Stochastic Analysis

Lecture 1

This Lecture

- Outline of module and class structure
- Brief review of Probability Theory
- Introduction to stochastic processes

Probability Review

- σ-algebras
- Probability measure
- Probability space
- Random variable
- Expectation
- Independence
- Conditional Expectation

σ-algebras

- Let Ω be any set, and denote the collection of all subsets of Ω by $\mathcal{P}(\Omega)$ "power set" of Ω
- $\mathcal{F}$ is called a σ-**algebra** if $\mathcal{F} \subset \mathcal{P}(\Omega)$ and the following hold:
  contains subsets of Ω
  1. $\emptyset \in \mathcal{F}$
     complement of A
  2. if $A \in \mathcal{F}$, then $A^C \in \mathcal{F}$
  3. if $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$
- The pair $(\Omega, \mathcal{F})$ is called a **measurable space**

Important Example

- An important example is the **Borel σ-algebra** on $\mathbb{R}$

  the standard topology
- Let τ be all open subsets of $\mathbb{R}$
- Let $\mathcal{B}$ be the smallest σ-algebra containing τ
  - $\mathcal{B}$ is the σ-algebra **generated** by τ
- $\mathcal{B}$ is called the **Borel σ-algebra** on $\mathbb{R}$
- An element $B \in \mathcal{B}$ is called a Borel set

- Borel sets include open & closed intervals.
- Making σ-algebra involves adding more subsets.

Probability Measure

$\mathbb{P}$ has input which is a subset of Ω

- Let $(\Omega, \mathcal{F})$ be a measurable space
- A **probability measure** is a function $\mathbb{P} : \mathcal{F} \to [0,1]$ with the following properties:

  $\mathbb{P}(A)$ is interpreted as the "size" of A.

  1. $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$
  2. if $A_1, A_2, \dots \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

sizes of subsets add up as you would expect.

- The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**
- We will always assume our probability space is **complete**:
  - if $A \in \mathcal{F}$ and $\mathbb{P}(A) = 0$, then $B \subset A$ implies $B \in \mathcal{F}$

Random Variables

- Let X be a function mapping Ω to $\mathbb{R}$

$$X : \Omega \to \mathbb{R}$$

- We call X a **random variable** if the preimage of all Borel sets is measurable

$$B \in \mathcal{B} \Rightarrow X^{-1}(B) \in \mathcal{F}$$

where

$$X^{-1}(B) = \{\omega : X(\omega) \in B\} \subset \Omega$$

- A random variable induces a probability measure on $\mathbb{R}$ by

$$\mu_X(B) = \mathbb{P}(X^{-1}(B))$$

$\mu_X : \mathcal{B} \to [0,1]$ is a probability measure.
(prove yourself)

CDF and pdf

- The **cumulative distribution function** (CDF) of a random variable X is defined by "$X \leq x$" $= \{\omega : \omega \in \Omega$ and $X(\omega) \leq x\}$

$$F(x) = \mu_X\Big((-\infty, x]\Big) = \mathbb{P}(X \leq x)$$

- If there exists a function f such that

$$F(x) = \int_{-\infty}^{x} f(u)du$$

then f is called the **probability density function** (pdf) of X

Important Example

- A random variable X has **Gaussian distribution** if it has a density function of the form

memorize this immediately

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

  for some parameters μ and $\sigma > 0$

- Then we will write $X \sim \mathcal{N}(\mu, \sigma^2)$

- Gaussian random variables (or random vectors) have many important and convenient properties

  - See Appendix A of Øksendal for more details

Expectation

- Given a random variable X, its **expectation** is defined by

Lebesgue integral

$$\mathbb{E}[X] = \int_\Omega X(\omega)d\mathbb{P}(\omega) = \int_{-\infty}^{\infty} x dF(x)$$

requires proof

- If X has a pdf f, then the above is equal to

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$

- In addition, if $Y = g(X)$ is also a random variable then

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

g must be nice function. g(X) is a composition of f'n as r.v. X is also a f'n.

Independence of Events

- Two events $A, B \in \mathcal{F}$ are said to be **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

- A collection of events $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ is said to be **mutually independent** if

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_{\alpha_i}\right) = \prod_{i=1}^{n} \mathbb{P}(A_{\alpha_i})$$

generalization of above

for any choice of $\alpha_1, \ldots, \alpha_n$ all distinct

- A collection of families of events $\{\mathcal{H}_\alpha\}_{\alpha \in \mathcal{A}}$ is said to be **independent** if

further generalization of above

$$\mathbb{P}\left(\bigcap_{i=1}^{n} H_{\alpha_i}\right) = \prod_{i=1}^{n} \mathbb{P}(H_{\alpha_i})$$

for any choice of $\alpha_1, \ldots, \alpha_n$ all distinct and with $H_{\alpha_i} \in \mathcal{H}_{\alpha_i}$

Intuition behind independent events.

Def: $A \perp B \iff \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

Given that A occurs, we gain no information about the probability of B occuring.

Ex. bowl of marbles: $P(A) = \frac{2}{100}$, $P(B) = \frac{10}{100}$

9 solid red

1 solid yellow

1 red/yellow stripe

89 green

A: any yellow colour

B: any red colour

$A \not\perp B$

Independence of Random Variables

- A collection of random variables $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ is said to be **independent** if the collection of σ-algebras they generate is independent

- If X and Y are independent random variables with $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|Y|] < \infty$ then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

- The converse is **false**

Interpretation of independent random variables

$X \perp\!\!\!\perp Y$ then any information given about the outcome of X gives no information about the outcome of Y.

Ex: $X \sim \mathcal{N}(0,1)$, $Z = \begin{cases} 1, & p=0.5 \\ -1, & p=0.5 \end{cases}$ $\quad X \perp\!\!\!\perp Z$

Let $Y = XZ$.

Check, however:

$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$

Question: $X \perp\!\!\!\perp Y$?

if $X \geq 1$ then $Y \leq -1$ or $Y \geq 1$

$\Rightarrow$ not independent.

Conditional Expectation

$\mathcal{H}$ has less information than $\mathcal{F}$

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X a random variable (integrable)
- Let $\mathcal{H} \subset \mathcal{F}$ be a σ-algebra "sub-σ-algebra"
- The **conditional expectation of** X **given** $\mathcal{H}$ is denoted $\mathbb{E}[X|\mathcal{H}]$ and is defined as the unique random variable satisfying

$\mathbb{E}[X] \in \mathbb{R}$

$Y = \mathbb{E}[X|H]$ is a random variable.

Y is H-measurable

1. $\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}$-measurable
2. $\int_H \mathbb{E}[X|\mathcal{H}] d\mathbb{P} = \int_H X d\mathbb{P}$ for all $H \in \mathcal{H}$

- Interpretation: $\mathbb{E}[X|\mathcal{H}]$ is the "best guess" of X given the information reflected in $\mathcal{H}$

$\mathcal{H}$-measurable means preimage of borel sets is an element of H. i.e. $Y^{-1}[B] \in \mathcal{H}$.

Properties of Conditional Expectation

- Let X and Y be integrable random variables
- Let $a, b \in \mathbb{R}$
- Let $\mathcal{G}$ and $\mathcal{H}$ be σ-algebras with $\mathcal{G} \subset \mathcal{H} \subset \mathcal{F}$

1. $\mathbb{E}[aX + bY|\mathcal{H}] = a\mathbb{E}[X|\mathcal{H}] + b\mathbb{E}[Y|\mathcal{H}]$
2. $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]] = \mathbb{E}[X]$
3. $\mathbb{E}[X|\mathcal{H}] = X$ if X is $\mathcal{H}$-measurable
4. $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[X]$ if X is independent of $\mathcal{H}$
5. $\mathbb{E}[XY|\mathcal{H}] = X\mathbb{E}[Y|\mathcal{H}]$ if X is $\mathcal{H}$-measurable
6. $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}]$

all of these are statements about random variables except #2

Stochastic Processes

- A **stochastic process** is a collection of random variables indexed by some indexing set: $\{X_\alpha\}_{\alpha \in \mathcal{A}}$
  - Here $\mathcal{A}$ is the indexing set and α is the index
  - For any particular $\alpha_0 \in \mathcal{A}$ the object X_{α_0} is a single random variable
  - That is, $X_{\alpha_0} : \Omega \to \mathbb{R}$
- We will almost always consider $\mathcal{A} = \mathbb{R}^+$ or $\mathcal{A} = [0, T]$ for some $T > 0$
- With this indexing set it is convenient to think of the index parameter as representing time
- Thus we can think of the collection $\{X_t\}_{t \in \mathbb{R}^+}$ as a function of two variables:

in Øksendal:

$X(t, \omega)$

$$X : \mathbb{R}^+ \times \Omega \to \mathbb{R}$$
$$X : (t, \omega) \mapsto X_t(\omega)$$

Other Perspectives

- As a function of two variables:

$$X : \mathbb{R}^+ \times \Omega \to \mathbb{R}$$
$$X : (t, \omega) \mapsto X_t(\omega)$$

- For fixed $t \in \mathbb{R}^+$, as a single random variable:

$$X_t(\cdot) : \Omega \to \mathbb{R}$$
$$X_t(\cdot) : \omega \mapsto X_t(\omega)$$

- For fixed $\omega \in \Omega$, as a function of time:

refer to this as a "path" of X

$$X_.(\omega) : \mathbb{R}^+ \to \mathbb{R}$$
$$X_.(\omega) : t \mapsto X_t(\omega)$$

- Usefulness of different perspectives will depend on context

Equality of Stochastic Processes

- There are different ways in which we may interpret two stochastic processes as being "the same"
- Let $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ be two stochastic processes (for either $t \in \mathbb{R}^+$ or $t \in [0,T]$)
- X is a **version of** Y (or a **modification of** Y) if

for fixed t, the r.v.s X & Y are equal a.s.

$$\mathbb{P}\{\omega : X_t(\omega) = Y_t(\omega)\} = 1\,, \quad \text{for all } t \geq 0$$

- X is **indistinguishable** from Y if

$$\mathbb{P}\{\omega : X_t(\omega) = Y_t(\omega) \text{ for all } t \geq 0\} = 1$$

All paths are equal i.e. path of X & Y are equal with prob. 1.

Important Example: Brownian Motion

- A stochastic process $\{B_t\}_{t \in \mathbb{R}^+}$ is called a **Brownian motion** or **Wiener process** starting at $x \in \mathbb{R}$ if it satisfies:
    1. $B_0 = x$ a.s.
    2. For $t_1 < t_2 < t_3$, the increment $B_{t_3} - B_{t_2}$ is independent of B_{t_1}
    3. For $t \geq 0$ and $u > 0$ the increments have distribution $B_{t+u} - B_t \sim \mathcal{N}(0, u)$
- Important question: does there exist a stochastic process that satisfies these assumptions?
- Øksendal takes a different (but equivalent) approach to defining Brownian motion

Continuity of Brownian Motion

- Take the following theorem as given:

Theorem

Let $\{X_t\}_{t\geq 0}$ be a stochastic process. Suppose that for all $T > 0$ there exists positive constants α, β, and D such that

$$\mathbb{E}\left[|X_t - X_s|^{\alpha}\right] \leq D\,|t-s|^{1+\beta} \quad 0 \leq s,t \leq T$$

Then there exists a continuous version of X

- It is not difficult to show that Brownian motion satisfies this condition

- We will always assume we work with a continuous version of Brownian motion

Stochastic Analysis

Lecture 2

This Lecture

- Motivation for stochastic integral from random dynamics
- Filtrations and adapted processes
- Elementary functions
- Successive approximation results
- Definition of stochastic integral

Motivation

- Differential equations are an important modeling tool in many areas of Applied Mathematics
- Typical form:

$$\frac{dV_t}{dt} = F(t, V_t)\,, \qquad\qquad V_0 = v\,.$$

- Would like to introduce random component to dynamic behaviour, formally:

$$\frac{dV_t}{dt} = F(t, V_t) + \text{“randomness”}\,, \qquad V_0 = v\,.$$

- It will help to consider differential equations in their differential form:

$$dV_t = F(t, V_t)\,dt + \text{“randomness”}\,, \qquad V_0 = v\,.$$

Discretization

- On the interval $[0, T]$ suppose we have $0 = t_0 < t_1 < \cdots < t_N = T$ with $\Delta t = t_{n+1} - t_n$

- The essentials of construction will come from discretization of the differential equation:

$$dV_t = F(t, V_t)\, dt \quad \longmapsto \quad V_{t_{n+1}} \approx V_{t_n} + F(t_n, V_{t_n})\, \Delta t$$

- The "randomness" is introduced in this discrete form:

$$V_{t_{n+1}} \approx V_{t_n} + F(t_n, V_{t_n})\, \Delta t + \underbrace{G(t_n, V_{t_n})\, \Delta B_{t_n}}_{\text{Random part.}},$$

where ΔB_t is some suitably chosen random variable

- We will choose $\Delta B_{t_n} = B_{t_{n+1}} - B_{t_n}$ where $\{B_t\}_{0 \le t \le T}$ is a Brownian motion

Return to Continuous Time

- Solution to the ODE is equivalent to solution to integral equation:

$$V_t = v + \int_0^t F(u, V_u)\, du$$

- Taking $\Delta t \to 0$ in stochastic difference equation, formally we have

$$V_t = v + \int_0^t F(u, V_u)\, du + \text{``}\int_0^t G(u, V_u)\, dB_u\text{''}$$

- Our next goal is to suitably define quantities of the form

$$\int_0^T f_t(\omega)\, dB_t(\omega)$$

such that is has desired properties relevant to our stochastic modeling

Illustration of the Difficulty

- Recall the construction of the Riemann integral
- Attempt similar construction for stochastic integral
- Compute

$$\int_0^T B_t \, dB_t$$

using two different approximations of B_t:

1. left endpoint approximation $\phi_t^{(1)}$
2. right endpoint approximation $\phi_t^{(2)}$

- This results in

$$\mathbb{E}\left[\int_0^T \phi_t^{(1)} \, dB_t\right] = 0 \,, \qquad \mathbb{E}\left[\int_0^T \phi_t^{(2)} \, dB_t\right] = T$$

- This difference will always remain regardless of how many subintervals are used

Here we integrate just like Reimann integral using piecewise const. fn.

Compute $\int_0^T B_t \, dB_t$

Subdivide $[0,T]$: let $t_k = 2^{-N} T k$

2^N subintervals.
$\Delta t = 2^{-N} T$

$\phi_t^{(1)}(\omega) = B_{t_k}(\omega)$ if $t \in [t_k, t_{k+1})$

$\phi_t^{(1)}(\omega) = \sum_{k=0}^{2^N - 1} B_{t_k}(\omega) \chi_{[t_k, t_{k+1})}(t)$

indicator function

left end pt approx.

$\phi_t^{(2)}(\omega) = B_{t_{k+1}}(\omega)$ if $t \in [t_k, t_{k+1})$

$\phi_t^{(2)}(\omega) = \sum_{k=0}^{2^N - 1} B_{t_{k+1}}(\omega) \chi_{[t_k, t_{k+1})}(t)$

Right end pt. approx.

ϕ is piecewise constant.

Approximate $\int_0^T B_t\,dB_t$ by $I^{(1)} = \int_0^T \phi_t^{(1)}\,dB_t$ and $I^{(2)} = \int_0^T \phi_t^{(2)}\,dB_t$

- $$I^{(1)} = \sum_{k=0}^{2^N-1} \phi_{t_k}^{(1)} \Delta B_k \quad \text{where} \quad \Delta B_k = B_{t_{k+1}} - B_{t_k}$$ for piecewise const. f'n.

$$= \sum_{k=0}^{2^N-1} B_{t_k}\left(B_{t_{k+1}} - B_{t_k}\right)$$

$$E\left[I^{(1)}\right] = \sum_k E\left[B_{t_k}\left(B_{t_{k+1}} - B_{t_k}\right)\right]$$

brownian motion increment independent

$$= \sum_k E\left[B_{t_k}\right] \underbrace{E\left[B_{t_{k+1}} - B_{t_k}\right]}_{0}$$

$$= 0$$

- $$I^{(2)} = \sum_{k=0}^{2^N-1} \phi_{t_k}^{(2)} \Delta B_k$$

$$= \sum_k B_{t_{k+1}}\left(B_{t_{k+1}} - B_{t_k}\right)$$

$$E\left[I^{(2)}\right] = \sum_k E\left[B_{t_{k+1}}\left(B_{t_{k+1}} - B_{t_k}\right)\right]$$

$$= \sum_k E\left[\left(B_{t_{k+1}} - B_{t_k} + B_{t_k}\right)\left(B_{t_{k+1}} - B_{t_k}\right)\right]$$

$$= \sum_k E\left[\left(B_{t_{k+1}} - B_{t_k}\right)^2\right] + \underbrace{E\left[B_{t_k}\left(B_{t_{k+1}} - B_{t_k}\right)\right]}_{0}$$

$$= \sum_{k=0}^{2^N-1} \Delta t = 2^N \Delta t = T$$

$B_{t_{k+1}} - B_{t_k} \sim N(0, \Delta t)$

(And using 2nd moment on variance of $(B_{t_{k+1}} - B_{t_k})^2$)

Comments

- The Riemann integral is constructed by approximation:

$$\int_0^T g(t)\,dt \approx \sum_{k=1}^{N} g(t_k^*)\Delta t$$

where $t_k^* \in [t_k, t_{k+1}]$ is arbitrary

- The previous example shows that for the stochastic integral the choice of t_k^* matters and can't be arbitrary

- If $t_k^* = t_k$ (left endpoint) we obtain the **Ito integral**

$$\int_0^T f_t\,dB_t$$

- If $t_k^* = \frac{t_k + t_{k+1}}{2}$ (midpoint) we obtain the **Stratonovich integral**

$$\int_0^T f_t \circ dB_t$$

we will no longer see this type. See Ø for more details.

Further Comments

- We will construct our integrals in the Ito sense (left endpoint)
- We will also require the integrand f to have certain properties
- One property is that f_t may only depend on the Brownian motion up to time t
  - That is, f_t may depend on B_s only if $s \leq t$
- This is properly formulated in terms of filtrations and adapted processes
- A filtration is interpreted as defining the history of a stochastic process

Filtration

- Let $\{\mathcal{N}_t\}_{t\geq 0}$ be a collection of σ-algebras such that
  1. $\mathcal{N}_t \subset \mathcal{F}$
  2. if $0 \leq s < t$ then $\mathcal{N}_s \subset \mathcal{N}_t$ "increasing"

  Then $\{\mathcal{N}_t\}_{t\geq 0}$ is called a **filtration** on $(\Omega, \mathcal{F})$

- Let $\mathcal{F}_t$ be the σ-algebra generated by the collection of random variables $\{B_s : 0 \leq s \leq t\}$
  - That is, $\mathcal{F}_t$ is the smallest σ-algebra containing all sets of the form

$$\{\omega : B_{t_1}(\omega) \in F_1, \ldots, B_{t_k}(\omega) \in F_k\}$$

    where each $t_i \leq t$ and each $F_i \subset \mathbb{R}$ is a Borel set

  Then $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration called the **filtration generated by** B

Adapted Processes

Borelset

$X_t^{-1}[B] \in \mathcal{N}_t$

- Let $\{\mathcal{N}_t\}_{t \geq 0}$ be a filtration on $(\Omega, \mathcal{F})$.
- Let $\{X_t\}_{t \geq 0}$ be a stochastic process. mean
- If X_t is $\mathcal{N}_t$-measurable for all $t \geq 0$ then we say $\{X_t\}_{t \geq 0}$ **is adapted to the filtration** $\{\mathcal{N}_t\}_{t \geq 0}$
- Example:

  yes, because $t/2$ is in the past with respect to t.

  1. the process $Y_t = B_{t/2}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$
  2. the process $Z_t = B_{2t}$ is not adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$

  no, because $2t$ is in the future w.r.t. t.

From previous example: $\phi^{(1)}$ is $\mathcal{F}_t$-adapted, $\phi^{(2)}$ is not $\mathcal{F}_t$-adapted.

Stochastic Integrable Functions

- We define the class of functions which may serve as stochastic integrands
- Denote by $\mathcal{V}(0,T)$ the class of functions (Valid functions)

$$f : [0,T] \times \Omega \to \mathbb{R}$$

⇐ stochastic process

with the following properties:

1. the map $(t,\omega) \mapsto f_t(\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable (where $\mathcal{B}$ is the Borel σ-algebra)
2. the stochastic process $\{f_t\}_{t\geq 0}$ is $\mathcal{F}_t$-adapted
3. $\underbrace{\mathbb{E}[\int_0^T f_t^2\, dt]}_{L^2([0,T]\times\Omega) \text{ of } f.}$ $< \infty$ ← This is double integral over 2 variables t & ω.

i.e. L^2 Norm

Defining the Ito Integral

Elementary

bounded & continuous.

bounded.

arbitrary.

