GM04/MM04 Exercise 2.

1. Consider the initial/boundary-value problem for the diffusion equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \text{ with } u = u(t,x),\ \ t \geq 0,\ \ 0 \leq x \leq 1, \tag{1}$$

$$u|_{t=0, 0\leq x \leq 1} = u_0(x), \tag{2}$$

$$u|_{x=1, t\geq 0} = q_1(t), \tag{3}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0, t\geq 0} = q_0(t). \tag{4}$$

(a) Show that the Crank-Nicolson approximation

$$\frac{u_{m+1,n} - u_{m,n}}{\Delta t} = \frac{1}{2}\left\{\frac{u_{m+1,n+1} - 2u_{m+1,n} + u_{m+1,n-1}}{(\Delta x)^2}\right. \tag{5}$$

$$\left.+\frac{u_{m,n+1} - 2u_{m,n} + u_{m,n-1}}{(\Delta x)^2}\right\}, \tag{6}$$

with $t_m = m\Delta t, x_n = n\Delta x$ is second-order accurate in time t, and derive an estimate for the local truncation error in x.

(b) Discuss organization of the computation and explain how the boundary condition at $x = 0$ could be approached to preserve the accuracy of the solution.

(c) Discuss the stability of the Crank-Nicolson algorithm in the von Neumann sense.

2. Consider the wave equation,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0. \tag{7}$$

Investigate stability (in the von Neumann sense) of a finite difference scheme based on a first-order forward approximation for the time derivative and a central, second-order approximation for the derivative with respect to x.

3. Consider the following elliptic equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0. \tag{8}$$

Write down a five-point finite-difference approximation of this equation of second-order accuracy on a uniform grid, $\Delta x = \Delta y = h$.

Assuming that the function u is specified on the sides of a unit square, $0 \leq x \leq 1, 0 \leq y \leq 1$, explain how the finite-difference equation can be written in a matrix form.

Explain the point Jacobi iteration method for solving this equation and investigate the convergence of this method.

Hand in your solutions before November 23.