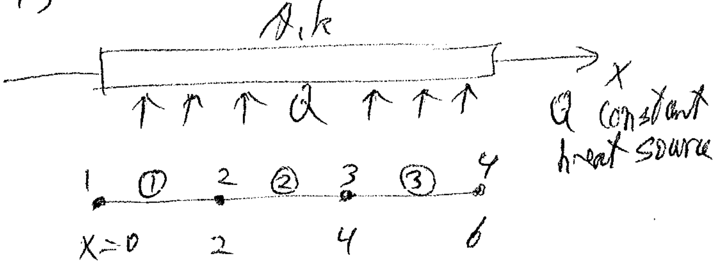
Example (3-element model for heat flow in a uniform fin):
(based on Ottosen & Petersson, p. 164)



Heat flow in a (uniform) fin:

$$\begin{cases} \frac{d}{dx}\left(Ak \frac{dT}{dx}\right) + Q = 0 \\ T(0) = T_0 \\ q(6) = \bar{q} \quad \left(q = -k\frac{dT}{dx}\right) \end{cases}$$

(completely insulated tip: $\frac{dT}{dx} = 0$)

FE eq: $K u = f = f_b + f_e$, $u = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix}$ (unknowns)

where $K_{ij} = \int_0^6 \frac{dN_i}{dx} Ak \frac{dN_j}{dx} dx$, in terms of global shape functions N_i (defined for all x)

Element shape functions:

$N_1^1(x) = -\frac{(x - x_2)}{L} = \frac{2-x}{2}$ $(L = 2)$

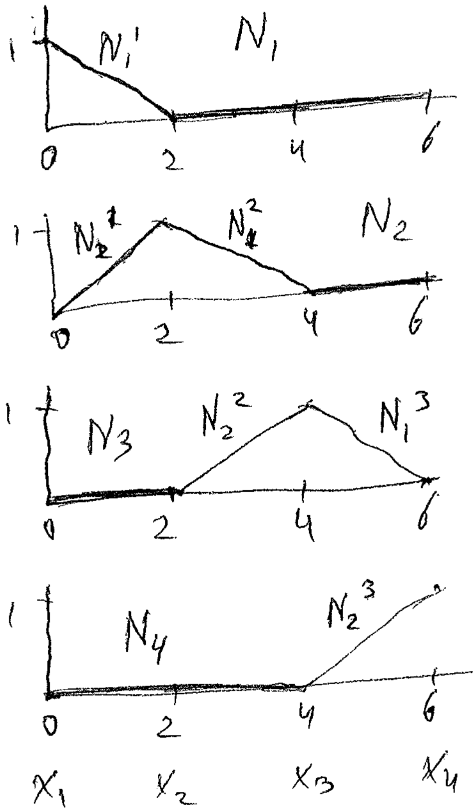
$N_2^1(x) = \frac{x - x_1}{L} = \frac{x}{L}$

$N_1^2(x) = -\frac{(x - x_3)}{L} = \frac{4-x}{L}$

$N_2^2(x) = \frac{x - x_2}{L} = \frac{x-2}{L}$

$N_1^3(x) = -\frac{(x - x_4)}{L} = \frac{6-x}{L}$

$N_2^3(x) = \frac{x - x_3}{L} = \frac{x-4}{L}$



thus:

$K_{11} = \int_0^6 \frac{dN_1}{dx} Ak \frac{dN_1}{dx} dx = \int_0^2 \frac{dN_1^1}{dx} Ak \frac{dN_1^1}{dx} dx = \int_0^2 \left(-\frac{1}{2}\right)^2 Ak\, dx = \frac{1}{2} Ak$

$K_{12} = \int_0^6 \frac{dN_1}{dx} Ak \frac{dN_2}{dx} dx = \int_0^2 \frac{dN_1^1}{dx} Ak \frac{dN_2^1}{dx} dx = \int_0^2 \left(-\frac{1}{2}\right) Ak \frac{1}{2}\, dx = -\frac{1}{2} Ak$

$K_{13} = \int_0^6 \frac{dN_1}{dx} Ak \frac{dN_3}{dx} dx = 0$ (no common region where both N_1 and N_3 are non-zero; nodes too far apart)

