Software Requirements Specification for Constraints in Chipmunk2D

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Contents

1 Table of Symbols

The table that follows summarizes the symbols used in this document along with their units. The symbols are listed in alphabetical order.

symbol	unit	description
$\mathbf{b}$		Error correction term
β		Ratio between two angular velocities
e_{c}		Error calculation
$\mathbf{g}_{\mathrm{a}}$	kgm^2	Position of the beginning of the groove
$\mathbf{g}_{\mathrm{b}}$	kgm^2	Position of the end of the groove
I_{i}	kgm^2	Moment of inertia of the ith object
$\mathbf{J}$	N s	Impulse
$\mathbf{k}$		Mass Matrix
k_{erp}		Error reduction parameter
ΔL		Change in angular momentum
m_{i}	kg	Mass of the ith object
m_{n}		Mass Normal
$\mathbf{n}$		Normal Vector between the two constraints
$\Delta\mathbf{P}$		Change in linear momentum
$\mathbf{p}_{\mathrm{i}}$	m	Position of the constraint of the ith object
ϕ_{i}	rad	Orientation of the ith object
$\mathbf{r}_{\mathrm{i}}$	m	Vector between the constraint and the position of the ith object
$\mathbf{r}_{\mathrm{i}\perp}$	m/s	Perpendicular vector of $\mathbf{r}_{\mathrm{i}}$
$\mathbf{r}_{\mathrm{v}}$	m/s	The relative velocity between two objects
ρ	rad	Ratchet
σ	rad	Phase
t	s	Time
Δt	s	Change in time between frames
$\mathbf{v}_{\mathrm{i}}$	m/s	Old velocity of the ith object
$\mathbf{v}'$	m/s	New velocity of the ith object
ω	rad/s	Old angular velocity of the ith object
ω'	rad/s	New angular velocity of the ith object
ω_{r}	rad/s	Relative angular velocity
$\mathbf{x}_{\mathrm{i}}$	m/s	Position of the ith object
ζ	unitless	Damping constant

2 Purpose of Document

Physics engines can have many different types of constraints that can be added to objects. Trying to make an abstract SRS that includes the constraints is difficult because different engines will have different types of constraints and implementations.

The SRS for the physics engine is located in the document: game physics library SRS.pdf. This document includes the data definitions and the instance models for the constraints that are in the Chipmunk2D physics library.

3 Solution Characteristics Specification

3.1 Assumptions

This section simplifies the original problem and helps in developing the theoretical model by filling in the missing information for the physical system. The numbers given in the square brackets refer to the data definition, or the instance model, in which the respective assumption is used.

A1: All objects are 2D rigid bodies

A2: The time between frames is small enough that gravity is negligible

3.2 General Definitions

This section collects the laws and equations that will be used in deriving the data definitions, which in turn are used to build the instance models.

Number	GD1
Label	**Change in Momentum**
Equation	$\Delta\ \mathbf{P} = m\ \Delta\ \mathbf{v}$ $\Delta\ L = I\ \Delta\ \omega$
Description	$\Delta\mathbf{P}$ = Change in linear momentum ΔL = Change in angular momentum m =Mass of the object I =Moment of inertia of the object $\Delta\mathbf{v}$ =Change in velocity of the object $\Delta\omega$ =Change in angular velocity of the object
Source	
Ref. By	IM1, IM7

3.3 Data Definitions

This section collects and defines all the data needed to build the instance models. The dimension of each quantity is also given.

Number	DD1
Label	**Normal vector between constraints**
Symbol	$\mathbf{n}$
SI Units	
Equation	$\mathbf{n} = \frac{\mathbf{p}_2 - \mathbf{p}_1}{\|\|\mathbf{p}_2 - \mathbf{p}_1\|\|}$
Description	$\mathbf{n}$ =The normal vector between the two constraints $\mathbf{p}_1$=The position the first object is constrained $\mathbf{p}_2$=The position the second object is constrained Refer to Figure 1 for more information
Sources	
Ref. By	IM2, IM6

Number	DD2
Label	**Constraint Location**
Symbol	$\mathbf{r}$
SI Units	
Equation	$\mathbf{r}_{\mathrm{i}} = \mathbf{p}_{\mathrm{i}} - \mathbf{x}_{\mathrm{i}}$
Description	$\mathbf{r}_{\mathrm{i}}$ =The vector between the ith object's constraint and its position $\mathbf{p}_{\mathrm{i}}$=The position the ith object is constrained $\mathbf{x}_{\mathrm{i}}$=The position of the ith object Refer to Figure 1 for more information
Sources	
Ref. By	DD3, DD5, DD6, IM1

3.3.1 Constraint between objects

In this section an image of a constraint between two objects is provided (Figure 1). The object's position, the constraints, the constraint locations (DD2), and the normal vector between the objects (DD1) are shown.

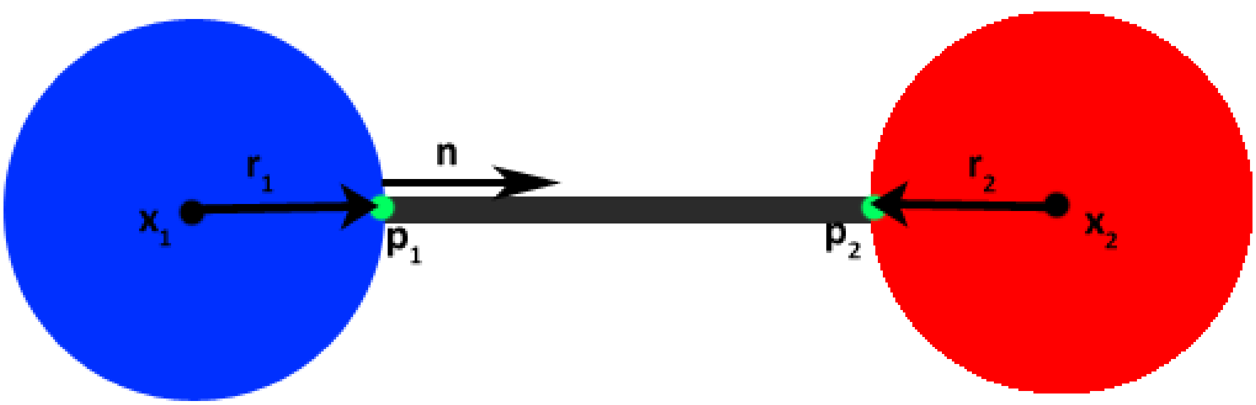

Figure 1: Constraint between two objects

Number	DD3
Label	**Mass Normal**
Symbol	m_{n}
SI Units	
Equation	$m_{\text{n}} = \frac{1}{\frac{1}{m_1}+\frac{1}{I_1}(\mathbf{r}_1\times\mathbf{n})^2+\frac{1}{m_2}+\frac{1}{I_2}(\mathbf{r}_2\times\mathbf{n})^2}$
Description	m_{n} = Mass normal I_{i}=The moment of inertia of the ith object m_{i}=The mass of the ith object $\mathbf{n}$ = The normal vector between the two constraints (DD3) $\mathbf{r}_{\text{i}}$ = The vector between the ith object's constraint and its position (DD2)
Sources	
Ref. By	IM2, IM6

Number	DD4
Label	**Velocity of a point on a rigid body**
Symbol	$\mathbf{v}_{\text{p}}$
SI Units	
Equation	$\mathbf{v}_{\text{p}} = \omega\mathbf{r}_{\perp} + \mathbf{v}$
Description	$\mathbf{v}_{\text{p}}$ = The velocity of a point on the rigid body ω=The angular velocity of the object $\mathbf{v}$=The velocity of the object $\mathbf{r}$ = The vector between the position of the point and the position of the object $\mathbf{r}_{\perp}$ = The vector r rotated counterclockwise by 90° (DD8)
Sources	
Ref. By	DD5

