lec I

- Bond (securitised loan) which can be sold unlike loans.
- Money Market → debt less than 1 yr in maturity
  Bond Market → " mor " 1yr " .

Interest Rates & Related Contracts

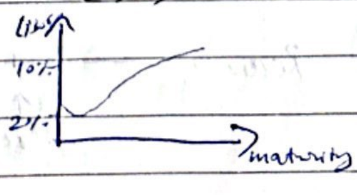
- ZCB (Zero Coupon Bond)
  T maturity ZCB pays 1 unit of currency at time T. The contract value at time $t < T$ is $P(t,T)$. The final payoff is called face value/notional value.
  $P(T,T) = 1$.

- Libor
  Rate at which Banks borrow from each other. Maturity ranges from overnight to 12 months.
  The Spot-Libor Rate at time t for maturity T is the constant rate at which an investment has to be made to produce 1 unit of currency at maturity starting from $P(t,T)$.

$$P(t,T)\left[1 + (T-t)\,L(t,T)\right] = 1$$

$$\boxed{L(t,T) = \frac{1 - P(t,T)}{(T-t)\,P(t,T)}}$$

The yield curve or zero-coupon curve is graph of Libor rates vs maturity. This f'n $t \to L(t,T)$ is called term structure of IR's at time t.



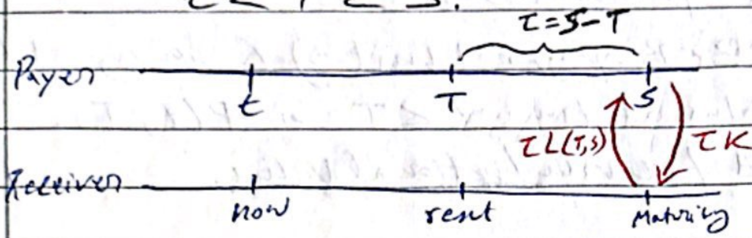
- Bonds
  → Issued by Government (Gilts, T-Bills) or Corporate bonds.
  Gilts are index-linked eg RPI.

- **FRA (Forward Rate Agreement)**

Allows one to lock-in Interest Rate between future times T & S. FRA is a contract that gives it's holder an IR payment for period T to S with fixed rate K at maturity S against floating rate $L(T,S)$.

t = Current time, T = reset time, S = maturity

$t < T < S$.



→ Receiver FRA : Receive Fixed K & Pay floating $L(T,S)$

→ Payer FRA : Pay fixed K & receive floating $L(T,S)$

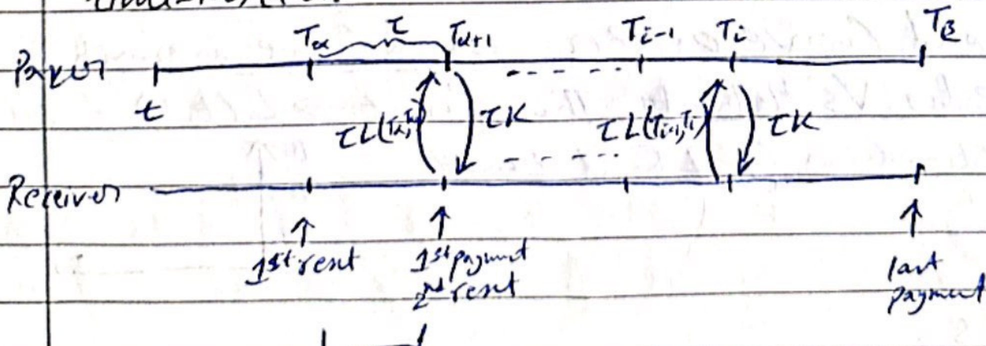
The Value K that makes FRA contract fair is called Forward Libor Interest Rate (K).

$$K = F(t;T,S) = \frac{1}{S-T}\left[\frac{P(t,T)}{P(t,S)} - 1\right]$$

Note: We could use any floating rate Libor or Euribor.

- **IRS (Interest Rate Swap)**

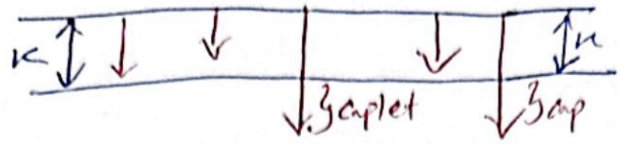
Contract that exchanges IR payments between two differently indexed legs starting from a future time-instant.



Fixed leg → τK

Float leg → $\tau L(T_\alpha, T_{\alpha+1})$

CAP :



Pays if over K.

So, IRS swaps IR, fixed: $\tau_i K$ with float: $\tau_i L(T_{i-1}, T_i)$ for various legs (reset time).

→ Receiver IRS : Receive K pay $L(T_{i-1}, T_i)$

→ Payer IRS : Pay K receive $L(T_{i-1}, T_i)$

- Interest Rate Cap

Allows a company to protect itself from future Libor rate increase.

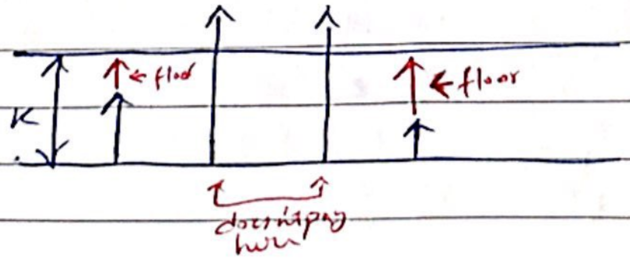
$$\boxed{\text{Payoff} = \tau_i \left(L(T_{i-1}, T_i) - K\right)^+}$$

So, the company only has to pay at most K. If Libor rate increases, the difference is covered by cap contract. Can be seen as Payer IRS where each exchange is executed only if it has +ve value. Payment is at $T_{\alpha+1}, \dots, T_\beta$ & reset are at $T_\alpha, \dots, T_{\beta-1}$.
Cap contract can be decomposed into caplets, one for each time moment.

- Interest Rate Floor

To meet ongoing cash obligation, one could buy floor contract to receive atleast K. It's like receiver IRS where exchange only happens if it's positive value.

$$\boxed{\text{Payoff} = \tau_i \left[K - L(T_{i-1}, T_i)\right]^+}$$



guarantees atleast K.

For each moment of time we have floorlets.

Other Contracts (Bonds)

- Fixed Coupon Bond (FCB)
  Bond that pays Nominal at maturity T_n and fixed coupon at sequence of settlement dates $T_1 < T_2 < \dots < T_n$. [Note: At maturity we get Coupon plus Nominal].

$$\tau_i = T_i - T_{i-1} \leftarrow \text{tenor structure of FCB.}$$

$$\left[\underset{\substack{\uparrow \\ \text{fixed cash coupon}}}{C_i} = \underset{\substack{\uparrow \\ \text{fixed rate}}}{K} \tau_i \underset{\substack{\uparrow \\ \text{Nominal}}}{N} \right]$$

  Can be decomposed into many ZCB's.

- Floating Rate Note (FRN)
  Similar to FCB but with variable rate at each settlement (reset) dates usually Libor.

$$\tau_i = T_i - T_{i-1} \leftarrow \text{tenor structure of FRN}$$

$$\left[C_i = \underset{\substack{\uparrow \\ \text{only different with FCB} \\ \text{is the rate.}}}{L(T_{i-1}, T_{i-1})} \tau_i N \right]$$

[Note: FRA & IRS are rate swaps & these contracts are independent on curve dynamics while the Caps/floors (like insurance) are dependent on curve dynamics]

Lec II

Black-Scholes Framework

Asset price have dynamics under $\mathbb{P}$:

$$\begin{bmatrix} dB_t = B_t r\,dt, \quad B_0 = 1, \\ dS_t = S_t[\mu\,dt + \sigma\,dW_t], \quad 0 \le t \le T \end{bmatrix}$$

μ: mean return of stock; σ: volatility of stock

- European Call Option

Option payoff is given as:

$$\left[Y = (S_T - K)^+ = \max(S_T - K, 0) \right]$$

where, S_T = stock price at maturity T

K = Strike price agreed earlier to buy stock at time T.

→ $V_t = V(t, S_t)$ is the value of call option at time t & can be found by solving Black-Scholes pde.

→ Using Feynman-Kac, the solution of Black-Scholes pde can be stated as discounted expectation s.t. the expectation is taken w.r.t the martingale measure $\mathbb{Q}$ i.e. prob. measure under which risky-asset price $\frac{S_t}{B_t}$ is martingale.

$$dS_t = S_t[r\,dt + \sigma\,dW_t^{\mathbb{Q}}]$$

$$V_{BS}(t) = \mathbb{E}^{\mathbb{Q}}\left(e^{-r(T-t)} Y \mid \mathcal{F}_t\right)$$

→ Note: We applied Girsanov's Thm to change measure from $\mathbb{P}$ to $\mathbb{Q}$ using Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

So, $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$ becomes

$$dS_t = r S_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}}$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\left(-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T - \frac{\mu - r}{\sigma} W_T\right)}$

Note: the drift rate μ becomes r (risk-free IR) in $\mathbb{Q}$, so investors with different outlook of world can agree on unique option price.

→ In B-S world S_t is risky asset & B_t is risk free asset, & $\frac{S_t}{B_t}$ is a fair game i.e. martingale.

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Stochastic Calculus Review

- Brownian Motion

  A adapted process W is called Brownian Motion if

  (i) $W_0 = 0$ (ii) $P(W_t - W_s) = P(W_{t-s})$ stationary increment

  (iii) $P(W_t - W_s \in A | \mathcal{F}_s) = P(W_t - W_s \in A)$ independent increment

  (iv) $W_t \sim N(0, t)$ (v) $t \to W_t$ CTS but not differentiable.

  $\forall 0 < s < t$.

- Quadratic Variation of stochastic process X is

$$\langle X \rangle_t = \lim_{\|\text{partition}\| \to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2$$

- Martingale (stochastic process representing fair Game)

  $\mathcal{F}_t$ adapted process M is $\mathcal{F}_t$-martingale if:

  - $\mathbb{E}(|M_t|) < \infty, \forall t > 0$ (integrable)
  - $\mathbb{E}(M_t | \mathcal{F}_s) = M_s, \forall s < t$

- Property of Stochastic Integral

  → Ito Isometry: $\mathbb{E}\left[\left(\int_0^t h_s \, dW_s\right)^2\right] = \mathbb{E}\left(\int_0^t |h_s|^2 ds\right)$

  → Quadratic Variation: $\left\langle \int_0^t h_s \, dW_s \right\rangle_t = \int_0^t |h_s|^2 ds$

- Ito Formula

$$df(X_{(t)}, t) = f_t(X_t, t) dt + f_x(X_t, t) dX_t + \frac{1}{2} f_{xx}(X_t, t) \phi^2 dt$$

- Stochastic Exponential of a Martingale M is: $\mathcal{E}(M)_t = \exp\{M_t - \frac{1}{2}\langle M \rangle_t\}$

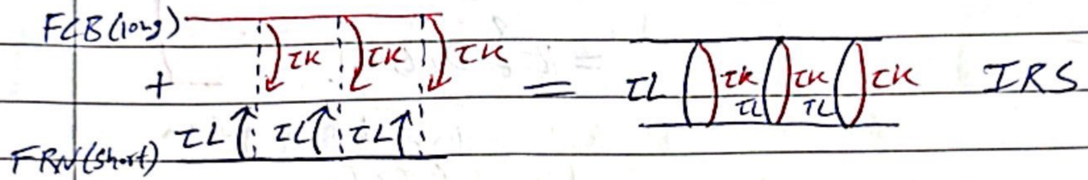
Lec 3

Replication

To replicate asset in Financial Market is done by building a portfolio of assets (buying & selling) with overall cashflows identical to those of the asset being replicated. This enables us to find the price of complicated contracts.

- IRS (IR Swaps) can be replicated by identical timing

$$\begin{cases} \text{Buy Fixed Coupon Bond (FCB) — long} \\ \text{sell Floating Rate Note (FRN) — short} \end{cases}$$



where $\tau = T_i - T_{i-1}$, K = fixed rate of FCB, L = Libor rate

- European Call option uses Risk Neutral pricing where a replicating portfolio & no-arbitrage argument express price as discounted expectation under risk-neutral measure.

- No Arbitrage

  Two ways to express no-arbitrage

  (i) every claim can be replicated by self-financing strategy.

  (ii) The underlying risky asset measured wrt Bank Account is a martingale under the measure used to price claims with expectations. This measure is Equivalent Martingale Measure (EMM).

- 1st fundamental Thm of Asset Pricing : If EMM exist ⇒ No arbitrage in market.
- 2nd " —— " —— : If unique EMM exists

  ⇒ the market is complete i.e. any claim can be replicated with a strategy.

Risk Neutral Valuation

- To value future unknown quantities now, discount at relevant interest rate & take Expectation.

$$\mathbb{E}_t^{Q}\Big[\underbrace{\frac{B(t)}{B(T)}}_{\text{discount}}\ \underbrace{H(\text{Asset})_T}_{\text{future stochastic payoff at future time } T.}\Big] = \mathbb{E}_t^{Q}\Big[\underbrace{e^{-\int_t^T r_s ds}}_{\text{discounting}} H(\text{Asset})_T\Big]$$

Pricing ZCB

$$P(t,T) = \mathbb{E}_t^{Q}\left[\frac{B(t)}{B(T)}\, 1\right] = \mathbb{E}_t^{Q}\left[e^{-\int_t^T r_s ds} \cdot 1\right]$$
$$= \mathbb{E}_t^{Q}\, \underbrace{D(t,T)}_{\text{discounted factor from } t \text{ to } T.}$$

Price of Receiver FRA (Forward rate Agreement)

i.e. receive fixed K & pay floating $L(T,S)$.

The no-arbitrage price is:

$$\boxed{FRA(t,T,S,K) = P(t,S)\overbrace{(S-T)}^{\tau = \text{year in fraction}}K - P(t,T) + P(t,S)}$$

The price of payer FRA is same with opposite sign.

The price above is calculated by taking risk neutral Expectation of FRA Discounted cash flows.

$$FRA(t,T,S,K) = \mathbb{E}_t\left[D(t,S)\tau K - D(t,S)\tau L(T,S)\right]$$

full calculation on pg 11 lec 3.
uses iterated conditioning of Expectation $\mathbb{E}_t\mathbb{E}_T(\cdot) = \mathbb{E}_t(\cdot)$ for $t<T$
& $D(t,S) = D(t,T)\,D(T,S)$
Also, $\mathbb{E}_t[D(t,S)] = P(t,S)$

- Notice: from above calculation we found

$$\mathbb{E}_t[D(t,S)L(T,S)] = P(t,S)\underbrace{F(t,T,S)}_{\text{forward rate}}$$

Discounted payoff at time $t < T_\alpha$ of a receiver IRS is

$$\sum_{i=\alpha+1}^{\beta} D(t,T_i)\,\underbrace{\tau_i\left(K - L(T_{i-1},T_i)\right)}_{\text{Payoff of Receiver IRS}}$$

Now take $\mathbb{E}$.

