Homework 2

EOC 6189 - Computational Fluid Dynamics
Florida Atlantic University

February 7, 2019

Collaboration policy: *You are strongly encouraged to discuss your work with fellow students, to brainstorm ideas, and to learn from each other. However, you are not allowed to copy solutions/code from each other, from the internet, or from any other source. All work shown must be your own. You are expected to follow the University Honor Code at all times.*

Instructions: Please create a single zip file containing all your PDFs, movies, etc., and upload it on Canvas by the deadline. Your code source files must be submitted via git repository. Try to keep your code as modular as possible, which involves writing separate functions, where one function does only one job (e.g., discretization, derivative computations, time integration, etc.). This will help you reuse your functions with minimal modification in future projects. Make sure to write short, but useful comments in the code and make regular git commits. Undocumented code will lose points.

1. **Runge-Kutta integrator**

   (a) Compute the analytical solution for the following ODE:

   $$\frac{dy}{dt} = -5y \tag{1}$$
   $$y(t=0) = 1 \tag{2}$$

   (b) Integrate the ODE numerically in time using two different methods: 1) the forward Euler method; and 2) the second order Runge-Kutta method (RK2). Set the time-step size to be $\Delta t = 0.1$, and find the solution for $0 \leq t \leq 2$. Make a single plot comparing the two numerical solutions to the analytical curve, and comment on the results.

   (c) Rerun your integration with $\Delta t = 0.01$, and compare the results.

2. **Implicit time-integration**

   Note: For the Crank-Nicolson scheme, the right hand side of the heat equation may be discretized as follows:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{2}\left[\frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{\Delta x^2} + \frac{T_{i+1}^{n} - 2T_i^{n} + T_{i-1}^{n}}{\Delta x^2}\right] \tag{3}$$

   Use the usual first order discretization for the time-derivative.

   (a) In this problem, you will redo one of the problems from Project 1, but using implicit time integration. Write code to solve the inhomogeneous 1D heat equation given below. Use the Crank-Nicolson scheme, which is **implicit** in time, with a second order discretization in space. Make sure to commit your changes to git as you build up your code.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \sin(5x) \tag{4}$$

   Use the following Initial and Boundary Conditions:

$$IC: \; T(x,0) = x(\pi - x) \tag{5}$$

$$BC: \; T(0,t) = T(L,t) = 0 \tag{6}$$

   Here, $\alpha = 0.25$ is the thermal diffusivity of the rod material, and $L = \pi$ is its length. Discretize the rod into 100 separate sections (i.e., using 101 points), and use a time step size of 10^{-2}. On paper, show all the steps needed to assemble the matrix system that we must solve for the implicit scheme.

   (b) Code up the solution for the implicit solver, and run your simulation up until $t = 50$. Plot the final temperature profile in the rod at $T(x, t = 50)$. Compare this to the analytical steady-state solution $T_{steady}(x) = \sin(5x)/25\alpha$. Are the results different from your previous attempt at using a fully explicit scheme with $\Delta t = 10^{-2}$? Why?

   (c) Rerun the simulation with $\Delta t = 1$ and $\Delta t = 3$. Comment on your steady state answer with regard to stability and accuracy. What is the speedup you get compared to using a fully explicit scheme? What is the disadvantage?