Handout.

Consistency and LTE.

When a finite-difference approximation is derived for a differential equation using standard formulae for the derivatives, the local truncation error is usually easy to estimate. But suppose you are given a numerical algorithm and you need to check whether it is consistent with the original differential equation (roughly speaking, consistency means that the finite-difference equation turns into the original equation when the step sizes tend to zero).

In general, let $L(y(t)) = 0$ be a differential equation (we consider an ODE for simplicity) and $N(y_i)$=0 its finite-difference approximation at the node y_i. To check for consistency, we replace y_i with the exact solution $y(t)$ at the corresponding grid points and see how accurately the equation $N(y(t)) = 0$ is satisfied. It will all become clearer from the following two examples.

Example 1. We are given the forward Euler algorithm,

$$\frac{y_{i+1} - y_i}{h} = f(y_i, t_i), \tag{0.1}$$

on the grid $y_i = ih$. Is it consistent with the ODE

$$\frac{dy}{dt} = f(y, t)? \tag{0.2}$$

To answer that, we replace y_i with $y(t)$ in the finite-difference scheme (0.1). Then $y(t_{i+1}) = y(t_i + h) = y(t_i) + hy'(t_i) + O(h^2)$, where the dash is the derivative, and (0.1) becomes

$$\frac{y(t_i) + hy'(t_i) + O(h^2) - y(t_i)}{h} = f(y(t_i), t_i), \tag{0.3}$$

or

$$y'(t_i) + O(h) = f(y(t_i), t_i). \tag{0.4}$$

This is the same as the ODE (0.2) with a local truncation error $O(h)$. Hence the forward Euler method is consistent and it is first-order accurate.

Example 2. Is the following scheme,

$$\frac{2y_{i+1} - 3y_i}{h} = f(y_i, t_i), \tag{0.5}$$

consistent with the ODE (0.2)?

Replace y_i with $y(t_i)$ as before. The equation (0.5) becomes,

$$\frac{2y(t_i) + 2hy'(t_i) + O(h^2) - 3y(t_i)}{h} = f(y(t_i), t_i). \tag{0.6}$$

On the left, we can simplify,

$$-\frac{y(t_i)}{h} + O(1) = f(y(t_i), t_i). \tag{0.7}$$

As $h \to 0$, the left-hand side in (0.7) looks nothing like the left-hand side in the ODE (0.2). Hence the scheme (0.5) is not consistent with (0.2).

The procedure for partial differential equations is similar.