Number	DD5
Label	**Relative Velocity**
Symbol	$\mathbf{r}_{\mathrm{v}}$
SI Units	
Equation	$\mathbf{r}_{\mathrm{v}} = \mathbf{v}_{\mathrm{p2}} - \mathbf{v}_{\mathrm{p1}}$ Using DD4 : $\mathbf{r}_{\mathrm{v}} = \omega_2 \mathbf{r}_{2\perp} + \mathbf{v}_2 - \omega_1 \mathbf{r}_{1\perp} - \mathbf{v}_1$
Description	$\mathbf{r}_{\mathrm{v}}$ = The relative velocity between the two objects $\mathbf{v}_{\mathrm{pi}}$ = The velocity of the point of constraint of the ith object ω_{i}=The angular velocity of the ith object $\mathbf{v}_{\mathrm{i}}$=The velocity of the ith object $\mathbf{r}_{\mathrm{i}}$ = The vector between the ith object's constraint and its center of mass (DD2) $\mathbf{r}_{\mathrm{i}\perp}$ = The vector $\mathbf{r}_{\mathrm{i}}$ rotated counterclockwise by 90° (DD8)
Sources	
Ref. By	IM2, IM4, IM6

Number	DD6
Label	**Mass Matrix**
Symbol	$\mathbf{k}$
SI Units	
Equation	$\mathbf{k} = \begin{bmatrix} \frac{1}{m_1} + \frac{1}{m_2} + \frac{r_{1\text{y}}^2}{I_1} + \frac{r_{2\text{y}}^2}{I_2} & \frac{-r_{1\text{x}} * r_{1\text{y}}}{I_1} + \frac{-r_{2\text{x}} * r_{2\text{y}}}{I_2} \\ \frac{-r_{1\text{x}} * r_{1\text{y}}}{I_1} + \frac{-r_{2\text{x}} * r_{2\text{y}}}{I_2} & \frac{1}{m_1} + \frac{1}{m_2} + \frac{r_{1\text{x}}^2}{I_1} + \frac{r_{2\text{x}}^2}{I_2} \end{bmatrix}$
Description	$\mathbf{k}$ = Mass Matrix $\mathbf{r}_\text{i}$ = The vector between the ith object's constraint and its position (DD2) I_i =The moment of inertia of the ith object m_i =The mass of the ith object r_ix= The x component of the vector $\mathbf{r}_\text{i}$ r_iy= The y component of the vector $\mathbf{r}_\text{i}$
Sources	
Ref. By	IM4

Number	DD7
Label	**Error Correction**
Symbol	b
SI Units	
Equation	$\mathbf{b} = \frac{k_{\text{erp}}}{\Delta t} e_{\text{c}}$
Description	The error correction term is used to fix any errors in the position/orientation of the object. An impulse is applied to the velocities and angular velocities of the object. The new position and orientation are then calculated. Due to numerical errors the position and orientation of the object can differ from the true location. $\mathbf{b}$ = The error correction k_{erp}= The error reduction parameter. This value is $0 \leq k_{\text{erp}} \leq 1$. A k_{erp} of 1 means to fix the error immediately, while a value of 0 means to not fix the error at all. Δt = The change in time between frames e_{c} = The calculation of the error. Each constraint calculates this value differently.
Sources	
Ref. By	IM2, IM3, IM4, IM9, IM10, IM11

Number	DD8
Label	**Perpindicular Vector**
Symbol	$\mathbf{v}_{\perp}$
SI Units	
Equation	$\mathbf{v}_{\perp} = \begin{bmatrix} -v_{\text{y}} \\ v_{\text{x}} \end{bmatrix}$
Description	The perpendicular vector is the vector $\mathbf{v}$ rotated counterclockwise by 90°
Sources	
Ref. By	IM2, IM4, IM6

Number	DD9
Label	**2D Cross Product**
Symbol	$\mathbf{u} \times \mathbf{v}$
SI Units	
Equation	$\mathbf{u} \times \mathbf{v} = u_{\text{x}} v_{\text{y}} - u_{\text{y}} v_{\text{x}}$
Description	For the purpose of this document, the cross product of two 2D vectors will result in a scalar quantity. $\mathbf{u}$ = A 2D vector $\mathbf{v}$ = A 2D vector $u_{\text{x}}, u_{\text{y}}$ = The x and y components of the vector $\mathbf{u}$ $v_{\text{x}}, v_{\text{y}}$ = The x and y components of the vector $\mathbf{v}$
Sources	
Ref. By	IM2, IM4, IM6

3.4 Instance Models

Number	IM1
Label	**Constraint on a 2D rigid body**
Input	$\mathbf{J}, \mathbf{p}_1, \mathbf{p}_2, \mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2, \mathrm{I}_1, \mathrm{I}_2, \mathrm{m}_1, \mathrm{m}_2, \omega_1, \omega_2$
Output	$\mathbf{v}'_1, \mathbf{v}'_2, \boldsymbol{\omega}'_1, \boldsymbol{\omega}'_2$ such that: $\mathbf{v}'_1 = \mathbf{v}_1 + \frac{-\mathbf{J}}{\mathrm{m}_1}$ $\mathbf{v}'_2 = \mathbf{v}_2 + \frac{\mathbf{J}}{\mathrm{m}_2}$ $\omega'_1 = \omega_1 + \frac{-\mathbf{r}_1 \times \mathbf{J}}{\mathrm{I}_1}$ $\omega'_2 = \omega_2 + \frac{\mathbf{r}_2 \times \mathbf{J}}{\mathrm{I}_2}$
Description	$\mathbf{J}$= The impulse acting on the constraint $\mathbf{p}_{\mathrm{i}}$=The point where the ith object is constrained $\mathbf{x}_{\mathrm{i}}$=The position of the ith object $\mathbf{v}_{\mathrm{i}}$=The old velocity of the ith object ω_{i}=The old angular velocity of the ith object $\mathbf{r}_{\mathrm{i}}$=The vector between the ith object's constraint and its position (DD2) I_{i}=The moment of inertia of the ith object m_{i}=The mass of the ith object $\mathbf{v}'_{\mathrm{i}}$=The new velocity of the ith object ω'_{i}=The new angular velocity of the ith object The following instance models use the same input and output as described above.The impulse for each constraint is calculated differently and is outlined in the following instance models.
Sources	
Ref. By	IM2, IM3, IM4, IM5, IM6

3.4.1 Derivation of the velocity and angular velocity

The impulse on an object is equal to the change in its linear momentum, that is:

$$\mathbf{J} = \Delta \mathbf{P}$$

Using GD1:

$$\mathbf{J} = m\Delta\mathbf{v} = m(\mathbf{v}' - \mathbf{v})$$

Rearrange:

$$\mathbf{v}' = \mathbf{v} + \frac{\mathbf{J}}{\mathrm{m}}$$

The angular momentum of a particle is equal to the cross product of the position vector of the particle and the linear momentum. That is:

$$\Delta L = \mathbf{r} \times \Delta\mathbf{P} = \mathbf{r} \times \mathbf{J}$$

Using GD1:

$$\mathbf{r} \times \mathbf{J} = I\Delta\omega = I(\omega' - \omega)$$

Rearrange:

$$\omega' = \omega + \frac{\mathbf{r} \times \mathbf{J}}{\mathrm{I}}$$

Newton's Third Law of motion says that the impulse felt by one object is equal and opposite the impulse felt by the other object.

$$\mathbf{v}_1' = \mathbf{v}_1 + \frac{-\mathbf{J}}{\mathrm{m}_1}$$

$$\mathbf{v}_2' = \mathbf{v}_2 + \frac{\mathbf{J}}{\mathrm{m}_2}$$

$$\omega_1' = \omega_1 + \frac{-\mathbf{r}_1 \times \mathbf{J}}{\mathrm{I}_1}$$

$$\omega_2' = \omega_2 + \frac{\mathbf{r}_2 \times \mathbf{J}}{\mathrm{I}_2}$$

3.4.2 Pin joint

A pin joint acts as a bar between two objects. The two constraints on the objects are always the same distance apart. The objects are able to rotate around the constraint. Figure 2 shows an example of a pin joint.