Also, noting receiver IRS can be valued as a collection of receiver FRAs. So, the value $K = S_{\alpha,\beta}(t)$ which makes $IRS = 0$ is called forward swap rate denoted $F_i(t) = F(t; T_{i-1}, T_i)$

$$\text{Receiver IRS}(t, [T_\alpha, \ldots, T_\beta], K) = \sum_{i=\alpha+1}^{\beta} FRA(t, T_{i-1}, T_i, K)$$

$$= \sum_{i=\alpha+1}^{\beta} \tau_i K P(t,T_i) - P(t,T_\alpha) + P(t,T_\beta)$$

$$= \sum_{i=\alpha+1}^{\beta} P(t,T_i)\tau_i\left(K - F(t;T_{i-1},T_i)\right)$$

(*) look next pg.

Cap (seen as payer IRS)

$$\text{Cap discounted payoff} = \sum_{i=\alpha+1}^{\beta} D(t,T_i)\tau_i\left(L(T_{i-1},T_i) - K\right)^+$$

$$= \sum_{i=\alpha+1}^{\beta} D(t,T_i)\tau_i\left(F_i(T_{i-1}) - K\right)^+$$

Cap price is risk neutral $\mathbb{E}$ of discounted payoff

Floor (seen as Receiver IRS)

$$\text{Floor discounted payoff} = \sum_{i=\alpha+1}^{\beta} D(t,T_i)\tau_i\left(K - L(T_{i-1},T_i)\right)^+$$

$$= \sum_{i=\alpha+1}^{\beta} D(t,T_i)\tau_i\left(K - F_i(T_{i-1})\right)^+$$

Floor price is risk neutral $\mathbb{E}$ of discounted payoff.

Interest Rate Derivatives

Swaptions

→ Options on IRS
→ A payer swaption is a contract giving the right to enter at future time a payer IRS.
→ Time of possible entrance is 1st reset date of IRS.

IRS value at first reset date T_α is

$$\text{Payer IRS}(T_\alpha, [T_\alpha, \ldots, T_\beta], K) = \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\, \tau_i \,(F(T_\alpha; T_{i-1}, T_i) - K)$$

$$= (S_{\alpha,\beta}(T_\alpha) - K) \underbrace{\sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)}_{C_{\alpha,\beta}(T_\alpha)}$$

Forward Swap rate

Option only exercised if IRS value is +ve, So, discounted payoff of payer swaption at t is

Payoff of Swaption

$$D(t, T_\alpha)\, C_{\alpha,\beta}(T_\alpha)\, (S_{\alpha,\beta}(T_\alpha) - K)^+ =$$
$$D(t, T_\alpha) \left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\, \tau_i \,(F(T_\alpha; T_{i-1}, T_i) - K) \right)^+$$

Forward Swap Rate

The value $K = S_{\alpha,\beta}(t)$ which makes the initial price of I.R. swap

$$IRS(t, [T_\alpha, \ldots, T_\beta], K) = 0$$

is called Forward Swap Rate.

Solving this eqn for $K = S_{\alpha,\beta}(t)$ gives 3 formulas one of them is $S_{\alpha,\beta}(t) = \dfrac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}$. Better to remember formula of price of IRS ⊛ on previous case & solve for K to get the formula.

~~Obtaining a Regression Tree (CART)~~

Interest Rate Modelling

→ Caps, Swaptions & more complex derivatives require curve dynamics & this amounts to specifying a stochastic model for I.R. The quantity to model might be short rate, libor $L(t,T)$, forward Libor $F_i(t) = F(t, T_{i-1}, T_i)$?

Short rate (r_t) (First choice for IR model)

→ Short rate r_t is interest charged on loan for next instant of time which you have to pay instantaneously i.e. Nominal + interest.

→ $dr_t = \text{local mean}(t, r_t)dt + \text{local St. Dev}(t, r_t) \times \text{Stoc. change}$

$$\Rightarrow dr_t = \underbrace{b(t,r_t)}_{\text{drift}}dt + \underbrace{\sigma(t,r_t)}_{\text{diffusion coefficient}}dW_t$$

Short Rate Models ($X_t = r_t$)

(i) Vasicek:

$$dX_t = \underbrace{K}_{\text{Speed of Mean reversion}}(\underbrace{\theta}_{\text{long term Mean}} - X_t)dt + \underbrace{\sigma}_{\text{Volatility}} dW_t, \quad \alpha = (K, \theta, \sigma)$$

(ii) Cox-Ingersoll-Ross (CIR):

$$dX_t = K(\theta - X_t)dt + \sigma\sqrt{X_t}\,dW_t, \quad \alpha = (K, \theta, \sigma)$$
$$2K\theta > \sigma^2$$

(iii) Dothan:

$$dX_t = aX_t dt + \sigma X_t dW_t$$
$$\left[X_t = X_0 e^{(a - \frac{1}{2}\sigma^2)t + \sigma W_t}, \quad \alpha = (a, \sigma)\right]$$

(iv) Exponential Vasicek:

$$X_t = \exp(Z_t), \quad dZ_t = K(\theta - Z_t)dt + \sigma dW_t$$
$$\alpha = (K, \theta, \sigma).$$

Vasicek Model (Pros & Cons)

$$dX_t = K(\theta - X_t)dt + \sigma dW_t, \quad r_t = X_t$$

Pros:
- Eqn linear & explicit soln.
- Joint Dist. of many important quantities are Gaussian.
- Mean reverting & Variance does not explode.

Cons:
- Rates can assume -ve values with +ve probabilities.
- Gaussian dist. for rates are not compatible with Market implied dist.

CIR Model (Positive)

$$dy_t = \kappa(\mu - y_t)\,dt + \nu\sqrt{y_t}\,dW_t, \quad r_t = y_t$$

→ Model implies +ve IR but the instantaneous rate is characterised by Non Central Chi-Square dist.
→ Mean reverting & Variance doesn't explode.
→ less Analytical tractability (Explicit soln) than Vasicek.
→ In Multifactor case, CIR is closer to Market implied dist. of rates than Vasicek.

- Both Vasicek & CIR says:
  → larger κ ⇒ faster process converges to mean. So κ kills volatility of IR.
  → large μ ⇒ higher long term mean, so model will tend to higher rates in future in average.
  → large ν ⇒ larger Volatility. κ & ν fight off each other.

Lecture 4

Discount factor, rates

- Bank Account/Money Market account:
  - → Pays interest Cont. Compounded & is locally risk free.
  - → Satisfies ODE: $dB(t) = r_t B(t)$, $B(0) = B_0$
    So, Sol'n is $B(t) = B_0 \exp(\int_0^t r_s ds)$.
  - → The appreciation of Bank account is proportional to the time interval Δt & r_t (instanteneous rate of appreciation, or instanteneous Spot rate or short rate)
    i.e.
$$\underbrace{\frac{B(t+\Delta t) - B(t)}{B(t)}}_{\text{appreciation of Bank account}} = r_t \Delta t$$
  - → Stochastic discount factor is the time t value of 1 unit currency payable at later time T.
$$\left[D(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_s ds\right) \right]$$

[Note: For B-S to equity or FX market r is assumed deterministic f'n of time while for IR products, r is stochastic.]

- Compounding Conventions
  - → Continuous Compounded Spot interest rate/Zero Coupon rate at time t for maturity T is the constant rate at which an investment $P(t,T)$ units at time t accrues continuously to yield 1 unit at time T:
$$\left[R(t,T) = -\frac{\ln P(t,T)}{(T-t)} \right]$$
$$\left[\text{ZCB price} := P(t,T) = e^{-R(t,T)(T-t)} \right]$$

→ Simply Compounded Spot Interest rate / Zero Coupon rate at time t for maturity T is the constant rate at which an investment of $P(t,T)$ units at time t accrues proportionally to investment time to yield 1 unit at time T:

$$\boxed{L(t,T) = \frac{1}{T-t}\left(\frac{1}{P(t,T)} - 1\right) \qquad \text{ZCB price: } P(t,T) = \frac{1}{1+(T-t)L(t,T)}}$$

This is simply the Libor rate which is simply compounded.

→ To learn about $P(t,T)$ we can model using Short rate (r_t), Libor rate $L(t,T)$, forward rate $F(t;T,S)$ (simply compounded) or forward interest rate or directly find $P(t,T)$.

Note: Forward interest rate is the instantaneous forward rate as maturity collapses close to expiry i.e.

$$\underbrace{f(t,T)}_{\text{Forward IR.}} = \lim_{S \to T^+} F(t;T,S) = \frac{\partial \ln P(t,T)}{\partial T}$$

$$\text{ZCB price: } P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right)$$

→ let learn abt $P(t,T)$ first using Short rate Models.

(One-factor) Short rate Model

We could write generalised version of Short rate model using parameters α, β & γ & depending on their values we get different historical models.

$$\boxed{dX_t = (\alpha + \beta X_t)\,dt + \sigma X_t^{\gamma}\,dW_t}$$

→ Merton : $\beta = \gamma = 0$ → Vasicek : $\gamma = 0$
→ CIR : $\gamma = 1/2$ → Dothan : $\alpha = \beta = 0, \gamma = 1$
→ Bren-Schwartz : $\gamma = 1$ → CIR Variable : $\alpha = \beta = 0, \gamma = 3/2$
→ Constant elastic of Variance (CEV) : $\alpha = 0$

level effect

[Note: From Market data we observe Volatility to be sensitive to the rate. For most model above when $\gamma > 0$, the volatility increases with the level of I.R. However, for Merton & Vasicek $\gamma = 0$ so, these model have weakness]

- Emperical Statistical Characteristic Captured by these Models:
  (i) Mean reversion.
  (ii) IR volatility depends on the level of the rate. (Very important)

- All of the above are endogenous Models & the initial term structure given by Model has too few parameters to be fitted to market data. The model zero curve & volatility surface may not fit well the market ones. We later improve this by considering Exogeneous Models. (Incorporates Market data)

Case Study: Vasicek

SDE: $dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^Q$

with parameters $\alpha = (\kappa, \theta, \sigma)$

Soln: $f(t,x) = e^{\kappa t} x \in C^{1,2} \Rightarrow$ Ito's formula can be used.

Ito formula gives

stochastic

$$d f(t,r_t) = \partial_t f(t,r_t)dt + \partial_x f(t,r_t)dr_t + \frac{1}{2}\partial_x^2 f(t,r_t)\, d\langle r\rangle_t$$

$$\begin{cases} \partial_t f(t,r_t) = \kappa e^{\kappa t} x = \kappa f(t,r_t) \\ \partial_x f(t,r_t) = e^{\kappa t} \\ \partial_{xx} = \partial_x^2 = 0 \end{cases}$$

$= \kappa f(t,r_t)dt + e^{\kappa t} dr_t + 0 \underbrace{dr_t}$

$= \kappa f(t,r_t)dt + e^{\kappa t}[\kappa(\theta - r_t)dt + \sigma dW_t] + 0$

$= e^{\kappa t}\kappa\theta\, dt + e^{\kappa t}\sigma\, dW_t$

Now, Integrate from s to t. for $s \le t$

$$e^{\kappa t} r_t - e^{\kappa s} r_s = \int_s^t e^{\kappa u}\kappa\theta\, du + \int_s^t e^{\kappa u}\sigma\, dW_u$$

Multiply both sides by $e^{-\kappa t}$,
Now, Solving for r_t we get

$$\boxed{r_t = e^{-\kappa(t-s)} r_s + \theta(1 - e^{-\kappa(t-s)}) + \sigma \underbrace{\int_s^t e^{-\kappa(t-u)} dW_u}}$$

Now this is Stochastic Integral
& Stoc. Int. is a Random Variable
which is Normally dist. with
Mean = zero & Var = σ^2.

Lets calculate Variance of this Stochastic Integral r_t
Since, mean is 0, Variance is just the 2^{nd} Moment.

$$Var\left[\int_s^t e^{-\kappa(t-u)} dW_u\right] = \mathbb{E}\left[\left(\int_s^t e^{-\kappa(t-u)} dW_u\right)^2\right] - \mu^2$$

Using Ito Isometry

Ito : Isometry: for deterministic $v(t)$ we have:
$Var(\int v(u)dW_u)$
$= \mathbb{E}[(\int v(u)dW_u)^2]$
$= \int v(u)^2 du$

$$= \mathbb{E}\left[\int_s^t e^{-2\kappa(t-u)} du\right] - 0^2$$

$$= \frac{1 - e^{-2\kappa(t-s)}}{2\kappa}$$

So, $r_t | r_s \sim N\Big(\underbrace{r_s e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)})}_{\text{Mean}},\ \underbrace{\sigma^2 \frac{1 - e^{-2\kappa(t-s)}}{2\kappa}}_{\text{Variance}}\Big)$

[Notes: Few drawbacks of this Model observed from Calculation:

- Vasicek has short comings since it can have -ve values with +ve prob. due to Normal dist. Even more it's unbounded from below.

- As $t \to \infty$, Mean $\to \theta \Rightarrow$ Mean reverting
- Variance doesn't depend on r_s. So constant in terms of r_s (no-sensitivity to levels of rates).]

