1 Applied Linear Systems - Elementary Review

1.1 In this session:

Motivation - **Introduction to partial differential equations**
Review of linear equations
Direct Methods for solution: Gaussian Elimination , LU Decomposition
Iterative Techniques: Jacobi Method, Gauss-Seidel Method
Vector Norms
Relaxation Techniques

The subject of systems of linear equations (or *Linear Systems*) is of interest due to its application in obtaining solutions of differential equations, and also forms a large branch of linear algebra.

Many partial differential equations (PDE's) can be reduced to *matrix inversion problems* of the form

$$A\underline{x} = \underline{b}.$$

A^{-1} must exist

So to solve the above, knowing the inverse of the matrix A gives

$$\underline{x} = A^{-1}\underline{b}$$

for the vector $\underline{x}$ of unknown terms $(x_1, x_2, \ldots\ldots\ldots, x_n)^T$.

Parabolic $\quad \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ (heat eq^n)

hyperbolic $\quad \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ (wave eq^n)

elliptic $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$ $\quad f(x,y) \neq 0$ Poisson

$f(x,y) = 0$ Laplace

1.2 Matrix Representation of Linear Equations - review of basics

We begin by considering a (two-by-two) set of equations for the unknowns x and y:

$$\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}$$

The solution is easily found. To get x, multiply the first equation by d, the second by b, and subtract to eliminate y:

$$(ad - bc)\,x = dp - bq.$$

Then find y:

$$(ad - bc)\,y = aq - cp.$$

This works and gives a unique solution *as long as* $ad - bc \neq 0$.

If $ad - bc = 0$, the situation is more complicated: there may be no solution at all, or there may be many.

Examples:

Here is a system with a unique solution:

$$\begin{aligned} x - y &= 0 \\ x + y &= 2 \end{aligned}$$

The solution is $x = y = 1$.

Now try

$$\begin{aligned} x - y &= 0 \\ 2x - 2y &= 2 \end{aligned}$$

Obviously there is no solution: from the first equation $x = y$, and putting this into the second gives $0 = 2$. Here $ad - bc = 1\,(-2) - (1-)\,2 = 0$.

Also note what is being said:

$$\left.\begin{aligned} x &= y \\ x &= 1 + y \end{aligned}\right\} \text{ Impossible.}$$

Lastly try

$$\begin{aligned} x - y &= 1 \\ 2x - 2y &= 2. \end{aligned}$$

The second equation is twice the first so gives no new information. Any x and y satisfying the first equation satisfy the second. This system has many solutions.

Note: If we have one equation for two unknowns the system is undetermined and has many solutions. If we have *three* equations for two unknowns, it is over-determined and in general has no solutions at all.

Then the general (2×2) system is written

$$\text{Coef matrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

or

$$\mathbf{A}\underline{x} = \underline{p}.$$

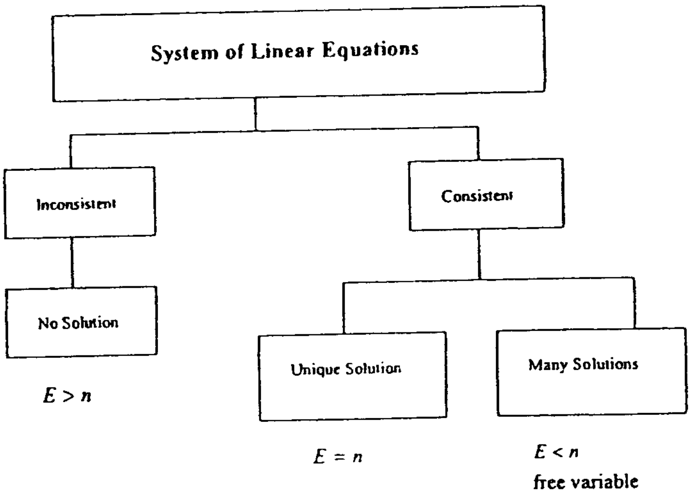
The equations can be solved if the matrix $\mathbf{A}$ is **invertible.** This is the same as saying that its **determinant**

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

is not zero.

