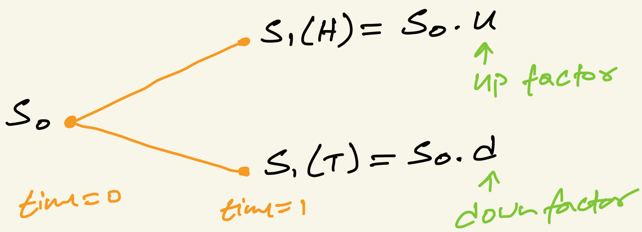
Binomial Model (one Period)

① Risky Asset

In a financial market, there is a risky asset S_t (stock). The value of stock at time $=0$ is S_0. We toss a coin & the value at time $=1$ is either $S_1(H)$ or $S_1(T)$.



$(0 < d < u)$

$\mathbb{P}(S_1 = S_0 u) = p \in (0,1)$
$\mathbb{P}(S_1 = S_0 d) = q \in (0,1)$
$p + q = 1$

This is a risky asset as its value can go either up or down.

② Non-Risky Asset

A Bank account or Bond is regarded as Non-risky asset.

$B_0 \longrightarrow$ the amount invested in Bank Account at time $t=0$.

$B_1 \longrightarrow B_0(1+r)$ initial investment plus interest

$r =$ interest rate ≥ 0 (in general).

£1 invested today $(t=0)$ will yield £$(1+r)$ at $t=1$.

- **Arbitrage**

A trading strategy that begins with no money, has a zero probability of losing money & has a strictly positive probability of making money.

- **Mathematical def^n of Arbitrage:**

Let V_t be the value of our Portfolio.

(i) $V_0 = 0$.
(ii) $V_1(\omega) \geq 0 \quad \forall \omega \in \{H, T\}$.
(iii) $\exists \omega \in \{H, T\}$ s.t. $V_1(\omega) > 0$.

Note:

- By Portfolio, I mean
  - No. of Risky Assets I buy or sell &
  - Money I invest in Bank Account or I borrow from Bank Account.
- Negative value in portfolio for No. of stocks means I short sell & -ve value for Bank Account means I borrow money.
  (-ve value $\Longrightarrow$ It's not mine)

In the Binomial model,
there is No Arbitrage $\Longleftrightarrow$ $\boxed{d < 1+r < u}$

We can construct an Arbitrage if either $d \geq 1+r$ or $u \leq 1+r$. Lets see an example of creating Arbitrage for the first case.

- case: $d \geq 1+r$

	time 0	time 1
• Borrow S_0 from the Bank	$-S_0$	$-S_0(1+r)$
• Buy stock.	$+S_0$	$S_1 \to S_1(H) = S_0.u$, $S_1 \to S_1(T) = S_0.d$
Portfolio Value (V_t)	0	(i) $S_0.u - S_0(1+r) = S_0[\underbrace{u-(1+r)}_{>0}]$ (since, $u > d \geq 1+r$) (ii) $S_0.d - S_0(1+r) = S_0[\underbrace{d-(1+r)}_{\geq 0}]$ (since, $d \geq 1+r$)

$$\left.\begin{cases} V_0 = 0 \\ V_1 = S_1 - S_0(1+r) \geq 0 \ \forall w \\ V_1(H) > 0 \end{cases}\right\} \Longrightarrow \text{ARBITRAGE.}$$

When V_t = Value of our portfolio

- Case: $u \le 1+r$

	time 0	time 1
• Short sell stock S_0.	$-S_0$	$-S_1 \nearrow -S_1(H) = -S_0 \cdot u$, $\searrow -S_1(T) = -S_0 \cdot d$
• Invest S_0 in risk-free asset.	$+S_0$	$S_0(1+r)$
Portfolio Value (V_t)	0	(i) $S_0(1+r) - S_0 \cdot u$ $S_0[\underbrace{(1+r)-u}_{\geq 0}]$ (since, $1+r \geq u$) (ii) $S_0(1+r) - S_0 \cdot d$ $S_0[\underbrace{(1+r)-d}_{>0}]$ (since, $1+r \geq u > d$)

$$\left.\begin{cases} V_0 = 0 \\ V_1 = S_0(1+r) - S_1 \geq 0 \ \forall \omega \\ V_1(T) > 0 \end{cases}\right\} \Rightarrow \text{ARBITRAGE.}$$

Where V_t = Value of our portfolio

- Exercise:

$$S_0 = 8 \quad \begin{array}{l} \nearrow S_1(H) = 16 \\ \searrow S_1(T) = 4 \end{array}$$

An investor has initial wealth $X_0 = £160$. What will be his loss if stock price goes down?

Ans: Investor can buy $\frac{160}{8} = 20$ units of stock at $t=0$.

At $t=1$:

Stock goes down so, price of stock is $S_1(T) = 4$.

Investors portfolio has now value $20 \times 4 = £80$.

$\Rightarrow$ $X_1 - X_0 = 80 - 160 = -£80$

$\Rightarrow$ Loss $= £80$.

- **Derivative Security**

This is a contract whose value depends on underlying asset.
This security pays some amount $V_1(H)$ or $V_1(T)$.

- European call option: Right but No obligation

$$\text{Payoff} = \boxed{V_1 = (S_1 - K)^+} \longrightarrow \text{to buy}$$

- European Put option:

$$\text{Payoff} = \boxed{V_1 = (K - S_1)^+} \longrightarrow \text{to sell}$$

- **Exercise:** $S_0 = 4$, $u = 2$, $d = 1/2$, $r = 1/4$

Call option with strike $K = 5$.
$X_0 = £1.20 \to$ No-Arbitrage price of the call option.

- If the price $> £1.20 \Rightarrow$ seller is doing Arbitrage.
- If the price $< £1.20 \Rightarrow$ Buyer is doing Arbitrage.

Lets consider a situation where Buyer is doing an Arbitrage.
Assume the price is £1.19.

	$t=0$	$t=1$
• Buy the option.	$+1.19$	$\omega = H \to (S_1(H) - K)^+ = (S_0 \cdot u - K)^+ = 3$ $\omega = T \to (S_1(T) - K)^+ = 0$
• Short sell ½ Stock.	-2	$-\frac{1}{2}S_1 \to H \to -\frac{1}{2} \cdot S_0 \cdot u = -4$ $-\frac{1}{2}S_1 \to T \to -\frac{1}{2} \cdot S_0 \cdot d = -1$
• Invest in the Bank Account separately 0.80 & 0.01.	(i) $+0.80$ (ii) $+0.01$	(i) $0.8(1+r) = 1$ (ii) $0.01(1+r) = 0.0125$ $+1.0125$
Portfolio Value (V_t)	0	$H \to 3 - 4 + 1.0125 = 0.0125$ $T \to 0 - 1 + 1.0125 = 0.0125$

In any case $\omega \in \{H, T\}$, the portfolio value at $t=1$ is:

$$V_1 = 0.0125 > 0 \Rightarrow \text{ARBITRAGE.}$$

[Note: We invested in two Bank Account just for better illustration of this example. We could have just used one Bank Account.]

One Central feature of Asset Pricing Theory is:

In an Equilibrium asset prices are such that "Arbitrage" is Not possible.

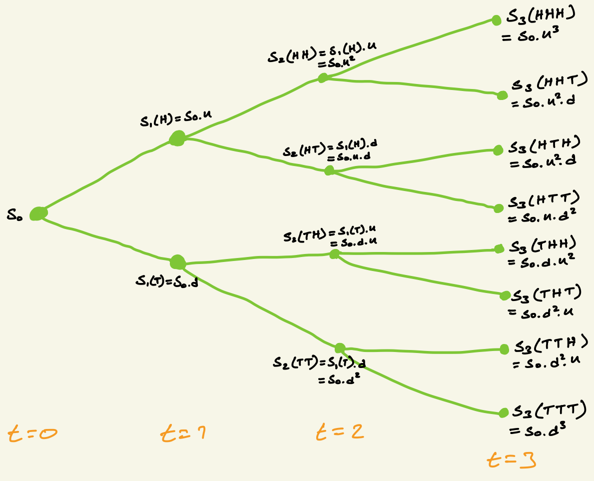
- An Arbitrageur purchases a set of financial assets at a low price & sells them at a high price simultaneously. In real world, there is rarely a case of pure arbitrage.

It is realistic to assume "No Arbitrage" in a financial market equilibrium because if there were some arbitrage opportunities, there would be eliminated immediately. More specifically, if asset prices allow for arbitrage opportunities, agents would immediately hone in on a portfolio position that yielded arbitrage profits.

Evolution of the Price of the Risky Asset (Multi-period)

- 3-period Binomial Model

  Assume $u > 1$, $d < 1$ to emphasize real gain or loss.



← (Evolution of Risky Asset)

Pricing & Hedging in the One-Period Binomial Model

We can invest in Risky Asset & Non-Risky Asset.

We are trying to price the option from a seller point of view. We compute this using replication. Let X_t be the replicating portfolio.

X_0 = initial wealth

If we receive fee X_0 from buyer for the option, we should be able by trading in risky & Non-risky asset to have at time = 1, the value of the option for all possible scenarios (i.e. the same payoffs as the option).

At $t=0$:

Invest Δ_0 units of risky asset S_0 & invest the remaining from X_0 in Bank account (Non-risky asset)

$$\underbrace{\Delta_0 . S_0}_{\text{risky asset}} + \underbrace{(X_0 - \Delta_0 . S_0)}_{\text{Bank}} = X_0$$

At $t=1$:

$$\Delta_0 . S_1(\omega) + (X_0 - \Delta_0 . S_0)(1+r) = X_1(\omega) \quad \text{for } \omega \in \{H, T\}$$

$$(1+r) X_0 + \Delta_0 [S_1(\omega) - (1+r) S_0] = X_1(\omega) \quad \text{——— (i)}$$

The replicating portfolio is called Self-financing portfolio because between $t=0$ & $t=1$, I don't bring new money or withdraw any money from the portfolio. The change in value of portfolio is only due to change in price of risky asset & the interest rate.

Replicating Portfolio:
• At $t=0$, X_0 must be same as V_0 i.e. option price has to be same as price of portfolio.
• At $t=1$, $X_1(\omega) = V_1(\omega) \; \forall \omega \in \{H, T\}$

Now, dividing eqⁿ (i) by $(1+r)$ & writing equations for each state $\omega \in \{H, T\}$, we get two equations:

$$\begin{cases} X_0 + \dfrac{\Delta_0}{1+r}\left[S_1(H) - (1+r)S_0\right] = \dfrac{X_1(H)}{1+r} & \text{—— (ii)} \\ X_0 + \dfrac{\Delta_0}{1+r}\left[S_1(T) - (1+r)S_0\right] = \dfrac{X_1(T)}{1+r} & \text{—— (iii)} \end{cases}$$

We get system of two equations with two unknowns X_0 & Δ_0. Since, this is a Replicating portfolio we must have

$$(*) \text{ —— } \begin{cases} X_1(H) = V_1(H) \\ X_1(T) = V_1(T) \end{cases}$$

where $V_1(\omega)$ is the payoff of option.

eg: For call option:

$$V_1(H) = (S_1(H) - K)^+$$
$$V_1(T) = (S_1(T) - K)^+$$

Now, subtracting (iii) from (ii) & substituting $(*)$ we get:

$$\boxed{\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}}$$

← Delta-Hedging formula i.e. No. of shares we need in portfolio so that we have a Replicating portfolio.

We will find X_0 using eqⁿ (ii) by substituting Δ_0:

$$X_0 = \frac{1}{1+r}\left[X_1(H) - \Delta_0\left(S_1(H) - (1+r)S_0\right)\right]$$

$$= \frac{1}{1+r}\left[X_1(H) - \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)}\left(S_1(H) - (1+r)S_0\right)\right]$$

(Regroup in terms of $X_1(H)$ & $X_1(T)$)

$$= \frac{1}{1+r}\left[X_1(H)\underbrace{\left(1 - \frac{S_1(H) - (1+r)S_0}{S_1(H) - S_1(T)}\right)}_{\tilde{p}} + X_1(T)\underbrace{\left(\frac{S_1(H) - (1+r)S_0}{S_1(H) - S_1(T)}\right)}_{\tilde{q}}\right]$$

- $$\frac{S_1(H)-(1+r)S_0}{S_1(H)-S_1(T)} = \frac{S_0.u-(1+r)S_0}{S_0.u-S_0.d} = \boxed{\frac{u-(1+r)}{u-d} := \tilde{q}}$$

- $$1-\frac{S_1(H)-(1+r)S_0}{S_1(H)-S_1(T)} = 1-\frac{u-(1+r)}{u-d} = \boxed{\frac{1+r-d}{u-d} := \tilde{p}}$$

① $\tilde{p}, \tilde{q}$ defines a Probability measure.

$$\tilde{p} = \tilde{\mathbb{P}}(S_1 = S_0.u)$$
$$\tilde{q} = \tilde{\mathbb{P}}(S_1 = S_0.d)$$

Due to No-Arbitrage Condition $\underline{d<1+r<u}$, we have $0<\tilde{p}<1$, $0<\tilde{q}<1$.
Moreover, $\underline{\tilde{p}+\tilde{q}=1}$. (Hence, $\tilde{p}, \tilde{q}$ is probability measure).

②
$$\boxed{\begin{aligned} X_0 &= \frac{1}{1+r}\left[\tilde{p}\,X_1(H)+\tilde{q}\,X_1(T)\right] \\ X_0 &= \frac{1}{1+r}\,\tilde{\mathbb{E}}[X_1] \end{aligned}}$$

Also,

$$S_0 = \frac{\tilde{\mathbb{E}}[S_1]}{1+r} = \frac{1}{1+r}\left[\tilde{p}\,S_1(H)+\tilde{q}\,S_1(T)\right]$$

$\tilde{p}, \tilde{q}$ are risk-Neutral Probability measure because in average, the average growth rate of risky asset under this new measure is the same as the one where we invest in the Bank Account (Non-Risky asset). So, the average growth is $(1+r)$.

Remark 1: Under the real Probability measure $\mathbb{P}$,

$$S_0 < \frac{1}{1+r}\left[p.S_1(H)+q.S_1(T)\right]$$

meaning we have a gain at time 1.

Remark 2:

Since, $X_1(\omega) = V_1(\omega)\ \forall\ \omega \in \{H,T\}$ & $X_0 = V_0$, we also get that

$$V_0 = \frac{1}{1+r}\left[\tilde{p}.V_1(H) + \tilde{q}.V_1(T)\right]$$

Where, V_0 = Price of the contract at time 0.

V_1 = the payoff of the contract.

■ $X_0 = V_0$ is the unique No-Arbitrage price of the financial contract with payoff V_1, since

- if $X_0 < V_0$ or $X_0 > V_0 \Rightarrow$ Arbitrage Opportunities.

→ Proof of $V_0 = X_0$ $\Rightarrow$ No Arbitrage in the market where we can trade Risky asset, Non-risky asset and the option.

