

MM04 Computational and Simulation Methods Notes

Based on the 2011 autumn lectures by Dr S N
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The Author has made every effort to copy down all the content on the board during lectures. The Author accepts no responsibility what so ever for mistakes on the notes or changes to the syllabus for the current year. The Author highly recommends that reader attends all lectures, making his/her own notes and to use this document as a reference only.

MMO4

Computational & Simulation Methods

PART 1 . Mostly FE (finite element) method

PART 2 Other numerical (integration) methods

PART I

THE FINITE ELEMENT METHOD

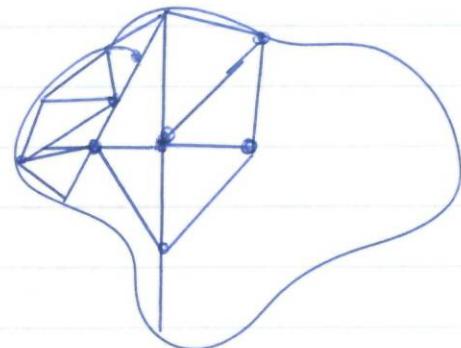
The finite element method is a method for solving differential eq's (ODEs or PDEs) approximately.

The complication in differential eq's is not only the eq' itself but also the boundary conditions.

The strategy of FE is the following 3-step procedure

(1) Divide the domain up into small regular parts ('finite elements') - This also approximates the boundary. Boundary conditions are more easily applied to the elements than on the entire domain.

(2) Choose points (nodes) where the solution will be computed
(discretisation step:
finite n° of degrees of freedom).



(3) Use interpolation to construct solution on full domain
to be done s.t. continuity of the sol' is guaranteed across element boundaries.

Thus the FE method has the following desirable properties:

(i) It's universal and robust - it applies to any DE and BCs

- (2) Can be successively refined (smaller elements, more nodes) to get any required precision. (\Rightarrow convergence to exact solution of the continuum problem.) ("usually OK").
- (3) Can to a very large extent use the Power of a Computer, both for the solution of the final system of eq's, and for the formulation of the approximation (steps 1 to 3)

1D Approximation Theory

Approximation techniques based on weighted residuals.

E.g. // $\frac{d^2u}{dx^2} + u + x = 0 \quad x \in [0, 1]$

b.c. $u(0) = 0 = u(1)$ (Dirichlet b.c. on u)

\uparrow
b.c. on u' called Neumann b.c.

- Take trial sol'n that satisfies the b.c.

$$\tilde{u}(x) = ax(1-x)$$

The error, or residual, we're making is

$$\begin{aligned} R(x) &= \frac{d^2\tilde{u}}{dx^2} + \tilde{u} + x \\ &= -2a + ax(1-x) + x \neq 0 \end{aligned}$$

\uparrow
free parameter a

We can adjust a to get "the best" sol? according to some criterion.

But which criterion to take?

The method of weighted residuals defines a weight w and asks that the weighted average

$$I = \int_0^1 w R \, dx = 0.$$

Three popular choices for w :

(1) Collocation method: $w(x) = \delta(x - x_i)$ \downarrow $x_i \in (0, 1)$

$$\text{Here, } I = \int_0^1 \delta(x - x_i) R(x) \, dx = R(x_i)$$

$$= -2a + ax_i(1-x_i) + x_i := 0$$

$$\text{Take } x_i = \frac{1}{2}. \text{ Find } I = -2a + \frac{1}{4}a + \frac{1}{2} := 0$$

$$\Rightarrow a = \frac{2}{7} \approx 0.2857$$

(2) Least squares method:

$$\text{Minimise } J = \int_0^1 R^2 \, dx$$

$$\text{Here } J = \frac{101}{30} a^2 - \frac{11}{6}a + \frac{1}{3}.$$

$$\text{To minimise, } \frac{dJ}{da} = 0 \Rightarrow a = \frac{55}{202} \approx 0.2723$$

$$\text{Note that } \frac{dJ}{da} = 2 \int_0^1 \frac{dR}{da} R \, dx = 0$$

i.e. this is of the form of a weighted residual
where

$$w(x) = \frac{dR}{da}$$

(3) Galerkin method

$$w(x) = \tilde{u}(x)$$

Choose the trial function itself as the weight f^n :

$$w(x) = x(1-x) \quad (\text{factor } a \text{ irrelevant})$$

$$\begin{aligned} \text{Hence } I &= \int_0^1 x(1-x) [-2a + ax(1-x) + x] dx \\ &= -\frac{3}{10}a + \frac{1}{12} = 0 \\ \Rightarrow a &= \frac{5}{18} \approx 0.2778 \end{aligned}$$

← To improve the approximation we can take more terms with more free parameters.

for instance, $x^2(1-x)$ also satisfies the BCs so we can take the following trial function

$$\tilde{u} = a_1 x(1-x) + a_2 x^2(1-x)$$

$$\begin{aligned} \text{Residual } R &= \frac{d^2 \tilde{u}}{dx^2} + \tilde{u} + x = -2a_1 + 2a_2 - 6a_2 x + a_1 x(1-x) \\ &\quad + a_2 x^2(1-x) + x \end{aligned}$$

Now need two weight f^n 's.

$$(1) \text{ Collocation: } w_1 = \delta(x - x_1) \quad x_{1,2} \in (0,1)$$

$$w_2 = \delta(x - x_2)$$

$$(2) \text{ Least squares: } w_1 = \frac{dR}{da_1} \Rightarrow a_1 = 0.1875$$

$$w_2 = \frac{dR}{da_2} \Rightarrow a_2 = 0.1695$$

$$(3) \text{ Galerkin: } w_1 = x(1-x)$$

$$w_2 = x^2(1-x)$$

$$\int_0^1 x(1-x) R(x) dx := 0 \Rightarrow a_1 = 0.1924$$

$$\text{and } \int_0^1 x^2(1-x) R(x) dx := 0 \Rightarrow a_2 = 0.1707$$

We could continue with a whole series of trial functions.
For instance we could have

$$\tilde{u}_n = a_n \sin(n\pi x) \quad \text{Fourier basis}$$

The exact sol. in this case is easily found, of course.

$$u'' + u = -x, \quad u(0) = 0 = u(1)$$

$$\text{By inspection, } u(x) = \frac{\sin x}{\sin 1} - x.$$

By taking the Fourier basis we'd be computing the Fourier series of this solution

$$\begin{aligned} u(x) &= \sum_{n=1}^{\infty} \tilde{u}_n \\ &= \sum_{n=1}^{\infty} a_n \sin n\pi x \end{aligned}$$

In more general form:

$$Lu = f \quad (L \text{ is a linear operator. In previous example } Lu = u'' + u)$$

Trial function φ_i , approx. $\tilde{u} = \sum_{i=1}^n a_i \varphi_i$

$$\text{Residual } R = L\tilde{u} - f = \sum_{i=1}^n a_i L\varphi_i - f$$

$$\text{Weighted residual: } \int_0^1 w_j R dx = 0 \quad (j=1, \dots, n)$$

$$\text{Least squares: } w_j = \frac{\partial R}{\partial a_j} = L\varphi_j$$

$$\Rightarrow 0 = \int_0^1 L\varphi_j R dx$$

$$= \int_0^1 L\varphi_j \left(\sum_{i=1}^n a_i L\varphi_i - f \right) dx$$

$$\Leftrightarrow \sum_{i=1}^n A_{ji} a_i = f_j$$

$$\text{where } A_{ji} = \int_0^1 L\varphi_j L\varphi_i dx$$

$$f_j = \int_0^1 f L\varphi_j dx$$

$$\text{i.e. } \mathbf{A}\mathbf{a} = \tilde{\mathbf{f}}$$

Note: matrix \mathbf{A} is symmetric

$$\tilde{\mathbf{f}} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Galerkin : $w_j = \varphi_j$

$$0 = \int_0^1 \varphi_j R dx = \sum_{i=1}^n a_i \int_0^1 \varphi_j L \varphi_i dx - \int_0^1 f \varphi_j dx = 0$$

$$\text{So } \underline{A}\underline{a} = \underline{\tilde{F}}$$

now $A_{ij} = \int_0^1 \varphi_i L \varphi_j dx$ not nec. symmetric

$$f_i = \int_0^1 f \varphi_i dx$$

Recall that the adjoint L^* of the linear operator L is defined by

$$\int_0^1 f L g dx = \int_0^1 L^* f g dx$$

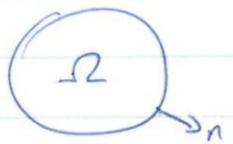
L is called self-adjoint if $L^* = L$

Conclusion: A is symmetric if L is self-adjoint.

2D Approximation Theory

Gauss' Divergence Thm:

$$\iint_{\Omega} \operatorname{div}(\underline{v}) dx dy = \oint_{\partial\Omega} \underline{v} \cdot \underline{n} dS$$



$$\text{If } \underline{v} = v_x \hat{x} + v_y \hat{y}$$

$$\nabla \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$

$$\begin{array}{c} \circlearrowleft \\ \text{div } \underline{v} > 0 \end{array}$$

$$\begin{array}{c} \circlearrowright \\ \text{div } \underline{v} < 0 \end{array}$$

$$\begin{array}{c} \circlearrowright \\ \text{div } \underline{v} = 0 \end{array}$$

"Sum of sources in Ω = flow through boundary $\partial\Omega$."

Integration by parts in 2D

Now change

$$\vec{v} \rightarrow \varphi \vec{v} \quad (\varphi \text{ scalar f.})$$

$$\nabla \cdot (\varphi \vec{v}) \quad \text{div}(\varphi \vec{v}) = \frac{\partial}{\partial x} (\varphi v_x) + \frac{\partial}{\partial y} (\varphi v_y)$$

$$= \varphi \frac{\partial v_x}{\partial x} + \frac{\partial \varphi}{\partial x} v_x + \varphi \frac{\partial v_y}{\partial y} + \frac{\partial \varphi}{\partial y} v_y$$

$$= \varphi \nabla \cdot \vec{v} + \nabla \varphi \cdot \vec{v}$$

So now taking the divergence thm,

$$\iint_{\Omega} \varphi \nabla \cdot \vec{v} \, dx dy = - \iint_{\Omega} \nabla \varphi \cdot \vec{v} \, dx dy + \oint_{\partial\Omega} \varphi v \cdot \vec{n} \, ds \quad (\text{Green's thm})$$

Now take $v_x = \psi$, $v_y = 0$ (holds $\forall \vec{v}$)

$$\iint_{\Omega} \varphi \frac{\partial \psi}{\partial x} \, dx dy = - \iint_{\Omega} \frac{\partial \varphi}{\partial x} \psi \, dx dy + \oint_{\partial\Omega} \varphi \psi n_x \, dx \quad (\text{Gauss-Green})$$

"integration by parts"

$$(v \cdot \vec{n} = v_x n_x + v_y n_y = \psi n_x) \text{ above}$$

In 1D:



$$\begin{aligned} n_x(a) &= -1 \\ n_x(b) &= 1 \end{aligned}$$

$$\text{then } \int_a^b \varphi \frac{d\psi}{dx} dx = - \int_a^b \frac{d\varphi}{dx} \psi dx$$

$$+ \underbrace{\varphi(a)\psi(a)n_x(a) + \varphi(b)\psi(b)n_x(b)}_{\text{sum over bdy}},$$

$$\varphi(b)\psi(b) - \varphi(a)\psi(a)$$

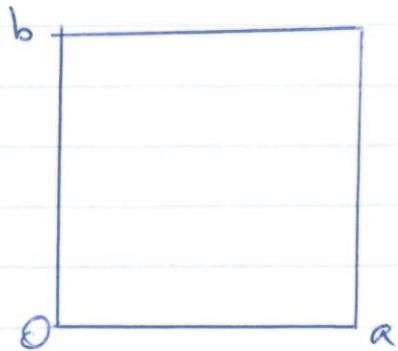
$$= [\varphi\psi]_a^b$$

giving the familiar integration by parts formula in 1D.

Example

Poisson's eqn on a rectangle

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -c$$



b.c. $u=0$ at $x=0, x=a, y=0, y=b$ (ie. on bdy).

Trial soln: $\tilde{u}(x,y) = dxy(x-a)(y-b)$

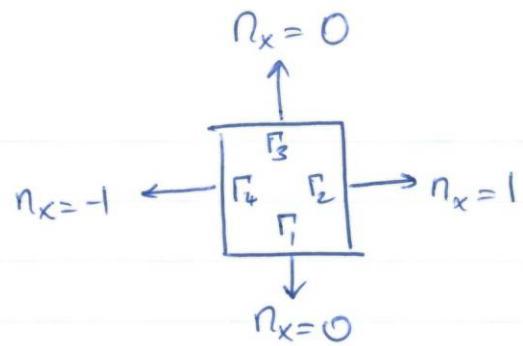
Operator $L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ $(Lu=f)$

$$f = -c$$

Proof that L is self-adjoint:

$$\iint_{\Omega} \varphi \frac{\partial^2 \psi}{\partial x^2} dx dy = - \iint_{\Omega} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial x} dx dy + \int_{\partial\Omega} \varphi \frac{\partial \psi}{\partial x} n_x ds$$

(Gauss-Green)



Boundary term:

$$\begin{aligned} & \oint_{\partial\Omega} \varphi \frac{\partial \psi}{\partial x} n_x dx \\ &= \int_{\Gamma_1} 0 dS + \int_{\Gamma_2} \varphi \frac{\partial \psi}{\partial x} dS + \int_{\Gamma_3} 0 dS - \int_{\Gamma_4} \varphi \frac{\partial \psi}{\partial x} dS \end{aligned}$$

Actually = 0 because $\varphi = 0$ on bdy

$$\Rightarrow \iint_{\Omega} \varphi \frac{\partial^2 \psi}{\partial x^2} dx dy = - \iint_{\Omega} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial x} dx dy$$

$$\text{Gauss-Green} = \iint_{\Omega} \frac{\partial^2 \varphi}{\partial x^2} \psi dx dy - \oint_{\partial\Omega} \frac{\partial \varphi}{\partial x} \varphi n_x dS \quad // 0$$

$$\text{Conclusion: } \iint_{\Omega} \varphi \frac{\partial^2 \psi}{\partial x^2} dx dy = \iint_{\Omega} \frac{\partial^2 \varphi}{\partial x^2} \psi dx dy.$$

Same for y-derivatives:

$$\iint_{\Omega} \varphi \frac{\partial^2 \psi}{\partial y^2} dx dy = \iint_{\Omega} \frac{\partial^2 \varphi}{\partial y^2} \psi dx dy$$

$$\text{Hence: } \iint_{\Omega} \varphi L \psi dx dy = \iint_{\Omega} \psi L \varphi dx dy.$$

$\Rightarrow L$ is self-adjoint.

$Aa = \bar{f}$, A is symmetric.

Take $n=1$, $Aa = \bar{f}$, $A = \int_L \varphi L \varphi dx dy$

where $\varphi = xy(x-a)(y-b)$

Exercise: verify that $A = -\frac{a^3 b^3}{90} (a^2 + b^2)$

$$\begin{aligned}\bar{f} &= c \int_0^b \int_0^a (x-a)(y-b) dx dy \\ &= -\frac{c}{36} a^3 b^3\end{aligned}$$

thus $\tilde{u}(x, y) = \frac{5c}{2(a^2+b^2)} xy(x-a)(y-b)$

L2 New Lect. —

This formulation in terms of the diff. eq? $Lu = f$ is called the strong formulation. To reduce the differentiability requirements, multiply by a test fn and integrate

$$\int_0^1 (u'' + u' + ux) v dx = 0$$

[Int. by parts: $\int_0^1 (-u'v' + uv + xv) dx + [vu']_0^1 = 0$ (Hr on first term $(u''v)$)

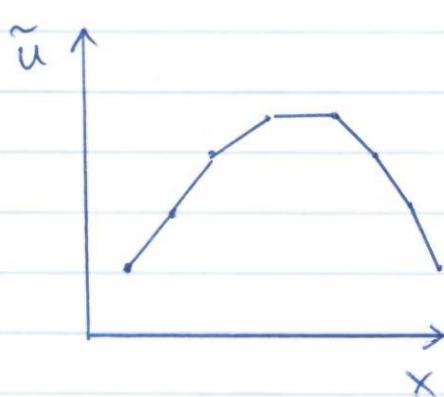
$$\left\{ \begin{array}{l} \int_0^1 (-u'v' + uv + xv) dx = 0 \quad (\text{Hr}) \\ u(0) = u(1) = 0 \end{array} \right.$$

This integral form of the problem is called the Weak formulation

(second derivative not req'd.)