- We wish to properly define

$$I[f](\omega) = \int_0^T f_t(\omega)\, dB_t(\omega)$$

where $f \in \mathcal{V}(0,T)$

- This will be done through a sequence of approximations:

1. Define $I[\phi](\omega)$ when $\phi \in \mathcal{V}(0,T)$ is an **elementary function**
2. When $g \in \mathcal{V}(0,T)$ is bounded and $g_\cdot(\omega)$ is continuous, approximate g by elementary functions
3. When $h \in \mathcal{V}(0,T)$ is bounded, approximate h by bounded continuous processes
4. When $f \in \mathcal{V}(0,T)$ is arbitrary, approximate f by bounded processes
5. Use these approximations to extend the definition from elementary functions

Three approx. results

Elementary Functions

- An **elementary function** is a stochastic process of the form

$$\phi_t(\omega) = \sum_{j \geq 0} e_j(\omega) \mathcal{X}_{[t_j, t_{j+1})}(t)$$

$\phi_t(\omega)$ is piecewise constant for each ω.

  where each e_j is a random variable and $\mathcal{X}$ denotes the indicator

- In order to have $\phi \in \mathcal{V}(0, T)$ we must have each e_j be $\mathcal{F}_{t_j}$-measurable

- We define the stochastic integral of an elementary function as

$$\int_0^T \phi_t(\omega)\, dB_t(\omega) = \sum_{j \geq 0} e_j(\omega)(B_{t_{j+1}}(\omega) - B_{t_j}(\omega))$$

this is by analogy to the Riemann integral.

The Ito Isometry for Elementary Functions

Lemma

If ϕ is bounded and elementary then

$$\mathbb{E}\Big[\underbrace{\Big(\int_0^T \phi_t \, dB_t\Big)}_{\text{object in } L^2(\Omega)}^2\Big] = \mathbb{E}\Big[\int_0^T \underset{\substack{\downarrow \\ \text{object in } L^2([0,T]\times\Omega)}}{\phi_t^2} \, dt\Big]$$

- Ito's lemma will be key in extending the definition of the stochastic integral from elementary functions to all functions in $\mathcal{V}(0,T)$
- It will allow us to use convergence of Cauchy sequences in L^2 spaces to give proper notion of convergence of our approximation procedure

- Let ϕ be bounded and elementary, then

$$\mathbb{E}\left[\left(\int_0^T \phi_t \, dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T \phi_t^2 \, dt\right]$$

Proof:

$$\phi_t(\omega) = \sum_k e_k(\omega) \, \chi_{(t_k, t_{k+1})}(t)$$

recall, if $\phi \in \mathcal{V}$ then $e_k \in \mathcal{F}_{t_k}$

also: ΔB_i is independent of $e_i e_k \Delta B_k$ if $i > k$

$$\mathbb{E}\left[\left(\int_0^T \phi_t \, dB_t\right)^2\right] = \mathbb{E}\left[\left(\sum_k e_k \Delta B_k\right)^2\right]$$

(finite sum squared results in double sum over each variables)

$$= \mathbb{E}\left[\sum_i \sum_k e_i e_k \Delta B_i \Delta B_k\right]$$

$$= \sum_i \sum_k \mathbb{E}\left[e_i e_k \Delta B_i \Delta B_k\right]$$

(only when $i = k$ for all $i > k$ or $k > i$, the $\mathbb{E}$ results in zero, as increments of B.M. $B_{t_{i+1}} - B_{t_i}$ has Expectation 0)

$$= \sum_k \mathbb{E}\left[e_k^2 (\Delta B_k)^2\right]$$

$$\Delta t_k = t_{k+1} - t_k$$

$$= \sum_k \mathbb{E}\left[e_k^2\right] \mathbb{E}\left[(\Delta B_k)^2\right] = \boxed{\sum_k \mathbb{E}\left[e_k^2\right] \Delta t_k}$$

(independent e_k & ΔB_k)

$$\mathbb{E}\left[\int_0^T \varphi_t^2\,dt\right] = \mathbb{E}\left[\int_0^T \left(\sum_k e_k \chi_{[t_k,t_{k+1})}(t)\right)^2 dt\right]$$

$$= \mathbb{E}\left[\int_0^T \sum_i \sum_k e_i e_k \chi_{[t_i,t_{i+1})}(t)\,\chi_{[t_k,t_{k+1})}(t)\,dt\right]$$

(Multiplying 2 indicator fn results in intersection of indicator fn, since these 2 sets are disjoint when we multiply them all are zero except for $i=k$.)

$$= \mathbb{E}\left[\int_0^T \sum_k e_k^2 \chi_{[t_k,t_{k+1})}(t)\,dt\right]$$

(Interchange integral & sum when sum is finite)

$$= \mathbb{E}\left[\sum_k \int_{t_k}^{t_{k+1}} e_k^2\,dt\right]$$

(Interchange sum & $\mathbb{E}$ when sum is finite)

$$= \mathbb{E}\left[\sum_k e_k^2 (t_{k+1}-t_k)\right]$$

$$= \boxed{\sum_k \mathbb{E}[e_k^2]\,\Delta t_k}$$

Approximation of Bounded Continuous Processes

Path of g CTS.
(ω fixed, fn of time)

Lemma

Let $g \in \mathcal{V}(0,T)$ be bounded such that $g_\cdot(\omega)$ is continuous for each ω. Then there exists a sequence of elementary functions $\phi^n \in \mathcal{V}(0,T)$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T (g_t - \phi_t^{(n)})^2 \, dt\right] = 0$$

[This means ϕ converges to g in L^2]

Proof: Let $g \in \mathcal{V}$ be bounded and $g_\cdot(\omega)$ continuous. Define:

$$\phi_t^{(n)}(\omega) = g_{t_k}(\omega) \quad \text{if } t \in [t_k, t_{k+1}) \qquad t_k = 2^{-n} T k$$

$$\Delta t_k = 2^{-n} T$$

then $$\int_0^T \left(\phi_t^{(n)}(\omega) - g_t(\omega)\right)^2 dt \longrightarrow 0 \quad \text{as } n \to \infty \text{ for each } \omega.$$

so by bounded convergence we have

$$\mathbb{E}\left[\int_0^T \left(\phi_t^{(n)} - g_t\right)^2 dt\right] \longrightarrow 0 \quad \text{as } n \to \infty$$

also, ϕ is $\mathcal{F}_t$-adapted.

Approximation of Bounded Processes

Lemma

Let $h \in \mathcal{V}(0,T)$ be bounded. Then there exists a sequence of bounded processes $g^{(n)} \in \mathcal{V}(0,T)$ such that $g_{\cdot}^{(n)}(\omega)$ is continuous for each ω and

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T (h_t - g_t^{(n)})^2 \, dt\right] = 0$$

Let $h \in \mathcal{V}$ bounded, say $|h| \leq M$.

Let $\{\varphi^{(n)}\}_{n=1}^{\infty}$ be a sequence $\varphi^{(n)}: \mathbb{R} \to \mathbb{R}$ such that

i) $\varphi^{(n)}(x) = 0$ if $x \notin (-\frac{1}{n}, 0)$

ii) $\varphi^{(n)}(x) \geq 0$

iii) $\int_{-\infty}^{\infty} \varphi^{(n)}(x)\, dx = 1$

$\{\varphi^{(n)}\}_{n=1}^{\infty}$ set is called an approximate identity

Define $g_t^{(n)}(\omega) = \int_0^{\frac{1}{n}} \varphi^{(n)}(-s)\, h_{t-s}(\omega)\, ds$

$\Rightarrow g_\cdot^{(n)}(\omega)$ is continuous for each ω.

$g_t^{(n)}(\omega)$ represents average of $h(\omega)$ between $t - \frac{1}{n}$ and t.

property of approximate identities:

$$\int_0^T \left(g_t^{(n)}(\omega) - h_t(\omega)\right)^2 dt \to 0 \quad \text{as } n \to \infty \text{ for each } \omega.$$

$$\Longrightarrow \quad \mathbb{E}\left[\int_0^T \left(g_t^{(n)} - h_t\right)^2 dt\right] \to 0 \quad \text{as } n \to \infty \text{ by bounded convergence.}$$

Approximation of Processes in $\mathcal{V}(0,T)$

Lemma

Let $f \in \mathcal{V}(0,T)$. Then there exists a sequence of processes $h^{(n)} \in \mathcal{V}(0,T)$ bounded for each n such that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T (f_t - h_t^{(n)})^2 \, dt\right] = 0$$

Let $f \in \mathcal{V}$. Define:

$$h_t^{(n)}(\omega) = \begin{cases} n & \text{if} \quad f_t(\omega) > n \\ f_t(\omega) & \text{if} \quad |f_t(\omega)| \leq n \\ -n & \text{if} \quad f_t(\omega) < -n \end{cases}$$

then $\left(f_t - h_t^{(n)}\right)^2 \leq \underbrace{\left(f_t - n\right)^2}$

integrable because $f \in \mathcal{V}$.

$$h_t^{(n)} \xrightarrow{n\to\infty} f_t \quad \text{a.s.}$$

$\Rightarrow$ by dominated convergence: $\mathbb{E}\left[\int_0^T \left(h_t^{(n)} - f_t\right)^2 dt\right] \to 0$ as $n \to \infty$.

Extension of Stochastic Integral to $\mathcal{V}(0, T)$

- Combining the previous three results means that any process $f \in \mathcal{V}(0, T)$ can be approximated by a sequence of elementary functions $\phi^{(n)} \in \mathcal{V}(0, T)$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T (f_t - \phi_t^{(n)})^2 \, dt\right] = 0$$

- Thus, for arbitrary $f \in \mathcal{V}(0, T)$ we take an approximating elementary sequence $\phi^{(n)}$ and define the stochastic integral as

$$I[f](\omega) = \int_0^T f_t(\omega) \, dB_t(\omega) = \lim_{n\to\infty} \int_0^T \phi_t^{(n)}(\omega) \, dB_t(\omega)$$

Comments on Approximation Procedure

- Our lemmas show that there exists an approximating elementary sequence $\phi^{(n)}$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T (f_t - \phi_t^{(n)})^2 \, dt\right] = 0$$

- The limit $\lim_{n\to\infty} \int_0^T \phi_t^{(n)} \, dB_t$ exists as an element of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ because it is a Cauchy sequence by the Ito isometry for elementary functions

- If $\psi^{(n)}$ is a different approximating elementary sequence, how do we know the following holds?

$$\lim_{n\to\infty} \int_0^T \phi_t^{(n)} \, dB_t = \lim_{n\to\infty} \int_0^T \psi_t^{(n)} \, dB_t$$

- Also comes from the Ito isometry for elementary functions

More details:

$f \in \mathcal{V}$ arbitrary, $\phi^{(n)}$ elementary, $\phi^{(n)} \xrightarrow{L^2([0,T]\times\Omega)} f$

$\int_0^T \phi_t^{(n)}\, dB_t$ well defined and in $L^2(\Omega)$.

Try exercise 3.1.7 in textbook

$$\mathbb{E}\left[\left(\int_0^T \phi_t^{(n)}\, dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T \phi_t^{(n)2}\, dt\right]$$

$$\Rightarrow \quad \mathbb{E}\left[\left(\int_0^T \left(\phi_t^{(n)} - \phi_t^{(m)}\right) dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T \left(\phi_t^{(n)} - \phi_t^{(m)}\right)^2 dt\right]$$

$\Rightarrow$ $\int_0^T \phi_t^{(n)}\, dB_t$ is Cauchy, $L^2(\Omega)$ is complete, thus it converges to something in $L^2(\Omega)$

the thing it converges to we will call $\int_0^T f_t\, dB_t$

Stochastic Analysis

Lecture 3

Re: Class Test

40 minutes

5 questions: 4 marks per correct

-1 mark per incorrect

0 for "I don't know".

Cauchy Sequence: $\{x_n\}_{n=1}^{\infty}$ is Cauchy if

$$\|x_n - x_m\| \to 0 \text{ as } n,m \to \infty.$$

Complete space: Let U be a set (with some notion of distance). Then U is complete if all Cauchy sequences converge to a point in U.

Example of a space which is not complete: $\mathbb{Q}$

Our important examples of complete spaces: $\mathbb{R}$, most L^2

$L^2([0,T] \times \Omega)$

Re: L^p spaces, L^p norms.

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Consider all r.v. such that $\mathbb{E}[X^2] < \infty$.

Call this set $L^2(\Omega, \mathcal{F}, \mathbb{P})$ (sometimes shorten to $L^2(\Omega)$ or $L^2(\mathbb{P})$)

There is a norm on this space: let $X \in L^2(\Omega)$

then $\|X\|_{L^2} = \mathbb{E}[X^2]^{1/2}$

another L^2 space we have seen often:

Lebesgue measure

$$L^2\left([0,T]\times\Omega,\ \mathcal{B}\otimes\mathcal{F},\ \ell\times\mathbb{P}\right)$$

an element of $L^2([0,T]\times\Omega)$ is a stochastic process with the condition: let $X\in L^2([0,T]\times\Omega)$

then: $\mathbb{E}\left[\int_0^T X_t^2\,dt\right] < \infty$.

$$\|X\|_{L^2([0,T]\times\Omega)} = \mathbb{E}\left[\int_0^T X_t^2\,dt\right]^{1/2}$$

remember: $\int_0^T f_t \, dB_t$ is a random variable.

this r.v. is an element of $L^2(\Omega)$, therefore it has norm: $\mathbb{E}\left[\left(\int_0^T f_t \, dB_t\right)^2\right]^{1/2}$ ← square root is taken after $\mathbb{E}$.

the object f is an element $L^2([0,T]\times\Omega)$ and has norm: $\mathbb{E}\left[\int_0^T f_t^2 \, dt\right]^{1/2}$

This Lecture

- The Ito Isometry
- Properties of the Ito integral
- Martingales
- Continuity of Ito integral
- Ito integrals and martingales

The Ito Isometry

- From the Ito isometry for elementary functions and from the definition of the Ito integral we get

Theorem

For all $f \in \mathcal{V}(0,T)$ *we have*

$$\mathbb{E}\left[\left(\int_0^T f_t\, dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T f_t^2\, dt\right]$$

- Further, this gives

Theorem

If $f \in \mathcal{V}(0,T)$ *and* $f^{(n)} \in \mathcal{V}(0,T)$ *such that* $f^{(n)} \to f$ *in* $L^2(\ell \times \mathbb{P})$, *then*

$$\lim_{n\to\infty} \int_0^T f_t^{(n)}\, dB_t = \int_0^T f_t\, dB_t \quad \textit{in } L^2(\mathbb{P})$$

want to prove: $$\mathbb{E}\left[\left(\int_0^T f_t\, dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T f_t^2\, dt\right]$$ for $f \in \mathcal{V}$.

from last week: $$\mathbb{E}\left[\left(\int_0^T \phi_t^{(n)}\, dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T (\phi_t^{(n)})^2\, dt\right]$$ for elementary $\phi^{(n)} \in \mathcal{V}$.

$$\Big\downarrow n \to \infty \qquad\qquad\qquad\qquad \Big\downarrow n \to \infty$$

$$\mathbb{E}\left[\left(\int_0^T f_t\, dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T f_t^2\, dt\right]$$

Properties of the Ito Integral

- Many basic properties of standard integrals extend to stochastic integrals
- Let $f, g \in \mathcal{V}$ and $0 \leq S < U < T$. Then
  1. $\int_S^T f_t \, dB_t = \int_S^U f_t \, dB_t + \int_U^T f_t \, dB_t$
  2. $\int_S^T (c \, f_t + g_t) \, dB_t = c \int_S^T f_t \, dB_t + \int_S^T g_t \, dB_t$ for $c \in \mathbb{R}$
  3. $\mathbb{E}[\int_S^T f_t \, dB_t] = 0$
  4. $\int_S^T f_t \, dB_t$ is $\mathcal{F}_T$-measurable

Martingales

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\{\mathcal{N}_t\}_{t\geq 0}$ be a filtration, and let $\{M_t\}_{t\geq 0}$ be a stochastic process
- We say M is a **martingale with respect to** $\mathcal{N}_t$ if the following hold:
  1. M_t is $\mathcal{N}_t$-measurable for all t (i.e. M is $\mathcal{N}_t$-adapted)
  2. $\mathbb{E}[|M_t|] < \infty$ for all t → different from $\mathbb{E}[M_s]$
  3. $\mathbb{E}[M_s|\mathcal{N}_t] = M_t$ for all $0 \leq t \leq s$
- Interpretation: given the path of a martingale up to time t, the expected value of the process in the future is M_t
- Example: let $\mathcal{F}_t$ be generated by $\{B_s\}_{s\leq t}$ where B is a Brownian motion
  - Then B is a martingale with respect to $\mathcal{F}_t$

Prove B is martingale wrt $\mathcal{F}_t$.

need: $\underline{\mathbb{E}[B_s / \mathcal{F}_t] = B_t}$ for $s \geq t$

$$\mathbb{E}[B_s / \mathcal{F}_t] = \mathbb{E}[B_s - B_t + B_t / \mathcal{F}_t]$$

$$= \mathbb{E}[B_s - B_t / \mathcal{F}_t] + \mathbb{E}[B_t / \mathcal{F}_t]$$

by independence of increments

$$= \mathbb{E}[B_s - B_t] + B_t$$

$$= 0 + B_t$$

Equality of Stochastic Processes

- There are different ways in which we may interpret two stochastic processes as being "the same"
- Let $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ be two stochastic processes (for either $t \in \mathbb{R}^+$ or $t \in [0, T]$)
- X is a **version of** Y (or a **modification of** Y) if

$$\mathbb{P}\{\omega : X_t(\omega) = Y_t(\omega)\} = 1\,, \quad \text{for all } t \geq 0$$

- X is **indistinguishable** from Y if

$$\mathbb{P}\{\omega : X_t(\omega) = Y_t(\omega) \text{ for all } t \geq 0\} = 1$$

Continuity of Ito Integral

- We will take the following theorem as given:

Theorem (Doob's martingale inequality)

If M_t is a martingale with continuous paths a.s. then for all $p \geq 1$, $T \geq 0$, and all $\lambda > 0$

$$\mathbb{P}\left[\sup_{0\leq t\leq T} |M_t| \geq \lambda\right] \leq \frac{1}{\lambda^p}\mathbb{E}[|M_T|^p]$$

- Doob's martingale inequality will be used to prove:

Theorem

almost surely

Let $f \in \mathcal{V}(0,T)$. Then there exists a t-continuous version of

$$\int_0^t f_s \, dB_s$$

Recall: Properties of Conditional Expectation

- Let $\{B_t\}_{t\in[0,T]}$ be a Brownian motion and $\{\mathcal{F}_t\}_{t\in[0,T]}$ the filtration it generates
- Let $X \in \mathcal{V}(0,T)$ and Y be a random variable

1. $\mathbb{E}[\mathbb{E}[X_s|\mathcal{F}_t]] = \mathbb{E}[X_s]$ for any $s,t \in [0,T]$
2. $\mathbb{E}[X_t|\mathcal{F}_t] = X_t$
3. $\mathbb{E}[Y|\mathcal{F}_t] = \mathbb{E}[Y]$ if Y is independent of $\mathcal{F}_t$
4. $\mathbb{E}[X_sY|\mathcal{F}_t] = X_s\mathbb{E}[Y|\mathcal{F}_t]$ for all $s \le t$
5. $\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[Y|\mathcal{F}_s]$ for all $s \le t$

$$\mathbb{E}\left[\mathbb{E}[Y|\mathcal{F}_s] \middle| \mathcal{F}_t\right] = \mathbb{E}[Y|\mathcal{F}_s]$$

remember: smaller filtration always wins.

- **There is an a.s. t-continuous version of $\int_0^t f_u \, dB_u$, $f \in \mathcal{V}$.** ← t is not fixed.

Proof: denote $I_t = \int_0^t f_u \, dB_u$.

Let $\phi^{(n)}$ be elementary, $\phi^{(n)} \in \mathcal{V}$, $\phi^{(n)} \to f$ in $L^2([0,T] \times \Omega)$

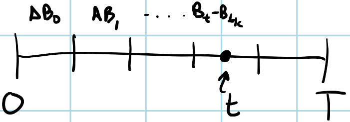
$$\left(\lim_{n \to \infty} E\left[\int_0^T (\phi_t^{(n)} - f_t)^2 \, dt \right] \to 0 \right)$$

denote $I_t^{(n)} = \int_0^t \phi_u^{(n)} \, dB_u$

1) $I_\bullet^{(n)}(\omega)$ is continuous for all ω, and any n — for different values of t

2) $I^{(n)}$ is a martingale wrt. $\{\mathcal{F}_t\}_{t \geq 0}$ for any n.

1) $I_t^{(n)} = \int_0^t \phi_u^{(n)} \, dB_u$



$$= \sum_{t_k \leq t_{k+1} \leq t} e_k^{(n)} \Delta B_{t_k}$$

$$= e_0^{(n)} \Delta B_{t_0} + e_1^{(n)} \Delta B_{t_1} + \cdots + e_{t_k}^{(n)} \Delta B_{t_k} \quad \text{for } t \in [t_k, t_{k+1})$$

ΔB_{t_k}: $B_t - B_{t_k}$

$\Longrightarrow I_\bullet^{(n)}(\omega)$ is continuous for all ω.