Assume that we have a new portfolio denoted by $\tilde{V}_t$ &

$$\begin{cases} \tilde{V}_0 = 0 \\ \tilde{V}_1(\omega) \geq 0\ \forall\ \omega \in \{H,T\} \end{cases}$$

At time $t=0$:

By assumption.

$$\tilde{V}_0 = \underbrace{\alpha S_0}_{\text{risky asset}} + \underbrace{B}_{\text{Bank}} + \underbrace{\gamma V_0}_{\text{options}} = 0,$$ where

α = No. of stocks

B = Amount into Bank

γ = No. of options in portfolio

At time $t=1$:

$$\tilde{V}_1 = \alpha S_1 + B(1+r) + \gamma V_1$$

Now, taking Expectation of $\tilde{V}_1$ under risk-Neutral Probability measure,

$$\begin{aligned}\tilde{\mathbb{E}}[\tilde{V}_1] &= \alpha\tilde{\mathbb{E}}[S_1] + B(1+r) + \gamma\tilde{\mathbb{E}}[V_1] \\ &= \alpha(1+r)S_0 + B(1+r) + \gamma(1+r)V_0 \\ &= (1+r)\underbrace{[\alpha S_0 + B + \gamma V_0]}_{\tilde{V}_0} = (1+r)\tilde{V}_0 \\ &= 0\end{aligned}$$

Hence, we have a random variable $\tilde{V}_1$ which takes positive values and its expectation is zero:

$$\begin{cases} \tilde{V}_1(H) \geq 0 \\ \tilde{V}_1(T) \geq 0 \\ \tilde{\mathbb{E}}[\tilde{V}_1] = 0 \end{cases} \Rightarrow \tilde{p}.\tilde{V}_1(H) + \tilde{q}.\tilde{V}_1(T) = 0$$

$0 < \tilde{p} < 1$ $\quad$ $0 < \tilde{q} < 1$

$$\Rightarrow \tilde{V}_1(H) = \tilde{V}_1(T) = 0$$

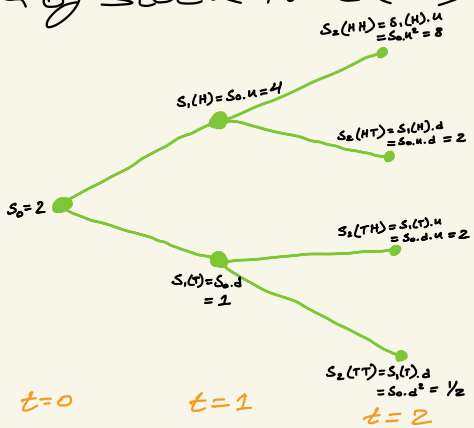
i.e. in both cases $\tilde{V}_1(w) = 0$

So, this means we started with portfolio with value zero ($\tilde{V}_0 = 0$) & at final time the portfolio value is still zero ($\tilde{V}_1(w) = 0$). So, with $V_0 = X_0$ (i.e. price of option being same as replicating portfolio value at $t=0$), we can't create an arbitrage in market where we trade risky, non-risky assets & options. ∎

Pricing And Hedging in Multi-Period Binomial Model

- Pricing and hedging of a Call Option with maturity 2 (two period Binomial model), strike $K=5$, $S_0=2$, $u=2$, $d=1/2$, $r=1/4$.

→ Evolution of Stock Price (S_n).



→ Pricing & Hedging Algorithm

We are trying to find the price of the option by replication and also the delta hedging strategy.

Idea: As a Seller of option, we receive V_0 (option price). This will be the initial amount from which we set up our portfolio such that final value of portfolio equals payoff of the option when we have a self-financing portfolio.

At $t=0$:

X_0 = initial value of the Replicating portfolio.

$$X_0 = \underbrace{\Delta_0 S_0}_{\substack{\text{Invested in} \\ \text{Risky asset}}} + \underbrace{(X_0 - \Delta_0 S_0)}_{\substack{\text{Invested in} \\ \text{Riskless Bank Account}}}$$

At $t=1$:

The value of the self-financing portfolio is:

$$X_1(\omega) = \Delta_0 S_1(\omega) + (X_0 - \Delta_0 S_0)(1+r) \quad \forall \omega \in \{H, T\}$$

We get system 1:

$$\Rightarrow \begin{bmatrix} X_1(H) = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) \text{ —— (i)} \\ X_1(T) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) \text{ —— (ii)} \end{bmatrix}$$

We can now readjust the hedge & hold Δ_1 share of stock (Δ_1 is random & depends on ω_1)

Now the rebalanced portfolio is:

$$X_1 = \Delta_1 S_1 + (X_1 - \Delta_1 S_1)$$

At $t=2$:

$$\underbrace{X_2(\omega) = V_2(\omega)}_{\text{(by replication)}} = \Delta_1(\omega) S_2(\omega) + (X_1(\omega) - \Delta_1(\omega) S_1(\omega))(1+r)$$

Here, $V_2(w) = (S_2(w) - K)^+$
$= X_2(w)$

Recall:
$V =$ Option price
$X =$ Replicating Portfolio

We now get System 2:

$$\boxed{\begin{aligned}
V_2(HH) &= X_2(HH) = \Delta_1(H) S_2(HH) + [X_1(H) - \Delta_1(H) S_1(H)](1+r) \quad \text{— (i)}\\
V_2(HT) &= X_2(HT) = \Delta_1(H) S_2(HT) + [X_1(H) - \Delta_1(H) S_1(H)](1+r) \quad \text{— (ii)}\\
V_2(TH) &= X_2(TH) = \Delta_1(T) S_2(TH) + [X_1(T) - \Delta_1(T) S_1(T)](1+r) \quad \text{— (iii)}\\
V_2(TT) &= X_2(TT) = \Delta_1(T) S_2(TT) + [X_1(T) - \Delta_1(T) S_1(T)](1+r) \quad \text{— (iv)}
\end{aligned}}$$

The unknowns are: $\Delta_0, \Delta_1(H), \Delta_1(T), X_0, X_1(H), X_1(T)$.

We now solve this going backwards, first solving System 2 & then solving System 1.

(iii) − (iv) from System 2:

$$V_2(TH) - V_2(TT) = \Delta_1(T) S_2(TH) - \Delta_1(T) S_2(TT)$$

$$\Rightarrow \boxed{\Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)}}$$

(i) − (ii) from System 2:

$$V_2(HH) - V_2(HT) = \Delta_1(H) S_2(HH) - \Delta_1(H) S_2(HT)$$

$$\Rightarrow \boxed{\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)}}$$

Substituting $\Delta_1(T)$ in (iii):

$$V_2(TH) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} \cdot S_2(TH) + \left[X_1(T) - \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} \cdot S_1(T)\right](1+r)$$

$$\Rightarrow \boxed{X_1(T) = \frac{1}{1+r}\left[\tilde{p}\, \underbrace{V_2(TH)}_{X_2(TH)} + \tilde{q}\, \underbrace{V_2(TT)}_{X_2(TT)}\right]}$$

Substituting $\Delta_1(H)$ in (i):

$$V_2(HH) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} \cdot S_2(HH) + \left[X_1(H) - \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} \cdot S_1(H)\right](1+r)$$

$$\Rightarrow \boxed{X_1(H) = \frac{1}{1+r}\left[\tilde{p}\,\underbrace{V_2(HH)}_{X_2(HH)} + \tilde{q}\,\underbrace{V_2(HT)}_{X_2(HT)}\right]}$$

where $\tilde{p}$ & $\tilde{q}$ are same as before in page 8.

(i)-(ii) from system 1:

$$\boxed{\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)}}$$

where $X_1(H)$ & $X_1(T)$ are already calculated.

Substituting Δ_0 into (i) from system 1:

$$\boxed{X_0 = \frac{1}{1+r}\left[\tilde{p}\, X_1(H) + \tilde{q}\, X_1(T)\right]} = V_0$$

- <u>Computational Considerations</u>

<u>eg:</u> If we consider a 3-period Binomial model & a Put option with strike K. The payoff is $(K-S_3)^+$.

$V_3 = (K-S_3)^+$ payoff coincides with option price at time 3.

$S_0 = 4$, $u = 2$, $d = 1/2$, $r = 1/4$, $K = 5$.

We have $2^3 = 8$ scenarios, so must do 8 computations.

$V_3(HHH) = 0$

$V_3(HHT) = V_3(HTH) = V_3(THH) = 0$

$V_3(HTT) = V_3(THT) = V_3(TTH) = 3$

$V_3(TTT) = 4.50$

$2^3 = 8$ computations

$V_3 = (K-S_3)^+$

Since, we have a recombining tree, we have repeated values for stock price at any given time. For eg. S_3 has 4 possible values & hence, 4 possible values for V_3. So, it should be enough to compute the prices of payoffs as a function of stock price. Thus, we don't have to calculate payoff & option price for each scenarios but rather compute all possible values of stock prices & see which is the price of option corresponding to stock price. This will reduce the no. of computations.

→ Denote the option price as a function of the stock price as $v_n(S_n)$.

For our eg,

$$v_3(32) = 0, \quad v_3(8) = 0, \quad v_3(2) = 3, \quad v_3(0.5) = 4.5$$

$$\Rightarrow \boxed{V_3(\omega) = v_3(S_3(\omega))}$$

- At time $t=2$:

$$v_2(s) = \frac{2}{5}\left[v_3(2 \cdot s) + v_3(s/2)\right]$$

We compute $v_2(16)$, $v_2(4)$ & $v_2(1)$.

- At time $t=1$:

$$v_1(s) = \frac{2}{5}\left[v_2(2s) + v_2(s/2)\right]$$

we compute $v_1(8)$ & $v_1(2)$.

- At time $t=0$:

$$v_0(4) = \frac{2}{5}\left[v_1(8) + v_1(2)\right]$$

- Delta Hedging:

$$\delta_n(s) = \frac{v_{n+1}(2s) - v_{n+1}(s/2)}{2s - s/2}$$

- **Path dependent/independent Options**

  - **Path-Independent Options:** Payoff depends only on the final stock price and not on the whole trajectory of stock prices. eg: Call, Put.
    $$\downarrow$$
    $$(S_3-K)^+$$

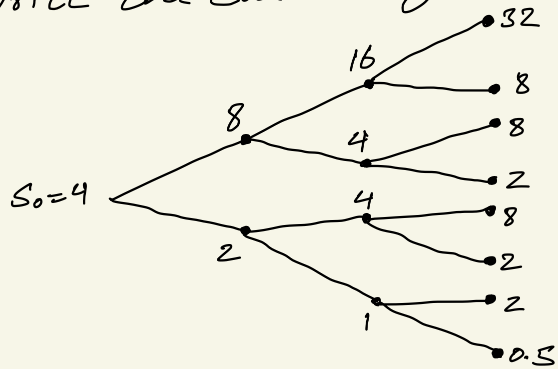
  - **Path-dependent Options:** Payoff depends on the whole trajectory of stock. eg. look back options, Asian Options.
    $$\downarrow$$
    $$\max_{0\le n\le 3}(S_n-S_3)$$

eg: Consider Path-dependent look back option with $S_0=4$, $u=2$, $d=1/2$, $r=1/4$, $\tilde{p}=\tilde{q}=1/2$ and payoff

$$V_3=\max_{0\le n\le 3}(S_n-S_3)$$

The steps for Pricing & Hedging Algorithm are:

- Write the evolution of stock Price.



- Compute the Payoffs.

$$V_3(HHH)=S_3(HHH)-S_3(HHH)=32-32=0$$
$$V_3(HHT)=S_2(HH)-S_3(HHT)=16-8=8$$
$$V_3(HTH)=S_1(H)-S_3(HTH)=8-8=0$$
$$V_3(HTT)=S_1(H)-S_3(HTT)=8-2=6$$
$$V_3(THH)=S_3(THH)-S_3(THH)=8-8=0$$

$$V_3(THT) = S_0 - S_3(THT) = 4-2 = 2$$
$$V_3(TTH) = S_0 - S_3(TTH) = 4-2 = 2$$
$$V_3(TTT) = S_0 - S_3(TTT) = 4-0.5 = 3.50$$

- Compute the price of option at other times using backward recursion:

<u>At time $t=2$:</u>

$$V_2(HH) = \frac{4}{5}\left[\frac{1}{2}V_3(HHH) + \frac{1}{2}V_3(HHT)\right] = 3.20$$
$$V_2(HT) = \frac{4}{5}\left[\frac{1}{2}V_3(HTH) + \frac{1}{2}(HTT)\right] = 2.40$$
$$V_2(TH) = \frac{4}{5}\left[\frac{1}{2}V_3(THH) + \frac{1}{2}(THT)\right] = 0.80$$
$$V_2(TT) = \frac{4}{5}\left[\frac{1}{2}V_3(TTH) + \frac{1}{2}(TTT)\right] = 2.20$$

<u>At time $t=1$:</u>

$$V_1(H) = \frac{4}{5}\left[\frac{1}{2}V_2(HH) + \frac{1}{2}V_2(HT)\right] = 2.24$$
$$V_1(T) = \frac{4}{5}\left[\frac{1}{2}V_2(TH) + \frac{1}{2}V_2(TT)\right] = 1.20$$

<u>At time $t=0$:</u>

$$V_0 = \frac{4}{5}\left[\frac{1}{2}V_1(H) + \frac{1}{2}V_1(T)\right] = 1.376$$

- The seller of this option can hedge the short position in the lookback option by buying Δ_0 shares of stock.

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = 0.17\dot{3}$$

- For Path-independent Options

$$\begin{cases} V_n \to \text{the price of an Option at time } n. \\ v_n(s) = \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(u.s) + \tilde{q}\, v_{n+1}(d.s)\right] \end{cases}$$

The relation between V_n and v_n is given by

$$\boxed{V_n(\omega_n) = v_n(S_n(\omega_n))}$$

This method of reducing computation steps can be extended to Path-dependent options as well.

Probability Theory on Coin toss Space

- Sample Space Ω is the set of all Possible outcomes.
  $\Omega = \{HHH, HHT, \dots, TTT\}$
- The probability of a head is p and the probability of a tail is $q = 1-p$.
- We assume that the coin tosses are <u>independent</u> and so the probabilities of the individual elements ω are given by

$$\mathbb{P}(HHH) = p^3;\ \mathbb{P}(HHT) = p^2 q; \dots$$

- The subsets of Ω are called Events. For eg: "The last two tosses are tails" $= \{HTT, TTT\} = B$

$$\mathbb{P}(B) = \sum_{\omega \in B} \mathbb{P}(\omega) = \mathbb{P}(HTT) + \mathbb{P}(TTT) = pq^2 + q^3$$

[Recall by def'n $\mathbb{P}(\Omega) = \sum_{\omega \in \Omega} \mathbb{P}(\omega)$]

- Two events are <u>Mutually Exclusive</u> if they do not occur at the same time $(A_1 \cap A_2 = 0)$

$$\Rightarrow \mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2)$$

- Independent Events

Def^n: Two events A and B are called independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A).\mathbb{P}(B)$

- Conditional Probability

Assume that you are not told ω, but you are told that $\omega \in B$. Suppose that $\mathbb{P}(B) > 0$

$$\boxed{\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}}$$

A & B are independent $\Leftrightarrow$ $\boxed{\mathbb{P}(A) = \mathbb{P}(A|B)}$

- Remarks:

1. Mutually Exclusive $\neq$ Independent.
2. Whether 2 sets are independent depends on the Probability measure $\mathbb{P}$.