For sufficiently smooth f 's, the strong and weak formulations are equivalent, but the weak formulation allows for a much wider class of trial f 's to approximate the real sol \approx .

For instance, it will be convenient to use piecewise linear approximations.

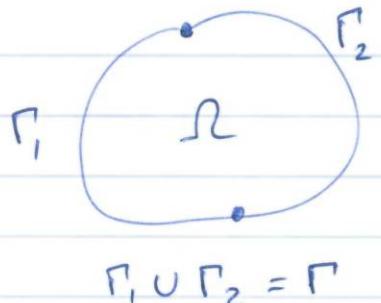


u' fine but u'' does not exist

Weak formulation in 2D

$$\text{Poisson eq: } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$$

$$\text{BC} \begin{cases} u=0 & \text{on } \Gamma_1 \\ \nabla u \cdot \underline{n} = g & \text{on } \Gamma_2 \quad (\text{given flux}) \end{cases}$$



$$\Gamma_1 \cup \Gamma_2 = \Gamma$$

$g, a f$.

$$\text{Weak formulation: } \int_{\Omega} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - f \right) v \, dx dy = 0$$

bring everything over to the LHS, then $\times v$ and integrate, set to zero.

$$\text{Gauss-Green: } - \int_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy$$

$$-\int_{\Omega} f v \, dx dy + \oint_{\Gamma} v \left(\underbrace{\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y}_{\nabla u \cdot \underline{n}} \right) dS = 0$$

$$\Leftrightarrow - \int_0^1 \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy$$

$$+\int_{\Gamma_1} \nabla u \cdot \underline{n} \, dS + \int_{\Gamma_2} rg \, dS - \int_{\Omega} f v \, dx dy = 0$$

$g = \nabla u \cdot \underline{n}$
 on Γ_2

Weak formulation:

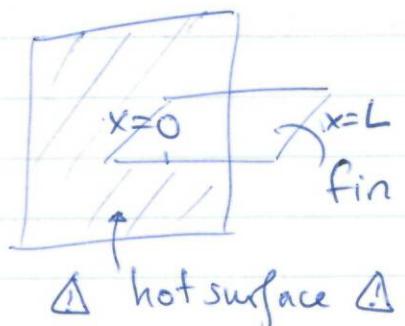
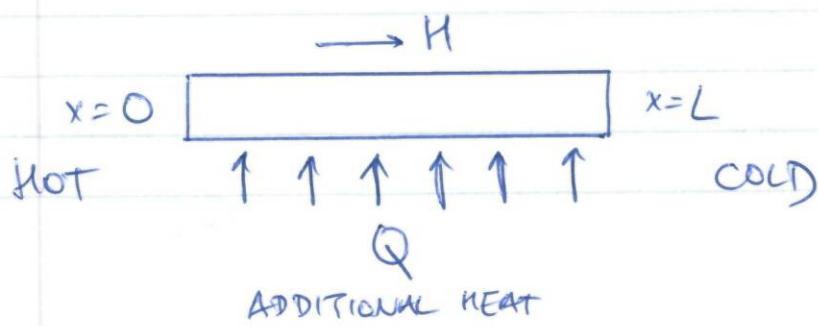
$$\int_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy + \int_{\Gamma_2} f v d\sigma - \int_{\Gamma_2} v g d\sigma = 0$$

$$\left\{ \begin{array}{l} u=0 \text{ on } \Gamma_1, \\ \text{(second b.c., } \cancel{\text{Dirichlet}} \text{ } \nabla u \cdot n = g \text{ on } \Gamma_2 \\ \text{has been absorbed}) \end{array} \right.$$

for all v satisfying BCs.

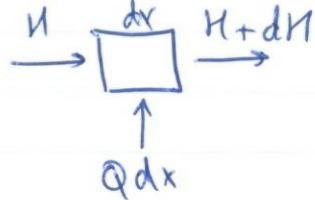
Heat flow modelling in 1D

Consider heat circulation in a thin fin:



Q = heat input / unit time / unit length
 H = heat flow per unit time along fin

Energy balance on infinitesimal element



$$H + Qdx = H + dH \Rightarrow \frac{dH}{dx} = Q$$

The heat flux q is defined by $q(x) = A(x)q(x)$
 where $A(x)$ is the x -sectional area.

Flux is related to temperature gradient

$$q = -k \frac{dT}{dx} \quad k = \text{thermal conductivity}$$

(Fourier's law). Example of a constitutive (depends on the material) relation. Minus sign \because heat flows from hot to cold. Put together:

$$\underbrace{\frac{d}{dx} \left(A(x) \frac{dT}{dx} \right)}_{\text{(1D heat eqn)}} + Q = 0 \quad (\text{1D heat eqn})$$

Possible b.c.s:

$$T(0) = T_0$$

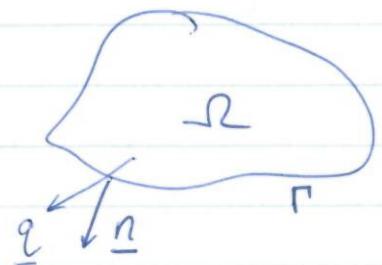
$$q(L) = \left. \left(-k \frac{dT}{dx} \right) \right|_{x=L} = q_L \quad (T_0, q_L \text{ given})$$

(e.g. if end $x=L$ insulated then no flux at $x=L$, so $q_L = 0$.)

Heat flow in 2D

Q = amount of heat supplied to the body / unit area / unit time

\underline{q} = heat flux vector



$$q_n = \underline{q} \cdot \underline{n} = \text{flux}$$

$$\text{Energy conservation: } \int_{\Omega} Q \, dx \, dy = \oint_{\Gamma} q_n \, dS$$

$$\text{Gauss integration formula: } \int_{\Omega} Q \, dx \, dy = \int_{\Omega} \nabla \cdot \underline{q} \, dx \, dy$$

$$\Rightarrow \text{each part of the region } \nabla \cdot \underline{q} = Q$$

q is related to temperature gradient ∇T :

$$\underline{q} = -D \nabla T \quad (\text{D conductivity matrix})$$

Since heat flows from hot to cold,

$$\nabla T \cdot \underline{q} < 0$$

$$\Rightarrow (\nabla T)^T D \nabla T > 0 \quad \forall \nabla T \neq 0$$

$\Rightarrow D$ is positive definite

$\Rightarrow \det D \neq 0$

$$\text{In practice often } D = \begin{pmatrix} k_{xx} & 0 \\ 0 & k_{yy} \end{pmatrix} \quad (\text{orthotropic material})$$

$$\star D = kI = k(1^0) \quad (\text{isotropic material})$$

Put together (for an isotropic material)

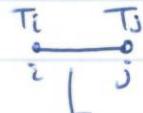
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{Q}{K} = 0 \quad (\text{Poisson's eqn})$$

If no source ($Q=0$), $\nabla^2 T = 0$ (Laplace's eqn)

Possible BCs: $T = g$ on Γ_g
 $q_n = n$ on Γ_n

L3 FE mesh in 1D

Consider a 2-node element



with an unknown field variable T , e.g. temperature

Approximate: $T(x) = \alpha_1 + \alpha_2 x$ (2 parameters).

For consistency, $T(x_i) = T_i$
 $T(x_j) = T_j$

\Rightarrow 2 eqns for 2 unknowns: $\alpha_1 + \alpha_2 x_i = T_i$
 $\alpha_1 + \alpha_2 x_j = T_j$

$$\Rightarrow \alpha_2 = \frac{T_j - T_i}{L} \quad (L = x_j - x_i)$$

$$\Rightarrow \alpha_1 = T_i - \frac{T_j - T_i}{L} x_i$$

Rearrange: $T(x) = N_i^e(x) T_i + N_j^e(x) T_j$

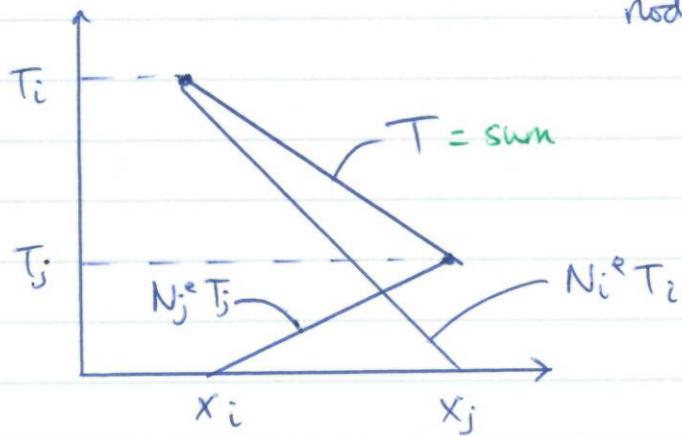
where $N_i^e(x) = \frac{x - x_i}{L}$ (element-shaped fⁿs.)

$$N_j^e(x) = \frac{x - x_j}{L}$$

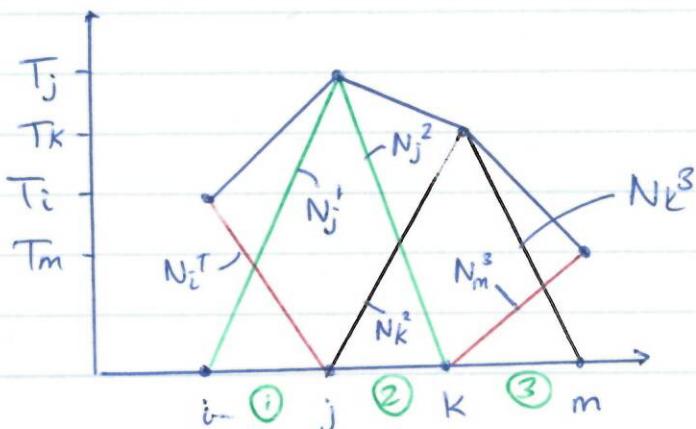
Note: $N_i^e(x_i) = 1, N_i^e(x_j) = 0$

$N_j^e(x_i) = 0, N_j^e(x_j) = 1$

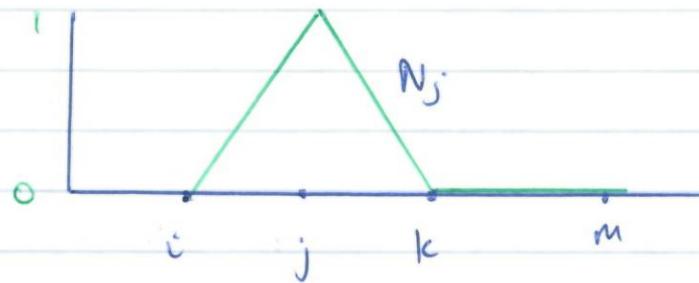
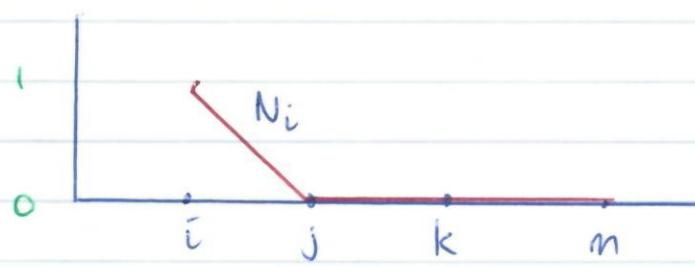
Graphically, this is simply linear interpolation between nodal values T_i, T_j



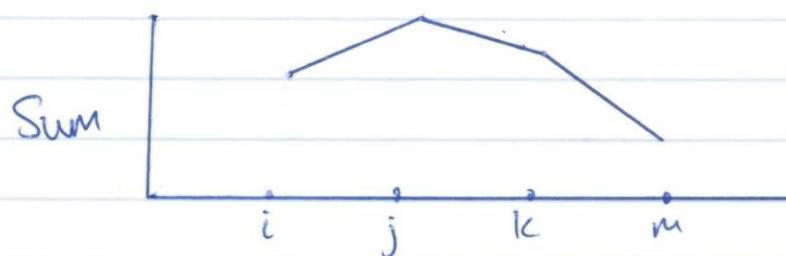
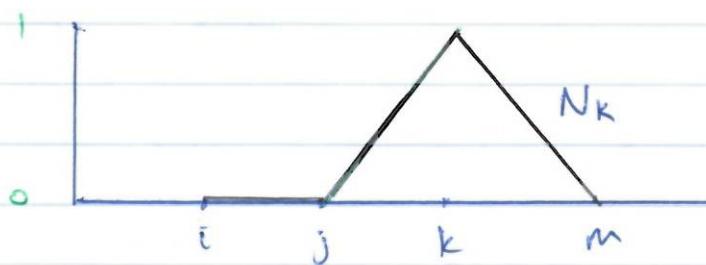
Consider 3 elements:



Introduce global shape functions defined over the whole domain and associated to the nodes



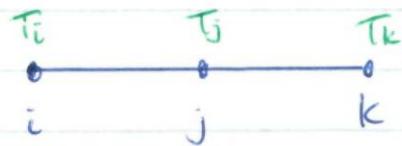
TENT
FNS



$$\text{We have } T(x) = N_i(x)T_i + N_j(x)T_j + N_k(x)T_k + N_m(x)T_m$$

(in terms of global shape functions)

Quadratic (3-node) element



Approximation : $T = \alpha_1 + \alpha_2 x + \alpha_3 x^2$

We require :
$$\begin{cases} T_i = \alpha_1 + \alpha_2 x_i + \alpha_3 x_i^2 \\ T_j = \alpha_1 + \alpha_2 x_j + \alpha_3 x_j^2 \\ T_k = \alpha_1 + \alpha_2 x_k + \alpha_3 x_k^2 \end{cases}$$

i.e.
$$\begin{pmatrix} 1 & x_i & x_i^2 \\ 1 & x_j & x_j^2 \\ 1 & x_k & x_k^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} T_i \\ T_j \\ T_k \end{pmatrix}$$

Solve by Cramer's rule

$$\alpha_1 = \frac{\begin{vmatrix} T_i & x_i & x_i^2 \\ T_j & x_j & x_j^2 \\ T_k & x_k & x_k^2 \end{vmatrix}}{\begin{vmatrix} 1 & x_i & x_i^2 \\ 1 & x_j & x_j^2 \\ 1 & x_k & x_k^2 \end{vmatrix}}$$

$$\alpha_2 = \frac{\begin{vmatrix} 1 & \frac{T_i}{x_i} & \frac{x_i^2}{x_i^2} \\ 1 & \frac{T_j}{x_j} & \frac{x_j^2}{x_j^2} \\ 1 & \frac{T_k}{x_k} & \frac{x_k^2}{x_k^2} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & \frac{x_i}{x_j} & \frac{x_i^2}{x_j^2} \\ 1 & \frac{x_k}{x_j} & \frac{x_k^2}{x_j^2} \end{vmatrix}}$$

$$\alpha_3 = \frac{\begin{vmatrix} 1 & x_i & T_i \\ 1 & x_j & T_j \\ 1 & x_k & T_k \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & x_j & T_j \\ 1 & x_k & T_k \end{vmatrix}}$$

Rearrange to write: $T(x) = N_i^e(x)T_i + N_j^e(x)T_j + N_k^e(x)T_k$

$$N_i^e(x) = \frac{2}{L^2} (x - x_i)(x - x_k)$$

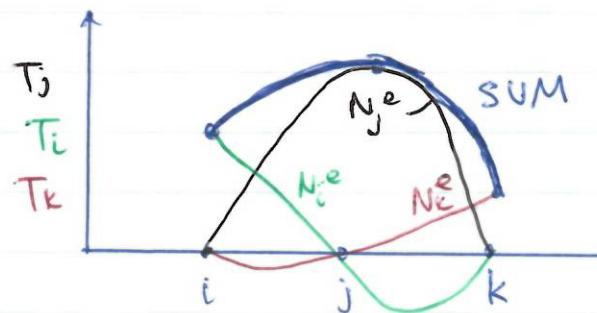
$$L = x_k - x_i$$

$$N_j^e(x) = -\frac{4}{L^2} (x - x_i)(x - x_k)$$

$$N_k^e(x) = \frac{2}{L^2} (x - x_k)(x - x_i)$$

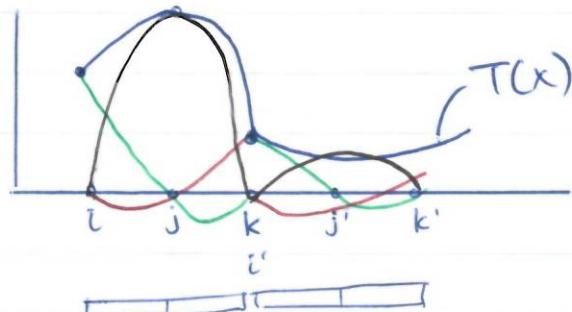
note $N_i^e(x_j) = \delta_{ij}$

Graphically,



Quadratic interpolation.