2) show $I^{(n)}$ is a martingale wrt $\mathcal{F}_t$

let $s \geq t$ $\mathbb{E}\left[I_s^{(n)} \mid \mathcal{F}_t\right] = \mathbb{E}\left[\int_0^s \phi_u^{(n)} \, dB_u \,\middle|\, \mathcal{F}_t\right]$

$$= \mathbb{E}\left[\int_0^t \phi_u^{(n)} \, dB_u + \int_t^s \phi_u^{(n)} \, dB_u \,\middle|\, \mathcal{F}_t\right]$$

$$= \int_0^t \phi_u^{(n)} \, dB_u + \mathbb{E}\left[\sum_{t \le t_k \le t_{k+1} \le s} e_k^{(n)} \Delta B_{t_k} \,\middle|\, \mathcal{F}_t \right]$$

$$= \int_0^t \phi_u^{(n)} \, dB_u + \sum_{t \le t_k \le t_{k+1} \le s} \mathbb{E}\left[e_k^{(n)} \Delta B_{t_k} \,\middle|\, \mathcal{F}_t \right]$$

$$= I_t^{(n)} + \sum_{t \le t_k \le t_{k+1} \le s} \mathbb{E}\left[\mathbb{E}\left[e_k^{(n)} \Delta B_{t_k} \,\middle|\, \mathcal{F}_{t_k} \right] \,\middle|\, \mathcal{F}_t \right]$$

$$= I_t^{(n)} + \sum_{t \le t_k \le t_{k+1} \le s} \mathbb{E}\left[e_k^{(n)} \, \underbrace{\mathbb{E}\left[\Delta B_{t_k} \,\middle|\, \mathcal{F}_{t_k} \right]}_{0} \,\middle|\, \mathcal{F}_t \right] \qquad \Delta B_{t_k} = B_{t_{k+1}} - B_{t_k}$$

$$= I_t^{(n)}$$

$\Rightarrow I^{(n)}$ is a continuous martingale.

$\Rightarrow$ $I^{(n)} - I^{(m)}$ is also a cont. mart.

$$\Rightarrow \quad \mathbb{P}\Big(\sup_{0 \le t \le T} \big| I_t^{(n)} - I_t^{(m)} \big| > \varepsilon \Big) \le \frac{1}{\varepsilon^2} \mathbb{E}\Big[\big(I_T^{(n)} - I_T^{(m)} \big)^2 \Big]$$

$$= \frac{1}{\varepsilon^2} \mathbb{E}\Big[\int_0^T \big(\phi_t^{(n)} - \phi_t^{(m)} \big)^2 dt \Big]$$

$$\to 0 \quad \text{as } n, m \to \infty.$$

there is a subsequence n_k such that

$$\mathbb{P}\Big(\sup_{0 \le t \le T} \big| I_t^{(n_{k+1})} - I_t^{(n_k)} \big| > 2^{-k} \Big) \le 2^{-k} \quad \text{for all } k.$$

$\sum 2^{-k} < \infty \to$ Borel-Cantelli lemma applies.

$$\mathbb{P}\left(\sup_{0 \le t \le T} \left| I_t^{(n_{k+1})} - I_t^{(n_k)} \right| > 2^{-k} \text{ for infinitely many } k \right) = 0.$$

$\to$ for a.e. ω there exists $k^*(\omega)$ where

$$\sup_{0 \le t \le T} \left| I_t^{(n_{k+1})}(\omega) - I_t^{(n_k)}(\omega) \right| \le 2^{-k} \quad \text{for } k \ge k^*(\omega)$$

$\Rightarrow$ $I_\bullet^{(n_k)}(\omega)$ converges uniformly for a.e. ω.

let $J_t(\omega) = \lim_{k \to \infty} I_t^{(n_k)}(\omega) \Rightarrow J_\bullet(\omega)$ continuous for a.e. ω.

but $I_t^{(n_k)} \longrightarrow I_t$ for each t in L^2 convergence.

$\Rightarrow$ $I_t = J_t$ a.s. for each t.

Ito Integral is Martingale

Theorem

Let $f \in \mathcal{V}(0,T)$ *and let*

$$M_t = \int_0^t f_s \, dB_s$$

for $t \in [0,T]$*. Then* M *is a martingale with respect to* $\mathcal{F}_t$

- Follows from parts of the proof of previous result
- From now on, we will always assume that we are working with a continuous version of the Ito integral

Stochastic Analysis

Lecture 4

This Lecture

- Ito Processes
- Ito's Lemma / Ito's Formula
- Multidimensional Ito's Formula

Computing Ito Integrals

- Much like standard calculus, computing an Ito integral directly from the definition is usually difficult
- Unlike standard calculus, Ito calculus does not benefit from tools of differentiation in the standard sense
- Ito's Formula is a result which will greatly assist in computing some Ito integrals, as well as carry out other important computations
- Ito's Formula can be thought of as a generalization of the chain rule from standard calculus

Ito Processes

- Let B be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- A stochastic process X is said to be an **Ito process** if it is of the form

$$X_t = X_0 + \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dB_s$$

Ito process

for some processes μ and σ which are $\mathcal{F}_t$-adapted

- To simplify notation, we will usually write the above in differential form:

$$dX_t = \mu_t\, dt + \sigma_t\, dB_t$$

Ito process

- The differential form makes no sense on its own, it is merely a placeholder for the properly defined integral form

Notation : letters.

Greek letter: coefficient in the differential form of a stochastic process. $(\alpha, \beta, \mu, \sigma, \eta)$

Capital Latin letter: stochastic process (X, Y, Z)

Lower case Latin letter: a standard (deterministic) function, possibly of multiple variables (f, g, h)

$f(t, x, y)$

Notation re: partial derivatives

Suppose $g(t,x)$ is given and $h(t)$ is given.

What does $\frac{\partial g}{\partial t}(t, h(t))$ mean?

Take the partial derivative wrt t of g, then substitute $(t, h(t))$ for (t,x) in the result

Ex. $g(t,x) = tx^2$ $\quad h(t) = t^4$

$$\frac{\partial g}{\partial t}(t, h(t)) = t^8$$

Correct

Let $f(t) = g(t, h(t))$

$= t^9$

$$\frac{\partial g}{\partial t}(t, h(t)) = 9t^8$$

Incorrect

Ito's Formula

- Sometimes called **Ito's Lemma**

Theorem (Ito's Formula)

Let X be an Ito process satisfying

$$dX_t = \mu_t \, dt + \sigma_t \, dB_t$$

and let $g \in C^2([0,T] \times \mathbb{R})$. Then $Y_t = g(t, X_t)$ is an Ito process with $g(t,x)$

$$dY_t = \left(\frac{\partial g}{\partial t}(t, X_t) + \mu_t \, \frac{\partial g}{\partial x}(t, X_t) + \frac{1}{2} \, \sigma_t^2 \, \frac{\partial^2 g}{\partial x^2}(t, X_t) \right) dt + \sigma_t \, \frac{\partial g}{\partial x}(t, X_t) \, dB_t$$

Partial derivatives now shown properly. Lecture capture video has incorrect notation.

- **Ito's Lemma:** $dX_t = \mu_t\,dt + \sigma_t\,dB_t, \qquad Y_t = g(t, X_t)$

prove:

$$Y_t = Y_0 + \int_0^t \underbrace{(\qquad)}_{\text{previous slide}} ds + \int_0^t \underbrace{(\qquad)}_{\text{previous slide}} dB_s$$

Proof:

$$g(t, X_t) = \overbrace{g(0, X_0)}^{\text{initial Point}} + \overbrace{\sum_k \Delta g_k}^{\text{all increments}} \qquad \Delta g_k = g(t_{k+1}, X_{t_{k+1}}) - g(t_k, X_{t_k})$$

Taylor Thm to approx. Δg_k

$$= g(0, X_0) + \sum_k \frac{\partial g}{\partial t}\Delta t_k + \frac{\partial g}{\partial x}\Delta X_k + \frac{1}{2}\left[\frac{\partial^2 g}{\partial t^2}(\Delta t_k)^2 + 2\frac{\partial^2 g}{\partial t\,\partial x}\Delta t_k \Delta X_k + \frac{\partial^2 g}{\partial x^2}(\Delta X_k)^2\right] + R_k$$

all derivatives evaluated at $(t, x) = (t_k, X_{t_k})$ $\qquad X_k = X_{t_k}$

where $R_k = o\left((\Delta t_k)^2 + (\Delta X_k)^2\right)$ $\qquad \Delta X_k = X_{t_{k+1}} - X_{t_k}$

i.e. R_k goes to 0 faster than RHS as Δt_k & $\Delta x_k \to 0$.

We may assume all derivatives are bounded, and that μ and σ are elementary functions.

Then as $\Delta t \to 0$ we have $\underline{\sum_k \frac{\partial g}{\partial t}(t_k, X_{t_k})\Delta t_k} \to \int_0^t \frac{\partial g}{\partial t}(s, X_s)\,ds$

and $\underline{\sum_k \frac{\partial g}{\partial x}(t_k, X_{t_k})\Delta X_k} \to \int_0^t \frac{\partial g}{\partial x}(s, X_s)\,dX_s$

Substituting dX $\longrightarrow$ $= \int_0^t \frac{\partial g}{\partial x}(s, X_s)\mu_s\,ds + \int_0^t \frac{\partial g}{\partial x}(s, X_s)\sigma_s\,dB_s$

Now for term,

$$\underline{\sum_k \frac{\partial^2 g}{\partial t \partial x}(t_k, X_{t_k})\Delta t_k(\mu_{t_k}\Delta t_k + \sigma_{t_k}\Delta B_k)}$$

$$= \sum_k (\quad)(\Delta t_k)^2 + \sum_k \frac{\partial^2 g}{\partial t \partial x}(t_k, X_{t_k})\sigma_{t_k}\Delta t_k \Delta B_k$$

but claim $\sum_k \frac{\partial^2 g}{\partial t \partial x}(t_k, X_{t_k})\, \sigma_{t_k}\, \Delta t_k\, \Delta B_k \xrightarrow{\Delta t \to 0} 0$ in $L^2(\Omega, \mathbb{P})$

$$\mathbb{E}\left[\left(\sum_k \frac{\partial^2 g}{\partial t \partial x}\, \sigma_k\, \Delta t_k\, \Delta B_k\right)^2\right] = \mathbb{E}\left[\sum_k \sum_j \left(\frac{\partial^2 g}{\partial t \partial x}\right)_k \left(\frac{\partial^2 g}{\partial t \partial x}\right)_j \sigma_k \sigma_j\, \Delta t_k\, \Delta t_j\, \Delta B_k\, \Delta B_j\right]$$

$$= \sum_k \mathbb{E}\left[\left(\frac{\partial^2 g}{\partial t \partial x}\right)^2 \sigma_k^2 (\Delta t_k)^2 (\Delta B_k)^2\right]$$

if $j \neq k$ the $\mathbb{E}$ of whole thing is 0 as ΔB_k & ΔB_j would be independent.

$$= \sum_k \mathbb{E}\left[\left(\frac{\partial^2 g}{\partial t \partial x}\right)^2 \sigma_k^2\right] (\Delta t_k)^3 \longrightarrow 0 \quad \text{as } \Delta t \to 0$$

Since ΔB_k is independent & $\mathbb{E}(\Delta B_k)^2 = \mathbb{E}(B_{k+1} - B_k)^2$

$= t_{k+1} - t_k$

$= \Delta t_k$

Last remaining term:

$$\underline{\sum_k \frac{\partial^2 g}{\partial x^2} (\Delta X_k)^2} = \sum_k \frac{\partial^2 g}{\partial x^2} (\mu_k \Delta t_k + \sigma_k \Delta B_k)^2$$

$$= \sum_k \frac{\partial^2 g}{\partial x^2} \left(\mu_k^2 (\Delta t_k)^2 + 2\mu_k \sigma_k \Delta t_k \Delta B_k + \sigma_k^2 (\Delta B_k)^2 \right)$$

$$\sum_k \frac{\partial^2 g}{\partial x^2} \left(\mu_k^2 (\Delta t_k)^2 + 2\mu_k \sigma_k \Delta t_k \Delta B_k \right) \to 0 \quad \text{in } L^2(\Omega, \mathbb{P})$$

claim $\sum_k \frac{\partial^2 g}{\partial x^2} \sigma_k^2 (\Delta B_k)^2 \xrightarrow{\Delta t \to 0} \int_0^t \frac{\partial^2 g}{\partial x^2}(s, X_s) \sigma_s^2 \, ds$ in $L^2(\Omega, \mathbb{P})$

consider $\sum_k \frac{\partial^2 g}{\partial x^2} \sigma_k^2 \Delta t_k$

Note all terms of taylor expansion goes to 0 except
— on first page giving us Ito's lemma.

$$\mathbb{E}\left[\left(\sum_k \frac{\partial^2 g}{\partial x^2}\sigma_k^2\left((\Delta B_k)^2-\Delta t_k\right)\right)^2\right] = \mathbb{E}\left[\sum_k \sum_j \alpha_k \alpha_j \sigma_k^2 \sigma_j^2 \left((\Delta B_k)^2-\Delta t_k\right)\left((\Delta B_j)^2-\Delta t_j\right)\right]$$

where $\alpha_k = \frac{\partial^2 g}{\partial x^2}\sigma_k^2$

$$\sum_k \mathbb{E}\left[\alpha_k^2 \sigma_k^4 \left((\Delta B_k)^2-\Delta t_k\right)^2\right] \longrightarrow 0 \quad \text{as } \Delta t \to 0$$

because $\mathbb{E}\left[(\Delta B_k)^4\right] = 3(\Delta t_k)^2$

$$\mathbb{E}\left[(\Delta B_k)^2 \Delta t_k\right] = (\Delta t_k)^2$$

$$\frac{\partial^2 g}{\partial t^2}(\Delta t_k)^2 \longrightarrow 0$$

Multidimensional Ito's Formula

Theorem (Multidimensional Ito's Formula)

Let X be an Ito process given by

$$dX_t = \mu_t\, dt + \sigma_t\, dB_t$$

where B is m-dimensional Brownian motion, μ takes values in $\mathbb{R}^n$, and σ takes values in $\mathbb{R}^{n\times m}$. Let $g \in C^2([0,T]\times\mathbb{R}^n, \mathbb{R}^d)$ and let $Y_t = g(t, X_t)$. Then Y is an Ito process with

$$dY_t^{(k)} = \frac{\partial g^{(k)}}{\partial t}(t, X_t)\, dt + \sum_{i=1}^{n} \frac{\partial g^{(k)}}{\partial x^{(i)}}\, dX_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 g^{(k)}}{\partial x^{(i)}\partial x^{(j)}}\, dX_t^{(i)}\, dX_t^{(j)}$$

where $dX_t^{(i)}\, dX_t^{(j)}$ is computed using the rules

$$dB_t^{(i)}\, dB_t^{(j)} = \delta_{i,j}\, dt\,, \qquad dB_t^{(i)}\, dt = dt\, dB_t^{(i)} = 0\,, \qquad dt^2 = 0$$

Partial derivatives now shown properly. Lecture capture video has incorrect notation.

Examples: $dX_t = \mu\, dt + \sigma\, dB_t$

μ, σ → constants

Compute $\cancel{d(tX_t^2)}$ let $Y_t = tX_t^2 \implies g(t,x) = tx^2$

$$\partial_t g = x^2$$
$$\partial_x g = 2tx$$
$$\partial_{xx} g = 2t$$

Incorrect way: $dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t$

$$= X_t^2 dt + 2tX_t \mu\, dt + 2tX_t \sigma\, dB_t$$

Correct Way: $dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t)^2$

$$= \left(X_t^2 + 2tX_t\mu + t\sigma^2\right)dt + 2tX_t\sigma\, dB_t$$

Ex.

compute $\int_0^T B_t \, dB_t$ Guess $\frac{B_T^2}{2}$

Let $X_t = B_t \implies dX_t = 0dt + 1dB_t$

$Y_t = \frac{B_t^2}{2} \implies g(t,x) = \frac{x^2}{2}$

$\partial_t g = 0$

$\partial_x g = x$

$\partial_{xx} g = 1$

$$dY_t = \cancel{\partial_t g \, dt} + \partial_x g \, dX_t + \frac{1}{2}\partial_{xx} g \, (dX_t)^2$$

$$= X_t\,(0dt + 1dB_t) + \frac{1}{2}(0dt + 1dB_t)^2$$

$$dY_t = B_t \, dB_t + \frac{1}{2} dt$$

$$\Rightarrow \quad Y_T = Y_0 + \frac{1}{2}\int_0^T dt + \int_0^T B_t \, dB_t$$

$$\Rightarrow \quad \int_0^T B_t \, dB_t = \frac{B_T^2}{2} - \frac{T}{2}$$

Ex.

compute: $\int_0^T t \sin(B_t) \, dB_t$ guess: $-T\cos(B_T)$

let $X_t = B_t$, $Y_t = -t\cos(B_t)$ $\Rightarrow$ $g(t,x) = -t\cos(x)$

$\partial_t g = -\cos(x)$

$\partial_x g = t\sin(x)$

$\partial_{xx} g = t\cos(x)$

$dX_t = 0\,dt + 1\,dB_t$

$$dY_t = \partial_t g\, dt + \partial_x g\, dX_t + \frac{1}{2}\partial_{xx} g\,(dX_t)^2$$

$$= -\cos(B_t)\,dt + t\sin(B_t)\,dB_t + \frac{1}{2} t\cos(B_t)\,dt$$

$$dY_t = \left(\frac{1}{2} t\cos(B_t) - \cos(B_t)\right)dt + t\sin(B_t)\,dB_t$$

$$Y_T = Y_0 + \int_0^T \frac{1}{2} t\cos(B_t) - \cos(B_t)\; dt + \int_0^T t\sin(B_t)\,dB_t$$

$$\int_0^T t\sin(B_t)\,dB_t = -T\cos(B_T) - \int_0^T \left(\frac{1}{2} t\cos(B_t) - \cos(B_t)\right)dt$$

Ex:

Let $Y_t = e^{\int_0^t B_s ds + \int_0^t B_s^2 dB_s}$

compute dY_t

Let $X_t = \int_0^t B_s ds + \int_0^t B_s^2 dB_s$

$$dX_t = B_t dt + B_t^2 dB_t$$

$$Y_t = e^{X_t} \Longrightarrow g(t,x) = e^x$$

$$dY_t = \partial_t g dt + \partial_x g dX_t + \frac{1}{2} \partial_{xx} g (dX_t)^2$$

$$= e^{X_t}(B_t dt + B_t^2 dB_t) + \frac{1}{2} e^{X_t} B_t^4 dt$$

$$dY_t = \left(e^{X_t} B_t + \frac{1}{2} e^{X_t} B_t^4\right) dt + e^{X_t} B_t^2 dB_t$$

Ex:

$$dX_t = B_t\,dt + dB_t$$

$$dY_t = t\,dt + A_t\,dA_t$$

A, B are BM. independent

let $Z_t = tX_t^9 \sin(Y_t) \implies g(t,x,y) = tx^9\sin(y)$

$$\partial_t g = x^9 \sin(y)$$
$$\partial_x g = 9tx^8 \sin(y)$$
$$\partial_y g = tx^9 \cos(y)$$
$$\partial_{xx} g = 72tx^7 \sin(y)$$
$$\partial_{yy} g = -tx^9 \sin(y)$$
$$\partial_{xy} g = 9tx^8 \cos(y)$$

$$dZ_t = \partial_t g\, dt + \partial_x g\, dX_t + \partial_y g\, dY_t + \frac{1}{2}\partial_{xx} g (dX_t)^2 + \partial_{xy} g\, dX_t\, dY_t + \frac{1}{2}\partial_{yy} g (dY_t)^2$$

$$dZ_t = X_t^9 \sin(Y_t)\,dt + 9tX_t^8 \sin(Y_t)\left(B_t\,dt + dB_t\right) + tX_t^9\cos(Y_t)\left(t\,dt + A_t\,dA_t\right) + \frac{1}{2}\,72tX_t^7\sin(Y_t)\,dt + 9tX_t^8\cos(Y_t)\cdot 0 + \frac{1}{2}\left(-tX_t^9\sin(Y_t)A_t^2\,dt\right)$$

This factor of $\frac{1}{2}$ is missing in the lecture capture video

Stochastic Analysis

Lecture 5

Module Evaluation

- Facilities and lecturer:
    - One person has "no strong opinion" regarding classroom facilities
    - Unanimous "agreement" with everything else
- The pace of lectures:
    - Four selected "a bit fast," two selected "a bit slow," and the rest selected "about right"
- Other comments:
    - Regarding posting of solutions to exercises and mock exams

Currently half way through Ch. 4 of $\emptyset$

Next week will cover ch. 5 of $\emptyset$

Ch. 6 skipped fully

- Girsanov's Theorem
- Feynman-Kac formula
- Stochastic control and the HJB equation

This Lecture

- The Ito Representation Theorem
- The Martingale Representation Theorem

Ito Integrals and Martingales

- Let $\{B_t\}_{t\in[0,T]}$ be a Brownian motion and let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be the filtration generated by $\{B_t\}_{t\in[0,T]}$
- Let $f \in \mathcal{V}(0,T)$ and let $\{M_t\}_{t\in[0,T]}$ be given by

$$M_t = M_0 + \int_0^t f_u \, dB_u$$

- Then $\{M_t\}_{t\in[0,T]}$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$
- There is a modification of $\{M_t\}_{t\in[0,T]}$ which is t-continuous

Martingale Representation Theorem - Main Idea

- The converse is also true
- Any martingale with respect to the filtration generated by a Brownian motion can be written as an Ito integral
- Proving this will require some preliminary technical results about approximating $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ random variables
- Main result then follows from proving it for approximations and using limiting arguments

First Approximation Result

ϕ is smooth & think of ϕ as f'n of vector x & when vector is large enough $\phi(x) = 0$.