1. Example: We consider the following experiment, Toss 1 coin.

- Sample Space: $\Omega = \{H, T\}$
- Event $A = \{\text{Heads comes up}\} = \{H\} \subset \Omega = \{H, T\}$
- Event $B = \{\text{Tails comes up}\} = \{T\} \subset \Omega = \{H, T\}$
- Events A and B are mutually exclusive because $A \cap B = \phi$.
- Events A and B are NOT independent as

$$\begin{cases} \mathbb{P}(A \cap B) = \mathbb{P}(\phi) = 0 \\ \mathbb{P}(A) = \mathbb{P}(\{H\}) = p > 0 \\ \mathbb{P}(B) = \mathbb{P}(\{T\}) = q > 0 \end{cases}$$

Recall, A independent of $B \iff \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

$$\Rightarrow 0 = p \cdot q$$

$$\Rightarrow \text{either } p = 0 \text{ or } q = 0$$

(impossible by defn of p & q)

This shows mutually exclusive is NOT same as Independent.

② The notion of Independence depends on the Probability measure $\mathbb{P}$.

<u>Example</u>: We toss a coin twice. The probability of getting a head is p & tail is q.

$\mathbb{P}(HH) = p^2$; $\mathbb{P}(HT) = \mathbb{P}(TH) = pq$; $\mathbb{P}(TT) = q^2$

Consider the events:

- $A = \{HH, HT\}$
- $B = \{HT, TH\}$
- $A \cap B = \{HT\}$
- We compute

$$\mathbb{P}(A) = p^2 + pq = p\underbrace{(p+q)}_{1} = p$$

$$\mathbb{P}(B) = pq + pq = 2pq$$

$$\mathbb{P}(A \cap B) = pq$$

A & B independent $\iff$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\iff pq = p \cdot 2pq$$

$$\iff 2p^2q - pq = 0$$

$$\iff pq(2p-1) = 0$$

Since, $p > 0$, $q > 0$

$$\Rightarrow 2p - 1 = 0$$

$$\Rightarrow p = 1/2 \Rightarrow q = 1/2$$

So, for $p = q = 1/2$, we have independence. But for other values of p & q we don't get independence. Hence, the notion of independence of events do depend on the Probability measure.

Random Variable, Probability distribution, Expectation / Conditional Expectation

We must remember the difference between Probability measure & Probability distribution function. Probability measure is a function defined on Sample space that takes values between 0 & 1. The probability distribution is a Probability measure associated to probability measure P on the Sample space plus random variable.

Probability	Probability distribution associated to a Random Variable (RV)
$(\Omega, \mathbb{P})$ $\mathbb{P}: \Omega \longrightarrow [0,1]$ $\mathbb{P}(\Omega) = 1$ $\mathbb{P}(\Omega) = \sum_{w \in \Omega} \mathbb{P}(w)$	• Random Variable $X: \Omega \longrightarrow \mathbb{R}$ Range / Set Space of a RV is $S_X = \{x \in \mathbb{R} : \exists w \text{ s.t. } X(w) = x\}$ = set of all possible values that RV can take.

Eg:

$$
\begin{tikzcd}[column sep=small]
 & & S_2(HH)=16 \\
 & S_1(H)=8 \arrow[ur] \arrow[dr] & \\
S_0 = 4 \arrow[ur] \arrow[dr] & & S_2(HT)=4 \\
 & S_1(T)=2 \arrow[r] \arrow[dr] & S_2(TH)=4 \\
 & & S_2(TT)=1
\end{tikzcd}
$$

The sets space of random variable S_0, S_1 & S_2 are:

$$S_{S_0} = \{4\}$$

$$S_{S_1} = \{2, 8\}$$

$$S_{S_2} = \{1, 4, 16\}$$

Probability Mass distribution of Random Variable X

- $P_X : S_X \longmapsto [0,1]$

  Pmf — set space

- $\sum_{x \in S_X} P_X(x) = 1$

$$\left(\sum_{x \in S_X} P_X(x) = \sum_{x \in S_X} \sum_{w \in X^{-1}(x)} P(w) = \sum_{w \in \Omega} P(w) = 1 \right)$$

$$\{ w \in X^{-1}(x) \} = \{ w : X(w) = x \}$$

- The notion of distribution depends on both probability measure & Random Variable.

- Two different RV can have same distribution.

  eg: $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

  X = Total number of heads

  Y = Total number of Tails

$X(HHH) = 3$

$X(HHT) = X(HTH) = X(THH) = 2$

$X(HTT) = X(THT) = X(TTH) = 1$

$X(TTT) = 0$

$Y(TTT) = 3$

$Y(HTT) = Y(THT) = Y(TTH) = 2$

$Y(HHT) = Y(HTH) = Y(THH) = 1$

$Y(HHH) = 0$

$P_X(0) = \mathbb{P}(\{X=0\}) = 1/8$ $\quad$ $P_Y(0) = \mathbb{P}(\{Y=0\}) = 1/8$

$P_X(1) = \mathbb{P}(\{X=1\}) = 3/8$ $\quad$ $P_Y(1) = \mathbb{P}(\{Y=1\}) = 3/8$

$P_X(2) = \mathbb{P}(\{X=2\}) = 3/8$ $\quad$ $P_Y(2) = \mathbb{P}(\{Y=2\}) = 3/8$

$P_X(3) = \mathbb{P}(\{X=3\}) = 1/8$ $\quad$ $P_Y(3) = \mathbb{P}(\{Y=3\}) = 1/8$

$\quad\Downarrow$

$\mathbb{P}(\{\omega \in \Omega : X(\omega) = 3\})$

So, X & Y are different Random Variables but they have same distribution.

Conditional Expectation

Let X be a Random Variable.

- Fix a number n & $\omega_1 \ldots \omega_n$.
- Random part will be $\omega_{n+1} \ldots \omega_N$.

We then define the Conditional Expectation of X based on information at time n as $\mathbb{E}_n[X]$:

$$\boxed{\tilde{\mathbb{E}}_n[X](\omega_1 \ldots \omega_n) = \sum_{\omega_{n+1} \ldots \omega_N} p^{\#H(\omega_{n+1} \ldots \omega_N)} q^{\#T(\omega_{n+1} \ldots \omega_N)} X(\omega_1 \omega_2 \ldots \omega_N)}$$

The Conditional Expectation is a Random Variable unlike normal Expectation which is just a number.

Martingales

$(M_0, \ldots, M_N) \longrightarrow$ Sequence of random variables indexed by time.

(i) $(M_0, \ldots, M_N)$ is an Adapted stochastic process, i.e. (M_n) only depends on $\omega_1 \ldots \omega_n$ for each n.

(ii) $\forall n$ between 0 & $(N-1)$, we have

$$M_n = \mathbb{E}_n[M_{n+1}]$$

i.e. $M_n(\omega_1 \ldots \omega_n) = \mathbb{E}_n[M_{n+1}](\omega_1 \ldots \omega_n)$ $\quad \forall (\omega_1 \ldots \omega_n)$

- **Examples: of Adapted Stochastic processes:**
  - stock Price $(S_n)_n$
  - Replicating Portfolio $(X_n)_n$
  - Option Price $(V_n)_n$
  - No. of stocks in Portfolio $(\Delta_n)_n$
- **Examples: of Martingales under Probability measure $\tilde{\mathbb{P}}$.**
  (i) Discounted Stock Price $\frac{S_n}{(1+r)^n}$.
  (ii) Discounted Wealth Process/replicating Portfolio $\frac{X_n}{(1+r)^n}$.
  (iii) Discounted Price of Option $\frac{V_n}{(1+r)^n}$.

(i) Show that the discounted stock Price $\left(\frac{S_n}{(1+r)^n}\right)$ is a martingale under $\tilde{\mathbb{P}}$.

Proof 1:

Defn of Conditional $\mathbb{E}$.

$$\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right](\omega_1\ldots\omega_n) = \frac{1}{(1+r)^n}\frac{1}{(1+r)}\left[\tilde{p}\,S_{n+1}(\omega_1\ldots\omega_n H) + \tilde{q}\,S_{n+1}(\omega_1\ldots\omega_n T)\right]$$

$$= \frac{1}{(1+r)^n}\frac{1}{(1+r)}\left[\tilde{p}\,u\,S_n(\omega_1\ldots\omega_n) + \tilde{q}\,d\,S_n(\omega_1\ldots\omega_n)\right]$$

$$= \frac{S_n(\omega_1\ldots\omega_n)}{(1+r)^n}\,\frac{\overbrace{\tilde{p}u+\tilde{q}d}^{1+r}}{1+r}$$

→ using

$$\boxed{\frac{u-(1+r)}{u-d} := \tilde{q}}$$

$$\boxed{\frac{1+r-d}{u-d} := \tilde{p}}$$

$$= \frac{S_n(\omega_1\ldots\omega_n)}{(1+r)^n} \qquad \boxed{\Sigma}$$

Proof 2: $\frac{S_{n+1}}{S_n}$ depends only on the $(n+1)$ coin toss i.e. the 1out coin toss.

$$\begin{cases} S_{n+1}(\omega_1\ldots\omega_n H) = S_n(\omega_1\ldots\omega_n)\cdot u \\ S_{n+1}(\omega_1\ldots\omega_n T) = S_n(\omega_1\ldots\omega_n)\cdot d \end{cases}$$

$$\frac{S_{n+1}}{S_n} = \begin{cases} u & \text{if } \omega_{n+1} = H \\ d & \text{if } \omega_{n+1} = T \end{cases}$$

$$\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] = \tilde{\mathbb{E}}_n\left[\frac{S_n}{(1+r)^{n+1}} \cdot \frac{S_{n+1}}{S_n}\right]$$

(Taking out what is known property of conditional $\mathbb{E}$) $\longrightarrow$

$$= \frac{S_n}{(1+r)^n}\tilde{\mathbb{E}}_n\left[\frac{1}{1+r} \cdot \frac{S_{n+1}}{S_n}\right]$$

(Independence of conditional $\mathbb{E}$, since $\frac{S_{n+1}}{S_n}$ only depends on $(n+1)$ coin toss & thus independent up to time n) $\longrightarrow$

$$= \frac{S_n}{(1+r)^n}\tilde{\mathbb{E}}\left[\frac{1}{1+r} \cdot \frac{S_{n+1}}{S_n}\right]$$

$$= \frac{S_n}{(1+r)^n}\frac{1}{(1+r)}\tilde{\mathbb{E}}\left[\frac{S_{n+1}}{S_n}\right]$$

$$= \frac{S_n}{(1+r)^n}\frac{1}{(1+r)} \cdot \underbrace{\tilde{p}u+\tilde{q}d}_{(1+r)}$$

$$= \frac{S_n}{(1+r)^n} \quad \boxed{\Sigma}$$

(ii) Show that the discounted wealth process / self financing replicating portfolio $\frac{X_n}{(1+r)^n}$ is martingale under $\tilde{\mathbb{P}}$.

$$X_{n+1} = \Delta_n S_{n+1} + (X_n - \Delta_n S_n)(1+r)$$

$$\tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] = \tilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}\right]$$

(Linearity of Cond. $\mathbb{E}$) $\longrightarrow$

$$= \tilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}}\right] + \tilde{\mathbb{E}}_n\left[\frac{X_n - \Delta_n S_n}{(1+r)^n}\right]$$

(Taking out what's known as Δ_n, X_n, S_n are adapted & known at time n) $\longrightarrow$

$$= \Delta_n \tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] + \frac{X_n - \Delta_n S_n}{(1+r)^n}$$

(Discounted stock price $\frac{S_n}{(1+r)^n}$ is Martingale under $\tilde{\mathbb{P}}$) $\longrightarrow$

$$= \Delta_n \frac{S_n}{(1+r)^n} + \frac{X_n - \Delta_n S_n}{(1+r)^n}$$

$$= \frac{X_n}{(1+r)^n} \quad \boxed{\Sigma}.$$

(iii) Show discounted Price of Option is Martingale under $\tilde{\mathbb{P}}$.

Since, $X_n = V_n \quad \forall n.$

$$\Rightarrow \tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right] = \frac{V_n}{(1+r)^n} \quad \forall n. \quad \boxed{3}$$

- <u>Exercise</u>: Find K such that $S_n = e^{\sigma M_n} K^n$ is a Martingale under a Probability measure $\mathbb{P}$ such that $\mathbb{P}(H) = \mathbb{P}(T) = 1/2$.

(Symmetric Random walk)

$$\begin{cases} M_n = \sum_{j=1}^{n} X_j, & n \geq 1 \\ M_0 = 0 \end{cases}$$

$$\begin{cases} X_j = 1 & \text{if } j^{th} \text{ toss results in Head.} \\ X_j = -1 & \text{if } j^{th} \text{ toss results in Tail.} \end{cases}$$

$\Rightarrow$ (i) S_n is adapted process for whatever value of K. S_n is adapted since M_n is adapted. So for each K, S_n is adapted.

(ii) $\mathbb{E}_n[S_{n+1}] = S_n$

$$\Rightarrow \mathbb{E}_n\left[e^{\sigma M_{n+1}} K^{n+1}\right] = e^{\sigma M_n} K^n$$

$$\Rightarrow \mathbb{E}_n\left[e^{\sigma (M_n + X_{n+1})} K^{n+1}\right] = e^{\sigma M_n} K^n$$

$$\Rightarrow \mathbb{E}_n\left[e^{\sigma M_n + \sigma X_{n+1}} K^{n+1}\right] = e^{\sigma M_n} K^n$$

(Taking out what's known) $\Rightarrow e^{\sigma M_n} \cdot K^{n+1} \mathbb{E}_n\left[e^{\sigma X_{n+1}}\right] = e^{\sigma M_n} K^n$

(Independence of X_{n+1} on $W_1 \dots W_n$) $\Rightarrow \cancel{e^{\sigma M_n}} \cdot K^{n+1} \mathbb{E}\left[e^{\sigma X_{n+1}}\right] = \cancel{e^{\sigma M_n}} K^n$

$$\Rightarrow K\, \mathbb{E}\left[e^{\sigma X_{n+1}}\right] = 1 \quad \text{or} \quad K = 0$$

$$\Rightarrow \quad K=0 \quad \text{or} \quad K = \frac{1}{\mathbb{E}[e^{\sigma X_{n+1}}]}$$

$$= \frac{1}{\frac{1}{2}e^{\sigma\cdot 1} + \frac{1}{2}e^{\sigma\cdot -1}}$$

$$= \frac{1}{\frac{1}{2}(e^{\sigma}+e^{-\sigma})} = \frac{2}{e^{\sigma}+e^{-\sigma}} \;//$$

For these values of K we get S_n martingale under measure $\mathbb{P}$.

