e^x$, so $g^{-1}(x) = \log x = \ln x$. Then the distribution function of Y is

$$\mathbb{P}(Y \leq y) \;=\; \mathbb{P}(g^{-1}(Y) \leq g^{-1}(y)) \;=\; \mathbb{P}(X \leq \log y) \;=\; \mathbb{P}(\frac{X-\mu}{\sigma} \leq \frac{\log y - \mu}{\sigma}) \;=\; \Phi(\frac{\log y - \mu}{\sigma})$$

where we have used that $Z = (X-\mu)/\sigma$ is a standard Normal random variable. The density of Y is given by

$$f_Y(y) \;=\; \frac{d}{dy}\mathbb{P}(Y \leq y) \;=\; \frac{1}{y\sigma}\Phi'(\frac{\log y - \mu}{\sigma}) \;=\; \frac{1}{y\sigma}\overset{\text{std. Normal density}}{n}(\frac{\log y - \mu}{\sigma}) \;=\; \frac{1}{y\sigma}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{\log y - \mu}{\sigma})^2}$$

for $y > 0$, where we have used that $\frac{d}{dy}\frac{\log y - \mu}{\sigma} = \frac{1}{\sigma}\frac{d}{dy}\log y = \frac{1}{y\sigma}$. This is the **lognormal distribution**. We can apply this result to the stock price S_t for the Black-Scholes model defined as $S_t = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ where $W_t \sim N(0,t)$, so S_t has a lognormal distribution (see Black-Scholes chapter).

- **Generating correlated Normal random variables**. Let Z_1, Z_2 be two i.i.d $N(0,1)$ random variables and set

$$\begin{aligned} X &= Z_1 \\ Y &= \rho Z_1 + \sqrt{1-\rho^2} Z_2 \end{aligned} \tag{10}$$

for some ρ with $-1 \le \rho \le 1$. This definition implies that $\mathbb{E}(Y|X) = \rho X$.

Then Y is also $N(0,1)$, and X and Y have **correlation** ρ, i.e.

$$\mathrm{Corr}(X,Y) \;:=\; \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} \;=\; \frac{\mathbb{E}((X-\mu_X)(Y-\mu_Y))}{\sigma_X\sigma_Y} \;=\; \mathbb{E}(XY) \;=\; \rho \tag{11}$$

where $\mu_X = \mathbb{E}(X)$, $\mu_Y = \mathbb{E}(Y)$ (which in this case are both zero), and σ_X, σ_Y denote the **standard deviation** of X and Y (which in this case are also both 1 since Z_1 and Z_2 are standard Normal RVs).

To prove that $\sigma_Y = 1$, we note that

independent

$$\begin{aligned} \mathrm{Var}(Y) \;=\; \mathrm{Var}(\rho Z_1 + \sqrt{1-\rho^2} Z_2) \;=\; \mathrm{Var}(\rho Z_1) + \mathrm{Var}(\sqrt{1-\rho^2} Z_2) &= \rho^2 \mathrm{Var}(Z_1) + (1-\rho^2)\mathrm{Var}(Z_2) \\ &= \rho^2 + 1 - \rho^2 \;=\; 1\,. \end{aligned}$$

Then

$$\mathbb{E}(XY) \;=\; \mathbb{E}(Z_1(\rho Z_1 + \sqrt{1-\rho^2} Z_2)) \;=\; \rho\mathbb{E}(Z_1^2) + \sqrt{1-\rho^2}\,\mathbb{E}(Z_1)\mathbb{E}(Z_2) \;=\; \rho\mathbb{E}(Z_1^2) \;=\; \rho$$

since $\mathrm{Var}(Z_1) = \mathbb{E}(Z_1^2) - \mathbb{E}(Z_1)^2 = \mathbb{E}(Z_1^2)$ as $\mathbb{E}(Z_1) = 0$, which verifies (11). To show that Y is a normal random variable, we appeal to the general result that any linear combination of two independent Normal random variables $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ is Normal, since

$$\mathbb{E}(e^{p(X_1+X_2)}) \;=\; e^{\mu_1 p + \frac{1}{2}\sigma_1^2 p^2} e^{\mu_2 p + \frac{1}{2}\sigma_2^2 p^2} \;=\; e^{(\mu_1+\mu_2)p + \frac{1}{2}(\sigma_1^2+\sigma_2^2)p^2}$$

which is the the m.g.f. of a $N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$ random variable.

- **The Gaussian copula**. We have show above in Eqs (7) and (8) that for a continuous random variable X with a strictly increasing distribution function $F_X(x)$, then $X \sim F_X^{-1}(U)$ so $F_X(X) \sim U[0,1]$ (this will be the case if X has a strictly positive density where it is defined, because then $F_X'(x) = f_X(x) > 0$). We will now use this fact to **generate two correlated random variables with given distribution functions**.

Applying std Normal dist f'n to X.

Using X, Y from (10) we know that $U = \Phi(X)$ and $V = \Phi(Y)$ are both $U[0,1]$ random variables (this follows from (8)), but U and V are not independent unless $\rho = 0$. Then from (7) above, if F and G are valid distribution functions, then $F^{-1}(U)$ and $G^{-1}(V)$ are two correlated random variables with distribution functions F and G respectively, and changing ρ in (10) will affect the extent of this correlation. This construction is known as the **Gaussian copula** - to summarize:

Algorithm

  - Let $X = Z_1$, $Y = \rho Z_1 + \sqrt{1-\rho^2} Z_2$, where Z_1, Z_2 are i.i.d. $N(0,1)$.
  - Set $U = \Phi(X)$ and $V = \Phi(Y)$. → U & V not independent unless $\rho = 0$, U & V both $U[0,1]$.
  - Then $F^{-1}(U)$ and $G^{-1}(V)$ are two correlated random variables with distribution functions F and G respectively. F & G are any dist. f'n we would like to generate.

- A general (not necessarily Gaussian) copula is a pair of random variables U, V with a **joint distribution function** $C(u,v)$ on the unit square $[0,1]\times[0,1]$ such that U and V are both $U \sim U[0,1]$, i.e.

$$C(u,v) \;=\; \mathbb{P}(U \le u, V \le v) \;\overset{\text{satisfying}}{,}\; \underbrace{\mathbb{P}(U \le u) = u \quad , \quad \mathbb{P}(V \le v) = v}_{\text{dist f'n of } U \text{ is } u \text{ \& } V \text{ is } v \text{ i.e. std. Uniform.}}\,.$$

In general U and V will not be independent.

They are other types of copula aside from the Gaussian one described above (e.g. **Frank**, **Gumbel**, **Archimedean** etc.) and copulas are used extensively in modelling/pricing **credit derivatives** for dealing with correlated **default** (i.e. bankruptcy) events, e.g the Financial meltdown that occurred in 2008.

Homework 0 solutions

1. Let $X \sim N(0,1)$ be a standard Normal random variable. Compute the distribution function and the density of $Y = X^2$.

Solution. We have to be careful in this example because $g(x) = x^2$ is not an increasing function for $x < 0$. The distribution function of Y is

$$\begin{aligned} \mathbb{P}(Y \leq x) \quad = \quad \mathbb{P}(X^2 \leq x) \quad &= \quad \mathbb{P}(-\sqrt{x} \leq X \leq \sqrt{x}) \\ &= \quad \Phi(\sqrt{x}) - \Phi(-\sqrt{x})\,. \end{aligned}$$

$$\boxed{\Phi'(x) = n(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$$

Differentiating and using the chain rule, we see that the density of Y is

$$f_Y(x) \quad = \quad \frac{d}{dx}\mathbb{P}(Y \leq x) \quad = \quad \frac{n(\sqrt{x})}{2\sqrt{x}} + \frac{n(-\sqrt{x})}{2\sqrt{x}} \quad = \quad \frac{e^{-x/2}}{2\sqrt{2\pi}\sqrt{x}} + \frac{e^{-x/2}}{2\sqrt{2\pi}\sqrt{x}} \quad = \quad \frac{e^{-x/2}}{\sqrt{2\pi}\sqrt{x}}$$

for $x \geq 0$, and zero otherwise. This is known as a **chi-square distribution** with 1 degree of freedom, and note that $\lim_{x\to 0} f_Y(x) = +\infty$, but the density $f_Y(x)$ still integrates to 1. Special case of chi-sq dist. since chi-sq dist. is the sum of squares of iid Normal r.v's.

2. If $W_t \sim N(0,t)$ and $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$, compute $\mathbb{E}(S_t^p)$ (hint: re-write S_t^p as $S_0^p e^{pX_t}$ where $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$). $p \in \mathbb{R}$

Solution.

$$\mathbb{E}(S_t^p) \quad = \quad S_0^p\,\mathbb{E}(e^{pX_t}) \quad = \quad S_0^p \mathbb{E}(e^{p[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t]})\,.$$

(for fixed t)

But $\underbrace{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}_{\text{Linear combination of Normal r.v. is Normal.}} \sim N(\mu_1, \sigma_1^2)$, where $\mu_1 = (\mu - \frac{1}{2}\sigma^2)t$ and $\sigma_1^2 = \sigma^2 t$. Thus

$$\mathbb{E}(S_t^p) \quad \underset{\text{mgf of Normal r.v.}}{=} \quad S_0^p e^{\mu_1 p + \frac{1}{2}\sigma_1^2 p^2} \quad = \quad S_0^p e^{(\mu - \frac{1}{2}\sigma^2)pt + \frac{1}{2}\sigma^2 t p^2}\,.$$

Note that $\mathbb{E}(S_t^p)$ for all $p > 1$, i.e. all moments of S_t are finite, i.e. less than infinity.
$\mathbb{E}(S_t^p) < \infty \ \forall p \in \mathbb{R}$,

3. Compute the moment generating function of a sum of n i.i.d. exponential random variables X_i with parameter λ. Compute the mgf of the average of these random variables (you may use that $\mathbb{E}(e^{pX_i}) = \frac{\lambda}{\lambda - p}$).

Solution.

$$\mathbb{E}(e^{p(X_1 + \ldots + X_n)}) \quad = \quad (\mathbb{E}(e^{pX_i}))^n \quad = \quad \frac{\lambda^n}{(\lambda - p)^n}$$

Then

$$\mathbb{E}(e^{p(X_1 + \ldots + X_n)/n}) \quad = \quad \frac{\lambda^n}{(\lambda - p/n)^n}\,.$$

look previous page Algorithm.

4. Explain how to use the Gaussian copula with $\rho = -1$ to generate two exponential random variables V and W with parameter 1. Explain the significance of the case $\rho = -1$.
$\lambda = 1$ in Exp dist. $F(x) = 1 - e^{-\lambda x}$ with $\lambda = 1$.

Solution. Let Z_1, Z_2 be iid $\sim N(0,1)$ and and $X = Z_1$ and $Y = \rho Z_1 + \sqrt{1-\rho^2} Z_2$ for some ρ with $-1 \leq \rho \leq 1$. Then from the lecture notes we know that Y is also $N(0,1)$, and X and Y have correlation ρ. V and W should both have distribution function $F(x) = 1 - e^{-x}$, so $F^{-1}(x) = -\log(1-x)$. Then $U_1 = \Phi(X)$ and $U_2 = \Phi(Y)$ are two (correlated) $U[0,1]$ random variables, and $V = F^{-1}(U_1)$ and $W = F^{-1}(U_2)$ are two (correlated) exponential random variables.

F & G are same in this question for simplicity (both Exp. r.v's)

In the extreme case $\rho = -1$, we see that $Y = -X$, and

$$U_1 = \Phi(X) \quad , \quad U_2 = \Phi(-X) = 1 - U_1$$

where we have used that $\Phi(x) = 1 - \Phi(-x)$ from the symmetry of the standard Normal distribution around $x = 0$, and

$$V = F^{-1}(U_1) \quad , \quad W = F^{-1}(1 - U_1) = F^{-1}(1 - F(V))$$

so we see that W is just a deterministic function of V, and in a sense U and V are maximally negatively correlated with each other.

$X = Z_1$
$Y = \rho Z_1 + \sqrt{1-\rho^2}\, Z_2$ since $\rho = -1$
$Y = -Z_1 + 0\, Z_2$
$Y = -X$

Here with $\rho = -1$, you only need one uniform r.v. U_1 to do the simulation.

[Note: look at Excel file: Gaussian Copula Exponential for this example 4 simulation using Gaussian Copula Algorithm]

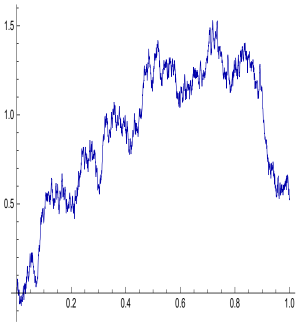
Cell F4 = RAND( ) → Generates Std Uniform r.v's

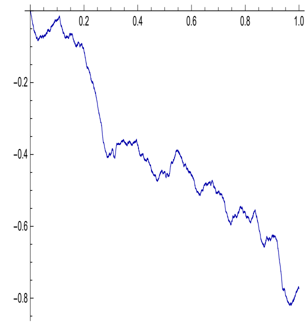
Z_1, Z_2 = NORMSINV(RAND( )) $= F_{Z_i}^{-1}(U) = \Phi^{-1}(U)$ generates $\sim N(0,1)$ i.i.d. (7) on Notes.

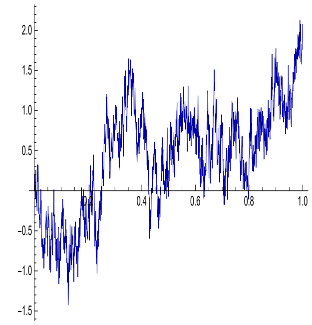
$X = Z_1$
$Y = \rho Z_1 + \sqrt{1-\rho^2}\, Z_2$ } Correlated Normal r.v.

U = NORMSDIST (D7)
V = " (D8) } Correlated Uniform r.v. $U = \Phi(X)$, $V = \Phi(Y)$

E_1 = −LN(1−D10) $= -\log(1-U)$
E_2 = −LN(1−D11) $= -\log(1-V)$ } Correlated Exp. r.v. $E_1 = F^{-1}(U)$, $E_2 = F^{-1}(V)$







7CCMFM02 Risk Neutral valuation: Pricing and Hedging Derivatives

Welcome to FM02. In this course you will learn the basic building blocks and mathematical techniques for pricing and hedging financial derivatives. In the first part of this course, you will learn about discrete-time models, in particular the **binomial model** where the stock price can only go up or down by a fixed amount at each discrete time step. At first glance this seems like a crude model, but many insight comes out of this model when we start pricing so-called **European options**, in particular the important notion of **risk-neutral pricing**, which is the backbone of the course and essentially says that the fair (or **no-arbitrage** price of an option does not depend on the real world **drift** of the stock price process. You will also learn how to price exotic **path-dependent options**, such as **American options** and **barrier options**, and this can easily be implemented in e.g. Excel or Matlab. In the second part of the course we will learn about **Brownian motion** and continuous time models which are more interesting but more mathematically sophisticated, and the financial models and instruments you will learn about are used on real life trading desks, where trillions of dollars worth of derivatives trade hands every day. I use to work in the city on Foreign Exchange and interest rate derivatives desks, so let me know if you have any careers or technical questions about that.

The course webpage is at `https://nms.kcl.ac.uk/martin.forde/FM02.html` (requires **Username: FM01Students, Password: Pointbreak**) and the course textbook is *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives*, by N.Bingham and R.Kiesel, which you can download at `https://bok.org`. Office hours and tutorials times will be announced in due course

Prerequisites for this course

- Some background in basic **probability**, e.g. calculating **expectations** of random variables, the **Normal distribution** etc.
- Some basic **linear algebra**, solving 2x2 **matrix** equations (also known as simultaneous equations).
- You should also attend FM01 lectures on **measure theory**, since we will be talking about **σ-algebras**, and if you're keen the undergraduate courses 388 (discrete time, Semester 1) and 338 (continuous time, Semester 2 taught by me)

Summary of course

On this course you will learn about:

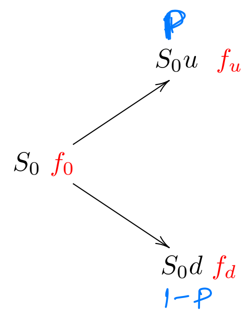
- **Discrete time models**- the **Binomial model** (see below) and general N-step discrete time models, risk-neutral measures, conditions for when we can and cannot make free money (known as **arbitrage**), **complete markets**, self-financing trading strategies, replicating options. Pricing/hedging of European, American and other types of exotic options under these models.
- **Brownian motion** (see first graph above), which is a continuous time random process which has independent, identically distributed increments with the property that $W_t - W_s \sim N(0, t-s)$ for $s < t$, i.e. Normally distributed with mean zero and variance $t-s$ (see the Excel worksheet **BrownianMotionMonteCarlo.xls** on the course website for Monte Carlo simulation). Brownian motion has a continuous sample paths (i.e. it does not jump) but is not differentiable with respect to t, so we cannot apply standard calculus techniques to W_t, so we have to develop a new calculus called **stochastic calculus**.

- **Ito's lemma** - how to do calculus on $f(W_t)$, where f is a twice differentiable function- we will see the rules are subtly different to standard 1st year calculus.
- The **Black-Scholes model/formula and PDE** for pricing a European call option with payoff $\max(S_T - K, 0)$ on a stock price process $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ where W_t is Brownian motion. μ is the **drift** of the process, which describes the overall upward/downward trend, and σ is the **volatility**, which describes its variability.

1 Discrete-time markets

1.1 The one-step binomial model

- We now consider a simple one step **binomial model** where the stock price at time zero is S_0, and at the final time T either goes up to $S_0 u$ with probability p, or down to $S_0 d$ with probability $1-p$, where $d < u$.



- We also assume the existence of a **riskless bond**, which has zero risk and grows in value from e^{-rT} at time zero to 1 at time T for $r \geq 0$, where r is the interest rate. We assume we can buy or sell any amount ϕ_1 of stock at time zero, and buy or sell any amount ϕ_0 of the bond, with zero transaction costs. In the language of FM01 and measure theory, we are working on the **sample space** $\Omega = \{u, d\}$ with **σ-algebra** $\mathcal{F} = \{\Omega, \emptyset, \{u\}, \{d\}\}$ and probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$ with $\mathbb{P}\{u\} = p$ and $\mathbb{P}\{d\} = 1-p$ (see FM01 module for definition and properties of σ-algebras).
- We now wish to see if we can choose (ϕ_0, ϕ_1) so as to **replicate** the payoff of an **option**, which pays f_u if S goes up, and f_d if S goes down. To achieve this, we just need to solve the **simultaneous equations**:

[The value of bond ϕ_0 at time $T = 1$. ϕ_0 & ϕ_1 are amount of bond & stock.
S_0u is the price of stock at time T if it goes up. S_0d is when price goes down.]

$$1 \times \phi_0 + \phi_1 S_0 u = f_u \qquad (1)$$
$$1 \times \phi_0 + \phi_1 S_0 d = f_d$$

Total value of Portfolio in up & down scenarios.

for the unknown ϕ_0 and ϕ_1. Subtracting the second equation from the first, we find that

$$\phi_1 = \frac{f_u - f_d}{S_0 u - S_0 d} = \frac{\Delta f}{\Delta S}$$

Note that the right hand side is a **finite difference derivative** of f, which gives an easy way to remember this formula. Then from (1) we get

$$\phi_0 = f_u - \phi_1 S_0 u \,.$$

- Then the initial cost of the option is the initial cost of **replicating** the option, i.e.

$$\begin{aligned} f_0 := \phi_0 e^{-rT} + \phi_1 S_0 &= (f_u - \phi_1 S_0 u)e^{-rT} + \phi_1 S_0 \\ &= (f_u - \frac{f_u - f_d}{S_0(u-d)} S_0 u)e^{-rT} + \frac{f_u - f_d}{S_0(u-d)} S_0 \\ &= (\frac{f_u(u-d)}{u-d} - \frac{f_u - f_d}{u-d} u)e^{-rT} + \frac{f_u - f_d}{u-d} \end{aligned}$$

- Collecting terms in f_u and f_d and taking out a factor of e^{-rT} in front of everything, we can re-write this expression as

$$f_0 = e^{-rT}[\frac{(u-d) - u + e^{rT}}{u-d} f_u + \frac{u - e^{rT}}{u-d} f_d] = e^{-rT}[q f_u + (1-q) f_d] \qquad (2)$$

where

$$q = \frac{e^{rT} - d}{u-d} \quad \Rightarrow \quad 1 - q = \frac{u - e^{rT}}{u-d} \qquad (3)$$

- Note that q (and hence $1-q$) does not depend on the option payoffs f_u and f_d. If $q \in (0,1)$, then we refer to q and $1-q$ as **risk-neutral probabilities**, and from (2) we see that

$$f_0 \;=\; e^{-rT}\mathbb{E}^{\mathbb{Q}}(f_T)$$

where f_T is a Bernoulli random variable (i.e. a discrete random variable with only two possible outcomes) with $f_T = f_u$ if $S_T = S_0u$ and $f_T = f_d$ if $S_T = S_0d$, and $\mathbb{Q} : \mathcal{F} \to [0,1]$ is the **probability measure** with $\mathbb{Q}\{u\} = q$ and $\mathbb{Q}\{d\} = 1-q$ (see the Excel sheet BinomialModelOneStep.xls on the course website to test numerical examples).

- Hence we see that the cost of replicating the option is equal to the **discounted expected terminal payoff**, but in a *fictitious* world where the probability of going up is q not p. We call this fictitious world the **risk-neutral** world. Note that the original probability p of going up does not affect the price of the option at all; this is because we are not pricing the option like a gambling game as a future expected payoff $\mathbb{E}^{\mathbb{P}}(f_T)$, but rather we are computing the cost required to **replicate** the option, i.e. **artificially re-create** the option payoff without actually buying/selling the option).

Let V_0 denote the total portfolio value at time zero and V_T denote the portfolio value at the final time T. Then an **arbitrage opportunity** for the 1-step binomial model is a trading strategy (ϕ_0, ϕ_1) for which $V_0 = 0$ and $V_T \geq 0$ with $\mathbb{P}(V_T > 0) > 0$, i.e. there is a non-zero probability of making something for nothing, with no downside risk.

