GM04: Exercise sheet 5, investigating stability

October 31, 2016

This sheet is deliberately open-ended so that you can focus on the coursework instead if you wish. However, stability investigation forms part of the coursework so it is useful to have had some practice at it.

1 The problem

Consider the ODE

$$y' = 1 + y^2, \qquad y(0) = 0$$

1. Find the exact solution
2. Use the forward Euler method with a step-size of $h = 0.1$ to compute an approximate solution over the range $x \in (0, 1)$.
3. Plot the approximate and exact solutions on a graph. What has gone wrong?
4. Try again with a smaller step size. Can you get the approximate solution to look anything like the exact solution? Refer to the section "Stability" in the lecture notes and try to find a value of h that will cause the method to work.
5. Try the equation $y' = 1 - y^2$ (use the same initial condition). How does changing h help here?

This shows some issues with the forward Euler method. Could you have predicted that the numerical solution would go badly, even if you didn't know the exact solution?

2 Further investigation

Now we turn our attention to the problem

$$y' = \alpha y, \qquad y(0) = 1$$

where α is real number.

1. What do you predict about the success of the forward Euler method when
   (a) The coefficient $\alpha > 0$
   (b) The coefficient $\alpha < 0$
2. Run some numerical experiments and see whether your predictions are correct
3. Set $\alpha = -10$. Can you predict the critical value of h at which the method becomes unstable? Test your predictions.

3 Extension

Retry the last question with the numerical methods listed below. This is meant to test your MATLAB, as they should be a bit more difficult to encode than the forward Euler. For each method, try and predict the boundary of critical stability and then verify

- RK4
- The Leap-frog method
- Backward Euler
- Crank-Nicolson