Calculating ZCB price in terms of Vasicek

- The price of pure-discount bond can be derived by computing the expectation (discounted)

$$\left[P(t,T) = \mathbb{E}_t \exp\left(-\int_t^T r_u \, du\right)\right]$$

We use the following property:

$$\left[\int_t^T r_u \, du \,\Big|\, r_t \text{ is a Gaussian r.v.}\right]$$

Recall:-

$$r(u) = r(t) e^{-\kappa(u-t)} + \theta(1 - e^{-\kappa(u-t)}) + \sigma \int_t^u e^{-\kappa(u-s)} dW_s$$

Integrating both sides gives from t to T

$$\int_t^T r(u) du = \int_t^T r(t) e^{-\kappa(u-t)} du + \int_t^T \theta(1 - e^{-\kappa(u-t)}) du + \sigma \int_t^T \left(\int_t^u e^{-\kappa(u-s)} dW(s)\right) du$$

(look extra page)

Vasicek Bond $\Rightarrow$ Put Option on a S-maturity ZCB

Vasicek Caplet

- A caplet can be seen as a put option on a zero bond.

or

$$\int_t^T r(u) du = \int_t^T r(t) e^{-\kappa(u-t)} du + \int_t^T \theta(1 - e^{-\kappa(u-t)}) du + \sigma \int_t^T \left(\int_t^T \mathbb{1}_{\{s \le u\}} e^{-\kappa(u-s)} dW(s)\right) du$$

If $X \sim N(M, V^2)$

$$\mathbb{E}[e^X] = e^{M + V^2/2}.$$

Switching order of integration between du & $dW(s)$ gives

$$\int_t^T r(u)\,du = \text{——— Same ———} + \sigma \int_t^T \underbrace{\left(\int_t^T \mathbb{1}_{\{s \le u\}} e^{-\kappa(u-s)}\,du\right)}_{g(s)} dW_s$$

where $g(s)$ is now a deterministic f'n.

Now, $r(t)$ has all deterministic except last integral on R.H.S which is Ito Integral $\therefore$ Gaussian R.V.

$\therefore \int_t^T r(u)\,du$ is Gaussian.

So, To Compute Bond price We need Mean & Variance:-

$$\mathbb{E}_t\left[\int_t^T r_u\,du\right] = \int_t^T \mathbb{E}[r_u]\,du = \int_t^T \left[r(t)\,e^{-\kappa(u-t)} + \theta(1 - e^{-\kappa(u-t)})\right] du$$

Now for Variance

$$\mathbb{E}_t\left[\left(\int_t^T r_u\,du\right)^2\right] = \mathbb{E}_t\left[\int_t^T \int_t^T r_u r_v\,du\,dv\right]$$

Plug value of r_u & r_v & using Isometry

$$\mathbb{E}\left[\int_t^u f(z)\,dW_z \int_t^v g(z)\,dW_z\right] = \int_t^{\min(u,v)} f(z)g(z)\,dz$$

One obtains

$$X = -\int_t^T r_u\,du \sim N(M, V^2)$$

Price of Bond $\rightarrow$ $\boxed{P(t,T) = A(t,T)\,e^{-B(t,T)\,r(t)}}$

where $A(t,T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2\kappa^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4\kappa} B(t,T)^2\right\}$

$B(t,T) = \frac{1}{\kappa}\left[1 - e^{-\kappa(T-t)}\right]$

Put Option on S-maturity ZCB (See Extra pg ①) Extra B1

Payoff at T (discounted back at t):

$$\left[\exp\left(-\int_t^T r_u\,du\right)[X-P(T,S)]^+\right]$$

where price at t of option with strike X, maturity T & written on pure discount bond maturing at time S is risk Neutral $\mathbb{E}$ of above payoff denoted $ZBP(t,T,S,X)$.

Price of Put option on S-maturity ZCB

$$ZBP(t,T,S,X) = \mathbb{E}\left[\exp\left(-\int_t^T r_u\,du\right)[X-P(T,S)]^+\right]$$

Pricing Caplet (see pg ~~...~~ 1-3) Extra ~~...~~ 4

- A Caplet can be seen as put option on Zero Bond

$$\text{Caplet Price}: Cpl(t,T,S,X,N) = \mathbb{E}\left(e^{-\int_t^S r_s\,ds} N\tau[L(T,S)-X]^+ \mid \mathcal{F}_t\right.$$

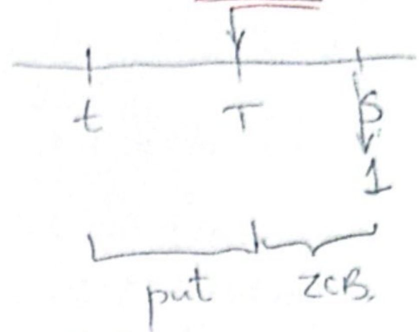
where $\tau = S-T$, N = Nominal

$$\text{Caplet Payoff} = N(S-T)[L(T,S)-K]^+$$

Scanned with CamScanner

Put option: European style

maturity T $(T<S)$



↳ if at maturity T we have $\boxed{K > P(T,S)}$ then

strike bond value

exercise the option: sell underlying for K

↳ ZCB with value $P(T,S)$

$\Rightarrow$ profit $K - P(T,S)$.

otherwise, we ~~lef~~ let the option expire unexercised.

Payoff of option at T: $\left(K - P(T,S)\right)^+$.

Pricing a caplet: payoff: $\tau_i \left(L(T_{i-1}, T_i) - K\right)^+$, $\tau_i = T_i - T_{i-1}$

In general, for a NOTIONAL amount N, for $T_{i-1} = T$, $T_i = S$

caplet payoff: $N(S-T)\left(L(T,S) - K\right)^+$.

Recall Libor rate: $L(T,S) = \dfrac{1 - P(T,S)}{(S-T)\,P(T,S)}$. $\Rightarrow$

$$\Rightarrow \text{caplet payoff}: \; N(\overset{>0}{\overbrace{S-T}})\left(\frac{1-P(T,S)}{(S-T)\,P(T,S)} - K\right)^+ =$$

$$= N\left(\frac{1-P(T,S)}{P(T,S)} - K(S-T)\right)^+ = N\left[\frac{1}{P(T,S)} - 1 - K(S-T)\right]^+$$

$$= N\underbrace{(1+K(S-T))}_{>0}\left(\frac{1}{\underbrace{P(T,S)[1+K(S-T)]}_{>0}} - 1\right)^+ =$$

$$= \frac{N(1+K(S-T))}{P(T,S)}\Bigg(\underbrace{\frac{1}{1+K(S-T)}}_{X} - P(T,S)\Bigg)^+$$

$(X - P(T,S))^+$ ← payoff of put option on bond

Caplet payoff at time S is:

$$N(S-T)\,(L(T,S)-K)^+ = \frac{N}{X\,P(T,S)}\,(X-P(T,S))^+$$

Construct the following portfolio:

- at $t<T$ buy $\frac{N}{X}$ units of put option with maturity T, strike $X=\frac{1}{1+K(S-T)}$ on S-bond
- at T portfolio value is: $\frac{N}{X}(X-P(T,S))^+$
- at T buy $\frac{N}{X}(X-P(T,S))^+\frac{1}{P(T,S)}$ units of S-bond, each worth $P(T,S)$.
- at S portfolio value is $\boxed{\frac{N}{X}(X-P(T,S))^+\frac{1}{P(T,S)}\cdot 1}$

same as caplet payoff

This portfolio replicates the caplet, so it is the

REPLICATING PORTFOLIO.

GOAL: price the caplet.

If there is NO-ARBITRAGE in the market (arbitrage-free) then the price of the caplet is EXACTLY equal to the value of the replicating portfolio.

At time t, 1 unit of put option on bond costs $ZBP(t,T,S,X)$, so the price of the caplet is

$$\boxed{Cpl(t,T,S,K)=\frac{N}{X}\,ZBP(t,T,S,X)}, \text{ where } X=\frac{1}{1+K(S-T)}.$$

Scanned with CamScanner

③

This formula agrees with risk-neutral pricing:

$$Cpl(t,T,S,K) = E_t\left[\exp\left(-\int_t^S r_u du\right) N(S-T)(L(T,S)-K)^+\right]$$

$$= E_t\left[\exp\left(-\int_t^T r_u du\right)\exp\left(-\int_T^S r_u du\right) N(S-T)(L(T,S)-K)^+\right]$$

= (iterated conditioning, tower property)

$$= E_t\left(E_T\left[\exp\left(-\int_t^T r_u du\right)\exp\left(-\int_T^S r_u du\right) N(S-T)(L(T,S)-K)^+\right]\right)$$

$L(T,S)$ is known at T, $\int_t^T r_u du$ is known at T

$$= E_t\left(\exp\left(-\int_t^T r_u du\right) N(S-T)(L(T,S)-K)^+ \underbrace{E_T \exp\left(-\int_T^S r_u du\right)}_{P(T,S)}\right)$$

$$= E_t\left[D(t,T)\frac{N}{X\cancel{P(T,S)}}(X-P(T,S))^+ \cdot \cancel{P(T,S)}\right]$$

$$= \frac{N}{X} E_t\left[\underbrace{D(t,T)}_{\text{discounted}} \underbrace{(X-P(T,S))^+}_{\text{payoff}}\right]$$

$$= \boxed{\frac{N}{X} ZBP(t,T,S,X)} \quad \text{with } X = \frac{1}{1+K(S-T)}.$$

Lec 5

Since, Endogenous Model don't fit well with market data we consider Exogenous Models (which are calibrated)

Exogenous Models for Short-Rate

→ Basic Strategy is modification of endogenous model by inclusion of "time-varying parameters" or shift.

eg: In Vasicek

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t)$$

$$\Downarrow$$

$$dr(t) = \kappa(\theta(t) - r(t))dt + \sigma dW(t)$$

→ Now define f'n interms of market curve $T \to L^M(0,T)$ s.t. model reproduces exactly curve itself at time 0.

- Exogenous Models (Short rate)

Tractable, simple, -ve IR

(i) Ho-Lee: $dX_t = \theta(t)dt + \sigma dW_t$

(ii) Hull-White (Extended Vasicek): $dX_t = \kappa(\theta(t) - X_t)dt + \sigma dW_t$

(iii) Hull-White (Extended CIR): $dX_t = \kappa(\theta(t) - X_t)dt + \sigma\sqrt{X_t}\,dW_t$

Tractable, easy, Possible -ve IR.

(iv) Black-Derman-Toy (Extended Dothan): $X_t = X_0\, e^{u(t) + \sigma(t)W_t}$

Not tractable, Bank Account Explosion

(v) Black-Karasinski (Ext. exp. Vasicek): $X_t = \exp(Z_t)$

$$dZ_t = \kappa[\theta(t) - Z_t]dt + \sigma dW_t$$

Not tractable only Numerical

(vi) CIR++: $r_t = X_t + \phi(t;\alpha)$

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}\,dW_t$$

Tractable, easy to implement & calibrate, used in industry.

[Note: Another way of getting Exogenous model is the inclusion of additional deterministic f'n called shifted as in the case for (vi) CIR++, ++ means shifted]

Calibration Procedure

→ Assume market zero coupon curve of IR at time 0
$T \mapsto L^M(0,T)$

→ Assume no. of vanilla option's volatilities. eg caps & swaptions.

→ Use time dependent parameters $\vartheta(t)$ or shift $\varphi(t)$ to fit the zero curve exactly.

→ Adjust parameters K, θ, σ, r_0 to best fit the vanilla option data

→ Best fit can be obtained by optimization method eg. gradient method.

Question we need to ask before Model choice

Does Model give:

- +ve IR?
- Suitable for Monte Carlo
- distribution of Model eg fat tail.
- Allow for explicit short rate dynamics
- Bond price / Bond option explicitly
- historical parameter estimation.
- Mean reverting while variance does not explode.
- how volatility structure look implied by Model.

Problem with ~~One factor~~ (short rate)

- Unrealistic correlation patterns between pts of curve with different maturities.

$$Corr(dF_i(t), dF_j(t)) = 1$$

- Poor calibration capability & lack of agreement with market data.
- Initial shock shift entire curve affecting long term IR which is not the same in actual life. eg. current fear subsides over long term.

What abt Instantaneous Forward rate $f(t,T)$

- $f(t,T)$ is Libor rate when S collapses to T $(S \to T)$

Thus we get instantaneous forward rate.

$$\boxed{f(t,T) = \lim_{S \to T} \underbrace{F(t;T,S)}_{\text{Forward libor rate}} \approx -\frac{\partial \ln P(t,T)}{\partial T}}, \lim_{T \to t} f(t,T) =$$

→ f's are not observed in market, so no improvement w.r.t. modelling r.

- HJM drift condition: Drift is completely determined once volatilities have been chosen.

$$\left[df(t,T) = \left[\sigma(t,T)\left(\int_t^T \sigma(t,s)ds\right)\right]dt + \sigma(t,T)\, dW_t \right]$$

So, setting $f(0,T) = f^M(0,T)$ we get

$$df(t,T) = \underbrace{\sigma(t,T)\left(\int_t^T \sigma(t,s)\,ds\right)}_{\text{drift}}dt + \sigma(t,T)\,dW_t$$

under risk neutral world measure if no arbitrage has to hold.

- Given volatility, no freedom in selecting drift.
- This can be useful to study arbitrage free properties of model. But for pricing & hedging most useful models coming out of HJM are already known Short rate Models. (Gaussian & Market models).

Advantage of HJM

- Extended Vasicek, Two factor Hull-White, CIR, Black-Kar. can be seen as special case of HJM framework.
- HJM may serve as unifying framework where all types of no-arbitrage IR Models can be expressed.

[Note: Due to perfect correlation between all maturities short rate Model is not good]

→ Thus use two factor Model

Multi factor Model

Two-factor G2 Model

Replace Gaussian Vasicek Model with 2-factor (G2)

$$r_t = X_t + Y_t$$

$$dX_t = K_x(\theta_x - X_t)\,dt + \sigma_x\,dW_1(t)$$

$$dY_t = K_y(\theta_y - Y_t)\,dt + \sigma_y\,dW_2(t)$$

$$dW_1 dW_2 = \rho\,dt$$

For this model Bond price is affine f'n, this time of 2 factors x & y.

$$P(t,T) = A(t,T)\ \exp\left(-B^x(t,T)\,x_t - B^y(t,T)\,y_t\right)$$

where short rate is given by x or y.

G2++ Model (Two factor) (look Extra pg 1-3)

→ Consider additive 2-factor model of form

$$r_t = x_t + y_t + \varphi(t), \quad r(0) = r_0$$

where $\varphi(t)$ is deterministic shift to fit zero coupon curve, as in case for one-factor case.

$$dx(t) = -a\,x(t)\,dt + \sigma\, dW_1(t), \quad x(0) = 0$$

$$dy(t) = -b\,y(t)\,dt + \eta\, dW_2(t), \quad y(0) = 0$$

$$dW_1(t)\,dW_2(t) = \rho\, dt \quad , \underbrace{r_0, a, b, \sigma, \eta}_{\text{constants}} > 0$$

$$-1 \leq \rho \leq 1$$

→ Assume dynamics of instantaneous short rate under risk-adjusted measure $\mathbb{Q}$.