- **How to make free money if the option price is not** f_0: if the option price in the market exceeds f_0, then we can sell the option for $> f_0$ and replicate it at cost f_0, and pocket the difference, at no risk. Conversely, if the option price in the market is less than f_0, then we can buy the option for $< f_0$ and replicate **minus** the original option payoff at a cost of $-f_0$, and again pocket the difference, at no risk. Replicating "minus the option payoff" just means doing the opposite of the replicating strategy above (i.e. holding $-\phi_0$ bonds and $-\phi_1$ units of stock) at a cost of $-f_0$. → i.e. the simultaneous eq^n in (1) is: $-\phi_0 - \phi_1 S_0 u = -f_u$, $-\phi_0 - \phi_1 S_0 d = -f_d$ [If something is expensive we sell it & if it's cheap we buy it.]

- **How to make free money if** $q \notin (0,1)$. If we set $\phi_0 = -S_0e^{rT}$ and $\phi_1 = 1$ then

$$V_0 \;=\; \phi_0 e^{-rT} + \phi_1 S_0 \;=\; 0 \qquad \text{(Since, } -S_0e^{rT}e^{-rT} + 1S_0 = 0\text{)}$$

and (since the bond price is 1 at time T) we see that

$$V_T = \begin{cases} \underbrace{1\times}_{\text{Bond price at } T} -S_0e^{rT} + S_0u \times 1 = & S_0(u - e^{rT}) \qquad \text{if } S \text{ goes up} \\ 1\times -S_0e^{rT} + S_0d \underbrace{\times 1}_{\text{Unit/Amount of stock}} = & S_0(d - e^{rT}) \qquad \text{if } S \text{ goes down} \end{cases} \tag{4}$$

and recall that

$$\text{risk Neutral Probability of Stock price going up} \longrightarrow q \;=\; \frac{e^{rT} - d}{u - d} \quad \Rightarrow \quad 1 - q \;=\; \frac{u - e^{rT}}{u - d}\,. \tag{5}$$

- **Case (i)** If $q \leq 0$ i.e. if $d \geq e^{rT}$, which also implies that $u \geq e^{rT}$ (since we are always assuming that $u > d$), then in (4) we see that $V_T \geq 0$ in both scenarios and $V_T > 0$ if S goes up; hence this is an arbitrage strategy.

- **Case (ii)** If $q \geq 1$ then from the expression for $1-q$ in (5) we see this implies that $u \leq e^{rT}$, which also implies that $d \leq e^{rT}$ (since $d < u$ by assumption), then $V_T \leq 0$ in both scenarios and $V_T < 0$ if S goes down (since $d < u$); we call this **negative arbitrage** (but we can then multiply ϕ_0 and ϕ_1 by -1 to convert this to a positive arbitrage strategy.

- If both condition (i) and (ii) are not satisfied, i.e. if $d < e^{rT} < u$, then $V_T > 0$ in one scenario and $V_T < 0$ otherwise, i.e. there is no arbitrage.

  Thus we have proved the following:

Theorem 1.1 *For the one-period binomial model, there is no arbitrage if and only if* $q \in (0,1)$ *i.e. if*

$$d \;<\; e^{rT} \;<\; u\,.$$

Numerical example 1 (see also BinomialModelOneStep.xls on course website): $S_0 = 1$, $u = 1.1$, $d = .9$, $r = .02$, $T = .25$. Then we find that $q = 0.525$ so there is no arbitrage (since $q \in (0,1)$) and $f_0 =$

$e^{-rT}(qf_u + (1-q)f_d) = 0.522$. Recall that f_0 is the cost of replicating the option with the underlying stock and the bond.

$$
\begin{tikzcd}
 & 1.1 \quad f_u = 1 \\
1 \quad f_0 = 0.522 \arrow[ur] \arrow[dr] & \\
 & 0.9 \quad f_d = 0
\end{tikzcd}
$$

Numerical example 2: constructing the arbitrage strategy for a one-step bionomial tree: Now change u to 1 and r to 0. Then we find that $q = 1$ which clearly is not in $(0, 1)$, so there is arbitrage. Specifically, holding $\phi_0 = -S_0 e^{rT}$ and $\phi_1 = 1$, our initial portfolio value is zero, and the final value is 0 or -0.10. Hence this is negative arbitrage, which we can turn into positive arbitrage by reversing the signs of ϕ_0 and ϕ_1.

$$
\begin{tikzcd}
 & 1 \quad f_u = 1 \\
1 \quad f_0 = 1 \arrow[ur] \arrow[dr] & \\
 & 0.95 \quad f_d = 0
\end{tikzcd}
$$

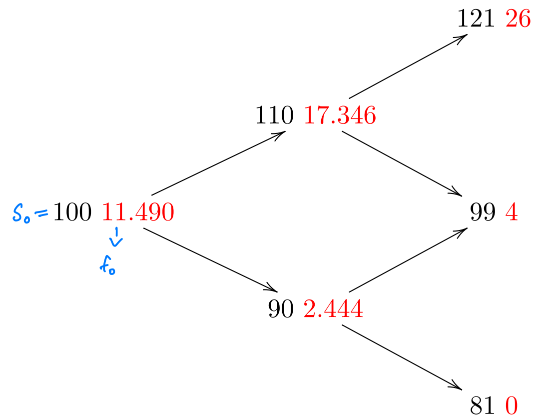
1.2 The two-period binomial model

We now consider pricing a general option under a two-period binomial model, where now the stock price value is multiplied by a factor of u or d at each successive time point. This kind of binomial tree is said to be **recombining**, because if S goes up and then down, S ends up at the same level as if S goes down and then up, since clearly $S_0ud = S_0du$. In this case, we can break the two-step model into three one-step models so the risk-neutral probability of going up at each node is now $q = \frac{e^{r\Delta T} - d}{u - d}$ (similar to above), but note that T has been replaced with ΔT here (where $\Delta T = T/2$) since we are considering three 1-step models each with maturity ΔT. The option with payoff f_{uu}, $f_{ud} = f_{du}$ or f_{dd} is then priced by **backwards induction**; specifically we **iterate** the procedure for the one-step model by breaking the two-step model into three one-step models, and working backwards, as this drawing demonstrates:

Backwards Induction

$$
\begin{tikzcd}
 & & S_0u^2 \quad f_{uu} \\
 & S_0u \quad f_u = e^{-r\Delta T}(qf_{uu} + (1-q)f_{ud}) \arrow[ur] \arrow[dr] & \\
S_0 \quad f_0 = e^{-r\Delta T}(qf_u + (1-q)f_d) \arrow[ur] \arrow[dr] & & udS_0 \quad f_{ud} = f_{du} \\
 & S_0d \quad f_d = e^{-r\Delta T}(qf_{ud} + (1-q)f_{dd}) \arrow[ur] \arrow[dr] & \\
 & & d^2S_0 \quad f_{dd}
\end{tikzcd}
$$

Numerical example 3: pricing a European call option with 2-step tree: A European call option is a financial instrument which pays $\max(S_T - K, 0)$ at the final time T for some K-value which is known as the **strike price**. Now consider $S_0 = 100$, $K = .95$, $u = 1.1$, $d = .9$, $r = .05$, and $\Delta T = .5$, so $T = 1$. Then we find that $q = 0.626$ so there is no arbitrage (since $q \in (0, 1)$) and $f_0 = 11.490$.

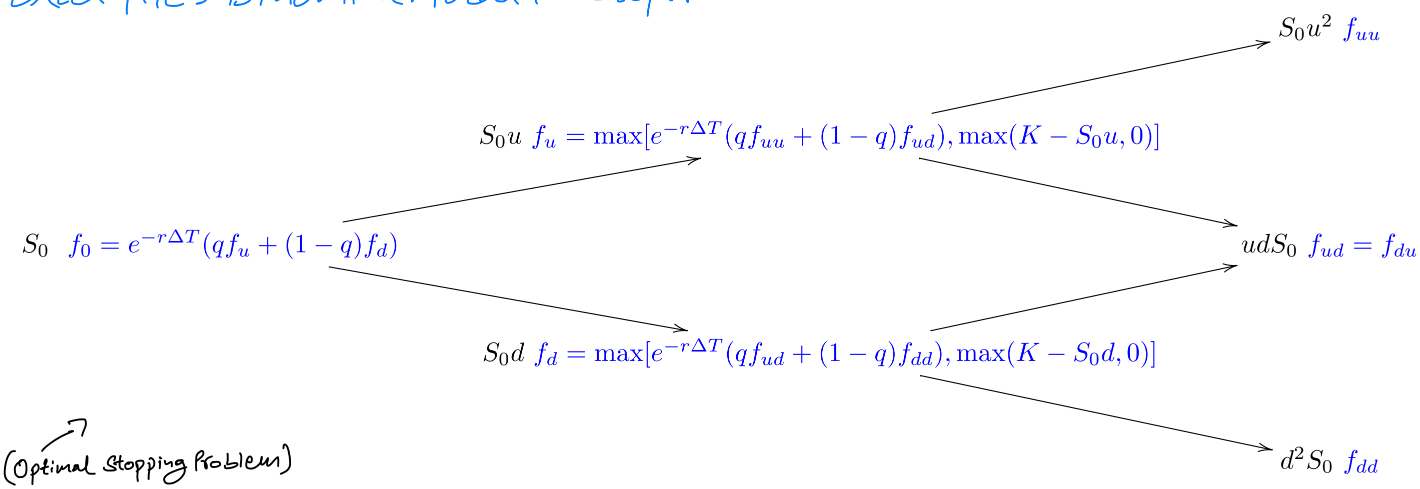


Excel file: Binomial Model Two Step

In the real world, an **American put option** gives the buyer the right but not the obligation to receive $\max(K - S_t, 0)$ any intermediate time $t \in [0, T]$ (we call $\max(K - S_t, 0)$ the **intrinsic value** of the option) ; if we take this option, we say that the option has been **exercised** early. For the two-step binomial model, we only allow the portfolio to be updated at the intermediate time, so in this case the American put can only be exercised at the intermediate time ΔT; if we do not exercise the option at this time, then the option just becomes a standard European put option which pays $\max(K - S_{2\Delta T}, 0)$ at the final time $T = 2\Delta T$. To price an American option we have to compare the price calculated at each node using our risk-neutral valuation formula, and take the maximum of this and the intrinsic value, as the following example demonstrates:

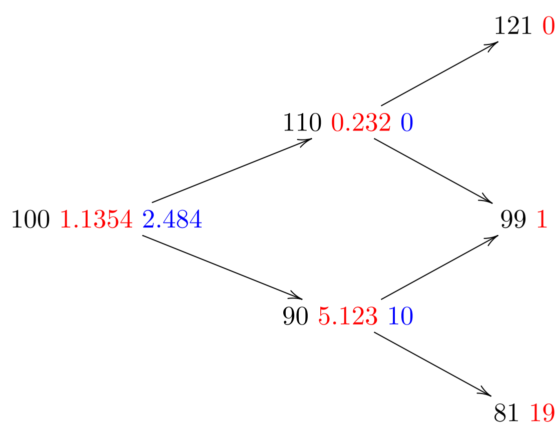
Numerical example 4: American put option with 2-step tree: For this example we let $S_0 = 100$, $K = 100$, $u = 1.1$, $d = .9$, $r = .10$, $\Delta T = .5$, so $T = 1$. Then we find that $q = 0.756$ so there is no arbitrage and $f_0^E = 0.752$ and $f_0^A = 1.159$, where the superscripts E and A refer to the European and American options respectively.

Excel file: BinomialModelTwoStepAmericanPut



(Optimal Stopping Problem)

Note in this example, $f_{uu} = \max(K - Su^2, 0)$, $f_{ud} = f_{du} = \max(K - Sud, 0)$ and $f_{dd} = \max(K - Sd^2, 0)$.



Red prices refer to the European put option and blue prices refer to the American put option. In this example we see that the American put is worth more since it is optimal to exercise early if S goes down over the first time interval, because in this scenario the intrinsic value of 5 exceeds its replication value of 3.245 if we continue to the final time.

1.3 Pricing path-dependent options under the two-step binomial model

We can also price path-dependent options such as **Asian** and **forward-starting** options; An Asian call option is a call option on the *average* stock price over some time period, e.g. paying $\max(\frac{1}{2}\sum_{n=1}^{2} S_{n\Delta t} - K, 0)$ at T at time T. A forward-starting call option is a call option on the change in the stock price from the intermediate to the final time, e.g. paying $\max(S_{2\Delta t}/S_{\Delta t} - K, -0)$ at T at time T. For these path-dependent we must use a **non-recombining tree** for the option price nodes in the tree since in this case $f_{ud} \neq f_{du}$ in general (see Excel sheets on course website for numerical examples, e.g. BinomialTwoStepAsianCallOption.xls and homework and past exam questions).

1.4 The equivalent martingale measure for the binomial model

Definition 1.1 *A discrete-time random process $X_0, X_1, X_2, ...$ said to be a* ***martingale*** *if*
(i) $\mathbb{E}(|X_n|) < \infty$ *for all* $n = 1, 2, ...$
(ii) $\mathbb{E}(X_n|X_1, .., X_m) = X_m$ *for* $0 \le m \le n-1$.

Remark 1.1 As an example, if X_n is your total wealth at time n in a gambling game, then if X_n is a martingale, it means the game is fair, i.e. your expected wealth at the future time n is equal to your current wealth at time m, e.g. if you have roulette wheel where red and black show up with equal probability one half in each spin, and your wealth increases or deceases by \$1 depending on whethere you choose correctly or not.

For the one-step binomial model, if we compute the discounted expected value of the final stock price using the risk-neutral probabilities, we find (after some algebra) that

[by Substituting q & $(1-q)$ by (5)]

$$e^{-rT}(qS_0u + (1-q)S_0d) \;=\; S_0\,.$$

discounted process is a Martingale.

This shows that under $\mathbb{Q}$ (i.e. in the risk-neutral world), the process $\tilde{S}_n = e^{-rn\Delta T}S_n$ in the one-step binomial model for $n = 0, 1$. Similarly, if we consider a n-step binomial model with the same u and d for each time step and node and recall that $\Delta T = T/N$, then (using the **tower property** of conditional expectations) we have (for $m \le n-1$)

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}(\tilde{S}_n|\mathcal{F}_m) &= \mathbb{E}(\mathbb{E}(\tilde{S}_n|\mathcal{F}_{n-1}|\mathcal{F}_m) \quad \text{(using the tower property)}\\
&= \mathbb{E}(\tilde{S}_{n-1}|\mathcal{F}_m) \quad \text{(by the martingale property)} \to \mathbb{E}(\tilde{S}_n|\mathcal{F}_{n-1}) = \tilde{S}_{n-1} \quad \text{Since it's martingale for 1-step shown above.}\\
&= \mathbb{E}(\mathbb{E}(\tilde{S}_{n-1}|\mathcal{F}_{n-2})|\mathcal{F}_m) \quad \text{(using the tower property again)} \quad m < n-2\\
&= \mathbb{E}(\tilde{S}_{n-2}|\mathcal{F}_m) \quad \text{(by the martingale property)}\\
&\quad ... \quad \leftarrow \text{Keep going until } \mathbb{E}(\tilde{S}_{m+1}|\mathcal{F}_m)\\
&= \mathbb{E}(\tilde{S}_{m+1}|\mathcal{F}_m)\\
&= \tilde{S}_m \quad \text{(by martingale Property)}
\end{aligned}$$

where $\mathcal{F}_m$ is all the information at time m (i.e. the observed values of S_1, ..., S_m) which in this case just means knowing the values of $S_1, ..., S_m$, i.e the stock price history up time m. Thus $\tilde{S}$ is a martingale under $\mathbb{Q}$ (condition (i) in Definition 1.1 is trivially satisfied since $S_n \le S_0u^n < \infty$. The notion of conditional expectation is made rigorous later in FM01 using a deep theorem of Kolmogorov.

1.5 The general N-period model

Risky Assets eg: Stock, Currency, commodity.

riskless Bond

Now consider an N-period discrete-time financial market with $d+1$ assets with price vector $S_n = (S_n^0, \underbrace{S_n^1, ..., S_n^d}_{\text{Risky Assets}})$ at discrete times $n = 0, 1, 2.., N$. Assume the zeroth asset here is a **riskless** bond so $S_n^0 = e^{rn}$, where $r > 0$ is the interest rate in each period (which is assumed constant for simplicity).

(S_n & φ_n are vectors)

A **trading strategy** $\varphi_n = (\varphi_n^0, \varphi_n^1, ..., \varphi_n^d)$ for this market is an $\mathbb{R}^{d+1}$-valued stochastic process such that φ_n^i is the number of units of asset i held at time n. φ_n is said to be **previsible** if $\varphi_n = f(S_1, ..., S_{n-1})$ for some function f, i.e. φ_n can only depend on the stock price history up to but not including time n.

This assumption is natural; it means that the trader must choose his holding φ_n in each of the assets once the prices are known at time $n-1$ but before he knows the prices of the assets at time n i.e. he cannot see into the future.

Definition 1.2 *A trading strategy is said to be* ***self-financing*** *if the total portfolio value V_n^φ at time n satisfies*

We are at time n, & we reallocate our portfolio at time n s.t. total portfolio value remains unchanged & wait to see what happens between time n & $n+1$.

$$V_n^\varphi \;=\; \varphi_n \cdot S_n \;=\; \varphi_{n+1} \cdot S_n \tag{6}$$

for all $n = 1..N$, where $\cdot$ here denotes the ***dot product****, so $\varphi_n \cdot S_n = \sum_{i=0}^d \varphi_n^i S_n^i$ and $\varphi_{n+1} \cdot S_n = \sum_{i=0}^d \varphi_{n+1}^i S_n^i$. This just means we cannot add or subtract any funds from any external sources, i.e. we can only* ***re-allocate our existing wealth*** *at each new time point.*

Definition 1.3 *In general, an option or* ***contingent claim*** *X with maturity date N is an $\mathcal{F}_N$-measurable random variable, which means that X is just some function of $S_1, .., S_N$.*

Example: a **European call option** which has payoff $X = (S_N - K)^+ = \max(S_N - K, 0)$. A **European put option** which has payoff $X = (K - S_N)^+ = \max(K - S_N, 0)$.

1.6 Arbitrage and risk-neutral measures

Suppose we have a sample space Ω and a σ-algebra $\mathcal{F}$ and two probability measures $\mathbb{P}$ and $\mathbb{Q}$.

Definition 1.4 *A probability measure $\mathbb{Q}$ is* ***equivalent*** *to $\mathbb{P}$ if $\mathbb{P}(A) > 0$ implies $\mathbb{Q}(A) > 0$ for all $A \in \mathcal{F}$ and vice versa. $\mathbb{Q}$ is called an* ***equivalent martingale measure*** *(EMM) if the* ***discounted vector of asset prices*** *$\tilde{S}_n = S_n/S_n^0$ is a* discrete time ***martingale*** *under $\mathbb{Q}$.* (Each compound of vector $\tilde{S}_n$ are martingales.)

Definition 1.5 *For a general N-step model, an* ***arbitrage*** *strategy is a self-financing trading strategy for which $V_0^\varphi = 0$ and $V_N^\varphi \geq 0$ and $\mathbb{P}(V_N^\varphi > 0) > 0$.* (Start with nothing but end up with something)

Recall that arbitrage means there's a *non-zero probability of making free money with zero risk.*

Theorem 1.2 *(No-arbitrage theorem). In a discrete-time market (where we only allow previsible self-financing trading strategies), if an equivalent martingale measure $\mathbb{Q}$ exists, then there is no arbitrage.*

Remark 1.2 With a few additional qualifications, we can also go the other way round, i.e. show that if the market has no arbitrage then an EMM $\mathbb{Q}$ exists, but we will not do this here, since the version stated in Theorem 1.2 is more useful in practice.

Proof. Assume that an equivalent martingale measure $\mathbb{Q}$ exists and let φ be a previsible self-financing trading strategy. Assume $B_n = e^{rn}$ and let $\tilde{S}_n = S_n/B_n$ denote the discounted vector of asset prices. Then

Discounted value of our portfolio at time n.

B_n = Bond price
$\tilde{S}_n$ = Discounted by Bond

$$\begin{aligned}
\tilde{V}_n^\varphi &= \varphi_n \cdot \tilde{S}_n \\
&= \varphi_0 \cdot \tilde{S}_0 + (\varphi_1 \cdot \tilde{S}_1 - \varphi_0 \cdot \tilde{S}_0) + (\varphi_2 \cdot \tilde{S}_2 - \varphi_1 \cdot \tilde{S}_1) + \ldots + (\varphi_n \cdot \tilde{S}_n - \varphi_{n-1} \cdot \tilde{S}_{n-1}) \\
&= \varphi_0 \cdot S_0 + \sum_{k=1}^n (\varphi_k \cdot \tilde{S}_k - \varphi_{k-1} \cdot \tilde{S}_{k-1}) \\
&= \varphi_0 \cdot S_0 + \sum_{k=1}^n \varphi_k \cdot \tilde{S}_k - \sum_{k=1}^n e^{-r(k-1)} \varphi_{k-1} \cdot S_{k-1} \\
&= \varphi_0 \cdot S_0 + \sum_{k=1}^n \varphi_k \cdot \tilde{S}_k - \sum_{k=1}^n e^{-r(k-1)} \varphi_k \cdot S_{k-1} \quad \text{(using the self-financing condition)} \\
&= \tilde{V}_0^\varphi + \sum_{k=1}^n \varphi_k \cdot \underbrace{(\tilde{S}_k - \tilde{S}_{k-1})}_{\text{Increment of vector of discounted asset prices of size D+1.}} . \qquad (7)
\end{aligned}$$

Writing as telescopic sum as everything else cancels to give us $\varphi_n \cdot \tilde{S}_n$

($\tilde{S}_0 = S_0$ at time 0.)

