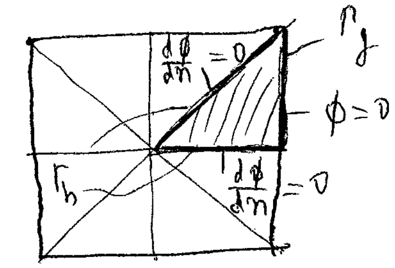
2D example (torsion of square shaft)

Governing equation: $\operatorname{div}(D\nabla\phi) + 2\theta = 0$, $\phi = 0$ on boundary

where $D = \begin{pmatrix} \frac{1}{G} & 0 \\ 0 & \frac{1}{G} \end{pmatrix} = \frac{1}{G} I$ (isotropic)

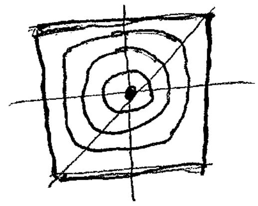
cross-section:



ϕ is the stress function
G is the shear modulus
θ is the twist rate

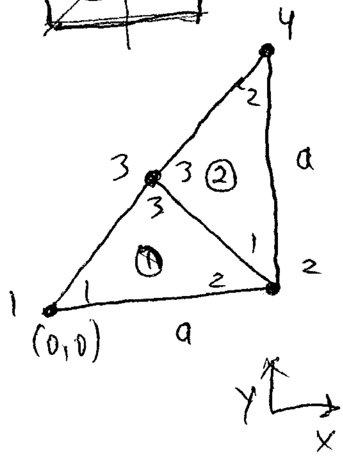
By symmetry we need to consider only an eighth of the cross-section, which we model by two three-node triangular elements.

potential lines normal to Γ_h (3 axes of symmetry):



On the triangular domain the problem becomes:

$$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + 2G\theta = 0 \\ \phi = 0 \quad \text{on } \Gamma_g \\ \dfrac{d\phi}{dn} = 0 \quad \text{on } \Gamma_h \qquad \left(\dfrac{d\phi}{dn} = \nabla\phi \cdot n\right) \end{cases}$$



Weak formulation:

$$\iint_\Delta \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + 2G\theta\right) v \, dx\,dy = 0$$

$$\Leftrightarrow \iint_\Delta \left(-\frac{\partial\phi}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial\phi}{\partial y}\frac{\partial v}{\partial y} + 2G\theta\right) dx\,dy + \int_\Gamma \underbrace{\left[v\frac{\partial\phi}{\partial x}n_x + v\frac{\partial\phi}{\partial y}n_y\right]}_{v\frac{d\phi}{dn}} ds = 0$$

$$\Leftrightarrow \iint_\Delta \left(\frac{\partial\phi}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial v}{\partial y}\right) dx\,dy = \int_{\Gamma_g} v\frac{d\phi}{dn} ds + 2\theta G \iint_\Delta v \, dx\,dy$$

interpolation: $\phi = \sum_{j=1}^{3} N_j \phi_j$, Galerkin: $v = N_i$, $i = 1,2,3$

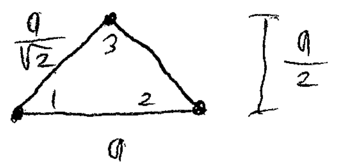
$$\Rightarrow \sum_{j=1}^{3} \iint_\Delta \left(\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x} + \frac{\partial N_j}{\partial y}\frac{\partial N_i}{\partial y}\right)\phi_j \, dx\,dy = \int_{\Gamma_g} N_i \frac{d\phi}{dn} ds + 2G\theta \iint_\Delta N_i \, dx\,dy$$

or $\sum_{j=1}^{3} K_{ij}\phi_j = f_{b_i} + f_{e_i}$ $\qquad (Ku = f_b + f_e)$

where $K_{ij} = \iint_{\Delta} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dx\,dy$, $\quad f_{e_i} = 2G\theta \iint_{\Delta} N_i\,dx\,dy$

$$f_{b_i} = \int_{\Gamma_g} N_i \frac{d\phi}{dn}\,ds$$

element ① :



shape functions: $N_1^e = \frac{1}{2A}\left[x_2y_3 - x_3y_2 + (y_2-y_3)x + (x_3-x_2)y\right]$

$A = \frac{a^2}{4}$ $\qquad N_2^e = \frac{1}{2A}\left[x_3y_1 - x_1y_3 + (y_3-y_1)x + (x_1-x_3)y\right]$

$$N_3^e = \frac{1}{2A}\left[x_1y_2 - x_2y_1 + (y_1-y_2)x + (x_2-x_1)y\right]$$

$$K_{11} = \iint_{\Delta}\left[(y_2-y_3)^2 + (x_3-x_2)^2\right]\frac{dx\,dy}{4A^2} = \frac{1}{a^2}\left[(y_2-y_3)^2+(x_3-x_2)^2\right] = \frac{1}{a^2}\left(\frac{a^2}{2}\right) = \frac{1}{2}$$