Cash flow Valuation

$C_0, C_1, \ldots, C_N$ be a sequence of random variables s.t. (C_n) is an Adapted Stochastic process.

$$V_n = \tilde{\mathbb{E}}_n\left[\sum_{k=n}^{N} \frac{C_k}{(1+r)^{k-n}}\right] \quad n = 0, 1, \ldots, N.$$

Net present value of sequence of payments $C_n, \ldots, C_N$

$$= \tilde{\mathbb{E}}_n\left[\frac{C_n}{(1+r)^{\overset{k=n}{n-n}}} + \sum_{k=n+1}^{N} \frac{C_k}{(1+r)^{k-n}}\right]$$

(Linearity) $$= \tilde{\mathbb{E}}_n[C_n] + \tilde{\mathbb{E}}_n\left[\sum_{k=n+1}^{N} \frac{C_k}{(1+r)^{k-n}}\right]$$

(Taking out what's known) $$= C_n + \tilde{\mathbb{E}}_n\left[\sum_{k=n+1}^{N} \frac{C_k}{(1+r)^{k-n}}\right]$$

(Iterated Conditioning) $$= C_n + \tilde{\mathbb{E}}_n\left[\frac{\tilde{\mathbb{E}}_{n+1}\left[\sum_{k=n+1}^{N} \frac{C_k}{(1+r)^{k-(n+1)}}\right]}{1+r}\right]$$

$$= C_n + \tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{1+r}\right]$$

$$V_n = C_n + \tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{1+r}\right]$$

$$V_n - C_n = \frac{1}{1+r}\left[\tilde{p}\, V_{n+1}(H) + \tilde{q}\, V_{n+1}(T)\right] \quad //$$

Markov Processes

Defn: $X_0, X_1, \ldots, X_N$ (adapted i.e. for each n, X_n depends on first n coin tosses) sequence of random variables indexed by time. We say that the process $(X_n)_n$ is markov process if for every function $f(x)$ and for each n $\exists$ a function $g(x)$ which also depends on f and n such that

$$\mathbb{E}_n[f(X_{n+1})] = g(X_n)$$

$$\forall n = 0, \ldots, N-1.$$

This means that the conditional expectation of $f(X_{n+1})$ can be written as $g(X_n)$ which only depends on the present value X_n and does not depend on any previous values.

example: In Finance, the stock price (S_n) is markov under both probability measure $\mathbb{P}$ & $\tilde{\mathbb{P}}$.

$$S_{n+1}(\omega_1 \ldots \omega_{n+1}) = \begin{cases} u\, S_n(\omega_1 \ldots \omega_n) & \text{if } \omega_{n+1} = H \\ d\, S_n(\omega_1 \ldots \omega_n) & \text{if } \omega_{n+1} = T \end{cases}$$

Under Real probability measure $\mathbb{P}$:

$$\mathbb{E}_n[f(S_{n+1})](\omega_1 \ldots \omega_n) = p f(uS_n) + q f(dS_n)$$
$$= g(S_n)$$

This shows S_n is markov.

Note: (Computational Consideration) The Markov property of the process (S_n) is essential for reducing the number of computations.

In the Risk-Neutral pricing formula:

$$\begin{cases} V_n = \tilde{\mathbb{E}}_n\left[\dfrac{V_N}{(1+r)^{N-n}}\right] & \text{where } V_N = \text{payoff of the option.} \\ \text{If } V_N = \underbrace{v_N}_{\text{function defined on } \mathbb{R}}(S_N) & \text{where } S_N = \text{final price of stock} \end{cases}$$

Then, by multistep Markov Property of process (S_n)

$$\exists\, v_n \text{ s.t. } V_n = \underbrace{v_n(S_n)}$$

Here, we compute the function v only along the possible values of stock & using the fact stock price is recombining tree. We have to do $(n+1)$ computations as opposed to 2^n computations.

At time $n = N-1$:

$$V_{N-1} = \tilde{\mathbb{E}}_{N-1}\left[\frac{V_N}{(1+r)}\right]$$

$$= \tilde{\mathbb{E}}_{N-1}\left[\frac{v_N(S_N)}{(1+r)}\right]$$

[Using Markov property of (S_n) by $f(x) = \dfrac{v_N(x)}{(1+r)}$]

$$= v_{N-1}(S_{N-1})$$

So, $V_{N-1} = v_{N-1}(S_{N-1})$

i.e. $V_{N-1}(\omega_1 \ldots \omega_{N-1}) = v_{N-1}(S_{N-1}(\omega_1 \ldots \omega_{N-1}))$

$\forall (\omega_1 \ldots \omega_{N-1})$.

- A Key tool in order to show that a process is Markov is Independence Theorem:

- Independence Theorem

  N period Binomial Model, n an integer number between 0 & N.

  $$\begin{cases} X \text{ only depends on the first } n \text{ coin tosses.} \\ Y \text{ only depends on the coin tosses from } (n+1) \text{ to } N. \end{cases}$$

  Define a function $g: \mathbb{R} \to \mathbb{R}$ as

  $$g(x) = \mathbb{E}[f(x, Y)]$$

  $x \in \mathbb{R}$ (a constant)

  Random ($\mathbb{E}$ only acts on Y)

  Then,

  $$\mathbb{E}_n[f(X, Y)] = g(X).$$

  eg: Fix ω_1:

  $$g(\underbrace{X(\omega_1)}_{\in \mathbb{R}}) = \mathbb{E}[f(X(\omega_1), Y)]$$

Independence Theorem is another way to show if a process is Markov.

Example: Showing stock price process is Markov.

Let f be an arbitrary function and n between 0 & $N-1$.

$$\mathbb{E}_n[f(S_{n+1})] = \mathbb{E}_n\Big[f\Big(\underbrace{S_n}_{X} \cdot \underbrace{\frac{S_{n+1}}{S_n}}_{Y}\Big)\Big]$$

Since, X depends only on first n coin toss & Y only on $(n+1)^{th}$ coin toss, by the Independence Theorem,

$$\mathbb{E}_n[f(S_{n+1})] = \mathbb{E}_n\Big[f\Big(\underbrace{S_n}_{X} \cdot \underbrace{\frac{S_{n+1}}{S_n}}_{Y}\Big)\Big]$$

(Independence Theorem) $\longrightarrow$ $= g(S_n)$.

$$\text{where, } g(x) = \mathbb{E}[f(xY)]$$
$$= f(xu)\cdot p + f(xd)\cdot q$$

- Example: Of Non-Markov process is the Maximum of Stock price: $\boxed{M_n = \max_{0 \le k \le n} S_k}$

- Example: The Couple Bi-dimensional process (M_n, S_n) is is a Markov process.

$$\begin{cases} S_{n+1} = S_n \cdot \left(\dfrac{S_{n+1}}{S_n}\right) & \text{—(i)} \\ M_{n+1} = M_n \vee S_{n+1} & \\ \quad\;\; = \underbrace{M_n}_{X^1} \vee \Big(\underbrace{S_n}_{X^2} \cdot \underbrace{\dfrac{S_{n+1}}{S_n}}_{Y}\Big) & \text{—(ii)} \end{cases}$$

Notation

$x \vee y = \max(x, y)$

Lets take an arbitrary function f.

$$\mathbb{E}_n[f(S_{n+1}, M_{n+1})] = \mathbb{E}_n[f(S_n \cdot Y, M_n \vee (S_n \cdot Y))]$$

(Independence Theorem with X^1, X^2 & Y) $\longrightarrow$ $= g(S_n, M_n)$

$$\text{where } g(\underbrace{s}_{\in\mathbb{R}}, \underbrace{m}_{\in\mathbb{R}}) = \mathbb{E}[f(sY, m \vee (sY))]$$

(The randomness only comes from Y)

$$= f(su, m \vee (su))\cdot p + f(sd, m \vee (sd))\cdot q \;//$$

This shows the process (M_n, S_n) is Markov.

- ## 2-Step Markov Property

Aim: $\forall$ function h, $\exists\, g$ s.t.

$$\mathbb{E}_n[h(X_{n+2})] = g(X_n) \quad \forall n.$$

Proof:

$\rightarrow$ Using One-step Markov property at time $(n+1)$:

$$\mathbb{E}_{n+1}[h(X_{n+2})] = f(X_{n+1}) \text{ ——— } (*)$$

$\rightarrow$ Take $\mathbb{E}_n$ on both sides of $(*)$:

$$\mathbb{E}_n[\mathbb{E}_{n+1}[h(X_{n+2})]] = \mathbb{E}_n[f(X_{n+1})]$$

(Iterated Conditioning) (one-step Markov Property)

$$\mathbb{E}_n[h(X_{n+2})] = g(X_n) \quad \boxed{\Sigma}$$

The Argument to extend this to Multi-step Property are the same.

- ## Application of the Markov Property of the Process (S_n, M_n) to pricing.

It's very useful for computational consideration in Lookback Option.

Eg: $V_N = (M_N - K)^+ \rightarrow$ Payoff

$$= v_N(M_N, S_N)$$

$$v_n(m, s) = (m-K)^+$$

Now, lets look at price of lookback option at time n, V_n.

$$V_n = \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right] = \tilde{\mathbb{E}}_n\left[\frac{v_N(M_N, S_N)}{(1+r)^{N-n}}\right]$$

(By Multi-step Markov property of the bi-dimensional process (M_n, S_n)) $\rightarrow = v_n(M_n, S_n)$

So, $V_n = v_n(M_n, S_n)$

Using One-step Markov Property, we can get a backward algorithm between v_{n+1} & v_n:

$$\tilde{\mathbb{E}}_n\left[\frac{v_{n+1}(sY, m\vee(sY))}{1+r}\right]$$

$$= \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(su, m\vee(su)) + \tilde{q}\, v_{n+1}(sd, m\vee(sd))\right]$$

$$= v_n(s, m)$$

Furthermore, $v_n(S_n, M_n) = V_n \rightarrow$ Price of option

---

Radon Nikodym Derivative

Aim: 2 Main probability measures in Finance.

(i) Actual (Real) Probability measure $\mathbb{P}$.

(ii) Risk-Neutral Probability measure $\tilde{\mathbb{P}}$.

Remarks:

(i) ($\mathbb{P}$ and $\tilde{\mathbb{P}}$ disagree): The 2 probability measures give different weights to the asset-price paths in the model.

(ii) $\mathbb{P}$ & $\tilde{\mathbb{P}}$ agree on which paths are possible.

Q: How to write the Pricing formula under the Real Probability measure?

Main tool to do this: Radon-Nikodym derivative

- <u>Defn</u>: In a Finite Sample space Ω with $\mathbb{P}$ & $\widetilde{\mathbb{P}}$,

  Define $Z(\omega) = \dfrac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$, $\forall \omega \in \Omega$

  Z = Radon-Nikodym derivative (<u>Random Variable</u>)

- <u>Main Theorem</u>:

  (i) $\mathbb{P}(Z>0) = 1$ (same as $Z(\omega) > 0 \ \forall \omega \in \Omega$)

  (ii) $\mathbb{E}[Z] = 1$

  (iii) For any random variable Y, we have

  $$\boxed{\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ]}$$

  ← Change of measure formula.

- How to write the pricing formula under the Real Probability measure?

  Using (iii), we get

  $$\widetilde{\mathbb{E}}\left[\frac{V_N}{(1+r)^N}\right] = \mathbb{E}\left[\frac{V_N}{(1+r)^N} \cdot Z\right]$$

  [we applied (iii) with $Y = \dfrac{V_N}{(1+r)^N}$]

- <u>Example</u>: 3-period Binomial model with $u=2, d=1/2$

  $S_0 = 4, r = 1/4, p = 2/3, q = 1/3$.

  Compute $Z = \dfrac{\widetilde{\mathbb{P}}}{\mathbb{P}}$.

$S_0 = 4$

$S_1(H) = 8$ $S_1(T) = 2$

$S_2(HH) = 16$ $S_2(HT) = 4$ $S_2(TH) = 4$ $S_2(TT) = 1$

$S_3(HHH) = 32$

$S_3(HHT) = 8$

$S_3(HTH) = 8$

$S_3(HTT) = 2$

$S_3(THH) = 8$

$S_3(THT) = 2$

$S_3(TTH) = 2$

$S_3(TTT) = 0.5$

Step 1: Write out the sample space (all possible paths).

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

We assume that $p = 2/3$ and $q = 1/3$.

Step 2: Compute $\mathbb{P}(\omega)$ $\forall \omega \in \Omega$.

$$\mathbb{P}(HHH) = p^3 = \frac{8}{27}$$

$$\mathbb{P}(HHT) = p^2 \cdot q = \frac{4}{27}$$

$$\mathbb{P}(TTH) = q^2 \cdot p = \frac{2}{27}$$

$$\vdots$$

$$\mathbb{P}(TTT) = q^3 = \frac{1}{27}$$

Step 3: Compute $\widetilde{\mathbb{P}}(\omega)$ $\forall \omega \in \Omega$.

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{1 + 1/4 - 1/2}{2 - 1/2} = 1/2$$

$$\tilde{q} = 1 - 1/2 = 1/2$$

$$\widetilde{\mathbb{P}}(HHH) = \tilde{p}^3 = (1/2)^3 = 1/8$$

$$\widetilde{\mathbb{P}}(HHT) = \tilde{p}^2 \cdot \tilde{q} = (1/2)^2 \cdot 1/2 = 1/8$$

$$\vdots$$

$$\widetilde{\mathbb{P}}(TTT) = \tilde{q}^3 = 1/8$$

Here, $\widetilde{\mathbb{P}}(\omega) = 1/8$, $\forall \omega \in \Omega$.

Step 4: Computation of Radon Nikodym derivative

$$Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} \quad \forall \omega \in \Omega.$$

$$Z(HHH) = \frac{\widetilde{\mathbb{P}}(HHH)}{\mathbb{P}(HHH)} = \frac{1/8}{8/27} = \frac{27}{64}$$

$$Z(HHT) = \frac{\widetilde{\mathbb{P}}(HHT)}{\mathbb{P}(HHT)} = \frac{1/8}{4/27} = \frac{27}{32}$$

⋮

$$Z(TTT) = \frac{\tilde{\mathbb{P}}(TTT)}{\mathbb{P}(TTT)} = \frac{1/8}{1/27} = \frac{27}{8}$$

- <u>Application to Pricing</u>

<u>Ex</u>: Call Option with maturity 3 and strike price $K=9$. Payoff: $V_3 = (S_3-9)^+$

Risk-Neutral Pricing formula: $V_0 = \tilde{\mathbb{E}}\left[\frac{V_3}{(1+r)^3}\right]$

$$= \tilde{\mathbb{E}}\left[\frac{(S_3-9)^+}{(1+r)^3}\right]$$

By using the Main Theorem property (iii), we have

$$V_0 = \mathbb{E}\left[\frac{(S_3-9)^+}{(1+r)^3}\cdot Z\right]$$

$$= \sum_{\omega\in\Omega} \frac{(S_3(\omega)-9)^+ \cdot Z(\omega)}{(1+r)^3}\cdot \mathbb{P}(\omega)$$

- <u>State Price density</u>:

$$\xi = \frac{Z(\omega)}{(1+r)^N}$$

For Conditional expectation, we need Radon-Nikodym derivative process.