($\tilde{S}_{k-1} = e^{-r(k-1)} S_{k-1}$)

($\varphi_{k-1} \cdot S_{k-1} = \varphi_k \cdot S_{k-1}$)

($\tilde{S}_{k-1} = e^{-r(k-1)} S_{k-1}$)

[By Assumption that EMM $\mathbb{Q}$ exists]

But $\tilde{S}_n$ is a $\mathbb{Q}$-martingale, which implies that our total wealth process $\tilde{V}_n^\varphi := \sum_{k=1}^n \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1})$ is a $\mathbb{Q}$-martingale because φ is previsible. To see this, note that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\Big(\sum_{k=1}^n \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}) | \mathcal{F}_{n-1}\Big) &= \sum_{k=1}^{n-1} \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}) + \mathbb{E}^{\mathbb{Q}}(\varphi_n \cdot (\tilde{S}_n - \tilde{S}_{n-1}) | \mathcal{F}_{n-1}) \\
&= \sum_{k=1}^{n-1} \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}) + \mathbb{E}^{\mathbb{Q}}\Big(\sum_{i=0}^d \varphi_n^i (\tilde{S}_n^i - \tilde{S}_{n-1}^i) | \mathcal{F}_{n-1}\Big) \\
&= \sum_{k=1}^{n-1} \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}) + \sum_{i=0}^d \varphi_n^i \underbrace{\mathbb{E}^{\mathbb{Q}}(\tilde{S}_n^i - \tilde{S}_{n-1}^i | \mathcal{F}_{n-1})}_{=0} \\
&= \sum_{k=1}^{n-1} \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1})
\end{aligned}$$

(Conditional IE at previous time n-1)

(info. up to (n-1) is known)

(rewriting dot product another way as sum)

(φ_n^i is previsible i.e. $\mathcal{F}_{n-1}$ measurable i.e. its value is known at time (n-1) so, comes out of IE)

Since, $\mathbb{E}^{\mathbb{Q}}(\tilde{S}_n^i | \mathcal{F}_{n-1}) = \tilde{S}_{n-1}^i$ as $\tilde{S}$ is Martingale under $\mathbb{Q}$.

where the final term has disappeared since $\tilde{S}$ is a martingale under $\mathbb{Q}$, and we can iterate this procedure to show that $\mathbb{E}^{\mathbb{Q}}(\sum_{k=1}^n \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}) | \mathcal{F}_m) = \sum_{k=1}^m \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1})$, i.e. $\tilde{V}_n^\varphi$ is a $\mathbb{Q}$-martingale. Thus we have

($m < n$)

$$\mathbb{E}^{\mathbb{Q}}(\tilde{V}_N^\varphi) = \tilde{V}_0^\varphi .$$

Proof in next page.

If φ is an arbitrage opportunity then $\tilde{V}_0^\varphi = 0$, so $\mathbb{E}^{\mathbb{Q}}(\tilde{V}_N^\varphi) = 0$. But if φ is an arbitrage opportunity we also know that $\tilde{V}_N^\varphi \geq 0$ $\mathbb{P}$-a.s. (and hence $\mathbb{Q}$-a.s.) and $\mathbb{P}(\tilde{V}_N^\varphi > 0) > 0$ which implies that $\mathbb{Q}(\tilde{V}_N^\varphi > 0) > 0$. But a non-negative random variable which has zero expectation must be zero a.s. Hence $\tilde{V}_N^\varphi = 0$ $\mathbb{Q}$ as (and thus as $\mathbb{P}$ a.s.), thus φ cannot be an arbitrage strategy. Thus we have shown that no arbitrage exists if an EMM exists. ■

$\mathbb{P}$-a.s. = (Probability almost surely) i.e with probability 1 under whatever measure we are considering)

Thm: Let X be a non-negative random variable. If $\mathbb{E}(X)=0$ then $X=0$ a.s. (i.e. $\mathbb{P}(X=0)=1$).

Proof:

For $n \geq 1$, we have

$$X \geq X \mathbb{1}_{\{X \geq 1/n\}} \geq \frac{1}{n} \mathbb{1}_{\{X \geq 1/n\}}$$

[Recall: if $X \leq Y \Rightarrow \mathbb{E}(X) \leq \mathbb{E}(Y)$]

$$0 = \mathbb{E}(X) \geq \mathbb{E}\left(\frac{1}{n} \mathbb{1}_{\{X \geq 1/n\}}\right) = \frac{1}{n} \mathbb{P}\left(X \geq \frac{1}{n}\right)$$

(by assumption)

So, $\mathbb{P}\left(X \geq \frac{1}{n}\right) = 0$.

Now,

$$\mathbb{P}(X > 0) = \mathbb{P}\left(\bigcap_{n \geq 1} \left\{X \geq \frac{1}{n}\right\}\right)$$

$$= \lim_{n \to \infty} \mathbb{P}\left(X \geq \frac{1}{n}\right)$$

$$= 0$$

$\boxed{\leq}$.

1.7 Market completeness

Definition 1.6 *The N-period market $\mathcal{M}$ is complete if every contingent claim can be replicated.*

Theorem 1.3 *(Completeness theorem). If $\mathcal{M}$ is arbitrage-free, then $\mathcal{M}$ is complete if and only if there is a unique probability measure $\mathbb{Q}$ under which the discounted vector of asset prices $\tilde{S}_n$ is a martingale.*

1.8 Forward contracts and no-arbitrage pricing

A forward contract is a contract to buy 1 unit of stock for a certain price F at a future time $T > 0$ (the price F is agreed at time zero, but no money or stock changes hands at time zero, only at time T). To replicate a forward contract, borrow S_0 in cash at time zero and use this to buy 1 unit of stock at time 0; you will then have 1 unit of stock at time T as required, and you will have to pay the bank back $S_0 e^{rT}$ at time T. Thus $F_0 = S_0 e^{rT}$ is the no-arbitrage price of the forward at time zero.

- If $F_0 > S_0 e^{rT}$, sell the forward contract, borrow S_0 at time zero and use this to buy 1 unit of stock at time zero. Then at time T, our net position in the stock is zero, and we receive F_0 for the stock sold, and we pay back $S_0 e^{rT} < F_0$ to the bank for the money borrowed, i.e. we make a riskless profit of $F_0 - S_0 e^{rT} > 0$.
- If $F_0 < S_0 e^{rT}$, buy the forward contract, **short** (i.e. sell) 1 unit of stock at time zero and invest the proceeds (i.e. S_0) in the riskless bank account until time T. Then at time T, our net position in the stock is zero, and we pay F_0 for the stock, and our riskless bank account is now worth $S_0 e^{rT} > F_0$, i.e. we make a riskless profit of $S_0 e^{rT} > F_0$.

1.9 Put-call parity

We now return to European call and put options. By considering both cases $S_T > K$ and $S_T < K$ we see that

$$(S_T - K)^+ + K \quad = \quad \max(K - S_T, 0) + S_T \,.$$

A portfolio of 1 call option and Ke^{-rT} dollars will be equal in value to the left hand side at T. Similarly, a portfolio of 1 put option and 1 share will be equal in value to the right hand side at T. Thus we obtain the **put-call parity**:

$$C + Ke^{-rT} \quad = \quad P + S_0$$

where C is the initial price of the call, where P is the initial price of the put.

2 Continuous time processes

2.1 Brownian motion

A continuous time stochastic process $(W_t)_{t \geq 0}$ is said to be a standard one-dimensional **Brownian motion** if it satisfies the following four properties:

(i) • $W_0 = 0$. → Starts at 0.

(ii) • W has **independent increments**, i.e.

$$W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2} \dots, W_{t_n} - W_{t_{n-1}}$$

are independent for all $0 \leq t_1 < t_2 < \dots < t_n$.

(iii) • The increments are **Normally distributed**: $W_t - W_s \sim N(0, t-s)$ for all $0 \leq s \leq t$.

(iv) • W_t is continuous as a function of t almost surely (i.e. with probability one), i.e. $W_{t+h} - W_t \to 0$ as $h \to 0$.

See graph on next page

Remark 2.1 We can work with $\mathcal{F}_t = \sigma(W_s; 0 \leq s \leq t)$, or often it is more convenient to work with a larger filtration called the augmented filtration (see FM04 for details, do not worry about the details of this filtration for this course).

Remark 2.2 For this course, the third property is the most important to remember. Setting $s = 0$, then since $W_s = 0$ by the first property, we see that

$$W_t \quad \sim \quad N(0, t)$$

and thus $\mathbb{P}(W_t > x) = \mathbb{P}(\frac{W_t - 0}{\sqrt{t}} > \frac{x-0}{\sqrt{t}}) = \mathbb{P}(Z > \frac{x}{\sqrt{t}}) = \Phi^c(\frac{x}{\sqrt{t}})$ where $\Phi^c(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$, and we are using that $Z = (W_t - 0)/\sqrt{t}$ is a standard Normal random variable. We will use this equation many times.

2.1.1 Covariance of Brownian motion

[$\mathrm{Cov}(X,Y) := \mathbb{E}((X - \mu_X)(Y - \mu_Y))$; For $\mathrm{Cov}(W_s, W_t)$ all terms evaluate to zero except $\mathbb{E}(W_s W_t)$] $\mu_X = 0$, $\mu_Y = 0$

Let $0 \leq s \leq t$. Then

$$\mathbb{E}(W_s W_t) \quad = \quad \mathbb{E}(W_s(W_s + W_t - W_s)) \quad = \quad \mathbb{E}(W_s^2) \quad = \quad \mathrm{Var}(W_s) \quad = \quad s$$

[If $0 \leq t \leq s$, then $\mathbb{E}(W_t W_s) = t$]

since $\mathbb{E}(W_s(W_t - W_s)) = \mathbb{E}((W_s - W_0)(W_t - W_s)) = 0$. This means that in general, for $s, t \geq 0$

(By Property ii & iii)

$$\text{Covariance} \rightarrow R(s,t) \quad := \quad \mathbb{E}(W_s W_t) \quad = \quad \min(s,t).$$

This is known as the **covariance function** of Brownian motion.

2.1.2 Lack of differentiability

Recall that a function f is said to be differentiable at x if the limit

$$f'(x) \quad = \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists.

Using our formula for $R(s,t)$ above, if $0 \leq s < t$, then for $h > 0$ sufficiently small the intervals $[s, s+h]$ and $[t, t+h]$ do not intersect, so using the second property of Brownian motion we see that

$$\mathbb{E}(\frac{W_{t+h} - W_t}{h} \frac{W_{s+h} - W_s}{h}) \quad = \quad 0$$

(Property ii & iii)

if $s < t$. Otherwise if $s = t$ then the answer is $\frac{1}{h^2}\mathbb{E}((W_{t+h} - W_t)^2) \quad = \quad \frac{1}{h^2} h = \frac{1}{h} \to \infty$ as $h \to 0$. Thus $\Delta_t^h := \frac{1}{h}(W_{t+h} - W_t)$ is a "process" with $\lim_{h \to 0} \mathbb{E}(\Delta_s^h \Delta_t^h) = \infty$ if $s = t$, and zero otherwise, which cannot be the covariance of a well defined process in the limit as $h \to 0$ because we have ∞ here.

Remark 2.3 We have shown that W is not differentiable, but it turns out that W is **continuous** a.s. (so it does not have jumps), and (using the **Kolmogorov continuity theorem** from FM04) we can make a stronger statement that W is **α-Hölder continuous** for $\alpha \in (0, \frac{1}{2})$, i.e. for $0 \leq s \leq t \leq T$

(stronger property than continuity)

$$|W_t - W_s| \quad \leq \quad c_1 |t - s|^\alpha$$

for some (random) constant c_1 which depends on $(W_t)_{0 \leq t \leq T}$ which is finite almost surely (a.s.). Note this implies that W is continuous a.s. since as $t - s \to$, $|W_t - W_s| \to 0$ since $\alpha > 0$.

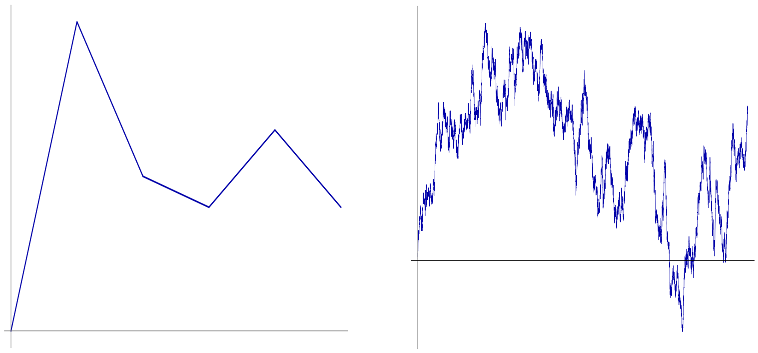


Figure 1: Euler scheme approximation for Brownian motion in Mathematica with large (left) and small (right) step size

W_t^n = Numerical approx. to B.M. (W_t).

2.2 Constructing and simulating Brownian motion (Euler Method)

- We can approximate Brownian motion numerically as follows: fix a small step size $\Delta t > 0$. Then

$$W^n_{(i+1)\Delta t} \quad = \quad W^n_{i\Delta t} + \sqrt{\Delta t}\, Z_i$$

  where Z_i is a sequence of i.i.d. standard $N(0,1)$ random variables, and $\Delta t = \frac{1}{n}$, and $W^n_0 = 0$.

- The W^n_t process starts at zero, has independent increments (more precisely the discrete time process $X_i = W^n_{i\Delta t}$ has independent increments) and we see that $W_{(i+1)\Delta t} - W_{i\Delta t} \sim N(0, \Delta t)$, which is consistent with the third property of Brownian motion that $W_t - W_s \sim N(0, t-s)$ (recall that for *any* random variable X, we have that $\mathrm{Var}(aX) = a^2 \mathrm{Var}(X)$).

  This procedure is known as the **Euler method**. If we then join the points $(W^n_{\Delta t}, W^n_{2\Delta t}, ...)$ with straight lines, this methods gives a *piecewise linear* approximation W^n_t to a true Brownian motion.

- Using a deeper result called **Donsker's theorem**, (need not be equidistant) it can be proved that this construction tends to a Brownian motion as the step size $\Delta t \to 0$, i.e. as $n \to \infty$ (recall that $\Delta t = \frac{1}{n}$). More precisely, for any bounded continuous function $f(x_1, ..., x_n)$ and $0 = t_0 < t_1 < t_2 < ... < t_k$, we have that

(The Numerical BM Converges to)

When random variables are independant the joint density is just the product of each individual densities.

(8)
$$\lim_{n\to\infty} \mathbb{E}\Big(f\Big(\frac{W^n_{t_1} - W^n_{t_0}}{\sqrt{t_1 - t_0}}, \frac{W^n_{t_2} - W^n_{t_1}}{\sqrt{t_2 - t_1}}, ..., \frac{W^n_{t_k} - W^n_{t_{k-1}}}{\sqrt{t_k - t_{k-1}}}\Big)\Big) \quad = \quad \mathbb{E}(f(Z_1, ..., Z_k))$$
$$= \quad \int_{z_1=-\infty}^{\infty} \int_{z_2=-\infty}^{\infty} .. \int_{z_k=-\infty}^{\infty} f(z_1, ..., z_k) \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z_i^2} dz_1 ... dz_k$$

  where $Z_1, ..., Z_k$ are iid standard Normal random variables, and the final equation is a k-dimensional integral of f with respect to the joint density of k iid Normal random variables. Note also that for any bounded continuous function f, $f(W^n_{t_1}, ..., W^n_{t_k})$ can easily be re-written in the form $\tilde{f}(\frac{W^n_{t_1} - W^n_0}{\sqrt{t_1 - t_0}}, \frac{W^n_2 - W^n_1}{\sqrt{t_2 - t_1}}, ..., \frac{W^n_k - W^n_{k-1}}{\sqrt{t_k - t_{k-1}}})$ for another function $\tilde{f}$, so the convergence result in (8) also tells us how to compute $\lim_{n\to\infty} \mathbb{E}(\tilde{f}(W^n_{t_1}, ..., W^n_{t_k}))$.

2.3 Quadratic variation of Brownian motion

(need not be equidistant, so may not have same step size)

A **partition** of the time interval $[0, t]$ is a set of the form $\Pi = \{t_0 = 0 < t_1 < ... < t_n = t\}$, and we define the size of the partition to be

$$|\Pi| \quad = \quad \max_{0 \le i \le n-1} (t_{i+1} - t_i)$$

i.e. equal to the largest interval of the partition. The **quadratic variation** of a process X over a fixed time interval $[0, t]$ is then defined as

$$\text{Covariation} \longrightarrow [X, X]_t \quad = \quad \langle X \rangle_t \quad = \quad \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 \,.$$

if this limit exists and does not depend on the choice of the sequence of partitions. In general the quadratic variation $[X, X]_t$ of a process X is a random process, but we will see that for Brownian motion W, $[W, W]_t = t$ a.s.

Let W denote a standard Brownian motion. We define $[W,W]^n_t = \sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2$ to be the **sampled quadratic variation**, for a single partition Π, where $n = n(\Pi)$ denotes the number of partitions in Π.

Proposition 2.1

$$\begin{aligned} \mathbb{E}([W,W]^n_t) &\to t \\ \mathrm{Var}([W,W]^n_t) = \mathbb{E}(([W,W]^n_t - t)^2) &\to 0 \end{aligned}$$

as $n \to \infty$.

(Swap $\mathbb{E}$ & Σ as it is finite sum. For ∞ sum need to use Fubini's Thm for justifying swap.)

Proof. We first note that

Summing all increments gives t.

$$\mathbb{E}([W,W]^n_t) = \mathbb{E}[\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2] = \sum_{i=0}^{n-1}\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] = \sum_{i=0}^{n-1}(t_{i+1} - t_i) = t$$

since $W_{t_{i+1}} - W_{t_i} \sim N(0, t_{i+1} - t_i)$. Thus the expected sampled quadratic variation is independent of partition under consideration, and trivially $\lim_{n\to\infty}\mathbb{E}([W,W]^n_t) = t$, since the expectation here does not depend on n. Moreover

$$\begin{aligned} \mathrm{Var}([W,W]^n_t) = \mathbb{E}(([W,W]^n_t - t)^2) &= \mathbb{E}((\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 - t)^2) \\ &= \mathbb{E}((\sum_{i=0}^{n-1}[(W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)])^2) \quad \leftarrow \text{writing } t \text{ as sum of increments} \\ &= \sum_{i=0}^{n-1}\mathbb{E}([(W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)]^2) \quad \leftarrow \text{swap } \mathbb{E} \,\&\, \Sigma. \\ &= \sum_{i=0}^{n-1}\mathbb{E}((W_{t_{i+1}} - W_{t_i})^4 - 2(t_{i+1} - t_i)(W_{t_{i+1}} - W_{t_i})^2 + (t_{i+1} - t_i)^2) \\ &= \sum_{i=0}^{n-1} 3(t_{i+1} - t_i)^2 - 2(t_{i+1} - t_i)^2 + (t_{i+1} - t_i)^2 \\ &\quad (\text{using that } \mathbb{E}(Z^4) = 3 \text{ where } Z \sim N(0,1) \text{ and } W_{t_{i+1}} - W_{t_i} \sim \sqrt{t_{i+1} - t_i}\, Z) \\ &= 2\sum_{i=0}^{n-1}(t_{i+1} - t_i)^2 \\ &\leq 2\sum_{i=0}^{n-1}(t_{i+1} - t_i)\,|\Pi| \\ &= 2|\Pi|t \end{aligned}$$

which clearly tends to zero if $|\Pi| \to 0$. ∎

We now recall some important notions of convergence of random variables:

- A sequence of random variables X_n is said to converge to a random variable X in L^2 if $\mathbb{E}((X - X_n)^2) \to 0$.
- A sequence of random variables X_n is said to converge to a random variable X **in probability** if $\lim_{n\to\infty}\mathbb{P}(|X - X_n| > K) \to 0$ for all $K > 0$.

Recall that Proposition 2.1 shows that $\mathbb{E}(([W,W]^n_t - t)^2) \to 0$ so we see that $[W,W]^n_t$ converges to t in L^2 where here $X_n = [W,W]^n_t$ and $X = t$. Moreover, using the **Chebychev inequality**: for any random variable X:

$$\mathbb{P}(|X| > K) = \mathbb{E}(1_{|X|>K}) \leq \frac{1}{K^2}\mathbb{E}(X^2)$$

(see plot at top of next page), this further implies that (letting $X = [W,W]^n_t - t$)

$$\lim_{|\Pi|\to 0}\mathbb{P}(|[W,W]^n_t - t| > K) \leq \lim_{|\Pi|\to 0}\frac{1}{K^2}\mathbb{E}(([W,W]^n_t - t)^2) = 0$$

for all $K > 0$, since we have just shown that $\lim_{|\Pi|\to 0}\mathbb{E}(([W,W]^{n(\Pi)}_t - t)^2) = 0$. This means that $[W,W]^n_t \to t$ **in probability** as well.

Remark 2.4 See Excel sheet BrownianMotionQuadraticVariation.xls to see this result confirmed numerically (note convergence is slow).

Remark 2.5 a.s. convergence implies convergence in probability but not necessarily the other way around

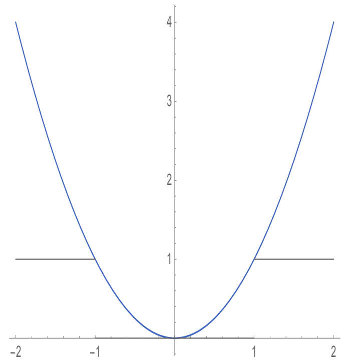
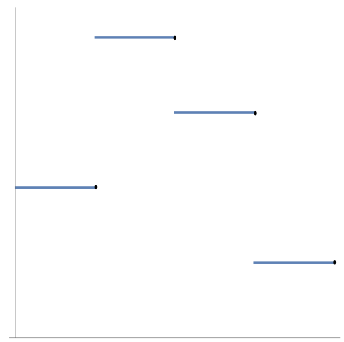


Figure 2: Graph to demonstrate the Chebychev inequality: the blue line is $\frac{1}{K^2}x^2$, and the grey line is $1_{|x|>K}$ for $K = 1$ here, and we see that the blue line is always above the grey line. We then replace a fixed value of x with a random variable X, and take expectations to obtain the Chebychev inequality.