$$K_{12} = \frac{1}{a^2}\left[(y_2-y_3)(y_3-y_1) + (x_3-x_2)(x_1-x_3)\right] = \frac{1}{a^2}\left(-\frac{a}{2}\frac{a}{2} + \frac{a}{2}\frac{a}{2}\right) = 0$$

$$K_{13} = \frac{1}{a^2}\left[(y_2-y_3)(y_3-y_1) + (y_1-y_2)(x_2-x_1)\right] = \frac{1}{a^2}\left(-\frac{a}{2}\frac{a}{2} + 0.a\right) = -\frac{1}{2}$$

$$K_{22} = \frac{1}{a^2}\left[(y_3-y_1)^2 + (x_1-x_3)^2\right] = \frac{1}{a^2}\left(\frac{a^2}{2}\right) = \frac{1}{2}$$

$$K_{23} = \frac{1}{a^2}\left[(y_3-y_1)(y_1-y_2) + (x_1-x_3)(x_2-x_1)\right] = \frac{1}{a^2}\left(\frac{a}{2}.0 - \frac{a}{2}.a\right) = -\frac{1}{2}$$

$$K_{33} = \frac{1}{a^2}\left[(y_1-y_2)^2 + (x_2-x_1)^2\right] = \frac{1}{a^2}.a^2 = 1$$

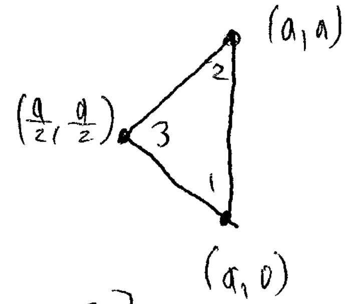
thus
$$\boxed{K^1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}}$$

To compute the load vector f_e note that $N_i^e = \eta_i$, the area coordinates, for which we have

$$\iint_{\Delta} \eta_1^n \eta_2^m \eta_3^p\,dx\,dy = \frac{m!\,n!\,p!}{(n+m+p+2)!}(2A)$$

so that, immediately, $\boxed{f_\ell^1 = \frac{2}{3} G\theta A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

element ②:



$$K_{11} = \frac{1}{a^2}\left[(y_2-y_3)^2 + (x_3-x_2)^2\right] = \frac{1}{2}$$

Note that because we use the same local node numbering as for element ① the matrix K^2 will be identical to K^1 as the computations only involve relative x and y distances. The load vector is clearly also the same, thus

$$\boxed{K^2 = K^1, \quad f_\ell^2 = f_\ell^1}$$

Assembly: element ①: $u^1 = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = A^1 u$

element ②: $u^2 = \begin{pmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = A^2 u$

(global nodal degrees of freedom)

$$K^1 u^1 = f^1 \Rightarrow K^1 A^1 u = f^1$$
$$\Rightarrow A^{1T} K^1 A^1 u = A^{1T} f^1$$

Now sum over elements:

$$\sum_e A^{eT} K^e A^e u = \sum_e A^{eT} f^e$$

$\Leftrightarrow K u = f$, where $K = \sum_e A^{eT} K^e A^e$ and $f = \sum_e A^{eT} f^e$

global eq.

Thus

$$K = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

or

$$\boxed{K = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ -1 & -2 & 4 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}}$$

Similarly (nodes 2 and 3 are common to both elements):

$$\boxed{f_f = \frac{2}{3} G\theta A \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}}$$

Boundary vector: $\quad f_{b_i} = \int_{\Gamma_g} N_i \frac{d\phi}{dn} ds$

Along boundary Γ_g the only non-zero global shape functions are N_2 and N_4 (element shape function N_k belonging to node k is zero along the element boundary defined by i and j).

Thus $\boxed{f_b = \begin{pmatrix} 0 \\ R_2 \\ 0 \\ R_4 \end{pmatrix}}$ where $R_2 = \int_{\Gamma_g} N_2 \frac{d\phi}{dn} ds$

$$R_4 = \int_{\Gamma_g} N_4 \frac{d\phi}{dn} ds$$

After applying the boundary conditions along Γ_g (i.e., $\phi_2 = 0 = \phi_4$) we obtain the FE equation

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ -1 & -2 & 4 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ 0 \\ \phi_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ R_2 \\ 0 \\ R_4 \end{pmatrix} + \frac{2}{3} G\theta A \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

We solve for ϕ_1 and ϕ_3 from the 1st and 3rd eq. and then for R_2 and R_4 from the 2nd and 4th:

$$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_3 \end{pmatrix} = \frac{2}{3} G\theta A \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} \phi_1 - \phi_3 &= \frac{4}{3} G\theta A \\ -\phi_1 + 4\phi_3 &= \frac{8}{3} G\theta A \\ \hline 3\phi_3 &= 4 G\theta A \\ \Rightarrow \quad \phi_1 &= \phi_3 + \frac{4}{3} G\theta A \end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned} \phi_3 &= \frac{4}{3} G\theta A \\ \phi_1 &= \frac{8}{3} G\theta A \end{aligned}}$$

then
$$\begin{aligned} -\phi_3 &= R_2 + \frac{4}{3} G\theta A \\ -\frac{1}{2}\phi_3 &= R_4 + \frac{2}{3} G\theta A \end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned} R_2 &= -\frac{8}{3} G\theta A \\ R_4 &= -\frac{4}{3} G\theta A \end{aligned}}$$