Two elements:

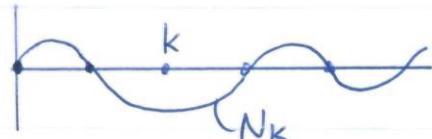


Higher order elements



$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \dots$$

looking for polynomials that are zero at all nodes except one where it is 1.



$$P_K^{n-1}(x) = \frac{(x-x_1)(x-x_2) \dots (x-x_{k-1})(x-x_{k+1}) \dots (x-x_n)}{(x_k-x_1)(x_k-x_2) \dots (x_k-x_{k-1})(x_k-x_{k+1}) \dots (x_k-x_n)}$$

Lagrange interpolation formula

$$\text{Clearly } P_K^{n-1}(x_i) = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$

Take for shape fⁱ's of n-node element

$$N_k^e(x) = P_{ik}^{n-1}(x) \quad k=1, \dots, n$$

for n=2 : $P_1^1(x) = \frac{x-x_2}{x_1-x_2} = \frac{-(x-x_2)}{L}$

$$P_2^1(x) = \frac{x-x_1}{x_2-x_1} = \frac{x-x_1}{L} \quad L = x_2 - x_1$$

Properties of shape fⁱ's

(1) $T(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_n x^{n-1} = \sum_{i=1}^n N_i(x) T_i$

Interpolation for any fⁱ T(x),
take special case $T(x) = T_0$ (const.)

$$\Rightarrow T_0 = \sum_{i=1}^n N_i(x) T_0$$

$$\Leftrightarrow \boxed{\sum_{i=1}^n N_i(x) = 1} \quad (\text{partition of unity})$$

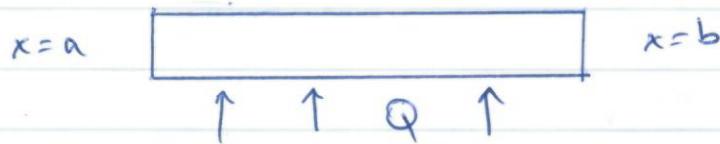
(2) $N_i(x) \neq 0$ only on those elements that contain node i



(3) $N_i(x_i) = 1$ at its own nodeⁱ and 0 at all other nodes)

First FE eqn in 1D :

$$\text{Heat: } \frac{d}{dx} \left(A k \frac{dT}{dx} \right) + Q = 0$$



$$T(x) = \sum_{i=1}^n N_i(x) T_i \quad (\text{interpolation})$$

(will play the role of a trial function in a Galerkin weighted residual method)

Pass from strong formulation to weak formulation.

$$\int_a^b \left(\frac{d}{dx} \left(A k \frac{dT}{dx} \right) + Q \right) v \, dx = 0$$

$$\Leftrightarrow \int_a^b \left(- \frac{dv}{dx} A k \frac{dT}{dx} + Qv \right) dx - [v A q]_a^b = 0$$

(using the relation $q = -k \frac{dT}{dx}$)

$$\Leftrightarrow \int_a^b \frac{dv}{dx} A k \frac{dT}{dx} \, dx = \int_a^b Qv \, dx - [v A q]_a^b$$

Galerkin: for test functions v take $N_i(x)$

and $T = \sum_j N_j T_j$

$$\Rightarrow \sum_{j=1}^n \int_a^b \frac{dN_i}{dx} A_k \frac{dN_j}{dx} dx T_j$$

$$= -[N_i A_q]_a^b + \int_a^b Q N_i dx$$

or $\sum_{j=1}^n K_{ij} T_j = f_{bi} + f_{ei}$

F.E. EQN

where $K_{ij} = \int_a^b \frac{dN_i}{dx} A_k \frac{dN_j}{dx} dx$

STIFFNESS MATRIX

$$f_{ei} = \int_a^b Q N_i dx$$

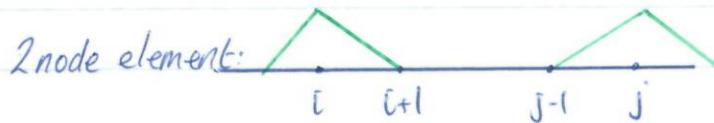
LOAD/FORCE VECTOR

$$f_{bi} = -[N_i A_q]_a^b$$

BOUNDARY VECTOR

Properties of the stiffness matrix

- (i) Bounded $\because N_i N_j$ will be nonzero only if i and j belong to the same element



$$N_i N_j = 0 \text{ if } |i-j| > 1$$

2-node element: $\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & & 0 \\ & * & * \\ & * & * \end{pmatrix}$ tridiagonal

for higher-order elements, the 'bandwidth' will be larger.

(2) Symmetric : $K_{ij} = K_{ji}$

(3) Singular : $\det K = 0$. K does not depend on Q , so consider the homogeneous eqn

$$\frac{d}{dx} \left(Ak \frac{dT}{dx} \right) = 0 \implies \text{F.E. Eqn} \quad Ku = 0,$$

$$u = \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix}$$

Any constant T is a soln so K must be singular

(4) Rows and columns sum up to zero : follows from partition of unity :

$$\sum_i N_i(x) = 1 \Rightarrow$$

$$\sum_i K_{ij} = \int_a^b \cancel{\frac{d(\sum N_i(x))}{dx}} Ak \frac{dN_i}{dx} dx =$$

$$= 0$$

Same for $\sum_j K_{ij} = 0$.

Property of the boundary vector

$$f_{bi} = -[N_i A_q]_a^b$$

$x=a$ corresponds to node 1
 $x=b$... n



$$\Rightarrow f_b = \begin{pmatrix} (A_q)|_{x=a} \\ 0 \\ \vdots \\ 0 \\ -(A_q)|_{x=b} \end{pmatrix}$$

Example (heat flow in a uniform fin)



model with 3 2-node elements

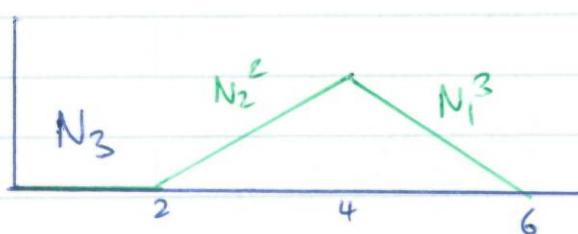
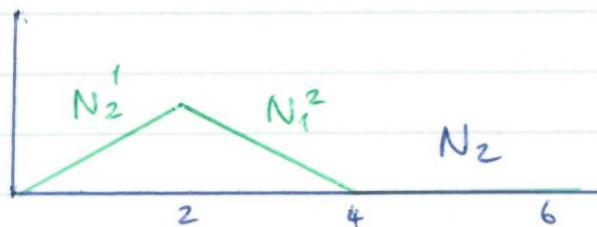
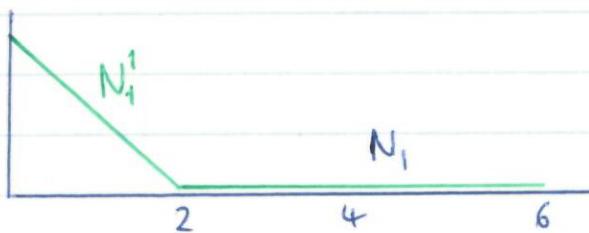


$$\frac{d}{dx} \left(A K \frac{dT}{dx} \right) + Q = 0$$

$$T(0) = T_0$$

$$q(6) = \bar{q}$$

Global shape fⁿs



Element shape functions:

$$N_1^1(x) = -\frac{(x-x_2)}{L} = \frac{2-x}{2} \quad (x_2=2, L=2)$$

$$N_2^1(x) = \frac{(x-x_1)}{L} = \frac{x}{2}$$

$$N_1^2(x) = \frac{-(x-x_3)}{L} = \frac{4-x}{2}$$

$$N_2^2(x) = \frac{(x-x_2)}{L} = \frac{x-2}{2}$$

$$N_1^3(x) = \frac{-(x-x_4)}{L} = \frac{6-x}{2}$$

$$N_2^3(x) = \frac{x-x_3}{L} = \frac{x-4}{2}$$

Stiffness matrix $K_{ij} = \int_0^6 \frac{dN_i}{dx} Ak \frac{dN_j}{dx} dx$

$$\begin{aligned} K_{11} &= \int_0^6 \frac{dN_1}{dx} Ak \frac{dN_1}{dx} dx \\ &= \int_0^2 \frac{dN_1^1}{dx} Ak \frac{dN_1^1}{dx} dx = \\ &= \int_0^2 \left(-\frac{1}{2}\right)^2 Ak dx \\ &= \frac{1}{2} Ak \end{aligned}$$

$$\begin{aligned} K_{12} &= \int_0^6 \frac{dN_1}{dx} Ak \frac{dN_2}{dx} dx \\ &= \int_0^2 \frac{dN_1^1}{dx} Ak \frac{dN_2^1}{dx} dx \\ &= \int_0^2 \left(-\frac{1}{2}\right) Ak \left(\frac{1}{2}\right) dx \\ &= -\frac{1}{2} Ak \end{aligned}$$

$$K_{13} = \int_0^6 \frac{dN_1}{dx} Ak \frac{dN_3}{dx} dx \\ = 0 \quad (\text{no overlap of } N_1 \text{ and } N_3)$$

$$K_{14} = 0 \quad (\text{same reason}).$$

$$K_{22} = \int_0^6 \frac{dN_2}{dx} Ak \frac{dN_2}{dx} dx \\ = \int_0^2 \frac{dN_2}{dx} Ak \frac{dN_2}{dx} dx + \int_2^4 \frac{dN_2}{dx} Ak \frac{dN_2}{dx} dx \\ = \int_0^2 \left(\frac{1}{2}\right)^2 Ak dx + \int_2^4 \left(-\frac{1}{2}\right)^2 Ak dx \\ = Ak$$

etc.

Get

$$K = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \frac{1}{2} Ak$$

Load vector: $f_{e_1} = \int_0^6 Q N_1 dx$

$$= Q \int_0^2 N_1^1 dx$$

$$= Q \int_0^2 \frac{2-x}{2} dx$$

$$= Q$$

$$\begin{aligned}
 f_{\ell_2} &= \int_0^6 Q N_2 \, dx \\
 &= Q \int_0^2 N_2^1 \, dx + Q \int_2^4 N_2^2 \, dx \\
 &= 2Q
 \end{aligned}
 \quad \rightarrow \quad \underline{f_{\ell} = Q \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}}$$

Boundary vector: $f_{b_i} = -[N_i A_q]_0^6$

$$\begin{aligned}
 &= -(N_i A_q)|_{x=6} + (N_i A_q)|_{x=0} \\
 &\quad \text{||} \qquad \qquad \qquad \text{||} \\
 &\quad A_q(6) \delta_{i4} \qquad \qquad \qquad A_q(0) \delta_{i1}
 \end{aligned}$$

or $f_b = \begin{pmatrix} A_q(0) \\ 0 \\ 0 \\ -A_q(6) \end{pmatrix}$

FE eq: $\frac{1}{2} A K \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix}$

$$= Q \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} A_q(0) \\ 0 \\ 0 \\ -A_q(6) \end{pmatrix} \quad (*)$$

Apply b.c.s : $T_1 = T_0$
 $q(6) = \bar{q}$

Exercise: show that the FE solⁿ is exact at the nodes.

$$\left\{ \begin{array}{l} \frac{d}{dx} \left(AK \frac{dT}{dx} \right) + Q = 0 \\ T(0) = T_0 \\ q(6) = \bar{q} \end{array} \right.$$

FE eq: $\frac{1}{2} AK \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} = T_0$

$$= Q \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} Aq(0) \\ 0 \\ 0 \\ -Aq(6) \end{pmatrix}$$
(*)

New stuff starts here

2-step soln

1. Solve reduced problem for (T_2, T_3, T_4)

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} 2Q + \frac{1}{2} AK T_0 \\ 2Q \\ Q - A\bar{q} \end{pmatrix}$$

$$\Rightarrow T_2 = \dots, T_3 = \dots, T_4 = \dots$$

$$2. \frac{1}{2} AK (T_0 - T_2) = Q + Aq(0) \quad [\text{first row of } (*)]$$

$$\Rightarrow q(0) = \bar{q} - \frac{6Q}{A} \quad (\text{using } T_2 \text{ solution})$$

Exact solution in this case

$$\frac{d^2T}{dx^2} = -\frac{Q}{Ak} \Rightarrow \frac{dT}{dx} = -\frac{Qx}{Ak} + C$$

$$BC: q(6) = -k \left. \frac{dT}{dx} \right|_{x=6} = \bar{q}$$

$$\Rightarrow C = -\frac{\bar{q}}{k} + \frac{6Q}{Ak}$$

$$\Rightarrow \frac{dT}{dx} = -\frac{\bar{q}}{k} - \frac{Q}{Ak} (x-6)$$

$$\Rightarrow T(x) = -\frac{\bar{q}}{k}x - \frac{Q}{Ak} \left(\frac{1}{2}x^2 - 6x \right) + d$$

$$BC: T(0) = T_0 = d$$

exact soln?

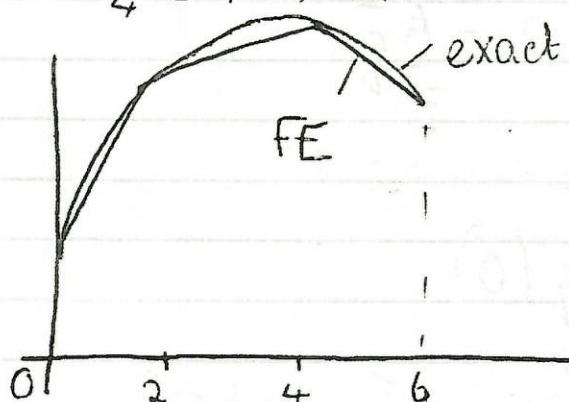


$$\Rightarrow \text{Solution. } T(x) = T_0 - \frac{\bar{q}}{k}x - \frac{Q}{2Ak}x(x-12)$$

$$\Rightarrow T_2 = T(2) = \dots$$

$$T_3 = T(4) = \dots$$

$$T_4 = T(6) = \dots$$



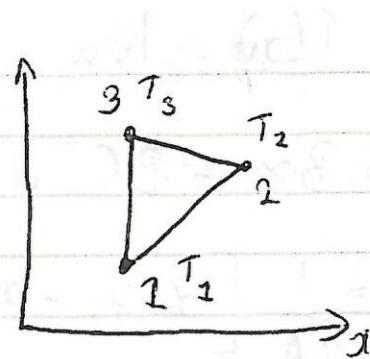
Conclusion

FE exact at nodes but linear interpolation in between.

(This is not a general rule)

Two-dimensional elements

3-node triangular element



$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$= \underline{B} \underline{\alpha}$$

where $B = (1 \ \alpha_1 \ y)$ $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

want to express $T(x) = N_1^e(y) T_1 + N_2^e(y) T_2 + N_3^e(y) T_3$

for consistency $T(x_1, y_1) = T_1, T(x_2, y_2) = T_2, T(x_3, y_3) = T_3$

$$T_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$T_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$T_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$\Leftrightarrow \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

Write $C\alpha = u$, $u = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$

Also $T(x,y) = Nu$ where $N = (N_1^e, N_2^e, N_3^e)$

$$\Rightarrow T = B\alpha = BC^{-1}u \text{ Hence } N = BC^{-1}$$

$$N_1^e(x,y) = \frac{1}{2A} [x_2y_3 - x_3y_2 + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$N_2^e(x,y) = \frac{1}{2A} [x_3y_1 - x_1y_3 + (y_3 - y_1)x + (x_1 - x_3)y]$$

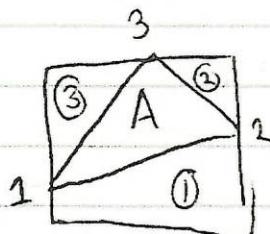
$$N_3^e(x,y) = \frac{1}{2A} [x_1y_2 - x_2y_1 + (y_1 - y_2)x + (x_2 - x_1)y]$$

This is a
Cyclic permutation!