These concepts generalise to systems of N equations in N unknowns. Now the matrix $\mathbf{A}$ is $N \times N$ and the vectors $\mathbf{x}$ and $\mathbf{p}$ have N entries.

If E = number of equations and n = unknowns, we can then summarise using the following:



Definition 1 *An upper triangular matrix* U *has zero elements below the principal diagonal, i.e.*

$$u_{ij} = 0 \quad \forall\ i > j.$$

For an upper triangular matrix U *consider the system* $U\underline{x} = \underline{b}$ *in component form*

$$\begin{array}{rcl}
u_{11}x_1 + u_{12}x_2 + \cdots\cdots\cdots\cdots\cdots\cdots + u_{1n}x_n & = & b_1 \\
u_{22}x_2 + \cdots\cdots\cdots\cdots\cdots\cdots + u_{2n}x_n & = & b_2 \\
u_{33}x_3 + \cdots\cdots\cdots\cdots\cdots + u_{3n}x_n & = & b_3 \\
\ddots \qquad\qquad\qquad\qquad \vdots & = & \vdots \\
\ddots \qquad\qquad\qquad \vdots & = & \vdots \\
\ddots \qquad\qquad \vdots & = & \vdots \\
u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n & = & b_{n-1} \\
u_{nn}x_n & = & b_n
\end{array}$$

The determinant of the matrix U, can now easily be obtained from

$$\det(U) = \prod_{i=1}^{n} u_{ii}.$$

Recall that U has an inverse if $\det(U) \neq 0$, hence for U to be non-singular requires the strict condition $u_{ii} \neq 0 \ \forall\, i$. This linear system can now easily be solved by **back-substitution**. The algorithm for this scheme becomes

$$
\begin{aligned}
x_n &= \frac{b_n}{u_{nn}} \\
x_i &= \frac{1}{u_{ii}} \left(b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) \quad \text{for } i = n-1,\ n-2, \ldots\ldots, 1
\end{aligned}
$$

Definition 2 *A lower triangular matrix L has zero elements above the principal diagonal, i.e.*

$$l_{ij} = 0 \quad \forall\, i < j.$$

Now suppose L is a lower triangular matrix and consider the system $L\underline{x} = \underline{b}$.

A *lower triangular* matrix L has zero elements above the main diagonal, i.e.

$$l_{ij} = 0 \quad \forall\, i < j.$$

Now suppose L is a lower triangular matrix and consider the system $L\underline{x} = \underline{b}$.

$$
\begin{array}{lllll}
l_{11}x_1 & & & = & b_1 \\
l_{21}x_1 + l_{22}x_2 & & & = & b_2 \\
\vdots & \vdots & \ddots & = & \vdots \\
\vdots & \vdots & \ddots & = & \vdots \\
\vdots & \vdots & \ddots & = & \vdots \\
l_{n1}x_1 + l_{n2}x_2 + \ldots\ldots\ldots + l_{nn}x_n & & & = & b_n
\end{array}
$$

Again, because

$$\det(L) = \prod_{i=1}^{n} l_{ii},$$

L is non-singular iff $l_{ii} \neq 0 \ \forall\, i$. We solve this system by **forward-substitution**. The scheme is given by

$$
\begin{aligned}
x_1 &= \frac{b_1}{l_{11}} \\
x_i &= \frac{1}{l_{ii}} \left(b_i - \sum_{j=1}^{i-1} l_{ij} x_j \right) \quad \text{for } i = 2, 3\ldots\ldots, n
\end{aligned}
$$

Definition 3 *An* $n \times n$ *matrix is called a* band matrix *if integers* p *and* q *exist such that* $p,\ q \in (1, n)$ *with the property that* $a_{ij} = 0$ *whenever* $i + p \le j$ *or* $j + q \le i$. *The band width* w *of a banded matrix is defined to be* $w = p + q - 1$.

So for example, the matrix

$$A = \begin{pmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & -5 & -6 \end{pmatrix}$$

is a banded matrix with $p = q = 2$ and band width $w = 3$.