- <u>Radon-Nikodym derivative Process</u>

- <u>Theorem</u>: Let Z be a Random Variable in an N-period Binomial model. Define a process

$$Z_n = \mathbb{E}_n[Z]$$

Then, $(Z_n)_n$ is a $\mathbb{P}$-Martingale. (General Result useful in also continuous time.)

Proof:

$$\mathbb{E}_n[Z_{n+1}]$$

(def^n of Z_{n+1}) $\longrightarrow$ $= \mathbb{E}_n[\mathbb{E}_{n+1}[Z]]$

(Iterated Conditioning) $\longrightarrow$ $= \mathbb{E}_n[Z]$

(def^n of Z_n) $\longrightarrow$ $= Z_n$ $\boxed{Z}$

- Relation of this theorem with the Radon-Nikodym process derivative:

  $\rightarrow$ If we take $Z(\omega) = \dfrac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$ then,

  (Z_n) is the Radon-Nikodym process and by the theorem, it is a Martingale under $\mathbb{P}$.

  $\rightarrow$ Since, $Z_n = \mathbb{E}_n[Z]$ $\forall n$ between 0 & N, we have:

  - At time $n=0$:

    $$Z_0 = \mathbb{E}_0[Z] = \mathbb{E}[Z] \overset{\uparrow}{=} 1$$

    (Main Thm, part (ii))

  - At time $n=N$:

    $$Z_N = \mathbb{E}_N[Z] \overset{\uparrow}{=} Z$$

    (Taking out what's known)

  So, we know the values of this process (Z_n) at initial time & terminal time.

- Some Properties:

  (1) If Y is a random variable which only depends on the information up to n (with n given) then

  $$\tilde{\mathbb{E}}[Y] = \mathbb{E}[Z_n Y]$$

  (Recall: $\forall Y$, $\tilde{\mathbb{E}}[Y] = \mathbb{E}[YZ]$)

(2) Change of Measure for Conditional Expectation:

$$\tilde{\mathbb{E}}_n[Y] = \frac{1}{Z_n} \mathbb{E}_n[Z_m Y]$$

where Y depends on the first m coin tosses.

- Application of Property (2) above to Pricing:

$$V_n = \tilde{\mathbb{E}}_n\Big[\underbrace{\frac{V_N}{(1+r)^{N-n}}}_{Y}\Big] \rightarrow \text{classical risk-neutral pricing formula}$$

(Applying Property (2) with $m=N$) $\rightarrow$ $= \frac{1}{Z_n} \mathbb{E}_n\left[Z_N \cdot \frac{V_N}{(1+r)^{N-n}}\right]$, with n between 0 & N.

This allows us to write change of measure for any time n, not just zero.

American Options

-

European Option	American Option
$\rightarrow$ Buyer can exercise his right at a given time called Maturity (N).	$\rightarrow$ Buyer can exercise his right at any time between 0 and the Maturity N.
$\rightarrow$ eg: Payoff Call: $(S_N - K)^+$ Put: $(K - S_N)^+$	$\rightarrow$ eg: Payoff is a Process. Call: $(S_n - K)^+$, $0 \le n \le N$ Put: $(K - S_n)^+$, $0 \le n \le N$ These are Path independent as values only depend on time n.

Price of American Option $\geq$ Price of an European Option.

An example of Look back type Path dependent American Option is $\rightarrow$ $\left(\max_{0 \le k \le n} S_k - K\right)^+$ where values depend on whole path.

• Pricing Algorithm

European Option

• Let $v_t(s)$ be price of European Option.

$g(s) \to$ Exercise Value.

eg: $g(s) = S-K$ for call option.

$\to$ Pricing Algorithm:

$$v_N(s) = \max\{g(s), 0\}$$

$$v_n(s) = \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(us) + \tilde{q}\, v_{n+1}(ds)\right]$$

where $0 \le n \le N-1$

$$\tilde{p} = \frac{1+r-d}{u-d}$$

$$\tilde{q} = (1-\tilde{p}) = \frac{u-1-r}{u-d}$$

The price of Option V_n for all scenarios $\bar{w}$ is

$$V_n(\bar{w}_n) = v_n(S_n(\bar{w}_n))$$

$$\bar{w}_n = (w_1 w_2 \dots w_n)$$

• $\left(\dfrac{v_n(S_n)}{(1+r)^n}\right)_n$ is a Martingale under $\tilde{\mathbb{P}}$ i.e.

$$\tilde{\mathbb{E}}_n\left[\frac{v_{n+1}(S_{n+1})}{(1+r)^{n+1}}\right] = \frac{v_n(S_n)}{(1+r)^n}$$

American Option

• Let $\tilde{v}_t(s)$ be price of American Option.

$\to$ Pricing Algorithm:

$$\tilde{v}_N(s) = \max\{g(s), 0\}$$

$$\tilde{v}_n(s) = \max\Big\{\underbrace{g(s)}_{\text{Exercise Value}}, \underbrace{\frac{1}{1+r}\left[\tilde{p}\,\tilde{v}_{n+1}(us) + \tilde{q}\,\tilde{v}_{n+1}(ds)\right]}_{\text{Continuation Value if buyer of Option doesn't Exercise. Seller must maintain this Value.}}\Big\}$$

where,

$$\tilde{p} = \frac{1+r-d}{u-d}$$

$$\tilde{q} = (1-\tilde{p}) = \frac{u-1-r}{u-d}$$

The price of American Option $\tilde{V}_n$ is $\tilde{V}_n = \tilde{v}_n(S_n)$.

• $\left(\dfrac{\tilde{v}_n(S_n)}{(1+r)^n}\right)_n$ is a SuperMartingale under $\tilde{\mathbb{P}}$ i.e.

$$\tilde{\mathbb{E}}_n\left[\frac{\tilde{v}_{n+1}(S_{n+1})}{(1+r)^{n+1}}\right] \le \frac{\tilde{v}_n(S_n)}{(1+r)^n}$$

Recall: The lower case v is the function which depends on possible values of stockprice S and is a real number. While capital V is computed for each possible scenarios. We used markov property to get relation between V & v so that the number of computation is reduced.

- <u>Example</u>: Pricing Americal put in a 2-period Binomial Model (Sellers Point of View). $S_0 = 4$, $u = 2$, $d = 1/2$, $r = 1/4$, $\tilde{p} = \tilde{q} = 1/2$, $K = 5$.

$$\text{Payoff} = (5 - S_n)^+, \quad 0 \le n \le N$$

$$\text{Exercise Value} = (5 - S_n)^+$$

The Algorith for Pricing gives:

$$v_2(s) = \max\{5 - s, 0\},$$

$$v_n(s) = \max\left[5 - s, \frac{2}{5}\left(v_{n+1}(2s) + v_{n+1}(s/2)\right)\right]$$

for $n = 1, 0$.

Price:

$$\begin{tikzcd}[column sep=small]
 & & v_2(16) = 0 \\
 & v_1(8) = 0.40 \arrow[ur] \arrow[dr] & \\
v_0(4) = 1.36 \arrow[ur] \arrow[dr] & & v_2(4) = 1 \\
 & v_1(2) = 3 \arrow[ur] \arrow[dr] & \\
 & & v_2(1) = 4
\end{tikzcd}$$

$\rightarrow$ If the price of the stock is 2 at time 1 (i.e. $\omega_1 = T$), then the price of Option $v_1(2)$ is $v_1(2) = g(2) = 5 - 2 = 3$.

$\Rightarrow$ It is <u>Optimal</u> for the buyer to Exercise.

If the Buyer <u>doesn't</u> exercise, the seller can continue the hedge with 2. The difference $3 - 2 = C_1$ is <u>consumed</u>. (Also called Consumption).

From this example, we see that the Wealth equation in the case of American Options is the same as in the case of European options except that we include the possibility of consumption.

- <u>Wealth Equation</u>

→ European Option:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

→ American Option:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \underbrace{C_n}_{\text{Consumption}} - \Delta_n S_n)$$

$$\text{with } C_n \geq 0$$

<u>Remark:</u>

If, $\frac{1}{1+r}\left[\tilde{V}_{n+1}(S_n u)\tilde{p} + \tilde{V}_{n+1}(S_n d)\tilde{q}\right] < \tilde{V}_n(S_n)$, then this implies its Optimal for buyer to exercise since $\tilde{V}_n(S_n)$ = exercise value is greater than Continuation Value.

If buyer doesn't exercise in this case then $C_n(S_n) > 0$ where

$$C_n = \tilde{V}_n(S_n) - \frac{1}{1+r}\left[\tilde{V}_{n+1}(S_n u)\tilde{p} + \tilde{V}_{n+1}(S_n d)\tilde{q}\right]$$

- <u>Hedging Equation</u>

$$\Delta_n = \frac{\tilde{V}_{n+1}(uS_n) - \tilde{V}_{n+1}(dS_n)}{(u-d)S_n}$$

which is same as that for European Option.

- **Buyer Point of View**

Q: When is it optimal to exercise?

A: $\tau^* = \min\{n : g(S_n) = \tilde{v}_n(S_n)\}$ i.e. the minimum value (first time) when exercise value equal price of option.

τ^* takes values in $\{0, 1, \ldots, N\} \cup \{\infty\}$.

- τ^* is a <u>stopping time</u> (random variable) that takes values in $\{0, 1, \ldots, N\} \cup \{\infty\}$ and satisfies the condition:

  if $\tau^*(w_1 \ldots w_n w_{n+1} \ldots w_N) = n$ then

  $\tau^*(w_1 \ldots w_n w'_{n+1} \ldots w'_N) = n$ for all $w'_{n+1} \ldots w'_N$.

  (i.e. τ^* only depends up to $w_1 \ldots w_n$ & independent of $w_{n+1} \ldots w_N$).

- If we could know from beginning the whole evolution of price (unrealistic) & consider stopping time δ which is like taking decision based on future evolution then δ is <u>NOT</u> a stopping time.

- **Stopped Process**

  Let $Y_n \rightarrow$ the initial process.

  $\tau \rightarrow$ a stopping time.

  We define stopped process as:

  $$Y_{n \wedge \tau} \rightarrow Y_{n \wedge \tau(w)}(w)$$

  $\boxed{n \wedge \tau = \min(n, \tau)}$

  (for n larger than $\tau(w)$, the process will be frozen).

• <u>Remark 1:</u>

Using the notion of stopped process,

$\dfrac{\tilde{v}_{n\wedge\tau^*}(S_{n\wedge\tau^*})}{(1+r)^{n\wedge\tau^*}}$ is a <u>Martingale</u> under $\tilde{\mathbb{P}}$, where

$$\tau^* = \min\{n : g(S_n) = \tilde{v}_n(S_n)\}.$$

• <u>Remark 2:</u>

For any stopping time τ,

$\dfrac{v_{n\wedge\tau}(S_{n\wedge\tau})}{(1+r)^{n\wedge\tau}}$ is a <u>SuperMartingale</u> under $\tilde{\mathbb{P}}$.

General American Options (Path-depend. & Path Indep.)

• <u>Price Definition:</u>

For each n, let G_n be a Random variable which depends on the first n coin tosses i.e. the payoff G_n is Adapted process. The price of General American Option V_n is also an Adapted process and is given as:

$$V_n = \max_{\tau\in S_n} \underbrace{\tilde{\mathbb{E}}_n\left[\mathbb{1}_{\tau\leq N}\frac{G_\tau}{(1+r)^{\tau-n}}\right]}_{\text{This is like price of European Option with maturity } \tau.}$$

where, τ = stopping time

$S_n = \{n, n+1, \dots\}$ set of stopping times.

So, V_n can be seen as maximum over all possible maturities τ of Price of European Option.

<u>Remark:</u> The price of an American Option is $\geq$ the price of an European Option.

<u>Proof:</u> Take $\tau^* = N$. Observe that $\tau^* \in S_n$.

$$\Rightarrow V_n^{American} = \max_{\tau \in S_n} \tilde{\mathbb{E}}_n\left[\mathbb{1}_{\tau \le N} \frac{G_\tau}{(1+r)^{\tau-n}}\right]$$

$$\ge \tilde{\mathbb{E}}_n\left[\mathbb{1}_{\tau^* \le N} \frac{G_{\tau^*}}{(1+r)^{\tau^*-n}}\right]$$

$$= \tilde{\mathbb{E}}_n\left[\frac{G_N}{(1+r)^{N-n}}\right] \quad \leftarrow (\text{since } \tau^* = N)$$

$$= V_n^{European}$$

$$\Rightarrow V_n^{American} \ge V_n^{European} \quad \boxed{\Sigma}.$$

- Recall:
  In the case of Path-Independent American Options, the price was defined through the following backward in time Algorithm:

$$\begin{cases} v_N(s) = \max\{g(s), 0\} \\ v_n(s) = \max\{g(s), \frac{1}{1+r}[\tilde{p}\, v_{n+1}(us) + \tilde{q}\, v_{n+1}(ds)] \\ \qquad \forall n = N-1, \dots, 0 \\ V_n(w_1 \dots w_n) = v_n(S_n(w_1 \dots w_n)) \ \forall w_1 \dots w_n \end{cases}$$

For Path dependent case, we don't have the function little v so, have to work directly with capital V_n and we have to compute for each path so it is computationally intensive.

- The pricing Algorithm for General case (including path-dependent American options) is:

$$V_N(w_1 \dots w_N) = \max\{G_N(w_1 \dots w_N), 0\}$$

$$V_n(w_1 \dots w_n) = \max\{G_n(w_1 \dots w_n), \frac{1}{1+r}[\tilde{p}\, V_{n+1}(w_1 \dots w_n H) + \tilde{q}\, V_{n+1}(w_1 \dots w_n T)]\}$$

$$\forall n = N-1, \dots, 0.$$

The price given by this Algorithm coincides with the price definition in terms of stopping time i.e.

$$V_n = \max_{\tau \in S_n} \tilde{\mathbb{E}}_n \left[\mathbb{1}_{\tau \in N} \frac{G_\tau}{(1+r)^{\tau-n}} \right]$$

To show this, we use the properties of the American option price (V_n).

- **Properties of American Option Price**

(i) $V_n \geq \max\{G_n, 0\}$

(ii) The discounted option price $\frac{V_n}{(1+r)^n}$ is a super Martingale under $\tilde{\mathbb{P}}$, i.e.

$$\tilde{\mathbb{E}}_n \left[\frac{V_{n+1}}{(1+r)^{n+1}} \right] \leq \frac{V_n}{(1+r)^n}$$

(iii) (Minimality Property)

If (Y_n) is another process satisfying $Y_n \geq \max\{G_n, 0\}$ and $\frac{Y_n}{(1+r)^n}$ is a SuperMartingale under $\tilde{\mathbb{P}}$ i.e. property (i) & (ii)

then $Y_n \geq V_n \;\; \forall n = 0, \ldots, N.$

Reminder: If we denote $\tilde{V}_n$ the solution of the pricing algorithm given in last page, we can show that $\tilde{V}_n$ satisfies property (i), (ii) & (iii) above.