① Extra 5

G2++ model (2 factor shifted model)

$$\underbrace{\text{2 factor}}_{X_t,\, Y_t} \qquad \text{shifted} \to \varphi(t)$$

$r_t = X_t + Y_t + \varphi(t)$

$$\begin{cases} dX_t = -aX_t\,dt + \sigma\, dW_1(t), & X_0 = 0 \\ dY_t = -bY_t\,dt + \eta\, dW_2(t), & Y_0 = 0. \\ dW_1(t)\,dW_2(t) = \rho\, dt \end{cases}$$

$\rho \to$ correlation

ZCB price: $P(t,T) = E\left[\exp\left(-\int_t^T r_u\,du\right)\right]$.

Need $\int_t^T r_u\,du$, which in this model is: $\underbrace{\int_t^T (X_u + Y_u)\,du}_{I(t,T)} + \underbrace{\int_t^T \varphi(u)\,du}_{\text{NOT RANDOM}}$

To find $I(t,T)$ need $\boxed{\int_t^T X_u\,du}$.

Lemma: $\int_t^T X_u\,du = \frac{1-e^{-a(T-t)}}{a} X_t + \frac{\sigma}{a}\int_t^T \left[1-e^{-a(T-u)}\right] dW_1(u)$.

Proof: Take $u \in [t,T]$ we have $dX_u = -aX_u\,du + \sigma\, dW_1(u)$, $u \geq t$.

Ito $f(t,x) = e^{at}\cdot x$ at X_u gives: $X_u = e^{-a(u-t)} X_t + \sigma \int_t^u e^{-a(u-v)}\, dW_1(v)$, $u \geq t$

Integration by parts (IBP)

$$d(X_t Y_t) = X_{t-}\,dY_t + Y_{t-}\,dX_t + d\langle X,Y\rangle_t.$$

Here: $TX_T = tX_t + \int_t^T X_u\,du + \int_t^T u\,dX_u + 0$

Additionally $\int_t^T T\,dX_u = TX_T - TX_t \Rightarrow TX_T = \int_t^T T\,dX_u + TX_t$

$$\int_t^T T\,dX_u + TX_t = tX_t + \int_t^T X_u\,du + \int_t^T u\,dX_u.$$

$$\int_t^T (T-u)\,dX_u + (T-t)X_t = \int_t^T X_u\,du$$

$$-a\int_t^T (T-u)X_u\,du + \sigma\int_t^T (T-u)\,dW_1(u) + (T-t)X_t = \int_t^T X_u\,du.$$

$$-a\int_t^T (T-u)e^{-a(u-t)}X_t\,du - a\int_t^T (T-u)\sigma\left(\int_t^u e^{-a(u-v)}\,dW_1(v)\right)du + \sigma\int_t^T (T-u)\,dW_1(u) + (T-t)X_t = \int_t^T X_u\,du. \quad (\times)$$

$$-a x_t \int_t^T (T-u) e^{-a(u-t)} du$$ $\boxed{\text{FIRST TERM}}$

Compute $d_u\left(e^{-a(u-t)}\right) = -a e^{-a(u-t)} du$

$$= -a x_t \int_t^T (T-u) \frac{d_u(e^{-a(u-t)})}{-a} = x_t \int_t^T (T-u) d_u(e^{-a(u-t)})$$

IBP
$$= x_t \Big[\cancel{(T-u) e^{-a(u-t)}}\Big|_{u=T}^{\to 0} - (T-u) e^{-a(u-t)}\Big|_{u=t} -$$

$$- \int_t^T e^{-a(u-t)} \underbrace{d_u(T-u)}_{-du} \Big]$$

$$= x_t \Big[-(T-t) + \int_t^T \underbrace{e^{-a(u-t)} du}_{\frac{d_u[e^{-a(u-t)}]}{-a}}\Big]. =$$

$$= x_t \Big[-(T-t) + \frac{e^{-a(u-t)}}{-a}\Big|_{u=T} - \frac{e^{-a(u-t)}}{-a}\Big|_{u=t}\Big]$$

$$= x_t \left[\frac{1-e^{-a(T-t)}}{a} - (T-t)\right] = -x_t(T-t) - \frac{x_t}{a}\left(e^{-a(T-t)} - 1\right).$$

$$-a \int_t^T (T-u)\, \sigma \left(\int_t^u e^{-a(u-v)} d W_1(v)\right) du$$ $\boxed{\text{SECOND TERM}}$

Compute $d_u\left(\int_t^u (T-v) e^{-av} dv\right) = (T-v) e^{-av}\Big|_{v=u} du = (T-u) e^{-au}$

$$-a \int_t^T \underbrace{(T-u) e^{-au}}\, \sigma \left(\int_t^u e^{av} d W_1(v)\right) du.$$

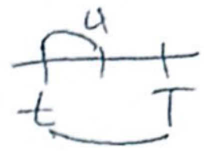
$$= -a \int_t^T d_u\left(\int_t^u (T-v) e^{-av} dv\right) \sigma \left(\int_t^u e^{av} d W_1(v)\right) du.$$

IBP
$$= -a\sigma \Big[\int_t^T e^{av} d W_1(v) \int_t^T (T-v) e^{-av} dv -$$

$$- \int_t^T \left(\int_t^u (T-v) e^{-av} dv\right) \underbrace{d_u\left(\int_t^u e^{av} d W_1(v)\right)}_{e^{au} d W_1(u)}\Big] =$$

$$= -a\sigma \left[\int_t^T \left(\int_t^T e^{-av}(T-v)\,dv\right) e^{au}\, dW_1(u) - \int_t^T \left(\int_t^u (T-v) e^{-av}\, dv\right) e^{au}\, dW_1(u)\right.$$

$$= -a\sigma \int_t^T \left(\int_u^T (T-v) \underbrace{e^{-av}\, dv}_{\frac{d_v(e^{-av})}{-a}}\right) e^{au}\, dW_1(u)$$



$$= \sigma \int_t^T \boxed{\left(\int_u^T (T-v)\, d_v(e^{-av})\right)} e^{au}\, dW_1(u).$$

$$\| \text{ IBP}$$

$$\underbrace{(T-v)e^{-av}\Big|_{v=T}}_{0} - \underbrace{(T-v)e^{-av}\Big|_{v=u}}_{-(T-u)e^{-au}} - \underbrace{\int_u^T e^{-av}\, d_v(T-v).}_{\underbrace{\int_u^T e^{-av}(-dv)}_{\| \; -\frac{e^{-aT}-e^{-au}}{-a}.}}$$

replace

to obtain expression for second term.

Then replace first and second term in (*) to obtain the result of the lemma.

Lec 6

Numeraire, T Forward Measure

- Recall in Risk Neutral Measure $\mathbb{Q}$ with numeraire B, $\mathbb{Q} = \mathbb{Q}^B$ (i.e. Stock discounted by numeraire is $\mathbb{Q}$-martingale).

- The T-forward measure with numeraire $N_t = P(t,T)$ (T-bond) is $\mathbb{Q}^N$ (i.e. $\mathbb{Q}^N$ Martingale) where,

$$\boxed{\frac{d\mathbb{Q}^N}{d\mathbb{Q}}\Big|\mathcal{F}_t = \frac{P(t,T)}{B_t P(0,T)}}$$

So, No-arbitrage Price = (T-forward pricing)

$$\boxed{\begin{aligned}\mathbb{E}_t^B\left[\frac{B(t)}{B(T)} H(\text{Asset})_T\right] &= \mathbb{E}_t^N\left[\frac{N(t)}{N(T)} H(\text{Asset})_T\right] \\ &= P(t,T)\,\mathbb{E}_t^N\left[H(\text{Asset})_T\right]\end{aligned}}$$

Expectation hypothesis

- Forward rates are $\mathbb{E}$ of short term rate under T-forward Measure

i.e. $f(t,T) = \mathbb{E}_t^N[r_T]$

Models of IR
- $\to$ short rate r_t (Endo. & Exo. Models)
- $\to$ forward rate (HJM) instantaneous $f(t,T)$
- $\to$ Forward Libor rate $F(t,T,S)$

- Short rates peak seasonally so better to use Forward rate.

•

Third Choice: Market Models

Recall: $F(t;T,S) = \frac{1}{S-T}\left[\frac{P(t,T)}{P(t,S)} - 1\right]$

is forward Libor rate at time t between T & S which makes FRA contract to lock in at time t, IR between T & S fair ($=0$).

The family of such rates for $(T,S) = (T_{i-1}, T_i)$ Spanning $T_0, T_1, \dots T_m$ is modelled in Libor Market Model.

Black's Pricing formula (Caplet) (Pg 1 Extra 6)

- T_2-Maturity Caplet resetting at time T_1 with strike X & Notional 1 with τ = Year fraction between T_1 & T_2.

$$\text{Payoff at } T_2 = \tau \left[L(T_1, T_2) - X \right]^+$$
$$= \tau \left[F_2(T_1) - X \right]^+$$

- Fact One: Price of any asset divided by reference asset (numeraire) is martingale (no drift) under the measure associated with that numeraire.

In particular,
$$F_2(t) = \frac{(P(t,T_1) - P(t,T_2)) / (T_2 - T_1)}{P(t,T_2)}$$
is a portfolio of 2 ZCB divided by ZCB $P(\cdot, T_2)$.

So, by Fact one, $F_2(t)$ is martingale with measure Q^2.

- Forward Libor rates F's are quantities modeled instead of r & f in LMM.

$$\left[dF_2(t) = \sigma_2(t)\, F_2(t)\, dW_2(t), \text{ mkt } F_2(0) \right]$$

Note drift = 0.

- Fact Two: The time-t risk neutral price

$P_t = \mathbb{E}_t^B \left[B(t) \frac{\text{Payoff}(T)}{B(T)} \right]$ is invariant to a change of numeraire So if N is another numeraire $\rightarrow \text{Price}_t = \mathbb{E}_t^N \left[N_t \frac{\text{Payoff}(T)}{N_T} \right]$

$\uparrow$ price_t

Replacing numeraire does not change price.

- Black caplet formula: $\mathbb{E}\left[\frac{B(0)}{B(T_2)}\tau\left(F(T_1;T_1,T_2)-X\right)^+\right]$
- No short rate is consistent with LMM.

Advantage of LMM over Short rate Model

- For LMM, (Forward libor rate)

$$corr\left(dF_k(t), dF_j(t)\right) = \rho_{kj} \leq 1$$

~~Whereas in one-factor short rate models dr~~ these correlations were fixed to 1.

- Intuitive calibration of caplets.
- Easy calibration of swaptions.
- Can calibrate high no's of market product
- Powerful diagnostics → can check future volatility.
- Can use Monte Carlo Simulation.

Swap Market Model (SMM) (look pg 2 Extra)

→ LMM good choice for caplets.

→ For Swaptions use SMM.

- Consider payer Swaption giving right to enter into swap first resetting in T_α & paying at $T_{\alpha+1}, T_{\alpha+2}, \ldots T_\beta$ for a fixed rate K.

Payout at Maturity T_α of this option is:

$$\left[\left[\underbrace{S_{\alpha,\beta}}_{\text{swap rate}}(T_\alpha)-K\right]^+\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)\right]$$

Changing numeraire & taking $\mathbb{E}$ yields Blacks formula for Swaptions (look extra pg 2)

- Taking long T_α bond & short T_β bond & discounting by $C_{\alpha,\beta}(t)$, it becomes Martingal (Fact one), so this has SDE of 0 drift:

$$dS_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t)\, S_{\alpha\beta}(t)\, dW_t^{\alpha\beta}, \quad Q^{\alpha\beta}\ (SMM)$$

→ Similar to IRS, but we have only one fixed leg at final time but still keep all the floating.

Zero Coupon Swaption (look pg 4 Extra 6)

- A payer (receiver) Zero-coupon swaption is a contract giving right to enter a payer (receiver) Zero-coupon IRS at future time.
  A zero coupon IRS is an IRS where a single fixed payment is due at unique (final) payment date T_β for the fixed leg in exchange for a stream of usual floating payments $\tau_i L(T_{i-1}, T_i)$ at times $T_{\alpha+1}, T_{\alpha+2}, \ldots T_\beta$.

- Discounted payoff of a payer Zero-coupon IRS is

$$\frac{B(t)}{B(T_\alpha)}\left[\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i F_i(T_\alpha) - P(T_\alpha, T_\beta)\tau_{\alpha,\beta} K\right]$$

where $t \leq T_\alpha$ ($\tau_{\alpha,\beta}$ is fraction of year between T_α & T_β)

Taking risk neutral $\mathbb{E}$ gives contract value as

$$P(t, T_\alpha) - P(t, T_\beta) - \tau_{\alpha,\beta} K P(t, T_\beta)$$

The value of K that renders the contract fair is obtained by equating to zero so, $\underline{K = F(t; T_\alpha, T_\beta)}$

The option to enter a payer Zero-coupon IRS is a payer Zero-coupon swaption & payoff is

$$* \quad \frac{B(t)}{B(T_\alpha)} \tau_{\alpha,\beta} P(T_\alpha, T_\beta)\left[F(T_\alpha; T_\alpha, T_\beta) - K\right]^+$$

- To price zero coupon swaption use
  (i) eqn $*$
  (ii) Black Caplet formula
  (iii) we Integrated % volatility, ~~we just get to get~~

~~ZCPS = Σ P(0, T_i) [F(0)~~

Constant Maturity Swaps (look pg. 5 Extras)

- Assume Unit Nominal. Denote $\{T_0, \dots, T_n\}$ a set of payment dates at which coupons are to be paid. At T_{i-1}, company A pays to B c-year (eg (5-year)) swap rate resetting at time T_{i-1}

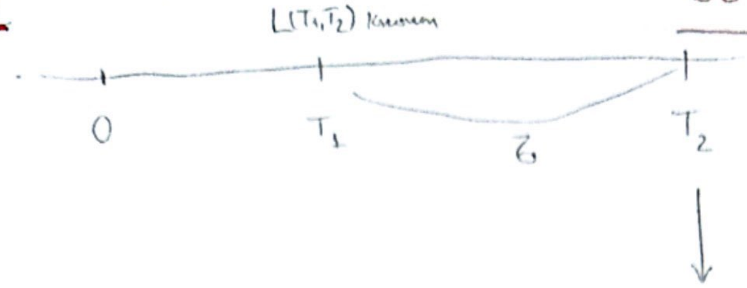
$$S_{i-1, i-1+c}(T_{i-1})\tau_i$$

in exchange for fixed rate K.

(• The payment in standard IRS would be instead $L(T_{i-1}, T_i)\tau_i = F_i(T_{i-1})\tau_i$)

- Just like IRS but pay swap rate instead of Libor rate.