Number	IM2
Label	**Pin Joint**
Input	IM1
Output	$\mathbf{J}$
Description	The impulse for a pin joint is calculated as: $\mathbf{J} = ((-b - \mathbf{r}_{\mathrm{v}} \cdot \mathbf{n})m_{\mathrm{n}})\mathbf{n}$ b = error correction: $\frac{k_{\mathrm{erp}}}{\Delta t}(\lvert \mathbf{p}_2 - \mathbf{p}_1 \rvert - \lvert (\mathbf{x}_2 + \mathbf{r}_2) - (\mathbf{x}_1 + \mathbf{r}_1) \rvert)$ (DD7) $\mathbf{r}_{\mathrm{v}}$= relative velocity (DD5) $\mathbf{n}$ = The normal vector between the two constraints (DD1) m_{n} = mass normal (DD3)
Sources	
Ref. By	

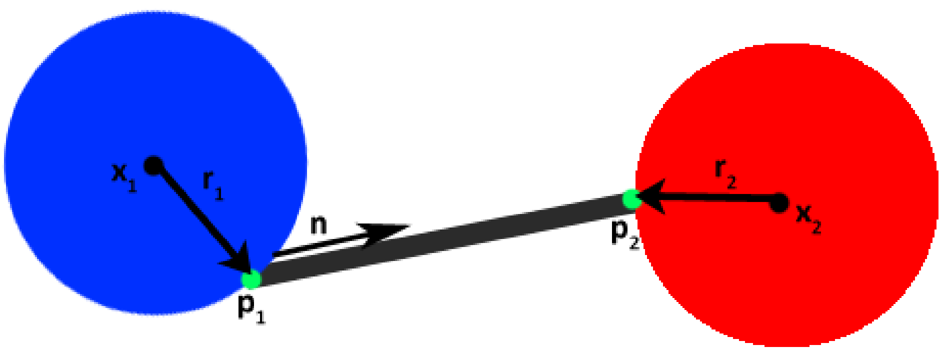

Figure 2: Pin joint between two objects

3.4.3 Derivation of the Impulse on a Pin Joint

A pin joint acts as a bar between the two objects. The relative velocities (DD5) between the two objects in the direction of the constraint should equal zero. That is:

$$(\mathbf{v}_2' + \omega_2' \mathbf{r}_{2\perp} - \mathbf{v}_1' - \omega_1' \mathbf{r}_{1\perp}) \cdot \mathbf{n} = 0 \tag{1}$$

The updated velocities and angular velocities should equal:

$$\mathbf{v}_1' = \mathbf{v}_1 + \frac{-\mathbf{J}}{\mathrm{m}_1}$$

$$\mathbf{v}_2' = \mathbf{v}_2 + \frac{\mathbf{J}}{\mathrm{m}_2}$$

$$\omega_1' = \omega_1 + \frac{-\mathbf{r}_1 \times \mathbf{J}}{\mathrm{I}_1}$$

$$\omega_2' = \omega_2 + \frac{\mathbf{r}_2 \times \mathbf{J}}{\mathrm{I}_2}$$

The impulse should only be applied in the direction of the constraint:

$$\mathbf{J} = J\mathbf{n} \tag{2}$$

Substituting the above equations into Eq1:

$$(\mathbf{v}_2 + \frac{J\mathbf{n}}{\mathrm{m}_2} + (\omega_2 + \frac{\mathbf{r}_2 \times J\mathbf{n}}{I_2})\mathbf{r}_{2\perp} - \mathbf{v}_1 + \frac{J\mathbf{n}}{\mathrm{m}_1} + (-\omega_1 + \frac{\mathbf{r}_1 \times J\mathbf{n}}{I_1})\mathbf{r}_{1\perp}) \cdot \mathbf{n} = 0$$

Factor out the impulse

$$J\mathbf{n} \cdot \mathbf{n}(\frac{\mathbf{n}}{\mathrm{m}_2} + \frac{(\mathbf{r}_2 \times \mathbf{n})\mathbf{r}_{2\perp}}{I_2} + \frac{\mathbf{n}}{\mathrm{m}_1} + \frac{(\mathbf{r}_1 \times \mathbf{n})\mathbf{r}_{1\perp}}{I_1}) \cdot \mathbf{n} + \mathbf{v}_2 \cdot \mathbf{n} + \omega_2 \mathbf{r}_{2\perp} \cdot \mathbf{n} - \mathbf{v}_1 \cdot \mathbf{n} - \omega_1 \mathbf{r}_{1\perp} \cdot \mathbf{n} = 0$$

Using DD8 and DD9

$$\mathbf{r}_{\mathrm{i}\perp} \cdot \mathbf{n} = \mathbf{r}_{\mathrm{i}} \times \mathbf{n}$$

Simplifying

$$J(\frac{1}{\mathrm{m}_2} + \frac{(\mathbf{r}_2 \times \mathbf{n})^2}{I_2} + \frac{1}{\mathrm{m}_1} + \frac{(\mathbf{r}_1 \times \mathbf{n})^2}{I_1}) + \mathbf{v}_2 \cdot \mathbf{n} + \omega_2 \mathbf{r}_{2\perp} \cdot \mathbf{n} - \mathbf{v}_1 \cdot \mathbf{n} - \omega_1 \mathbf{r}_{1\perp} \cdot \mathbf{n} = 0$$

Rearrange:

$$J = \frac{(\mathbf{v}_1 + \omega_1 \mathbf{r}_{1\perp} - \mathbf{v}_2 - \omega_2 \mathbf{r}_{2\perp}) \cdot \mathbf{n}}{\frac{1}{\mathrm{m}_2} + \frac{(\mathbf{r}_2 \times \mathbf{n})^2}{I_2} + \frac{1}{\mathrm{m}_1} + \frac{(\mathbf{r}_1 \times \mathbf{n})^2}{I_1}}$$

Using DD5 and DD3 the above equation simplifies to:

$$J = (-\mathbf{r}_{\mathrm{v}} \cdot \mathbf{n}) m_{\mathrm{n}}$$

Substituting the above equation into Eq2:

$$\mathbf{J} = ((-\mathbf{r}_{\mathrm{v}} \cdot \mathbf{n}) m_{\mathrm{n}})\mathbf{n}$$

As mentioned in DD7 an error correction term must be added to the velocity to fix potential issues with the position:

$$\mathbf{J} = ((-b - \mathbf{r}_{\mathrm{v}} \cdot \mathbf{n}) m_{\mathrm{n}})\mathbf{n}$$

3.4.4 Slide joint

A slide joint is similar to a pin joint but has a minimum and maximum distance. Figure 3 shows an example of a slide joint

Number	IM3
Label	**Slide Joint**
Input	IM1, max, min
Output	$\mathbf{J}$
Description	Impulse is calculated the same as IM2. Except $b = \begin{cases} \frac{k_{\text{erp}}}{\Delta t}(\lvert\mathbf{p}_2 - \mathbf{p}_1\rvert - max) & \text{if } \lvert\mathbf{p}_2 - \mathbf{p}_1\rvert > max \\ \frac{k_{\text{erp}}}{\Delta t}(min - \lvert\mathbf{p}_2 - \mathbf{p}_1\rvert) & \text{if } \lvert\mathbf{p}_2 - \mathbf{p}_1\rvert < min \\ \text{no impulse} & \text{otherwise} \end{cases}$ b = Error correction: (DD7) min = The minimum distance the two objects can be from each other max = The maximum distance the two objects can be from each other
Sources	
Ref. By	

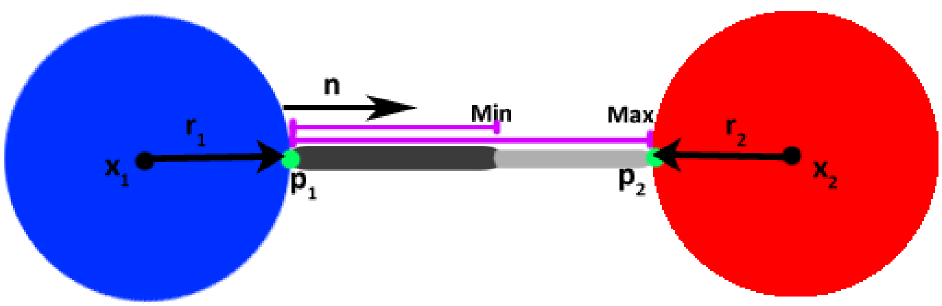

Figure 3: Slide joint between two objects

3.4.5 Pivot joint

A pivot joint holds two constraint points together. The rotational point is not fixed. Figure 4 shows an example of a pivot joint