From the (iii) condition & since both $\tilde{V}_n$ & V_n satisfy this condition we get,

$$\begin{cases} V_n \geq \tilde{V}_n & \text{(Since } \tilde{V}_n \text{ satisfy (iii))} \\ \tilde{V}_n \geq V_n & \text{(} V_n \text{ is the smallest process satisfying (i) \& (ii))} \end{cases}$$

$$\Rightarrow V_n = \tilde{V}_n$$

- ## Optimal Exercise Time for Buyer of Option

(1)
$$\max_{\tau \in S_0} \tilde{\mathbb{E}}\left[\mathbb{1}_{\tau \le N} \frac{G_\tau}{(1+r)^\tau}\right] = \tilde{\mathbb{E}}\left[\mathbb{1}_{\tau^* \le N} \frac{G_{\tau^*}}{(1+r)^{\tau^*}}\right]$$

τ^* such that (1) holds is an Optimal Stopping Time.
we have seen that:

$$\tau^* = \min\{n : V_n = G_n\}$$

Remark: $G_n \longrightarrow$ Exercise Value. eg: Put $(K-S_n)^+$, Call $(S_n-K)^+$

Note, that in the pricing Algorithm, we can use either G_n or $\max\{G_n, 0\}$. Why?

$$V_n = \max\left\{G_n, \frac{1}{1+r}\left[\tilde{p} V_{n+1}(H) + \tilde{q} V_{n+1}(T)\right]\right\}$$

$$= \max\left\{\max(G_n, 0), \frac{1}{1+r}\left[\tilde{p} V_{n+1}(H) + \tilde{q} V_{n+1}(T)\right]\right\}$$

(because of Property (ii) $V_n \ge 0, \forall n$)

But, In the definition of the optimal stopping time, you can only take G_n.

- ## Replication of Path-dependent American Option

→ Define hedging strategy:

$$\Delta_n(\omega_1 \ldots \omega_n) = \frac{V_{n+1}(\omega_1 \ldots \omega_n H) - V_{n+1}(\omega_1 \ldots \omega_n T)}{S_{n+1}(\omega_1 \ldots \omega_n H) - S_{n+1}(\omega_1 \ldots \omega_n T)}$$

→ Define Consumption process:

$$C_n(\omega_1 \ldots \omega_n) = V_n(\omega_1 \ldots \omega_n) - \frac{1}{1+r}\left[\tilde{p} V_{n+1}(\omega_1 \ldots \omega_n H) + \tilde{q} V_{n+1}(\omega_1 \ldots \omega_n T)\right]$$

→ Define value of self-financing Portfolio:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n)$$

$$X_0 = V_0$$

Then, we have

$$X_n(\omega_1 \dots \omega_n) = V_n(\omega_1 \dots \omega_n) \quad \forall n, \forall \omega_1 \dots \omega_n.$$

Furthermore, we have

$$X_n \geq G_n \,, \forall n.$$

Reminder

→ $\frac{V_n}{(1+r)^n}$ is a $\tilde{\mathbb{P}}$ Super-Martingale.

→ By the Sampling Theorem, for all stopping times τ, $\left(\frac{V_{n\wedge\tau}}{(1+r)^{n\wedge\tau}}\right)$ is a $\tilde{\mathbb{P}}$- Super Martingale.
→ Stopped process.

→ In the case when τ is τ^*, then
optimal stopping time

$$\left(\frac{V_{n\wedge\tau^*}}{(1+r)^{n\wedge\tau^*}}\right) \text{ is a } \tilde{\mathbb{P}}\text{-Martingale.}$$
→ Optimal Stopped process

Random Walk Construction

We repeatedly toss a fair coin (P is probability of head & q is probability of tail).

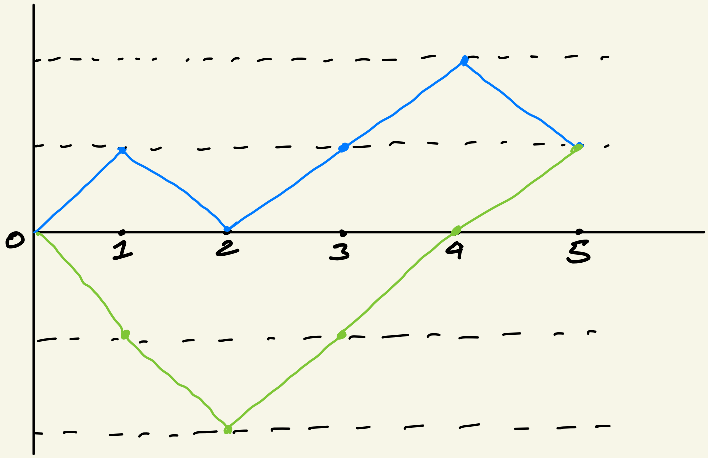
$$X_j = \begin{cases} 1 & \text{if } \omega_j = H \\ -1 & \text{if } \omega_j = T \end{cases}$$

Define, Symmetric Random Walk as:

$$M_0 = 0$$
$$M_n = \sum_{j=1}^{n} X_j \,, \quad n = 1, 2, \dots$$

Note: Symmetric random walk is when $P = q = 1/2$
& a Non-Symmetric is when $P \neq q$.

eg:



- First Passage Time

  Let m be given

  $$\tau_m = \min\{n : M_n = m\}$$

  eg: For $m=1$ for above figure,

  $\tau_m = 1$

  $\tau_m = 5$

Binomial Model for Interest Rate

- Up to now, the Interest Rate was assumed to be a constant r.

  Ω = set of 2^N possible outcomes $w_1 \ldots w_N$ of N tosses of a coin.

  $\tilde{\mathbb{P}}$ is a probability measure on Ω which gives a strictly positive "weights" to each scenario. $\tilde{p} > 0, \tilde{q} > 0$.

- The Interest Rate process is defined as a sequence of random variables $R_0, R_1, \ldots, R_{N-1}$ where R_0 is the only non-random variable and for each $n = 1, \ldots, N-1$, R_n only depends on the first n coin tosses (i.e. R_n is adapted).

- £1 invested in Bank Account at time n grows to $(1+R_n)$ at time $n+1$.
- We assume that Interest Rate process

$$R_n(\omega_1 \ldots \omega_n) > 0, \quad \forall \omega_1 \ldots \omega_n, \ \forall n.$$

- The <u>discount process</u> is defined as:

$$D_n = \frac{1}{(1+R_0)(1+R_1)\ldots(1+R_{n-1})}, \quad n = 1, 2, \ldots, N$$

$$D_0 = 1$$

[In the case of a constant interest rate r, we have

$$D_n = \frac{1}{(1+r)^n} \quad]$$

- The <u>Risk-Neutral Pricing Formula</u> is given by

$$\tilde{\mathbb{E}}[D_m X] \longrightarrow (*) \quad \text{(time 0 price)}$$

where X is the payment received at time m.

- We define the <u>time zero price of a ZCB</u> as being

$$\tilde{\mathbb{E}}[D_m] \quad \text{(i.e. formula } (*) \text{ with } X = 1\text{)}$$

Conditional Expectations in the Case of Non-independent Coin tosses

- <u>Def^n</u>: (Conditional Probability)

Let $\tilde{\mathbb{P}}$ be a probability measure on the space Ω of all possible sequence of N coin tosses. Assume that every sequence $\omega_1 \ldots \omega_N$ in Ω has a positive probability under $\tilde{\mathbb{P}}$.

Let $1 \leq n \leq N-1$ and let $\bar{\omega}_1 \ldots \bar{\omega}_N$ be a sequence of N coin tosses.

We define conditional probability:

$$\tilde{\mathbb{P}}\{w_{n+1}=\bar{w}_{n+1}\dots w_N=\bar{w}_N \mid w_1=\bar{w}_1\dots w_n=\bar{w}_n\}$$

$$=\frac{\tilde{\mathbb{P}}(\bar{w}_1\dots\bar{w}_N)}{\tilde{\mathbb{P}}(\bar{w}_1\dots\bar{w}_n)}$$

(Recall $\mathbb{P}(A|B)=\dfrac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$)

- Let X be a random variable. We define the conditional Expectation of X based on the information at time n as:

$$\tilde{\mathbb{E}}_n[X](w_1\dots w_n)=\sum_{\bar{w}_{n+1}\dots\bar{w}_N} X(\bar{w}_1\dots\bar{w}_n\bar{w}_{n+1}\dots\bar{w}_N).\tilde{\mathbb{P}}\{w_{n+1}=\bar{w}_{n+1}\dots w_N=\bar{w}_N \mid w_1=\bar{w}_1\dots w_n=\bar{w}_n\}$$

Zero-Coupon Bond Price at any time n

- $\tilde{\mathbb{P}}$: probability measure on the space Ω.
- Assume that $\tilde{\mathbb{P}}(w)>0\ \forall w\in\Omega$.
- $R_0,\dots,R_{N-1}$ an Interest Rate Process.
- $\begin{cases} D_n=\dfrac{1}{(1+R_0)\dots\dots(1+R_{n-1})} \\ D_0=1 \end{cases}$ discount process.
- The price of ZCB with maturity m at time n is:

$$B_{n,m}=\tilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}\right]$$

Remark:

If the face-value (or the final payment at the maturity of a ZCB) is C, then the price is $C.B_{n,m}$. $\left[\tilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}.C\right]=C.\tilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}\right]=C.B_{n,m}\right]$

Recall: $\frac{1}{(1+r)^n} \cdot S_n$ is a Martingale under $\tilde{\mathbb{P}}$.

We have a similar result in the case of ZCB's.

- Thm: $D_n B_{n,m}$ is a Martingale under $\tilde{\mathbb{P}}$.

Proof: By using the def'n of the price, we have

$$D_n B_{n,m} = D_n \tilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}\right]$$

$$= D_n \cdot \frac{1}{D_n} \tilde{\mathbb{E}}_n[D_m] \quad \text{(taking out what's known)}$$

$$D_n B_{n,m} = \tilde{\mathbb{E}}_n[D_m] \quad \text{——} \quad (*)$$

From this relation $(*)$, for $k \le n$ we get,

$$\tilde{\mathbb{E}}_k[D_n B_{n,m}] = \tilde{\mathbb{E}}_k[\tilde{\mathbb{E}}_n[D_m]] \quad (\text{Iterated conditioning } \& \ (*))$$

$$= \tilde{\mathbb{E}}_k[D_m]$$

$$= D_k B_{k,m} \quad (\text{using } (*))$$

So, $\tilde{\mathbb{E}}_k[D_n B_{n,m}] = D_k B_{k,m}$ (the Martingale Property)

$\boxed{\xi}$

Coupon Paying Bond

- Sequence of constant (i.e. Non-random) quantities $C_0, C_1, \ldots, C_m$.
- At a given n, C_n is the coupon payment made at time n (which can be zero).
- The final payment C_m includes the face-value and the coupon due at time m.

- The price of such a contract is:

$$\sum_{k=0}^{m} C_k B_{0,k} = \tilde{\mathbb{E}}\left[\sum_{k=0}^{m} C_k D_k\right]$$

- The price at time n is:

$$\sum_{k=n}^{m} C_k B_{n,k} = \tilde{\mathbb{E}}_n\left[\sum_{k=n}^{m} C_k \frac{D_k}{D_n}\right]$$

- <u>Example:</u> $N=3$ (3 period Binomial Model)

$\Omega = \{HHH, HHT, \ldots, TTT\} \to 2^3 = 8$ possible scenarios.

Assume that:

$\tilde{\mathbb{P}}(HHH) = 2/9$; $\tilde{\mathbb{P}}(HHT) = 1/9$; $\tilde{\mathbb{P}}(HTH) = 1/12$; $\tilde{\mathbb{P}}(HTT) = 1/12$;
$\tilde{\mathbb{P}}(THH) = 1/6$; $\tilde{\mathbb{P}}(THT) = 1/12$; $\tilde{\mathbb{P}}(TTH) = 1/8$; $\tilde{\mathbb{P}}(TTT) = 1/8$

Define the sets where 1st two coin tosses are fixed:

$A_{HH} = \{\omega_1 = H ; \omega_2 = H\} = \{HHH, HHT\}$

$A_{HT} = \{\omega_1 = H ; \omega_2 = T\} = \{HTH, HTT\}$

$\vdots$

A_{TH}

A_{TT}

Similarly, define sets when 1st toss is fixed:

$A_H = \{\omega_1 = H\} = \{HHT, HHH, HTH, HTT\}$

$A_T = \{\omega_1 = T\} = \{THH, TTH, THT, TTT\}$

$\tilde{\mathbb{P}}(A_{HH}) = 2/9 + \frac{1}{9} = 1/3$

$\tilde{\mathbb{P}}(A_{HT}) = 1/12 + 1/12 = 1/6$

Now, lets compute the conditional probabilities.

$$\tilde{\mathbb{P}}(\omega_3 = H \mid \omega_1 = T; \omega_2 = H) = \frac{\tilde{\mathbb{P}}(THH)}{\tilde{\mathbb{P}}(A_{TH})} = 2/3$$

$$\tilde{\mathbb{P}}(\omega_3 = T \mid \omega_1 = T; \omega_2 = H) = \frac{\tilde{\mathbb{P}}(THT)}{\tilde{\mathbb{P}}(A_{TH})} = 1/3$$

Let X be a random variable. We compute the Conditional Expectation as:

$$\tilde{\mathbb{E}}_2[X](TH) = X(THT) \cdot \tilde{\mathbb{P}}(\omega_3 = T \mid \omega_1 = T; \omega_2 = H) + X(THH) \cdot \tilde{\mathbb{P}}(\omega_3 = H \mid \omega_1 = T; \omega_2 = H)$$

ZCB

- Consider an agent who can trade in the ZCB's of every maturity and in the Bank Account.
- We define by $\Delta_{n,m}$ the number of m-maturity ZCB's held by the agent between times n and $n+1$.
- We denote by X_n the wealth at time n.

At time $n+1$, we have:

$$X_{n+1} = \underbrace{\sum_{m=n+1}^{N} \Delta_{n,m} B_{n+1,m}}_{\text{Investment in all maturity ZCB's}} + \overbrace{\left(X_n - \sum_{m=n+1}^{N} \Delta_{n,m} B_{n,m}\right)(1+R_n)}^{\text{Investment in Bank Account}}$$

$$= \Delta_{n,n+1} \underbrace{B_{n+1,n+1}}_{=1} + \sum_{m=n+2}^{N} \Delta_{n,m} B_{n+1,m} + \left(X_n - \sum_{m=n+1}^{N} \Delta_{n,m} B_{n,m}\right)(1+R_n)$$

$$= \Delta_{n,n+1} + \sum_{m=n+2}^{N} \Delta_{n,m} B_{n+1,m} + \left(X_n - \sum_{m=n+1}^{N} \Delta_{n,m} B_{n,m}\right)(1+R_n)$$

- **Thm:** The discounted wealth process $(D_n X_n)$ is a Martingale under $\tilde{\mathbb{P}}$.