Caplet (1) BLACK'S CAPLET FORMULA



Payoff $= \tau\left(L(T_1,T_2)-K\right)^+$

T_2-forward measure Q^2 defined $\left.\frac{dQ^2}{dQ}\right|_{\mathcal{F}_t} = \frac{P(t,T_2)}{B_t\,P(0,T_2)}$

No-arbitrage price $E^{Q^2}\left[\frac{P(t,T_2)}{\underbrace{P(T_2,T_2)}_{=1}}\text{payoff}\right] = \underbrace{P(t,T_2)}_{\text{known at time } t \text{ so comes out of } \mathbb{E}.} E^{Q^2}\left[\tau(L(T_1,T_2)-K)^+\right]$ (1)

Define libor rates

$$L(T_1,T_2) = F(T_1;T_1,T_2) =: F_2(T_1).$$

$$F(t;T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)}-1\right) \Rightarrow F(T_1;T_1,T_2) = \frac{1}{\tau}\left(\frac{1}{P(T_1,T_2)}-1\right)$$

$$= \frac{\left(P(T_1,T_1)-P(T_1,T_2)\right)/(T_2-T_1)}{P(T_1,T_2)}$$

$F(T_1;T_1,T_2) = F_2(T_1)$ is a portfolio of two ZCBs (one long, one short) discounted by the numeraire $P(T_1,T_2)$.

Fact one $\Rightarrow$ it is a Q^2-mart. (its SDE has no drift under Q^2) ↑ T-2 forward measure

$$\Rightarrow \boxed{dF_2(t) = 0\cdot dt + \sigma_2 F_2(t)\,dW_t^2} \quad \text{LMM}$$

(1) becomes $P(t,T_2)\,\tau\, E^{Q^2}\left[\underbrace{(F_2(T_1)-K)^+}_{\text{payoff of } T_1\text{-maturity call option}}\right]$

with strike X, underlying vol. σ_2, and 0 risk-free rate.

Swaption (option on IRS) ② BLACK'S SWAPTION FORMULA

Payoff at T_α: $\left(S_{\alpha,\beta}(T_\alpha)-K\right)^+ \underbrace{\sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha,T_i)}$ ← linear combination of Bond prices of different Maturity

$C_{\alpha,\beta}(T_\alpha)$ annuity numeraire (DV01)

Price: $E^{C_{\alpha,\beta}}\left[\underbrace{\frac{C_{\alpha,\beta}(0)}{C_{\alpha,\beta}(T_\alpha)}}_{\text{discount}}\left(S_{\alpha,\beta}(T_\alpha)-K\right)^+ C_{\alpha,\beta}(T_\alpha)\right]$

$$= C_{\alpha,\beta}(0)\, E^{C_{\alpha,\beta}}\left[\left(S_{\alpha,\beta}(T_\alpha)-K\right)^+\right] \quad (2)$$

Underlying $S_{\alpha,\beta}(t) = \dfrac{P(t,T_\alpha)-P(t,T_\beta)}{C_{\alpha,\beta}(t)}$ is a portfolio of two ZCBs discounted by numeraire DV01

Factone $\Rightarrow$ it is a $Q^{C_{\alpha,\beta}}$-mart., where $\dfrac{dQ^{C_{\alpha,\beta}}}{dQ}\Big|\mathcal{F}_t = \dfrac{C_{\alpha,\beta}(t)}{B_t C_{\alpha,\beta}(0)}$

(its SDE has no drift under $Q^{C_{\alpha,\beta}}$) FORWARD SWAP MEASURE $Q^{C_{\alpha,\beta}} = Q^{\alpha\beta}$

$$\Rightarrow \boxed{dS_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t)\, S_{\alpha,\beta}(t)\, dW_t^{(\alpha,\beta)} + 0\,dt} \quad \text{SMM}$$

(2) becomes $C_{\alpha,\beta}(0)\, E^{\alpha,\beta}\left[\underbrace{\left(S_{\alpha,\beta}(T_\alpha)-K\right)^+}\right]$

payoff of T_α-maturity call option with strike price K, underlying vol. $\sigma^{\alpha,\beta}(t)$, and 0 risk-free rate

③

LMM $\quad dF_2(t) = 0\,dt + \sigma_2(t) F_2(t)\, dW_t^2$ under Q^2

lognormal forward-Libor model under the measure associated with numeraire $P(\cdot, T_2)$ (T_2-forward measure)

SMM $\quad dS_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t)\, S_{\alpha,\beta}(t)\, dW_t^{\alpha,\beta}$ under $Q^{\alpha,\beta}$

lognormal forward-swap model under forward swap measure

model	LMM	$\sim$	SMM
	$\downarrow$		$\downarrow$
market	IR caps	ρ	IR swaptions

$\sim$ approx (drift freezing)

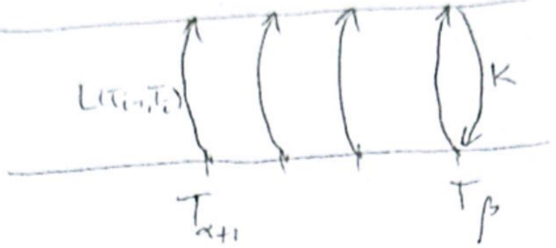
ρ needs model

(need to model correlation for swaptions)

Zero-coupon swaption pricing (ZCPS = ?)

IRS

ZCIRS



single payment fixed leg.

Zero-coupon swaption = option to enter ZCIRS at future time, strike K

↳ express it (payoff) as a caplet

Use Black's caplet pricing formula

– needs integrated percentage volatility of forward rate $F(t, T_\alpha, T_\beta) =: F(t)$??

– $F_i(t) = F(t; T_{i-1}, T_i)$

Compute
$$dF(t) = \frac{1+\tau F(t)}{\tau} \sum_{i=\alpha+1}^{\beta} \frac{\tau_i \, dF_i(t)}{1+\tau_i F_i(t)} + (\ldots)\,dt$$

don't [illegible] as it doesn't contribute to variance

drift freezing by setting to 0.

$$\mathrm{VAR}_t\left(\frac{dF(t)}{F(t)}\right) = \left(\frac{1+\tau \cancelto{0}{F(t)}}{\tau \cancelto{0}{F(t)}}\right)^2 \sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i \tau_j \rho_{ij} \sigma_i(t) \sigma_j(t) \cancelto{0}{F_i(t)} \cancelto{0}{F_j(t)}}{(1+\tau_i \cancelto{0}{F_i(t)})(1+\tau_j F_j(t))}\, dt$$

$$\int_0^{T_\alpha} \mathrm{VAR}_t\left(\frac{dF(t)}{F(t)}\right) dt = \left(\frac{1+\tau F(0)}{\tau F(0)}\right)^2 \sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i \tau_j \rho_{ij} F_i(0) F_j(0)}{(1+\tau_i F(0))(1+\tau_j F(0))} \underbrace{\int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\,dt}_{\text{market}}$$

↓ time-averaged quadratic vol.

$$\frac{1}{T_\alpha}\int_0^{T_\alpha} (\quad)\, dt = \ldots \xrightarrow{} \text{Black's caplet formula}$$

plug in

Constant maturity swaps (CMS)

In FRN, CAPS/FLOORS (CAPLETS) the underlying rates are ZCB rates (rates based on one payment).

In CMS the underlying rate is a swap rate (rate based on multiple payments)

Eg: FRN(6 month Libor) pays semi-annual coupons based on semi-annual fixings of 6 month Libor

- has info about short rates

CMS pays semi-annual coupons based on 10-year annual swap rate

- has info about overall level of yield curve

CMS pricing : { - payoff
- change of numeraire

use swap rates = weighted averages of forward rates

$$S_{\alpha,\beta}(T_\alpha) = \sum_{i=\alpha+1}^{\beta} w_i(0)\, F_i(T_\alpha) \approx \sum_{i=\alpha+1}^{\beta} w_i(0)\, F_i(T_\alpha)$$

like drift-freezing

Q^i: $dF_i(t) = \sigma_i(t) F_i(t)\, dW_t^i$

Q^k: $dF_i(t) = \bar{\mu}_{i,k} F_k(t)\,dt + \sigma_k(t)\, dW_k^{\gamma}(t)$ under T_γ-forward adjusted Q^{γ}.

$\alpha = i-1,\ \beta = i-1+c$ → very fast approximation

lec7

- last time: • LMM (Forward libor rates)
  • SMM (" Swap rates)

Fourth Moment in LMM (Question answered in pg 1-3 Extra 7)

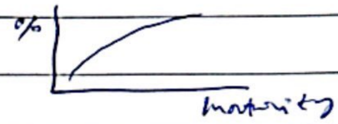
Consider contract paying T_4 (4yrs) 4^{th} power libor rate rent in T_3 plus a +ve strike $K > 0$.
Assume constant volatility LMM:

$$\left[dF_4(t) = \sigma_4 F_4(t)\, dW_4(t) \quad , \text{under } Q^4 \right]$$

Q^4 is T_4 forward means associated with numeraire T_4 bond.

Gilts => Government Bonds

Yield Curves (derived from Gilts)



- Shape is important
- Expectation hypothesis: long term rate is geometric average of expected future short rate.

- If long term rate is high investors expect short term rate rise (upward curve).

- Liquidity Preference: Investors demand higher return for long-term maturity to compensate for lost liquidity.

- Libor has credit risk whereas Overnight Index ~~Swap (OIS) doesnot~~ rate does not.
- Swap built around Overnight rate are called OIS (overnight Indexed swap)

At maturity T for OIS, swap parties calculate final payment as difference between accrued interest of fixed rate K & geometric average $L(0,T)$ of floating index rates $O(t_{i-1}, t_i)$ on the swap notional for t_i ranging from

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initial time $t=0$ to swap maturity T. Since the net difference is exchanged, rather than actual rates, OIS have little counterparty credit risk.

- Notes: For libor swap, we pay L & receive K. If counterparty defaults we pay L & lose whole K. But with OIS we lose only $(K-L)$ if +ve.

①

Extra 7

Numeraire T_4-bond (bond with maturity $T_4 = 4y$)

- price at time t of T_4-bond is $N_t = P(t, T_4)$.

Associated EMM: Q^N defined via Radon-Nikodym derivative

$$\left.\frac{dQ^N}{dQ}\right|_{\mathcal{F}_t} = \frac{P(t,T_4)}{B_t \; P(0,T_4)}$$

where Q was the risk-neutral measure associated with numeraire bank account ($dB_t = r_t B_t dt$) $Q = Q^B$.

No-arbitrage T_4-forward pricing of payoff H_{T_4} is

$$E_t^{Q^N}\left[\frac{N(t)}{\underbrace{N(T_4)}_{P(T_4,T_4)=1}} H_{T_4}\right] = P(t,T_4) \cdot E_t^N(H_{T_4}).$$

Denote $F_4(t) = F(t; T_3, T_4)$ forward Libor rate at time t with reset time T_3 and maturity T_4

$$F_4(t) = \frac{1}{T_4 - T_3}\left(\frac{P(t,T_3)}{P(t,T_4)} - 1\right)$$

Recall $F_4(T_3) = \frac{1}{T_4 - T_3}\left(\frac{\overset{1}{\overset{\parallel}{P(T_3,T_3)}}}{P(T_3,T_4)} - 1\right) =$

$$= \frac{1}{T_4 - T_3}\left(\frac{1}{P(T_3,T_4)} - 1\right) = L(T_3, T_4).$$

②

(LMM) $\quad dF_4(t) = \sigma_4 F_4(t)\, dW_4(t)$ under Q^4 (T_4-forward measure)
const. vol.

Contract
- maturity T_4 (4 years)
- payoff $(L(T_3,T_4)^4+K)^+$
- reset T_3 (3 years)

a. Apply Ito formula to $f(x)=x^4$, at $F_4(t)$:

$d\langle F_4\rangle_t = \sigma_4^2 F_4^2(t)\,dt$

$$d(F_4)^4 = f'(F_4)\,dF_4 + \frac{1}{2} f''(F_4)\, d\langle F_4\rangle_t$$

$$= 4F_4^3 \sigma_4 F_4\, dW_4(t) + \frac{1}{2} 4\cdot 3 \cdot F_4^2 \sigma_4^2 F_4^2(t)\, dt$$

$$= \underbrace{4\sigma_4}_{\text{vol.}} F_4^4\, dW_4(t) + \underbrace{6\sigma_4^2}_{\mu\ (\text{drift})} F_4^4\, dt$$

GBM with drift $6\sigma_4^2$ and vol. $4\sigma_4$.

$Y_t = F_4^4(t) \Rightarrow dY_t = 6\sigma_4^2 Y_t\, dt + 4\sigma_4 Y_t\, dW_4(t)$.

b. PRICING: change from risk-neutral to T_4-forward measure:

$$E^B\left[\frac{B(0)}{B(T_4)}\left(F_4(T_3)^4+K\right)^+\right] = E^4\left[\frac{P(0,T_4)}{\underbrace{P(T_4,T_4)}_{1}}\left(F_4(T_3)^4+K\right)^+\right]$$

$$= E^4\left(F_4(T_3)^4+K\right)\cdot P(0,T_4) = P(0,T_4)\left[K+E^4\left(F_4(T_3)^4\right)\right].$$

If $\underline{Y_t = F_4^4(t)}$ then $dY_t = Y_t\,(a\,dt + b\,dW_4(t))$.

GBM
$\Rightarrow Y_t = Y_0 \exp\left[\left(a-\frac{b^2}{2}\right)t + b W_4(t)\right]$, where $W_4(t) \sim N(0,t)$ under Q^4

(*) look for calc.

Use mgf of normal to show $E[Y_T] = Y_0\, e^{aT}\Big|_{a=6\sigma_4^2}$.