So the definition of a banded matrix forces such matrices to have all its non-zero elements to be clustered close to the main diagonal.

p is then called the *upper band-width* and q the *lower band-width.*

Solving equations with banded matrices can be much more efficient than with full matrices because the triangular factors in the LU decomposition are also banded. This results in greater efficiency in terms of storage and amount of computations required. Banded matrices arise from the discretization of differential equations, because the finite difference method use to approximate derivatives only involve values at nearby mesh points.

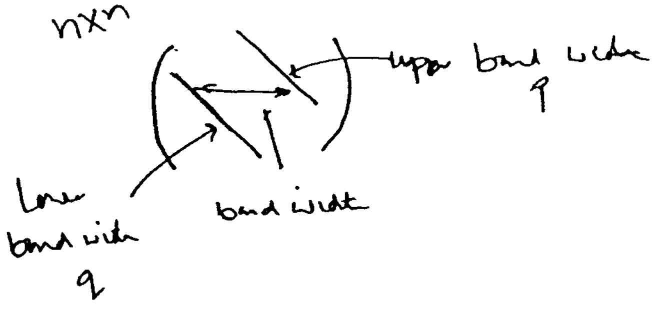
From an applied mathematics perspective, the special case occurring when $p = q = 2$ and band width $w = 3$ is the most useful for us. These matrices are called *tridiagonal* since they have the form

band width

$$\begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & \cdots & 0 \\ \boxed{a_{21} & a_{22} & a_{23}} & \ddots & & \vdots \\ 0 & a_{32} & a_{33} & a_{34} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 & a_{n,n-1} & a_{nn} \end{pmatrix}$$

General Structure of tridiagonal matrix.
$p = q = 2$, $w = 3$

upper bandwidth



$1 < p < n$
$1 < q < n$.

Tri diagonal matrix $\rightarrow$ $\boxed{\begin{matrix} p, q = 2 \\ w = 3 \end{matrix}}$

1.3 Gaussian Elimination

Two non-singular systems of equations are said to be *equivalent* if they have the same solution. *Gaussian elimination* is the systematic reduction of a general system of equations into an equivalent upper triangular system, which can then be solved by back-substitution.

The procedure entails subtracting multiples of one equation from the equations below it in order to introduce zeroes below the main diagonal.

The usual notation for systems of linear equations is that of matrices and vectors. Consider the simple system

$$\begin{aligned} ax + by + cz &= p \\ dx + ey + fz &= q \\ gx + hy + iz &= r \end{aligned} \tag{1}$$

comprising of three equations in three unknowns.

We gather the unknowns x, y and z and the given p, q and r into vectors:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

and put the coefficients into a matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

A is called the *coefficient matrix* of the linear system (1) and the special matrix formed by

$$\left(\begin{array}{ccc|c} a & b & c & p \\ d & e & f & q \\ g & h & i & r \end{array}\right)$$

is called the *augmented matrix*.

So as an example the augmented matrix for the system

$$\begin{aligned} 2x - 3y + 6z &= -1 \\ 3x + 6y + z &= 4 \end{aligned}$$

is

$$\left(\begin{array}{ccc|c} 2 & -3 & 6 & -1 \\ 3 & 6 & 1 & 4 \end{array}\right)$$

Now consider a linear system consisting of n equations in n unknowns which can be written in augmented form as

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & .. & .. & .. & a_{1n} & b_1 \\ a_{21} & a_{22} & .. & .. & .. & a_{2n} & b_2 \\ \vdots & & & & & \vdots & \vdots \\ \vdots & & & & & \vdots & \vdots \\ \vdots & .. & .. & .. & .. & \vdots & \vdots \\ a_{n1} & a_{n2} & - & - & - & a_{nn} & b_n \end{array}\right).$$

We can perform a series of row operations on this matrix and reduce it to a simplified matrix of the form

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & .. & .. & .. & a_{1n} & b_1 \\ 0 & a_{22} & .. & .. & .. & a_{2n} & b_2 \\ 0 & 0 & & & & \vdots & \vdots \\ 0 & 0 & 0 & & & \vdots & \vdots \\ \vdots & .. & .. & .. & .. & \vdots & \vdots \\ 0 & 0 & .. & .. & 0 & a_{nn} & b_n \end{array}\right).$$

Such a matrix is said to be of *echelon form* if the number of zeros preceding the first nonzero entry of each row increases row by row.