$K_{14} = 0$

$K_{22} = \int_0^6 \frac{dN_2}{dx} Ak \frac{dN_2}{dx} dx = \int_0^2 \frac{dN_2^1}{dx} Ak \frac{dN_2^1}{dx} dx + \int_2^4 \frac{dN_1^2}{dx} Ak \frac{dN_1^2}{dx} dx$

$= \int_0^2 \left(\frac{1}{2}\right)^2 Ak\, dx + \int_2^4 \left(-\frac{1}{2}\right)^2 Ak\, dx = Ak$

$$K_{23} = \int_0^6 \frac{dN_2}{dx} Ak \frac{dN_3}{dx}\,dx = \int_2^4 \frac{dN_1^2}{dx} Ak \frac{dN_2^2}{dx}\,dx = \int_2^4 (-\tfrac{1}{2}) Ak (\tfrac{1}{2})\,dx = -\tfrac{1}{2}Ak$$

$$K_{24} = 0$$

$$K_{33} = \int_0^6 \frac{dN_3}{dx} Ak \frac{dN_3}{dx}\,dx = \int_2^4 \frac{dN_2^2}{dx} Ak \frac{dN_2^2}{dx}\,dx + \int_4^6 \frac{dN_1^3}{dx} Ak \frac{dN_1^3}{dx}\,dx$$

$$= \int_2^4 (+\tfrac{1}{2})^2 Ak\,dx + \int_4^6 (-\tfrac{1}{2})^2 Ak\,dx = Ak$$

$$K_{34} = \int_0^6 \frac{dN_3}{dx} Ak \frac{dN_4}{dx}\,dx = \int_4^6 \frac{dN_1^3}{dx} Ak \frac{dN_2^3}{dx}\,dx = \int_4^6 (-\tfrac{1}{2}) Ak (\tfrac{1}{2})\,dx = -\tfrac{1}{2}Ak$$

$$K_{44} = \int_0^6 \frac{dN_4}{dx} Ak \frac{dN_4}{dx}\,dx = \int_4^6 \frac{dN_2^3}{dx} Ak \frac{dN_2^3}{dx}\,dx = \int_4^6 (\tfrac{1}{2})^2 Ak\,dx = \tfrac{1}{2}Ak$$

or

$$K = \tfrac{1}{2}Ak \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

(tridiagonal, as expected!)

Load vector:

$$f_{l_1} = \int_0^6 N_1 Q\,dx = Q\int_0^2 N_1^1\,dx = Q\int_0^2 \frac{2-x}{2}\,dx = \frac{Q}{2}\left[2x - \tfrac{1}{2}x^2\right]_0^2$$

$$= \frac{Q}{2}(4-2) = Q$$

$$f_{l_2} = \int_0^6 N_2 Q\,dx = Q\int_0^2 N_2^1\,dx + Q\int_2^4 N_1^2\,dx = Q\int_0^2 \frac{x}{2}\,dx + Q\int_2^4 \frac{4-x}{2}\,dx$$

$$= \frac{Q}{2}\left[\tfrac{1}{2}x^2\right]_0^2 + \frac{Q}{2}\left[4x - \tfrac{1}{2}x^2\right]_2^4 = Q + \frac{Q}{2}(16 - 8 - 8 + 2) = 2Q$$

$$f_{l_3} = \int_0^6 N_3 Q\,dx = \int_2^4 N_2^2 Q\,dx + \int_4^6 N_1^3 Q\,dx = 2Q$$

$$f_{l_4} = \int_0^6 N_4 Q\,dx = Q\int_4^6 N_2^3\,dx = Q$$

thus: $f_l = Q\begin{pmatrix}1\\2\\2\\1\end{pmatrix}$

Boundary vector:

$$f_{b_i} = -\left[N_i A q\right]_0^6 = \underbrace{-(N_i A q)|_{x=6}}_{= A q(6)\,\delta_{i4}} + \underbrace{(N_i A q)|_{x=0}}_{= A q(0)\,\delta_{i1}}$$

because $x=0$, i.e., the first node, lies only in element 1, so only N_1 is non-zero, and is equal to 1; similarly, at $x=6$, i.e., at node 4, only N_4 contributes

Thus,
$$f_b = \begin{pmatrix} A q(0) \\ 0 \\ 0 \\ -A q(6) \end{pmatrix}$$