Explicit methods for the first-order wave equation

November 14, 2016

We will consider the PDE

$$\begin{gathered} u_t + a u_x = 0 \\ x \in [-1, 1], \quad 0 < t < 2 \\ u(x, 0) = u_0(x), \quad u(\pm 1, t) = 0. \end{gathered}$$

Exercise 1. *First-order algorithm*
Write a code that solves the equation using first-order forward differencing in time and space.

Use spacing $\Delta x = 0.02$ and $\Delta t = 0.01$ and take $a = -1$, $u_0(x) = \exp(-30x^4)$.

Your code should create a matrix, u, where the i-th row represents the approximation to the solution at the i-th time-step.

To visualise your solution, use `waterfall` or write a code that involving the `pause` command that animates the results.

What happens if you set $\Delta x = 0.01$ and $\Delta t = 0.02$? Verify that instability occurs when the CFL condition is violated.

Exercise 2. *A more efficient code?*
Modify your code so that it updates all values of u together at each time step via matrix multiplication. Care is required to ensure the boundary conditions are correctly applied.

How would you use matrix multiplication if the boundary conditions were $u_t(\pm 1, t) = 0$?

We will look more at matrix methods when we study implicit formulations next week.

Exercise 3. *Lax-Friedrichs method*

Now write code that solves the equation using the Lax-Friedrichs method, which is second order in space. You should notice something very different about your results!

Try instead the initial conditions

$$
\begin{aligned}
u_0(x) =& 0 \qquad |x| > 0.25 \\
=& 1 \qquad |x| < 0.25.
\end{aligned}
$$

Which of your two methods treats this problem better?

The Lax-Friedrichs method introduces what is known as numerical dissipation. Calculate the value of the diffusion coefficient D (see part 3 of the lecture notes).

What equations would this method be suitable for? Re-run your code with $a = -500$. Is this any better? Why?