Idea of Proof:

$$\tilde{\mathbb{E}}_n[D_{n+1} X_{n+1}] = D_n X_n$$

known at time n

$$\tilde{\mathbb{E}}_n[X_{n+1}] = \underbrace{\Delta_{n,n+1}}_{\text{known at time } n} + \sum_{m=n+2}^{N} \Delta_{n,m}\, \tilde{\mathbb{E}}_n[B_{n+1,m}] + \underbrace{(1+R_n)\left(X_n - \sum_{m=n+1}^{N} \Delta_{n,m} B_{n,m}\right)}_{\text{known at time } n}$$

Since, D_{n+1} is known at time n, we can write

$$\tilde{\mathbb{E}}_n[B_{n+1,m}] = \frac{\tilde{\mathbb{E}}_n[D_{n+1} B_{n+1,m}]}{D_{n+1}}$$

$$\tilde{\mathbb{E}}_n[X_{n+1}] = \Delta_{n,n+1} + \sum_{m=n+2}^{N} \frac{\Delta_{n,m}}{D_{n+1}} \tilde{\mathbb{E}}_n[D_{n+1} B_{n+1,m}] + \underbrace{\frac{D_n}{D_{n+1}}}_{(1+R_n)} \left(X_n - \sum_{m=n+1}^{N} \Delta_{n,m} B_{n,m}\right)$$

(Martingale Property of Discounted ZCB price)

$$= \Delta_{n,n+1} + \sum_{m=n+2}^{N} \frac{\Delta_{n,m}}{D_{n+1}} D_n B_{n,m} + \frac{D_n}{D_{n+1}} X_n - \frac{D_n}{D_{n+1}} \sum_{m=n+1}^{N} \Delta_{n,m} B_{n,m}$$

$$= \Delta_{n,n+1} + \frac{D_n}{D_{n+1}} X_n - \frac{D_n}{D_{n+1}} \Delta_{n,n+1} B_{n,n+1} \qquad (*)$$

But: $B_{n,n+1} = \tilde{\mathbb{E}}_n\left[\frac{D_{n+1}}{D_n}\right] = \frac{D_{n+1}}{D_n}$

as D_n & D_{n+1} are known at time n.

Now (*) becomes:

$$= \Delta_{n,n+1} + \frac{D_n}{D_{n+1}} X_n - \frac{D_n}{D_{n+1}} \Delta_{n,n+1} \frac{D_{n+1}}{D_n}$$

$$= \frac{D_n}{D_{n+1}} X_n$$

Thus, $\tilde{\mathbb{E}}_n[X_{n+1}] = \frac{D_n X_n}{D_{n+1}}$

$$\Rightarrow \tilde{\mathbb{E}}_n[D_{n+1} X_{n+1}] = D_{n+1} \cdot \frac{D_n X_n}{D_{n+1}}$$

$$\tilde{\mathbb{E}}_n[D_{n+1} X_{n+1}] = D_n X_n$$

This shows $D_n X_n$ is a Martingale under $\tilde{\mathbb{P}}$. $\square$

- The Martingale Property of $(D_n X_n)$ under $\tilde{\mathbb{P}}$ implies that there is No-Arbitrage. Why?

$$\tilde{\mathbb{E}}[D_n X_n] = X_0, \quad n = 0, 1, \ldots, N.$$

If one could construct an Arbitrage by trading in the ZCB's and the Bank Account, then there is a portfolio which begins with $X_0 = 0$ and at some future time n results in $X_n \geq 0$, whatever the outcome of the coin tossing is and further results in $X_n > 0$ for some of the outcomes $(\mathbb{P}(X_n > 0) > 0)$.

In this case, we have

$$X_0 \longrightarrow \underbrace{\tilde{\mathbb{E}}[D_n X_n]} > 0 \Rightarrow X_0 > 0 \ \lightning$$

- <u>Exam Paper 2018</u>

(1) $S_0 = 7,\ u = 2,\ d = 1/2,\ p = 2/3,\ q = 1/3,\ r = 0.$

(a)

(i)
$$\begin{array}{lll} & & S_2 = 7.2^2 \\ & S_1 = 7.2^1 & \\ S_0 = 7 & & \\ & S_1 = 7.\frac{1}{2} & \end{array}$$

The Largest value stock price has at time $N = S_0 . u^N$
$$= 7.\ 2^N$$

(ii) No-Arbitrage condition is
$$d < 1+r < u$$
$$1/2 < 1+0 < 2/3 \quad \rightarrow \text{Condition is satisfied}$$
so, yes.

(iii)
$$\mathbb{E}[S_n] = S_1(H).\,p + S_1(T).\,q$$
$$= (S_0.u).\,p + (S_0.d).\,q$$
$$= S_0(up + qd)$$

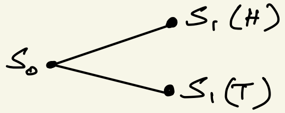
(c) $(S_N - K)^+,\ K = 4,\ N = 2,\ r = 0,\ S_0 = 3,\ u = 2,\ d = 1/2$

Using Algorithm of pricing:
$$V_N(s) = (s-K)^+$$
for each $n = N-1, \dots, 0$
$$V_n(s) = \frac{1}{1+r}\left[\tilde{p}\, V_{n+1}(us) + \tilde{q}\, V_{n+1}(ds)\right]$$

(d) $\mathbb{E}_2[S_2] = S_2$
$$\mathbb{E}_1[S_2](w_1) = S_2(w_1 H).\,p + S_2(w_1 T).\,q$$
$$\mathbb{E}_1[S_2](H) = S_2(HH).\,p + S_2(HT).\,q$$
$$\mathbb{E}_1[S_2](T) = S_2(TH).\,p + S_2(TT).\,q$$

- Exam Paper 2019

(1)



$g(S_1) \longrightarrow$ payoff at time $t=1$.

(a) (i) let Δ = No. of shares of stock.

X_0 = Initial value of the portfolio.

At $t=0$: $\Delta S_0 + (X_0 - \Delta S_0) = X_0$

At $t=1$: $\Delta S_1 + (X_0 - \Delta S_0)(1+r) = g(S_1)$

So, for $t=1$, we have two equations for two scenarios:

$$\begin{cases} \Delta S_1(H) + (X_0 - \Delta S_0)(1+r) = g(S_1(H)) & \text{— (i)} \\ \Delta S_1(T) + (X_0 - \Delta S_0)(1+r) = g(S_1(T)) & \text{— (ii)} \end{cases}$$

The unknowns in the above eq's are X_0 & Δ. Solving (i) & (ii):

$$\boxed{\Delta = \frac{g(S_1(H)) - g(S_1(T))}{S_1(H) - S_1(T)}}$$

(ii) From the system, we derive that

$$X_0 - \Delta S_0 = \frac{1}{1+r}\left[g(S_1(H)) - g(S_1(T))\right]$$

we replace Δ & get X_0.

(c) $\tilde{p}, \tilde{q}$ are risk neutral probabilities.

$$\tilde{p} = \frac{1+r-d}{u-d}$$

$$S_0 . \tilde{p} = \frac{S_0(1+r) - S_0 . d}{S_0 . u - S_0 . d} = \frac{S_0(1+r) - S_1(T)}{S_1(H) - S_1(T)}$$

(2) $M_n = \sum_{j=0}^{n} X_j$ where $X_j = \begin{cases} 1 \text{ if } \omega_j = H & \text{with prob. } p = 1/2 \\ -1 \text{ if } \omega_j = T & \text{with prob. } q = 1/2 \end{cases}$

$M_0 = 0$

(a) To show M_n is a Martingale we need to satisfy:

- (M_n) is an adapted process.
- $\forall n, \ \mathbb{E}_n[M_{n+1}] = M_n$

$$\mathbb{E}_n[M_{n+1}] = \mathbb{E}_n\left[\sum_{j=0}^{n+1} x_j\right]$$

(Linearity + Taking out what is known.) $\rightarrow$ $= \sum_{j=0}^{n} x_j + \mathbb{E}_n[X_{n+1}]$

(X_{n+1} is independent of n) $\rightarrow$ $= M_n + \mathbb{E}[X_{n+1}]$

($\mathbb{E}[X_{n+1}] = 0$) $\longrightarrow$ $= M_n + 0$

$= M_n$.

(b) Showing $S_n = K^n e^{\sigma M_n}$ is a Martingale.

For $K=0$, $S_n = 0$ is a constant process which is adapted & is a Martingale.

- S_n depends on first n tosses $\Rightarrow$ Adapted.
- $\mathbb{E}_n[S_{n+1}] = \mathbb{E}_n[K^{n+1} e^{\sigma M_{n+1}}]$

$= \mathbb{E}_n[K^{n+1} e^{\sigma M_n + \sigma X_{n+1}}]$

(Taking out what's known) $\rightarrow$ $= K^{n+1} e^{\sigma M_n} \mathbb{E}_n[e^{\sigma X_{n+1}}]$

(Independence of $n+1$) $\rightarrow$ $= K^{n+1} e^{\sigma M_n} \mathbb{E}[e^{\sigma X_{n+1}}]$

Lets calculate $\mathbb{E}[e^{\sigma X_{n+1}}]$:

$$\mathbb{E}[e^{\sigma X_{n+1}}] = e^{\sigma \cdot 1} \cdot \mathbb{P}(X_{n+1}=1) + e^{\sigma \cdot -1} \cdot \mathbb{P}(X_{n+1}=-1)$$

$$= \frac{e^{\sigma}+e^{-\sigma}}{2}$$

$$\mathbb{E}_n[S_{n+1}] = K^{n+1} e^{\sigma M_n} \cdot \frac{e^{\sigma}+e^{-\sigma}}{2}$$

$$= \underbrace{K^n e^{\sigma M_n}}_{S_n} \cdot \frac{K(e^{\sigma}+e^{-\sigma})}{2}$$

$$\mathbb{E}_n[S_{n+1}] = S_n \cdot \frac{K(e^{\sigma}+e^{-\sigma})}{2}$$

$$\Rightarrow K=0 \text{ or } \frac{K(e^{\sigma}+e^{-\sigma})}{2} = 1$$

$$K = \frac{2}{e^{\sigma}+e^{-\sigma}}$$

(c) $I_0 = 0$

$$I_n = \sum_{k=0}^{n-1} M_j (M_{j+1} - M_j) \quad \text{for } n \geq 1.$$

$(M_0 = 0)$

Show that $I_n = \frac{1}{2} M_n^2 - \frac{n}{2}$.

We proceed by Induction.

$\underline{n=1}$: $I_1 = M_0 (M_1 - M_0) = 0(M_1 - 0) = 0$

$$I_1 = \frac{1}{2} M_1^2 - \frac{1}{2} = \frac{1}{2} X_1^2 - \frac{1}{2} = 0$$

since, $X_1 = \begin{cases} 1 \\ -1 \end{cases}$

$n \longrightarrow n+1$

Assume that $I_n = \frac{1}{2} M_n^2 - \frac{n}{2}$ & prove

$$I_{n+1} = \frac{1}{2} M_{n+1}^2 - \frac{n+1}{2}$$

$$I_{n+1} = I_n + M_n \underbrace{(M_{n+1} - M_n)}_{X_{n+1}}$$

$$= \left(\frac{1}{2} M_n^2 - \frac{n}{2}\right) + M_n \cdot X_{n+1}$$

$$= \left[\frac{1}{2} \underbrace{(M_{n+1} - X_{n+1})}_{M_n}^2 - \frac{n}{2}\right] + (M_{n+1} - X_{n+1}) X_{n+1}$$

$$= \frac{1}{2} M_{n+1}^2 - M_{n+1} \cdot X_{n+1} + \frac{X_{n+1}^2}{2} - \frac{n}{2} + M_{n+1} \cdot X_{n+1} - X_{n+1}^2$$

$$= \frac{1}{2} M_{n+1}^2 + \frac{X_{n+1}^2}{2} - \frac{n}{2} - X_{n+1}^2$$

$$= \frac{1}{2} M_{n+1}^2 - \frac{n+1}{2} \quad \left(\text{Since, } X_{n+1}^2 = 1\right)$$

(d) (X_n) is <u>Markov</u> if it is Adapted & $\forall f$ and $\forall n$, $\exists g$ such that

$$\mathbb{E}_n[f(X_{n+1})] = g(X_n)$$

(e) $$\mathbb{E}_n[f(I_{n+1})] = \mathbb{E}_n\left[f\left(\frac{1}{2} M_{n+1}^2 - \frac{n+1}{2}\right)\right]$$

$$= \mathbb{E}_n\left[f\left(\frac{1}{2}(M_n + X_{n+1})^2 - \frac{n+1}{2}\right)\right]$$

$f(I_{n+1})$ is a function f_1 of M_n and X_{n+1} i.e.

$$f(I_{n+1}) = f_1(M_n, X_{n+1})$$

$$\text{where } f_1(m, x) = f\left(\frac{1}{2}(m+x)^2 - \frac{n+1}{2}\right)$$

I_{n+1} is written this way so that we have two random variables, one which is known at time n & other is independent. Doing this allows us to use Independence lemma & prove the process is Markov.

M_n depends on tosses $1, 2, \ldots, n$ and X_{n+1} depends on toss $(n+1)$.

$\Rightarrow$ By the Independence lemma

$$\exists\, g_1(m) = \mathbb{E}[f_1(m, X_{n+1})]$$

$$\text{s.t. } \mathbb{E}_n[f_1(M_n, X_{n+1})] = g_1(M_n)$$

Here, $$g_1(m) = \sum_{x \in S_{X_{n+1}}} f_1(m, x) \cdot \mathbb{P}(X_{n+1} = x)$$

$$= f_1(m, 1) \cdot \frac{1}{2} + f_1(m, -1) \cdot \frac{1}{2}$$

Since, $$\frac{1}{2} M_n^2 - \frac{n}{2} = I_n$$

$$\Rightarrow M_n = \sqrt{2I_n + n}$$

$$\Rightarrow \mathbb{E}_n[\underbrace{f_1(M_n, X_{n+1})}_{f(I_n)}] = g_1(\sqrt{2I_n + n})$$

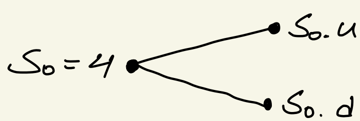
We have shown that $\forall f, n \;\exists\, g$ s.t.

$$\mathbb{E}_n[f(I_{n+1})] = g(I_n)$$

where, $$g(x) = g_1(\sqrt{2x + m})$$

$$= f_1(\sqrt{2x+m}, 1) \cdot \frac{1}{2} + f(\sqrt{2x+m}, -1) \cdot \frac{1}{2}$$

(3) Asian Option $\rightarrow$ Path dependent type option.