Lemma

Fix $T > 0$, let $\{B_t\}_{t\in[0,T]}$ be a Brownian motion and let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be its generated filtration. The set of random variables

$$\{\underbrace{\phi(B_{t_1},\dots,B_{t_k})}_{\text{random variable}} : t_i \in [0,T],\ \phi \in C_0^\infty(\mathbb{R}^k),\ k = 1,2,\dots\} \qquad (*)$$

compact support

is dense in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

- This means that for any $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ there is a sequence of random variables X_n in the set $(*)$ such that

$$X_n \xrightarrow{L^2} X$$

- The proof is a collection of other approximation and limit results

A Useful Lemma

- We will use the following result in our proof:

Lemma

$\mathcal{H}_n \subset \mathcal{H}_{n+1}$

Let $\mathcal{H}_n \subset \mathcal{F}$ be an increasing sequence of σ-algebras, and let $\mathcal{H}$ be the smallest σ-algebra containing each $\mathcal{H}_n$ ($\mathcal{H}$ is generated by $\{\mathcal{H}_n\}_{n=1}^{\infty}$). Let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and define

$$Y_n = \mathbb{E}[X|\mathcal{H}_n]\,,$$
$$Y = \mathbb{E}[X|\mathcal{H}]\,.$$

Then $\lim_{n\to\infty} Y_n = Y$ where this limit is taken both in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and almost surely.

proof is in ϕ Appendix C.

$$\left\{ \phi(B_{t_1}, \dots, B_{t_k}) : t_i \in [0,T],\ \phi \in C_0^\infty(\mathbb{R}^k),\ k = 1, 2, \dots \right\}$$

is dense in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

proof:

let $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Let $\{t_i\}_{i=1}^\infty$ be a dense subset of $[0,T]$. Define $\mathcal{H}_n$ as the σ-algebra generated by $\{B_{t_1}, \dots, B_{t_n}\}$. Then $\mathcal{H}_n \subset \mathcal{H}_{n+1}$. Also, $\mathcal{F}_T$ is the smallest σ-algebra containing each $\mathcal{H}_n$.

Define $Y_n = \mathbb{E}[X \mid \mathcal{H}_n]$ and note $\mathbb{E}[X \mid \mathcal{F}_T] = X$.

Then $Y_n \to X$ in $L^2(\Omega)$ and a.s.

But $\mathbb{E}[X \mid \mathcal{H}_n] = g_n(B_{t_1}, \dots, B_{t_n})$ $\quad$ g_n is a Borel function.

let $U \subset \mathbb{R}$ be a Borel set.

Then $g_n^{-1}(U)$ is measurable

A Borel function g_n can be approximated in L^2 by smooth compactly supported functions.

$\Rightarrow$ X can be approximated by $\phi(B_{t_1}, \dots, B_{t_k})$ with $\phi \in C_0^\infty(\mathbb{R}^k)$

Second Approximation Result

(Stochastic exponential $\mathcal{E}$)

Lemma

The linear span of random variables of the form

$$\exp\left\{\int_0^T h_t\,dB_t - \frac{1}{2}\int_0^T h_t^2\,dt\right\} \qquad (*)$$

with $h \in L^2([0,T])$ *deterministic is dense in* $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$*.*

- Idea of proof is to show that if X is orthogonal to all functions of the form $(*)$ then it is orthogonal to a dense subset of $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

$$\exp\left\{\int_0^T h_t\,dB_t - \frac{1}{2}\int_0^T h_t^2\,dt\right\} \qquad h \in L^2[0,T] \qquad (*)$$

<u>proof:</u> Let $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Suppose X is orthogonal to all r.v.'s of the form $(*)$.

inner product in L^2 space $=0$ if orthogonal.

$$\Rightarrow \mathbb{E}\left[X \exp\left\{\int_0^T h_t\,dB_t - \frac{1}{2}\int_0^T h_t^2\,dt\right\}\right] = 0.$$

$$\Rightarrow \mathbb{E}\left[e^{\lambda_1 B_{t_1} + \cdots + \lambda_n B_{t_n}} X\right] = 0 \quad \text{for any } \lambda \in \mathbb{R}^n$$

define $G(\lambda) = \mathbb{E}\left[e^{\lambda_1 B_{t_1} + \cdots + \lambda_n B_{t_n}} X\right]$.

Then G is a real <u>analytic</u> function.

$\hookrightarrow$ G can be written as a power series.

$\Rightarrow$ G can be extended analytically to complex space:

$G : \mathbb{C}^n \to \mathbb{C}$.

But $G(z) = 0$ for $z \in \mathbb{R}^n \Rightarrow G(z) = 0$ for $z \in \mathbb{C}^n$

In particular: $\mathbb{E}\left[e^{i(y_1 B_{t_1} + \cdots + y_n B_{t_n})} X\right] = 0$.

Now, let $\phi \in C_0^\infty(\mathbb{R}^n)$, consider

$$\mathbb{E}\left[\phi(B_{t_1}, \ldots, B_{t_n}) X\right] = \int_\Omega \phi(B_{t_1}(\omega), \ldots, B_{t_n}(\omega)) X(\omega) \, dP(\omega).$$

taking inner product to check orthogonality

Fourier inversion theorem:

$$\hat{\phi}(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\vec{y}\cdot\vec{x}} \phi(\vec{x}) d\vec{x}$$

then

$$\phi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\vec{y}\cdot\vec{x}} \hat{\phi}(\vec{y}) d\vec{y}$$

$$\mathbb{E}\left[\phi(B_{t_1},\dots,B_{t_n})X\right] = \int_\Omega (2\pi)^{-n/2}\int_{\mathbb{R}^n} \hat{\phi}(y) e^{i(y_1 B_{t_1}(\omega)+\dots+y_n B_{t_n}(\omega))} dy \, X(\omega) \, d\mathbb{P}(\omega)$$

$$= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \hat{\phi}(y) \int_\Omega e^{i(y_1 B_{t_1}(\omega)+\dots+y_n B_{t_n}(\omega))} X(\omega)\, d\mathbb{P}(\omega)\, dy$$

$$= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \hat{\phi}(y)\, \underbrace{\mathbb{E}\left[e^{i(y_1 B_{t_1}+\dots+y_n B_{t_n})}X\right]}_{=0 \text{ shown previously}} dy$$

$$= 0$$

$$\Rightarrow X \perp \phi(B_{t_1},\dots,B_{t_n})$$

$$\Rightarrow X = 0$$

$\Rightarrow$ X is orthogonal to ϕ which are dense in L^2.

Ito's Representation Theorem for $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ Random Variables

Theorem

Let $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Then there exists a unique stochastic process $f \in \mathcal{V}(0, T)$ such that

$$X = \mathbb{E}[X] + \int_0^T f_t \, dB_t$$

for $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, there exists unique $f \in \mathcal{V}(0,T)$ such that

$$X = \mathbb{E}[X] + \int_0^T f_t \, dB_t$$

<u>proof:</u> Suppose $X = \exp\left\{\int_0^T h_t \, dB_t - \frac{1}{2}\int_0^T h_t^2 \, dt\right\}$ ✱ h deterministic.

Let $Y_t = \exp\left\{\int_0^t h_u \, dB_u - \frac{1}{2}\int_0^t h_u^2 \, du\right\}$

using Itô's lemma calculation.

$$dY_t = h_t Y_t \, dB_t$$

$$\Rightarrow \quad Y_T = Y_0 + \int_0^T h_t Y_t \, dB_t$$

$$X = 1 + \int_0^T f_t \, dB_t$$

$f_t = h_t Y_t$

Let $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ be arbitrary. Let X_n be a linear combination of r.v.'s of the form ✱ such that

$$X_n \xrightarrow{L^2(\Omega)} X.$$

$$X_n = \mathbb{E}[X_n] + \int_0^T f_t^{(n)}\, dB_t$$ because stochastic integration is linear.

consider

$$\underbrace{\mathbb{E}\left[(X_n - X_m)^2\right]}_{\substack{\downarrow \\ 0 \\ \text{as } n,m\to\infty}} = \mathbb{E}\left[\left(\mathbb{E}[X_n] + \int_0^T f_t^{(n)}\, dB_t - E[X_m] - \int_0^T f_t^{(m)}\, dB_t\right)^2\right]$$

$$= \mathbb{E}\Big[\left(\mathbb{E}[X_n - X_m]\right)^2 + \left(\int_0^T f_t^{(n)} - f_t^{(m)}\, dB_t\right)^2 + 2\mathbb{E}[X_n - X_m]\int_0^T f_t^{(n)} - f_t^{(m)}\, dB_t\Big]$$

know this by construction of sequence X_n

$$= \underbrace{\left(\mathbb{E}[X_n - X_m]\right)^2}_{\substack{\downarrow \\ 0 \\ \text{as } n,m\to\infty}} + \mathbb{E}\left[\left(\int_0^T f_t^{(n)} - f_t^{(m)}\, dB_t\right)^2\right]$$

$$\Longrightarrow \mathbb{E}\left[\left(\int_0^T f_t^{(n)} - f_t^{(m)}\, dB_t\right)^2\right] \longrightarrow 0 \quad \text{as } n,m \to \infty.$$

$$\mathbb{E}\left[\int_0^T \left(f_t^{(n)} - f_t^{(m)}\right)^2 dt\right] \longrightarrow 0 \quad \text{as } n,m \to \infty.$$

$\Longrightarrow$ $f^{(n)}$ is Cauchy in $L^2([0,T]\times\Omega)$

$\Longrightarrow$ $f^{(n)} \longrightarrow f$ in $L^2([0,T]\times\Omega)$.

$\Longrightarrow$ there is a subsequence which converges for a.e. (t,ω)

$\Longrightarrow$ f can be altered on a set of t-measure 0, so that it is adapted to $\mathcal{F}_t$. $\Longrightarrow$ $f \in \mathcal{V}(0,T)$.

$$X_n = \mathbb{E}[X_n] + \int_0^T f_t^{(n)}\, dB_t$$

$$\downarrow \qquad\quad \downarrow \qquad\qquad \downarrow$$

$$X = \mathbb{E}[X] + \int_0^T f_t\, dB_t$$ shows existence.

suppose

$$\cancel{\mathbb{E}[X]} + \int_0^T f_t\, dB_t = \cancel{\mathbb{E}[X]} + \int_0^T g_t\, dB_t$$

$$\mathbb{E}\left[\left(\int_0^T f_t - g_t\, dB_t\right)^2\right] = 0$$

$$\mathbb{E}\left[\int_0^T (f_t - g_t)^2\, dt\right] = 0 \implies f_t = g_t \text{ a.e. } (t,\omega)$$

Martingale Representation Theorem

Theorem

Let $\{M_t\}_{t\geq 0}$ be a martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ with $\mathbb{E}[M_t^2] < \infty$ for all $t \geq 0$. Then there exists a unique f such that $f \in \mathcal{V}(0,t)$ for all $t \geq 0$ and

$$M_t = M_0 + \int_0^t f_u \, dB_u$$

almost surely, for all $t \geq 0$.

proof:

Apply Ito Representation for fixed t to M_t.

$$M_t = \mathbb{E}[M_t] + \int_0^t f_u^{(t)}\, dB_u$$

now use martingale definition: let $t \geq s$

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$$

$$\mathbb{E}\left[\mathbb{E}[M_t] + \int_0^t f_u^{(t)}\, dB_u \,\Big|\, \mathcal{F}_s\right] = \mathbb{E}[M_s] + \int_0^s f_u^{(s)}\, dB_u$$

$$\cancel{\mathbb{E}[M_0]} + \mathbb{E}\left[\int_0^t f_u^{(t)}\, dB_u \,\Big|\, \mathcal{F}_s\right] = \cancel{\mathbb{E}[M_0]} + \int_0^s f_u^{(s)}\, dB_u$$

Since M is Martingale.

$$\int_0^s f_u^{(t)}\, dB_u = \int_0^s f_u^{(s)}\, dB_u$$

$$\implies f_u^{(t)} = f_u^{(s)} \quad \text{for all } u \leq s \leq t \quad \text{a.e. } \omega.$$

Define: $f_t = f_t^{(N)}$ where $N = \lceil t \rceil$

$$\Rightarrow \quad M_t = E[M_0] + \int_0^t f_u^{(t)} \, dB_u = M_0 + \int_0^t f_u^{(N)} \, dB_u$$

$$\boxed{M_t = M_0 + \int_0^t f_u \, dB_u}$$

uniqueness by Itô's isometry.

Stochastic Analysis

Lecture 6

→ Class Test Revision.

$$\phi_t(\omega) = \sum_j e_j(\omega)\, \chi_{[t_j, t_{j+1})} \qquad \phi_t(\omega) = Z\, \chi_{[t_j, t_{j+1})}$$

$$\mathbb{V}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] \quad \text{if } \mathbb{E}[X] = 0$$

$$= \mathbb{E}\left[\left(\int_0^T B_t\, dB_t\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^T B_t^2\, dt\right]$$

$$= \int_0^T \mathbb{E}[B_t^2]\, dt$$

$$= \int_0^T t\, dt$$

$$= \frac{T^2}{2}$$

This Lecture

- Stochastic Differential Equations
  - Examples and solutions
  - Existence and uniqueness of solutions

Stochastic Differential Equations

- A **stochastic differential equation** (SDE) is one of the form:

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t \qquad (*)$$

- Recall that this equation doesn't mean anything on its own. It simply stands for the integral equation:

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s \qquad (**)$$

- Terms in the integral form are rigourously defined, but often the differential form is more intuitive and easier to work with computationally

- When we speak of a **solution** to $(*)$ we mean a stochastic process which satisfies $(**)$

Solutions to SDEs

- Much like ODEs and PDEs, the solution of an SDE can usually be checked easily
- Also like ODEs and PDEs, it is often difficult to find the solution to an SDE
- There is no single method or algorithm which will produce solutions to any SDE
- Some methods are essentially clever manipulations of the SDE, others are basically guess-and-check with some guidance from experience

Checking a Solution - Examples

- Verify that the process $X_t = e^{B_t}$ is a solution to:

$$dX_t = \frac{1}{2} X_t \, dt + X_t \, dB_t$$

- Verify that the two dimensional process

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \cos(B_t) \\ \sin(B_t) \end{bmatrix}$$

$X_t = \cos(B_t)$

$Y_t = \sin(B_t)$

is a solution to:

$$dX_t = -\frac{1}{2} X_t \, dt - Y_t \, dB_t$$
$$dY_t = -\frac{1}{2} Y_t \, dt + X_t \, dB_t$$

Ex.1

$$dX_t = \frac{1}{2} X_t\, dt + X_t\, dB_t \quad , \quad X_t = e^{B_t}$$

$Y = B_t$

$f(t,y) = e^{y}$

$dY_t = dB_t$

$$dX_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial y} dY_t + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} (dY_t)^2$$

using Ito lemma

$$dX_t = 0\, dt + e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt$$

$$dX_t = \frac{1}{2} X_t\, dt + X_t\, dB_t$$

$$X_t = \cos(B_t) \quad , \quad Y_t = \sin(B_t)$$

using Ito lemma

$$dX_t = -\sin(B_t)\, dB_t + \frac{1}{2}\left(-\cos(B_t)\right) dt = -Y_t\, dB_t - \frac{1}{2} X_t\, dt$$

$$dY_t = \cos(B_t) dB_t + \frac{1}{2}\left(-\sin(B_t)\right) dt = X_t\, dB_t - \frac{1}{2} Y_t\, dt$$

Finding a Solution - Examples

- Find a solution to the following SDEs:

  i) $dX_t = r\,dt + \alpha\,dB_t$ $X_0 = x$

  ii) $dX_t = \mu\,X_t\,dt + \sigma\,X_t\,dB_t$ $X_0 = x$ $x \in \mathbb{R}$

  iii) $dX_t = \kappa(\theta - X_t)\,dt + \eta\,dB_t$ $X_0 = x$

i) $dX_t = r\,dt + \alpha\,dB_t$ $\qquad X_0 = x$

$$X_t = X_0 + \int_0^t r\,du + \int_0^t \alpha\,dB_u$$

$$X_t = x + rt + \alpha(B_t - B_0)$$

---

ii) $dX_t = \mu X_t\,dt + \sigma X_t\,dB_t$

"$\mu(t,x) = \mu x$"

"$\sigma(t,x) = \sigma x$"

$$\frac{dX_t}{X_t} = \mu dt + \sigma dB_t$$

$$d\log(X_t) = \mu\,dt + \sigma\,dB_t$$

$$\log(X_t) = \log(x) + \mu t + \sigma(B_t - B_0)$$

$$\cancel{X_t = x\,e^{\mu t + \sigma(B_t - B_0)}}$$

$Y_t = f(X_t)$

$dY_t = \cdots\cdots$

$d\log(X_t)$ $\quad dX_t^2$ are confusing.

$Y_t = \log(X_t)$, compute dY_t

$$dY_t = \frac{1}{X_t}\,dX_t - \frac{1}{2}\frac{1}{X_t^2}(dX_t)^2 = \cdots\cdots$$

Can be done via Ito lemma on $Y_t = \log(X_t)$

---

ii) done properly.

Define $Y_t = X_t e^{-\sigma B_t}$

S.t. we only have dt term

get rid of diffusion term

$$dY_t = e^{-\sigma B_t}\,dX_t - \sigma e^{-\sigma B_t} X_t\,dB_t + \frac{1}{2}\sigma^2 e^{-\sigma B_t} X_t (dB_t)^2$$
$$- \sigma e^{-\sigma B_t}\,dX_t\,dB_t$$

Applying Multivariate Ito on X_t & B_t.

$$dY_t = e^{-\sigma B_t}(\mu X_t\,dt + \sigma X_t\,dB_t) - \sigma e^{-\sigma B_t} X_t\,dB_t + \frac{1}{2}\sigma^2 e^{-\sigma B_t} X_t\,dt$$

$$- \sigma e^{-\sigma B_t} \sigma X_t\,dt$$

$$dY_t = \mu e^{-\sigma B_t} X_t\,dt - \frac{1}{2}\sigma^2 e^{-\sigma B_t} X_t\,dt$$

$$dY_t = \left(\mu - \frac{1}{2}\sigma^2\right) Y_t\,dt$$ note this is an ODE.

$$Y_t = Y_0\, e^{(\mu - \frac{1}{2}\sigma^2)t}$$

$$e^{-\sigma B_t} X_t = e^{-\sigma B_0} x\, e^{(\mu - \frac{1}{2}\sigma^2)t}$$

$$\boxed{X_t = x\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma(B_t - B_0)}}$$

iii) $dX_t = \kappa(\theta - X_t)\,dt + \eta\, dB_t \quad , \quad X_0 = x$

define $Y_t = e^{\kappa t} X_t$ ← (s.t. we contract $(-\kappa X_t dt)$ part which is like exponential decay with expo. growth.)

$$dY_t = \kappa e^{\kappa t} X_t\, dt + e^{\kappa t} dX_t$$ ← Applying Ito lemma

$$= \kappa e^{\kappa t} X_t\, dt + e^{\kappa t}\left(\kappa(\theta - X_t)\,dt + \eta\, dB_t\right)$$

$$dX_t = \kappa\theta e^{\kappa t}\, dt + \eta e^{\kappa t}\, dB_t$$

$$Y_t = Y_0 + \kappa\theta \int_0^t e^{\kappa u}\, du + \eta \int_0^t e^{\kappa u}\, dB_u$$

$$e^{\kappa t} X_t = x + \theta\left(e^{\kappa t} - 1\right) + \eta \int_0^t e^{\kappa u}\, dB_u$$

$$X_t = e^{-\kappa t} x + \theta\left(1 - e^{-\kappa t}\right) + \eta \int_0^t e^{-\kappa(t-u)}\, dB_u$$

Existence and Uniqueness Theorem - Comments

- Some ODEs and PDEs do not have solutions, or solutions may exist only locally, or solutions may not be unique
- The same issues can occur with SDEs
- Similarly to ODEs and PDEs, we may specify conditions on an SDE which guarantees that a solution exists and is unique
- Knowing that an SDE has a solution which is unique is beneficial from a modeling perspective and for applying numerical methods

Existence and Uniqueness Preliminaries

- Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a measurable function
- We say f satisfies the **linear growth condition** if there exists $C > 0$ such that

$$|f(t, x)| \leq C\,(1 + |x|)$$

  for all $t \in [0, T]$ and $x \in \mathbb{R}$
- We say f is **spatially Lipschitz** if there exists $D > 0$ such that

$$|f(t, x) - f(t, y)| \leq D\,|x - y|$$

  for all $t \in [0, T]$ and $x, y \in \mathbb{R}$

An Existence and Uniqueness Theorem

Theorem

Let $T > 0$, and let $\mu, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ be measurable functions which both satisfy the linear growth condition and are both spatially Lipschitz. Then the stochastic differential equation

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t\,, \quad X_0 = x$$

has a unique solution with the following properties:

- *X is t-continuous almost surely*
- *X is adapted to $\mathcal{F}_t$*
- $\mathbb{E}[\int_0^T X_t^2 dt] < \infty$

proof outline: 1) uniqueness: assume it's not unique,

$X_t^{(1)}, X_t^{(2)}$ both satisfy SDE.