← left continuous

Figure 3: Example of a simple random process

2.4 Stochastic integrals with respect to Brownian motion

We now want to make sense of an integral of the form $\int_0^t \alpha_s dW_s$ for some stochastic process α_t which is adapted to $\mathcal{F}_t$, i.e. depends on the history of W up to time t. Such an integral is known as a **stochastic integral**. These cannot be defined as a conventional integral as in FM01 because Brownian motion has **infinite variation**, which means that

$$\lim_{N\to\infty} \sum_{i=1}^{N} |W_{t_i} - W_{t_{i-1}}| \quad = \quad \infty \tag{8}$$

a.s. Note that we computing the sum of $|W_{t_i} - W_{t_{i-1}}|$ not $|W_{t_i} - W_{t_{i-1}}|^2$ now.

Let $0 < t_1 < t_2 < ....t_N = T$. Then we can define the **stochastic integral** of a **simple random process** of the form $\alpha_t = \sum_{i=1}^N \alpha_i 1_{t\in(t_{i-1},t_i]}$ (where α_i is random but can only depend on the history of W up to time t_{i-1}), in the natural way as

$$\int_0^T \alpha_s dW_s \quad = \quad \sum_{i=1}^{N} \alpha_i (W_{t_i} - W_{t_{i-1}})\,.$$

To generalize the stochastic integral to more general (non-simple) process α_t, we approximate the process to arbitrary accuracy with a simple process and use arguments involving L^2-convergence which will be done rigourously in FM04.

2.5 Stochastic differential equations and Ito's lemma

We now want to assign a rigorous meaning to the **stochastic differential equation** (SDE):

$$dX_t \quad = \quad b(X_t)dt + \sigma(X_t)dW_t\,. \tag{9}$$

If $b(x) = 0$ and $\sigma(x) = 1$, then $dX_t = dW_t$, so $X_t = c + W_t$ for some arbitrary constant c.

Definition 2.1 *A strong solution to the SDE* (9) *with $X_0 = x_0$ is a process X_t with a continuous sample path which satisfies the integral equation*

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s\,. \tag{10}$$

b is known as the drift, and σ is known as the volatility (or diffusion coefficient).

Remark 2.6 Note if $\sigma \equiv 0$, then (9) reduces to the Ordinary Differential equation (ODE):

$$\frac{dx(t)}{dt} = b(x(t))$$

which we can solve be re-writing as $\frac{dx}{b(x)} = dt$ and integrating both sides.

X in (10) is implicitly defined in terms of itself, so how do we construct a solution rigorously? We first simulate the Brownian sample path W. We then start with the constant solution $X_t^0 = x_0$, feed it into the SDE, and then we keep feeding the answer back into the SDE as follows:

$$\begin{aligned}
X_t^{(1)} &= x_0 + \int_0^t b(X_0)ds + \int_0^t \sigma(X_0)dW_s\,, \\
X_t^{(2)} &= x_0 + \int_0^t b(X_s^{(1)})ds + \int_0^t \sigma(X_s^{(1)})dW_s \\
&\quad \ldots \\
X_t^{(n+1)} &= x_0 + \int_0^t b(X_s^{(n)})ds + \int_0^t \sigma(X_s^{(n)})dW_s \\
&= \ldots
\end{aligned}$$

With some work we can then show that the sequence of sample paths $\{X_s^{(n)}; 0 \le s \le t\}$ converges to some continuous function X_t which we call the solution to (9), i.e. $\lim_{n\to\infty} \max_{0\le s\le t}|X_s^n - X_s)| \to 0$ (see chapter 5 in Karatzas&Shreve book for details). This is called the **Picard iteration method**.

A function f is **Lipschitz** continuous if $|f(x) - f(y)| \le K|y - x|$ for some non-negative constant $K < \infty$.

Proposition 2.2 *If b and σ are bounded and Lipschitz continuous, then a strong solution exists for T sufficiently small.*

On a computer, we typically do not use the Picard method to numerically approximate the SDE, but rather we use an extension of the Euler scheme where we essentially "freeze" the coefficients over each time step:

$dX_t = b(X_t)dt + \sigma(X_t)dW_t$

$$X_{t+\Delta t}^n = X_t^n + b(X_t^n)\Delta t + \sigma(X_t^n)\sqrt{\Delta t}\, Z_i \tag{11}$$

where Z_i is a sequence of i.i.d. standard Normals as before and $\Delta t = \frac{1}{n}$ as before, and under certain conditions we can show that the approximate solution X_t^n tends to true solution X_t to the SDE as $n \to \infty$ (i.e. as the step size $\Delta t = \frac{1}{n} \to 0$).

2.6 Ito's lemma

Theorem 2.3 *For a process X satisfying the SDE in* (9) *such that $f(x,t)$ is twice differentiable in x and once in t, we have*

$$\begin{aligned}
f(X_t,t) &= f(X_0,0) + \int_0^t f_x(X_s,s)dX_s + \int_0^t [f_t(X_s,s) + \frac{1}{2}f_{xx}(X_s,s)\sigma(X_s)^2]ds \\
&= f(X_0,0) + \int_0^t f_x(X_s,s)[b(X_s)ds + \sigma(X_s)dW_s] + \int_0^t [f_t(X_s,s) + \frac{1}{2}f_{xx}(X_s,s)\sigma(X_s)^2]ds\,.
\end{aligned} \tag{12}$$

As shorthand, we write (12) in the differential form as

$$df(X_t,t) = f_t(X_t,t)dt + f_x(X_t,t)dX_t + \frac{1}{2}f_{xx}(X_t,t)\sigma(X_t)^2dt\,.$$

Note that we get a surprising additional second order term here (the term involving f_{xx}) which we do not get if X_t is just a non-random function which differentiable in x and in t), and this extra terms is the main difference between ordinary calculus and **stochastic calculus**.

THE RULE FOR THE GENERALIZED ITO LEMMA IS: WHATEVER IS IN FRONT OF dW_t IN THE ORIGINAL SDE GETS SQUARED AND MULTIPLIED BY ONE-HALF

2.7 Important examples

Note that W satisfies the trivial SDE:

$$dX_t \quad = \quad 0dt \, + \, 1dW_t \, .$$

1. Let $f(x,t) = x^2 - t$ so $f(W_t,t) = W_t^2 - t$. Then $f_x(x,t) = 2x$, $f_{xx}(x,t) = 2$ and $f_t(x,t) = -1$. Thus from Ito's lemma we have

$$\begin{aligned} df(W_t,t) &= f_t(W_t,t)dt + f_x(W_t,t)dW_t + \frac{1}{2} f_{xx}(W_t,t)dt \\ &= -dt \, + \, 2W_t dW_t \, + \, \frac{1}{2} \cdot 2dt \\ &= 2W_t dW_t \, . \end{aligned}$$

Writing this in integrated form we have $f(W_t,t) - f(W_0,0) = W_t^2 - t = \int_0^t 2W_s dW_s$, and we can easily verify that $M_t = W_t^2 - t$ is a martingale, i.e. $\mathbb{E}(W_t^2 - t | (W_u)_{0 \le u \le s}) = W_s^2 - s$.

2. Let $f(x,t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma x}$, so $S_t := f(W_t,t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$. This is the famous **Black-Scholes** model for a stock price process. Then $f_x(x,t) = \sigma f(x,t)$, $f_{xx}(x,t) = \sigma^2 f$ and $f_t(x,t) = (\mu - \frac{1}{2}\sigma^2)f$. Thus from Ito's lemma we have

$$\begin{aligned} dS_t &= f_t(W_t,t)dt + f_x(W_t,t)dW_t + \frac{1}{2} f_{xx}(W_t,t)dt \\ &= (\mu - \frac{1}{2}\sigma^2)S_t dt \, + \, \sigma S_t dW_t \, + \, \frac{1}{2}\sigma^2 S_t dt \\ &= \mu S_t dt \, + \, \sigma S_t dW_t \qquad (13) \\ &= b(S_t)dt \, + \, \tilde{\sigma}(S_t)dW_t \end{aligned}$$

where $b(S) = \mu S$ and $\tilde{\sigma}(S) = \sigma S$. Note that b and σ are both linear functions of S, so this process is known as a **Geometric Brownian motion**.

We can show that S_t is a $\mathcal{F}_t$-martingale iff $\mu = 0$, and this is an example of a stochastic differential equation. μ describes the overall trend of the process, i.e. its tendency to go up or down in the long run (it can be shown using the mgf of a Normal distribution that $\mathbb{E}(S_t) = S_0 e^{\mu t}$, and σ is the volatility which controls the variability of the stock price (see Excel sheet on website). Note that S_t is always positive.
↳ BlackScholesMonteCarlo.xls

3. **pth power of stock price for Black-Scholes model**. Assume that $dS_t = S_t \sigma dW_t$ and let $M_t = S_t^p$ for $p > 1$. Apply Ito's lemma to M_t, and write dM_t as an SDE in terms of M_t itself.

Set $M_t = f(S_t,t)$, where $f(S,t) = S^p$. Applying Ito's lemma to $dS_t = S_t \sigma dW_t$, we have that $f_t(S,t) = 0$, $f_S(S,t) = pS^{p-1}$, $f_{SS}(S,t) = p(p-1)S^{p-2}$ and

$$\begin{aligned} dM_t = df(S_t,t) &= f_t(S_t,t)dt \, + \, f_S(S_t,t)dS_t \, + \, \frac{1}{2} f_{SS}(S_t,t) S_t^2 \sigma^2 dt \\ &= pS_t^{p-1} dS_t \, + \, \frac{1}{2} p(p-1) S_t^{p-2} S_t^2 \sigma^2 dt \\ &= pS_t^{p-1} S_t \sigma dW_t \, + \, \frac{1}{2} p(p-1) S_t^p \sigma^2 dt \\ &= \frac{1}{2} p(p-1)\sigma^2 M_t dt \, + \, p\sigma M_t dW_t \\ &= M_t(\frac{1}{2} p(p-1)\sigma^2 dt \, + \, p\sigma dW_t) \end{aligned}$$

i.e. the drift and volatility for M_t are linear functions of M_t, so M is a Geometric Brownian motion like S in (13).

4. The Ornstein-Uhlenbeck process. Consider the Ornstein-Uhlenbeck (OU) process which satisfies

$$dY_t \quad = \quad -\kappa Y_t dt + \sigma dW_t \qquad (14)$$

for $\kappa, \sigma > 0$. The drift term means the process tends to "mean-revert" back to zero if Y goes too far away from zero. Let $Z_t = e^{\kappa t} Y_t = f(t, Y_t)$ where $f(t,y) = e^{\kappa t} y$. Then $f_t = \kappa f$, $f_y = e^{\kappa t}$ and $f_{yy} = 0$.

$$\begin{aligned} dZ_t \; = \; f_t dt + f_y dY_t \, + \, \frac{1}{2} f_{yy} \sigma^2 dt &= \kappa e^{\kappa t} Y_t dt \, + \, e^{\kappa t} dY_t + 0 \\ &= \kappa e^{\kappa t} Y_t dt \, + \, e^{\kappa t}(-\kappa Y_t dt + \sigma dW_t) \\ &= \sigma e^{\kappa t} dW_t \, . \end{aligned}$$

Integrating this equation, we see that $Z_t = Z_0 + \int_0^t \sigma e^{\kappa s} dW_s$. Multiplying by $e^{-\kappa t}$ and noting that $Z_0 = Y_0$, we obtain

$$Y_t \;=\; e^{-\kappa t} Z_0 \,+\, e^{-\kappa t}\int_0^t \sigma e^{\kappa s} dW_s \;=\; e^{-\kappa t} Y_0 \,+\, \int_0^t \sigma e^{-\kappa(t-s)} dW_s\,.$$

This is the solution to the SDE in (14). It turns out that for any non-random continuous function ϕ, $\int_0^t \phi(s) dW_s \sim N(0, \int_0^t \phi(s)^2 ds)$ (proof not required). In this case, setting $\phi(s) = \sigma e^{-\kappa(t-s)}$, we find that

$$\begin{aligned} Y_t \;&\sim\; e^{-\kappa t} Y_0 \,+\, N(0, \int_0^t \phi(s)^2 ds) \\ &=\; e^{-\kappa t} Y_0 \,+\, N(0, \sigma^2 \frac{1 - e^{-2\kappa t}}{2\kappa}) \end{aligned}$$

From this we see that $Y_\infty = \lim_{t\to\infty} Y_t \sim N(\frac{\sigma^2}{2\kappa})$. The OU process is often used in practice to model volatility or an interest rate process (this is the well known **Vasicek model**, see FM07 for more details on this and the CIR process).

5. The Black-Scholes model, and foreign exchange rates. Set $Y_t = f(S_t)$ where S_t satisfies the Black-Scholes SDE $dS_t = \sigma S_t dW_t$ with $\mu = 0$ and $f(S) = 1/S$. Applying the generalized Ito lemma to $dS_t = \sigma S_t dW_t$ we have $f_S = -\frac{1}{S^2} = -Y^2$, $f_{SS} = \frac{2}{S^3} = 2Y^3$ and

$$dY_t \;=\; -\frac{1}{S_t^2} dS_t + \frac{1}{2}\frac{2}{S_t^3} S_t^2 \sigma^2 dt \;=\; Y_t[\sigma^2 dt \,-\, \sigma dW_t]\,.$$

If S_t is e.g. the GBP/USD Exchange rate i.e. the cost of a pound in dollars, then $Y_t = 1/S_t$ is the USD/GBP exchange rate, i.e. the cost of a dollar in pounds.

Hitting time problems

Using that $Z_t = e^{\lambda W_t - \frac{1}{2}\lambda^2 t}$ is an $\mathcal{F}_t^W$-martingale with $Z_0 = 1$, using the monotone and dominated convergence theorems (see FM14 notes for details) we can show that $\mathbb{E}(e^{\lambda W_{\tau_b} - \frac{1}{2}\lambda^2 \tau_b}) = \mathbb{E}(e^{\lambda b - \frac{1}{2}\lambda^2 \tau_b}) = 1$ so

$$\mathbb{E}(e^{-p\tau_b}) \;=\; e^{-b\sqrt{2p}}\,.$$

Moreover (common sense) we know that $\{M_t \geq b\} = \{\tau_b \leq t\}$, so $\mathbb{P}(M_t > b) = \mathbb{P}(\tau_b \leq t)$. Then using something called the **reflection principle** in FM14, we will see that

$$\mathbb{P}(M_t \geq b) \;=\; \mathbb{P}(\tau_b \leq t) \;=\; 2\mathbb{P}(W_t > b) \;=\; 2\Phi^c(\frac{b}{\sqrt{t}})\,.$$

We can then differentiate the right hand side with respect to b using the chain rule to get the density of M_t as

$$p_t(b) \;=\; \frac{2}{\sqrt{2\pi t}}\, e^{-\frac{b^2}{2t}} 1_{b>0} \tag{15}$$

which is a *one-sided* Normal density (in fact $p_t(b)$ is twice the density of a standard Brownian motion at timr t but only one-sided, since $M_t \geq 0$ a.s.). Similarly, we can instead differentiate wrt t to get the density of τ_b as

$$f_{\tau_b}(t) \;=\; \frac{b}{\sqrt{2\pi t^3}}\, e^{-\frac{b^2}{2t}} 1_{t>0}\,.$$

3 Continuous time models

3.1 The Black-Scholes option pricing model

As mentioned above, the Black-Scholes model for a stock price process is defined as

$$S_t \;=\; S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{16}$$

where W is a standard Brownian motion. From Ito's lemma, we have seen that S_t satisfies the SDE

$$dS_t \;=\; S_t(\mu dt \,+\, \sigma dW_t)$$

which is known as **Geometric Brownian motion**. μ is the **drift** of the process which describes the overall upward/downward trend, and σ is the **volatility**, which describes its variability.

3.2 The terminal stock price distribution

- Re-arranging (16) we see that

$$\log S_t - \log S_0 = \log \frac{S_t}{S_0} = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\,.$$

  Thus we can calculate the distribution of $\log \frac{S_t}{S_0}$ as $\log \frac{S_t}{S_0} \sim N((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$ or equivalently $\log S_t \sim N(\log S_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$.

- For some fixed $T > 0$, we can then compute $\mathbb{P}(S_T > K)$ as follows:

$$\begin{aligned}
\mathbb{P}(S_T > K) &= \mathbb{P}(\log \frac{S_T}{S_0} > \log \frac{K}{S_0}) \\
&= \mathbb{P}(\frac{\log \frac{S_T}{S_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} > \frac{\log \frac{K}{S_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) \\
&= \mathbb{P}(Z > \frac{\log \frac{K}{S_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) \\
&= \Phi^c(z)
\end{aligned}$$

  where $z = \frac{\log \frac{K}{S_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, because $Z = \frac{\log \frac{S_T}{S_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ is a standard $N(0,1)$ random variable, and $\Phi^c(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$.

- Example: set $S = 1, K = 1.1, \sigma = .1, T = .25, \mu = .05$. Then we have $z = 1.681153596$ and $\mathbb{P}(S_T > K) = \Phi^c(z) = 1 - \Phi(z) = 0.046361687633$. Use Normsdist(.) to calculate Φ and Φ^c in Excel.
  Φ

- Note that

$$\mathbb{P}(S_t \le S) = \mathbb{P}(\log S_t \le \log S) = F(\log S)$$

  where F is the distribution function of $\log S_t$. Differentiating both sides with respect to S and using the chain rule, we see that the density $p_{S_t}(S)$ of S_t is given by

$$p_{S_t}(S) = \frac{d}{dS}\mathbb{P}(S_t \le S) = \frac{1}{S}F'(\log S) = \frac{1}{S}p_{X_t}(x)$$

  where $x = \log S$ and $p_{X_t}(x)$ is the density of $X_t = \log S_t$ which is given by $p_{X_t}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{(x - x_0 - (\mu - \frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t}}$ since $X_t \sim N(X_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$, and the density of a general $N(\mu_1, \sigma_1^2)$ is given by

$$\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}$$

  so

$$p_{S_t}(S) = \frac{1}{S\sigma\sqrt{2\pi t}}\, e^{-\frac{[\log \frac{S}{S_0} - (\mu - \frac{1}{2}\sigma^2)t]^2}{2\sigma^2 t}}$$

  for $S > 0$. S_t has what is known as a **lognormal distribution**. Note the presence of the $\frac{1}{S}$ pre-factor, and this pdf is only defined for $S > 0$ because the stock price cannot go negative.

- $p_{S_t}(S)$ is a probability density function and thus must integrate to 1, i.e. $\int_0^\infty p_{S_t}(S)dS = 1$.

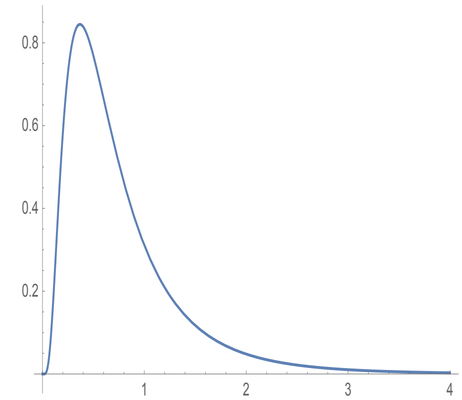


Figure 4: Here we have plotted the stock price density $f_{S_t}(S)$ for $\sigma = 1$ and $t = 1$.

3.3 The Black-Scholes PDE

[Like Heat Eqn: $f_t + \frac{1}{2} f_{xx} = 0$]

Proposition 3.1 *There is a unique solution $C(S,t)$ to the Black-Scholes partial differential equation:*

$$C_t(S,t) + rSC_S(S,t) + \frac{1}{2}\sigma^2 S^2 C_{SS}(S,t) \quad = \quad rC(S,t) \tag{17}$$

with terminal boundary condition $C(S,T) = f(S)$.

We will not prove this on this course, but the solution for the case when $f(S) = \max(S-K,0)$ is given below as the famous **Black-Scholes formula**.

Now suppose a stock follows the Black-Scholes model in (16) and a trader begins with **initial wealth** X_0 and at each time instant t, holds ϕ_t shares of stock. ϕ_t can be random but must be **adapted**, which means its value can only depend on what happened up to time t, i.e. the trader cannot look into the future to determine the value of ϕ_t, as we would expect. His remaining wealth $X_t - \phi_t S_t$ is placed in a risk-free bank account which earns $(X_t - \phi_t S_t)rdt$ in each infinitesimal time instant dt (note that if $X_t - \phi_t S_t$ is negative then we are **borrowing** money so are paying $(X_t - \phi_t S_t)rdt$ in interest). This is what we call a **self-financing** trading strategy for a continuous model - there are no injections or withdrawals of funds. Then the trader's total wealth X_t evolves as

$$\begin{aligned} dX_t &= \phi_t dS_t + (X_t - \phi_t S_t)rdt \qquad (18) \\ &= \phi_t S_t(\mu dt + \sigma dW_t) + (X_t - \phi_t S_t)rdt \\ &= (rX_t + \phi_t S_t(\mu - r))dt + \phi_t S_t \sigma dW_t \,. \qquad (19) \end{aligned}$$

Note that this is really just shorthand for

$$X_t - X_0 \quad = \quad \int_0^t (\phi_u dS_u + (X_u - \phi_u S_u)r du) \tag{20}$$

Applying Ito's lemma to $C(S_t,t)$, where $C(S,t)$ is the solution to the PDE (17), we see that

$$\begin{aligned} dC(S_t,t) &= C_t(S_t,t)dt + C_S(S_t,t)dS_t + \frac{1}{2}C_{SS}(S_t,t)S_t^2\sigma^2 dt \\ &= C_t(S_t,t)dt + C_S(S_t,t)S_t(\mu dt + \sigma dW_t) + \frac{1}{2}C_{SS}(S_t,t)S_t^2\sigma^2 dt \\ &= (C_t(S_t,t) + \mu C_S(S_t,t)S_t + \frac{1}{2}C_{SS}(S_t,t)S_t^2\sigma^2)dt + C_S(S_t,t)S_t\sigma dW_t \,. \qquad (21) \end{aligned}$$

We now wish to find a ϕ_t such that its corresponding wealth process $X_t = X_t^\phi$ satisfies $X_T = f(S_T)$, i.e. our self-financing trading strategy ϕ_t perfectly replicates the payoff $f(S_T)$ at time T. To this end, we now guess that $\phi_t = C_S(S_t,t)$ for all $t \in [0,T]$. Then if we **equate the drift terms** in Eqs (19) and (21) with $\phi_t = C_S(S_t,t)$ we see that

$$C_t(S_t,t) + \mu C_S(S_t,t)S + \frac{1}{2}C_{SS}(S_t,t)S_t^2\sigma^2 \quad = \quad rX_t + C_S(S_t,t)S_t(\mu - r)$$

which (after cancelling the μ terms) we can re-write as

$$C_t(S_t,t) + rC_S(S_t,t)S + \frac{1}{2}C_{SS}(S_t,t)S_t^2\sigma^2 \quad = \quad rC(S_t,t) \quad = \quad rX_t$$

where we used the Black-Scholes PDE to obtain the first equality here. Thus we see that $X_t = C(S_t, t)$ and in particular $X_T = C(S_T, T) = f(S_T)$ (from the boundary condition of the Black-Scholes PDE). Since S_t can take any value in $(0, \infty)$, we must have that

$$C_t(S,t) + rC_S(S,t)S + \frac{1}{2}C_{SS}(S,t)S^2\sigma^2 \quad = \quad rC(S,t)$$

Note that the PDE and boundary condition are independent of μ, and hence the no-arbitrage price of the option is also independent of μ, as in the discrete-time setup.