(Here, $2A = \det C$)

Claim: A is the area of the triangle.

Proof



$$A = (x_2 - x_1)(y_3 - y_1) - ① - ② - ③$$

$$① = \frac{1}{2} (x_2 - x_1)(y_2 - y_1)$$

$$② = \frac{1}{2} (x_2 - x_3)(y_3 - y_2)$$

$$③ = \frac{1}{2} (x_3 - x_1)(y_1 - y_3)$$

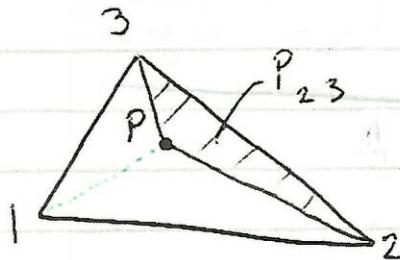
$$\Rightarrow A = \frac{1}{2} [(x_2y_3 - x_3y_2) + (x_1y_2 - x_2y_1) + (x_3y_1 - x_1y_3)]$$

$$= \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2} \det C$$

$$\Leftrightarrow 2A = \det C$$

Area coordinates



$$\gamma_1 = \frac{\text{area}(P_{23})}{\text{area}(123)} \quad \text{total area}$$

$$\gamma_2 = \frac{\text{area}(P_{31})}{\text{area}(123)}$$

$$\gamma_3 = \frac{\text{area}(P_{12})}{\text{area}(123)}$$

$$\text{Clearly } \gamma_1 + \gamma_2 + \gamma_3 = 1$$

If P coincides with node 1 then $\gamma_1 = 1, \gamma_2 = 0, \gamma_3 = 0$

$$\text{In fact } \gamma_i(x_j, y_j) = \delta_{ij}$$

Fact $\gamma_i(x, y) = N_i(x, y)$ (shape functions are the area coordinates)

Proof

$$\eta_1 = \frac{\text{area } (P_{23})}{\text{area } (123)}$$

$$= \frac{1}{2} \det \begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

A

$$= \dots$$

$$= N_1(x, y)$$

$$\text{Inverse relationship: } x = x_1 \eta_1 + x_2 \eta_2 + x_3 \eta_3$$

$$y = y_1 \eta_1 + y_2 \eta_2 + y_3 \eta_3$$

$$(\eta_3 = 1 - \eta_1 - \eta_2)$$

The area coordinates are convenient when computing integrals of type $I_{nm} = \int x^n y^m dx dy$ that appear in

stiffness matrices and load vectors

$$(\text{Recall } K_{ij} = \int \nabla N_i \cdot \nabla N_j dx dy \quad f_{Li} = \iint Q N_i(x, y) dx dy)$$

Idea: transform $(x, y) \rightarrow (\eta_1, \eta_2, \eta_3)$

$$\Rightarrow \int \eta_1^m \eta_2^n \eta_3^p dx dy$$

$$= \frac{m! n! p!}{(m+n+p+2)!} (2A)$$

Example

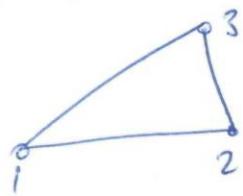
$$1. I_{00} = \int dx dy = A$$

$$2. I_{10} = \int x dx dy = x_1 \int \eta_1 dx dy + x_2 \int \eta_2 dx dy + x_3 \int \eta_3 dx dy$$
$$= \frac{A}{3} (x_1 + x_2 + x_3)$$

$$3. I_{11} = \int xy dx dy$$

$$= \dots = (x_1 + \dots + x_n) A \dots$$

Triangular elements



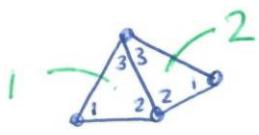
Shape fns N_1^e, N_2^e, N_3^e

$$\text{Property: } N_i(x_j, y_j) = \delta_{ij}$$

$$\Rightarrow N_i(x, y) = 0 \text{ along entire edge } 2-3.$$

$$(T = \alpha_1 + \alpha_2 x + \alpha_3 y \\ = N_1^e T_1 + N_2^e T_2 + N_3^e T_3)$$

Conformity (or compatibility)



A mesh of elements is conforming if the shape functions, and hence the soln T_e , is guaranteed to be continuous across element boundaries.

Need to show $N_2^1 = N_2^2$ along boundary 2-3

We can write for the boundary $y = c_1 + c_2 x$

$$\Rightarrow N_2^1(x, y) = \beta_1 + \beta_2 x + \beta_3 y \\ = a_1 + b_1 x \quad (= y = f(x))$$

$$\text{and } N_2^2(x, y) = a_2 + b_2 x$$

But we have 4 conditions:

$$N_2^1(x_2, y_2) = 1 = N_2^2(x_2, y_2) \\ N_2^1(x_3, y_3) = 0 = N_2^2(x_3, y_3)$$

\Rightarrow can uniquely determine coefficients a_1, a_2, b_1, b_2

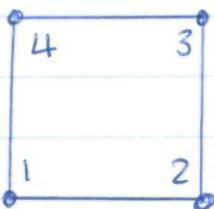
\Rightarrow we find $a_1 = a_2$ and $b_1 = b_2$

$\Rightarrow N_2^1 = N_2^2$ along common boundary

and $N_3^1 = N_3^2$ along common boundary similarly.

\Rightarrow mesh of arbitrary triangular elements is conforming.

4-node rectangular element



to keep symmetry
in x and y

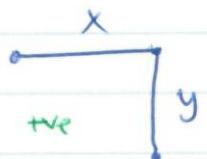
$$T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$\text{We want again } T = \sum_{i=1}^4 N_i T_i.$$

Can use direct method to construct shape functions.
But we can also obtain them by taking products of
1D Lagrange polynomials.

$$N_1(x, y) = P_1'(x) P_1'(y)$$

$$= \frac{x - x_2}{x_1 - x_2} \cdot \frac{y - y_4}{y_1 - y_4}$$



$$\text{area} = \frac{1}{4ab} (x - x_2)(y - y_4)$$

$$N_2(x, y) = P_2'(x) P_1'(y) = -\frac{1}{4ab} (x - x_1)(y - y_3) \quad \text{--- ve}$$

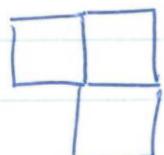
$$N_3(x, y) = P_2'(x) P_2'(y) = \frac{1}{4ab} (x - x_4)(y - y_2) \quad \text{--- +ve}$$

$$N_4(x, y) = P_1'(x) P_2'(y) = -\frac{1}{4ab} (x - x_3)(y - y_1) \quad \text{--- -ve}$$

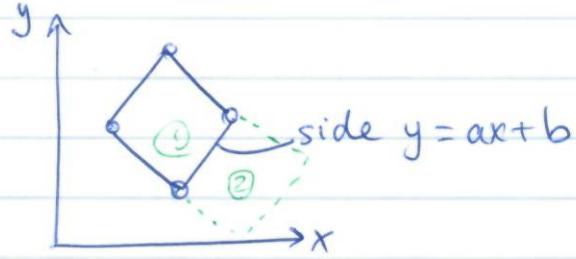
$$(\text{note: } N_i(x_j, y_j) = \delta_{ij})$$

A mesh of these elements is conforming (proof similar to triangular case: N uniquely determined by nodal values).

However, it does require that the sides are parallel to the x- and y-axes.



Otherwise, consider



$$\Rightarrow T = \alpha_1 + \alpha_3 b + (\alpha_2 + \alpha_3 a + \alpha_4 b) x + \alpha_4 a x^2 \\ = \beta_1 + \beta_2 x + \beta_3 x^2$$

Quadratic

3 coefficients to find $\beta_1, \beta_2, \beta_3$
2 nodes of info T_1, T_2

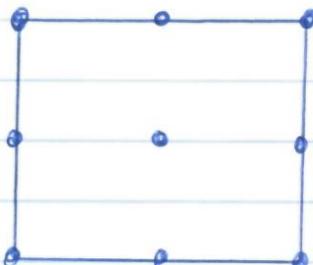


- not uniquely defined.
- no guarantee that ① and ② give same values.

We can solve this by modifying the mesh (at the cost of more nodes).

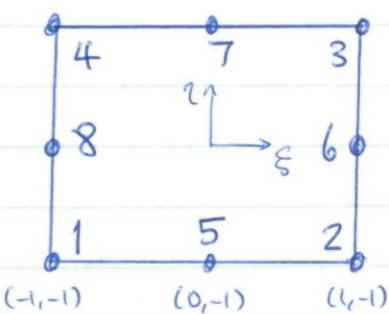


9-node rectangular element



Shape f_i's are products of quadratic Lagrange elements along x and y.

8-node rectangular element



Not Lagrange!

We can construct shape fns for this element by using the property:

$$N_i(x_j, y_j) = \delta_{ij}$$

Introduce local coordinates (ξ, η)

Consider mid-side node:

$$N_5(x, y) = \alpha (\xi + 1)(\xi - 1)(\eta - 1)$$

$$\text{where } \alpha = \frac{1}{2} \therefore N_5(0, 1) = 1$$

Consider corner node:

$$N_1(x, y) = \alpha (\xi - 1)(\eta - 1)(1 + \xi + \eta)$$

(Turns out you can require N_1 to be 0 along entire line connecting 5 & 8).

$$\text{where } \alpha = -\frac{1}{4}$$

This element is also known as the "Serendipity element"

Approximation in this case:

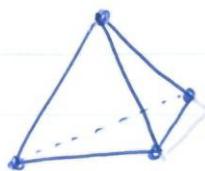
$$T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 xy^2$$

A mesh of such elements is conforming provided the sides are parallel to the x- and y-axes.

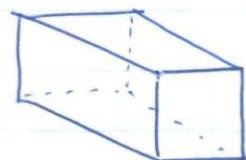
3D elements

4-node tetrahedral element:

$$T_1 = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$$

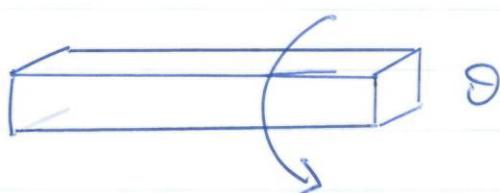


8-node prism/brick element

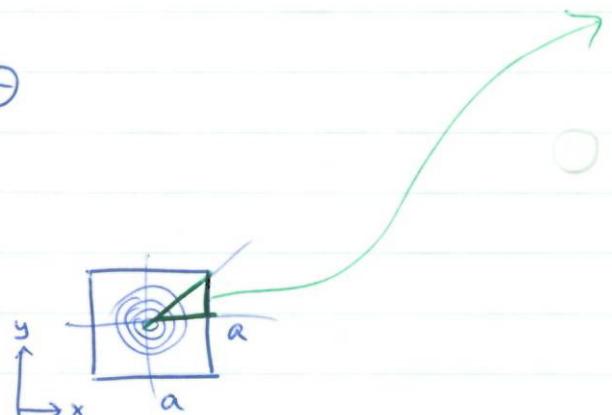


2D example (the big one!)

Torsion of square cross-section shaft



Consider square cross-section



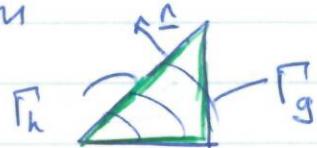
Governing eqⁿ for stress in shaft:

$$\left\{ \begin{array}{l} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2G\theta = 0 \\ \phi = 0 \text{ on boundary} \end{array} \right.$$

(shaft only loaded at the end)

ϕ stress function
 G shear
 θ twist rate

by symmetry about $(x, y, y=x)$ we need only consider an eighth of the section



On this reduced domain we have

$$\left\{ \begin{array}{l} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2G\theta = 0 \\ \phi = 0 \text{ on } \Gamma_g \quad (\text{original outer boundary}) \\ \frac{\partial \phi}{\partial n} := \nabla \phi \cdot \underline{n} = 0 \text{ on } \Gamma_h \quad \text{by symmetry} \\ \quad (\nabla \phi \text{ parallel to boundary}) \end{array} \right.$$

Weak formulation:

$$\iint_{\Delta} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2G\theta \right) v \, dx \, dy = 0$$

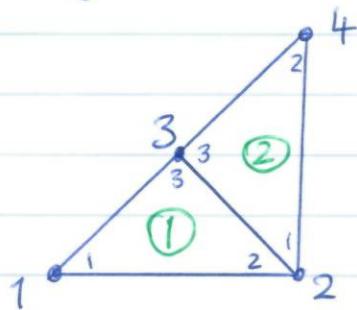
$$\stackrel{G-G}{\iff} \iint_{\Delta} \left(-\frac{\partial \phi}{\partial x} \frac{\partial v}{\partial x} - \frac{\partial \phi}{\partial y} \frac{\partial v}{\partial y} + 2G\theta v \right) \, dx \, dy$$

$$+ \iint_{\Gamma} \left[v \frac{\partial \phi}{\partial x} n_x + v \frac{\partial \phi}{\partial y} n_y \right] \, ds = 0$$

$$\xleftarrow{G-G} \iint_{\Delta} \left(\frac{\partial \phi}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial v}{\partial y} \right) dx dy$$

$$= \int_{\Gamma_g} v \frac{\partial \phi}{\partial n} ds + 2G\theta \iint_{\Delta} v dx dy$$

Choose mesh of two 3-node triangular elements.



For single element we introduce the interpolation

$$\phi = \sum_{j=1}^3 N_j^e \phi_j$$

$$\text{Galerkin: } v = N_i^e \quad (i=1,2,3)$$

$$\Rightarrow \sum_{j=1}^3 \iint_{\Delta} \left(\frac{\partial N_j^e}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial N_j^e}{\partial y} \frac{\partial v}{\partial y} \right) \phi_j dx dy$$

$$= \int_{\Gamma_g} N_i^e \frac{dv}{dn} ds + 2G\theta \iint_{\Delta} N_i^e dx dy$$

$$\text{FE eq: } \sum_{j=1}^3 K_{ij} \phi_j = f_{e_i} + f_{b_i} \quad \text{where}$$

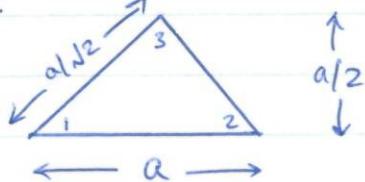
$$\text{where } K_{ij} = \iint_{\Delta} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dx dy$$

(Stiffness matrix)

$$f_{e_i} = 2G\theta \iint_{\Delta} N_i dx dy \quad (\text{load vector})$$

$$f_{b_i} = \int_{\Gamma} N_i \frac{d\phi}{dn} ds$$

Element ① :



$$A = \text{area} \\ = \frac{a^2}{4}$$

$$\text{Shape function: } N_1^e = \frac{1}{2A} [x_2 y_3 - x_3 y_2 + (y_2 - y_3)x + (x_3 - x_2)y]$$

N_2^e, N_3^e cyclic.

$$\Rightarrow K_{11} = \iint_{\Delta} [(y_2 - y_3)^2 + (x_3 - x_2)^2] \frac{1}{4A^2} dx dy$$

$$K_{12} = 0$$

$$\Rightarrow K = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

note rows + cols add to 0, as they should.
So you only need to compute 2 and you
know the 3rd!

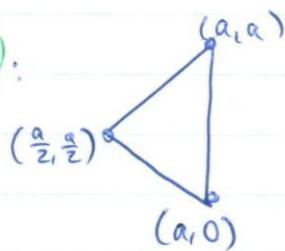
Load vector: remember that $N_i = n_i$ (area coords)

$$\text{Integration formula: } \iint_{\Delta} n_i^n n_j^m n_k^p dx dy$$

$$= \frac{m! n! p! (2A)}{(m+n+p+2)!}$$

$$\Rightarrow f_e^1 = \frac{2}{3} \text{GOA}(1)$$

Element ②:



Element congruent to element ① with equivalent local node numbering $\Rightarrow K^2 = K^1$.

$$\text{Also } f_e^2 = f_e^1$$

Q: How to 'add up' these local element data into global data?