$\Rightarrow X^{(1)} = X^{(2)}$

2) existence: (same as for ODEs).

Define a recursive sequence of processes

$X^{(n)} \rightarrow X^{(n+1)}$ called Picard iterates.

Then show $X^{(n)}$ is Cauchy $\Rightarrow$ it converges to something.

Show that $\lim_{n\to\infty} X^{(n)}$ satisfies the SDE.

Proof of uniqueness: Suppose $X^{(1)}$ and $X^{(2)}$ are both solutions to the SDE which are both continuous.

Compute:
$$\mathbb{E}\left[\left(X_t^{(1)} - X_t^{(2)}\right)^2\right] = \mathbb{E}\left[\left(\int_0^t \mu(u, X_u^{(1)}) - \mu(u, X_u^{(2)})\, du + \int_0^t \sigma(u, X_u^{(1)}) - \sigma(u, X_u^{(2)})\, dB_u\right)^2\right]$$

$$= \mathbb{E}\left[\left(\int_0^t \alpha_u\, du + \int_0^t \lambda_u\, dB_u\right)^2\right]$$

$\alpha_t = \mu(t, X_t^{(1)}) - \mu(t, X_t^{(2)})$

$\lambda_t = \sigma(\bullet) - \sigma(\bullet)$

$\vdots$ computations which are detailed and somewhat tedious

$$\leq K \int_0^t \mathbb{E}\left[\left(X_u^{(1)} - X_u^{(2)}\right)^2\right] du$$

this constant depends on T and the Lipschitz constant

$$\mathbb{E}\left[\left(X_t^{(1)} - X_t^{(2)}\right)^2\right] \leq K \int_0^t \mathbb{E}\left[\left(X_u^{(1)} - X_u^{(2)}\right)^2\right] du$$

let $f(t) = \mathbb{E}\left[\left(X_t^{(1)} - X_t^{(2)}\right)^2\right]$

ch.5 exercise in ϕ.

then $f(t) \leq K \int_0^t f(u)\,du \implies$ by the Gronwall inequality

this implies $f(t) = 0$ for all t.

$$\mathbb{E}\left[\left(X_t^{(1)} - X_t^{(2)}\right)^2\right] = 0 \quad \text{for all } t \in [0,T]$$

$\mathbb{E}\left[(X-Y)^2\right] = 0$

$$\mathbb{P}\left(X_t^{(1)} = X_t^{(2)}\right) = 1 \quad \text{for all } t \in [0,T]$$

$\implies X = Y$ a.s.

$\implies \mathbb{P}\left(X_t^{(1)} = X_t^{(2)} \text{ for all } t \in [0,T]\right) = 1$ because $X^{(1)}$ and $X^{(2)}$ are assumed continuous.

proof of existence: Construct Picard iterates.

let $X_t^{(0)} = x$, $X_t^{(n+1)} = x + \int_0^t \mu(u, X_u^{(n)})\,du + \int_0^t \sigma(u, X_u^{(n)})\,dB_u$

we will show that $X^{(n)}$ is a Cauchy sequence in $L^2([0,T]\times\Omega)$.

$\mathbb{E}\left[\left(X_t^{(n+1)} - X_t^{(n)}\right)^2\right]$ ..... detailed computations

$\leq \bar{K} \int_0^t \mathbb{E}\left[\left(X_u^{(n)} - X_u^{(n-1)}\right)^2\right] du$ (**) for $n \geq 1$

$\mathbb{E}\left[\left(X_t^{(1)} - X_t^{(0)}\right)^2\right]$ ..... detailed computation

$\leq A t$ (*)

can show by induction using (*) and (**) that

$$\mathbb{E}\left[\left(X_t^{(n+1)} - X_t^{(n)}\right)^2\right] \leq \frac{\tilde{K}^{n+1} t^{n+1}}{(n+1)!}$$

want to show $\{X^{(n)}\}_{n=0}^{\infty}$ is Cauchy in $L^2([0,T]\times\Omega)$.

$$\|X^{(n)} - X^{(m)}\|_{L^2([0,T]\times\Omega)} = \left\|\sum_{k=n}^{m-1} X^{(k+1)} - X^{(k)}\right\|_{L^2([0,T]\times\Omega)}$$

Telescopic Sum, only first & last term survive.

$$\leq \sum_{k=n}^{m-1} \|X^{(k+1)} - X^{(k)}\|_{L^2([0,T]\times\Omega)}$$

$$= \sum_{k=n}^{m-1} \left\{ \mathbb{E}\left[\int_0^T (X_t^{(k+1)} - X_t^{(k)})^2 dt\right]\right\}^{1/2}$$

$$= \sum_{k=n}^{m-1} \left\{\int_0^T \mathbb{E}\left[(X_t^{(k+1)} - X_t^{(k)})^2\right] dt\right\}^{1/2}$$

$$\leq \sum_{k=n}^{m-1} \left(\int_0^T \frac{\tilde{K}^{k+1} t^{k+1}}{(k+1)!} dt\right)^{1/2} = \sum_{k=n}^{m-1} \left(\tilde{K}^{k+1} \frac{T^{k+2}}{(k+2)!}\right)^{1/2}$$

$$\longrightarrow 0 \quad \text{as } n, m \to \infty.$$

$$\Longrightarrow \{X^{(n)}\}_{n=0}^{\infty} \text{ is Cauchy.}$$

$\Rightarrow \quad X^{(n)} \longrightarrow X$ in $L^2([0,T] \times \Omega)$

need to verify that X solves the original SDE.

$$X_t^{(n+1)} = x + \int_0^t \mu(u, X_u^{(n)})\,du + \int_0^t \sigma(u, X_u^{(n)})\,dB_u$$

by some Famous inequalities

$$X_t = x + \int_0^t \mu(u, X_u)\,du + \int_0^t \sigma(u, X_u)\,dB_u$$

Don't know if X is continuous, but there is a modification $\tilde{X}$ which is continuous. $X_t = \tilde{X}_t$ a.s. for each $t \in [0,T]$

$$\tilde{X}_t = x + \int_0^t \mu(u, X_u)\,du + \int_0^t \sigma(u, X_u)\,dB_u$$

$$\tilde{X}_t = x + \int_0^t \mu(u, \tilde{X}_u)\, du + \int_0^t \sigma(u, \tilde{X}_u)\, dB_u$$

Stochastic Analysis

Lecture 7

This Lecture

- Girsanov's Theorem (Section 8.6 from Øksendal textbook)

Girsanov's Theorem and Stochastic Dynamics

- Girsanov's Theorem establishes a connection between the dynamics of a process in two different probability measures when the relation between the measures takes a specific form
- In particular, if $\{B_t\}_{t\geq 0}$ is a Brownian motion under probability measure $\mathbb{P}$, it may not be Brownian motion under probability measure $\mathbb{Q}$
- If the change from $\mathbb{P}$ to $\mathbb{Q}$ is done in a particular way, then Girsanov's Theorem will tell us how to alter $\{B_t\}_{t\geq 0}$ so that it is a Brownian motion in $\mathbb{Q}$
- One interpretation of Girsanov's Theorem is that changing the drift of a process is essentially equivalent to changing the probability measure we work with

Preliminaries Regarding Brownian Motion

- There are multiple definitions of Brownian motion, all of them are essentially equivalent
- The following characterization of Brownian motion will be useful in proving Girsanov's Theorem

Theorem (Levy Characterization of Brownian Motion)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* $\{X_t\}_{t \geq 0}$ *be a continuous stochastic process. Suppose:*

i) $\{X_t\}_{t \geq 0}$ *is a martingale with respect to* $\mathbb{P}$ *and its own filtration,*

ii) $\{X_t^2 - t\}_{t \geq 0}$ *is a martingale with respect to* $\mathbb{P}$ *and its own filtration.*

Then $\{X_t\}_{t \geq 0}$ *is a Brownian motion with respect to* $\mathbb{P}$*.*

Preliminaries Regarding Measure Changes

- Let $(\Omega, \mathcal{F})$ be a measurable space
- Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on $(\Omega, \mathcal{F})$
- We say $\mathbb{Q}$ is **absolutely continuous with respect to** $\mathbb{P}$ if

$$\mathbb{P}(E) = 0 \quad \Longrightarrow \quad \mathbb{Q}(E) = 0$$

for all $E \in \mathcal{F}$

- We denote this relation by $\mathbb{Q} \ll \mathbb{P}$ does not mean $\mathbb{P}(E) > \mathbb{Q}(E)$ for all $E \in \mathcal{F}$
- By the Radon-Nikodym theorem, if $\mathbb{Q} \ll \mathbb{P}$ then there exists a random variable Z such that

$$\mathbb{Q}(E) = \int_E Z(\omega)\, d\mathbb{P}(\omega) = \mathbb{E}^{\mathbb{P}}[Z\, \mathcal{X}_E]\,,$$

we call Z the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ and we write

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z$$

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[X Z]$$

Measure Changes in a Filtered Probability Space

- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space
- Let $T > 0$ and let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F}_T)$
- We can't say $\mathbb{Q} \ll \mathbb{P}$ because $\mathbb{P}$ and $\mathbb{Q}$ have different domains (if $E \in \mathcal{F}$ we may not have $E \in \mathcal{F}_T$)
- However, we have $\mathcal{F}_T \subset \mathcal{F}$ so we can restrict $\mathbb{P}$ to $\mathcal{F}_T$
- Thus, it is possible to have $\mathbb{Q} \ll \mathbb{P}|_{\mathcal{F}_T}$
- As before, this implies there is a random variable Z_T which is $\mathcal{F}_T$-measurable such that

$$\frac{d\mathbb{Q}}{d(\mathbb{P}|_{\mathcal{F}_T})} = Z_T$$

Some Important Lemmas - 1 of 2

Lemma

Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{F})$ with $\mathbb{Q} \ll \mathbb{P}$. Let X be a random variable such that

$$\mathbb{E}^{\mathbb{Q}}[|X|] = \int_{\Omega} |X(\omega)|\, Z(\omega)\, d\mathbb{P}(\omega) < \infty\,,$$

where

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}\,.$$

Let $\mathcal{G} \subset \mathcal{F}$, then

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\, \mathbb{E}^{\mathbb{P}}[Z|\mathcal{G}] = \mathbb{E}^{\mathbb{P}}[Z\, X|\mathcal{G}]$$

$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[XZ]$

$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] = \mathbb{E}^{\mathbb{P}}[XZ|\mathcal{G}]$ incorrect

Recall definition of conditional expectation:

$\mathbb{E}^{\mathbb{P}}[Y \mid \mathcal{G}]$ is a random variable, $\mathcal{G}$-measurable, and satisfies

$$\int_A \mathbb{E}^{\mathbb{P}}[Y \mid \mathcal{G}]\, d\mathbb{P} = \int_A Y\, d\mathbb{P} \quad \text{for all } A \in \mathcal{G}.$$

$$\left.\begin{aligned} \int_A \mathbb{E}^{\mathbb{P}}[ZX \mid \mathcal{G}]\, d\mathbb{P} &= \int_A ZX\, d\mathbb{P} \\ \int_A \mathbb{E}^{\mathbb{P}}[Z \mid \mathcal{G}]\, d\mathbb{P} &= \int_A Z\, d\mathbb{P} \\ \int_A \mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{G}]\, d\mathbb{Q} &= \int_A X\, d\mathbb{Q} \end{aligned}\right\} \text{for all } A \in \mathcal{G}$$

"$d\mathbb{Q} = Z\, d\mathbb{P}$"

$$\int_A \mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{G}]\, Z\, d\mathbb{P} = \int_A XZ\, d\mathbb{P} \quad \text{for all } A \in \mathcal{G}$$

by definition: $\mathbb{E}^{\mathbb{P}}[XZ|\mathcal{G}] = \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]Z = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]Z \,\middle|\, \mathcal{G}\right]$

$$= \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\,\mathbb{E}^{\mathbb{P}}[Z|\mathcal{G}]$$

✓

Usually this relation will be applied in this form:

$$\mathbb{E}^{\mathbb{Q}}[X\,|\,\mathcal{G}] = \frac{\mathbb{E}^{\mathbb{P}}[XZ|\mathcal{G}]}{\mathbb{E}^{\mathbb{P}}[Z|\mathcal{G}]}$$

Some Important Lemmas - 2 of 2

Lemma

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *be a filtered probability space. Suppose* $\mathbb{Q} \ll \mathbb{P}|_{\mathcal{F}_T}$ *and let* $Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}|_{\mathcal{F}_T}}$*. Then*

$$\mathbb{Q}|_{\mathcal{F}_t} \ll \mathbb{P}|_{\mathcal{F}_t} \,,$$

for all $t \in [0, T]$*, and if we define* $\{Z_t\}_{t\in[0,T]}$ *by*

$$Z_t = \frac{d(\mathbb{Q}|_{\mathcal{F}_t})}{d(\mathbb{P}|_{\mathcal{F}_t})} \,,$$

Then $\{Z_t\}_{t\in[0,T]}$ *is a martingale with respect to* $\{\mathcal{F}_t\}_{t\in[0,T]}$ *and* $\mathbb{P}$*.*

- $\mathbb{Q} \ll \mathbb{P}\big|_{\mathcal{F}_T}$.

prove: $\mathbb{Q}\big|_{\mathcal{F}_t} \ll \mathbb{P}\big|_{\mathcal{F}_t}$.

Let $\underbrace{A \in \mathcal{F}_t}$ such that $\mathbb{P}(A) = 0$.

$\hookrightarrow$ $A \in \mathcal{F}_T$ combined with $\mathbb{Q} \ll \mathbb{P}\big|_{\mathcal{F}_T} \Rightarrow \mathbb{Q}(A) = 0$

so if $A \in \mathcal{F}_t$ and $\mathbb{P}(A) = 0$ then $\mathbb{Q}(A) = 0$ ✓

---

let $Z_t = \dfrac{d(\mathbb{Q}|_{\mathcal{F}_t})}{d(\mathbb{P}|_{\mathcal{F}_t})}$. Prove Z is an $\mathcal{F}_t, \mathbb{P}$ martingale.

Recall: let $A \in \mathcal{F}_t$ then $\int_A d\mathbb{Q} = \mathbb{Q}(A) = \int_A Z_t \, d\mathbb{P}$.

what exactly do we need to prove?

$$\mathbb{E}^{\mathbb{P}}[Z_t \mid \mathcal{F}_s] = Z_s \quad \text{for } s<t.$$

by definition:

$$\int_A \mathbb{E}^{\mathbb{P}}[Z_t \mid \mathcal{F}_s]\, d\mathbb{P} = \int_A Z_t \, d\mathbb{P} \qquad \text{for all } A \in \mathcal{F}_s$$

so what we try to prove is:

$$\int_A Z_t \, d\mathbb{P} = \int_A Z_s \, d\mathbb{P} \qquad \text{for } s<t \text{ and } A \in \mathcal{F}_s$$

begin proof:

let $A \in \mathcal{F}_s$: $\int_A Z_s \, d\mathbb{P} = \mathbb{Q}(A)$ but $A \in \mathcal{F}_s \Rightarrow A \in \mathcal{F}_t$

$$= \int_A Z_t \, d\mathbb{P}$$

Girsanov's Theorem

Theorem

Fix $T > 0$ finite. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{B_t\}_{t\in[0,T]}$ be a Brownian motion that generates $\{\mathcal{F}_t\}_{t\in[0,T]}$, and assume $\mathcal{F} = \mathcal{F}_T$. Let $\{\widehat{B}_t\}_{t\in[0,T]}$ be an Ito process of the form

$$d\widehat{B}_t = \alpha_t\, dt + dB_t\,, \qquad \widehat{B}_0 = 0\,.$$

Define $\{M_t\}_{t\in[0,T]}$ by

$$M_t = \exp\left\{- \int_0^t \alpha_u\, dB_u - \frac{1}{2}\int_0^t \alpha_u^2\, du\right\},$$

and assume that $\{M_t\}_{t\in[0,T]}$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ and $\mathbb{P}$. Define the measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = M_T\,.$$

Then $\mathbb{Q}$ is a probability measure and $\{\widehat{B}_t\}_{t\in[0,T]}$ is a Brownian motion with respect to $\mathbb{Q}$.

first: need $\mathbb{Q}(\Omega) = 1$.

$$\mathbb{Q}(\Omega) = \int_\Omega M_T \, d\mathbb{P} = \mathbb{E}^{\mathbb{P}}[M_T]$$

$$= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}[M_T \mid \mathcal{F}_0]\right]$$

$$= \mathbb{E}^{\mathbb{P}}[M_0]$$

$$= 1$$

Need to show $\hat{B}$ is a $\mathbb{Q}$-Brownian motion. We will do this using the Levy characterization of Brownian motion.

First compute: $dM_t = -\alpha_t M_t \, dB_t$

we will assume all integrands are in $\mathcal{V}(0,T)$.

now consider $X_t = M_t \hat{B}_t$

Itô's lemma: $dX_t = M_t \, d\hat{B}_t + \hat{B}_t \, dM_t + dM_t \, d\hat{B}_t$

$= M(\alpha dt + dB) + \hat{B}(-\alpha M dB) + (-\alpha M dB)(\alpha dt + dB)$

$= \alpha M dt + M dB - \alpha M \hat{B} dB - \alpha M dt$

$$dX_t = M_t (1 - \alpha_t \hat{B}_t) dB_t$$

$\Rightarrow$ X is a $\mathbb{P}$-martingale

$\mathbb{E}^{\mathbb{P}}[X_t | \mathcal{F}_s] = X_s$ for $t > s$

$\mathbb{E}^{\mathbb{P}}[M_t \hat{B}_t | \mathcal{F}_s] = M_s \hat{B}_s$

$\mathbb{E}^{\mathbb{Q}}[\hat{B}_t | \mathcal{F}_s] \, \mathbb{E}^{\mathbb{P}}[M_t | \mathcal{F}_s] = M_s \hat{B}_s$

$\mathbb{E}^{\mathbb{Q}}[X | \mathcal{G}] \mathbb{E}^{\mathbb{P}}[Z | \mathcal{G}] = \mathbb{E}^{\mathbb{P}}[XZ | \mathcal{G}]$

$$\mathbb{E}^{Q}\left[\hat{B}_t \mid \mathcal{F}_s\right] \cancel{M_s} = \cancel{M_s}\, \hat{B}_s$$

$$\mathbb{E}^{Q}\left[\hat{B}_t \mid \mathcal{F}_s\right] = \hat{B}_s \implies \hat{B} \text{ is a } Q, \mathcal{F}_t \text{ martingale.}$$

Let $Y_t = (\hat{B}_t^2 - t) M_t$

$$dY_t = (\quad) \, dB_t$$ details omitted.

$\implies$ Y is a $\mathbb{P}$-martingale.

$\implies$ rest of computations are identical

$\implies$ $\hat{B}_t^2 - t$ is a Q martingale

$\implies$ $\hat{B}$ is a Q Brownian motion.

Comments on Girsanov's Theorem

- Girsanov's Theorem establishes the connection between changing measures and changing the drift of a Brownian motion

- The definition of $\{M_t\}_{t\in[0,T]}$ shows that $M_t > 0$, so we also have

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{1}{M_T}$$

- This also implies that the measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent:

$$\mathbb{P}(E) > 0 \iff \mathbb{Q}(E) > 0$$

- If we restrict to events $E \in \mathcal{F}_t$, then

$$\frac{d(\mathbb{Q}|_{\mathcal{F}_t})}{d(\mathbb{P}|_{\mathcal{F}_t})} = M_t$$

Measure Change

$$\boxed{\mathbb{E}^{\mathbb{Q}}[X_t \mid \mathcal{F}_s] = \frac{\mathbb{E}^{\mathbb{P}}[X_t M_t \mid \mathcal{F}_s]}{\mathbb{E}^{\mathbb{P}}[M_t \mid \mathcal{F}_s]}}$$

<u>NOT</u>

$$\mathbb{E}^{\mathbb{Q}}[X_t \mid \mathcal{F}_s] = \mathbb{E}^{\mathbb{P}}[X_t M_t \mid \mathcal{F}_s]$$

We can prove something is Martingale by

(i) $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s \quad , s < t$

(ii) Showing drift term $= 0$ in ito process.