$u = 2$; $r = 1/4$
$d = 1/2$

$$Y_n = S_0 + S_1 + \dots + S_n$$

$$N=2, \quad K=4, \quad \left(\tfrac{1}{3} Y_2 - 4\right)^+ \rightarrow \text{Payoff}$$

In order to reduce the complexity of our problem, we need Markovian case to reduce no. of computation. Since, we are not in Markovian case, we introduce additional process to get Markovian. So instead of looking at 1-Dimension process, we look at 2-Dim. process which will be S_n & Y_n.

$v_n(s, y) \rightarrow$ Price of Option at time n for stock price $= s$ & $Y_n = y$.

At time $n=2$:

$$v_2(s,y) = \left(\tfrac{1}{3} y - 4\right)^+$$

(a) $v_n(s,y) = v_n(S_n = s, Y_n = y)$

$$V_n(\omega_1 \dots \omega_n) = \frac{1}{1+r}\left[\tilde{p}\, \underbrace{V_{n+1}(\omega_1 \dots \omega_n H)} + \tilde{q}\, \underbrace{V_{n+1}(\omega_1 \dots \omega_n T)}\right]$$

$$V_{n+1}(\omega_1 \dots \omega_n H) = v_{n+1}(S_{n+1}(H), Y_{n+1}(H))$$

$$V_{n+1}(\omega_1 \dots \omega_n T) = v_{n+1}(S_{n+1}(T), Y_{n+1}(T))$$

$$v_{n+1}(S_{n+1}(\omega_1 \dots \omega_n H), Y_{n+1}(\omega_1 \dots \omega_n H))$$

$$= v_{n+1}(S_n(\omega_1 \dots \omega_n) . 2, \; Y_n(\omega_1 \dots \omega_n) + S_{n+1}(\omega_1 \dots \omega_n H))$$

$$= v_{n+1}(2 S_n, \; Y_n + S_n . 2)$$

$$= v_{n+1}(2 S_n, \; Y_n + 2 S_n)$$

Since, $S_{n+1} = \begin{cases} S_n . u & \text{if } \omega_{n+1} = H \\ S_n . d & \text{if } \omega_{n+1} = T \end{cases}$

Similarly,

$$v_{n+1}(S_{n+1}(w_1 \dots w_n T), Y_{n+1}(w_1 \dots w_n T)) = v_{n+1}\left(\frac{S_n}{2}, Y_n + \frac{S_n}{2}\right)$$

Thus,

$$V_n(w_1 \dots w_n) = \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(2S_n, Y_n + 2S_n) + \tilde{q}\, v_{n+1}\left(\frac{S_n}{2}, Y_n + \frac{S_n}{2}\right)\right]$$

$$v_n(s,y) = \frac{2}{5}\left[v_{n+1}(2s, y+2s) + v_{n+1}\left(\frac{s}{2}, y + \frac{s}{2}\right)\right]$$

(c) $\delta_n(s,y)$

$$\delta_n(s,y) = \frac{V_{n+1}(w_1 \dots w_n H) - V_{n+1}(w_1 \dots w_n T)}{S_n(w_1 \dots w_n)(u-d)}$$

$$= \frac{v_{n+1}(2s, y+2s) - v_{n+1}\left(\frac{s}{2}, y + \frac{s}{2}\right)}{2s - \frac{s}{2}}$$

(4)

(a) It follows by Jensen inequality.

- Y_n is a Martingale $\Rightarrow$ Y_n is Adapted.
  $\Rightarrow$ $\phi(Y_n)$ is Adapted.

- Jensen Inequality: $\mathbb{E}_n[g(X)] \geq g(\mathbb{E}_n[X])$

  To remember the Jensen inequality we could think of following:

$$\left(\begin{aligned} &\int x\,dx \leq \left(\int x^2\,dx\right)^{1/2} \\ &\Rightarrow \left(\int x\,dx\right)^2 \leq \int x^2\,dx \end{aligned}\right)$$

with $g(x) = x^2$

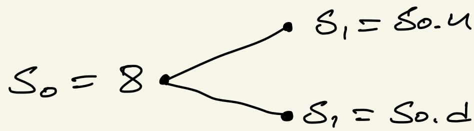
Y_n is a Martingale.

$$\mathbb{E}_n[\phi(Y_{n+1})] \geq \phi(\mathbb{E}_n[Y_{n+1}]) \quad \text{(Jensen inequality)}$$

$$= \phi(Y_n) \quad \text{(Martingale Property of } Y_n\text{)}.$$

<u>Remark</u>: In case if ϕ is <u>concave</u>, we get a <u>SuperMartingale</u>.

(b)



(i) $P(HH) = p^2 = (2/3)^2 = 4/9$

$P(HT) = pq = 2/3 \cdot 1/3 = 2/9$

⋮

$$\tilde{p} = \frac{1+r-d}{u-d} = 1/3$$

$$\tilde{q} = 1 - 1/3 = 2/3$$

$$\tilde{P}(HT) = \tilde{p} \cdot \tilde{q} = 1/3 \cdot 2/3 = 2/9$$

⋮

(ii) Z_n, $n = 0, 1, 2$.

$Z_2 = Z$ ← This is used for change of measure with Expectation.

The process (Z_n) is used for the change of measure with Conditional Expectation

$$Z(HH) = \frac{\tilde{P}(HH)}{P(HH)}$$

⋮

- Reminder

$$\begin{cases} \bullet\ Z_n = \mathbb{E}_n[Z] \quad \text{— (1)} \\ \bullet\ (Z_n) \text{ is a Martingale under } \mathbb{P}. \\ \bullet\ Z_n(w_1 \dots w_n) = \dfrac{\tilde{\mathbb{P}}(w_1 \dots w_n)}{\mathbb{P}(w_1 \dots w_n)} \end{cases}$$

By (1), we have:

$$Z_1(H) = \mathbb{E}_1[Z](H) = Z(HH) \cdot p + Z(HT) \cdot q$$

$$Z_1(T) = \mathbb{E}_1[Z](T) = Z(TH) \cdot p + Z(TT) \cdot q$$

$$Z_0 = \mathbb{E}[Z_1] = Z_1(H) \cdot p + Z_1(T) \cdot q$$

Alternatively, we could find Z_0 as:

$$Z_0 = \mathbb{E}[Z] = Z(HH) \cdot p^2 + Z(HT)\, p \cdot q + Z(TH) \cdot qp + Z(TT)\, q^2$$

(iii) State Price density Process (ξ_n)

$$\xi_n = \frac{Z_n}{(1+r)^n}$$

(iv) $\{G_k\}_{k=0}^2$, $\{V_k\}_{k=0}^2$

$\{\xi_j V_j\}_{j=0}^2$ Super Martingale under $\mathbb{P}$.

Reminder:

- The discounted price of an American option is a SuperMartingale under $\tilde{\mathbb{P}}$. i.e.

$$\frac{V_n}{(1+r)^n} \geq \tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right]$$

by change of Measure result $\Longrightarrow$ $\dfrac{V_n}{(1+r)^n} \geq \mathbb{E}_n\left[\dfrac{Z_{n+1}}{Z_n}\dfrac{V_{n+1}}{(1+r)^{n+1}}\right]$

$$\frac{V_n Z_n}{(1+r)^n} \geq \mathbb{E}_n\left[\frac{Z_{n+1}}{(1+r)^{n+1}} V_{n+1}\right]$$

$$V_n \xi_n \geq \mathbb{E}_n\left[V_{n+1}\xi_{n+1}\right]$$

$\Rightarrow (V_n \xi_n)$ is a Super Martingale under $\mathbb{P}$.

Reminder:

① V_n is the price process of an European Option,

$$\frac{V_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right]$$

② If $\tau^* = \min\{n : V_n = G_n\}$ is the optimal stopping time of the buyer then,

$\dfrac{V_{n\wedge\tau^*}}{(1+r)^{n\wedge\tau^*}}$ is a Martingale under $\tilde{\mathbb{P}}$.

The difference between Q3 (Asian Option) & M_n (notes is:

$$(S_n, Y_n) \longrightarrow (S_n, M_n)$$

$$V_{n+1}(\omega_1 \ldots \omega_n H) = v_{n+1}(S_{n+1}(H), M_{n+1}(H))$$
$$= v_{n+1}(S_n \cdot u, M_n \vee S_{n+1}(H))$$

$$v_n(s,m) = \frac{1}{1+r}\left[\tilde{p} \cdot v_{n+1}(s \cdot u, m \vee (u \cdot s)) + \tilde{q} \cdot v_{n+1}(s \cdot d, m \vee (d \cdot s))\right]$$

- Exercise 4.5 (HW9)

$$\tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau\le 2\}}\left(\tfrac{4}{5}\right)^{\tau} G_\tau\right] \leftarrow$$ What is the τ that maximises this?

Step 1: Which are the possible stopping times.

$\tau(HH)$, $\tau(HT)$, $\tau(TH)$, $\tau(TT)$ $\rightarrow$ possible scenarios.

$\tau \backslash \omega$	HH	HT	TH	TT
	0	0	0	0
	1	1	1 2 2 ∞ ∞	1 2 ∞ 2 ∞
	2	2	1 2 2 ∞ ∞	1 2 ∞ 2 ∞
	2	∞	1 2 2 ∞ ∞	1 2 ∞ 2 ∞
	∞	2	1 2 2 ∞ ∞	1 2 ∞ 2 ∞
	∞	∞	1 2 2 ∞ ∞	1 2 ∞ 2 ∞

Step 2: Compute the Expectation for each one and identify the maximiser.

Recall : Computing $\mathbb{E}$ and Conditional $\mathbb{E}$.

$X(HH), X(HT), X(TH), X(TT)$

$$\underbrace{\mathbb{E}[X]}_{\in\mathbb{R}} = p^2 \cdot X(HH) + pq \cdot X(HT) + qp \cdot X(TH) + q^2 \cdot X(TT)$$

$$\underbrace{\mathbb{E}_1[X]}_{\text{Random Variable}}(H) = X(HH) \cdot p + X(HT) \cdot q$$

$$\mathbb{E}_1[X](T) = X(TH) \cdot p + X(TT) \cdot q$$

$\mathbb{E}_2[X] = X$ (Since, X is 2-period Binomial Model it's known at time 2)

The buyer of American Option wants to exercise at stopping time that maximises

$$\max_{\tau} \mathbb{E}\left[\mathbb{1}_{\{\tau \le N\}} \frac{G_N}{(1+r)^N}\right]$$

which is at optimal stopping time τ^*.

$$\tau^* = \min\{n : V_n = G_n\}$$

Eg: American Put Option.

$$G_n = K - S_n \quad \leftarrow \text{Exercise value}$$

when computing optimal stopping time take $(K-S_n)$

But in Pricing Algorithm you take either $(K-S_n)$ or +ve part of $(K-S_n)^+$. i.e.

$$V_n = \max\left\{G_n, \frac{1}{1+r}\tilde{\mathbb{E}}_n[V_{n+1}]\right\}$$

or

$$V_n = \max\left\{G_n \vee 0, \frac{1}{1+r}\tilde{\mathbb{E}}_n[V_{n+1}]\right\}$$

Why? We can see this by induction (backward in time) since the terminal value $V_N = \max\{G_N, 0\}$.

- **Properties of American Options:**

(1) $\dfrac{V_n}{(1+r)^n}$ is a $\widetilde{\mathbb{P}}$-Super Martingale.

(2) $\dfrac{V_{n\wedge\tau}}{(1+r)^{n\wedge\tau}}$ is a $\widetilde{\mathbb{P}}$- SuperMartingale $\forall\,\tau$.

(3) $\dfrac{V_{n\wedge\tau^*}}{(1+r)^{n\wedge\tau^*}}$ is a $\widetilde{\mathbb{P}}$- Martingale.

- **2019 Exam (Bond Price)**

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
 & & R_2(HH) \\
 & R_1(H) \arrow[ur] \arrow[dr] & \\
R_0 \arrow[ur] \arrow[dr] & & R_2(HT) \\
 & R_1(T) \arrow[r] \arrow[dr] & R_2(TH) \\
 & & R_2(TT)
\end{tikzcd}
$$

(3 period Binomial Model)

$\widetilde{\mathbb{P}}(HHH) = \widetilde{\mathbb{P}}(HHT) = \dots = \widetilde{\mathbb{P}}(TTT) = 1/8$

$\omega_1\omega_2$	$\frac{1}{1+R_0}$	$\frac{1}{1+R_1}$	$\frac{1}{1+R_2}$	D_1	D_2	D_3
HH	a	b	c	a	ab	abc
HT						
TH						
TT						

Complete the above table then:

$$B_{n,m} = \widetilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}\right] = \frac{1}{D_n}\widetilde{\mathbb{E}}_n\left[D_m\right]$$

eg: $B_{1,2}(H)$, $B_{1,2}(T)$.

- **Hedging & Pricing**

$G \rightarrow$ Payoff at time 1.

Determine $\begin{cases} X_0 = V_0 \\ \Delta = \text{No. of Shares of Stock in the portfolio} \end{cases}$

$\underline{t=0}$: $\Delta S_0 + (X_0 - \Delta S_0) = X_0$

$\underline{t=1}$:

$$\begin{cases} \Delta S_1(H) + (X_0 - \Delta S_0)(1+r) = X_1(H) = G(H) \\ \Delta S_1(T) + (X_0 - \Delta S_0)(1+r) = X_1(T) = G(T) \end{cases}$$

<u>European Option</u>:

$$V_n = \tilde{\mathbb{E}}_n\left[\frac{G}{(1+r)^N}\right] = \tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)}\right]$$

<u>American Option</u>:

$$\tilde{V}_n = \max\left\{ G_n, \frac{1}{1+r}\tilde{\mathbb{E}}_n[V_{n+1}] \right\}$$

<u>eg</u>:

$$X_n = \underbrace{M_n}_{\text{Martingale part}} - A_n$$

$$M_n = \sum_{j=1}^{n}\left(X_j - \mathbb{E}_{j-1}[X_j]\right) + X_0$$

To show M_n is Martingale:

$\rightarrow$ M_n is adapted.

$$\rightarrow \mathbb{E}_n[M_{n+1}] = \mathbb{E}_n\left[\sum_{j=1}^{n}(X_j - \dots) + X_{n+1} - \mathbb{E}_n[X_{n+1}]\right]$$

$$= \mathbb{E}_n[M_n]$$

$$= M_n$$

Since, $\mathbb{E}_n\left[X_{n+1} - \mathbb{E}_n[X_{n+1}]\right]$
$= \mathbb{E}_n[X_{n+1}] - \mathbb{E}_n[X_{n+1}]$
$= 0$

For A_n, use SuperMartingale process of X_n & Martingale process of M_n, and show:

$$A_{n+1} - A_n \geq 0$$

- <u>Exam 2019, Q4(b)(ii)</u>

$Z_n(\omega) = \mathbb{E}_n[Z](\omega)$. Show Z_n is Martingale.

$\rightarrow$ Z_n is adapted.

Iterated Conditioning

$$\rightarrow \mathbb{E}_n[Z_{n+1}] = \mathbb{E}_n\left[\mathbb{E}_{n+1}[Z]\right] = \mathbb{E}_n[Z] = Z_n.$$