Number	IM4
Label	**Pivot Joint**
Input	IM1
Output	$\mathbf{J}$
Description	The impulse for a pivot joint is calculated as: $\mathbf{J} = (-\mathbf{b} - \mathbf{r}_\mathrm{v})\mathbf{k}^{-1}$ $\mathbf{b}$ = Error correction: $\frac{k_\mathrm{erp}}{\Delta t}(\mathbf{p}_2 - \mathbf{p}_1)$ (DD7) $\mathbf{r}_\mathrm{v}$= Relative velocity (DD5) $\mathbf{k}$ = Mass matrix(DD6)
Sources	
Ref. By	

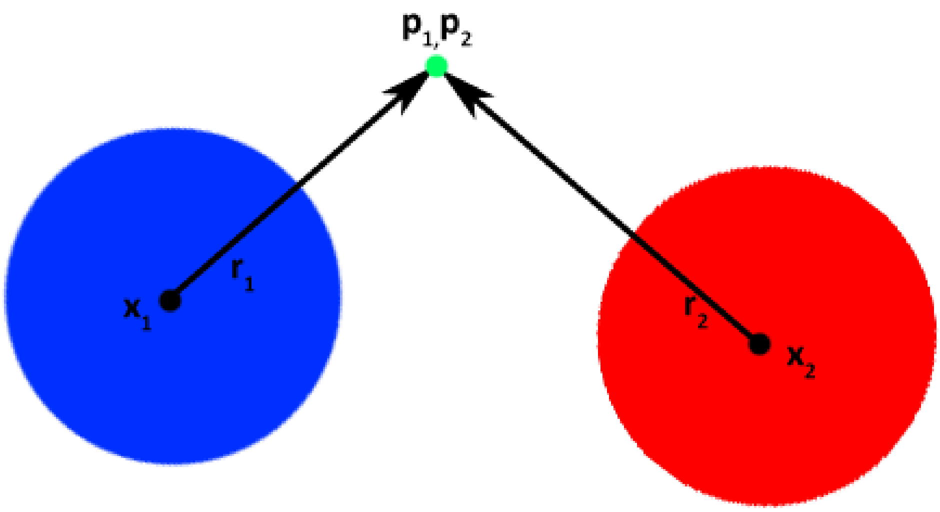

Figure 4: Pivot joint between two objects

3.4.6 Derivation of the Impulse on a Pivot Joint

The relative velocities (DD5) between the two objects should equal zero. That is

$$(\mathbf{v}_2' + \omega_2'\mathbf{r}_{2\perp} - \mathbf{v}_1' - \omega_1'\mathbf{r}_{1\perp}) = 0 \tag{3}$$

The updated velocities and angular velocities should equal:

$$\mathbf{v}_1' = \mathbf{v}_1 + \frac{-\mathbf{J}}{\mathrm{m}_1}$$

$$\mathbf{v}_2' = \mathbf{v}_2 + \frac{\mathbf{J}}{\mathrm{m}_2}$$

$$\omega_1' = \omega_1 + \frac{-\mathbf{r}_1 \times \mathbf{J}}{I_1}$$

$$\omega_2' = \omega_2 + \frac{\mathbf{r}_2 \times \mathbf{J}}{I_2}$$

Substituting the above equations into Eq3:

$$\mathbf{v}_2 + \frac{\mathbf{J}}{\mathrm{m}_2} + (\omega_2 + \frac{\mathbf{r}_2 \times \mathbf{J}}{I_2})\mathbf{r}_{2\perp} - \mathbf{v}_1 + \frac{\mathbf{J}}{\mathrm{m}_1} + (-\omega_1 + \frac{\mathbf{r}_1 \times \mathbf{J}}{I_1})\mathbf{r}_{1\perp} = 0$$

Rearrange

$$\mathbf{J}(\frac{1}{\mathrm{m}_2} + \frac{1}{\mathrm{m}_1}) + \frac{(\mathbf{r}_2 \times \mathbf{J})}{I_2}\mathbf{r}_{2\perp} + \frac{(\mathbf{r}_1 \times \mathbf{J})}{I_1}\mathbf{r}_{1\perp} = \mathbf{v}_1 + \omega_1\mathbf{r}_{1\perp} - \mathbf{v}_2 - \omega_2\mathbf{r}_{2\perp}$$

Expand cross product and perpendicular vector using DD9 and DD8

$$\mathbf{J}(\frac{1}{\mathrm{m}_2} + \frac{1}{\mathrm{m}_1}) + \frac{(r_{2\mathrm{x}}J_{\mathrm{y}} - r_{2\mathrm{y}}J_{\mathrm{x}})}{I_2}\begin{bmatrix} -r_{2\mathrm{y}} \\ r_{2\mathrm{x}} \end{bmatrix} + \frac{(r_{1\mathrm{x}}J_{\mathrm{y}} - r_{1\mathrm{y}}J_{\mathrm{x}})}{I_1}\begin{bmatrix} -r_{1\mathrm{y}} \\ r_{1\mathrm{x}} \end{bmatrix} = \mathbf{v}_1 + \omega_1\mathbf{r}_{1\perp} - \mathbf{v}_2 - \omega_2\mathbf{r}_{2\perp}$$

Multiply the cross product by the perpendicular vector

$$\mathbf{J}(\frac{1}{\mathrm{m}_2} + \frac{1}{\mathrm{m}_1}) + \frac{\begin{bmatrix} J_{\mathrm{x}}r_{2\mathrm{y}}^2 - J_{\mathrm{y}}r_{2\mathrm{x}}r_{2\mathrm{y}} \\ J_{\mathrm{y}}r_{2\mathrm{x}}^2 - J_{\mathrm{x}}r_{2\mathrm{x}}r_{2\mathrm{y}} \end{bmatrix}}{I_2} + \frac{\begin{bmatrix} J_{\mathrm{x}}r_{1\mathrm{y}}^2 - J_{\mathrm{y}}r_{1\mathrm{x}}r_{1\mathrm{y}} \\ J_{\mathrm{y}}r_{1\mathrm{x}}^2 - J_{\mathrm{x}}r_{1\mathrm{x}}r_{1\mathrm{y}} \end{bmatrix}}{I_1} = \mathbf{v}_1 + \omega_1\mathbf{r}_{1\perp} - \mathbf{v}_2 - \omega_2\mathbf{r}_{2\perp}$$

Factor out the impulse

$$\mathbf{J}\begin{bmatrix} \frac{1}{\mathrm{m}_2} + \frac{1}{\mathrm{m}_1} & 0 \\ 0 & \frac{1}{\mathrm{m}_2} + \frac{1}{\mathrm{m}_1} \end{bmatrix} + \mathbf{J}\begin{bmatrix} \frac{r_{2\mathrm{y}}^2}{I_2} & \frac{-r_{2\mathrm{x}}r_{2\mathrm{y}}}{I_2} \\ \frac{-r_{2\mathrm{x}}r_{2\mathrm{y}}}{I_2} & \frac{r_{2\mathrm{x}}^2}{I_2} \end{bmatrix} + \mathbf{J}\begin{bmatrix} \frac{r_{1\mathrm{y}}^2}{I_1} & \frac{-r_{1\mathrm{x}}r_{1\mathrm{y}}}{I_1} \\ \frac{-r_{1\mathrm{x}}r_{1\mathrm{y}}}{I_1} & \frac{r_{1\mathrm{x}}^2}{I_1} \end{bmatrix} = \mathbf{v}_1 + \omega_1\mathbf{r}_{1\perp} - \mathbf{v}_2 - \omega_2\mathbf{r}_{2\perp}$$