To sum up, we can replicate a terminal payoff of $f(S_T)$ under the Black-Scholes model by:

Replicating portfolio of payoff of $f(S_T)$

- Holding $X_0 = C(S_0, 0)$ dollars in cash at time zero. (Start with X_0 amount of Cash)
- Dynamically trading the stock from 0 to T, holding $\phi_t = C_S(S_t, t)$ units of stock at each time instant t. $C_S(S_t, t)$ is known as the **Delta** (Δ) of the option at time t. (Hold ϕ_t of Stock)
- Hold our remaining wealth $X_t - C_S(S_t, t)S_t$ at each time instant t in the risk-free bank account, so our total wealth X_t evolves as in (19) with $X_T = f(S_T)$. (Put remaining cash in Bank account)

If the option price in the market $C^{mkt}(S,t) > C(S,t)$, then there is **arbitrage**. Specifically, we can sell the option in the market for C^{mkt} and replicate it using the arguments above at a cost of $C(S,t)$ to realize a riskless profit. Conversely, if the option is too cheap in the market, we can buy it, and replicate $-f(S_T)$ at a cost of $-C(S,t)$, and again realize a riskless profit.

$C_{SS}(S_t, t)$ is known as the **Gamma** (Γ) and $C_t(S_t, t)$ is known as the **Theta** (Θ) of the option, and as we shall see, these partial derivatives have explicit formulae for the case when $f(S) = \max(S_K, 0) = (S-K)^+$, i.e. a European call option.

3.4 Solving the Black-Scholes PDE - the Black-Scholes formula

For a standard European call option, we know that $f(S) = \max(S - K, 0) = (S-K)^+$ and in this case, the Black-Scholes PDE has an explicit solution given by the famous **Black-Scholes formula**:

$$C(S,t) \quad = \quad C^{BS}(S, K, \sigma, T-t, r) \quad = \quad S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$$

where $\tau = T - t$ and

$$d_1 \quad = \quad \frac{\log\frac{S}{K} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \quad , \quad d_2 \quad = \quad d_1 - \sigma\sqrt{\tau}$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-z^2/2}}{\sqrt{2\pi}}dz$ is the standard cumulative Normal distribution function. Note that the call price C^{BS} actually depends on five parameters, but only S and t dynamically change over time.

Thus $C(S_t, t) = C^{BS}(S_t, K, \sigma, T-t, r)$ and $\phi_t = C_S(S_t, t)$, which can be calculated explicitly using the formula in (22) below. We can also compute $C(S,t)$ explicitly if the terminal payoff is $\log S_T$, $(\log S_T)^2$ or S_T^p (see mock exams and homeworks), in which case the argument is exactly the same except the boundary condition for the PDE will change and thus $C(S,t)$ will change, and hence so will $\phi_t = C_S(S_t, t)$.

3.4.1 Numerical example

Assume current stock price is 1, volatility is .10 and interest rate is .05. Price a call option at time zero with strike 1.1 with maturity 1: take $S = 1, K = 1.1, \sigma = .1, \tau = 1, r = .05$ and $t = 0$ and $\tau = T$. Plugging these numbers into the BS formula we obtain

$$\begin{aligned} d_1 &= -0.403101798043249\,, \\ d_2 &= -0.503101798043249 \end{aligned}$$

and the call price

$$C \quad = \quad 0.021739503382137$$

(the Excel sheet "BlackScholesModel.xls" on the course website implements this formula in Visual Basic).
BlackScholesFormula+Greeks.xls

$C(S,t) \to \max(S-K, 0)$ as $t \to T$, and (less obvious) $C(S,t) \to S$ if $T \to \infty$. It can be shown from something called **Jensen's inequality** that $C(S,t)$ is increasing in $\tau = T - t$.

Remark 3.1 The initial (i.e. $t = 0$) cost of the replicating strategy at time zero is $C(S_0, 0)$, and at each time instant, $C(S_t, t)$ is the unique no-arbitrage price of the call option (see next subsection to see why this is so).

3.5 The Greeks

We can compute partial derivatives of the BS formula with respect to each of the parameters

$$\begin{aligned}\Delta &= \frac{\partial C}{\partial S} = \Phi(d_1) > 0, \\ \Gamma &= \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \frac{n(d_1)}{S\sigma\sqrt{\tau}} > 0\,, \\ \Lambda &= \frac{\partial C}{\partial \sigma} = Sn(d_1)\sqrt{\tau} > 0\end{aligned} \tag{22}$$

where $n(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$ is the standard Normal density. Δ, Γ and Λ are known as the Delta, Gamma and Vega respectively of the option. The proof of these expressions for the Greeks are very tedious and not examinable.

- Delta measures the responsiveness of the call option price to small changes in the underlying stock price.
- Vega measures the responsiveness of the call option price to small changes in the volatility.
- Gamma measures the responsiveness of the Delta to small changes in the underlying stock price.

For the numerical example above, we obtain $\Delta = 0.343435379743$, $\Gamma = 3.677756432729$ and $\Lambda = 0.367811626137$.

3.6 The Feynman Kac formula

It turns out that $C(S,t)$ (the price of a option which pays $f(S_T)$ at time T under the Black-Scholes model) also has the following *probabilistic* representation:

$$C(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(f(S_T)\,|\,S_t = S)$$

where $\mathbb{Q}$ is a new probability measure under which S satisfies

$$dS_t = S_t(rdt + \sigma dW_t) \tag{23}$$

and W is a standard Brownian motion under $\mathbb{Q}$ (this is a special case of a more general result called the **Feynman-Kac** formula). In words: **the option price (i.e. the cost of replicating the option) is the discounted expected value of $f(S_T)$ in the risk-neutral world $\mathbb{Q}$ where the drift is r not μ.** We refer to this world as the **risk neutral measure**. Note that (23) this is the same as the Black-Scholes SDE $dS_t = S_t(\mu dt + \sigma dW_t)$, but the real-world μ has been replaced by r, as for the discrete case.

Example. Price a digital call option under the Black-Scholes model which pays 1 if $S_T > K$ and zero otherwise.

Solution:

$$\begin{aligned}P(S,t) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(1_{S_T>K}\,|\,S_t = S) \\ &= e^{-r(T-t)}\mathbb{Q}(S_T > K\,|\,S_t = S)\end{aligned}$$

where (again) S_t satisfies the SDE $dS_t = S_t(rdt+\sigma dW_t)$ under the probability measure $\mathbb{Q}$, and $\mathbb{Q}(A)$ denotes the probability of an event A under the probability measure $\mathbb{Q}$, and we have used that for any random variable X, $\mathbb{P}(X > K) = \mathbb{E}(1_{X>K})$. If we now let $t = 0$, then we can compute $\mathbb{Q}(S_T > K)$ similar to before as:

$$\begin{aligned}\mathbb{Q}(S_T > K) &= \mathbb{Q}(\log\frac{S_T}{S_0} > \log\frac{K}{S_0}) \\ &= \mathbb{Q}(\frac{\log\frac{S_T}{S_0} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} > \frac{\log\frac{K}{S_0} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) \\ &= \mathbb{Q}(Z > \frac{\log\frac{K}{S_0} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) \\ &= \Phi^c(z)\end{aligned}$$

where $z = \frac{\log\frac{K}{S_0} - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ (note that we have now just replaced μ with r). Similarly

$$\mathbb{Q}(S_T > K\,|\,S_t = S) = \Phi^c(z)$$

where now $z = \frac{\log \frac{K}{S_t} - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$. Note that

$$S_T \;=\; S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T} \;=\; S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t} e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)} \;=\; S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}$$

so at time t we have $\log S_T \sim N(\log S_t + (r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))$.

Example. Write down the Black-Scholes PDE and boundary condition for the price of a **time-window barrier option** at time $t \leq T$, which pays $\max(S_T - K, 0)$ at time T if S_t stays above $B < K$ over the interval $[0, T_2]$ where $0 < T_2 < T$. [Barrier Option: if you hit B in [0, T_2] then the option dies] ← cheaper than Call Option.

Solution: From the Black-Scholes hedging/replication arguments above, we know that the no-arbitrage price $P(S,t)$ satisfies the Black-Scholes PDE:

$$P_t(S,t) + rP_S(S,t)S + \frac{1}{2}P_{SS}(S,t)S^2\sigma^2 \;=\; rP(S,t)$$

stock price

with terminal condition $P(S,T) = (S-K)^+$, but we now have the additional **boundary condition** $P(B,t) = 0$ for $t \in [0, T_2]$.

To replicate the contract, we replicate the payoff $(S_T - K)^+ 1_{\underline{S}_{T_2} > B}$ at time T as follows:

- Let $(X_t)_{t \geq 0}$ denote total wealth process of the hedging strategy at time t and we require initial wealth $X_0 = P(S_0, 0)$.
- Hold $\phi_t = P_S(S_t, t)$ units of stock until time $\overbrace{T \wedge H_B}^{\text{min}}$ where H_B is the hitting time of S to B, and place remaining wealth $X_t - \phi_t S_t$ in the riskless bank account.
- X then evolves as

$$dX_t \;=\; \phi_t dS_t + (X_t - \phi_t S_t) r dt$$

  whose solution is then given by $X_t = P(S_t, t)$ (proof not required, comes from the usual Black-Scholes hedging argument), so in particular $X_T = (S_T - K)^+ 1_{\underline{S}_{T_2} > B}$, which hedges a long position in the original contract.

$X_t = P(t, S_t)$, we see in particular that if S hits the barrier at time $\tau_B \in [0, T_2]$, than at this exact moment $X_{\tau_B} = P(\tau_B, B) = 0$ (from the boundary condition we have imposed for the PDE), so we have also replicated the Time Window Barrier option in this scenario.

3.6.1 Implied volatility

The Vega $\frac{\partial C}{\partial \sigma} = Sn(d_1)\sqrt{\tau}$ of a call option under Black-Scholes is positive, so C is monotonically increasing as a function of σ. Thus, given an observed call price C^{obs} in the market, we can extract a unique σ value consistent i.e. such that

$$C(S, K, \hat{\sigma}, \tau, r) \;=\; C^{obs}.$$

if $\max(S_0 - Ke^{-rT}, 0) \leq C^{obs} < S_0$. This σ is known as the **implied volatility** of the option, and is a very important concept in practice.

$T_1, T_2, \ldots$ are exponential r.v. generated with parameter λ. T_i are length of time between jumps.
N_t = No. of jumps at time t.

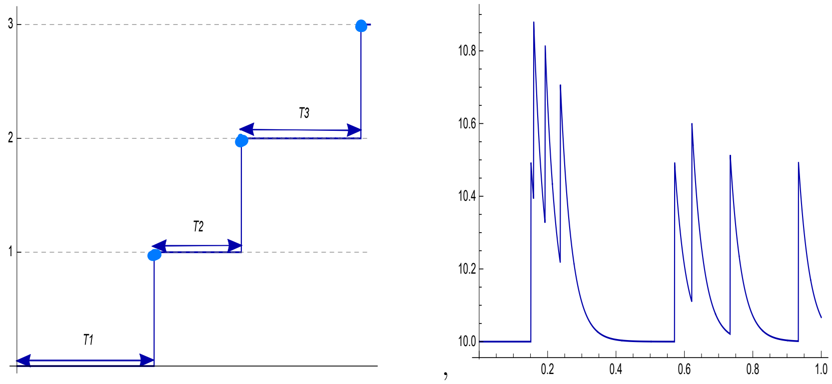


Figure 5: On the left we see a Monte Carlo simulation of a Poisson process and labelled the times $T_1, T_2, \ldots$ between jumps, which are i.i.d. $\text{Exp}(\lambda)$ random variables. In the middle we have simulated the intensity process λ_t of a Hawkes process with $\lambda_0 = 10$ and kernel $\phi(t) = \frac{1}{2}e^{-30t}$

.

Exponential Distribution

pdf → $f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

The cumulative distribution function is given by

cdf → $F(x;\lambda) = \begin{cases} 1 - e^{-(\lambda x)} & x \geq 0, \\ 0 & x < 0. \end{cases}$

4 Jump-diffusion models and hitting time densities

- Let $T_1, T_2, T_3, \ldots$ denote an i.i.d. sequence of exponential $\text{Exp}(\lambda)$ random variables. Then a Poisson process $(N_t)_{t\geq 0}$ is a piecewise constant right continuous process with $N_0 = 0$ which increases by $+1$ at times $T_1, T_1 + T_2, T_1 + T_2 + T_3 \ldots$ etc (see Figure 5 above). Then we see that

mgf of arrival time →
$$M_n(p) \quad := \quad \mathbb{E}(e^{p(T_1+\ldots T_n)}) \underset{\text{i.i.d}}{=} \mathbb{E}(e^{pT_1})^n \quad = \quad (\int_0^\infty e^{pt}\lambda e^{-\lambda t}dt)^p \quad = \quad (\frac{\lambda}{\lambda - p})^n$$

for $p \in (-\infty, \lambda)$, and $M_n(p) = +\infty$ otherwise. It can be shown that the density which has $M_n(p)$ as its moment generating function is the **Erlang distribution** with density $f_n(t) = \frac{t^{n-1}\lambda^n e^{-\lambda t}}{(n-1)!}$. Thus

Pmf →

$$\underset{\text{Poisson r.v.}}{\mathbb{P}(N_t = n)} = \mathbb{P}(T_1 + \ldots T_n = s \text{ for some } s \in [0,t] \cap T_{n+1} > t - s) = \int_0^t f_n(s)\,\mathbb{P}(T_{n+1} > t - s)ds$$
$$= \int_0^t f_n(s)\,e^{-\lambda(t-s)}ds = \frac{1}{\lambda}\int_0^t f_n(s)\,\lambda e^{-\lambda(t-s)}ds$$
$$= \frac{1}{\lambda}f_{n+1}(t) = \frac{t^n\lambda^n e^{-\lambda t}}{n!} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}.$$

But we recognize this as the **probability mass function** of a **Poisson distribution** with parameter λt (not λ!), i.e. $N_t \sim \text{Po}(\lambda t)$.

- To simulate a Poisson process, we just need to be able simulate iid $\text{Exp}(\lambda)$ random variables, and we already know how to do this using the $F^{-1}(U)$ method, where $U \sim U[0,1]$.

- We also see that

Conditional Probability →
$$\mathbb{P}(T_i \in [t, t+dt]\,|\,T_i > s) = \frac{\mathbb{P}(T_i \in [t, t+dt] \cap T_i > s)}{\mathbb{P}(T_i > s)} \underset{t>s}{=} \frac{\mathbb{P}(T_i \in [t, t+dt])}{\mathbb{P}(T_i > s)} = \frac{\lambda e^{-\lambda t}dt}{\underbrace{e^{-\lambda s}}_{\text{complimentary cdf.}}} = \lambda e^{-\lambda(t-s)}dt$$

defined for $t \in [s, \infty)$, which means the conditional density of T_i given $T_i > s$ is $\lambda e^{-\lambda(t-s)}$, i.e. a shifted exponential distribution defined on $[s, \infty)$. This is the so-called **lack of memory property** of Exponential random variables: it says that the conditional distribution of T_i given $T_i > s$ is the same as the distribution of $s + T_i$. The lifetime of a light bulb is said to be an exp random variables, so this result means that an light bulb which hasn't blown yet is just as good as a new one, i.e. has the same lifetime distribution.

- From this, we see that $\mathbb{P}(N_t - N_s = n\,|\,(N_u)_{u\in[0,s]}) = \mathbb{P}(N_{t-s} = n) \sim \text{Po}(\lambda(t-s))$ since (from the previous bullet point) we know that the conditional distN of $T_{N_s+1} - s$ given that $T_1 + T_2 + \ldots + T_{N_s} < s$ and $T_1 + T_2 + \ldots + T_{N_s} + T_{N_s+1} > s$ is the same as T_1, so $N_t - N_s$ is a new Poisson process starting afresh at time s, and independent of $(N_u)_{0\leq u\leq t}$ (i.e. the process N has independent increments, like Brownian motion).

- We also see that

$$\mathbb{P}(N_{t+\delta} - N_t > 0) \quad = \quad \mathbb{P}(T_1 < \delta) \quad = \quad 1 - e^{-\lambda\delta} \underset{\text{Taylor Expansion}}{=} \lambda\delta + O(\delta^2)$$

as $\delta \to 0$, so the probability of at least one jump occurring over a small time interval dt is approximately λdt.

(*) **Explanation of $\mathbb{P}(N_t = n)$:**

T_i are time interval between jumps. $N_t = n$ is the no. of jumps at time t is n.

$P(N_t = n)$

$= \mathbb{P}(T_1 + \dots T_n = s \text{ for some } s \in [0,t] \cap T_{n+1} > t-s)$

$N_t = n$ means the sum of n time interval is s which is less than t. But it also has to be true that $(n+1)$ time value (first value after n) is greater than $(t-s)$. i.e. we have atleast n jumps but not $(n+1)$ jumps.

$$= \int_0^t f_n(s)\, \mathbb{P}(T_{n+1} > t-s)\, ds$$

Integrate density over all admissible s values. Multiply the density by idependant $\mathbb{P}(T_{n+1} > t-s)$

$$= \int_0^t f_n(s)\, e^{-\lambda(t-s)}\, ds \quad \text{(using complimentary cdf)}$$

$$= \frac{1}{\lambda} \underbrace{\int_0^t f_n(s)\, \lambda e^{-\lambda(t-s)} ds}_{\text{convolution}} \quad \text{(rewriting)}$$

$$= \frac{1}{\lambda} f_{n+1}(t)$$

(Note: $\lambda e^{-\lambda(t-s)}$ is pdf of Exp. So we are adding one more density)

$\vdots$

$$= \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad \leftarrow \text{pmf of Poisson dist.}$$

- We can also consider an extension of the Poisson process called a **Hawkes process** where this arrival rate λ is no longer constant, but rather depends on the history of N itself as

$$\lambda_t \;=\; \mu + \int_0^t \phi(t-s)dN_s \;=\; \mu + \sum_{n=1}^{N_t}\phi(t-\sum_{i=1}^{n}T_i)\,.$$

  This type of process is said to be **self-exciting** because the history of N **feeds back** into the dynamics of the **intensity** process λ_t (see simulation of λ_t above). In this case, the times between jumps are still independent, but not iid and not exponentially distributed. We will say more about these kind of processes later.

4.1 A general jump-diffusion model

γ (drift), σ (volatility) $\in$ Constants
ξ_i = jump sizes

Now consider a general **jump-diffusion** process $X_t = \gamma t + \sigma W_t + Y_t$, where $Y_t = \sum_{i=1}^{N_t}\xi_i$ and the ξ_i's are independent and identically distributed (i.i.d) random variables with density $\mu(x)$ and N_t is a Poisson process with rate $\lambda > 0$, and W is a standard Brownian motion, and W, N and the ξ_i are all independent of each other. Y is known as a **compound Poisson process**. Using independence we have

mgf when p is replaced by iu

$$\phi(u) \;\doteq\; \mathbb{E}(e^{iuX_t}) \;=\; \mathbb{E}(e^{iu(\gamma t+\sigma W_t)})\,\mathbb{E}(e^{iuY_t}) \;=\; e^{i\gamma u t - \frac{1}{2}\sigma^2u^2t}\,\mathbb{E}(e^{iuY_t}) \tag{24}$$

where $i = \sqrt{-1}$ and $u \in \mathbb{R}$. But (Think as simulating N_t first then consider the conditional dist. of sum of ξ_i.)

$$\begin{aligned}
\mathbb{E}(e^{iuY_t}) \;&=\; \sum_{n=0}^{\infty}\mathbb{E}(e^{iuY_t}\,|\,N_t=n)\,\mathbb{P}(N_t=n) \;=\; \sum_{n=0}^{\infty}\mathbb{E}(e^{iu(\xi_1+\ldots\xi_n)})\,\mathbb{P}(N_t=n)\\
&\overset{\xi_i\text{ independent}}{=}\; \sum_{n=0}^{\infty}\mathbb{E}(e^{iu\xi_1})\mathbb{E}(e^{iu\xi_2})\ldots\mathbb{E}(e^{iu\xi_n})\,\mathbb{P}(N_t=n)\\
&\overset{\xi_i\text{ i.i.d}}{=}\; \sum_{n=0}^{\infty}\mathbb{E}(e^{iu\xi_1})^n\,\mathbb{P}(N_t=n)
\end{aligned}$$

where we have used that $\xi_1, \xi_2, \ldots$ are i.i.d. Now let $\psi(u) = \mathbb{E}(e^{iu\xi_1})$. Then we have

$$\begin{aligned}
\mathbb{E}(e^{iuY_t}) \;&=\; \sum_{n=0}^{\infty}\psi(u)^n\underbrace{\frac{(\lambda t)^n e^{-\lambda t}}{n!}}_{\text{Pmf of Poisson}} \;=\; e^{-\lambda t}\sum_{n=0}^{\infty}\frac{z^n}{n!} \qquad (\text{where } z=\lambda t\psi(u))\\
&=\; e^{-\lambda t}e^{z}\\
&=\; \exp(-\lambda t+\psi(u)\lambda t)\\
&=\; \exp(\lambda t(-1+\int_{-\infty}^{\infty}e^{iux}\mu(x)dx))\\
&=\; \exp[\lambda t\int_{-\infty}^{\infty}(e^{iux}-1)\mu(x)dx]
\end{aligned}\tag{25}$$

using that $\int_{-\infty}^{\infty}\mu(x)dx = 1$ because $\mu(x)$ is a density. Combining (24) and (25), we see that

$$\phi(u) \;=\; \mathbb{E}(e^{iuX_t}) \;=\; e^{i\gamma ut - \frac{1}{2}\sigma^2u^2t + \lambda t\int_{-\infty}^{\infty}(e^{iux}-1)\mu(x)dx}\,.$$

If we set $iu = p$ with $p \in \mathbb{R}$, then we obtain the moment generating function of X_t as:

$$\mathbb{E}(e^{pX_t}) \;=\; e^{\gamma pt+\frac{1}{2}\sigma^2p^2t+\lambda t\int_{-\infty}^{\infty}(e^{px}-1)\mu(x)dx} \;=\; e^{V(p)t} \tag{26}$$

where $V(p) = \gamma p + \frac{1}{2}\sigma^2p^2 + \lambda\int_{-\infty}^{\infty}(e^{px}-1)\mu(x)dx$, and note that $\mathbb{E}(e^{pX_t}) = \infty$ if $\int_{-\infty}^{\infty}(e^{px}-1)\mu(x)dx = \infty$.