Assembly: Element ① :

$$u^1 = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

local degrees of freedom \rightarrow *global degrees of freedom*

$$=: A^1 u$$

$$\text{Element ②: } u^2 = \begin{pmatrix} \phi_2 \\ \phi_4 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

$$=: A^2 u$$

$$K^1 u^1 = f_e^1 \Rightarrow K^1 A^1 u = f_e^1$$

$$\Rightarrow A^{1T} K^1 A^1 u = A^{1T} f_e^1$$

$$\text{Also, for element } 2, \quad A^{2T} K^2 A^2 u = A^{2T} f_e^2$$

Now sum over elements,

$$\sum_e A^{eT} K^e A^e u = \sum_e A^{eT} f_e^e$$

or

$$Ku = f_e$$

$$\text{where } K = \sum_e A^{eT} K^e A^e \quad (\text{global stiffness matrix})$$

$$f_e = \sum_e A^{eT} f_e^e \quad (\text{global load vector})$$

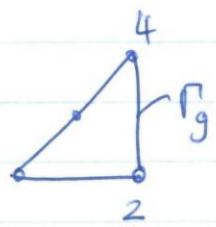
We find

$$K = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ -1 & -2 & 4 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$f_e = \frac{2}{3} QAD \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{Boundary vector: } f_{b_i} = \int_{\Gamma_g} N_i \frac{d\phi}{dn} ds$$

$\therefore \Gamma_g$ along 2-4 edge only, only
2nd + 4th component will be nonzero



Along Γ_g the only nonzero global shape f^n's are N_2 and N_4 .

Thus $f_b = \begin{pmatrix} 0 \\ R_2 \\ 0 \\ R_4 \end{pmatrix}$ where $R_2 = \int_{\Gamma_g} N_2 \frac{d\phi}{dn} ds$

$$\Rightarrow FE \text{ eq: } \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ -1 & -2 & 4 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ R_2 \\ 0 \\ R_4 \end{pmatrix} + \frac{2}{3} G\Delta A \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

Boundary conditions: $\phi = 0$ on Γ_g

$$\Rightarrow \phi_2 = 0 = \phi_4$$

Reduced system for (ϕ_1, ϕ_3) :

$$\frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_3 \end{pmatrix} = \frac{2}{3} G\Delta A \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \phi_1 = \frac{8}{3} G\Delta A \quad \Rightarrow \begin{cases} R_2 = -\frac{8}{3} G\Delta A \\ R_4 = -\frac{4}{3} G\Delta A \end{cases}$$

General approach

1. Formulate boundary-valued problem
2. Weak formulation
3. Mesh (discretisation)
4. Interpolation [$\phi = \sum \phi_i N_i$]
5. Compute element data
6. Assembly into global data
7. Apply BCs
8. Solution for ϕ_i 's.

let's see if this result makes any sense.

(i) Torsional stiffens $C = \frac{M}{\theta}$,

where the twisting moment M is given

by

$$M = 2 \iint_D \phi \, dx \, dy$$

$$\text{we have } M = 8 \left[2 \int_{\textcircled{1}} \phi \, dx \, dy + 2 \int_{\textcircled{2}} \phi \, dx \, dy \right]$$

$$\begin{aligned} &= 8 \left[2 \int_{\textcircled{1}} (N_1^e \phi_1 + N_2^e \phi_2 + N_3^e \phi_3) \, dx \, dy \right. \\ &\quad \left. + 2 \int_{\textcircled{2}} (N_1^e \phi_2 + N_2^e \phi_4 + N_3^e \phi_3) \, dx \, dy \right] \end{aligned}$$

$$= 16 \left[\frac{A}{3} (1 \ 1 \ 1) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} + \frac{A}{3} (1 \ 1 \ 1) \begin{pmatrix} \phi_2 \\ \phi_4 \\ \phi_3 \end{pmatrix} \right]$$

$$\text{because } \iint_D N_i(x, y) \, dx \, dy = \frac{A}{3}.$$

Plugging in our values for ϕ :

$$\frac{16}{3} A (\phi_1 + 2\phi_3) = \frac{256}{9} G \theta A^2$$

$$\begin{aligned}\Rightarrow C = \frac{M}{\theta} &= \frac{256}{9} G A^2 \\ &= \frac{256}{9} G \left(\frac{\bar{A}}{16}\right)^2 \\ &= \frac{1}{9} G \bar{A}^2 \\ &= 0.1111 G \bar{A}^2\end{aligned}$$

split our element into 2
we have $\frac{1}{8}$
 2×8

where $\bar{A} = 16A$, total
cross-sect. area of square rod.

Exact result: $C = k_1 G \bar{A}^2$ with $k_1 = 0.1406$,
so we have a

21% error.

(2) Compute the stress! over the entire domain (equipotential lines)

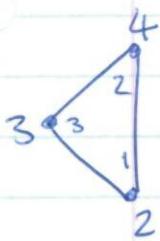
Element-wise:

$$\begin{aligned}\textcircled{1}: \quad \phi &= N_1 \phi_1 + N_2 \phi_2 + N_3 \phi_3 \quad (\text{in terms of global shapefns}) \\ &= N_1^e \phi_1 + N_2^e \phi_2 + N_3^e \phi_3\end{aligned}$$

$$N_1^e = \frac{1}{2A} [x_2 y_3 - x_3 y_2 + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$N_3^e = \dots$$

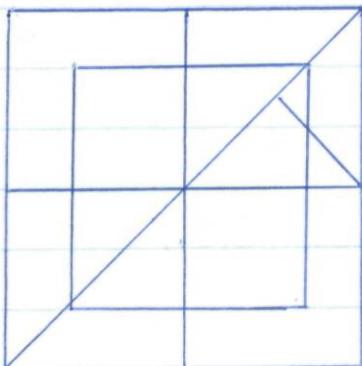
$$\Rightarrow \phi = \frac{8}{3} G \theta A \left(1 - \frac{x}{a}\right)$$



$$\begin{aligned}
 ②: \phi &= N_2\phi_2 + N_3\phi_3 + N_4\phi_4 && (\text{globally}) \\
 &= N_1^e\phi_2 + N_3^e\phi_3 + N_2^e\phi_4 && (\text{locally}) \\
 &= \frac{8}{3} G \Delta A \left(1 - \frac{x}{a}\right)
 \end{aligned}$$

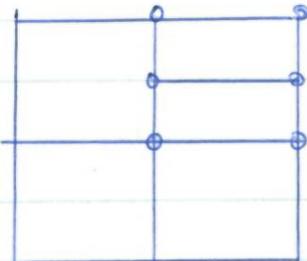
Note: $\phi^1 = \phi^2$ (in particular, cts across element bdy)
 ϕ^1 and ϕ^2 satisfy the b.c. at $x=a$ (zero)

Constant ϕ curves: $\phi = \text{const} \Rightarrow x = \text{const}$



Makes sense to this
linear order of approx.

Exercise:



Consider a mesh of 2
rectangular elements for a
quarter of the shaft's
X-section.

Assembly: $K = A^{1T} K^1 A^1 + A^{2T} K^2 A^2$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+ \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} K^2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

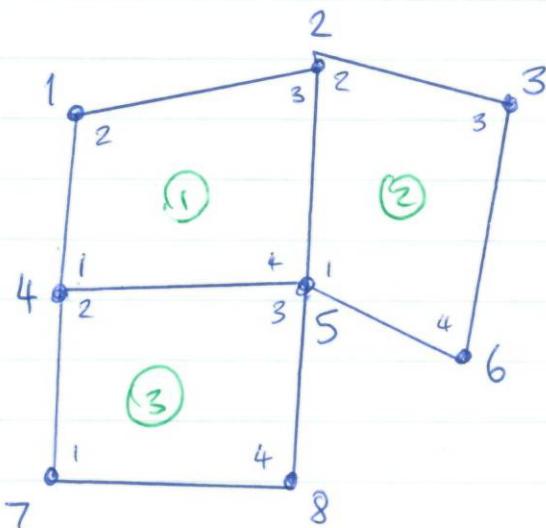
$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

More convenient way to encode the network topology is to use the connectivity matrix / table.

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \end{pmatrix}$$

↑ global nodes in order of local nodes
↓ vertically elements

Example



(as on sheet)

An Example of the Assembly Process

In Figure 2.12 is shown a small mesh containing three elements all of which have properties defined by (2.51). The problem is to assemble the element matrices into the complete system matrix. Assuming that there is only one unknown at each node the global node numbers will correspond to the global freedom numbers and the element node numbers will correspond to the local freedom numbers.

Each element possesses local freedom numbers (shown in parentheses) which follow the standard scheme, namely a consistent clockwise numbering.

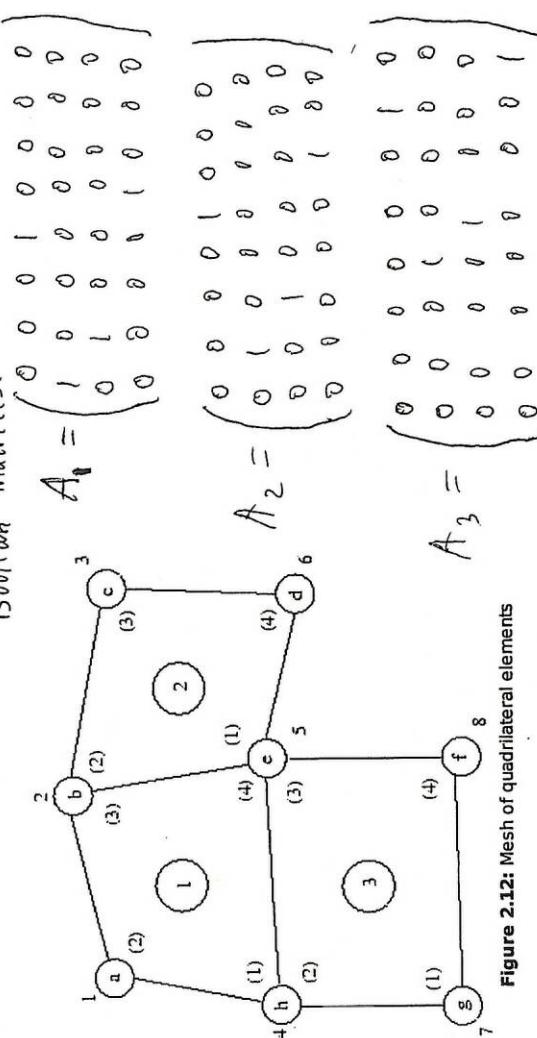


Figure 2.12: Mesh of quadrilateral elements

Each individual element equation can then be written as

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} \quad (2.52)$$

However, in the global freedom numbering system (not in parentheses) freedom 4 at node (h) corresponds to local freedom (1) in element 1 and local freedom (2) in element 3. In the assembled system matrix, term K_{11} from element 1 and K_{33}

(1) in element 3 would be added together and would appear in location (4,4) of the system matrix and so on. Similarly the contributions to global freedom 5 at node (e) will come from three elements. The assembly process is essentially additive. The only complication is in ensuring that the correct element contributions are added together. The total system matrix for Figure 2.12 is given in Table 2.3 where the superscripts refer to element numbers.

Table 2.3: System stiffness matrix for mesh Figure 2.12

K_{23}^1	K_{23}^0	0	K_{21}^1	K_{24}^1	0	0	0
$K_{32}^1 + K_{22}^2$	K_{22}^2	K_{31}^1	$K_{34}^1 + K_{21}^2$	K_{24}^2	0	0	0
0	K_{32}^2	K_{35}^1	0	K_{31}^2	K_{34}^2	0	0
K_{12}^1	K_{13}^1	0	$K_{11}^1 + K_{22}^3$	$K_{14}^1 + K_{23}^3$	0	K_{21}^3	K_{22}^3
K_{42}^1	$K_{43}^1 + K_{32}^2$	K_{13}^2	$K_{41}^1 + K_{32}^3$	$K_{44}^1 + K_{31}^2 + K_{33}^3$	K_{14}^2	K_{31}^3	K_{34}^3

0	K_{42}^2	K_{43}^2	0	K_{41}^2	K_{44}^2	0	0
0	0	0	K_{42}^3	K_{43}^3	0	K_{41}^3	K_{44}^3
0	0	0	K_{42}^3	K_{43}^3	0	K_{41}^3	K_{44}^3

This matrix will be symmetric if its constituent matrices are symmetric, and also possesses the useful property of bandedness. That is, the terms making up Table 2.3 are concentrated around the *leading diagonal* which stretches from top left to the lower right of the table. In fact no term in any row can be more than 4 locations removed from the leading diagonal so the system is said to have a *semi-bandwidth* of 5. The semi-bandwidth is usually denoted by HBAND.

The semi-bandwidth can be obtained by inspection of Figure 2.10 by subtracting lowest freedom number from the highest each element and adding one. Complex meshes have variable bandwidths and computer programs make use of banded when storing the system matrices.

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Next: Programming the Element Matrix Up: Assembly of System Matrices Previous: Assembly of System Matrices
Chris Greenough (c.greenough@rl.ac.uk): September 2001

Next | Up | Previous

Connectivity matrix:

$$C = \begin{pmatrix} 4 & 1 & 2 & 5 \\ 5 & 2 & 3 & 6 \\ 7 & 4 & 5 & 8 \end{pmatrix}$$

Suppose $K^1 = \begin{pmatrix} K_{11}^1 & \dots & K_{14}^1 \\ \vdots & \ddots & \vdots \\ K_{41}^1 & \dots & K_{44}^1 \end{pmatrix}$

The rows in C tell us where the entries K_{ij}^1 go in the global 8×8 stiffness matrix K .

$$K_{11}^1 \rightarrow K_{44} \text{ slot of } K$$

$$K_{12}^1 \rightarrow K_{41}$$

$$K_{13}^1 \rightarrow K_{42}$$

$$K_{14}^1 \rightarrow K_{45}$$

$$K_{21}^1 \rightarrow K_{14} \quad (\text{just relabels local to global node n's})$$

(see diagram opposite)

Algorithm for assembly

Let $\underline{\underline{K}} = \underline{\underline{0}}$ be the zero matrix

for $e = 1:N$

$(N = \# \text{elements})$

for $i = 1:n$

$(n = \# \text{nodes per element})$

for $j = i:n$ (by symmetry of $\underline{\underline{K}}$)

assumed const.

find global nodes p, q

corresponding to local nodes i, j

using C .

compute/retrieve K_{ij} and add
 this to K_{pq} (and K_{qp} if $p \neq q$)
 end
 end
 end

Note on accuracy and efficiency of the FE method

1D: $T = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \dots + \alpha_n x^{n-1}$
 for an n -node element

Taylor expansion of real T on element $[x_i, x_{i+1}]$
 about x_i :

$$\begin{aligned}
 T &= T(x_i) + \left. \frac{\partial T}{\partial x} \right|_{x=x_i} (x - x_i) + \frac{1}{2} \left. \frac{\partial^2 T}{\partial x^2} \right|_{x=x_i} (x - x_i)^2 \\
 &\quad + \dots + \frac{1}{(n+1)!} \left. \frac{\partial^{n+1} T}{\partial x^{n+1}} \right|_{x=\xi} (x - x_i)^{n+1}
 \end{aligned}$$

$\xi \in [x_i, x_{i+1}]$

Conclude: The error we're making is $O(h^{p+1})$ if we include all terms up to order p , and h is the size of the element. ($h = x_i - x_j$)

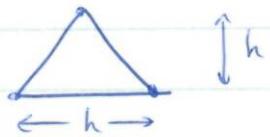
2D: $T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \dots$

Taylor expansion of real T on element
 about (x_i, y_i) :

$$\begin{aligned}
 T &= T(x_i, y_i) + \left. \frac{\partial T}{\partial x} \right|_{(x_i, y_i)} (x - x_i) + \left. \frac{\partial T}{\partial y} \right|_{(x_i, y_i)} (y - y_i) \\
 &\quad + \frac{1}{2} \left. \frac{\partial^2 T}{\partial x^2} \right|_{(x_i, y_i)} (x - x_i)^2 + \frac{1}{2} \left. \frac{\partial^2 T}{\partial y \partial x} \right|_{(x_i, y_i)} (x - x_i)(y - y_i) + \dots
 \end{aligned}$$

Conclude: The error we're making is $O(h^p)$ if all terms of order p are included, where h is a measure for the size of the element

(h is really about order, not too bothered about exact value)

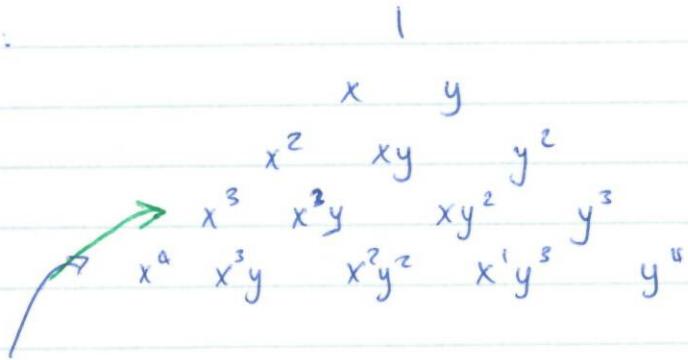


Triangular elements

3-node		$T = \alpha_1 + \alpha_2 x + \alpha_3 y$	error = $O(h^2)$
6-node		$T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$	error = $O(h^3)$