Add the matrices together:

$$\mathbf{J}\begin{bmatrix} \frac{1}{\mathrm{m}_2} + \frac{1}{\mathrm{m}_1} + \frac{r_{1\mathrm{y}}^2}{I_1} + \frac{r_{2\mathrm{y}}^2}{I_2} & \frac{-r_{1\mathrm{x}}*r_{1\mathrm{y}}}{I_1} + \frac{-r_{2\mathrm{x}}*r_{2\mathrm{y}}}{I_2} \\ \frac{-r_{1\mathrm{x}}*r_{1\mathrm{y}}}{I_1} + \frac{-r_{2\mathrm{x}}*r_{2\mathrm{y}}}{I_2} & \frac{1}{\mathrm{m}_2} + \frac{1}{\mathrm{m}_1} + \frac{r_{1\mathrm{x}}^2}{I_1} + \frac{r_{2\mathrm{x}}^2}{I_2} \end{bmatrix} = \mathbf{v}_1 + \omega_1\mathbf{r}_{1\perp} - \mathbf{v}_2 - \omega_2\mathbf{r}_{2\perp}$$

Rearrange:

$$\mathbf{J} = (\mathbf{v}_1 + \omega_1\mathbf{r}_{1\perp} - \mathbf{v}_2 - \omega_2\mathbf{r}_{2\perp})\begin{bmatrix} \frac{1}{\mathrm{m}_2} + \frac{1}{\mathrm{m}_1} + \frac{r_{1\mathrm{y}}^2}{I_1} + \frac{r_{2\mathrm{y}}^2}{I_2} & \frac{-r_{1\mathrm{x}}*r_{1\mathrm{y}}}{I_1} + \frac{-r_{2\mathrm{x}}*r_{2\mathrm{y}}}{I_2} \\ \frac{-r_{1\mathrm{x}}*r_{1\mathrm{y}}}{I_1} + \frac{-r_{2\mathrm{x}}*r_{2\mathrm{y}}}{I_2} & \frac{1}{\mathrm{m}_2} + \frac{1}{\mathrm{m}_1} + \frac{r_{1\mathrm{x}}^2}{I_1} + \frac{r_{2\mathrm{x}}^2}{I_2} \end{bmatrix}^{-1}$$

Simplifying using DD6 and DD5:

$$\mathbf{J} = (-\mathbf{r}_{\mathrm{v}})\mathbf{k}^{-1}$$

As mentioned in DD7 an error correction term must be added to the velocity to fix potential issues with the position:

$$\mathbf{J} = (-\mathbf{b} - \mathbf{r}_{\mathrm{v}})\mathbf{k}^{-1}$$

3.4.7 Groove joint

A groove joint is similar to a pivot joint, except one of the constraints is a line segment that the pivot can slide in. Figure 5 shows an example of a groove joint.

Number	IM5
Label	**Groove Joint**
Input	IM1, $\mathbf{g}_{\mathrm{a}}, \mathbf{g}_{\mathrm{b}}$
Output	$\mathbf{J}$
Description	Impulse is calculated the same as 4 except: $\mathbf{r}_1 = \begin{cases} \mathbf{g}_{\mathrm{a}} - \mathbf{x}_1 & \text{if } \mathbf{p}_2 \times \mathbf{n} \leq \mathbf{g}_{\mathrm{a}} \times \mathbf{n} \\ \mathbf{g}_{\mathrm{b}} - \mathbf{x}_1 & \text{if } \mathbf{p}_2 \times \mathbf{n} \geq \mathbf{g}_{\mathrm{b}} \times \mathbf{n} \\ -(\mathbf{p}_2 \times \mathbf{n})\mathbf{n}_{\perp} + \mathbf{n}(\mathbf{g}_{\mathrm{a}} \cdot \mathbf{n}) - \mathbf{x}_1 & \text{otherwise} \end{cases}$ $\mathbf{g}_{\mathrm{a}}$ = The point the line segment begins $\mathbf{g}_{\mathrm{b}}$ = The point the line segment ends $\mathbf{n}$ = The normalized vector perpendicular to the line segment
Sources	
Ref. By	

3.4.8 Damped Spring

A spring between two objects. The spring tries to stay at its rest length (d). Figure 6 shows an example of a damped spring.

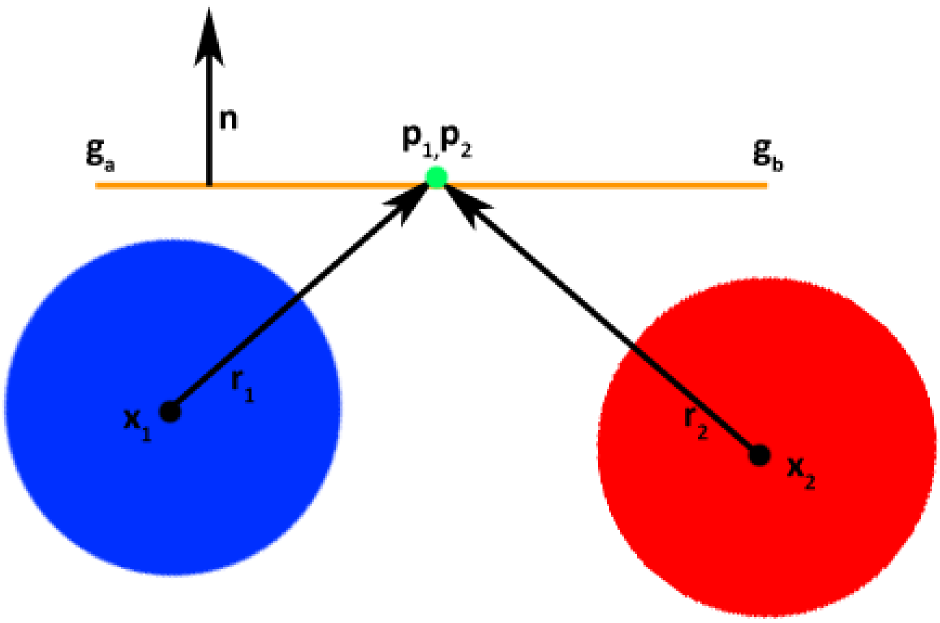

Figure 5: Groove joint between two objects

Number	IM6
Label	**Damped Spring**
Input	IM1,ζ, k , d
Output	$\mathbf{J}$
Description	$\mathbf{J}$= $(k(\lvert\mathbf{p}_2 - \mathbf{p}_1\rvert - d)\Delta t)\mathbf{n}$ - $((\mathbf{r}_\mathrm{v} \cdot \mathbf{n}(1 - e^{-\zeta \frac{1}{m_\mathrm{n}} t})m_\mathrm{n})\mathbf{n})$ k= Spring stiffness $\mathbf{p}_\mathrm{i}$= Position of the constraint of the ith object d= Rest length of the spring $\mathbf{r}_\mathrm{v}$= Relative velocity (DD5) $\mathbf{n}$ =The normal vector between the two constraints (DD1) m_n = Mass normal (DD3) t = Time Δt= Time in between frames ζ =Damping coefficient
Sources	
Ref. By	

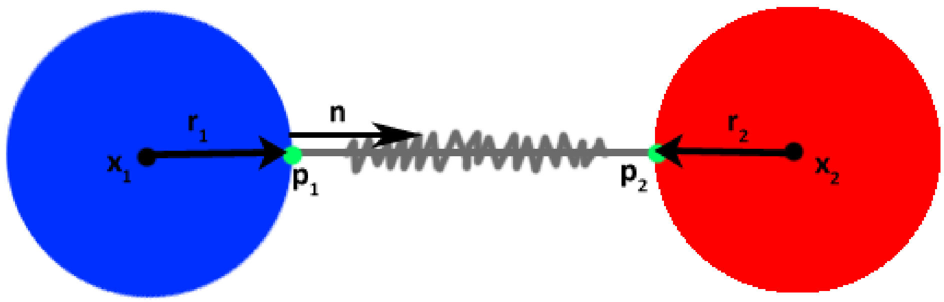

Figure 6: Damped spring between two objects

3.4.9 Derivation of the Impulse on a Damped Spring

Acts as a spring between the two objects. Hooke's law states:

$$\mathbf{F} = k\mathbf{X}$$

Where x is the amount the spring is displaced from its rest length.