Stochastic Analysis

Lecture 8

This Lecture

- The Markov Property (Section 7.1 from Øksendal textbook)
  - We shall use a different (but equivalent) definition which is more directly useful for our purposes
- The Feynman-Kac Formula (Sections 8.1, 8.2, 9.1, 9.2, and 9.3 from Øksendal textbook are relevant)
  - What we will call "Feynman-Kac" is what Øksendal refers to as the Poisson and Dirichlet problems
  - What Øksendal calls "Feynman-Kac" is a highly related concept but in the opposite time direction

Ito Diffusions and PDEs

- The Feynman-Kac Formula establishes a connection between diffusion processes and partial differential equations
- It is a result which gives a representation of the solution to a PDE in terms of a conditional expectation
- In practice:
  - Sometimes a particular PDE is difficult to solve (even numerically) but an associated conditional expectation is easier to compute (possibly numerically)
  - Sometimes a particular conditional expectation is difficult to compute (even numerically) but an associated PDE is easier to solve (possibly numerically)

The Markov Property

- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space and let $\{X_t\}_{t \in [0,T]}$ be a stochastic process adapted to $\mathcal{F}_t$

- We say $\{X_t\}_{t \geq 0}$ has the **Markov property** (or is **Markovian** or is a **Markov process**) if:

  For $T > 0$, and for any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$, there exists a function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that

  $$\mathbb{E}[f(X_T) | \mathcal{F}_t] = g(t, X_t) \qquad \text{for all } t \in [0, T]$$

- There are many different (but equivalent) definitions of the Markov property, but this one is most directly applicable to deriving the Feynman-Kac formula

Markov Property - Comments

For $T > 0$, and for any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$, there exists a function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}[f(X_T)|\mathcal{F}_t] = g(t, X_t) \qquad \text{for all } t \in [0, T]$$

- The function g is allowed to depend on the choice of the function f and the constant T
- In English: suppose $\{X_t\}_{t \geq 0}$ is Markovian and fix $T > 0$. If we need to estimate the future value of a function of a process, then this will only depend on the current value of the process (and on time)
- Alternate interpretation: the distribution of the process in the future only depends on the value of the process at the present (and on the present time), it does not depend on the historical path of the process

Ito Diffusions and the Markov Property

- An **Ito diffusion** is a process which satisfies an equation of the form

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t\,,$$

where $\mu, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$

Reminder: Ito process: $dX_t = \nu_t dt + \eta_t dB_t$

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\{B_t\}_{t\geq 0}$ be a Brownian motion,and let $\{X_t\}_{t\geq 0}$ be an Ito diffusion. Then $\{X_t\}_{t\geq 0}$ has the Markov property.

- Intuition: if $\{X_t\}_{t\geq 0}$ is an Ito diffusion, then from the perspective at time t what could the future distribution of the process possibly depend on other than t and X_t?

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$$

$$X_T - X_t = \int_t^T \mu(s, X_s)\,ds + \int_t^T \sigma(s, X_s)\,dB_s$$

$$t = t_0 < t_1 < \dots < t_N = T$$

$$X_{t_N} \approx X_{t_0} + \sum_{k=0}^{N-1} \mu(t_k, X_{t_k})\,\Delta t + \sum_{k=0}^{N-1} \sigma(t_k, X_{t_k})\,\Delta B_{t_k}$$

$$X_{t_1} = X_{t_0} + \mu(t_0, X_{t_0})\,\Delta t + \sigma(t_0, X_{t_0})(B_{t_1} - B_{t_0})$$ ← one time step

assume that $X_{t_n} = h_n(t_{n-1}, X_{t_{n-1}}, B_0, B_{t_1}, \dots, B_{t_n})$

$$X_{t_{n+1}} = X_{t_n} + \mu(t_n, X_{t_n})\,\Delta t + \sigma(t_n, X_{t_n})(B_{t_{n+1}} - B_{t_n})$$

substitute $X_{t_n} = h_n$

$$X_{t_{n+1}} = h_n(\cdot) + \mu(t_n, h_n(\cdot))\,\Delta t + \sigma(t_n, h_n(\cdot))(B_{t_{n+1}} - B_{t_n})$$

$\vdots$

$$X_{t_N} = \varphi(X_{t_0}, B_{t_0}, B_{t_1}, \dots, B_{t_N})$$

$$\mathbb{E}\left[f(X_{t_N}) \,\middle|\, \mathcal{F}_{t_0}\right] = \mathbb{E}\left[f(\varphi(X_{t_0}, \underbrace{B_{t_0}, \dots, B_{t_N}}))\,\middle|\,\mathcal{F}_{t_0}\right]$$

condition on $\mathcal{F}_{t_0}$ this has a known multivariate density: p

$$= \int_{\mathbb{R}^N} f(\varphi(X_{t_0}, z_1, \dots, z_N))\, p(z_1, \dots, z_N)\, d\vec{z}$$

$$= g(t_0, x)\Big|_{x = X_{t_0}}$$

X_t is markov if it is Ito diffusion.

The Feynman-Kac Formula - Part 1

Theorem

Suppose $\{X_t\}_{t\in[0,T]}$ *satisfies*

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t\,.$$

← Ito diffusion

Let $\psi : \mathbb{R} \to \mathbb{R}$ *be a bounded Borel function, and let* $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be the function which satisfies*

← markov property

$$v(t, X_t) = \mathbb{E}[\psi(X_T)|\mathcal{F}_t]\,.$$

If $v \in C^{1,2}([0, T) \times \mathbb{R})$ *then* v *satisfies the PDE with terminal conditions*

$$\frac{\partial v}{\partial t} + \mu(t, x)\, \frac{\partial v}{\partial x} + \frac{1}{2}\, \sigma^2(t, x)\, \frac{\partial^2 v}{\partial x^2} = 0\,,$$

$$v(T, x) = \psi(x)\,.$$

$$Y_t = v(t, X_t) = \mathbb{E}\left[\varphi(X_T) \mid \mathcal{F}_t\right]$$ ⟸ This exist as Ito diffusion has Markov property.

Y is a martingale.

$t > s$:

$$\mathbb{E}\left[Y_t \mid \mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{E}\left[\varphi(X_T) \mid \mathcal{F}_t\right] \mid \mathcal{F}_s\right]$$

$$= \mathbb{E}\left[\varphi(X_T) \mid \mathcal{F}_s\right]$$

$$= Y_s \quad \checkmark$$

Now, compute dY_t

$$dY_t = \left(\frac{\partial v}{\partial t}(t, X_t) + \mu(t, X_t)\frac{\partial v}{\partial x}(t, X_t) + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 v}{\partial x^2}(t, X_t)\right) dt$$

$$+ \sigma(t, X_t)\frac{\partial v}{\partial x}(t, X_t)\, dB_t$$

⟸ Stochastic integral is martingale

but Y is a martingale. Hence, the underline ——— term must be zero.

$\Rightarrow$ "dt" term must be zero.

$$\left.\begin{aligned} &\frac{\partial v}{\partial t} + \mu \frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} = 0 \\ &\qquad\qquad\qquad v(T, x) = \varphi(x) \end{aligned}\right\}$$

$\boxed{\xi}$

Ex: $\mathbb{E}\left[\varphi(X_T)\right] = v(0, X_0)$

The Feynman-Kac Formula - Part 2

T is fixed but t is variable & depending on it we get different r.v.

Theorem

Let $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ be the function which satisfies

$$v(t, X_t) = \mathbb{E}\left[e^{-\int_t^T r(s,X_s)\,ds} \underbrace{\psi(X_T)}_{\text{cash flow at } T} + \int_t^T \underbrace{e^{-\int_t^u r(s,X_s)\,ds}}_{\text{discount back to } t.} \underbrace{\gamma(u, X_u)}_{\text{cash flow received at } u} \, du \,\middle|\, \mathcal{F}_t \right].$$

If $v \in C^{1,2}([0, T) \times \mathbb{R})$ then v satisfies the PDE with terminal conditions

$$\frac{\partial v}{\partial t} + \mu(t, x)\,\frac{\partial v}{\partial x} + \frac{1}{2}\,\sigma^2(t, x)\,\frac{\partial^2 v}{\partial x^2} - r(t, x)\,v + \gamma(t, x) = 0\,,$$

$$v(T, x) = \psi(x)\,.$$

$$\underbrace{e^{-\int_t^T r(s,X_s)\,ds}}_{\text{discount to time } t} = e^{-r(T-t)}$$

$$V_{t_{n-1}} = V_{t_n} - \partial_t V \,\Delta t$$

In general: $M_t = \mathbb{E}[Z \mid \mathcal{F}_t]$ this is a martingale

$N_t = \mathbb{E}[f(t,Z) \mid \mathcal{F}_t]$ not a martingale in general

Let $Y_t = v(t, X_t)$

getting rid of dependence on t.

Let $Z_t = Y_t + \int_0^t e^{-\int_t^u r(s,X_s)\,ds} \gamma(u, X_u)\,du$

$$Z_t = \mathbb{E}\left[e^{-\int_t^T r(s,X_s)\,ds} \varphi(X_T) + \int_0^T e^{-\int_t^u r(s,X_s)\,ds} \gamma(u,X_u)\,du \,\Big|\, \mathcal{F}_t \right]$$

getting rid of dependence out.

Let $A_t = Z_t \, e^{-\int_0^t r(s,X_s)\,ds}$

$$A_t = \mathbb{E}\left[e^{-\int_0^T r(s,X_s)\,ds} \varphi(X_T) + \int_0^T e^{-\int_0^u r(s,X_s)\,ds} \gamma(u,X_u)\,du \,\Big|\, \mathcal{F}_t \right]$$

$\to$ A is a martingale. (as done for part I) look previous

$r_t = r(t, X_t)$

$\gamma_t = \gamma(t, X_t)$

$$A_t = e^{-\int_0^t r_s\,ds} Z_t$$

$$= e^{-\int_0^t r_s\,ds}\left(Y_t + \int_0^t e^{-\int_t^u r_s\,ds}\gamma_u\,du\right)$$

writing A_t in terms of V.

$$A_t = \underline{e^{-\int_0^t r_s\,ds} v(t, X_t)} + \underline{\int_0^t e^{-\int_0^u r_s\,ds}\gamma_u\,du}$$

$\leftarrow$ product rule & Ito lemma for $V(t,X_t)$

Computed A_t:

$$dA_t = \left[-r_t e^{-\int_0^t r_s ds} v(t,X_t) + e^{-\int_0^t r_s ds}\left(\partial_t v + \mu\partial_x v + \frac{1}{2}\sigma^2\partial_{xx} v\right)\right]dt$$

$$+ e^{-\int_0^t r_s ds}\sigma\partial_x v\,dB_t + e^{-\int_0^t r_s ds}\gamma_t\,dt$$

$$dA_t = e^{-\int_0^t r_s ds}\left(\partial_t v + \mu\partial_x v + \frac{1}{2}\sigma^2\partial_{xx} v - r v + \gamma\right)dt$$

$$+ e^{-\int_0^t r_s ds}\sigma\partial_x v\,dB_t$$

but A is a martingale $\to$ "dt" term must be zero.

$$e^{-\int_0^t r_s\,ds}\left(\partial_t v + \mu\,\partial_x v + \frac{1}{2}\sigma^2 \partial_{xx} v - r v + \gamma\right) = 0$$

$$\left.\begin{array}{ll} & \partial_t v + \mu\,\partial_x v + \frac{1}{2}\sigma^2 \partial_{xx} v - r v + \gamma = 0 \\ \text{terminal conditions} & v(T, x) = \varphi(x) \end{array}\right\}$$

$\boxed{3}$

Ex. 1 consider $dX_t = \mu dt + \sigma dB_t$

compute: $\mathbb{E}\left[e^{X_T} \mid \mathcal{F}_t\right]$

$\Rightarrow$ let $v(t, X_t) = \mathbb{E}\left[e^{X_T} \mid \mathcal{F}_t\right]$

$v(T, x) = e^{\overset{a(T)}{0} + \overset{b(T)}{1}\, x}$

v solves:

$$\partial_t v + \mu \partial_x v + \frac{1}{2}\sigma^2 \partial_{xx} v = 0, \quad v(T, x) = e^x$$

guess: $v(t,x) = e^{a(t) + b(t)x} \Rightarrow a(T) = 0, \; b(T) = 1$

sub. this into PDE:

$$(a' + b'x)e^{a+bx} + \mu b e^{a+bx} + \frac{1}{2}\sigma^2 b^2 e^{a+bx} = 0$$

$$a' + b'x + \mu b + \frac{1}{2}\sigma^2 b^2 = 0$$

$$\left(a'(t) + \mu b(t) + \frac{1}{2}\sigma^2 b^2(t)\right) + \left(b'(t)\right) x = 0$$

$\leftarrow$ group in powers of x.

$$\begin{cases} a'(t) + \mu\, b(t) + \frac{1}{2}\sigma^2 b^2(t) = 0 \\ b'(t) = 0 \implies b(t) = 1 \end{cases}$$

$$a'(t) + \mu + \frac{1}{2}\sigma^2 = 0 \implies a(t) = \left(\mu + \frac{1}{2}\sigma^2\right)(T - t)$$

$$v(t,x) = e^{\left(\mu + \frac{1}{2}\sigma^2\right)(T-t) + x}$$

$$\mathbb{E}\left[e^{X_T} \mid \mathcal{F}_t\right] = e^{\left(\mu + \frac{1}{2}\sigma^2\right)(T-t) + X_t}$$

$$a'(t) = -\left(\mu + \frac{1}{2}\sigma^2\right)$$

$$\int_t^T a'\,du = \int_t^T -\left(\mu + \frac{1}{2}\sigma^2\right)du$$

$$a(T) - a(t) = -\left(\mu + \frac{1}{2}\sigma^2\right)(T - t)$$

$$a(t) = \left(\mu + \frac{1}{2}\sigma^2\right)(T - t)$$

Ex. 2 $\quad dX_t = \mu X_t\,dt + \sigma X_t\,dB_t$

$$v(t, X_t) = \mathbb{E}\left[e^{-r(T-t)} X_T^2 \,\middle|\, \mathcal{F}_t \right]$$

$$\partial_t v + \mu x \partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_{xx} v - r v = 0$$

$$v(T, x) = x^2$$

guess: $v(t,x) = a(t) + b(t)x + c(t)x^2$, $\implies a(T) = 0$

$b(T) = 0$

$c(T) = 1$

$$a' + b'x + c'x^2 + \mu x(b + 2cx) + \frac{1}{2}\sigma^2 x^2 (2c) - r(a + bx + cx^2) = 0$$

$$a' - ra = 0 \qquad a(T) = 0 \implies a(t) = 0$$

$$b' + \mu b - rb = 0 \qquad b(T) = 0 \implies b(t) = 0$$

$$c' + 2\mu c + \sigma^2 c - rc = 0 \qquad c(T) = 1$$

$$\implies c(t) = e^{(2\mu + \sigma^2 - r)(T-t)}$$

$$\mathbb{E}\left[e^{-r(T-t)} X_T^2 \mid \mathcal{F}_t \right] = e^{(2\mu + \sigma^2 - r)(T-t)} X_t^2$$

Two Guesses you must remember to solve a PdE is exponential & quadratic.

$$\mathbb{E}^{\mathbb{Q}}\left\{X_t \mid \mathcal{F}_s\right\} = \frac{\mathbb{E}^{\mathbb{P}}\left[X_t M_t \mid \mathcal{F}_s\right]}{\mathbb{E}^{\mathbb{P}}\left[M_t \mid \mathcal{F}_s\right]}$$

$$\mathbb{E}^{\mathbb{Q}}\left[X_t \mid \mathcal{F}_s\right] = \mathbb{E}^{\mathbb{P}}\left[X_t M_t \mid \mathcal{F}_s\right]$$

incorrect.

Stochastic Analysis

Lectures 9 and 10

This Lecture

- Stochastic Optimal Control
- Problem Formulation
- Dynamic Programming Principle
- The Hamilton-Jacobi-Bellman Equation

Stochastic Optimal Control

- Stochastic optimal control is an application which appears in a wide range of fields
- Main idea is that we will consider a large class of stochastic processes and try to find the one which minimizes or maximizes a given expectation
- Throughout this lecture we will not deal with some technical details so that we can focus on the main idea
  - We will simply assume that all SDEs we consider have solutions and all expressions are integrable when necessary

Controlled Process

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{B_t\}_{t\in[0,T]}$ be a Brownian motion which generates the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$
- Let $\{\pi_t\}_{t\in[0,T]}$ be an arbitrary stochastic process adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$
- Let $\{X_t^\pi\}_{t\in[0,T]}$ be a process which satisfies

$$dX_t^\pi = \mu(t, X_t^\pi, \pi_t)\,dt + \sigma(t, X_t^\pi, \pi_t)\,dB_t\,, \quad X_0 = x$$

- We call $\{X_t^\pi\}_{t\in[0,T]}$ a **controlled process** or **controlled diffusion** and we refer to $\{\pi_t\}_{t\in[0,T]}$ as the **control process**

SDE: $dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$

Markov Controls

$$dX_t^\pi = \mu(t, X_t^\pi, \pi_t)\, dt + \sigma(t, X_t^\pi, \pi_t)\, dB_t\,, \quad X_0 = x \quad (*)$$

- In general the process given by $(*)$ is not an Ito diffusion
- If the control is of the form $\pi_t = \pi(t, X_t^\pi)$ then $(*)$ becomes

Ito diffusion

$$dX_t^\pi = \mu(t, X_t^\pi, \pi(t, X_t^\pi))\, dt + \sigma(t, X_t^\pi, \pi(t, X_t^\pi))\, dB_t\,, \quad X_0 = x$$

- This is now an Ito diffusion, and therefore $\{X_t\}_{t\in[0,T]}$ has the Markov property
- A control of the form $\pi_t = \pi(t, X_t)$ is called a **Markov control**
- We will consider only Markov controls
- The set of Markov controls which act between times s and t is denoted $\mathcal{A}_{s,t}$

$$A_{s,t} = \{\pi : [s,t] \times \mathbb{R} \to \mathbb{R}\}$$

Performance Criterion and Value Function

- Let $\{\pi_t\}_{t\in[0,T]}$ be one particular control with $\pi_t = \pi(t, X_t^\pi)$
- Given a bounded continuous function $\psi : \mathbb{R} \to \mathbb{R}$ we define the **performance criterion** to be

Starting point of process X

$$H^\pi(x) = \mathbb{E}[\psi(X_T^\pi)]$$

Terminal value

- Our objective is to maximize the performance criterion over all Markov controls
- We define the **value function** to be

$$H(x) = \sup_{\pi\in\mathcal{A}_{0,T}} H^\pi(x) = \sup_{\pi\in\mathcal{A}_{0,T}} \mathbb{E}[\psi(X_T^\pi)]$$

Value function that maximizes performance criterion over all Markov controls.

- We would also like to find π^* such that

$$\underbrace{H^{\pi^*}(x)}_{\text{Performance criterion}} = \underbrace{H(x)}_{\text{Value f'n}}$$

if such a π^* exists

Dynamic Value Function

- Our strategy will be to embed the optimization problem within a large class of related optimization problems
- One of the optimizations in this large class will be the same as the original problem
- Define the **dynamic value function** $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ through the relation

$$H(t, X_t) = \sup_{\pi \in \mathcal{A}_{t,T}} \mathbb{E}[\psi(X_T^\pi) | \mathcal{F}_t],$$

- Note that this gives

$$\underbrace{H(0,x)}_{\text{dynamic value f'n}} = \sup_{\pi \in \mathcal{A}_{0,T}} \mathbb{E}[\psi(X_T^\pi) | \mathcal{F}_0] = \sup_{\pi \in \mathcal{A}_{0,T}} \mathbb{E}[\psi(X_T^\pi)] = \underbrace{H(x)}_{\text{value f'n}}$$

Dynamic Programming Principle

- One reason we extend to the dynamic value function is because we can establish a relation between the value function at different times
- This relation is called the **dynamic programming principle**

Theorem (Dynamic Programming Principle)

Let $0 \le s \le t \le T$. *Then*

$$H(s, X_s) = \sup_{\pi \in \mathcal{A}_{s,t}} \mathbb{E}[H(t, X_t^\pi) | \mathcal{F}_s]$$

- If we already act optimally between times t and T, then to act optimally between times s and t we must only optimize with respect to the value function itself at time t

- $$H(s,X_s) = \sup_{\pi \in \mathcal{A}_{s,t}} \mathbb{E}\left[H(t, X_t^{\pi}) \mid \mathcal{F}_s\right]$$

Proof: let π° be an arbitrary Markov control.

$$H^{\pi^\circ}(s,X_s) = \mathbb{E}\left[\varphi(X_T^{\pi^\circ}) \mid \mathcal{F}_s\right]$$
dynamic performance criterion

$$H^{\pi^\circ}(s,X_s) = \mathbb{E}\left[\mathbb{E}\left[\varphi(X_T^{\pi^\circ}) \mid \mathcal{F}_t\right] \mid \mathcal{F}_s\right]$$
reverse of "smallest always wins"

$$H^{\pi^\circ}(s,X_s) = \mathbb{E}\left[H^{\pi^\circ}(t,X_t^{\pi^\circ}) \mid \mathcal{F}_s\right]$$
again by def of dynamic performance criterion.

$$H^{\pi^\circ}(s,X_s) \leq \mathbb{E}\left[\sup_{\pi \in \mathcal{A}_{t,T}} H^{\pi}(t,X_t^{\pi^\circ}) \mid \mathcal{F}_s\right]$$
the function obviously gets bigger.

$$H^{\pi^\circ}(s,X_s) \leq \mathbb{E}\left[H(t,X_t^{\pi^\circ}) \mid \mathcal{F}_s\right]$$
definition of H.

$$\sup_{\pi \in \mathcal{A}_{s,t}} H^{\pi}(s,X_s) \leq \sup_{\pi \in \mathcal{A}_{s,t}} \mathbb{E}\left[H(t,X_t^{\pi}) \mid \mathcal{F}_s\right]$$
just think about it.

$$H(s,X_s) \leq \sup_{\pi \in \mathcal{A}_{s,t}} \mathbb{E}\left[H(t,X_t^{\pi}) \mid \mathcal{F}_s\right] \quad (*)$$ definition of H

Let π° be arbitrary Markov control.

$$H^{\pi^{\circ}}(s,X_s) = \mathbb{E}\left\{\mathbb{E}\left[\Psi(X_T^{\pi^{\circ}}) \mid \mathcal{F}_t\right\} \mid \mathcal{F}_s\right]$$ from previous part of proof

$$H^{\pi^{\circ}}(s,X_s) = \mathbb{E}\left[H^{\pi^{\circ}}(t,X_t^{\pi^{\circ}}) \mid \mathcal{F}_s\right]$$ def. of dynamic performance.