These elements are called order-complete (all terms up to certain order are included)

Pascal's triangle:

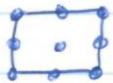


you need a 10-node element to be order complete up to x^4

Rectangular element

4-node		$T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$	error = $O(h^2)$
			\therefore not all quadratic terms are included

9-node
(Lagrange)



cannot contain all terms up
to cubic order \Rightarrow error = $O(h^3)$

note central node is useless

- (1) It doesn't increase accuracy (error still $O(h^3)$)
- (2) doesn't help conformity (i.e. for guaranteeing continuity across element boundaries) because it's not on the boundary

it only increases the dimension of the problem. Such a node is called parasitic and hence it is often left out and the 8-node serendipity element is considered instead

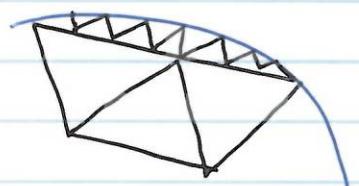
Conclude: There are two ways of increasing accuracy:

(1) h -method : reduce h , i.e. finer mesh, taking smaller elements

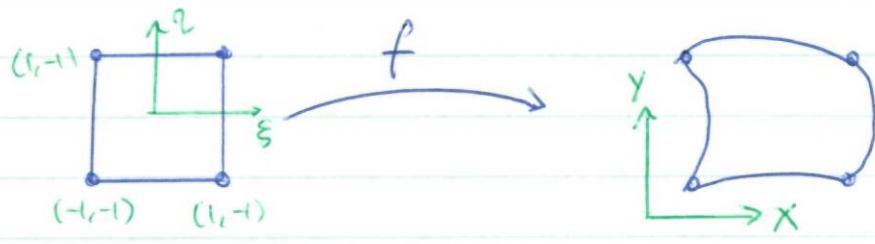
(2) p -method : introduce more nodes per element and construct higher-order shape functions (increasing p)

Iso-parametric elements

To deal with curved boundaries of the domain we could



- (1) refine the mesh (at computational cost)
- (2) construct elements with curved edges (isoparametric elements)



parent
or natural
element

distorted
or physical
element

We require that f is:

- (1) continuous
- (2) locally one-to-one, i.e. Jacobian must be non-singular

$$X = X(\xi, \eta)$$

$$Y = Y(\xi, \eta)$$

$$|\mathcal{J}| = \begin{vmatrix} \frac{\partial X}{\partial \xi} & \frac{\partial X}{\partial \eta} \\ \frac{\partial Y}{\partial \xi} & \frac{\partial Y}{\partial \eta} \end{vmatrix} \neq 0$$

- (3) globally one-to-one

(usually OK if element not too distorted, and we'll assume it to be OK)

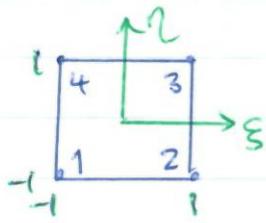
How to choose this mapping ??



use SHAPE FUNCTIONS

(hence the name isoparametric: same functions used for interpolation and transformation).

4-node isoparametric quadrilateral element



$$N_1^e = \frac{1}{4}(\xi - 1)(\eta - 1)$$

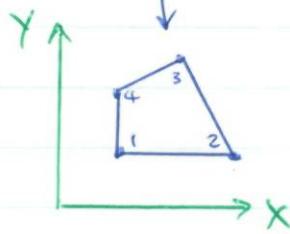
$$N_2^e = -\frac{1}{4}(\xi + 1)(\eta - 1)$$

$$N_3^e = \frac{1}{4}(\xi + 1)(\eta + 1)$$

$$N_4^e = -\frac{1}{4}(\xi - 1)(\eta + 1)$$

these are special cases of the 4-node rectangular element

$$N_i^e(x_j, y_j) = \delta_{ij}$$



Transformation f :

$$X = X(\xi, \eta) = \sum_{i=1}^4 N_i^e(\xi, \eta) X_i$$

$$Y = Y(\xi, \eta) = \sum_{i=1}^4 N_i^e(\xi, \eta) Y_i$$

Consider node 1: $X(-1, 1) = X, \quad \left. Y \right|_{(-1, 1)} = Y, \quad \left. \right\} \text{ and some for other nodes}$

maps nodes onto nodes

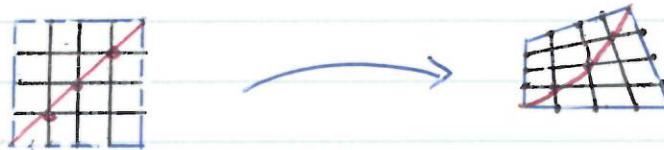
Consider edge $\xi = -1$: $X = X(-1, \eta) = -\frac{1}{2}(\eta - 1)X_1 + \frac{1}{2}(\eta + 1)X_4$
 $Y = Y(-1, \eta) = -\frac{1}{2}(\eta - 1)Y_1 + \frac{1}{2}(\eta + 1)Y_4$

Eliminating η gives a linear relationship between X and Y ($Y = aX + b$).

and some for other edges

\Rightarrow edges map onto edges

Warning: horizontal + vertical lines are mapped onto straight lines but diagonal lines are not.



this is because shape fn's are bilinear but not linear.

The isoparametric element is conforming because the square parent element is. (It has its edges parallel to the ξ and η axes). The mapping preserves this by continuity.

In particular this solves the problem that nonaligned rectangular elements are not conforming

Now (with these isoparametric shape functions) a mesh of such elements is conforming

Question: What about the uniqueness of this mapping?

$$X = N_1^e X_1 + N_2^e X_2 + N_3^e X_3 + N_4^e X_4$$

$$Y = N_1^e Y_1 + N_2^e Y_2 + N_3^e Y_3 + N_4^e (\xi, \eta) Y_4$$

$$\rightarrow X = \frac{1}{4} [(-\xi - \eta + 1 + \xi\eta) X_1 + (\xi\eta + 1 - \xi\eta) X_2]$$

$$Y = \frac{1}{4} [\dots]$$

$$J = \begin{pmatrix} \frac{\partial X}{\partial \xi} & \frac{\partial X}{\partial \eta} \\ \frac{\partial Y}{\partial \xi} & \frac{\partial Y}{\partial \eta} \end{pmatrix} \Rightarrow \det J = c_1 + c_2 \xi + c_3 \eta$$

↗
note quadratic terms cancel

Consequence: If $\det J$ has the same sign at all 4 nodes then $\det J$ cannot vanish on the element \Rightarrow uniqueness!

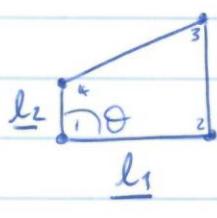
At node 1 we have $\xi = -1, \eta = -1$

$$\begin{aligned} \frac{\partial X}{\partial \xi} &= \frac{1}{4}(-x_1 + x_2 + x_3 + x_4) + \frac{1}{4}(x_1 - x_2 + x_3 + x_4)\eta \\ \left. \begin{array}{l} \eta = -1 \\ \xi = -1 \end{array} \right\} \Rightarrow &= \frac{1}{2}(x_2 - x_1) \end{aligned}$$

Similarly $\frac{\partial Y}{\partial \eta} = \frac{1}{2}(y_4 - y_1)$

$$\det J = \frac{1}{4} [(x_2 - x_1)(y_4 - y_1) - (y_2 - y_1)(x_4 - x_1)]$$

Geometric interpretation:



$$\underline{l}_1 = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

$$\underline{l}_2 = \begin{pmatrix} x_4 - x_1 \\ y_4 - y_1 \end{pmatrix}$$

$$\text{Then } \det J = \frac{1}{4} \underline{l}_1 \times \underline{l}_2$$

$$= \frac{1}{4} |\underline{l}_1| |\underline{l}_2| \sin \theta > 0 \text{ if } \theta < \pi.$$

Conclude:

$\det J$ is nonvanishing on the element
 \Updownarrow

the quadrilateral is convex, i.e. if all the angles are less than π .

Example of a singular Jacobian

Isoparametric transformation:

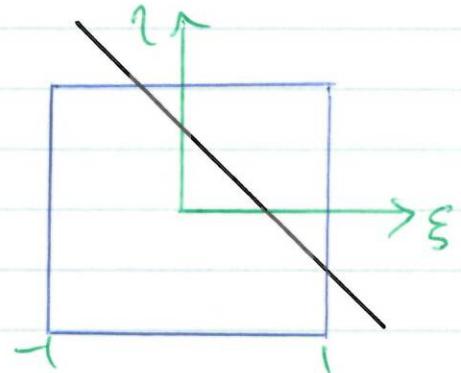
$$X = \sum_{i=1}^N N_i^e(\xi, \eta) X_i$$

$$Y = \sum_{i=1}^N N_i^{-\alpha} (\xi_{i,n}) Y_i$$

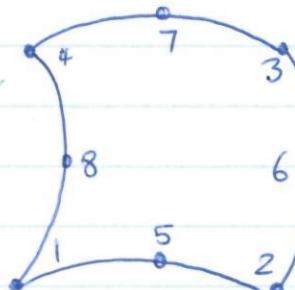
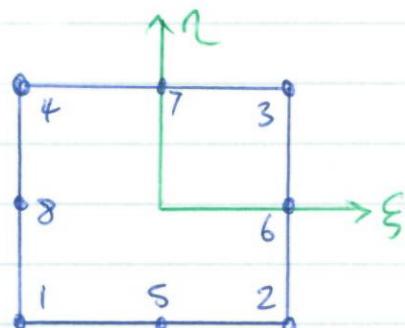
$$J = \begin{pmatrix} \frac{\partial X}{\partial \xi} & \frac{\partial X}{\partial \eta} \\ \frac{\partial Y}{\partial \xi} & \frac{\partial Y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4}(1-3\eta) & \frac{1}{4}(1-3\xi) \end{pmatrix}$$

$$\Rightarrow \det J = \frac{1}{8} (2 - 3\xi - 3\eta) \quad \leftarrow \text{this changes sign on } -1 \leq \xi, \eta \leq 1$$

namely on $\eta = -\xi + \frac{2}{3}$



8-node isoparametric quadrilateral element

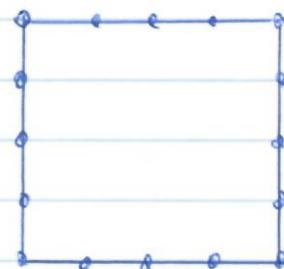


Shape functions:

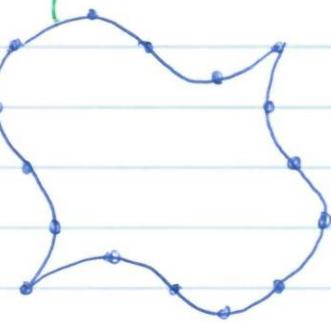
$$N_i^e = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \quad \text{biquadratic.}$$

We can use this 8-node element to map quadratic boundary exactly:

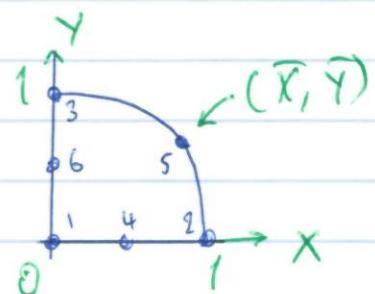
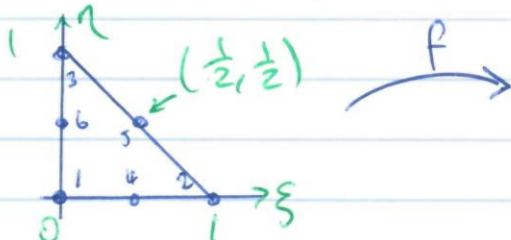
higher order:



cubic boundaries



Restriction on node placement



6-node triangular parent element

$$\begin{aligned} f : X &= \xi + (4\bar{X} - 2)\xi_2 & (= \sum N_i^e X_i) \\ Y &= \eta + (4\bar{Y} - 2)\xi_2 & (= \sum N_i^e Y_i) \end{aligned}$$

note: $X(\frac{1}{2}, \frac{1}{2}) = \bar{X}$
 $Y(\frac{1}{2}, \frac{1}{2}) = \bar{Y}$

$$\det J = \begin{vmatrix} 1 + (4\bar{X}-2)\eta & (4\bar{X}-2)\xi \\ (4\bar{Y}-2)\eta & 1 + (4\bar{Y}-2)\xi \end{vmatrix}$$

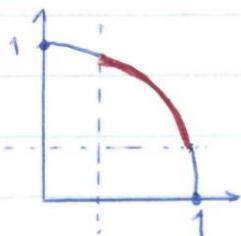
$$= 1 + (4\bar{x} - 2)\eta + (4\bar{y} - 2)\xi \quad \text{linear!}$$

at node 1: $\xi = 0, \eta = 0 \Rightarrow \det J = 1$

at node 2: $(1, 0) : 4\bar{y} - 2 > -1 \Rightarrow \bar{y} > \frac{1}{4}$

at node 3: $(0, 1) : 4\bar{x} - 2 > -1 \Rightarrow \bar{x} > \frac{1}{4}$

node needs to be placed in sector for well-behaved map



② Transformation of the stiffness matrix

$$K_{ij} = \iint \left[\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right] dx dy$$

Now, $N_i^- = N_i(\xi, \eta)$ so we need to transform $(x, y) \rightarrow (\xi, \eta)$

[comes from $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = c$ (Poisson's eqn)]

More generally,

$$a_{11} \frac{\partial^2 \phi}{\partial x^2} + 2a_{12} \frac{\partial^2 \phi}{\partial x \partial y} + a_{22} \frac{\partial^2 \phi}{\partial y^2}$$

Weak formulation

$$\Rightarrow K_{ij} = \iint \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y} \right) D \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} dx dy$$

$$\text{where } D = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \quad D = D(x, y)$$

Jacobian of isoparametric transformation

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \xi}$$

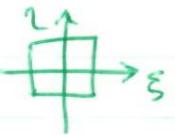
$$\frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \eta}$$

$$\begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} = J^T \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix}$$

Invert:

$$\begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} = (J^T)^{-1} \begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix}$$

$$\Rightarrow K_{ij} = \iint_{-1-1}^1 \left(\frac{\partial N_i}{\partial \xi}, \frac{\partial N_i}{\partial \eta} \right) J^{-1} D (J^T)^{-1} \begin{pmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{pmatrix} |\det J| d\xi d\eta$$

↑ assuming a square 

Transformation of the load vector

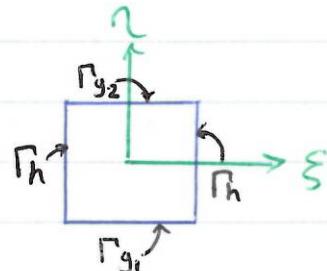
$$f_{e_i} = \iint_e N_i(\xi, \eta) Q(x(\xi, \eta), y(\xi, \eta)) |\det J| d\xi d\eta$$

Transformation of the boundary vector

[for Poisson: $f_{b_i} = \int_{\Gamma} N_i \frac{d\phi}{dn} ds$]

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta$$

Consider square element



along Γ_g : $d\eta = 0 \Rightarrow dx = \frac{\partial x}{\partial \xi} d\xi$

$$dy = \frac{\partial y}{\partial \xi} d\xi$$

$$\Rightarrow \int_{\Gamma_g} f(x, y) ds = \int_{-1}^1 f(x(\xi, \eta), y(\xi, \eta)) \left[\left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 \right]^{1/2} d\xi$$

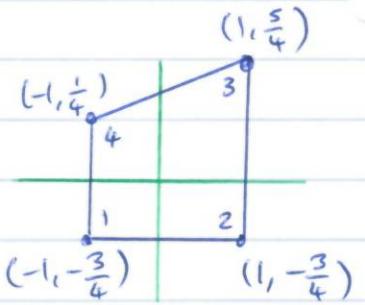
along Γ_h :

$$\int_{\Gamma_h} f(x, y) ds = \int_{-1}^1 f(x(\xi, \eta), y(\xi, \eta)) \left[\left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2 \right]^{1/2} d\eta$$