$$\mathbf{F} = k(|\mathbf{p}_2 - \mathbf{p}_1| - d)\mathbf{n}$$

Impulse is a force acting over an interval of time:

$$\mathbf{J} = k\Delta t(|\mathbf{p}_2 - \mathbf{p}_1| - d)\mathbf{n} \tag{4}$$

With no damping ($\zeta = 0$), the above equation would be the impulse. With damping, an impulse to calculate the damping is needed. The updated relative velocities between the two objects in the direction of the constraint should equal the old relative velocities multiplied by a damping value (γ). That is:

$$(\mathbf{v}_2' + \omega_2'\mathbf{r}_{2\perp} - \mathbf{v}_1' - \omega_1'\mathbf{r}_{1\perp}) \cdot \mathbf{n} = \gamma(\mathbf{v}_2 + \omega_2\mathbf{r}_{2\perp} - \mathbf{v}_1 - \omega_1\mathbf{r}_{1\perp}) \cdot \mathbf{n} \tag{5}$$

The updated velocities and angular velocities should equal:

$$\mathbf{v}_1' = \mathbf{v}_1 + \frac{-\mathbf{J}}{\mathrm{m}_1}$$

$$\mathbf{v}_2' = \mathbf{v}_2 + \frac{\mathbf{J}}{\mathrm{m}_2}$$

$$\omega_1' = \omega_1 + \frac{-\mathbf{r}_1 \times \mathbf{J}}{I_1}$$

$$\omega_2' = \omega_2 + \frac{\mathbf{r}_2 \times \mathbf{J}}{I_2}$$

The impulse should only be applied in the direction of the constraint:

$$\mathbf{J} = J\mathbf{n} \tag{6}$$

Substituting the above equations into Eq5:

$$(\mathbf{v}_2 + \frac{J\mathbf{n}}{\mathrm{m}_2} + (\omega_2 + \frac{\mathbf{r}_2 \times J\mathbf{n}}{I_2})\mathbf{r}_{2\perp} - \mathbf{v}_1 + \frac{J\mathbf{n}}{\mathrm{m}_1} + (-\omega_1 + \frac{\mathbf{r}_1 \times J\mathbf{n}}{I_1})\mathbf{r}_{1\perp}) \cdot \mathbf{n} = \gamma(\mathbf{v}_2 + \omega_2\mathbf{r}_{2\perp} - \mathbf{v}_1 - \omega_1\mathbf{r}_{1\perp}) \cdot \mathbf{n}$$

Factor out the impulse

$$J\mathbf{n} \cdot \mathbf{n}(\frac{\mathbf{n}}{\mathrm{m}_2} + \frac{(\mathbf{r}_2 \times \mathbf{n})\mathbf{r}_{2\perp}}{I_2} + \frac{\mathbf{n}}{\mathrm{m}_1} + \frac{(\mathbf{r}_1 \times \mathbf{n})\mathbf{r}_{1\perp}}{I_1}) \cdot \mathbf{n} + \mathbf{v}_2 \cdot \mathbf{n} + \omega_2\mathbf{r}_{2\perp} \cdot \mathbf{n} - \mathbf{v}_1 \cdot \mathbf{n} - \omega_1\mathbf{r}_{1\perp} \cdot \mathbf{n} = \gamma(\mathbf{v}_2 + \omega_2\mathbf{r}_{2\perp} - \mathbf{v}_1 - \omega_1\mathbf{r}_{1\perp}) \cdot \mathbf{n}$$

Using DD8 and DD9

$$\mathbf{r}_{\mathrm{i}\perp} \cdot \mathbf{n} = \mathbf{r}_{\mathrm{i}} \times \mathbf{n}$$

Simplifying

$$J(\frac{1}{\mathrm{m}_2} + \frac{(\mathbf{r}_2 \times \mathbf{n})^2}{I_2} + \frac{1}{\mathrm{m}_1} + \frac{(\mathbf{r}_1 \times \mathbf{n})^2}{I_1}) + \mathbf{v}_2 \cdot \mathbf{n} + \omega_2\mathbf{r}_{2\perp} \cdot \mathbf{n} - \mathbf{v}_1 \cdot \mathbf{n} - \omega_1\mathbf{r}_{1\perp} \cdot \mathbf{n} = \gamma(\mathbf{v}_2 + \omega_2\mathbf{r}_{2\perp} - \mathbf{v}_1 - \omega_1\mathbf{r}_{1\perp}) \cdot \mathbf{n}$$

Rearrange:

$$J = \frac{(\mathbf{v}_1 + \omega_1\mathbf{r}_{1\perp} - \mathbf{v}_2 - \omega_2\mathbf{r}_{2\perp}) \cdot \mathbf{n} + \gamma(\mathbf{v}_2 + \omega_2\mathbf{r}_{2\perp} - \mathbf{v}_1 - \omega_1\mathbf{r}_{1\perp}) \cdot \mathbf{n}}{\frac{1}{\mathrm{m}_2} + \frac{(\mathbf{n} \times \mathbf{r}_2)^2}{I_2} + \frac{1}{\mathrm{m}_1} + \frac{(\mathbf{n} \times \mathbf{r}_1)^2}{I_1}}$$

Simplifying:

$$J = \frac{(1 - \gamma)(\mathbf{v}_1 + \omega_1\mathbf{r}_{1\perp} - \mathbf{v}_2 - \omega_2\mathbf{r}_{2\perp}) \cdot \mathbf{n}}{\frac{1}{\mathrm{m}_2} + \frac{(\mathbf{n} \times \mathbf{r}_2)^2}{I_2} + \frac{1}{\mathrm{m}_1} + \frac{(\mathbf{n} \times \mathbf{r}_1)^2}{I_1}} \tag{7}$$

The damping value that is used in chipmunk is:

$$\gamma = e^{-\zeta(\frac{1}{m_1} + \frac{1}{I_1}(\mathbf{r}_1 \times \mathbf{n})^2 + \frac{1}{m_2} + \frac{1}{I_2}(\mathbf{r}_2 \times \mathbf{n})^2)t}$$

Substituting the above damping value into Eq7:

$$J = \frac{(1 - e^{-\zeta(\frac{1}{m_1} + \frac{1}{I_1}(\mathbf{r}_1 \times \mathbf{n})^2 + \frac{1}{m_2} + \frac{1}{I_2}(\mathbf{r}_2 \times \mathbf{n})^2)t})(\mathbf{v}_1 + \omega_1\mathbf{r}_{1\perp} - \mathbf{v}_2 - \omega_2\mathbf{r}_{2\perp}) \cdot \mathbf{n}}{\frac{1}{\mathrm{m}_2} + \frac{(\mathbf{n} \times \mathbf{r}_2)^2}{I_2} + \frac{1}{\mathrm{m}_1} + \frac{(\mathbf{n} \times \mathbf{r}_1)^2}{I_1}}$$

Using DD5 and DD3 the above equation simplifies to:

$$J = (-\mathbf{r}_{\mathrm{v}} \cdot \mathbf{n}(1 - e^{-\zeta \frac{1}{m_{\mathrm{n}}} t}) m_{\mathrm{n}})$$

Substituting the above equation into Eq6:

$$\mathbf{J} = (-\mathbf{r}_{\mathrm{v}} \cdot \mathbf{n}(1 - e^{-\zeta \frac{1}{m_{\mathrm{n}}} t}) m_{\mathrm{n}})\mathbf{n} \tag{8}$$

Add Eq4 and Eq8:

$$\mathbf{J} = (k(|\mathbf{p}_2 - \mathbf{p}_1| - d)\Delta t)\mathbf{n} - ((\mathbf{r}_{\mathrm{v}} \cdot \mathbf{n}(1 - e^{-\zeta \frac{1}{m_{\mathrm{n}}} t}) m_{\mathrm{n}})\mathbf{n})$$

Number	IM7
Label	**Constraint on a 2D rigid body - Rotation only**
Input	$J, \phi_1, \phi_2, I_1, I_2, \omega_1, \omega_2$
Output	ω_1', ω_2' such that: $\omega_1' = \omega_1 + \frac{-J}{I_1}$ $\omega_2' = \omega_2 + \frac{J}{I_2}$
Description	J= The impulse acting on the constraint ϕ=The orientation of the object I=The moment of inertia ω=The initial angular velocity of the object ω'=The new angular velocity of the object The following instance models use the same input and output as described above. The impulse for each constraint is calculated differently and is outlined below.
Sources	
Ref. By	IM8, IM9, IM10, IM11, IM12

3.4.10 Derivation of the angular velocity

The impulse on an object is equal to the change in its angular momentum, that is:

$$\mathbf{J} = \Delta L$$

Using GD1:

$$\mathbf{J} = I\Delta\omega = I(\omega' - \omega)$$

Rearrange:

$$\omega' = \omega + \frac{J}{I}$$

Newton's Third Law of motion says that the impulse felt by one object is equal and opposite the impulse felt by the other object.

$$\omega_1' = \omega_1 + \frac{-J}{I_1}$$

$$\omega_2' = \omega_2 + \frac{J}{I_2}$$

3.4.11 Rotary Damped Spring

Similar to a damped spring, instead of the spring trying to remain at its rest length, a rotary damped spring tries to remain at its rest angle (θ) - the difference between the two object's orientations. Figure 7 shows an example of a rotary damped spring.