Let $\hat{\pi}$ be ε-optimal control.

i.e. $\quad H(t,x) \geq H^{\hat{\pi}}(t,x) \geq H(t,x) - \varepsilon$

Define $\quad \tilde{\pi}_u = \begin{cases} \hat{\pi}_u & u \geq t \\ \pi^{\circ}_u & u < t \end{cases}$

$$H^{\tilde\pi}(s,X_s) = \mathbb{E}\left[H^{\tilde\pi}(t,X_t^{\tilde\pi}) \,\middle|\, \mathcal{F}_s\right]$$

$$H^{\tilde\pi}(s,X_s) = \mathbb{E}\left[H^{\hat\pi}(t,X_t^{\tilde\pi}) \,\middle|\, \mathcal{F}_s\right]$$

because $H^{\tilde\pi}(t,\cdot)$ only depends on $\tilde\pi$ after time t.

$$H^{\tilde\pi}(s,X_s) \geq \mathbb{E}\left[H(t,X_t^{\tilde\pi}) \,\middle|\, \mathcal{F}_s\right] - \varepsilon$$

because $\hat\pi$ is ε-optimal

$$\sup_{\pi\in\mathcal{A}_{s,t}} H^{\pi}(s,X_s) \geq \mathbb{E}\left[H(t,X_t^{\tilde\pi}) \,\middle|\, \mathcal{F}_s\right] - \varepsilon$$

function always less than its sup

$$H(s,X_s) \geq \mathbb{E}\left[H(t,X_t^{\tilde\pi}) \,\middle|\, \mathcal{F}_s\right] - \varepsilon$$

definition of H

$$H(s,X_s) \geq \mathbb{E}\left[H(t,X_t^{\pi^\circ}) \,\middle|\, \mathcal{F}_s\right] - \varepsilon$$

$X_t^{\tilde\pi}$ only depends on $\tilde\pi$ between s and t.

$$H(s,X_s) \geq \sup_{\pi\in\mathcal{A}_{s,t}} \mathbb{E}\left[H(t,X_t^{\pi}) \,\middle|\, \mathcal{F}_s\right] - \varepsilon$$

if $\geq$ holds for arbitrary π° then it holds for the sup.

$$H(s,X_s) \geq \sup_{\pi\in\mathcal{A}_{s,t}} \mathbb{E}\left[H(t,X_t^{\pi}) \,\middle|\, \mathcal{F}_s\right] \quad (**)$$

if $\geq$ true for all $\varepsilon>0$ then true for $\varepsilon=0$.

$\circledast + \circledast\circledast \longrightarrow \quad H(s, X_s) = \sup_{\pi \in \mathcal{A}_{s,t}} \mathbb{E}\left[H(t, X_t^{\pi}) \mid \mathcal{F}_s \right]$

$\boxed{\xi}$

Lecture 10 begins here

Ito process: $dX_t = u_t\,dt + v_t\,dB_t$ for admissible u and v.

Ito diffusion: $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)\,dB_t$ where μ, σ are functions.

- if X is an Ito diffusion then it is an Ito process.

The converse is not true. (i.e. Ito diffusion $\Rightarrow$ Ito process)

Ex: Suppose X satisfies $dX_t = B_{t-1} \chi_{t \geq 1}\,dt + dB_t$

then X is an Ito process but not an Ito diffusion.

- SDE:

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$$

$$X_t = X_0 + \int_0^t \mu(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s$$

$$X_T = X_t + \int_t^T \mu(s, X_s)\,ds + \int_t^T \sigma(s, X_s)\,dB_s$$

Regarding path dependence: consider $dX_t = \mu X_t\,dt + \sigma X_t\,dB_t, \quad X_0 = x$

$$dY_t = \kappa(\theta - X_t)\,dt + \eta\,dB_t, \quad Y_0 = y$$

$$\Rightarrow \quad X_t = x e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma(B_t - B_0)}$$

$$Y_t = y e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \eta \int_0^t e^{-\kappa(t-u)}\,dB_u$$

Hamilton-Jacobi-Bellman Equation - Part 1

Theorem

Let $\{X_t^\pi\}_{t\in[0,T]}$ *be a controlled Markov diffusion given by*

$$dX_t^\pi = \mu(t, X_t^\pi, \pi(t, X_t^\pi))\, dt + \sigma(t, X_t^\pi, \pi(t, X_t^\pi))\, dB_t\,,$$

and let H *be the dynamic value function given by*

$$H(t, X_t) = \sup_{\pi\in\mathcal{A}_{t,T}} \mathbb{E}[\psi(X_T^\pi)|\mathcal{F}_t]\,,$$

for a bounded continuous function ψ*. If* $H \in C^{1,2}([0,T]\times\mathbb{R})$ *then* H *satisfies the PDE with terminal conditions*

$$\sup_{\pi\in\mathbb{R}}\left\{\frac{\partial H}{\partial t} + \mu(t,x,\pi)\,\frac{\partial H}{\partial x} + \frac{1}{2}\,\sigma^2(t,x,\pi)\,\frac{\partial^2 H}{\partial x^2}\right\} = 0\,, \qquad (*)$$

$$H(T,x) = \psi(x)\,.$$

If $\pi^* : [0,T]\times\mathbb{R}\to\mathbb{R}$ *obtains the supremum in* $(*)$ *then* π^* *is an optimal control.*

proof: introduce operator $\mathcal{L}^{\pi} = \mu^{\pi} \partial_x + \frac{1}{2}(\sigma^{\pi})^2 \partial_{xx}$

let π be arbitrary. Let $Y_t = H(t, X_t^{\pi})$. Apply Itô lemma to Y_t:

$$dY_t = \left(\partial_t H(t, X_t^{\pi}) + \mathcal{L}^{\pi} H(t, X_t^{\pi})\right) dt + \sigma(t, X_t^{\pi}, \pi(t, X_t^{\pi}))\, \partial_x H(t, X_t^{\pi})\, dB_t$$

$$Y_{t+h} - Y_t = \int_t^{t+h} \partial_t H(s, X_s^{\pi}) + \mathcal{L}^{\pi} H(s, X_s^{\pi})\, ds + \int_t^{t+h} \sigma^{\pi}\, \partial_x H(s, X_s^{\pi})\, dB_s$$

$$\mathbb{E}\left[Y_{t+h} \mid \mathcal{F}_t\right] - Y_t = \mathbb{E}\left[\int_t^{t+h} \partial_t H(s, X_s^{\pi}) + \mathcal{L}^{\pi} H(s, X_s^{\pi})\, ds \,\Big|\, \mathcal{F}_t\right] + 0$$

$$\mathbb{E}\left[H(t+h, X_{t+h}^{\pi}) \mid \mathcal{F}_t\right] - H(t, X_t) = \mathbb{E}\left[\int_t^{t+h} \partial_t H(s, X_s^{\pi}) + \mathcal{L}^{\pi} H(s, X_s^{\pi})\, ds \,\Big|\, \mathcal{F}_t\right]$$

$$0 \geq \mathbb{E}\left[\int_t^{t+h} \partial_t H(s, X_s^{\pi}) + \mathcal{L}^{\pi} H(s, X_s^{\pi})\, ds \,\Big|\, \mathcal{F}_t\right]$$

$$0 \geq \mathbb{E}\left[\frac{1}{h}\int_t^{t+h} \partial_t H(s, X_s^\pi) + \mathcal{L}^\pi H(s, X_s^\pi)\, ds \;\middle|\; \mathcal{F}_t\right]$$

$\lim h \searrow 0$

like taking derivative

$$0 \geq \partial_t H(t, X_t^\pi) + \mathcal{L}^\pi H(t, X_t^\pi)$$

$$0 \geq \partial_t H(t, x) + \mu(t, x, \pi)\, \partial_x H(t, x) + \frac{1}{2}\sigma^2(t, x, \pi)\, \partial_{xx} H(t, x) \quad *$$

Let $\hat{\pi}$ be εh-optimal from time t to $t+h$:

recall: $$H(t, X_t) = \sup_{\pi \in \mathcal{A}_{t,t+h}} \mathbb{E}\left[H(t+h, X_{t+h}^\pi) \middle| \mathcal{F}_t\right]$$

$$\Longrightarrow \quad H(t, X_t) \geq \mathbb{E}\left[H(t+h, X_{t+h}^{\hat{\pi}}) \middle| \mathcal{F}_t\right] \geq H(t, X_t) - \varepsilon h$$

proceed as before: let $Y_t = H(t, X_t^{\hat{\pi}})$ Now Apply Itô & write in integral form as before to get:

$$Y_{t+h} - Y_t = \int_t^{t+h} \partial_t H(s, X_s^{\hat{\pi}}) + \mathcal{L}^{\hat{\pi}} H(s, X_s^{\hat{\pi}})\, ds + \int_t^{t+h} \sigma^{\hat{\pi}} \partial_{xx} H(s, X_s^{\hat{\pi}})\, dB_s$$

$$\mathbb{E}\left[H(t+h, X_{t+h}^{\hat{\pi}}) \,|\, \mathcal{F}_t\right] - H(t, X_t) = \mathbb{E}\left[\int_t^{t+h} \partial_t H(s, X_s^{\hat{\pi}}) + \mathcal{L}^{\hat{\pi}} H(s, X_s^{\hat{\pi}})\, ds \,\Big|\, \mathcal{F}_t\right]$$

$$-\varepsilon h \leq \mathbb{E}\left[\int_t^{t+h} \partial_t H(s, X_s^{\hat{\pi}}) + \mathcal{L}^{\hat{\pi}} H(s, X_s^{\hat{\pi}})\, ds \,\Big|\, \mathcal{F}_t\right]$$

$$-\varepsilon \leq \mathbb{E}\left[\frac{1}{h}\int_t^{t+h} \partial_t H(s, X_s^{\hat{\pi}}) + \mathcal{L}^{\hat{\pi}} H(s, X_s^{\hat{\pi}})\, ds \,\Big|\, \mathcal{F}_t\right]$$

lim $h \searrow 0$

$$-\varepsilon \leq \partial_t H(t, X_t^{\hat{\pi}}) + \mathcal{L}^{\hat{\pi}} H(t, X_t^{\hat{\pi}})$$

$$-\varepsilon \leq \partial_t H(t,x) + \mu(t,x,\hat{\pi}(t,x))\partial_x H(t,x) + \frac{1}{2}\sigma^2(t,x,\hat{\pi}(t,x))\partial_{xx} H(t,x) \quad (**)$$

(*) for any π:

$$0 \geq \partial_t H + \mu(t,x,\pi)\partial_x H + \frac{1}{2}\sigma^2(t,x,\pi)\partial_{xx} H$$

(**) for any $\varepsilon > 0$, there is a $\hat{\pi}$ (which may depend on t and x)

$$-\varepsilon \leq \partial_t H + \mu(t,x,\hat{\pi})\partial_x H + \frac{1}{2}\sigma^2(t,x,\hat{\pi})\partial_{xx} H$$

$\Longrightarrow$ (*) + (**)

$$\sup_{\pi \in \mathbb{R}} \left\{ \partial_t H + \mu(t,x,\pi)\partial_x H + \frac{1}{2}\sigma^2(t,x,\pi)\partial_{xx} H \right\} = 0$$

Hamilton-Jacobi-Bellman Equation - Part 2

Theorem

Let $\{X_t^\pi\}_{t\in[0,T]}$ be a controlled Markov diffusion given by

$$dX_t^\pi = \mu(t, X_t^\pi, \pi(t, X_t^\pi))\, dt + \sigma(t, X_t^\pi, \pi(t, X_t^\pi))\, dB_t\,,$$

and let H be the dynamic value function given by

$$\begin{aligned} H(t, X_t) = \sup_{\pi\in\mathcal{A}_{t,T}} \mathbb{E}\Big[& e^{-\int_t^T r(s, X_s^\pi, \pi(s, X_s^\pi))\, ds}\psi(X_T^\pi) \\ & + \int_t^T e^{-\int_t^u r(s, X_s^\pi, \pi(s, X_s^\pi))\, ds}\gamma(u, X_u^\pi, \pi(u, X_u^\pi))\, du \Big| \mathcal{F}_t \Big]\,, \end{aligned}$$

for bounded continuous functions ψ, γ, and r. If $H \in C^{1,2}([0,T]\times\mathbb{R})$ then H satisfies the PDE with terminal conditions

$$\sup_{\pi\in\mathbb{R}}\left\{ \frac{\partial H}{\partial t} + \mu(t,x,\pi)\,\frac{\partial H}{\partial x} + \frac{1}{2}\,\sigma^2(t,x,\pi)\,\frac{\partial^2 H}{\partial x^2} - r(t,x,\pi)\,H + \gamma(t,x,\pi)\right\} = 0\,, \qquad (*)$$

$$H(T,x) = \psi(x)\,.$$

If $\pi^* : [0,T]\times\mathbb{R}\to\mathbb{R}$ obtains the supremum in $(*)$ then π^* is an optimal control.

Comments on HJB Equation and Value Function

- In most cases the HJB equation will be highly non-linear PDE
- In general it does not have a classical solution
  - New type of weak solution definition required: **viscosity solution**
- The HJB method of solving for H and π^* results in a Markov control
  - In many cases, π^* will still be optimal even when non-Markov controls are allowed

Example 1:

$$dX_t^\pi = (\mu + \pi_t)\,dt + \sigma \pi_t \, dB_t$$

$$H(t, X_t) = \sup_{\pi \in \mathcal{A}_{t,T}} \mathbb{E}\left[X_T^\pi - \frac{\beta}{2} (X_T^\pi)^2 \,\Big|\, \mathcal{F}_t \right]$$

$\Rightarrow$

$$\boxed{\begin{aligned} &\sup_{\pi} \left\{ \partial_t H + (\mu+\pi)\partial_x H + \frac{1}{2}\sigma^2 \pi^2 \partial_{xx} H \right\} = 0 \qquad \longleftarrow (i) \\ &H(T,x) = x - \frac{\beta}{2} x^2 \end{aligned}}$$

Part I
HJB
PDE

$$\partial_\pi : \quad \partial_x H + \sigma^2 \pi \partial_{xx} H = 0 \qquad \Leftarrow (\text{Maximising wrt } \pi)$$

$$\left[\pi^* = \frac{-\partial_x H}{\sigma^2 \partial_{xx} H} \right]$$

Sub π^* in (i):

$$\partial_t H + \mu \partial_x H - \frac{(\partial_x H)^2}{\sigma^2 \partial_{xx} H} + \frac{1}{2}\sigma^2 \frac{(\partial_x H)^2}{\sigma^4 (\partial_{xx} H)^2} \partial_{xx} H = 0$$

$$\star \left[\partial_t H + \mu \partial_x H - \frac{1}{2} \frac{(\partial_x H)^2}{\sigma^2 \partial_{xx} H} = 0 \right., \quad \left. H(T,x) = x - \frac{\beta}{2} x^2 \right]$$

$$\left.\begin{aligned}
&\underline{\text{guess 1:}} \quad H(t,x) = f(t)\,g(x) \\
&\underline{\text{guess 2:}} \quad H(t,x) = f_0(t) + f_1(t)\,g(x) \\
&\underline{\text{guess 3:}} \quad H(t,x) = f_0(t) + f_1(t)\,x + f_2(t)\,x^2
\end{aligned}\right\} \text{3 guesses to solve PDE in HJB problem}$$

$$\left.\begin{aligned}
\partial_t H &= f_0' + f_1' x + f_2' x^2 \\
\partial_x H &= f_1 + 2 f_2 x \\
\partial_{xx} H &= 2 f_2
\end{aligned}\right\} \text{Trying guess 3}$$

$$H(T,x) = f_0(T) + f_1(T)x + f_2(T)x^2 = x - \frac{\beta}{2}x^2$$

$$\Longrightarrow \quad f_0(T) = 0, \quad f_1(T) = 1, \quad f_2(T) = -\frac{\beta}{2}$$

Using Terminal Condition

substitute into PDE ⊛ :

$$f_0' + f_1' x + f_2' x^2 + \mu(f_1 + 2f_2 x) - \frac{1}{2\sigma^2}\frac{(f_1 + 2f_2 x)^2}{2f_2} = 0$$

$$f_0' + f_1' x + f_2' x^2 + \mu f_1 + 2\mu f_2 x - \frac{1}{2\sigma^2}\frac{f_1^2 + 4f_1 f_2 x + 4f_2^2 x^2}{2f_2} = 0$$

$$f_0' + f_1' x + f_2' x^2 + \mu f_1 + 2\mu f_2 x - \frac{f_1^2}{4\sigma^2 f_2} - \frac{f_1}{\sigma^2}x - \frac{f_2}{\sigma^2}x^2 = 0$$

$$f_0' + \mu f_1 - \frac{f_1^2}{4\sigma^2 f_2} + \left(f_1' + 2\mu f_2 - \frac{f_1}{\sigma^2}\right)x + \left(f_2' - \frac{f_2}{\sigma^2}\right)x^2 = 0$$

writing eq^n in powers of x

$$\left\{\begin{array}{lll} f_0' + \mu f_1 - \dfrac{f_1^2}{4\sigma^2 f_2} = 0 & \text{— (i)} & f_0(T) = 0 \\ f_1' - \dfrac{f_1}{\sigma^2} + 2\mu f_2 = 0 & \text{— (ii)} & f_1(T) = 1 \\ f_2' - \dfrac{f_2}{\sigma^2} = 0 & \text{— (iii)} & f_2(T) = -\dfrac{\beta}{2} \end{array}\right\}$$

Solve this from bottom to top eqⁿ.

Solving (iii):

$$f_2(t) = -\frac{\beta}{2} e^{-\frac{1}{\sigma^2}(T-t)}$$

Solving (ii):

$$f_1' - \frac{f_1}{\sigma^2} - \beta\mu e^{-\frac{1}{\sigma^2}(T-t)} = 0 \qquad f_1(T) = 1$$

$$f_1' - \frac{f_1}{\sigma^2} = \beta\mu e^{-\frac{1}{\sigma^2}(T-t)}$$

Solving using integrating factor method. Multiply both sides by $e^{-\frac{1}{\sigma^2}t}$

$$e^{-\frac{1}{\sigma^2}t} f_1' - \frac{1}{\sigma^2} e^{-\frac{1}{\sigma^2}t} f_1 = \beta\mu e^{-\frac{1}{\sigma^2}T}$$

$$\left(e^{-\frac{1}{\sigma^2}t} f_1\right)' = \beta\mu e^{-\frac{1}{\sigma^2}T}$$ (LHS written as derivative of product)

$$\int_t^T \left(e^{-\frac{1}{\sigma^2}s} f_1(s)\right)' ds = \beta\mu e^{-\frac{1}{\sigma^2}T}(T-t)$$ (Integrating between t & T)

$$e^{-\frac{1}{\sigma^2}T} f_1(T) - e^{-\frac{1}{\sigma^2}t} f_1(t) = \beta\mu e^{-\frac{1}{\sigma^2}T}(T-t)$$ (FTC on LHS)

$$-e^{-\frac{1}{\sigma^2}t} f_1(t) = \beta\mu e^{-\frac{1}{\sigma^2}T}(T-t) - e^{-\frac{1}{\sigma^2}T}$$ Sub. $f_1(T) = 1$

$$f_1(t) = e^{-\frac{1}{\sigma^2}(T-t)} - \beta\mu(T-t)e^{-\frac{1}{\sigma^2}(T-t)}$$

Solving (i):

$$f_0' + \mu f_1 - \frac{f_1^2}{4\sigma^2 f_2} = 0 \implies f_0(t) = \int_t^T \mu f_1(s) - \frac{f_1^2(s)}{4\sigma^2 f_2(s)}\, ds$$

No need to solving for $f_0(t)$ as this term will vanish in $\pi^*(t,x)$ as we do $\partial_x H$ & $\partial_{xx} H$.

Now, optimal control is:

$$\pi^*(t,x) = \frac{-\partial_x H}{\sigma^2 \partial_{xx} H} = -\left(\frac{f_1(t) + 2f_2(t)x}{\sigma^2 \; 2f_2(t)}\right)$$ //

(Can sub. value for $f_1(t)$ & $f_2(t)$).