"The calculations are straightforward but you need to know what you're doing"

Example :

Q: Compute stiffness matrix for Laplacian $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ on this element



$$N_1 = \frac{1}{4}(\xi-1)(\eta-1)$$

Transformation :

$$x = \sum_{i=1}^4 N_i x_i \quad y = \sum_{i=1}^4 N_i (\xi, \eta) y_i$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{4}(\eta+1) & \frac{1}{4}(\xi+3) \end{pmatrix}$$

$$\Rightarrow \det J = \frac{1}{4}(\xi+3)$$

$$J^{-1} = \frac{4}{\xi+3} \begin{pmatrix} \frac{1}{4}(\xi+3) & 0 \\ -\frac{1}{4}(\eta+1) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\eta+1}{\xi+3} & \frac{4}{\xi+3} \end{pmatrix}$$

$$\Rightarrow K_{11} = \iint_D \left(\frac{\partial N_1}{\partial x}, \frac{\partial N_1}{\partial y} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{pmatrix} dx dy$$

$D \leftarrow$ from Laplacian

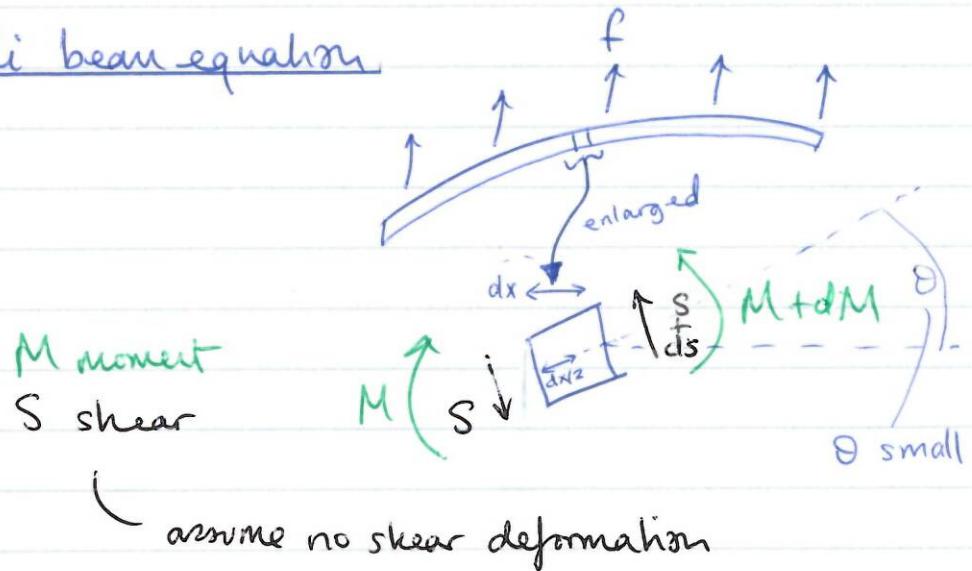
$$= \iint_{-1-1}^{1-1} \left(\frac{\partial N_1}{\partial \xi}, \frac{\partial N_1}{\partial \eta} \right) J^{-1} D(J^T)^{-1} \begin{pmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{pmatrix} |\det J| d\xi d\eta$$

$$= \int_{-1}^1 \left[\left[\frac{1}{4}(\eta-1) - \frac{(\eta+1)(\xi-1)}{4(\xi+3)} \right]^2 + \left[\frac{\xi-1}{\xi+3} \right]^2 \right] \frac{\xi+3}{4} d\xi d\eta$$

polynomial in η_1 then rational in ξ .

FOURTH ORDER EQUATIONS

Euler-Bernoulli beam equation



force balance in vertical direction:

$$(S+ds) \cos\theta - S \cos\theta + f dx = 0$$

force per unit length

$$\rightarrow \boxed{\frac{dS}{dx} + f = 0} \quad \text{if } \theta \text{ is small}$$

Moment balance (about axis through centre of mass)

$$(M+dM) - M + \underbrace{S \frac{dx}{2}}_{\text{Force dist}} + (S+ds) \frac{dx}{2} = 0$$

$$\rightarrow \boxed{\frac{dM}{dx} + S = 0}$$

$$\Rightarrow \boxed{\frac{d^2M}{dx^2} = f}$$

Constitutive relation:

$$M = EI \frac{d^2y}{dx^2}$$

bending \nwarrow
 Young's modulus \downarrow
 area moment of inertia \searrow
 curvature

Conclusion: $\frac{d^2}{dx^2} \left(EI \frac{d^2y}{dx^2} \right) = f$

Euler-Bernoulli Beam Equation
statics case

Possible b.c.s:

(1)



"Simply supported"
hinged
pinned

$$y(0) = 0$$

$$y(L) = 0$$

$$\frac{d^2y}{dx^2}(0) = 0$$

(no bending moment)

$$\frac{d^2y}{dx^2}(L) = 0$$

(2)



"cantilever beam"

fixed/clamped/welded at $x=0$

$$y(0) = 0$$

free at $x=L$

$$\frac{dy}{dx}(0) = 0$$

$$\frac{d^2y}{dx^2}(L) = 0 \quad (\text{no bending moment at end})$$

[Shear force $S = -\frac{dM}{dx}$]

$$\frac{d^3y}{dx^3}(L) = 0 \quad (\text{no force} \Rightarrow -\frac{d^3}{dx^3}S = -\frac{dM}{dx}).$$

Weak formulation

$$\int_0^L \left(\frac{d^2M}{dx^2} v - fv \right) dx = 0 \quad (v \text{ test f?})$$

by parts {

$$-\int_0^L \left[\frac{dM}{dx} \frac{dv}{dx} + fv \right] dx + \left[v \frac{dM}{dx} \right]_0^L = 0$$

by parts {

$$\int_0^L \left[M \frac{d^2v}{dx^2} - fv \right] dx + \left[v \frac{dN}{dx} - \frac{dv}{dx} M \right]_0^L = 0$$

\uparrow
 $-S$

$$\text{Interpolation: } y = \sum_j N_j(x) u_j \text{ require}$$

$$\text{Galerkin: } v = N_i$$

\Rightarrow FE equation becomes

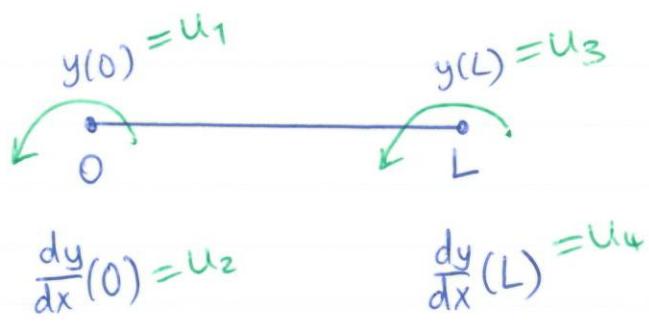
$$\sum_j \int_0^L \frac{d^2N_i}{dx^2} EI \frac{d^2N_j}{dx^2} dx u_j$$

$$= \int_0^L f N_i dx + \left[\frac{dN_i}{dx} M \right]_0^L + [N_i S]_0^L$$

This makes sense if we require $\frac{dN_i}{dx}$ to be continuous and piecewise differentiable.



Idea: interpolate derivatives as well as the function itself



Two degrees of freedom per node

- displacement y
- slope $\frac{dy}{dx}$

Approximation: $y(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$ (4 data points)

For consistency: $y(0) = u_1$ $y(L) = u_3$
 $\frac{dy}{dx}(0) = u_2$ $\frac{dy}{dx}(L) = u_4$

4 equations for α_i , $i=1, \dots, 4$.

$$y(x) = \sum_{i=1}^4 N_i(x) u_i$$

Shape fns: $N_1^e = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}$

$$N_2^e = x - \frac{2x^2}{L} + \frac{x^3}{L^2}$$

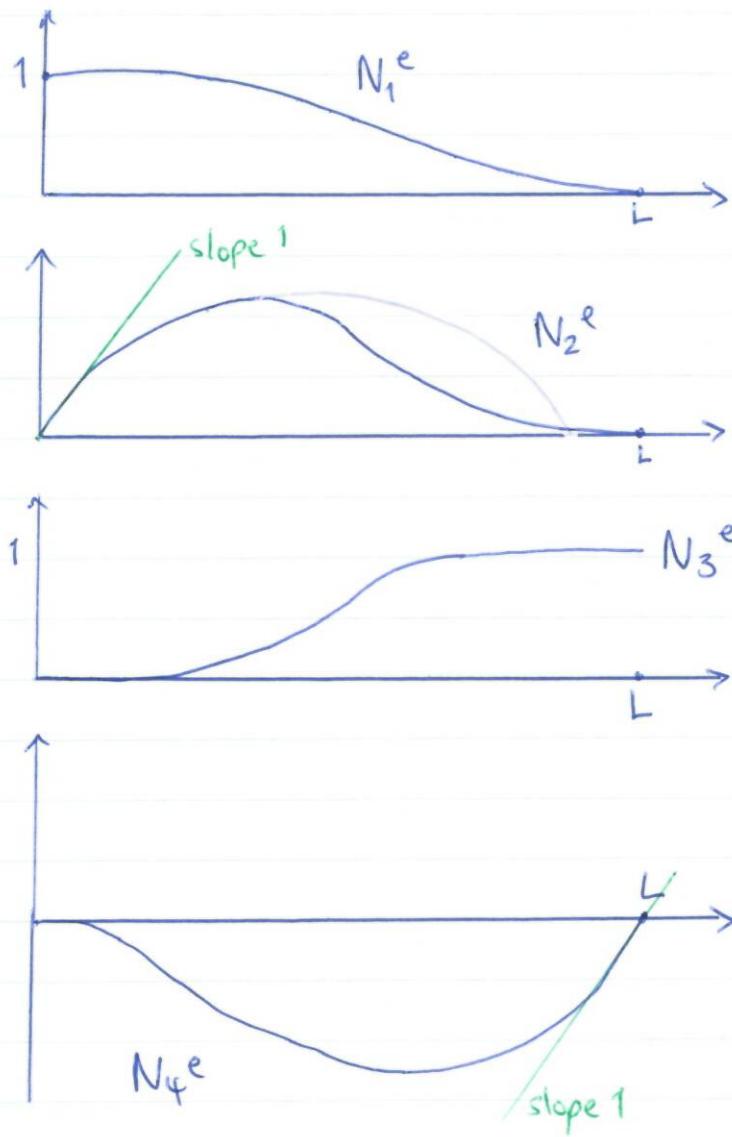
$$N_3^e = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$$

$$N_4^e = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

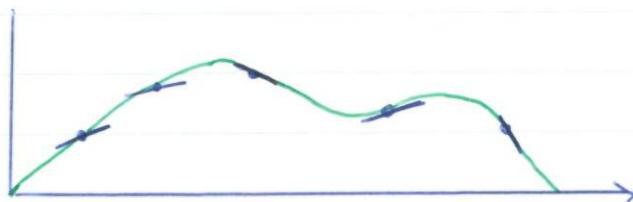
Note that $N_{odd}^e(x_j) = \delta_{ij}$ where $N_{odd}^e = N_{2i-1}^e$

$$\frac{dN_{even}^e}{dx}(x_j) = \delta_{ij} \quad \text{where } N_{even}^e = N_{2i}^e$$

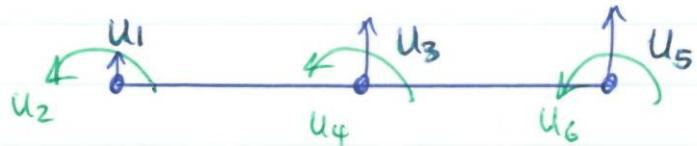
Graphically,



Polynomial approximations of f^n 's that fulfil not only the f^n values at given pts but also the slopes at these pts are called Hermite polynomials.



Higher-order shape fns



$$y(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 x^4 + \alpha_6 x^5$$

Evaluation of the element data (for uniform beam,
ie $EI = \text{const}$, $f = \text{const}$)

Stiffness matrix

$$K_{ij} = \int_0^L \frac{d^2 N_i}{dx^2} EI \frac{d^2 N_j}{dx^2} dx$$

$$\begin{aligned} K_{12} &= EI \int_0^L \left(-\frac{6}{L^2} + \frac{12x}{L^3} \right) \left(-\frac{4}{L} + \frac{6x}{L^2} \right) dx \\ &= \frac{6EI}{L^2} \end{aligned}$$

etc

to find $K = \frac{EI}{L^3} \begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}$

Load vector

$$f_{ei} = \int_0^L f N_i dx$$

to find

$$f_e = \frac{fL}{12} \begin{pmatrix} 6 \\ L \\ 6 \\ -L \end{pmatrix}$$

boundary vector

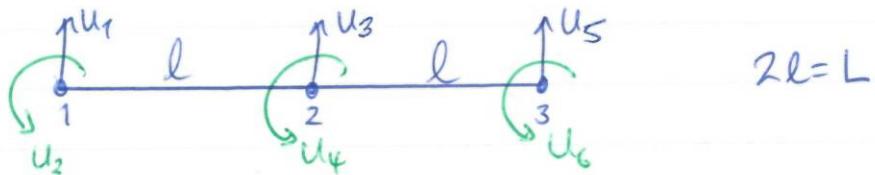
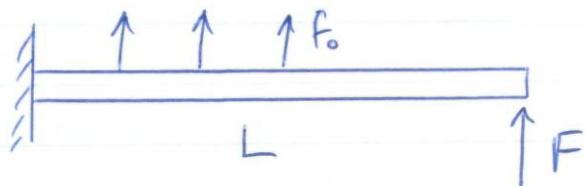
$$f_{b_i} = [N_i S]_0^L + \left[\frac{dN_i}{dx} M \right]_0^L$$

to find

$$f_b = \begin{pmatrix} -S(0) \\ -M(0) \\ S(L) \\ M(L) \end{pmatrix}$$

Lots of these shapes
 f_b 's are 0 or have
0 derivative at $x=0$ or L .

Example (2 element model for a cantilever beam
with an end force)



BCs : $y(0) = 0$
 $y'(0) = 0$
 $M(L) = 0$
 $S(L) = F$

$$K^1 = K^2 = \frac{EI}{l^3} \begin{pmatrix} 12 & 6l & \dots & \dots \\ 6l & 4l^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \end{pmatrix}$$

Assembly

Connectivity matrix

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{pmatrix} \quad \begin{matrix} \leftarrow e=1 \\ \leftarrow e=2 \end{matrix}$$

Note: we count the degrees of freedom rather than the nodes.

$$K = \frac{EI}{l^3} \begin{pmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l^2 & 12+12 & -6l+6l & -12 & 6l \\ 6l & 2l^2 & -6l+6l & 4l^2+4l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{pmatrix}$$

↑ global stiffness matrix ↑

load vector: $f_e^1 = f_e^2 = \frac{fl}{12} \begin{pmatrix} 6 \\ l \\ 6 \\ -l \end{pmatrix}$

$$\rightarrow f_e = \begin{pmatrix} 6 \\ l \\ 6+6 \\ -l+l \\ 6 \\ l \end{pmatrix}$$

boundary vector:

$$f_b = \begin{pmatrix} -S(0) \\ -M(0) \\ 0+0 \\ 0+0 \\ S(L) \\ M(L) \end{pmatrix}$$

makes sense
node 2 is not on the body.

Solving the FE eqn:

$$Ku = f_e + f_b \quad \text{ie}$$

$$\frac{EI}{l^3} \begin{pmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & 6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = \frac{fl}{12} \begin{pmatrix} 6 \\ l \\ 12 \\ 0 \\ 6 \\ -l \end{pmatrix} + \begin{pmatrix} -S_0 \\ -N_0 \\ 0 \\ 0 \\ S(L) = F \\ M(L) = 0 \end{pmatrix}$$

equivalent (by reducing) to 4x4 problem:

$$\frac{EI}{l^3} \begin{pmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} fl \\ 0 \\ \frac{1}{2}fl + F \\ -\frac{fl^2}{12} \end{pmatrix}$$

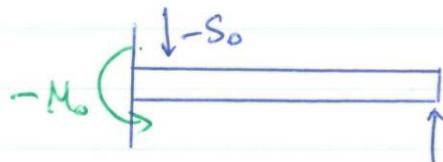
Solve to give u_3, u_4, u_5, u_6 .