A matrix A is said to be *row equivalent* to a matrix B, written $A \sim B$ if B can be obtained from A from a finite sequence of operations called *elementary row operations* of the form:

[ER$_1$]: Interchange the i^{th} and j^{th} rows: $R_i \leftrightarrow R_j$

[ER$_2$]:Replace the i^{th} row by itself multiplied by a nonzero constant k : $R_i \rightarrow kR_i$

[ER$_3$]:Replace the i^{th} row by itself plus k times the j^{th} row: $R_i \rightarrow R_i + kR_j$

These have no affect on the solution of the of the linear system which gives the augmented matrix.

Examples:

Solve the following linear systems:

$$\left.\begin{array}{r} 2x + y - 2z = 10 \\ 3x + 2y + 2z = 1 \\ 5x + 4y + 3z = 4 \end{array}\right\} \equiv A\underline{x} = \underline{b} \text{ with } A = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix}$$

$$\text{and } \underline{b} = \begin{pmatrix} 10 \\ 1 \\ 4 \end{pmatrix}$$

The augmented matrix for this system is

$$\left(\begin{array}{ccc|c} 2 & 1 & -2 & 10 \\ 3 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4 \end{array}\right) \xrightarrow[R_3 \to 2R_3 - 5R_1]{R_2 \to 2R_2 - 3R_1} \left(\begin{array}{ccc|c} 2 & 1 & -2 & 10 \\ 0 & 1 & 10 & -28 \\ 0 & 3 & 16 & -42 \end{array}\right) \xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 - 3R_2}$$

$$\left(\begin{array}{ccc|c} 2 & 0 & -12 & 38 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & -14 & 42 \end{array}\right)$$

$$\begin{aligned} -14z &= -42 \to z = -3 \\ y + 10z &= -28 \to y = -28 + 30 = 2 \\ x - 6z &= 19 \to x = 19 - 18 = 1 \end{aligned}$$

Therefore solution is unique with $\underline{x} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$.

$$\left.\begin{array}{r} x + 2y - 3z = 6 \\ 2x - y + 4z = 2 \\ 4x + 3y - 2z = 14 \end{array}\right\}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & 14 \end{array}\right) \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 10 & -10 \end{array}\right) \xrightarrow[R_2 \to -0.5R_2]{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Number of equations is less than number of unknowns.

$$\begin{aligned} y - 2z &= 2 \text{ so } z = a \text{ is a free variable} \Rightarrow y = 2(1 + a) \\ x + 2y - 3z &= 6 \to x = 6 - 2y + 3z = 2 - a \Rightarrow x = 2 - a; \; y = 2(1 + a); \; z = a \end{aligned}$$

Therefore there are many solutions $\underline{x} = \begin{pmatrix} 2-a \\ 2(1+a) \\ a \end{pmatrix}$

$$\left.\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{aligned}\right\}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right) \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

The last line reads $0 = -3$. Also middle iteration shows that the second and third equations are inconsistent. Hence no solution exists.

Suppose we have the $(n \times n)$ linear system $A\underline{x} = \underline{b}$ with augmented matrix

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & \cdot\cdot & \cdot\cdot & \cdot\cdot & a_{1n} & b_1 \\ 0 & a_{22} & \cdot\cdot & \cdot\cdot & \cdot\cdot & a_{2n} & b_2 \\ 0 & 0 & & & & \vdots & \vdots \\ 0 & 0 & 0 & & & \vdots & \vdots \\ \vdots & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \vdots & \vdots \\ 0 & 0 & - & - & 0 & a_{nn} & b_m \end{array}\right).$$

then the method of Gaussian elimination is given by

$$x_i = \frac{b_i}{a_{ii}} - \sum_{j=i+1}^{n} \left(\frac{a_{i\,j} x_j}{a_{ii}}\right) \qquad \text{for each} \quad i = n-1,\ n-2,\ \ldots\ldots,2,\ 1.$$