Number	IM8
Label	**Rotary Damped Spring**
Input	IM7,ζ, k , θ
Output	J
Description	$J = k(\lvert\phi_2 - \phi_1\rvert - \theta)\Delta t - \frac{(\omega_2-\omega_1)(1-e^{-\zeta(\frac{1}{I_1}+\frac{1}{I_2})t})}{\frac{1}{I_1}+\frac{1}{I_2}}$ k = Spring stiffness ϕ_{i} = The orientation of the ith object θ = The difference between the two object's orientations (the rest angle) t = Time Δt = Time in between frames ω_{i}= The angular velocity of the ith object I_{i}= The moment of inertia of the ith object ζ =Damping coefficient
Sources	
Ref. By	

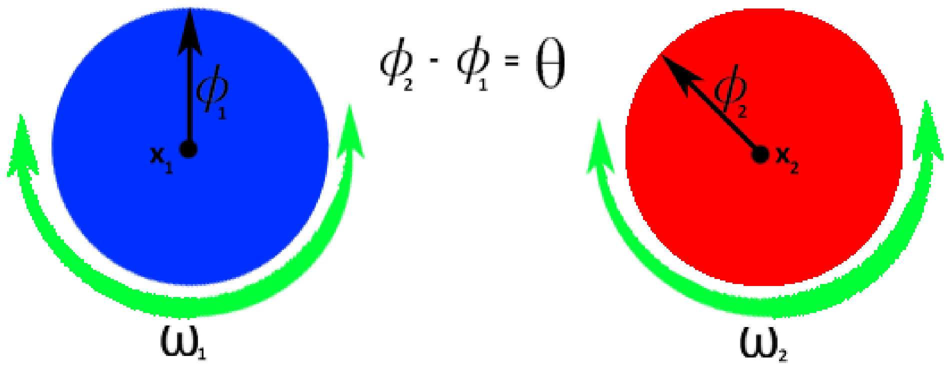

Figure 7: Rotary Damped spring between two objects

3.4.12 Derivation of the Impulse on a Damped Rotary Spring

Hooke's law states:

$$F = kX$$

Where X is the amount the rotary spring is displaced from its rest angle.

$$F = k(|\phi_2 - \phi_1| - \theta)$$

Impulse is a force acting over an interval of time:

$$J = k\Delta t(|\phi_2 - \phi_1| - \theta) \tag{9}$$

With no damping ($\zeta = 0$), the above equation would be the impulse. With damping, an impulse to calculate the damping is needed. The updated relative angular velocities between the two objects in the direction of the constraint should equal the old relative angular velocities multiplied by a damping value (γ). That is:

$$\omega_2' - \omega_1' = \gamma(\omega_2 - \omega_1) \tag{10}$$

The updated angular velocities should equal:

$$\omega_1' = \omega_1 + \frac{-J}{I_1}$$

$$\omega_2' = \omega_2 + \frac{J}{I_2}$$

Substituting the above equations into Eq10:

$$\omega_2 + \frac{J}{I_2} - \omega_1 + \frac{J}{I_1} = \gamma(\omega_2 - \omega_1)$$

Factor out the impulse

$$J(\frac{1}{I_2} + \frac{1}{I_1}) + \omega_2 - \omega_1 = \gamma(\omega_2 - \omega_1)$$

Rearrange:

$$J = \frac{\omega_1 - \omega_2 + \gamma(\omega_2 - \omega_1)}{\frac{1}{I_2} + \frac{1}{I_1}}$$

Simplifying:

$$J = -\frac{(1-\gamma)(\omega_2 - \omega_1)}{\frac{1}{I_2} + \frac{1}{I_1}} \tag{11}$$

The damping value that is used in chipmunk is:

$$\gamma = e^{-\zeta(\frac{1}{I_1} + \frac{1}{I_2})t}$$

Substituting the above damping value into Eq11:

$$J = \frac{(1 - e^{-\zeta(\frac{1}{I_1} + \frac{1}{I_2})t})(\omega_1 - \omega_2)}{\frac{1}{I_2} + \frac{1}{I_1}} \tag{12}$$

Add Eq9 and Eq12:

$$J = k(|\phi_2 - \phi_1| - \theta)\Delta t - \frac{(\omega_2 - \omega_1)(1 - e^{-\zeta(\frac{1}{I_1} + \frac{1}{I_2})t})}{\frac{1}{I_1} + \frac{1}{I_2}}$$

3.4.13 Rotary Limit Joint

A rotary limit joint forces two bodies to be in a constant angular range to each other [min,max] Figure 8 shows an example of a rotary joint.

Number	IM9
Label	**Rotary Limit Joint**
Input	IM7 , min , max
Output	J
Description	$J = \frac{-b+\omega_1-\omega_2}{\frac{1}{I_1}+\frac{1}{I_2}}$ $\text{b} = \begin{cases} \frac{k_{\text{erp}}}{\Delta t}(max - \phi_2 - \phi_1) & \text{if } \phi_2 - \phi_1 > max \\ \frac{k_{\text{erp}}}{\Delta t}(min - \phi_2 - \phi_1) & \text{if } \phi_2 - \phi_1 < min \\ 0 \text{ , no impulse} & \text{otherwise} \end{cases}$ b = Error correction: (DD7) min = The minimum angle the two objects can be from each other max = The maximum angle the two objects can be from each other
Sources	
Ref. By	

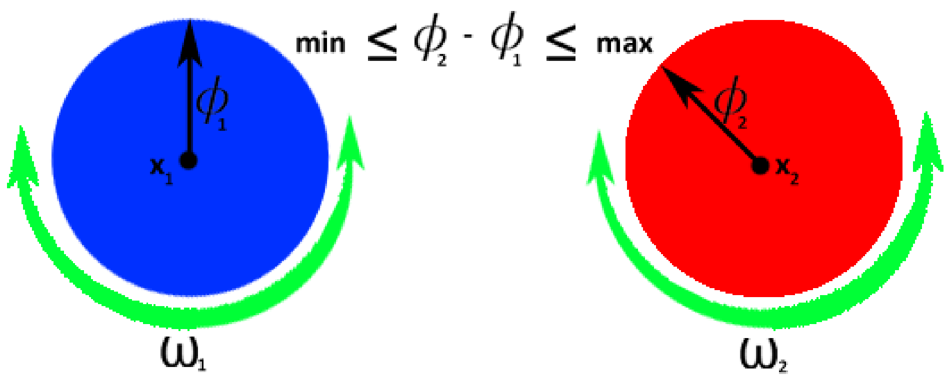

Figure 8: Rotary Joint between two objects

3.4.14 Derivation of the Impulse on a Rotary Limit Joint

A rotary limit joint constrains the objects to be within a certain orientation of each other. Only apply impulse if the object's orientations are not within [min, max] of each other. When the impulse is applied the relative angular velocities should be zero. That is:

$$\omega_2' - \omega_1' = 0 \tag{13}$$

The updated angular velocities should equal:

$$\omega_1' = \omega_1 + \frac{-J}{I_1}$$

$$\omega_2' = \omega_2 + \frac{J}{I_2}$$

Substituting the above equations into Eq13:

$$\omega_2 + \frac{J}{I_2} - \omega_1 + \frac{J}{I_1} = 0$$

Factor out the impulse:

$$J(\frac{1}{I_2} + \frac{1}{I_1}) + \omega_2 - \omega_1 = 0$$

Rearranging:

$$J = \frac{\omega_1 - \omega_2}{\frac{1}{I_2} + \frac{1}{I_1}}$$

As mentioned in DD7 an error correction term must be added to the angular velocity to fix potential issues with the orientation:

$$J = \frac{-b + \omega_1 - \omega_2}{\frac{1}{I_2} + \frac{1}{I_1}}$$

3.4.15 Ratchet Joint

Acts like a ratchet. Forces one body to only follow one direction of rotation from the other body. Figure 9 shows an example of ratchet joint.