Then the first 2 eqns (of the 6x6 system) give:

$$\left. \begin{aligned} \frac{EI}{l^3} (-12u_3 + 6lu_4) &= \frac{1}{2}fl - S_o \\ \frac{EI}{l^3} (-6lu_3 + 2l^2u_4) &= \frac{1}{12}fl^2 - M_o \end{aligned} \right\} \Rightarrow \begin{array}{l} \text{solve} \\ \text{to give} \\ S_o, M_o \end{array}$$

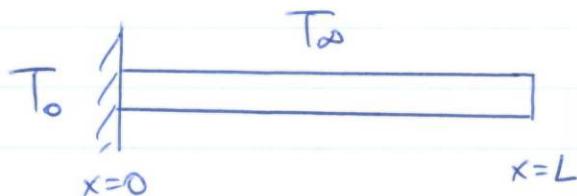
S_o is the force exerted by the beam on the wall at $x=0$, which is balanced by an equal and opposite reaction force $-S_o$ on the beam.

$-M_o$ is the reaction moment (similarly) on the beam at $x=0$.
(the fixed end)



Nonlinear problems

Consider radiative heat transfer in a fin, which goes as T^4 .



$$Ak \frac{d^2T}{dx^2} - A_s \sigma (T^4 - T_\infty^4) + Q = 0$$

$$T_0 = T(0).$$

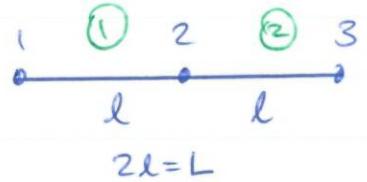
$$q(L) = -k \frac{dT}{dx}(L) = \sigma (T^4(L) - T_\infty^4)$$

where σ is the Stefan-Boltzmann constant

T_{∞} is the ambient reference temperature

A_s is the total lateral surface area

Take mesh of two 2-node elements.



Weak formulation:

$$\int_0^L \left(-\frac{dT}{dx} AK \frac{dv}{dx} - A_s \sigma T^4 v + \bar{Q}v \right) dx - [Aqv]_0^L = 0$$

$$\text{where } \bar{Q} = Q + A_s \sigma T_{\infty}^4.$$

For single element:

$$T = \sum_j N_j T_j, \quad v = N_i$$

$$\Rightarrow \text{FE eqn: } [Aqv]_0^L - \sum_j \int_0^L AK \frac{dN_i}{dx} \frac{dN_j}{dx} T_j dx \\ + \int_0^L \bar{Q}N_i dx - \int_0^L A_s \sigma N_i \left(\sum_j N_j T_j \right)^4 dx = 0.$$

$$\left. \begin{array}{l} N_1 = 1 - \frac{x}{l} \\ N_2 = \frac{x}{l} \end{array} \right\} \Rightarrow K = \frac{AK}{l} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$f_e = \frac{\bar{Q}l}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus:

$$-\frac{Ak}{l} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \frac{\bar{Q}l}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - A_s \sigma \left(\begin{array}{l} \int_0^l N_1 (N_1 T_1 + N_2 T_2)^4 dx \\ \int_0^l N_2 (N_1 T_1 + N_2 T_2)^4 dx \end{array} \right) + \begin{pmatrix} [-A_s N_1]_0^l \\ [-A_s N_2]_0^l \end{pmatrix} = 0$$

Extra nonlinear term

$$-A_s \sigma \begin{pmatrix} \int N_1^5 & 4 \int N_1^4 N_2 & 6 \int N_1^3 N_2^2 & 4 \int N_1^2 N_2^3 & \int N_1 N_2^4 \\ \int N_1^4 N_2 & 4 \int N_1^3 N_2^2 & 6 \int N_1^2 N_2^3 & 4 \int N_1 N_2^4 & \int N_2^5 \end{pmatrix} \begin{pmatrix} T_1^4 \\ T_1^3 T_2 \\ T_1^2 T_2^2 \\ T_1 T_2^3 \\ T_2^4 \end{pmatrix}$$

$$\text{Use integration formula } \int_0^l N_1^n N_2^m dx = \frac{n! m!}{(n+m+1)!} l$$

$$= -l A_s \sigma \begin{pmatrix} \frac{1}{6} & \frac{2}{15} & \frac{1}{10} & \frac{1}{15} & \frac{1}{30} \\ \frac{1}{30} & \frac{1}{15} & \frac{1}{10} & \frac{2}{15} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} T_1^4 \\ T_1^3 T_2 \\ T_1^2 T_2^2 \\ T_1 T_2^3 \\ T_2^4 \end{pmatrix}$$

Assembly

$$\frac{Ak}{l}$$

Assembly

$$\frac{4k}{l} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} + lA_{S0}$$

$$+ lA_{S0} \begin{pmatrix} \frac{1}{6} & \frac{2}{15} & \frac{1}{10} & \frac{1}{15} & \frac{1}{30} & 0 & 0 & 0 & 0 \\ \frac{1}{30} & \frac{1}{15} & \frac{1}{10} & \frac{2}{5} & \frac{1}{6} + \frac{1}{6} & \frac{2}{5} & \frac{1}{10} & \frac{1}{15} & \frac{1}{30} \\ 0 & 0 & 0 & 0 & \frac{1}{30} & \frac{1}{15} & \frac{1}{10} & \frac{2}{15} & \frac{1}{6} \end{pmatrix}$$

$$\begin{pmatrix} T_1^4 \\ T_1^3 T_2 \\ T_1^2 T_2^2 \\ T_1 T_2^3 \\ T_2^4 \\ T_2^3 T_3 \\ T_2^2 T_3^2 \\ T_2 T_3^3 \\ T_3^4 \end{pmatrix}$$

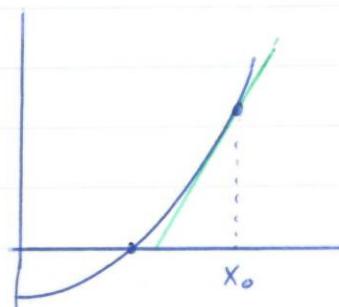
$$= \frac{\bar{Q}l}{2} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} Aq(0) \\ 0 \\ -Aq(L) \end{pmatrix}$$

b.c.s $T_1 = T_0$

$$q(L) = \sigma(T_3^4 - T_a^4)$$

Strategy: Solve 2nd + 3rd eqⁿ for T_2, T_3
then the first eqⁿ for $q(0)$.

Can solve nonlinear system of algebraic eqⁿ's using
Newton-Raphson.



tangent at stage k of the iteration is determined by the slope $f'(x_k)$ and the point $(x_k, f(x_k))$.

$$y - f(x_k) = f'(x_k)(x - x_k)$$

x_{k+1} is then defined as that x s.t. $y=0$.

$$0 - f(x_k) = f'(x_k)(x_{k+1} - x_k)$$

Solve for x_{k+1} :

$$x_{k+1} = x_k - \underbrace{\frac{f(x_k)}{f'(x_k)}}$$

It can be shown that, if \bar{x} is the exact solⁿ,

$$|\bar{x} - x_{k+1}| \leq \alpha |\bar{x} - x_k|^2$$

where α is a constant independent of k , so the convergence is quadratic

\Rightarrow in each step, you double the n° of correct digits.

Time-dependent problems

Heat flow: $\rho c A \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} (A k \frac{\partial T}{\partial x}) - \underbrace{\alpha (T - T_a)}_{\text{convective term}} + Q$

ρ mass density

c specific heat

T_a ambient temperature

A area of cross-section

Weak formulation:

$$\int_0^L \left[-Ak \frac{\partial T}{\partial x} \frac{\partial v}{\partial x} - \alpha T v + \bar{Q} v - \rho c A \frac{\partial T}{\partial t} v \right] dx + \left[Ak \frac{\partial T}{\partial x} v \right]_0^L = 0$$

$$T = \sum_j N_j T_j, \quad v = N_i \quad (\text{Galerkin})$$

$$\bar{Q} = Q + \alpha T_a$$

2-node element.

FE eqn:

$$\begin{aligned} & -Ak \sum_j \int_0^L \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dx T_j - \alpha \sum_j \int_0^L N_i N_j dx T_j + \bar{Q} \\ & + \bar{Q} \int_0^L N_i dx - \rho c A \sum_j N_i N_j dx \dot{T}_j \\ & + [-A_q N_i]_0^L = 0 \end{aligned}$$

$$Ak \int_0^L \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dx \rightarrow \frac{Ak}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\int_0^L N_i N_j dx \rightarrow \begin{pmatrix} \int N_1^2 & \int N_1 N_2 \\ \int N_1 N_2 & \int N_2^2 \end{pmatrix} = \frac{L}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\bar{Q} \int_0^L N_i dx \rightarrow \frac{\bar{Q}L}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus:

$$\begin{aligned} \frac{gcAL}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \frac{Ak}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \\ + \frac{\alpha L}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \frac{QL}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -Aq(0) \\ Aq(L) \end{pmatrix} \end{aligned}$$

$$\text{or } Mu + Ku = f_e + f_b$$

where M is the "mass matrix"

$$M = \frac{gcAL}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} .$$

and K is the stiffness matrix plus the advective term

$$K = \frac{Ak}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{\alpha L}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

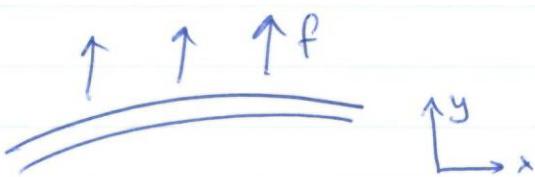
and u contains our unknown temperatures

$$u = (T_1, T_2) .$$

Beam dynamics

$$\underbrace{\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right)}_{\tilde{M}} + gA \frac{\partial^2 y}{\partial t^2} = f$$

Euler-Bernoulli
beam eqⁿ



Weak formulation (requires 2 integrations by parts):

$$\begin{aligned} & \int_0^L EI \frac{\partial^2 y}{\partial x^2} \frac{\partial^2 v}{\partial x^2} dx + \int_0^L gA \frac{\partial^2 y}{\partial t^2} dx \\ &= \int_0^L f v dx + [Sv]_0^L + [\tilde{M} \frac{\partial v}{\partial x}]_0^L \end{aligned}$$

$$y = \sum_j N_j u_j, \quad v = N_i$$

Extra dynamics term:

$$gA \sum_j \int_0^L N_i N_j dx \ddot{u}_j$$

Take single 2-node element



$$N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}$$

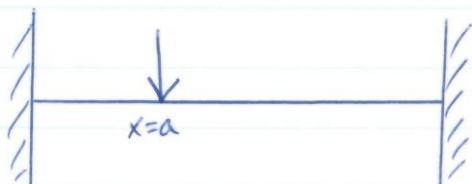
\Rightarrow mass matrix M is

$$M = \frac{\rho AL}{420} \begin{pmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{pmatrix}$$

\Rightarrow FE eqⁿ:

$$M\ddot{u} + Ku = f_e + f_b \quad (2^{\text{nd}} \text{ order ODE}).$$

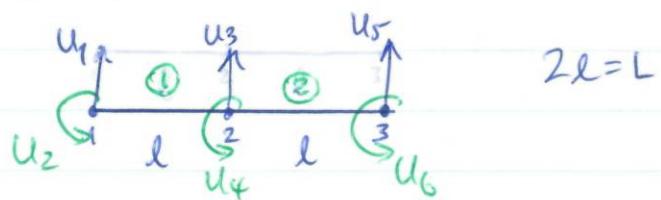
Example (periodically forced fixed-fixed beam)



$$f(x,t) = P \sin \omega t \delta(x-a)$$

(assume $a < l$, i.e. f acts on the first element)

Consider a 2-element mesh



← mass matrix

$$K^1 = K^2, \quad M^1 = M^2$$

$$f_{e_i}^1 = \int_0^l f N_i dx$$

$$= P \sin \omega t \int_0^l \delta(x-a) N_i(x) dx$$

$$= P N_i(a) \sin \omega t$$

$$f_{\text{ext}}^2 = 0$$

$$K^1 = K^2 = \frac{EI}{l^3} \begin{pmatrix} 12 & 6l & \dots & \dots \\ 6l & 4l^2 & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Assembly:

$$\begin{pmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & \begin{pmatrix} 156 & 22L \\ +156 & -22L \end{pmatrix} & 54 & -13L \\ -13L & -3L^2 & \begin{pmatrix} 22L & 4L^2 \\ -22L & +4L^2 \end{pmatrix} & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{pmatrix} \quad \begin{array}{l} u_1 \neq 0 \\ u_2 = 0 \\ u_3 \\ u_4 \\ u_5 = 0 \\ u_6 \neq 0 \end{array}$$

$$= P \sin \omega t \begin{pmatrix} N_1(a) \\ N_2(a) \\ N_3(a) \\ N_4(a) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -S(0) \\ -M(0) \\ 0 \\ 0 \\ S(L) \\ M(L) \end{pmatrix}$$

BCs: fixed-fixed : $u_1 = 0 = u_2$
 $u_5 = 0 = u_6$ (green above)

allows us to cross out first and last 2 rows + columns of the matrix :

Reduced eqⁿ:

$$M = \frac{gAl}{420} \begin{pmatrix} 156+156 & -22l+22l \\ 156+156 & 4l^2+4l^2 \\ -22l+22l & 4l^2+4l^2 \end{pmatrix}$$

$$= \frac{gAl}{105} \begin{pmatrix} 78 & 0 \\ 0 & 2l^2 \end{pmatrix}$$

$$K = \frac{EI}{l^3} \begin{pmatrix} 12+12 & -6l+6l \\ -6l+6l & 4l^2+4l^2 \end{pmatrix}$$

$$= \frac{8EI}{l^3} \begin{pmatrix} 3 & 0 \\ 0 & l^2 \end{pmatrix}$$

Thus:

$$\frac{gAl}{105} \begin{pmatrix} 78 & 0 \\ 0 & 2l^2 \end{pmatrix} \begin{pmatrix} \ddot{u}_3 \\ \ddot{u}_4 \end{pmatrix} + \frac{8EI}{l^3} \begin{pmatrix} 3 & 0 \\ 0 & l^2 \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = P_s \sin \omega t \begin{pmatrix} N_3(a) \\ N_4(a) \end{pmatrix}$$

Solve by trial solⁿ's $\begin{pmatrix} u_3(t) \\ u_4(t) \end{pmatrix} = \begin{pmatrix} \bar{u}_3 \\ \bar{u}_4 \end{pmatrix} \sin \omega t$.

Advection-reaction dispersion eqⁿ:

$$\frac{\partial C}{\partial t} = -v \underbrace{\frac{\partial C}{\partial x}}_{\substack{\text{advection} \\ \text{②}}} + D_L \frac{\partial^2 C}{\partial x^2} + \underbrace{\rho C(1-C)}_{\text{reaction}}$$

$$C = \sum N_i c_i \quad (\text{2 node element}).$$

$$\textcircled{3} \quad \int_0^L \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dx \rightarrow \frac{1}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\textcircled{1} \quad \int_0^L N_i N_j dx \rightarrow \frac{L}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\textcircled{2} \quad \int_0^L N_i \frac{\partial N_j}{\partial x} dx \rightarrow \frac{1}{2} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\textcircled{4} \quad \text{(nonlinear)} \quad \int_0^L (N_1 c_1 + N_2 c_2)^2 N_i dx$$

$$\rightarrow \int_0^L \begin{pmatrix} N_1^3 & 2N_1^2 N_2 & N_1 N_2^2 \\ N_1^2 N_2 & 2N_1 N_2^2 & N_2^3 \end{pmatrix} dx \begin{pmatrix} c_1^2 \\ c_1 c_2 \\ c_2^2 \end{pmatrix}$$

$$= L \begin{pmatrix} \frac{1}{4} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} c_1^2 \\ c_1 c_2 \\ c_2^2 \end{pmatrix}$$

\Rightarrow after multiplying by M^{-1} :

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \left[-\frac{v}{L} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} - \frac{2D_L}{L^2} \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \right.$$

$$\left. + P \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$- \frac{P}{6} \begin{pmatrix} 5c_1^2 + c_1 c_2 - c_2^2 \\ -c_1^2 + c_1 c_2 + 5c_2^2 \end{pmatrix} - \frac{2}{L} \begin{pmatrix} -2J(0) - J(L) \\ J(0) + 2J(L) \end{pmatrix}$$

constitutive rel.ⁿ

where $J = -D_L \frac{\partial c}{\partial x}$ (Fick's law)

a 2D nonlinear ODE.