In general, if we know the characteristic function $\phi(u) = \mathbb{E}(e^{iuX})$ of a random variable X and if $\int_{-\infty}^{\infty}|\phi(u)|du < \infty$ i.e. $\phi \in L^1$, then X has a density $p(x)$ and we can compute $p(x)$ using an **inverse Fourier transform**:

$$p(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iux}\phi(u)du\,.$$

If we now assume a stock price process $S_t = e^{X_t}$, then setting $iu = p = 1$ i.e. $u = -i$, we see that

$$\mathbb{E}(S_t) \;=\; \mathbb{E}(e^{X_t}) \;=\; \exp[t\,[\gamma+\frac{1}{2}\sigma^2+\lambda\int_{-\infty}^{\infty}(e^x-1)\mu(x)dx)]$$

and to exclude arbitrage we must have $\mathbb{E}(S_t) = S_0e^{rt}$ as for the Black-Scholes model which one can show implies that S_te^{-rt} is a martingale, so taking logs of both sides and dividing by t, we have the following no-arbitrage condition for γ:

$$\gamma + \frac{1}{2}\sigma^2 + \lambda\int_{-\infty}^{\infty}(e^x-1)\mu(x) \;=\; \gamma + \frac{1}{2}\sigma^2 + \lambda\mathbb{E}(e^{\xi_i}-1) \;=\; r\,.$$

4.2 Hitting times and barrier options under jump models

- Let X be a jump-diffusion process $X_t = \gamma t + \sigma W_t + Y_t$ where $Y_t = \sum_{i=1}^{N_t} \xi_i$ as above and assume that X has only negative jumps and for simplicity we assume that $X_0 = 0$ throughout this subsection.

- From (26), we see that

$$\mathbb{E}(e^{pX_t - V(p)t}) \quad = \quad 1 \tag{27}$$

and from this we can show in fact that $e^{pX_t - V(p)t}$ is a martingale. By a famous result called the **optional stopping theorem**, if $V(p) > 0$ we can replace t by the random time $\tau_a = \min\{t : X_t \geq a\}$, for $a > X_0 = 0$ to get

$$\mathbb{E}(e^{pX_{\tau_a} - V(p)\tau_a}) \quad = \quad \mathbb{E}(e^{pa - V(p)\tau_a}) \quad = \quad 1 \tag{28}$$

i.e. because X has only negative jumps, we know that X cannot jump over the barrier a, so $X_{\tau_a} = a$. We can re-arrange this expression as

$$\mathbb{E}(e^{-V(p)\tau_a}) \quad = \quad e^{-pa}\,.$$

If we now set $V(p) = q$ and set $\Phi(q)$ equal to the positive solution p of $V(p) = q$, and we have

$$\mathbb{E}(e^{-q\tau_a}) \quad = \quad \mathbb{E}(e^{-q\tau_a} 1_{\tau_a < \infty}) \quad = \quad e^{-a\Phi(q)} \tag{29}$$

for $q > 0$ (note the first equality may not be true if $q = 0$). This is the moment generating function of τ_a but with a negative exponent, which we call the **Laplace transform** of τ_a. Letting $q \to 0$ we see that

$$\mathbb{E}(1_{\tau_a < \infty}) \quad = \quad \mathbb{P}(\tau_a < \infty) \quad = \quad e^{-a\Phi(0_+)} \tag{30}$$

where $\Phi(0_+) = \lim_{q \to 0} \Phi(q)$, so we see that $\tau_a < \infty$ with probability 1 if $\Phi(0_+) = 0$ and $\mathbb{P}(\tau_a < \infty) < 1$ if $\Phi(0_+) > 0$ (see graphs below). Now let $\bar{X}_t = \max_{0 \leq s \leq t} X_s$. Then the event $\{\bar{X}_\infty \geq a\}$ is the same as the event $\tau_a < \infty$, so we have

$$\mathbb{P}(\bar{X}_\infty \geq a) \quad = \quad \mathbb{P}(\tau_a < \infty) \quad = \quad e^{-\Phi(0_+)a}\,. \tag{31}$$

Thus if $\Phi(0_+) > 0$ then $\bar{X}_\infty$ has an exp distribution with parameter $\Phi(0_+)$.

- To compute the density of τ_a, we set $-q = ik$ for $k \in \mathbb{R}$, and compute the inverse Fourier transform as before:

$$f_{\tau_a}(t) \quad = \quad \frac{1}{2\pi} \int_{k=-\infty}^{\infty} e^{-ikt} \mathbb{E}(e^{ik\tau_a}) dk \quad = \quad \frac{1}{2\pi} \int_{k=-\infty}^{\infty} e^{-ikt} e^{-a\Phi(-ik)} dk\,. \tag{32}$$

- **Example**: Consider the **Kou model** where **minus** the jump sizes have a exponential distribution with parameter $\eta > 0$, which means that $\mu(x) = \eta e^{-\eta|x|} = \eta e^{\eta x}$ for $x < 0$ and zero otherwise. Then (26) takes the form

$$\mathbb{E}(e^{pX_t}) \quad = \quad e^{\gamma pt + \frac{1}{2}\sigma^2 p^2 t + \lambda t \int_{-\infty}^{0} (e^{px} - 1)\eta e^{\eta x} dx} \quad = \quad e^{\gamma pt + \frac{1}{2}\sigma^2 p^2 t + \lambda t (\frac{\eta}{\eta + p} - 1)}$$

so $V(p) = \gamma p + \frac{1}{2}\sigma^2 p^2 + \lambda(\frac{\eta}{\eta+p} - 1) = \gamma p + \frac{1}{2}\sigma^2 p^2 - \lambda \frac{p}{\eta + p}$. Solving $V(p) = q$ leads to a cubic equation in p, which can be solved explicitly but is messy, so we will only consider two special cases: $\lambda = 0$ or $\sigma = 0$.

- If $\lambda = 0$, there are no jumps, and the equation $V(p) = q$ becomes $\frac{1}{2}\sigma^2 p^2 + \gamma p = q$ which we can solve as

$$p \quad = \quad \Phi(q) \quad = \quad \frac{-\gamma + \sqrt{2\sigma^2 q + \gamma^2}}{\sigma^2} \tag{33}$$

and letting $q \to 0$, we see that $\lim_{q \to 0} \Phi(q) = \Phi(0_+) = (-\gamma + |\gamma|)/\sigma^2$, which is zero if $\gamma > 0$ and > 0 if $\gamma < 0$. Thus if $\gamma < 0$, X may never hit a and the probability that X does hit a is $e^{-a\Phi(0_+)} = e^{-2|\gamma|/\sigma^2 a}$, i.e.

$$\mathbb{P}(\bar{X}_\infty > a) \quad = \quad \mathbb{P}(\tau_a < \infty) \quad = \quad e^{-\frac{2|\gamma|}{\sigma^2} a}\,.$$

i.e. $\bar{X}_\infty \sim \text{Exp}(\frac{2|\gamma|}{\sigma^2})$. (33) means the density $f_{\tau_a}(t)$ of τ_a satisfies

$$\mathbb{E}(e^{-q\tau_a}) \quad = \quad \int_0^\infty e^{-qt} f_{\tau_a}(t) dt \quad = \quad e^{-a\frac{-\gamma + \sqrt{2\sigma^2 q + \gamma^2}}{\sigma^2}} \tag{34}$$

and it turns out the unique $f_{\tau_a}(t)$ which satisfies this equation for all $q > 0$ is given by

$$f_{\tau_a}(t) \quad = \quad \frac{a}{\sqrt{2\pi t^3}}\, e^{-(a-\tilde{\gamma}t)^2/2t}\, 1_{t>0} \tag{35}$$

where $\tilde{\gamma} = \gamma/\sigma$ (see also graphs on next page). For the case $\gamma < 0$, we have seen there is a non-zero probability that $\tau_a = \infty$, hence the density $f_{\tau_a}(t)$ does not integrate to 1 in this case. From the density of τ_a we can then compute $\mathbb{P}(\tau_a \leq t) = \mathbb{P}(\bar{X}_t \geq a)$ and the price of a One-Touch option with barrier $B > S_0$ on S

$$\underbrace{e^{-rT}\int_0^T f_{\tau_a}(t)dt}_{\text{discounted value of Prob. that we hit barrier before time T.}} \quad \text{where } a = \log\frac{B}{S_0}.$$

- If $\sigma = 0$ then $V(p) = p\mu - \frac{\lambda p}{\eta+p}$ we can also solve for $\Phi(q)$ explicitly as

$$\Phi(q) \quad = \quad \frac{1}{2\mu}[q + \lambda - \eta\mu + \sqrt{4q\eta\mu + (q+\lambda-\eta\mu)^2}]$$

and recall that $\mathbb{E}(e^{-q\tau_a}) = e^{-\Phi(q)a}$ for $q > 0$ so this gives us the Laplace transform of τ_a which can then invert using the inverse Fourier transform as above. Letting $q \searrow 0$, we see that

$$\Phi(0_+) \quad = \quad \lim_{q\to 0}\Phi(q) \quad = \quad \frac{1}{2\mu}[\lambda - \eta\mu + \sqrt{(\lambda-\eta\mu)^2}]$$

which is 0 if and only if $\lambda \leq \eta\mu$. Hence for the Kou model with negative only jumps with $\sigma = 0$, we see that $\tau_a < \infty$ a.s. iff $\lambda \leq \eta\mu$. This shows that the arrival rate λ of the jumps (which are all negative) and the drift μ are competing against each other to determine whether $\tau_a < \infty$ or not for any barrier $a > X_0$.

If X is a log stock price process and interest rates are zero, recall that to exclude arbitrage opportunities, μ must be chosen so that $\mathbb{E}(e^{X_t}) = \mathbb{E}(S_t) = 1$, which implies that the stock price process $S_t = e^{X_t}$ is a martingale. Thus we must impose that $V(1) = 0$, which (because we also have $V(0) = 0$ and V is convex) means that $V'(0) < 0$ so $\mathbb{P}(\tau_a < \infty) < 1$. If interest rates are non-zero, we instead impose that $\mathbb{E}(e^{X_t}) = \mathbb{E}(S_t) = e^{rt}$, and thus $V(1) = r$. For the Kou model this equation becomes

$$V(1) \quad = \quad \mu + \frac{1}{2}\sigma^2 - \frac{\lambda}{1+\eta} \quad = \quad r\,.$$

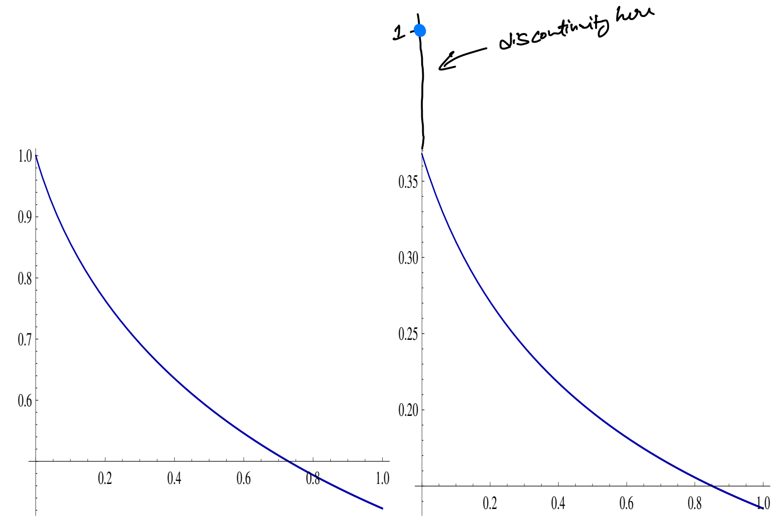


Figure 6: Here we have plotted $\mathbb{E}(e^{-q\tau_a}1_{\tau_a<\infty}) = \mathbb{E}(e^{-q\tau_a}) = e^{-\Phi(q)a}$ for the Kou model with $a = 1, \sigma = 1, \lambda = 1, \eta = 1$ and $\mu = 1$ (left) and $\mu = -.5$ (right)

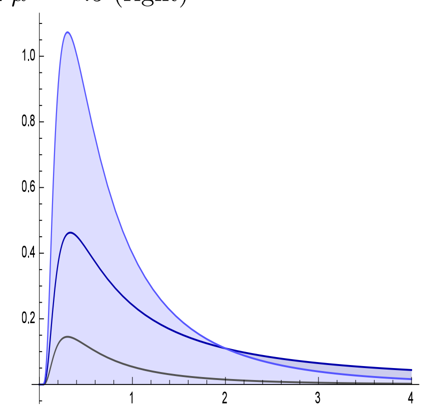


Figure 7: Here we have plotted the hitting time density of τ_a for $X_t = \gamma t + \sigma W_t$ in (35) for $\sigma = 1$ and $\gamma = -1$ (grey), $\gamma = 0$ (dark blue) and $\gamma = 1$ (light blue).

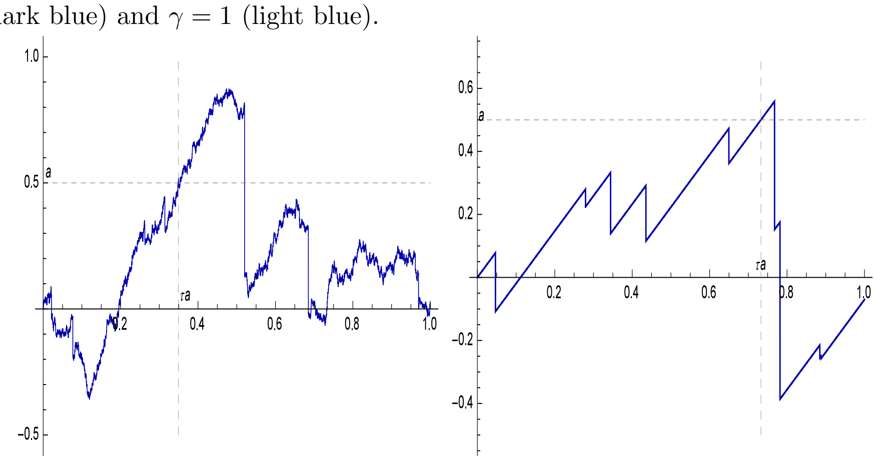


Figure 8: Here are two Monte Carlo simulations of the log stock price process for the Kou model hitting an upper barrier $a = 0.5$ for $\lambda = 10$, $\eta = 5$ and $T = 1$ for $\sigma = 0.4$ (left plot) and $\sigma = 0$ (right plot), with interest rate $r = 0$ in both cases.

.

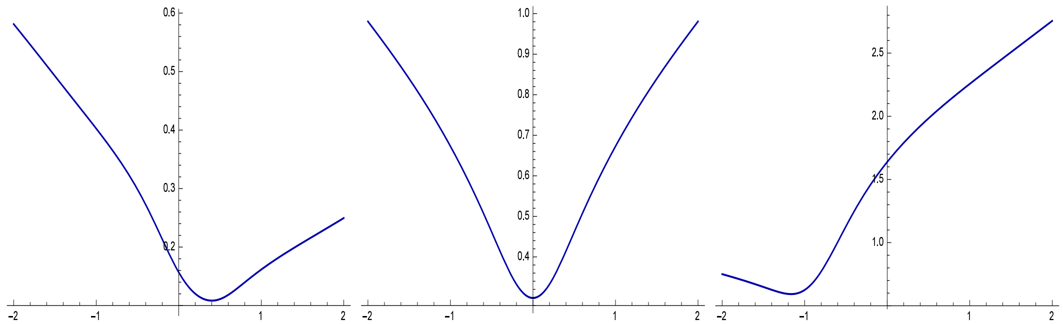


Figure 9: Here we have plotted the square of the implied volatility smile for the Merton model as a function of $k = \log \frac{K}{S_0}$ obtained using (**??**) for $\sigma = .2$, $r = 0$, $\lambda = .5$, $\delta = 1$, $T = 1$ and expected jump size $\alpha = -1, -0.5, .5, 1$ (left to right). Recall that if $\lambda = 0$, the smile is flat.

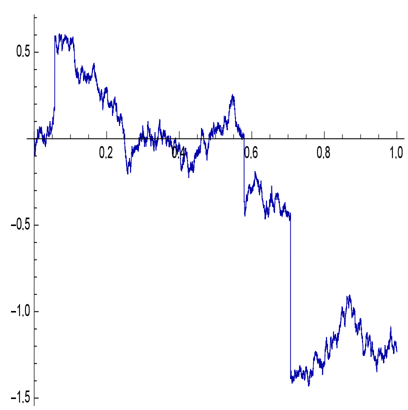


Figure 10: Here we have plotted a Monte Carlo simulation of the Merton model, and for this simulation, there were three jumps of size 0.427431 at time .0567, -0.384758 at time 0.5789 and -0.956973 at time .7069.

FM02 Homework 1 solutions

1. A European call option pays $\max(S_T - K, 0)$ at time T. Show how to price and replicate a European call option under the 1-step binomial model with $S_0 = 1$, $u = 1.1$, $d = .9$, $r = .02$, $T = 1$ and $K = 1$. If we change u to 1, describe how to construct an arbitrage strategy using just the stock and the bond (assume the bond grows from e^{-rT} at time zero to 1 at the final time T).

Solution. $f_0 = .0589$, $q = .601$. If $u = 1$ then $u < e^{rT}$, hence from the notes we set $\phi_0 = S_0 e^{rT}$ and $\phi_1 = -1$; then $V_0 = 0$ and $V_T = S_0 e^{rT} - S_0 u = .020$ or $S_0 e^{rT} - S_0 d = .102$.

2. Price the same European option as Q1. but using a 2-step binomial model with $T = 2$ (i.e. two time periods of length 1 year each).

Solution. q remains unchanged at .601, and we find that $f_u = .123$, $f_d = 0$ and $f_0 = .0729$.

3. Consider a two-step binomial model with $S_0 = 1$. Assume S goes up to 1.1 or down to .9 at the first time step, and from 1.1 assume S can then go up to 1.2 or 1, and from 0.9 assume S can go up to 1 or down to 0.8. Calculate all risk-neutral probabilities and use these to price an Asian option which pays $\max(\frac{1}{2}(S_{.5} + S_1) - K, 0)$ with $K = 1$ if $r = 0.02$ if each of the two time steps is of length 0.5.

Solution. Set $\Delta T = .5$. Then $f_{uu} = .155$, $f_{ud} = 0.045$, $f_{du} = 0$ and $f_{dd} = 0$. The risk-neutral probability of going up from S_0 is .550, the risk neutral probability of going up from S_u is .555 and risk-neutral probability of going up from S_d is .545. Working backwards through the tree, we find that $f_u = .104$, $f_d = 0$ and $f_0 = .0569$.

FM02 Homework 2 solutions

1. Consider a discrete-time market with a single stock which evolves under the physical probability measure $\mathbb{P}$ as

$$S_n \quad = \quad S_{n-1}e^{\mu + \sigma Z_n}$$

for $0 \le n \le N$, and a riskless bond which evolves as $B_n = e^{rn}$, where Z_n is a sequence of iid $N(0,1)$ random variables. Does this market admit arbitrage? Is the market complete?

Solution. Define the probability measure $\tilde{\mathbb{P}}(A) = \mathbb{P}((S_1,..,S_n) \in A)$ on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$. Let $\tilde{\mathbb{Q}}$ denote the corresponding probability measure on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ when μ is replaced with $r - \frac{1}{2}\sigma^2$. Then we can easily verify that $\tilde{S}_n := e^{-rn}S_n$ is a martingale under $\mathbb{Q}$. It can be shown that $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{Q}}$ are equivalent probability measures on $\mathbb{R}^N$ (proof not required here), so from the first Fundamental Theorem of Asset pricing, there is no arbitrage.

The market is not complete, since we can replace the Z_n's here with any sequence of iid non-Gaussian random variables with a density on $\mathbb{R}$, and solve for the unique μ such that

$$\mathbb{E}(e^{\mu+\sigma Z_n}) \quad = \quad e^r$$

which defines another risk-neutral measure on $\mathbb{R}^N$, as long as the quantity on the left is finite.

2. Let N_t and $\tilde{N}_t$ be two independent Poisson processes, for which $N_t \sim \text{Po}(\lambda t)$ and $\tilde{N}_t \sim \text{Po}(\lambda t)$. Using that a standard $\text{Po}(\lambda)$ random variable has moment generating function $\mathbb{E}(e^{pN}) = e^{\lambda(e^p-1)}$, show that if λ is replaced by $\frac{\lambda}{\varepsilon}$ then $X^\varepsilon_t = \sqrt{\varepsilon}(N^\varepsilon_t - \tilde{N}^\varepsilon_t)$ has the same distribution as a multiple of Brownian motion W_t as $\varepsilon \to 0$. Note that X^ε_t has a very large number of a very small (positive and negative) jumps when ε is small, and this type of process is often used to model incoming buy and sell orders for a stock over small-time scales, e.g. seconds or minutes.