$\Rightarrow \underline{\text{price}} = \left(E^4[Y(T_3)]+K\right) P(0,T_4) = P(0,T_4)\left[\underbrace{F_4(0)}_{Y_0}\, e^{6\sigma_4^2 T_3} + K\right]$

(3)

GBM $dY_t = aY_t\,dt + bY_t\,dW_t \Rightarrow Y_t = Y_0 \exp\left[\left(a-\frac{b^2}{2}\right)t + bW_t\right]$

$E(Y_t) = Y_0 \exp\left[\left(a-\frac{b^2}{2}\right)t\right] \cdot E\left[e^{bW_t}\right]$

Recall W_t is BM $\Rightarrow W_t \sim N(0,t) \Rightarrow W_t = \sqrt{t}\cdot Z$, where $Z \sim N(0,1)$,

$$E\left[e^{bW_t}\right] = E\left[e^{b\sqrt{t}Z}\right]$$

But mgf of standard normal r.v. Z is: $\Rightarrow$

$$E\left[e^{tZ}\right] = e^{\frac{t^2}{2}}$$

$$\Rightarrow E\left[e^{bW_t}\right] = e^{\frac{b^2t}{2}}$$

$$E(Y_t) = Y_0 \exp\left[\left(a-\frac{b^2}{2}\right)t + \frac{b^2t}{2}\right] = Y_0 \exp at = Y_0 e^{at}$$

$$\frac{N(t)}{N(T_4)} = \frac{P(t,T_4)}{\underbrace{P(T_4,T_4)}_{1}} = P(t,T_4) \text{ is } \mathcal{F}_t\text{-measurable}$$

$\Rightarrow$ comes out of $E_t[\ldots]$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; \| $

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad E[\ldots|\mathcal{F}_t]$

Yield curves (BoE)

- UK gilts and GC repo (COUPONS) (risk-free)
- Interbank loans (no collateral) Libor (credit risk)
- overnight swap rates (SONIA, OIS) (limited credit risk)

ZERO COUPON CURVE

(Lecture 1).

Expectation hypothesis $E_t^{Q^S}[L(T,S)] = F(t,T,S)$

(see Lecture 3)

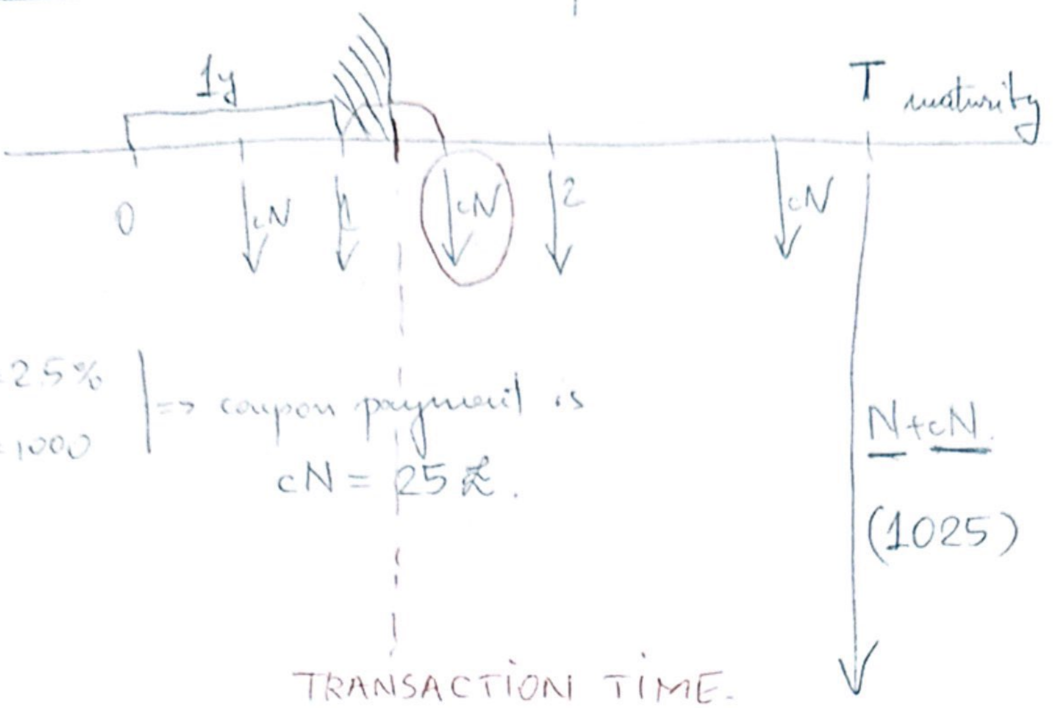
- long term rate is higher when investors expect short-term rates to rise.

Liquidity preference: investors prefer liquidity, in its long-term absence they demand higher rates of return

Market segmentation theory: bond prices as result of demand-offer, certain maturities are (dis) advantaged.

(5)

Gilt semi-annual coupons cN, nominal value N
£1000



$c = 2.5\%$
$N = 1000$ $\Rightarrow$ coupon payment is
$cN = 25$ £.

Question part (c)

(c) $V = P(0, T_4)\left[F_4^4(0)\, e^{6\sigma_4^2 T_3} + K\right]$

Yes, the price is sensitive to volatility as σ_4^2 is in the price formula.

(d) $\nu = \dfrac{\partial V}{\partial \sigma_4} = P(0, T_4)\, F_4^4(0)\, e^{6\sigma_4^2 T_3} \cdot 12\sigma_4$

lec 8

Credit Risk

Defaultable ZCB

- τ = default time, $\bar{P}(t,T)$ = price of defaultable Bond

Note: Bond Price = $\mathbb{E}$ [Discount × Payoff]

Price of ZCB → $P(t,T) = \mathbb{E}[D(t,T)\, 1 \mid \mathcal{F}_t]$

Price of Defaultable ZCB
$$\mathbb{1}_{\{\tau > t\}} \bar{P}(t,T) = \mathbb{E}[D(t,T)\, \mathbb{1}_{\{\tau > T\}} \mid \mathcal{G}_t\}$$

where $\mathcal{G}_t$ is larger filtration includes market variables & information on whether default occurred before t.

So, Payoff ignoring recovery is $\mathbb{1}_{\{\tau > T\}}$
i.e. 1 if no default & 0 if default.

- With Recovery discounted payoff is
  (i) Recovery REC paid at default τ

$$\left[D(t,T)\, \mathbb{1}_{\{\tau > T\}} + REC\ D(t,\tau)\, \mathbb{1}_{\{\tau \leq T\}} \right]$$

  (ii) REC paid at maturity T

$$\left[D(t,T)\, \mathbb{1}_{\{\tau > T\}} + REC\ D(t,T\}\, \mathbb{1}_{\{\tau \leq T\}} \right]$$

Price is obtained by taking $\mathbb{E}[\cdot \mid \mathcal{G}_t]$ of (i) or (ii).

Credit Default Swaps (CDS)

- Basic protection contracts against default.
- Very liquid in market.

Defn of CDS (like insurance against default)

2 Companies A (Protection buyer), B (Protection seller) agree on following:

If a 3rd Company C (reference credit) defaults at time τ with $T_a < \tau < T_b$ then B pays A a certain deterministic cash LGD.
In turn A pays B at rate R at times $T_{a+1}, \ldots, T_b$ or until defaults.

Protection leg →
Premium leg ←

$$\boxed{\text{Protection seller}}\ B \xrightarrow{\;LGD \text{ at } \tau_c \text{ if } T_a < \tau_c < T_b\;} \boxed{\text{Protection Buyer}}\ A$$

$$B \xleftarrow{\;\text{rate } R \text{ at } T_{a+1}, \ldots, T_b \text{ or until default } \tau_c\;} A$$

Typically, [LGD = notional] or [Notional − Recovery = 1 − REC.]

- Note for us we define τ_c as Credit Event ?
  This are as follows:
  (i) Bankruptcy of C (ii) Failure to pay of C
  (iii) C is requested to pay debt earlier due to not meeting terms of loan
  (iv) Restructuring of C.

- If C defaults B can pay A in two ways:

  (i) Cash Settlement: A receives loss in value of the reference instrument eg C issued Bond. ie gets cash.
  (ii) Physical Settlement: A receives cash payment & Seller takes possession of defaulted instrument or bond.
  But Most Physical Settle, allows buyer to choose deliverables from a pool of defaulted bonds with equal seniority
  If not enough bond to match insured face value, Credit event Auction occurs, in this case payment is significantly less

- Typically hard to estimate Recovery rate. Typically around $Rec = 20\%$ to 40%.
- Higher credit risk $\Rightarrow$ low Rec.

Running CDS discounted Payoff to Seller B at $t < T_a$ is

$$\boxed{\Pi_{RCDS_{a,b}}(t) = D(t,\tau)(\tau - T_{\beta(\tau)-1}) R \mathbb{1}_{\{T_a < \tau < T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R \mathbb{1}_{\{\tau > T_i\}} - \mathbb{1}_{\{T_a < \tau < T_b\}} D(t,\tau) LGD}$$

Where $T_{\beta(\tau)}$ is the 1^{st} of T_i's following τ
& $\alpha_i = T_i - T_{i-1}$

CDS payout to Protection Seller (Three payout)

Also called receiver CDS

(i) Discounted Accrued rate at default

This covers the protection seller for protection provided from last T_i before default until τ:

$$D(t,\tau)(\tau - T_{\beta(\tau)-1}) R \mathbb{1}_{\{T_a < \tau < T_b\}}$$

(ii) CDS Rate premium payment if no default (premium)

$$\sum_{i=a+1}^{b} D(t,T_i)\alpha_i R \mathbb{1}_{\{\tau > T_i\}}$$

(iii) Payment of Protection at default if happens before final T_b

$$-\mathbb{1}_{\{T_a < \tau \leq T_b\}} D(t,\tau) LGD$$

- Denote $CDS_{a,b}(t,R,LGD)$ the time t price of Running CDS payoffs.

$\Downarrow$ CDS price at $t=0$.

- $CDS_{a,b}(0,R,LGD) = \mathbb{E}\{\Pi_{RCDS_{a,b}}(0)\}$

- Calculating CDS price using Model Independent pricing given survival probabilities $Q(\tau \geq \cdot)$.

Model indep. $\because$ we don't assume any distribution on default time τ.

Total (Receiver) CDS price:

$$CDS_{a,b}(t,R,LGD;Q(\tau>\cdot)) = R\ \text{Premium leg}\ \Pi_{a,b}(P(0,\cdot),Q(\tau>\cdot)) - LGD\ \text{Protection leg}\ \Pi_{a,b}(Q(\tau>\cdot))$$

* Continue last pg.

(look page 2-3 Extra 8)

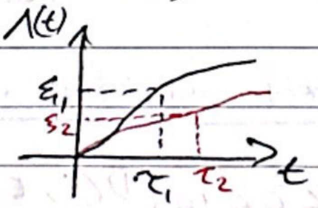
- We use "CDS stripping" to get survival probabilities to price CDS.

Intensity Models for CDS & Defaultable Bonds

- Assume default time τ is exponentially distributed.
- λ_t = default intensity (hazard rate) given for Bond issuer.
- Cumulative intensity is $t \to \int_0^t \lambda_s ds = \Lambda_t$

  Since $\lambda > 0$, Λ is increasing in time
- Default time is inverse of cumulative intensity on exponential R.V. ξ with mean $=1$ & independent of λ.

$$\text{So,} \quad \tau = \Lambda^{-1}(\xi)$$



$Q(\xi > u) = e^{-u}$

$Q(\xi < u) = 1 - e^{-u}$

$\mathbb{E}(\xi) = 1$

- Prob of surviving time t:

$$Q(\tau > t) = Q(\Lambda^{-1}(\xi) > t) = Q(\xi > \Lambda(t)).$$

$\therefore \mathbb{E}[\mathbb{Q}(\xi > \Lambda(t) | \Lambda(t)] = \mathbb{E}[e^{-\Lambda(t)}] = \mathbb{E}[e^{-\int_0^t \lambda_s ds}]$

looks like bond price replacing r by λ.

- Price of defaultable ZCB under Intensity Model: $\bar{P}(0,T) = \mathbb{E}[D(0,T) \mathbb{1}_{\{\tau > T\}}] = \mathbb{E}[e^{-\int_0^T r_s ds} \mathbb{1}_{\{\Lambda^{-1}(\xi) > T\}}] = \mathbb{E}[e^{-\int_0^T (r_s + \lambda_s) ds}]$
- So λ is instantaneous Credit Spread or local default prob.
- ξ is pure jump to default risk.

- Exponential Structure of τ in Intensity Models makes modeling of Credit risk similar to IR Models.

Intensity Model When λ_t is Constant, time dependent/Stochastic

- ~~Constant λ_t~~: $\lambda_t = \gamma$ deterministic constant credit spread.

- Time dependent deterministic intensity: $\lambda_t = \gamma(t)$
  This is model with a term structure of credit spread but without credit spread volatility.

- Time dependent Stochastic Intensity λ_t
  In this case λ_t is Stochastic process & allows to model term structure of Credit Spreads but also their volatility

For Constant $\lambda_t = \gamma$ case, replace original formula of CDS with $T_a = 0$.
Inserting $R = R_{0,b}^{MKT\ MID}(0)$ in premium leg & solving $CDS_{a,b}(t, R, LGD; \mathbb{Q}(\tau > \cdot)) = 0$ in R we get

$$\boxed{\gamma = \frac{R_{0,b}^{MKT\ MID}(0)}{LGD}}$$

In taking $\mathbb{E}$, we assumed stochastic Discount factor $D(s,t)$ to be independent of default time τ. Replacing τ by T we introduced dirac δ, $\delta_\tau(t)dt = \mathbb{1}_{\{\tau \in [t, t+dt)\}}$ & $\mathbb{E}$ of $\mathbb{1}$ fn gives survival prob.

(*) $CDS_{a,b}(t, R, LGD; Q(\tau > \cdot))$

$$= R\left[-\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})\, d_t Q(\tau \geq t) + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i Q(\tau \geq T_i)\right] + LGD\left[\int_{T_a}^{T_b} P(0,t)\, d_t Q(\tau \geq t)\right]$$

using $d_t Q(\tau > t) = d_t(1 - Q(\tau \leq t)) = -d_t Q(\tau \leq t)$

$$= R\left[\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})\, d_t Q(\tau \leq t) + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i Q(\tau \geq T_i)\right] - LGD\left[\int_{T_a}^{T_b} P(0,t)\, d_t Q(\tau \leq t)\right]$$

To find survival prob., we get $R = R_{0,b}^{MKT\,MID}(0)$ i.e. the market quoted spread by taking middle of ask & bid price & solving

$$CDS_{0,b}(t, R_{0,b}^{MKT\,MID}, LGD; Q(\tau > \cdot)) = 0$$

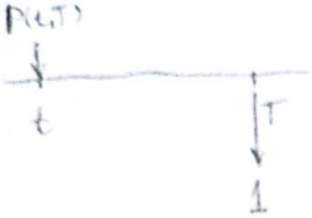
for different maturities $T_b \in \{1y, 2yr, 3yr, \ldots, 10y\}$ & solve for $\{Q(\tau > t), t < 1y\}$ etc. This method of getting survival prob. is called CDS stripping, in a model independent way.