We can use the solution to estimate the torsional stiffness.
First, the torque is defined by $M = 2 \iint_{\square} \phi \, dx\, dy$

This can be computed as

$$M = 8\left(2\int_{①} \phi \, dx\,dy + 2\int_{②} \phi \, dx\,dy \right)$$

$$= 8\left[2\int_{①} (N_1^e \phi_1 + N_2^e \phi_2 + N_3^e \phi_3)\, dx\,dy + 2\int_{②} (N_1^e \phi_2 + N_2^e \phi_4 + N_3^e \phi_3)\, dx\,dy \right]$$

$$= 16\left[\frac{A}{3} (1 \; 1 \; 1) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} + \frac{A}{3} (1 \; 1 \; 1) \begin{pmatrix} \phi_2 \\ \phi_4 \\ \phi_3 \end{pmatrix} \right]$$

$$= \frac{16}{3} A (\phi_1 + 2\phi_3) \qquad \left(\text{since } \iint_{\Delta} N_i^e \, dx\,dy = \frac{A}{3}\right)$$

with the above solution, this gives

$$M = \frac{16A}{3} \frac{16}{3} G\theta A = \frac{256}{9} G\theta A^2$$

The stiffness is then obtained as $C = \dfrac{M}{\theta}$.

Thus $C = \frac{256}{9} G A^2 = \frac{256}{9} G \left(\frac{\bar{A}}{16}\right)^2 = \frac{1}{9} G \bar{A}^2 = 0.1111\, G\bar{A}^2$

where $\bar{A}$ is the full area of the shaft $(\bar{A} = 16A)$

This result may be compared against the exact result:

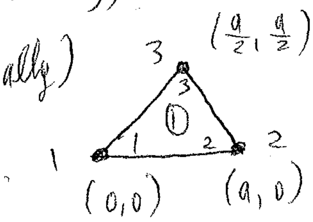
$C = k_1 G \bar{A}^2$ with $k_1 = 0.1406$ for a square shaft

(21% error)

We can also find the stress function over the entire cross-section (element-wise):

element ①: $\phi = N_1 \phi_1 + N_2 \phi_2 + N_3 \phi_3$ (globally)

$= N_1^1 \phi_1 + N_2^1 \underset{=0}{\phi_2} + N_3^1 \phi_3$ (locally)



where $N_1^1(x,y) = \frac{1}{2A}\left[x_2 y_3 - x_3 y_2 + (y_2 - y_3)x + (x_3 - x_2)y\right]$

$= \frac{1}{2A}\left(\frac{a^2}{2} - 0 - \frac{a}{2}x - \frac{a}{2}y\right)$

$= 1 - \frac{x}{a} - \frac{y}{a}$

$N_3^1(x,y) = \frac{1}{2A}\left[x_1 y_2 - x_2 y_1 + (y_1 - y_2)x + (x_2 - x_1)y\right]$

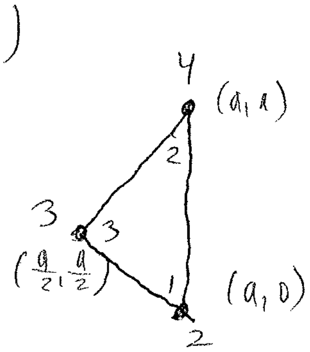
$= \frac{1}{2A}(0 + 0 + ay) = \frac{2y}{a}$

Thus $\phi = \left(1 - \frac{x}{a} - \frac{y}{a}\right)\frac{8}{3} G\theta A + \frac{2y}{a}\frac{4}{3} G\theta A = \underline{\frac{8}{3} G\theta A \left(1 - \frac{x}{a}\right)}$

element ②: $\phi = N_2 \phi_2 + N_3 \phi_3 + N_4 \phi_4$ (globally)

$= N_1^2 \underset{=0}{\phi_2} + N_3^2 \phi_3 + N_2^2 \underset{=0}{\phi_4}$

$= N_3^2 \phi_3$

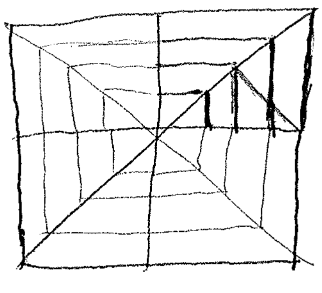


where $N_3^2(x,y) = \frac{1}{2A}\left[a^2 - ax\right] = 2\left(1 - \frac{x}{a}\right)$

Thus $\phi = 2\left(1 - \frac{x}{a}\right)\frac{4}{3} G\theta A = \underline{\frac{8}{3} G\theta A \left(1 - \frac{x}{a}\right)}$

(note: satisfies BC at $x = a$!)

Conclusion: $\phi|_{\text{element 1}} = \phi|_{\text{element 2}}$



straight contour lines (ϕ = const.) makes sense at this level of approximation!