Solution

$$\begin{aligned}
\mathbb{E}(e^{p\sqrt{\varepsilon}(N^\varepsilon_t - \tilde{N}^\varepsilon_t)}) \quad &= \quad \mathbb{E}(e^{p\sqrt{\varepsilon}N^\varepsilon_t})\,\mathbb{E}(e^{-p\sqrt{\varepsilon}\tilde{N}^\varepsilon_t}) \quad = \quad \exp[\frac{\lambda t}{\varepsilon}(e^{p\sqrt{\varepsilon}}-1) + \frac{\lambda t}{\varepsilon}(e^{-p\sqrt{\varepsilon}}-1)] \\
&= \quad \exp[\frac{\lambda t}{\varepsilon}(p\sqrt{\varepsilon} + \frac{1}{2}p^2\varepsilon + ...) \; + \; \frac{\lambda t}{\varepsilon}(-p\sqrt{\varepsilon} + \frac{1}{2}p^2\varepsilon + ...)] \\
&\to \quad e^{\lambda p^2 t}
\end{aligned}$$

as $\varepsilon \to 0$. But this is the mgf of $\sqrt{2\lambda}\,W_t$. This is a (less common) alternative to the Euler method for numerically approximating Brownian motion (see graphs above).

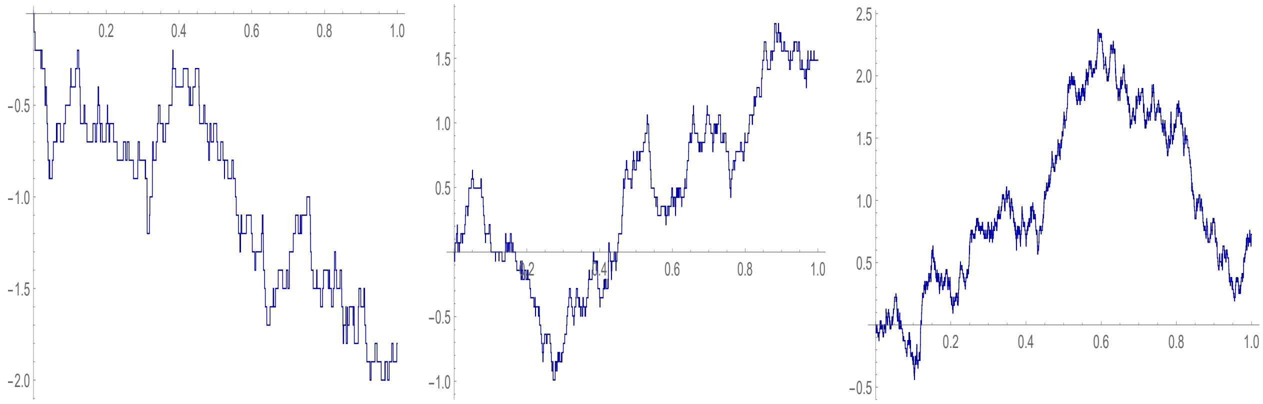


Figure 1: Poisson approximation to Brownian motion in q2 with $\varepsilon = .01$, $.005$ and $.001$.

3. Describe how to simulate (i) a Po(λ) random variable N using a standard Uniform random variable U, and (ii) two correlated Po(λ) random variables using the Gaussian copula.

Solution (i) Let $F : \mathbb{N} \cup \{0\} \to [0,1]$ denote $F(n) = \mathbb{P}(N \le n)$, i.e. the distribution function of N. Then we can define an inverse $F^{-1} : [0,1] \to \mathbb{N} \cup \{0\}$ of F as $F^{-1}(x) = \max\{n : F(n) < x\}$. Then

$$\mathbb{P}(F^{-1}(U) \le n) \quad = \quad \mathbb{P}(U \le F(n)) \quad = \quad F(n)$$

i.e. $F^{-1}(U) \sim N$, as required.
(ii) Let X, Y be two standard Normal RVs with correlation ρ. Then $F^{-1}(\Phi(X))$ and $F^{-1}(\Phi(Y))$ are two correlated Po(λ) random variables (see Applied Probability notes for details on Gaussian copula).

Homework 3 solutions

1. Consider the **Ornstein-Uhlenbeck** process which satisfies

$$dY_t = -\alpha Y_t dt + \sigma dW_t$$

for $\alpha > 0$. Write down the SDE for $Z_t = e^{\alpha t} Y_t$. What do you notice?

Solution. Let $f(t, y) = e^{\alpha t} y$. Then $Z_t = f(t, Y_t)$ satisfies

$$\begin{aligned} dZ_t &= \alpha e^{\alpha t} Y_t dt + e^{\alpha t} dY_t + 0 \\ &= \alpha e^{\alpha t} Y_t dt + e^{\alpha t}(-\alpha Y_t dt + \sigma dW_t) \\ &= \sigma e^{\alpha t} dW_t\,. \end{aligned}$$

Thus we see that Z has zero drift (we can also show that $Z_t - z_0 \sim N(0, \sigma^2 \int_0^t e^{2\alpha u} du)$ but this is not examinable).

2. Consider the **Bessel process** which satisfies

$$dR_t = \frac{2\delta - 1}{R_t} dt + dW_t$$

for $\delta \geq 0, R_0 > 0$. Compute the SDE for $Z_t = R_t^2$.
Solution.

$$\begin{aligned} dZ_t &= 2R_t dR_t + \frac{1}{2} \cdot 2dt \\ &= 2[(2\delta - 1)dt + R_t dW_t] + dt \\ &= (4\delta - 1)dt + 2\sqrt{Z_t} dW_t\,. \end{aligned}$$

3. Consider a process X_t satisfying the SDE

$$dX_t = X_t^2 dW_t\,.$$

Compute the SDE for $R_t = 1/X_t$ in terms of R_t.

Solution:

$$\begin{aligned} dR_t &= -\frac{1}{X_t^2} dX_t + \frac{1}{2} \frac{2}{X_t^3} X_t^4 dt \\ &= -\frac{1}{X_t^2} X_t^2 dW_t + \frac{1}{2} \frac{2}{X_t^3} X_t^4 dt \\ &= -dW_t + \frac{1}{R_t} dt \end{aligned}$$

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

Then from Ito's lemma, we know that the log stock price $X_t = \log S_t$ satisfies

$$dX_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

which we can integrate to obtain that

$$X_t = X_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$$

$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$

Since, W is CTS a.s., W_t can't be $\pm\infty$ for some finite t, so we see that $S_t > 0 \ \forall t \in [0,\infty)$

FM02 Homework 4 solutions

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion throughout.

1. Do W_t/t and $W_t/\sqrt{t}$ tend to a constant as $t \to \infty$?

Solution. $\mathbb{E}((W_t/t - 0)^2) = t/t^2 \to 0$ so $W_t/t \to 0$ in L^2, and hence also in probability as $t \to \infty$ a.s. $W_t/\sqrt{t} \sim N(0,1)$ and hence does not tend to a constant

2. Compute the conditional distribution of W_t given W_s for $s,t > 0$. You may use that for two correlated Normal random variables X and Y with $\mathrm{Corr}(X,Y) = \rho$, $Y|X \sim N(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X), (1-\rho^2)\sigma_Y^2)$, and you do not have to assume that $s < t$ or vice versa. [For Normal dist. $\rho = 0 \Rightarrow$ independant]

Solution. In this case, $X = W_s$, $Y = W_t$, $\mu_X = 0$, $\mu_Y = 0$, $\sigma_X = \sqrt{s}$, $\sigma_Y = \sqrt{t}$ and $\rho = \min(s,t)/\sqrt{st}$, where we are using that

$$\mathrm{Corr}(X,Y) = \frac{\mathbb{E}((X-\mu_X)(Y-\mu_Y)}{\sigma_X \sigma_Y}$$

and recall that we have previously shown that $\mathbb{E}(W_s W_t) = \min(s,t)$.

3. Under what condition(s) does $S_t \to 0$ a.s. as $t \to \infty$ for the Black-Scholes model under the physical probability measure $\mathbb{P}$?

Solution. For simplicity we assume $X_0 = 0$. Then from Ito's lemma we know that the log stock price $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$. Then from q1, $X_t/t = (\mu - \frac{1}{2}\sigma^2) + \sigma W_t/t \to \mu - \frac{1}{2}\sigma^2$ as $t \to \infty$, so if $\mu - \frac{1}{2}\sigma^2 \neq 0$ $X_t \sim (\mu - \frac{1}{2}\sigma^2)t$ as $t \to \infty$, so if $\mu - \frac{1}{2}\sigma^2 < 0$, $S_t \to 0$ as $t \to \infty$ a.s., if $\mu - \frac{1}{2}\sigma^2 > 0$, $S_t \to \infty$ as $t \to \infty$ a.s. If $\mu - \frac{1}{2}\sigma^2 = 0$, then $X_t = \sigma W_t$ is undefined as $t \to \infty$ since its variance tends to infinity, so S_t is also undefined.

4. Let Z_i be a sequence of i.i.d. random variables with zero expectation and finite variance and let $S_n = \sum_{i=1}^n Z_i$. Donsker's theorem states that

$$X_t^n = \frac{S_{[nt]}}{\sqrt{n}} \qquad (t \in [0,1])$$

tends to a Brownian motion on $[0,1]$ as $n \to \infty$, where $[x]$ denotes the smallest integer less than or equal to x. Explain how this is connected to the Euler scheme for simulating Brownian motion. If the Z_i's now have a mean $\mu \neq 0$, what can we say about

$$Y_t^n = \frac{S_{[nt]}}{n} \qquad (t \in [0,1])\,.$$

Solution. The Euler scheme is a special case of Donsker's theorem for the special case when the Z_i's are $N(0,1)$ random variables. For the 2nd part

$$Y_t^n = \frac{\mu[nt]}{n} + \frac{1}{\sqrt{n}}\frac{\tilde{S}_{[nt]}}{\sqrt{n}} \qquad (t \in [0,1])$$

where $\tilde{S}_n = \sum_{i=1}^n (Z_i - \mu)$. Hence we see that $Y_t^n \to \mu t$, since $nt - [nt] \leq 1$.

for Black-Scholes in this case $\mu = 0$

5. Assume that S_t **is a continuous non-negative martingale** (not necessarily the Black-Scholes model) and assume that $S_\infty = \lim_{t\to\infty} S_t = 0$ with probability one, and consider a barrier level $B > S_0$. Compute the price of an **infinite-maturity** One-Touch option which pays 1 if S hits a barrier $B > S_0$ at any time $t \in (0,\infty)$, and zero otherwise.

Solution. Buy $\frac{1}{B}$ units of stock at time zero when the stock price is S_0, and then if S hits B, sell these $\frac{1}{B}$ units of stock for a total price of $\frac{1}{B} * B = 1$ at the exact instant that we hit. Otherwise, if S never hits B, the stock price will eventually tend to zero. Thus we can *replicate* the infinite maturity One-Touch payoff for an initial price of $\frac{S_0}{B} < 1$.

Let $\bar{S}_\infty = \max_{0 \le t < \infty} S_t$ denote the **ultimate maximum** of S, i.e. the maximum of S for all time. Then since interest rates are zero, from risk-neutral valuation we know the price is equal to the expected payoff:

$$\mathbb{E}(1_{\bar{S}_\infty \ge B}) \quad = \quad \mathbb{P}(\bar{S}_\infty \ge B) \quad = \quad \frac{S_0}{B} \tag{1}$$

and note that $\frac{S_0}{B} < 1$, so with probability $q = 1 - \frac{S_0}{B} > 0$, S will never hit B.

As a side-note, note that we can re-write (1) as

$$\mathbb{P}(\log \bar{S}_\infty \ge \log B) \quad = \quad \mathbb{P}(\log \bar{S}_\infty \ge b) \quad = \quad \frac{S_0}{B} \quad = \quad \frac{S_0}{e^b} \quad \le \quad 1$$

where $b = \log B$. For $S_0 = 1$, this means that $\mathbb{P}(\log \bar{S}_\infty \ge b) = e^{-b}$, so $\log \bar{S}_\infty$ has an exponential distribution with parameter 1. Alternatively, differentiating (1) with respect to B and multiplying by -1, we see that the density of $\bar{S}_\infty$ is $\frac{S_0}{B^2}$, defined for $B \ge S_0$, because we must have that $\bar{S}_\infty \in [S_0, \infty)$.

6. Let $(S_t)_{t \ge 0}$ denote a stock price process governed by the Black-Scholes model

$$dS_t \quad = \quad S_t(\mu dt + \sigma dW_t)$$

under the physical measure $\mathbb{P}$, for $\mu \in \mathbb{R}, \sigma > 0$ and $S_0 > 0$. Compute the no-arbitrage price $P(S,t)$ and the delta at time $t \in [0,T]$ of a option which pays $(\log S_T)^2$ at time T, and describe how to dynamically hedge this option.

Solution From the Black-Scholes hedging argument, we know that the no-arbitrage price $P(S,t)$ satisfies the Black-Scholes PDE:

$$P_t(S,t) \, + \, rP_S(S,t)S + \frac{1}{2}P_{SS}(S,t)S^2\sigma^2 \quad = \quad rP(S,t)$$

with terminal condition $P(S,T) = (\log S)^2$. From the Feynman-Kac formula, we know that $P(S,t)$ also has the probabilistic representation:

$$P(S,t) \quad = \quad e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}((\log S_T)^2 | S_t = S)$$

where S satisfies

$$dS_t \quad = \quad S_t(rdt + \sigma dW_t) \tag{2}$$

under the risk-neutral measure $\mathbb{Q}$. We know that the solution to (2) is

$$S_t \quad = \quad S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

and taking logs we see that $\log S_t \sim N(\log S_0 + (r - \frac{1}{2}\sigma^2)t, \sigma^2 t)$ so

$$S_T \quad = \quad S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T} \quad = \quad S_t e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}$$

so the conditional distribution of $\log S_T$ at t is

$$\log S_T \quad \sim \quad N(\log S_t \, + \, (r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))\,.$$

Hence (using the risk-neutral valuation formula which comes from the Feynman-Kac formula):

$$\begin{aligned} P(S_t,t) \quad &= \quad e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}((\log S_T)^2 \mid S_t) \\ &= \quad e^{-r(T-t)}[(\log S_t + (r - \frac{1}{2}\sigma^2)(T-t))^2 + \sigma^2(T-t)] \end{aligned}$$

where we have used that $\mathbb{E}(X^2) = \mathbb{E}(X)^2 + \mathrm{Var}(X)$ for any random variable X. The Delta of the contract at time $t \in [0,T]$ is obtained using the chain rule as

$$P_S(S_t,t) \quad = \quad 2e^{-r(T-t)}\frac{1}{S_t}(\log S_t + (r - \frac{1}{2}\sigma^2)(T-t))\,.$$

To hedge the contract, we have to replicate the payoff $-(\log S_T)^2$, which is achieved as follows:

- Hold $\phi_t = -P_S(S_t, t)$ units of stock at each time t, and place remaining wealth $X_t - \phi_t S_t$ in the riskless bank account.
- Start with initial wealth $X_0 = -P(S_0, 0)$,
- Total wealth then evolves as

$$dX_t \;=\; \phi_t dS_t + (X_t - \phi_t S_t) r dt$$

whose solution is then given by $X_t = -P(S_t, t)$ (proof not required), so in particular $X_T = -(\log S_T)^2$, as required.

FM02 Homework 5 solutions

1. Show that $f_n(t) = \frac{t^{n-1}\lambda^n e^{-\lambda t}}{(n-1)!}$ is the density of $T_1 + T_2 + .. + T_n$ for the Poisson process, by computing its mgf. You may use the fact that $f_n(t)$ is a density.

Solution

$$\begin{aligned}\int_0^\infty e^{pt} f_n(t)dt &= \int_0^\infty e^{pt}\frac{t^{n-1}\lambda^n e^{-\lambda t}}{(n-1)!}dt = \int_0^\infty \frac{t^{n-1}\lambda^n e^{-(\lambda-p)t}}{(n-1)!}dt \\ &= (\frac{\lambda}{\lambda-p})^n \int_0^\infty \frac{t^{n-1}(\lambda-p)^n e^{-(\lambda-p)t}}{(n-1)!}dt \\ &= (\frac{\lambda}{\lambda-p})^n\end{aligned}$$

since the remaining integral term in the second-to-last line is still $f_n(t)$ but with λ replaced by $\lambda - p$.

2. Use the strong law of large numbers from the AppliedProbabilityRevision.pdf document on the course website to show that $N_t/t \to \lambda$ a.s. as $t \to \infty$, for $t \in \mathbb{N}$.

Solution. For $t \in \mathbb{N}$, $N_t = N_1 + (N_2 - N_1) + ... + (N_t - N_{t-1})$ is a sum of iid Po(λ) random variables which have expectation λ (proof of latter not required). Thus from the SLLN we have

$$\mathbb{P}(\frac{N_t}{t} \to \lambda) = 1\,.$$

FM02 Homework 6 solutions

1. In the lecture notes on jump-diffusion hitting times, we know that for a jump diffusion X with negative-only jumps

$$\mathbb{E}(e^{-q\tau_a}) \quad = \quad \mathbb{E}(e^{-q\tau_a} 1_{\tau_a<\infty}) \quad = \quad e^{-a\Phi(q)}$$

for $q > 0$, where $\Phi(q)$ is equal to the positive solution p of $V(p) = q$, where $\mathbb{E}(e^{pX_t}) = e^{V(p)t}$. Explain why the limit of this expression as $q \to 0$ is $\mathbb{E}(1_{\tau_a<\infty})$ and not 1 as we might expect (see graphs of $e^{-a\Phi(q)}$ in the notes). ← look Kou Model lec Notes

Solution Letting $q \to 0$ in the middle expression, we see that $e^{-q\tau_a} 1_{\tau_a<\infty} \to 1_{\tau_a<\infty}$ a.s., and we then apply the bounded convergence theorem. Note that $q\tau_a$ may not tend to 0 a.s. as $q \to 0$ since we may have that $\tau_a = \infty$, in which case $q\tau_a = \infty \to \infty$ as $q \to 0$, so even in this case $e^{-q\tau_a} \to 1_{\tau_a<\infty}$.

2. For the case when $X_t = W_t$ is just Brownian motion, we have seen that $\mathbb{E}(e^{-q\tau_b}) = e^{-b\sqrt{2q}}$. Describe the relationship between τ_b (considered now as a *process* indexed by b as the time variable) and the process $M_t = \max_{0\le s\le t} W_s$. (t = current time) → look (23) r = 0 σ = 1

Running Max of B.M.

Solution If we consider the family of random variables $(\tau_b)_{b\ge0}$, then τ_b is a *stochastic process*, and trivially we note that

$$\tau_{b_1} \quad \le \quad \tau_{b_2}$$

b = time

for $b_1 < b_2$. Hence τ_b is non-decreasing in b. $\tau_b = \min\{t : M_t \ge b\}$, so in this sense τ_b is the *inverse* of the process M (rotate the graph of M_t, see below).
(Fig 1 next pg)

3. The process τ_b (indexed by b) is a special case of a more general type of increasing pure-jump process called an **α-stable Lévy process** Z for which $\mathbb{E}(e^{-qZ_t}) = e^{-ctq^{\alpha}}$ for $q > 0$, $\alpha \in (0, 1)$ and some constant $c > 0$ (not that τ_b has $\alpha = \frac{1}{2}$ and $c = \sqrt{2}$). We can then consider the inverse process of Z, i.e. $A_t = \min\{s : Z_s \ge t\}$ (or equivalently $A_b = \min\{s : Z_s \ge b\}$), and then A_t will have flat periods like M_t above since Z has jumps. Explain how we can compute the density of A_t.

Solution Similar to Brownian motion $\{A_b \le t\} = \{Z_t \ge b\}$ so

$$\mathbb{P}(A_b \le t) \quad = \quad \mathbb{P}(Z_t \ge b)\,.$$

The density $f_{Z_t}(x)$ of Z_t is obtained from the inverse Fourier transform of $\phi_t(k) = \mathbb{E}(e^{ikZ_t}) = e^{-ct(-ik)^{\alpha}}$ as $f_{Z_t}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\phi_t(k)dk$. Then we see that $\frac{d}{dt}\mathbb{P}(A_x \le t) = \frac{d}{dt}\mathbb{P}(Z_t > x)$. The cdf of a random variable can be recovered from its Fourier transform $\phi_X(k) = \mathbb{E}(e^{ikX})$ using the inversion formula:

$$F_X(x) \quad = \quad \frac{1}{2} - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}(e^{-ikx}\phi_X(k))}{k} dk$$

Im = Take imaginary part

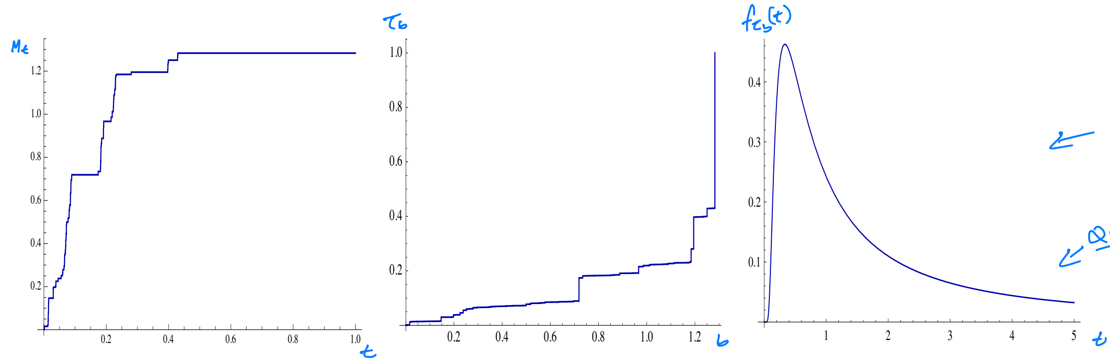
if $\int |\phi_X(k)|dk < \infty$. Hence

$$\mathbb{P}(Z_t > x) \quad = \quad 1 - (\frac{1}{2} - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}(e^{-ikx}e^{-ct(-ik)^{\alpha}})}{k} dk)$$

which we can then differentiate wrt t to obtain the density of A_x at t. We can then swap the variables t and x to obtain the density of A_t at x. We can verify that the desnity of Z_t agrees with the known exact formula for the special case when $\alpha = \frac{1}{2}$.