1

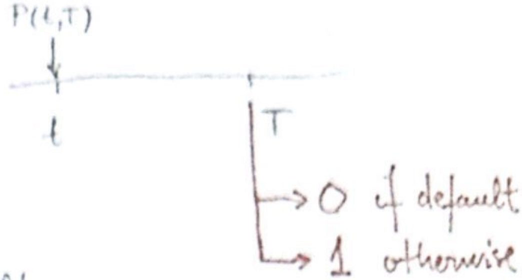
Extra 8

ZCB

DEFAULT-FREE



DEFAULTABLE



τ = time of default

STOCHASTIC DISCOUNT

$$D(t,T) = \exp\left(-\int_t^T r_s\,ds\right), \text{ where } r_s \text{ is SHORT RATE.}$$

is stochastic

ZCB PRICES - NO RECOVERY

$$P(t,T) = E\left[D(t,T)\cdot 1 \mid \mathcal{F}_t\right]$$

$$\bar{P}(t,T)\cdot \underbrace{1_{\{\tau>t\}}}_{} = E\left[D(t,T)\cdot 1_{\{\tau>T\}} \mid \mathcal{G}_t\right] \quad \text{(no recovery)}$$

price makes sense only if there was no default yet

$$\left(\mathcal{G}_t = \underset{\substack{\uparrow \\ \text{market filtration} \\ (r_t \text{ etc})}}{\mathcal{F}_t} \vee \underset{\substack{\uparrow \\ \text{default info}}}{\underbrace{\sigma\left(\{\tau\le u\}, 0\le u\le t\right)}} \right)$$

(Recovery Gen) →

$$= E\left[D(t,T)\, 1_{\{\tau>T\}} + \underline{REC}\cdot D(t,\tau)\cdot 1_{\{\tau\le T\}} \mid \mathcal{G}_t\right]$$

↑ recovery paid at default τ

$$= E\left[D(t,T)\, 1_{\{\tau>T\}} + \underline{REC}\cdot D(t,T)\cdot 1_{\{\tau\le T\}} \mid \mathcal{G}_t\right]$$

↑ recovery paid at maturity T

2

Credit Default Swap (CDS): contract in which two parties swap two streams of payments:

- premium leg: rate R at $T_{a+1}, \dots, T_b$ or until default τ_C
- protection leg: protection LGD (loss given default) at default τ_C if $T_a < \tau_C \le T_b$.

linked to the default of a third party (reference credit).

Usually $LGD = 1 - Rec$ if considered as fractions

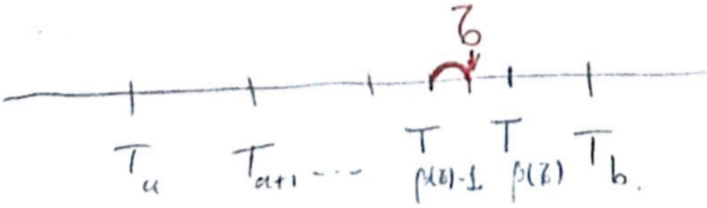
Default (at time τ_C)

- bankruptcy
- failure to pay
- obligation acceleration
- restructuring

CREDIT EVENTS

PAYMENTS at default

- cash settlement : LGD is paid
- physical settlement : LGD is paid in exchange for distressed asset (eg bond).
  - cheapest to deliver (from equal pool)
  - auction.



(3)

CDS price:

$$CDS_{a,b}(t,R,LGD) = E\left[\Pi_{RCDS_{a,b}}(t)\right]$$

$$= R \cdot \underline{\text{PremiumLeg}1_{a,b}(Q(\tau > \cdot))} \leftarrow \text{PremiumLeg}_{a,b}(R; P(0,\cdot), Q(\tau>\cdot))$$

$$- LGD \cdot \text{ProtecLeg}1_{a,b}(Q(\tau > \cdot)). \leftarrow \text{ProtecLeg}_{a,b}(LGD; P(0,\cdot), Q(\tau>\cdot))$$

↳ "unit-premium" premium leg (DV01)

↳ "unit-notional" protection leg

Running CDS discounted payoff to "B" at time $t < T_a$

$$\Pi_{RCDS_{a,b}}(t) = D(t,\tau)(\tau - T_{\beta(\tau)-1}) R \, 1_{\{T_a < \tau < T_b\}}$$

discounted accrued rate at default

$$+ \sum_{i=a+1}^{b} D(t,T_i) \alpha_i R \, 1_{\{\tau > T_i\}}$$

α_i ↳ $T_i - T_{i-1}$

CDS rate premiums if no default

PREMIUM LEG

$$- 1_{\{T_a < \tau \le T_b\}} D(t,\tau) LGD$$

protection payment at default

PROTECTION LEG.

$$CDS_{a,b}(t,R,LGD; Q(\tau \le \cdot)) = +LGD \int_{T_a}^{T_b} P(0,t) \, d_t Q(\tau \le t)$$

$$+ R\left[\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) \, d_t Q(\tau \le t) + \sum_{i=a+1}^{b} P(0,T_i) \alpha_i Q(\tau \ge T_i)\right]$$

This price is 0 (par) for market quoted $R^M_{0,b}$ =>

=> implicit eq'n in survival prob. $Q(\tau \ge t)$ CDS stripping

$$T_b = 1y \overset{R_{1y}}{\Longrightarrow} \{Q(\tau \ge t), t \le 1y\}$$

$$T_b = 2y \overset{R_{2y}}{\Longrightarrow} \{Q(\tau \ge t), 1y < t \le 2y\} \text{ etc}$$

(3)

CDS price:

$$CDS_{a,b}(t,R,LGD) = E\left[\Pi_{RCDS_{a,b}}(t)\right]$$

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$$\Pi_{RCDS_{a,b}}(t) = D(t,\tau)(\tau - T_{\beta(\tau)-1}) R\, 1_{\{T_a<\tau<T_b\}}.$$

discounted accrued rate at default

$$+ \sum_{i=a+1}^{b} D(t,T_i)\, \underbrace{\alpha_i}_{T_i - T_{i-1}}\, R\, 1_{\{\tau > T_i\}}$$

CDS rate premiums if no default

PREMIUM LEG

$$- 1_{\{T_a<\tau\le T_b\}} D(t,\tau)\, LGD$$

protection payment at default

PROTECTION LEG.

$$CDS_{a,b}(t,R,LGD; Q(\tau\le\cdot)) = +LGD\int_{T_a}^{T_b} P(0,t)\, d_t\, Q(\tau\le t)$$

$$+R\left[\int_{T_a}^{T_b} P(0,t)(t-T_{\beta(t)-1})\, d_t\, Q(\tau\le t) + \sum_{i=a+1}^{b} P(0,T_i)\,\alpha_i\, Q(\tau\ge T_i)\right]$$

This price is 0 (par) for market quoted $R^M_{0,b}$ =>

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$$T_b = 1y \overset{R_{1y}}{\Longrightarrow} \{Q(\tau\ge t), t\le 1y\}$$

$$T_b = 2y \overset{R_{2y}}{\Longrightarrow} \{Q(\tau\ge t), 1y<t\le 2y\} \text{ etc}$$

Lec 9

Poisson X: $\mathbb{Q}(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$; $\mathbb{E}(X)=\lambda$

- CDF of exp. dist $F(x,\lambda) = \begin{cases} 1-e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$$\therefore \mathbb{Q}(\tau > 10) = 1 - \mathbb{Q}(\tau \leq 10)$$
$$= 1-(1-e^{-\lambda x})$$
$$= e^{-\lambda x}$$
$$= e^{-\gamma 10}$$

$\lambda = \gamma$ for constant intensity.

Standard Poisson Process

- No. of arrivals in a queue is modelled by $\{N_t\}_{t \geq 0}$
  - (i) $N_0 = 0$
  - (ii) independent increment
  - (iii) no. of pts in an interval of length t is Poisson(λt) R.V. ($\Rightarrow N_{t+s} - N_s \sim$ Poisson(λt))

- When λ = Constant $\Rightarrow$ Process is homogeneous (Stationary) Poisson point process.

- Standard Poisson Process: Unit intensity $\lambda = 1$.

- Arrival of defaults is event counted as Poisson process. Inter arrival time $\tau_1 - \tau_0 = \tau_1 - 0$ is exponential

$$\mathbb{Q}(\tau_1 > t) = P(N_t = 0) = e^{-\lambda t} \quad \text{— ①}$$

$\tau_1 \sim \exp(\lambda) \Leftrightarrow \lambda \tau_1 \sim \exp(1)$ since $\mathbb{E}(\tau_1) = 1/\lambda$

For standard ($\lambda = 1$) Poisson process $\tau_1 \sim \exp(1)$.

- Simulating defaults
  - from ① cdf of 1st default time is $1-e^{-\lambda t}$
  - Simulate cdf with $U \sim$ Uniform$(0,1) = U(0,1)$
  - Equivalently $1-U \sim U(0,1)$
  - Default time $\Rightarrow \left[\tau_1 = -\ln U/\lambda\right]$ — ②

$1-e^{-\lambda t} = 1-U \xrightarrow{\text{Solve for } \tau_1}$

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→ link with last time

- hazard f'n $\Lambda_t = \int_0^t \lambda_s ds$
- $\lambda_t = \gamma$ gives $\Lambda_t = \gamma t$
- $\Lambda_\tau = \gamma\tau \sim exp(1)$
- denote $\xi = \Lambda_\tau \sim exp(1)$ exponential trigger
- simulate $\xi = -\ln U$ from ②
- Inverse $\Lambda^{-1}(t) = t/\gamma$
- Default time $\tau = \Lambda^{-1}(\xi) = \xi/\gamma = -\ln U/\gamma$

Intensity with Time Dependent Intensity $\lambda_t = \gamma(t)$ CDS

- $\lambda_t = \gamma(t)$
- denote: $\Gamma(t) = \int_0^t \gamma(u)du$ → cumulated intensity / hazard rate.

$$
\begin{aligned}
Q(s < \tau \leq t) &= Q\{s < \Gamma^{-1}(\xi) \leq t\} \\
&= Q\{\Gamma(s) < \xi \leq \Gamma(t)\} \\
&= Q(\xi > \Gamma(s)) - Q\{\xi > \Gamma(t)\} \\
&= \exp(-\Gamma(s)) - \exp(-\Gamma(t))
\end{aligned}
$$

i.e. prob. of default between time s & t is

$$e^{-\int_0^s \gamma(u)du} - e^{-\int_0^t \gamma(u)du} \approx \int_s^t \gamma(u)du$$

CIR++ Stochastic Intensity λ

$$\lambda_t = y_t + \psi(t;\beta), \quad t \geq 0$$

Here $\lambda(t) = \lambda_t$ is stochastic process & $\Gamma(t) = \int_0^t \gamma(u)du$ is replaced by $\Lambda(t) = \int_0^t \lambda(u)du$.

For y we consider Jump-CIR Model:

$$dY_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}\, dZ_t + dJ_t.$$

$$B = (\kappa, \mu, \nu, y_0) \quad , \quad 2\kappa\mu > \nu^2$$

- With no jumps, y follows non-central Chi-Sq. dist.
- κ = Speed of mean reversion

  μ = long term mean reversion level

  ν = volatility

$$\mathbb{E}[\lambda_t] = \lambda_0 e^{-\kappa t} + \mu(1 - e^{-\kappa t})$$

$$Var(\lambda_t) = \lambda_0 \frac{\nu^2}{\kappa}(e^{-\kappa t} - e^{-2\kappa t}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa t})^2$$

As $t \to \infty$, mean $= \mu$

$$Var = \frac{\mu \nu^2}{2\kappa}.$$

lec10

Risk Measures (under $\mathbb{P}$)

Two risk measures

(i) Value at Risk (VAR)

(ii) Expected shortfall (ES)

VAR

- L_H = Value of difference between value of portfolio today & in future H.
- L_H = Portfolio$_0$ − Portfolio$_H$
- $\pi(t,T)$ = Sum of future cash flows discounted back at t.
- Price of Portfolio$_t$ = $\mathbb{E}_t^{\mathbb{Q}}[\pi(t,T)]$
- Defn: $VaR_{H,\alpha}$ with horizon H & confidence level α is defined as that number s.t.

$$\mathbb{P}[L_H < VaR_{H,\alpha}] = \alpha$$

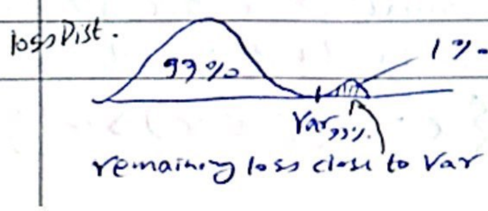
or

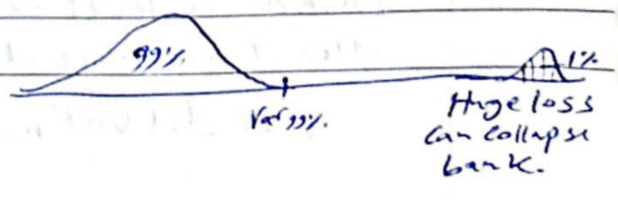
$$P[\mathbb{E}_0^{\mathbb{Q}}[\pi(0,T)] - \mathbb{E}_H^{\mathbb{Q}}(\pi(H,T)] < VaR_{H,\alpha}] = \alpha$$

- Var is α $\mathbb{P}$-percentile of loss dist. over T.
- Calculate VaR using Monte Carlo Simulation

Drawbacks of VAR

- Does not take into account the tail structure beyond percentile.





- Var is not Sub-additive on portfolio

Sub Additive $\Rightarrow$ $Var(P_1 + P_2) \leq Var(P_1) + Var(P_2)$

Portfolio's

Not true for Var.

Expected Shortfall (ES)

- To resolve sub-additivity & partly 1st drawback (ES) has been introduced.

- ES requires Var to be computed first & then takes the expected value on Tail of loss dist. for values larger than Var conditional on loss being larger than value at risk.

- ES for portfolio at confidence level α & risk Horizon H is

$$\boxed{ES_{H,\alpha} = \mathbb{E}^P[L_H \mid L_H > VaR_{H,\alpha}]}$$

where $L_H = Portfolio_0 - Portfolio_H$

- By defn $ES > Var$.

- ES is also called CVaR, AVar, ETL.

Credit VaR

For Loss L Credit $VaR_{H,\alpha}$ with Horizon H & Confidence level α is the smallest x s.t. we are atleast α-percent confident loss will not exceed level x.

$$Credit\ VaR_{H,\alpha} = \inf\{x : P(L \leq x) > \alpha\}$$

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- Think of loss L as L_H before but this time due to particular credit risk.

Credit Value Adjustment (CVA)

look Extra p2

Credit VaR Vs CVA

- Both related to credit risk.
- Credit VaR measures a potential loss due to counterparty default.
- CVA is a price Adjustment. CVA is obtained by pricing the counterparty risk component of a deal similarly to how one would price credit derivative.