- Example 2

$$dX_t = \pi_t\,dt + \sigma\,dB_t$$

$$H(t, X_t) = \sup_{\pi \in \mathcal{A}_{t,T}} \mathbb{E}\left[(X_T^\pi)^2 - \int_t^T c\,\pi_u^2\,du \;\middle|\; \mathcal{F}_t \right] \qquad c > 0.$$

$\Rightarrow$ Using HJB Part II, with $\gamma = -c\pi^2$ & $\Psi(x) = x^2$:

$$\sup_{\pi} \left\{ \partial_t H + \pi\,\partial_x H + \frac{1}{2}\sigma^2 \partial_{xx} H - c\pi^2 \right\} = 0 \qquad H(T,x) = x^2 \qquad *$$

Now differentiate eq^n inside sup wrt π:

$$\partial_\pi: \quad \partial_x H - 2c\pi = 0$$

$$\pi^* = \frac{\partial_x H}{2c}$$

Sub π^* back to same eq^n $*$:

$$\partial_t H + \frac{(\partial_x H)^2}{2c} + \frac{1}{2}\sigma^2 \partial_{xx} H - \frac{(\partial_x H)^2}{4c} = 0$$

$$\boxed{\partial_t H + \frac{1}{2}\sigma^2 \partial_{xx} H + \frac{(\partial_x H)^2}{4c} = 0 \qquad H(T,x) = x^2} \ *$$

assume: $H(t,x) = f_0(t) + f_1(t)x + f_2(t)x^2$

Guessing solution to PDE to be quadratic again.

Using terminal condition $H(T,x) = x^2$ we get:

$\longrightarrow \quad f_0(T) = 0, \quad f_1(T) = 0, \quad f_2(T) = 1$

Sub guess solution to PDE *:

$$f_0' + f_1' x + f_2' x^2 + \frac{1}{2}\sigma^2(2f_2) + \frac{(f_1 + 2f_2 x)^2}{4c} = 0$$

$$f_0' + f_1' x + f_2' x^2 + \sigma^2 f_2 + \frac{f_1^2}{4c} + \frac{f_1 f_2 x}{c} + \frac{f_2^2 x^2}{c} = 0$$

$$f_0' + \sigma^2 f_2 + \frac{f_1^2}{4c} + \left(f_1' + \frac{f_1 f_2}{c}\right)x + \left(f_2' + \frac{f_2^2}{c}\right)x^2 = 0 \leftarrow \text{(writing in powers of } x\text{)}$$

$$f_0' + \sigma^2 f_2 + \frac{f_1^2}{4c} = 0, \qquad f_0(T) = 0 \quad \text{(i)}$$

$$f_1' + \frac{f_1 f_2}{c} = 0, \qquad f_1(T) = 0 \quad \text{(ii)}$$

$$f_2' + \frac{f_2^2}{c} = 0, \qquad f_2(T) = 1 \quad \text{(iii)}$$

Solve bottom to top eq^n

Solving (iii)

$$f_2' = \frac{f_2^2}{c}$$

$$\frac{f_2'}{f_2^2} = \frac{1}{c}$$

$$\left(\frac{1}{f_2}\right)' = -\frac{1}{c}$$

L.H.S is $\left(\frac{1}{f_2}\right)'$

$$\frac{1}{f_2(T)} - \frac{1}{f_2(t)} = \frac{-(T-t)}{c}$$

Integrating from t to T

$$1 + \frac{T-t}{c} = \frac{1}{f_2(t)}$$

$f_2(T) = 1$

$$f_2(t) = \frac{c}{c+(T-t)}$$

Solving (ii) & (i)

$f_1(t) = 0$ for all t. $\Rightarrow$ $f_0(t) = \int_t^T \sigma^2 f_2(s)\,ds$

$$\Rightarrow H(t,x) = \int_t^T \frac{\sigma^2 c}{c+(T-s)} \, ds + \frac{c x^2}{c+T-t} \quad //$$

Now for optimal control:

$$\pi^*(t,x) = \frac{\partial_x H}{2c} = \frac{x}{c+(T-t)} \quad //$$

$$\frac{\partial_x H}{2c} = \frac{f_1 + 2 f_2 x}{2c} = \frac{0 + 2 \frac{c}{c+(T-t)} x}{2c}$$

$$= \frac{x}{c+(T-t)}$$

Stochastic Analysis

Lectures 11

This Lecture

Discontinuous Processes

- Motivation
- The Poisson Process
- Stochastic Integrals with respect to Poisson Processes
- Compound Processes and Stochastic Integrals
- Ito's Formula for Compound Poisson Processes
- Feynman-Kac formula for jump processes

Motivation

In finance, assets prices jumps up or down in a short amount of time like jumps & can no longer be assumed continuous.

- Recall from a previous lecture our motivation to have random dynamics

$$\frac{dV_t}{dt} = F(t, V_t) + \text{“randomness”}\,, \qquad V_0 = v$$

- This eventually led to our formulation of SDEs

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t\,, \qquad X_0 = x$$

- All such resulting processes $\{X_t\}_{t\geq 0}$ were continuous

- Some applications require modeling of discontinuities, random or not, in the dynamics

- Formally, we may want to make sense of something written as

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t + \text{“}\beta(t, X_t)\,dJ_t\text{”}$$

where dJ_t is a term which contributes jumps into X

Integral Representation

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t\,, \qquad\qquad X_0 = x$$

- This SDE with respect to Brownian motion does not have meaning on its own
- It must be interpreted rigorously in terms of the integral form:

$$X_t = x + \int_0^t \mu(u, X_u)\,du + \int_0^t \sigma(u, X_u)\,dB_u$$

  where we had to assign meaning to the stochastic integral
- Similarly, our jump dynamics will require us to assign meaning to integrals of the form

$$\int_0^t \beta_u\,dJ_u$$

  for appropriate integrand processes $\{\beta_t\}_{t\geq 0}$ and integrator processes $\{J_t\}_{t\geq 0}$

Poisson Process

- Up to now we have used Brownian motion as our building block for diffusion processes
  - This led to the theory of stochastic integration with respect to Brownian motion
- The building block of our discontinuous processes will be the Poisson process analogous to Brownian motion for SDE before.
- Let $\{N_t\}_{t\geq 0}$ be a stochastic process satisfying
  i) $N_0 = 0$
  ii) For $t_1 \leq t_2 \leq t_3$, the increment $N_{t_3} - N_{t_2}$ is independent of N_{t_1}
  iii) For $t \geq 0$ and $u > 0$ the increment $N_{t+u} - N_t$ has distribution $Pois(\lambda u)$

  Then we say $\{N_t\}_{t\geq 0}$ is a **Poisson process with intensity λ**

Properties of Poisson Process

- Let $\{\mathcal{G}_t\}_{t\geq 0}$ be any filtration such that $\{N_t\}_{t\geq 0}$ is adapted
  - $\{N_t\}_{t\geq 0}$ is **not** a martingale with respect to $\{\mathcal{G}_t\}_{t\geq 0}$ → Note: While it was martingale in case of B.M.
  - If $M_t = N_t - \lambda t$ then $\{M_t\}_{t\geq 0}$ is a martingale which we call the **compensated Poisson process**
- We can (and will always) choose our Poisson processes to be **right-continuous**
- Define a sequence of random variables, for $k \in \mathbb{N}$

$$\tau_k = \inf\{t \in \mathbb{R}^+ : N_t \geq k\}$$

$\tau_0 = 0$
$\tau_1 =$ "first time that $N_t = 1$"

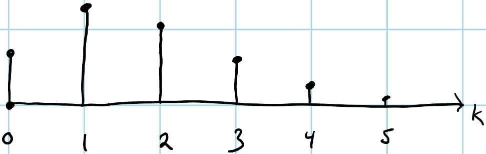
and let $\hat{\tau}_k = \tau_k - \tau_{k-1}$ — τ first time our stochastic process reaches value k.

- Then the $\hat{\tau}_k$'s are independent and $\hat{\tau}_k \sim Exp(\lambda)$
- The sequence $\{\hat{\tau}_k\}_{k\in\mathbb{N}}$ is called the sequence of **inter-arrival times** of $\{N_t\}_{t\geq 0}$

Third property of Poisson process:

iii) $t \geq\ ,\ u > 0$ then $N_{t+u} - N_t \sim \mathrm{Pois}(\lambda u)$

$$\mathbb{P}(N_{t+u} - N_t = k) = \frac{(\lambda u)^k}{k!} e^{-\lambda u}, \quad k = 0, 1, 2, \dots$$



Showing Poisson process is not martingale:

- $$\mathbb{E}[N_t \mid \mathcal{G}_s] = \mathbb{E}[N_t - N_s + N_s \mid \mathcal{G}_s] = \mathbb{E}[N_t - N_s \mid \mathcal{G}_s] + \mathbb{E}[N_s \mid \mathcal{G}_s]$$
$$= \mathbb{E}[N_t - N_s] + N_s$$
$$= \lambda(t-s) + N_s \neq N_s$$

$\Longrightarrow$ N is not a martingale

- Showing compensated Poisson is martingale:

$$M_t = N_t - \lambda t$$

$$\mathbb{E}[M_t | \mathcal{G}_s] = \mathbb{E}[N_t - \lambda t | \mathcal{G}_s] = \mathbb{E}[N_t - N_s + N_s - \lambda t | \mathcal{G}_s]$$

$$= \lambda(t-s) + N_s - \lambda t$$

$$= N_s - \lambda s = M_s$$

$\Longrightarrow$ M is a martingale

Interarrival time is exponentially dist.:

$$\hat{\tau}_k \sim Exp(\lambda) \Longrightarrow \hat{\tau}_k \text{ has CDF : } F(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}$$

- larger λ gives larger value of process N_t but will gives us lower value of interarrival times.

Desired Properties of Stochastic Integral

- Want to assign value to expression of the form

$$\int_0^t \beta_u \, dN_u$$

such that it has some desirable properties

- Stochastic integral with respect to $\{B_t\}_{t\geq 0}$ yields martingale
  - $\{B_t\}_{t\geq 0}$ is martingale, so "infinitesimal increment" dB_t has zero mean

$dB_t \approx \Delta B_{t_k} = B_{t_{k+1}} - B_{t_k}$

- Poisson process is not a martingale, shouldn't expect the above expression to be one either

- However, we should choose the definition such that

$$\int_0^t \beta_u \, dM_u$$

$M_t = N_t - \lambda t$

is a martingale where $\{M_t\}_{t\geq 0}$ is compensated process

One Approach and a Problem

$$\int_0^t \beta_u \, dN_u$$

- A natural selection for the above quantity is

$$\int_0^t \beta_u \, dN_u = \sum_{0 \le u \le t} \beta_u \, \Delta N_u = \sum_{k=1}^{N_t} \beta_{\tau_k}$$

where $\Delta N_u = N_u - N_{u^-}$ $\quad N_{u^-} = \lim_{h \nearrow 0} N_{u+h} = \lim_{h \searrow 0} N_{u-h}$

- In words, if N jumps at time t, then the integral jumps by β_t
- Consistent with how we would like to interpret $dX_t = \beta_t \, dN_t$
- Problem: consider integrating the process $\beta_t = \Delta N_t$
  - Then compensated integral will not be a martingale

↳ lets show this with example, next pg.

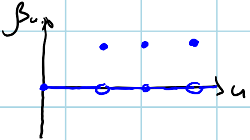
- Example: compute $\int_0^t \beta_u \, dM_u$ where $\beta_t = \Delta N_t$

$$= N_t - N_{t-}$$

$$\int_0^t \beta_u \, dM_u = \int_0^t \beta_u \, dN_u - \int_0^t \beta_u \lambda \, du$$

$$= \sum_{0 \le u \le t} \beta_u \Delta N_u - \int_0^t \beta_u \lambda \, du$$

$$\beta_u = \begin{cases} 0, & \text{if no jump @ time } u \\ 1, & \text{if jump @ time } u. \end{cases}$$



$$= \sum_{0 \le u \le t} (\Delta N_u)^2 - 0 \quad (a.s) \qquad (\beta_u = \Delta N_u)$$

$$= \sum_{0 \le u \le t} \Delta N_u$$

$$= N_t \implies \int_0^t \Delta N_u \, dM_u = N_t$$

$\implies$ not a <u>martingale</u>.

Solution to this Problem

- Evaluate integrand at its left limit at jump times
- Emphasize by explicitly writing left limit in all jump integrals
- Now if we define

$$\int_0^t \beta_{u^-}\, dN_u = \sum_{0\le u\le t} \beta_{u^-}\, \Delta N_u = \sum_{k=1}^{N_t} \beta_{\tau_k^-}$$

then

$$\int_0^t \beta_{u^-}\, dM_u = \sum_{0\le u\le t} \beta_{u^-}\, \Delta N_u - \int_0^t \lambda\, \beta_u\, du$$

is a martingale

Compound Poisson Process

- Let $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $\mathbb{E}[|Z_i|] < \infty$, and let $\{N_t\}_{t\geq 0}$ be a Poisson process with intensity λ

$J_0 = 0$
$J_{\tau_1} = Z_1$
$J_{\tau_2} = Z_1 + Z_2$
$\vdots$

- Let $\{J_t\}_{t\geq 0}$ be a process defined by

$$J_t = \sum_{k=1}^{N_t} Z_k$$

- We call $\{J_t\}_{t\geq 0}$ a **compound Poisson process**

- Paths of $\{J_t\}_{t\geq 0}$ are right-continuous, jump at the same times as $\{N_t\}_{t\geq 0}$, but jumps will be of random size Z_k

- Can similarly define the jump integral

$\Delta J_u = J_u - J_{u-}$

$$\int_0^t \beta_{u^-}\, dJ_u = \sum_{0\leq u\leq t} \beta_{u^-}\, \Delta J_u = \sum_{k=1}^{N_t} \beta_{\tau_k^-}\, Z_k$$

Ito's Formula (for Compound Poisson Process)

Theorem

Suppose $\{X_t\}_{t \geq 0}$ *satisfies*

$$dX_t = \beta_{t^-}\, dJ_t$$

where $\{J_t\}_{t \geq 0}$ *is a compound Poisson process, and let* $Y_t = g(t, X_t)$*. Then*

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)\, dt + \Big(g(t, X_{t^-} + \beta_{t^-} Z_{N_t}) - g(t, X_{t^-}) \Big) dN_t$$

- Intuition: if a jump in N occurs at time τ:
  - X changes in value by: $\beta_{\tau^-} Z_{N_\tau}$
  - Y changes in value by: $g(\tau, X_{\tau^-} + \beta_{\tau^-} Z_{N_\tau}) - g(\tau, X_{\tau^-})$

Ito formula for compound Poisson process:

- **Example:** solve for Y.

$$* \quad dY_t = \eta Y_{t^-} \, dN_t \quad , \quad Y_0 = 1. \quad \eta > 0 \text{ constant.}$$

guess: $Y_t = A e^{BN_t} \implies A = 1$

let $X_t = N_t \implies dX_t = dN_t$

$Y_t = g(t, X_t)$ where $g(t,x) = e^{Bx}$

$$dY_t = \partial_t g(t, X_t) dt + \left(g(t, X_{t^-} + 1) - g(t, X_{t^-}) \right) dN_t$$

$$dY_t = \left(e^{B(X_{t^-}+1)} - e^{BX_{t^-}} \right) dN_t$$

$$dY_t = e^{BX_{t^-}} (e^B - 1) dN_t \qquad \text{choose } B = \log(1+\eta)$$

$$dY_t = Y_{t^-} \eta \, dN_t$$

matching with *

$$\Rightarrow \quad Y_t = e^{\log(1+\eta) N_t} \quad \text{satisfies the SDE.}$$

$$Y_t = (1+\eta)^{N_t}$$

$$Y_0 = 1$$

$$Y_{\tau_1} = Y_0 + Y_0 \eta = Y_0 (1+\eta)$$

$$Y_{\tau_2} = Y_{\tau_1} + Y_{\tau_1} \eta = Y_{\tau_1} (1+\eta) = Y_0 (1+\eta)^2$$

$$\vdots$$

$$Y_{\tau_k} = Y_0 (1+\eta)^k$$

Example 2: $dY_t = \mu Y_t\, dt + \eta\, dN_t \qquad Y_0 = y$ constant.

define $Z_t = e^{-\mu t} Y_t$ → kills exponential dynamics & only have $\eta\, dN_t$ dynamics in Z_t

$$dZ_t = -\mu e^{-\mu t} Y_t\, dt + e^{-\mu t} dY_t$$ → Applying Ito's lemma for CT's version.

$$= -\mu e^{-\mu t} Y_t\, dt + e^{-\mu t}(\mu Y_t\, dt + \eta\, dN_t)$$ → sub dY_t

$$dZ_t = \eta e^{-\mu t} dN_t$$

$$Z_t = Z_0 + \eta \int_0^t e^{-\mu u}\, dN_u$$

$$Z_t = Z_0 + \eta \sum_{k=1}^{N_t} e^{-\mu \tau_k}$$

$$e^{-\mu t} Y_t = Y_0 + \eta \sum_{k=1}^{N_t} e^{-\mu \tau_k}$$

$$Y_t = e^{\mu t} y + \eta \sum_{k=1}^{N_t} e^{\mu(t-\tau_k)}$$

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Feynman-Kac Formula for Compound Poisson Processes

Theorem

Suppose Suppose $\{X_t\}_{t\geq 0}$ satisfies

$$dX_t = \beta(t, X_{t^-})\, dJ_t$$

where $\{J_t\}_{t\geq 0}$ is a compound Poisson process, and let ψ be a bounded Borel-measurable function. Let $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ be the function which satisfies

$$u(t, X_t) = \mathbb{E}[\psi(X_T) | \mathcal{F}_t]\,.$$

Then u satisfies the partial integro-differential equation

$$\frac{\partial u}{\partial t} + \lambda \mathbb{E}\Big[u(t, x + \beta(t,x)Z) - u(t,x)\Big] = 0\,,$$
$$u(T, x) = \psi(x)\,.$$

Outline of derivation:

$$dX_t = \beta(t, X_{t-})\, dJ_t$$

$$u(t, X_t) = \mathbb{E}\left[\varphi(X_T) \mid \mathcal{F}_t\right]$$

let $Y_t = u(t, X_t)$.

Then Y is a martingale. $\Rightarrow$ expected changes in Y are zero.

Using Ito formula for compound process:

$$dY_t = \partial_t u(t, X_t)\, dt + \left(u(t, X_{t-} + \beta(t, X_{t-}) Z_{N_t}) - u(t, X_{t-})\right) dN_t$$

Taking conditional $\mathbb{E}$ of both sides:

$$\mathbb{E}\left[dY_t \mid \mathcal{F}_t\right] = \partial_t u(t, X_t)\, dt + \mathbb{E}\left[\left(u(t, X_{t-} + \beta(t, X_{t-}) Z_{N_t}) - u(t, X_{t-})\right) dN_t \mid \mathcal{F}_t\right]$$

$$= \partial_t u(t, X_t)\, dt + \mathbb{E}\left[dN_t\right] \mathbb{E}\left[u(t, X_{t-} + \beta(t, X_{t-}) Z_{N_t}) - u(t, X_{t-}) \mid \mathcal{F}_t\right]$$

$$= \partial_t u(t, X_t)\, dt + \lambda\, \mathbb{E}\left[u(t, X_{t-} + \beta(t, X_{t-}) Z) - u(t, X_{t-}) \mid \mathcal{F}_t\right] dt = 0$$

$$\Rightarrow \quad \left.\begin{aligned} &\partial_t u(t,x) + \lambda\, \mathbb{E}\left[u(t, x+\beta(t,x)Z) - u(t,x)\right] = 0 \\ &u(T,x) = \varphi(x) \end{aligned}\right\} \quad \boxed{\xi}$$

- Example: $\quad dX_t = dN_t \qquad X_0 = 0.$

  Compute $\mathbb{E}\left[\varphi(X_T)\right]$.

  let u satisfy $\quad u(t, X_t) = \mathbb{E}\left[\varphi(X_T) \mid \mathcal{F}_t\right]$

  $\Rightarrow \mathbb{E}\left[\varphi(X_T)\right] = u(0,0)$

  by Feynman-Kac:

$$\left.\begin{aligned} &\partial_t u + \lambda\big(u(t,x+1) - u(t,x)\big) = 0 \\ &u(T,x) = \varphi(x) \end{aligned}\right\} \text{ for } x \in \mathbb{N}$$

alternate perspective: $u_n = u(\cdot, n)$

$$\left.\begin{aligned} u_n' + \lambda(u_{n+1} - u_n) &= 0 \\ u_n(T) &= \varphi_n \end{aligned}\right\} \text{infinite system of coupled ODE's.}$$

Related Further Topics

- Some applications may require models where the intensity of jumps is not constant
  - Leads to construction of **doubly stochastic Poisson processes**
- The intensity of a (doubly stochastic) Poisson process can be interpreted in a similar fashion to the drift component of an Ito process
  - Also in a similar fashion to Ito processes, there is a Girsanov theorem for doubly stochastic Poisson processes
  - The resulting idea is the same: changing the intensity is equivalent to changing the underlying probability measure