As a point of interest, we can let $S_t = B_{A_t}$ (where B is a Brownian motion independent of Z) as a **high frequency trading** model of how a stock evolves over small time periods of the order of seconds or milliseconds (this is known as a **subdiffusive Brownian motion**, see final plot on next page) (Fig 2)

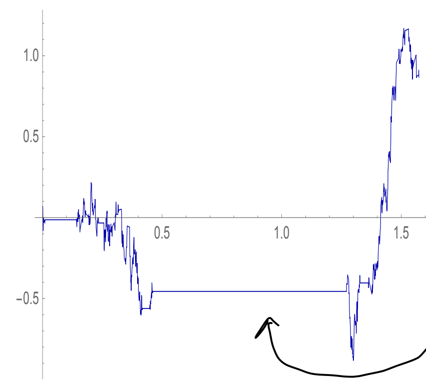
Inversegraph $\left(\begin{smallmatrix} X \to Y \\ Y \to X \end{smallmatrix}\right)$



← density of τ_b graph

← Q4: why this has a fat tail

Figure 1: Here we have plotted a Monte Carlo simulation of M_t (left) and its inverse process τ_b (middle), and the hitting time density $f_{\tau_b}(t)$ for $b = 1$.



High freq. trading (Milliseconds)

No trade happening here since these are very short periods.

Waiting for trade to happen.

Figure 2: A simulation of $S_t = B_{A_t}$ for $\alpha = .9$

~ Mean both quantities $\to 0$ but the ratio of $f_{\tau_b}(t)$ to $\frac{b}{\sqrt{2\pi t^3}}$ goes to 1.

4. Show that $\mathbb{E}(\tau_b) = \infty$ when X is Brownian motion.

Solution Recall that for $X_t = W_t + \gamma t$, the hitting time density is

$$f_{\tau_b}(t) \quad = \quad \frac{b}{\sqrt{2\pi t^3}}\, e^{-(b-\gamma t)^2/2t}\, 1_{t>0} \tag{1}$$

$\mathbb{E}(e^{-\lambda \tau_a})$

which simplifies to

$$f_{\tau_b}(t) \quad = \quad \frac{b}{\sqrt{2\pi t^3}}\, e^{-\frac{b^2}{2t}}\, 1_{t>0} \tag{2}$$

for $\gamma = 0$, and in this case we note that $f_{\tau_b}(t) \sim \frac{b}{\sqrt{2\pi t^3}}$ as $t \to \infty$, and recall that

$$\mathbb{E}(\tau_b) \quad = \quad \int_0^\infty t f_{\tau_b}(t) dt$$

For any $c > 0$ we see that

$$\int_c^\infty t \frac{b}{\sqrt{2\pi t^3}} dt \quad = \quad \infty$$

for $c > 0$, which suggests that $\mathbb{E}(\tau_b) = \infty$. To rigourize this, we just note that any $\delta \in (0, 1)$, $e^{-\frac{b^2}{2t}} > 1 - \delta$ for t sufficiently large, and use this as a lower bound for $f_{\tau_b}(t)$.

King's College London

University Of London

FOLLOW THE INSTRUCTIONS YOU HAVE BEEN GIVEN ON HOW TO UPLOAD YOUR SOLUTIONS

MSc Examination

7CCMFM02T (CMFM02) Risk Neutral Valuation

January 2021

Time Allowed: Two Hours

Full marks will be awarded for complete answers to all FOUR questions.

Within a given question, the relative weights of the different parts are indicated by a percentage figure.

You may consult lecture notes.

1. **a.** Consider a two-step standard binomial model with a riskless bond with initial price $e^{-2r\Delta T}$ and a single risky asset S with initial price $S_0 = 1$, with two equal time steps of length $\Delta T = 0.5$. Assume that at each time step, S is multiplied by $u = 1.1$ with probability p, or multiplied by $d = .91$ with probability $1-p$ and the bond price is multiplied by $e^{r\Delta T}$, and the interest rate $r = 0.02$. Price an American put option with strike $K = 1.01$. **[40%]**

b. Let X be a random variable with a Cauchy distribution which has density $f_X(x) = \frac{1}{\pi(1+x^2)}$ for $x \in \mathbb{R}$. Describe how we can simulate X (you may use that $\int f_X(x)dx = \frac{1}{\pi}\tan^{-1}(x)$ where $\tan^{-1}$ is the inverse of the tan function, and that $\tan(0) = 0$). **[30%]**

Solution.

$$\int_{-\infty}^{x} \frac{1}{\pi(1+z^2)}dz \quad = \quad \frac{1}{\pi}\tan^{-1}(x) - \frac{1}{\pi}\tan^{-1}(-\infty) \quad = \quad \frac{1}{\pi}\tan^{-1}(x) + \frac{1}{2}$$

Set $X = F_X^{-1}(U)$, where $F_X(x) = \frac{1}{\pi}\tan^{-1}(x)+\frac{1}{2}$ is the distribution function of X, so $F_X^{-1}(x) = \tan(\pi(x - \frac{1}{2}))$.

c. Let $C(u,v)$ be a general copula on $\overbrace{[0,1] \times [0,1]}^{\text{Square}}$. Explain how to use $C(.)$ with two standard uniform random variables to simulate two correlated standard Normal random variables X and Y. Write down an integral expression for the correlation between X and Y.

Solution. Let U,V be two random variables with joint cdf $C(u,v)$. Then from the definition of a copula, U and V are standard Uniform random variables. Then set $X = \Phi^{-1}(U)$, $Y = \Phi^{-1}(V)$, and **[30%]**

$$\rho \quad = \quad \mathbb{E}(XY) \quad = \quad \int_0^1 \int_0^1 \Phi^{-1}(u)\Phi^{-1}(v)C_{uv}(u,v)dudv$$

where $C_{uv}(u,v)$ is the joint density of U and V.

In general X & Y will have Non-zero Correlation.
Cuv is joint cdf obtained by differentiating w.r.t u & v.
X & Y are N(0,1). This is not a Gaussian Copula
even though we end up with 2 correlated Normal r.v's.
Since, we have not used 2 correlated r.v's as the building
blocks here.

 See Next Page

2. **a.** Let $(W_t)_{t\geq 0}$ denote a standard Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathcal{F}_t$ satisfying the usual conditions. Which of these statements is true

i. W_t is twice differentiable W is not differentiable (see notes)

ii. A Hölder continuous random process is continuous Yes, Holder continuity implies continuity, see notes

iii. W_s, W_t and W_u have a multivariate normal distribution
Yes in general, $W_{t_1}, ..., W_{t_n}$ has a n-dimensional multivariate normal distribution with joint pdf $\frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}x\Sigma^{-1}\mathbf{x}}$, where $\Sigma_{i,j} = \mathbb{E}(W_{t_i} W_{t_j}) = \min(t_i, t_j)$

iv. The conditional mean of the process $X_t = \int_0^t (t-u)^{H-\frac{1}{2}} dW_u$ at time $s < t$ is zero because stochastic integrals have zero expectation
$\mathbb{E}(X_t|\mathcal{F}_s) = \int_0^s (t-u)^{H-\frac{1}{2}} dW_u$ which is not zero in general

v. $|W_t| \to \infty$ as $t \to \infty$
No, since W returns to zero infinitely often

vi. If $W_t = x > 0$, W will return to zero with probability 1 in finite time. **[30%]**

b. Let U_i be a sequence of iid random variables which are ± 1 with probabilty $\frac{1}{2}$, and let $S_n = \sum_{i=1}^n U_i$. Consider the random function

$$X_t^n = \frac{S_{[nt]}}{\sqrt{n}} \qquad (t \in [0,1])$$

where $[x]$ denotes the largest integer less than or equal to x. What can we say about X^n in the limit as $n \to \infty$? What is the maximum jump size of X_t^n? What is the quadratic variation process of X_t^n? What is $\mathbb{E}(X_s^n X_t^n)$? **[35%]**

Solution. X_t^n process tends to a Brownian motion from Donskers theorem. The jump sizes of X_t^n are $\frac{1}{\sqrt{n}}$. To compute its covariance, we note that

$$X_s^n X_t^n = \frac{S_{[ns]}}{\sqrt{n}} \frac{S_{[nt]}}{\sqrt{n}} = \frac{1}{n}(U_1 + ... + U_{[ns]})(U_1 + ... + U_{[nt]})$$

Taking expectations of this expression, all cross terms vanish since all U_i's are independent with zero mean and all diagonal terms are 1, so we get $[ns]/n$ if $s \leq t$, hence the covariance is $\min(\frac{[ns]}{n}, \frac{[nt]}{n})$ which tends to $\min(s,t)$ as $n \to \infty$ (i.e. the Covariance of Brownian motion).

c. Consider the following stock price process:

$$dS_t \;=\; \sigma(\beta S_t + 1 - \beta)dW_t$$

with $S_0 > 0$, $\beta \in (0,1)$ and $\sigma > 0$. By setting $X_t = \beta S_t + 1 - \beta$ and applying Ito's lemma, compute the exact distribution of S_t. Can S go negative? **[35%]**

Solution.

$$dX_t \;=\; \beta dS_t \;=\; \beta\sigma X_t dW_t\,.$$

So X is GBM with volatility $\beta\sigma$. Then

$$\begin{aligned}\mathbb{P}(S_t \leq S) &= \mathbb{P}(X_t \leq \beta S + 1 - \beta)\\ &= \mathbb{P}\Big(\frac{\log(\frac{X_t}{X_0}) + \frac{1}{2}\beta^2\sigma^2 t}{\beta\sigma\sqrt{t}} \leq \frac{\log(\frac{\beta S+1-\beta}{X_0}) + \frac{1}{2}\beta^2\sigma^2 t}{\beta\sigma\sqrt{t}}\Big) \;=\; \Phi\Big(\frac{\log(\frac{\beta S+1-\beta}{X_0}) + \frac{1}{2}\beta^2\sigma^2 t}{\beta\sigma\sqrt{t}}\Big)\end{aligned}$$

For the final part, we note that

$$\mathbb{P}(S_t \leq 0) \;=\; \mathbb{P}(X_t \leq 1 - \beta) \;>\; 0$$

3. Let $(W_t)_{t\geq 0}$ denote a standard Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathcal{F}_t$. Consider the Black-Scholes model

$$dS_t \;=\; S_t(\mu dt + \sigma dW_t)$$

for a Stock price process S.

a. Write down the Black-Scholes PDE and boundary condition for the price of an option which pays 1 at time T if S hits a barrier level $B > S_0$ at any time $t \in [0, T]$, and otherwise pays 1 at time T. What is the no-arbitrage price of this option? **[30%]**

Solution. P satisfies usual Black-Scholes PDE, but with boundary conditions $P(B,t) = e^{-r(T-t)}$ and $P(S,T) = 1$. Option pays 1 in both scenarios at time T, so its value is just $e^{-r(T-t)}$.

b. Assume $r = 0$, and consider an up-and-out put option with strike K and barrier level $B = K > S_0$ which pays $\max(K - S_T, 0)$ at time T if S stays below B for all $t \in [0, T]$, and pays zero otherwise. Show that the no-arbitrage price of this contract at time at time $t \in [0, T]$ is $K - S_t$. What is the vega of this option? **[35%]**

Solution. $P(S,t) = K - S$ satisfies the BS PDE with boundary condition $P(B,t) = K - B = 0$ and $P(S,T) = K - S$. Vegas is zero since $K - S$ does not depend on σ.

c. Show that the price of a European call option under the Black-Scholes model is increasing in the maturity T (Hint: use conditional Jensen's inequality and the tower property).

Solution. Let $0 < T_1 < T_2$. Then

$$\begin{aligned}
& e^{-rT_2}\mathbb{E}((S_{T_2} - K)^+) \\
= \; & e^{-rT_2}\mathbb{E}(\mathbb{E}(S_{T_2} - K)^+ | S_{T_1}) \quad \text{(from the tower property from FM01)} \\
\geq \; & e^{-rT_2}\mathbb{E}(\mathbb{E}(S_{T_2} - K | S_{T_1})^+) \\
& \quad \text{(from conditional Jensen inequality with convex function } f(S) = (S-K)^+\text{)} \\
= \; & e^{-rT_2}\mathbb{E}((S_{T_1} e^{r(T_2 - T_1)} - K)^+) \\
= \; & e^{-rT_1}\mathbb{E}((S_{T_1} - K e^{-r(T_2 - T_1)})^+) \\
\geq \; & e^{-rT_1}\mathbb{E}((S_{T_1} - K)^+)
\end{aligned}$$

where all expectations are under the risk-neutral measure $\mathbb{Q}$.

[35%]

W_t = B.M.
ξ = jump sizes
N_t = no. of jumps

4. Consider a jump-diffusion model where the log stock price $X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} \xi_i$, where W_t is standard Brownian motion, the ξ_i's are i.i.d $N(\alpha, \delta^2)$ random variables and N_t is a Poisson process with $N_0 = 0$ and arrival rate $\lambda > 0$, and W, the ξ_i's and N_t are all independent of each other.

a. Compute $\mathbb{E}(e^{pX_t})$. **[30%]**

mgf of Normal r.v

Solution. $V(p) = \mu p + \frac{1}{2}\sigma^2 p^2 + \lambda \int_{-\infty}^{\infty}(e^{px} - 1)\nu(x)dx = \mu p + \frac{1}{2}\sigma^2 p^2 + \lambda(\mathbb{E}(e^{p\xi_i}) - 1) = \mu p + \frac{1}{2}\sigma^2 p^2 + \lambda(e^{\alpha p + \frac{1}{2}\delta^2 p^2} - 1)$, where $\nu(x)$ is the (Normal) jump size density. Then $\mathbb{E}(e^{pX_t}) = e^{tV(p)}$. This model is known as the Merton jump diffusion model

mgf of jumpsize

b. For the Black-Scholes model, compute the no-arbitrage price of an option which pays 1 at the hitting time H_B if S hits a barrier level $B > S_0$ at any time $t \in [0, T]$, and otherwise pays 1 at time T. **[40%]**

← No jumps in this model.

Solution. No-arbitrage price is

$$\mathbb{E}(e^{-r\tau_a} 1_{\tau_a \le T}) + e^{-rT}\mathbb{E}(1_{\tau_a > T}) = \int_0^T e^{-rt} f_{\tau_a}(t)dt + e^{-rT}\int_T^\infty f_{\tau_a}(t)dt$$

where

$$f_{\tau_a}(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-(a-\tilde{\gamma}t)^2/2t} 1_{t>0}$$

(see Eq 35 in lecture notes) is the hitting time density of $\frac{X_t - X_0}{\sigma}$ to the level $a = \frac{\log B - X_0}{\sigma}$, and $X_t = \log S_t$, and $\tilde{\gamma} = \mu/\sigma$ (see part c) as well).

drift of log stock price

lower barrier

c. If $r - \frac{1}{2}\sigma^2 > 0$ for part b), compute the probability that S does not hit $B < S_0$ in finite time, where $B > 0$. What is the distribution of the ultimate minimum: $\min_{0 \le u < \infty} S_t$? **[30%]**

Solution.

$$dS_t = S_t(rdt + \sigma dW_t)$$

and recall that $dX_t = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t = \mu dt + \sigma dW_t$, where we define $\mu := r - \frac{1}{2}\sigma^2$. Then $X_t \to \infty$ as $t \to \infty$ and may not ever hit a lower barrier $a < X_0$. Moreover, we note that

$$-\min_{0 \le s \le t}(X_s - X_0) \sim \max_{0 \le s \le t}(\tilde{X}_s - \tilde{X}_0)$$

where $\tilde{X}_t = -\mu t + W_t$ and recall that $b > 0$. Then $\underline{S}_\infty := \min_{0 \le u < \infty} S_u$ and

ultimate min. of log fn   ultimate Max. of $\tilde{X}_\infty$

$$\mathbb{P}(\underline{S}_\infty < B) \;=\; \mathbb{P}(\log \frac{\underline{S}_\infty}{S_0} < a) \;=\; \mathbb{P}(\underline{X}_\infty - X_0 < a) \;=\; \mathbb{P}(\bar{\tilde{X}}_\infty > -a) \;=\; e^{-\frac{2|\mu|}{\sigma^2}|a|}$$

where $a = \log \frac{B}{S_0} < 0$, and we have used Eq 35 in the lecture notes (updated 6th Jan 2021).

Get rid of initial stock price & work on log stock price.

King's College London

UNIVERSITY OF LONDON

FOLLOW THE INSTRUCTIONS YOU HAVE BEEN GIVEN ON HOW TO UPLOAD YOUR SOLUTIONS

MSC EXAMINATION

7CCMFM02T (CMFM02) RISK NEUTRAL VALUATION

JANUARY 2021

TIME ALLOWED: TWO HOURS

FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO ALL FOUR QUESTIONS.

WITHIN A GIVEN QUESTION, THE RELATIVE WEIGHTS OF THE DIFFERENT PARTS ARE INDICATED BY A PERCENTAGE FIGURE.

YOU MAY CONSULT LECTURE NOTES.

1. **a.** Construct a n-dimensional Gaussian copula with correlation matrix ρ_{ij} and show how we can use this simulate n correlated standard Exponential random variables $E_1, ..., E_n$.

Solution. Recall that the joint density of n Normal random variables $X_1, ..., X_n$ is $\frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}\mathbf{x}^T \Sigma^{-1} \mathbf{x}}$, where here $\Sigma_{i,j} = \mathbb{E}(X_i X_j)$, but in this case here the Z_i's will have variance of 1, so $\Sigma_{i,j} = \rho_{ij} = \mathbb{E}(Z_i Z_j)$. Σ is assumed to be given here, but has to positive definite to ensure that the Var of $c_1 Z_1 + ... + c_n Z_n$ is non-negative for any vector $\mathbf{c} = (c_1, .., c_n)$ with $c_i \in \mathbb{R}$ (this variance is given by $\mathbf{c}^T \Sigma \mathbf{c}$). We know from the Applied Probability Revision notes that $U_i := \Phi(Z_i) \sim U([0, 1])$. We then set $E_i = F^{-1}(U_i)$ where $F(x) = 1 - e^{-x}$ is the distribution function of an Exp(1) random variable.

$\underbrace{c^T \Sigma c}_{\text{Variance}} \geq 0$

has to be ≥ 0.

$\tilde{S}_T$ = Max of stock price at final time T.

2. **a.** Consider a jump-diffusion model for a log stock price process X_t as in the lecture notes with negative only jumps. Explain how we can price a call option on $\bar{S}_T$, where $\bar{S}_t = \max_{0 \le u \le t} S_u$ is the running maximum process of S (example of a lookback option). **[30%]**

Solution. For pricing options, recall we first have to choose μ so that $\mathbb{E}(S_t) = \mathbb{E}(e^{X_t}) = e^{V(1)t} = e^{rt}$ (so as to make the discounted stock price a martingale) so we must impose that $V(1) = r$. To see this note that

$$\begin{aligned}
\mathbb{E}(e^{-rt}S_t|\mathcal{F}_s) \quad = \quad e^{-rt}\mathbb{E}(e^{X_t}|\mathcal{F}_s) &= e^{-rt}\mathbb{E}(e^{(X_s+X_t-X_s)}|\mathcal{F}_s) \\
(s \le t) \qquad\qquad\qquad\qquad &= e^{-rt}e^{X_s}\mathbb{E}(e^{X_t-X_s}|\mathcal{F}_s) \quad \text{increment independent} \\
&= e^{-rt}e^{X_s}\mathbb{E}(e^{X_t-X_s}) \quad \text{stationary process} \\
&= e^{-rt}e^{X_s}\mathbb{E}(e^{X_{t-s}}) \\
&= S_s e^{-rt}e^{V(1)(t-s)} = e^{-rs}S_s
\end{aligned}$$

where we have used that X has independent increments i.e. $X_t - X_s$ is independent of $(X_u)_{0 \le u \le s}$, and X is a stationary process, i.e. $X_t - X_s$ has the same distribution as X_{t-s}.

From the notes we also know that $\mathbb{E}(e^{-q\tau_a}) = e^{-a\Phi(q)}$ for $q > 0$, where $\Phi(q)$ is the largest inverse of $V(p) = \log \mathbb{E}(e^{pX_1}) = \frac{1}{t}\log \mathbb{E}(e^{pX_t})$. To compute the density of τ_a, recall that we set $-q = ik$ for $k \in \mathbb{R}$, and compute the inverse Fourier transform of $\mathbb{E}(e^{ik\tau_a})$ as

$$f_{\tau_a}(t) \quad = \quad \frac{1}{2\pi}\int_{k=-\infty}^{\infty} e^{-ikt}\,\mathbb{E}(e^{ik\tau_a})dk \quad = \quad \frac{1}{2\pi}\int_{k=-\infty}^{\infty} e^{-ikt}e^{-a\Phi(-ik)}dk\,.$$

Then

$$\mathbb{P}(\bar{X}_t \ge a) \quad = \quad \mathbb{P}(\tau_a \le t)$$

so the density $f_{\bar{X}_t}(a)$ of $\bar{X}_t$ is given by

$$f_{\bar{X}_t}(a) \quad = \quad -\frac{d}{da}\mathbb{P}(\bar{X}_t \ge a) \quad = \quad -\frac{d}{da}\mathbb{P}(\tau_a \le t) \quad = \quad -\frac{d}{da}\int_0^t f_{\tau_a}(s)ds$$

Pricing option payoff by discounting $\longrightarrow$

$$e^{-rT}\mathbb{E}((\bar{S}_T - K)^+) \quad = \quad e^{-rT}\mathbb{E}((e^{\bar{X}_T} - K)^+) \quad = \quad e^{-rT}\int_{a=0}^{\infty} f_{\bar{X}_T}(a)(e^a - K)^+ da$$

$a = \bar{X}_T$ dummy variable

Exponential of Max. log stock price since log f'n is well behaved i.e. CTS & Monotonic.

assuming $X_0 = 0$.

Final Page