→ Credit Var says how much can I lose this portfolio within a year at ~~[illegible]~~ confidence level of 99% due to default risk exposure.

→ ~~CVA~~ CVA says how much discount do I get on price ~~[illegible]~~ to counterparty risk.

① Extra Io

Interest rate and credit risk → ASSETS → PRICING MODELS

↓

under $\mathbb{Q}$

- → Market risk → unexpected moves in financial assets (stocks)
- → Credit risk CCR
- → Liquidity risk
- → Others: operational, fraud etc
- → Model risk

Timeline

- 1922 : capital requirements NYSE
- 1952 : Markowitz mean-variance : max mean, min varian
  - variance ← measure of RISK
- Sharpe ratio : measure of risk (mean, volatility)
- 1970's · leverage (eg: ATM IRS)
- 1990's : large losses (Barings, long term oil contracts, LTCM)
  - losses ↓ search for Indicator to announce/avert loss
- 1993 : VaR, ES (Basel Committee, G-30 paper)
  - ES ↑ coherent risk measure

Value at risk : (VaR/V@R)

- max loss expected over a time horizon with certain confidence
- indicator of value at risk over next period

2

Portfolio contains some assets => cash flows

Denote $\underline{\Pi(t,T)}$ = sum of all discounted cash flows occurring in $[t,T]$, discounted back to t

$\underline{\text{Portfolio}_t}$ = time t value of the portfolio

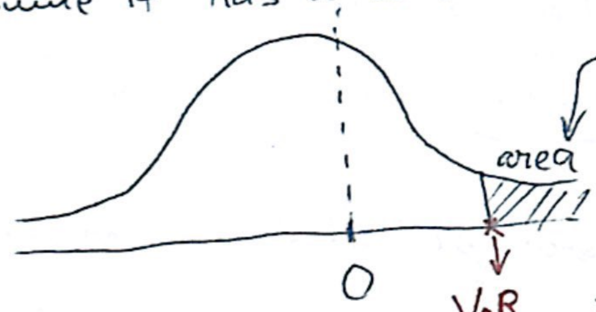
$$\text{Portfolio}_t = E_t^Q\left[\Pi(t,T)\right]$$ risk-neutral pricing

(Unknown) loss ~~with~~ with time horizon H;

$$\underline{L_H} = \text{Portfolio}_0 - \text{Portfolio}_H \quad (\text{r.v.})$$

Assume it has a continuous distribution:

Given confidence level α



$\text{VaR}_{H,\alpha}$ = α-percentile of loss distrib. (UNDER $\mathbb{P}$)

profit ⟵|⟶ loss

$\underline{\text{Statement}}$ I'm α% confident my losses at horizon H will not be larger than $\text{VaR}_{H,\alpha}$.

$\underline{\text{Consequence}}$: I will be wrong $1-\alpha$ % of the times

$\underline{\text{Pb}}$. Can I survive the loss in the case I'm wrong?

VaR doesn't tell that.

VaR disadvantages

1. No info on the tail structure.
2. Not always improved for diversified portfolios.

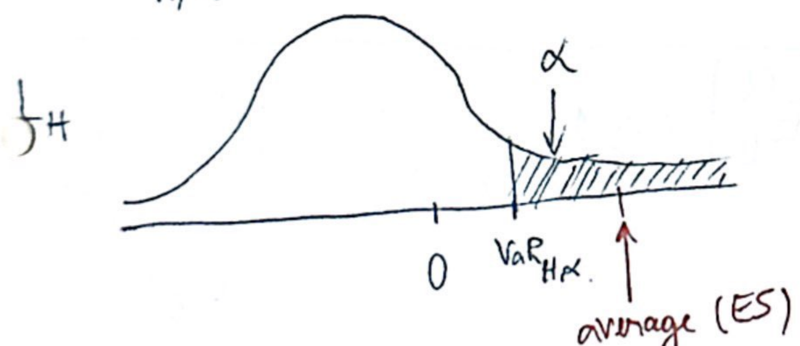
(not sub-additive $\Rightarrow$ not coherent risk measure

$$VaR(P_1+P_2) \leq VaR(P_1) + VaR(P_2).$$

ES $\begin{cases} 1 \text{ gives average loss in the tail (one number)} \\ 2 \text{ always improved for diversified portfolios (sub-additi} \end{cases}$

Def: Expected value of the loss conditional on loss being larger than VaR is Expected Shortfall

$$ES_{H,\alpha} = E^{\mathbb{P}}[L_H \mid L_H > VaR_{H,\alpha}]. \quad (= CVaR)$$



Credit $VaR_{H\alpha}$ with horizon H and confidence level α, for a loss L,

$$= \inf\{x : P(L_H \leq x) > \alpha\}.$$

(4)

CVA = Credit Valuation Adjustment to the price of a contract between two parties B and C (bank, counterparty) due to CCR

- Unilateral CVA: $\tau_c < \infty$, $\tau_B = \infty$ (never default).
- Bilateral CVA: $\tau_c < \infty$, $\tau_B < \infty$ (both defaultable)

Risk-neutral price of contract:

- in the absence of defaults: $V_t = E_t^Q[\Pi_B(t,T)]$,
- in the presence of CCR: $\bar{V}_t = E_t^Q[\Pi_B^D(t,T)]$

$\Pi_B(t,T)$ = discounted cash flows for B in $[t,T]$ w/o. default

$\Pi_B^D(t,T)$ = —————— // ——— with —//—

$$\bar{V}_t = V_t \mathbb{1}_{\{\tau_c > t\}} - \boxed{E_t^Q\Big[LGD_c \mathbb{1}_{\{t < \tau_c \le T\}} D(t,\tau_c) [\underbrace{NPV_B(\tau_c)}_{= E_{\tau_c}[\underbrace{\Pi(\tau_c,T)}_{\text{lost cashflow}}]}]^+\Big]}$$

UCVA = time t value of unrecoverable losses (LGD = 1 − Rec). due to default of counterparty

Example European call option price is £8. Value of (potential) losses due to default: $\underset{\uparrow \text{CVA}}{£2}$. Adjusted price: $£8 - \underbrace{£2}_{\text{price discount for CCR}} = £6$.

5

	Credit VaR	CVA
Measure	$\mathbb{P}$ (historical)	$\mathbb{Q}$ (pricing)

Example: Portfolio = ZCB + European call option

Risk factors: ZCB ← short rate; European call option ← stock

short rate:

$\mathbb{P}$: $dr_t = K_r(\bar{\theta} - r_t)dt + \sigma_r d\bar{Z}_t$

$\mathbb{Q}$: $dr_t = K_r(\theta - r_t)dt + \sigma_r dZ_t$

stock:

$dS_t = \cancel{\mu_t} S_t dt + \sigma_s S_t d\bar{W}_t$: $\mathbb{P}$

$dS_t = \underbrace{r_t}_{\text{no-arbitrage}} S_t dt + \sigma_s S_t dW_t$: $\mathbb{Q}$ (corr between r_t and dW_t)

$$\pi(t,T) = D(t,2y) N_B + D(t,2y)(S_{2y} - K_c)^+$$

$$L_{1y} = E_0^{\mathbb{Q}}[\pi(0,2y)] - E_{1y}^{\mathbb{Q}}[\pi(1y,2y)].$$

- simulate r_{1y}, S_{1y} under $\mathbb{P}$
- use Black-Scholes formula for call price, involving S_{2y}, r_{2y}
- simulate S_{2y}, r_{2y} starting from 1y under $\mathbb{Q}$

$\mathrm{corr}(dr, dS) = \rho \quad (dZ_t dW_t = \rho\, dt)$

$r \nearrow \quad \rho = -1 \Rightarrow S \searrow$

Bond $\downarrow$

both go down.

① Extra 11

Structural models of default (Credit Risk)

Default event occurs when firm's assets reach a sufficiently low level compared to its liabilities.

Merton model (1974)

Assets: $dA_t = A_t(\mu dt + \sigma dW_t)$ under $\mathbb{P}$.

Debt: equivalent to a single bond with face value K and maturity T

Default at T: if $A_T < K$ liquidate and receive a recovery

No default at T: if $A_T > K$ pay debt (bond) and distribute excess to shareholders / invest

Generally $\tau = \inf\{t : \underbrace{A_t}_{\text{assets}} < \underbrace{D_t}_{\text{debt (liabilities)}}\}$ BLACK-COX (1976) $D_t = K(t)$ deterministic

- first passage time of A_t (GBM) reduces to first passage time of Brownian motion

Alternative: reduced-form models (intensity models)

(2)

HJM - drift condition restricts the model

- eg: Pb 2 Set 5 Ho-Lee model $dr_t = \theta(t)dt + \sigma dW_t$

$\boxed{\text{forward rate } f(t,T) = -\partial_T \ln P(t,T)}$ has dynamics

$$df(t,T) = \underbrace{\sigma^2(T-t)}_{\text{drift}} dt + \sigma dW_t .$$

$$r_t = f(t,t) = f(0,t) + \frac{\sigma^2 t^2}{2} + \sigma W_t$$

$$\boxed{\text{HJM: } \quad df(t,T) = \underbrace{\mu(t,T)}_{\text{fixed once } \sigma \text{ is known (drift condition)}} dt + \sigma(t,T)\, dW_t \text{ under } \mathbb{Q}}$$

Recall SHORT-RATE r_t s.t.

$$P(t,T) = E^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \right].$$

Short-rate r_t = continuously compounded (annualized) interest rate at which an entity can borrow money for an infinitesimal amount of time starting at t.

$\boxed{r_t = f(t,t)}$, instantaneous forward rate, where $f(t,T) = -\partial_T \ln P(t,T)$ is the forward rate.

Note $\quad r_t = \lim_{T \to t^+} R(t,T) = \lim_{T \to t^+} L(t,T)$ (infinitesimally all rates are the same around present time t)

Also $f(t,T) = E_t^{\mathbb{Q}^T}[r_T]$

r_t cannot be directly observed (proxy: 1 mo spot rate)

③ Libor market model (LMM)

Forward Libor rate (simply compounded, prevailing at t, for expiry T and maturity $S>T$).

$$F(t,T,S)=\frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)}-1\right).$$

It is the value of the fixed rate K in a FRA with ~~maturity~~ expiry T and maturity S that makes the FRA a fair contract at t (i.e. value is 0)

$$\boxed{\text{LMM:}\quad dF(t;T_1,T_2)=\sigma_2\, F(t;T_1,T_2)\, dW_t^2 \text{ under } \mathbb{Q}^2}$$

$\mathbb{Q}^2$ = T_2-forward measure for bond price numeraire $P(t,T_2)$

Recall $L(T_1,T_2)=F(T_1;T_1,T_2)=F_2(T_1)$, so

equivalently $$\boxed{dF_2(t)=\sigma_2(t)\,F_2(t)\,dW_t^2 \text{ under } \mathbb{Q}^2}$$

Pricing under change of measure ($\equiv$ change of numeraire)

Fact 2.

$$\text{Price}_t=E_t^B\left[\frac{B(t)}{B(T)}\text{Payoff}(T)\right]=E_t^N\left[\frac{N(t)}{N(T)}\text{Payoff}(T)\right]$$

and take $N_t=P(t,T_2)$.

④

Pricing caplets in LMM

Caplet payoff $\big(\underbrace{F(T_1;T_1,T_2)}_{\text{GBM with 0 drift under } \mathbb{Q}^2} - X\big)^+$

$\Rightarrow$ price Cpl = equivalent Black-Scholes formula.

Swap market model (SMM)

Forward swap rate

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha)-P(t,T_\beta)}{\underbrace{\sum_{i=\alpha+1}^{\beta} \tau_i \, P(t,T_i)}_{N}} = \frac{P(t,T_\alpha)-P(t,T_\beta)}{\underbrace{C_{\alpha,\beta}(t)}_{N}}$$

It is the value of the fixed rate K of the underlying IRS (strike price of swaption) that makes the swaption contract (option on IRS) fair at time t (IRS has price 0)

$C_{\alpha,\beta}$ numeraire $\Rightarrow$ $\mathbb{Q}^{\alpha,\beta}$ measure

$$\boxed{\text{SMM:} \quad dS_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t)\, S_{\alpha,\beta}(t)\, dW_t^{(\alpha,\beta)} \text{ under } \mathbb{Q}^{\alpha,\beta}}$$

Credit risk

Counterparty credit risk: Basel definition

Default triggered by a credit event (definition)

Products $\begin{cases} \text{- defaultable bond} \\ \text{- CDS} < \begin{matrix} \text{premium leg} & \text{(R-rate)} \\ \text{protection leg} & \text{(LGD)} \end{matrix} \end{cases}$

Pricing formulas

CDS spread (rate): value of R that makes the contract fair (price of contract is 0).

Take it from the market and invert pricing formula to obtain survival prob. (CDS stripping)

INVERT

NO MODEL

MODEL (INTENSITY)

$$\mathbb{Q}(\tau \in [t, t+dt) \mid \tau > t, \mathcal{F}_t) = \lambda_t \, dt$$

default intensity

$\Lambda_t = \int_0^t \lambda_s \, ds$

cumulated intensity

$\xi \sim \exp(1)$ exponential trigger

$\tau = \Lambda^{-1}(\xi)$ default time

$$\begin{cases} -\lambda_t = \gamma = \frac{R}{LGD} \\ -\lambda_t = \gamma(t) \\ -\lambda_t = \gamma_t \end{cases}$$

Risk measures

Loss: $L_H = V_0 - V_H$

$VaR_{H,\alpha}$ is the number x such that

$$P(L_H < x) = \alpha \qquad (x \text{ is } \mathbb{P}\text{-quantile})$$

$ES_{H,\alpha}$ is the average of the tail beyond VaR:

$$ES_{H,\alpha} = E^{\mathbb{P}}\left[L_H \mid L_H > VaR_{H,\alpha}\right]$$

Credit $VaR_{H,\alpha}$ is the smallest value of the loss that we are α-percent confident will not be exceeded at future horizon H:

$$\text{Credit } VaR_{H,\alpha} = \inf\{x : P(L_H \le x) > \alpha\}.$$

CVA is price adjustment due to default risk:

$$CVA = E_t\left[LGD\, \mathbb{1}_{\{t < \tau_c \le T\}}\, D(t,\tau_c)\, \underbrace{\left[NPV_B(\tau_c)\right]^+}_{\text{residual NPV of contract at } \tau_c}\right]$$

residual NPV of contract at τ_c