Stiffness matrices and load vectors

<u>second-order equation</u> (1D heat equation):

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)+Q=0, \qquad T(0)=T_0, \qquad q(L)=-k\frac{dT}{dx}(L)=q_L$$

weak formulation:

$$\int_0^L \frac{dv}{dx}Ak\frac{dT}{dx}dx=-\left[vAq\right]_0^L+\int_0^L vQdx, \qquad T(0)=T_0, \qquad q(L)=q_L$$

nodal discretisation (n nodes):

$$T(x)=\sum_{i=1}^{n}N_i(x)T_i \qquad (T_i \text{ the nodal temperatures})$$

Galerkin approach:

$$v=N_i \qquad (\text{take the } n \text{ shape functions for test functions})$$

finite element equation:

$$Ku=f_l+f_b$$

where

$$K_{ij}=\int_0^L \frac{dN_i}{dx}Ak\frac{dN_j}{dx}dx, \qquad f_{l_i}=\int_0^L N_iQdx, \qquad f_{b_i}=-\left[N_iAq\right]_0^L$$

2-node element stiffness matrix and load vector in the uniform case (i.e., A, k, Q constant):

$$K=\frac{Ak}{L}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}, \qquad f_l=\frac{QL}{2}\begin{pmatrix}1\\ 1\end{pmatrix}$$

<u>fourth-order equation</u> (Euler-Bernoulli beam equation):

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right)=f, \qquad M=EI\frac{d^2y}{dx^2}$$

possible boundary conditions:

$$y(0)=0=\frac{dy}{dx}(0), \qquad y(L)=0=M(L) \qquad (\text{fixed-pinned beam})$$

weak formulation ($S=-dM/dx$ is the shear force):

$$\int_0^L \frac{d^2v}{dx^2}Mdx=\int_0^L vfdx+\left[vS\right]_0^L+\left[\frac{dv}{dx}M\right]_0^L$$

nodal discretisation (n nodes, two variables per node):

$$y(x)=\sum_{i=1}^{2n}N_i(x)u_i \qquad (u_{2i-1} \text{ and } u_{2i} \text{ the displacement, } y\text{, and slope, } dy/dx\text{, at the } i^{\text{th}} \text{ node})$$

Galerkin approach:

$$v = N_i \qquad \text{(take the } 2n \text{ shape functions for test functions)}$$

finite element equation:

$$Ku = f_l + f_b$$

where

$$K_{ij} = \int_0^L \frac{d^2 N_i}{dx^2} EI \frac{d^2 N_j}{dx^2} dx, \qquad f_{li} = \int_0^L N_i f \, dx, \qquad f_{bi} = [N_i S]_0^L + \left[\frac{dN_i}{dx} M \right]_0^L$$

2-node element stiffness matrix and load vector in the uniform case (i.e., E, I, f constant):

$$K = \frac{EI}{L^3} \begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}, \qquad f_l = \frac{fL}{12} \begin{pmatrix} 6 \\ L \\ 6 \\ -L \end{pmatrix}$$