Number	IM10
Label	**Ratchet Joint**
Input	IM7, $\sigma(phase), \rho(ratchet)$
Output	J
Description	Impulse is calculated the same as 9 except: $b = \begin{cases} \frac{k_{\text{erp}}}{\Delta t}(\theta - (\phi_2' - \phi_1')) & \text{if } (\theta - (\phi_2' - \phi_1'))\rho > 0 \\ \text{no impulse, } \theta = \lfloor \frac{(\phi_2' - \phi_1' - \sigma)}{\rho} \rfloor \rho + \sigma & \text{otherwise} \end{cases}$ b = Error correction (DD7) ρ = Ratchet. The angles which an impulse will be applied to the other object. An impulse will be applied every kρ where k = 0,1,2,3..... σ =Phase. The angle offset. An impulse will be applied every k$\rho + \sigma$ where k = 0,1,2,3.....
Sources	
Ref. By	

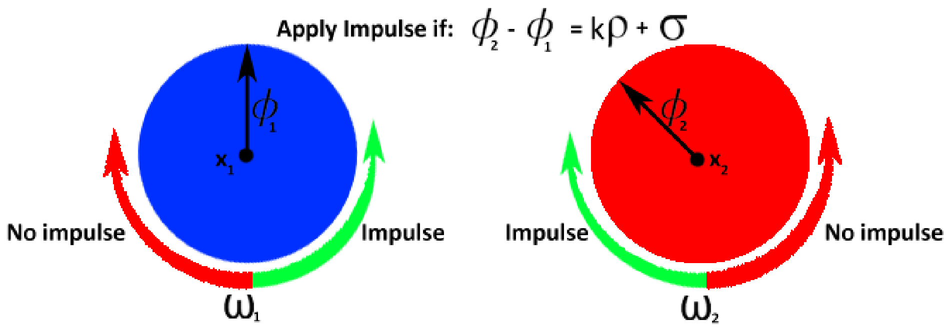

Figure 9: Ratchet joint between two objects

3.4.16 Gear Joint

Maintains a specific angular velocity ratio between the two objects. Uses the ratio (β) to determine how fast the opposing body will rotate. Figure 10 shows an example of a gear joint.

Number	IM11
Label	**Gear Joint**
Input	IM7, $\sigma(phase), \beta(ratio)$
Output	J
Description	The updated angular velocity is calculated differently than in IM7: $\omega_1' = \omega_1 + \frac{-J}{I_1}\frac{1}{\beta}$ $\omega_2' = \omega_2 + \frac{J}{I_2}$ The impulse is calculated as follows: $J = \frac{-b-\omega_2\beta+\omega_1}{\frac{1}{I_1}\frac{1}{\beta}+\frac{\beta}{I_2}}$ b = Error correction: $\frac{k_{\text{erp}}}{\Delta t}(\phi_2\beta - \phi_1 - \sigma)$ (DD7) β = The ratio between the two angular velocities σ = Phase. The angle offset.
Sources	
Ref. By	

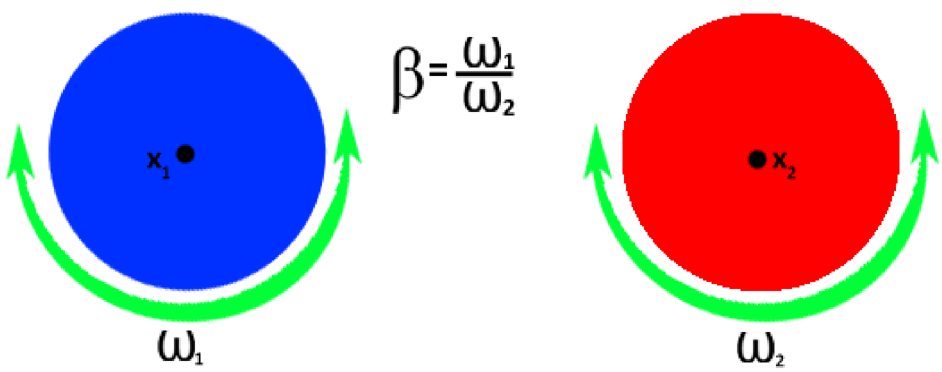

Figure 10: Gear joint between two objects

3.4.17 Derivation of the Impulse on a Gear Joint

A gear joint maintains a specific angular velocity ratio. That is:

$$\beta\omega_2' - \omega_1' = 0 \tag{14}$$

The updated angular velocities should equal:

$$\omega_1' = \omega_1 + \frac{-J}{I_1}\frac{1}{\beta}$$

$$\omega_2' = \omega_2 + \frac{J}{I_2}$$

Substituting the above equations into Eq14:

$$\beta\omega_2 + \beta\frac{J}{I_2} - \omega_1 + \frac{J}{I_1}\frac{1}{\beta} = 0$$

Factor out the impulse:

$$J(\frac{\beta}{I_2} + \frac{1}{I_1}\frac{1}{\beta}) + \beta\omega_2 - \omega_1 = 0$$

Rearranging:

$$J = \frac{\omega_1 - \beta\omega_2}{\frac{\beta}{I_2} + \frac{1}{I_1}\frac{1}{\beta}}$$

As mentioned in DD7 an error correction term must be added to the angular velocity to fix potential issues with the orientation:

$$J = \frac{-b + \omega_1 - \beta\omega_2}{\frac{\beta}{I_2} + \frac{1}{I_1}\frac{1}{\beta}}$$

3.4.18 Simple Motor

Maintains a constant angular relative velocity between the two objects. Figure 11 shows an example of a simple motor..

Number	IM12
Label	**Simple Motor**
Input	IM7, ω_{r}
Output	J
Description	$J = \frac{\omega_1 - \omega_2 - \omega_{\mathrm{r}}}{\frac{1}{I_1} + \frac{1}{I_2}}$ ω_{r}= The specific angular relative velocity
Sources	
Ref. By	

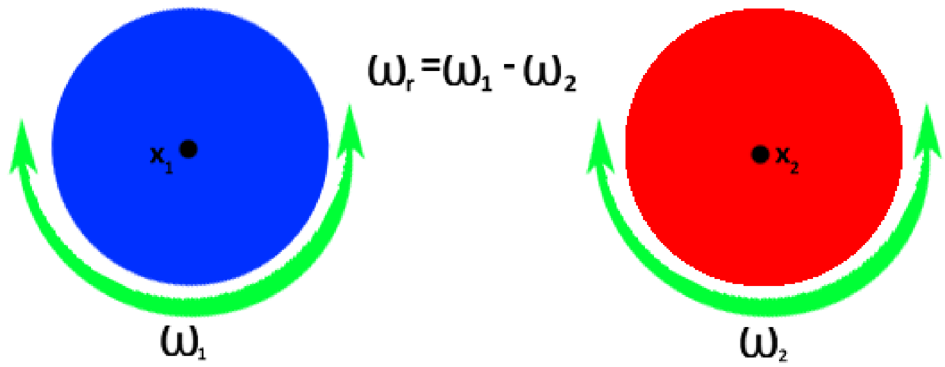

Figure 11: Simple motor between two objects

3.4.19 Derivation of the Impulse on a Simple Motor

A simple motor maintains a specific angular relative velocity. That is:

$$\omega_2' - \omega_1' = -\omega_\mathrm{r} \tag{15}$$

The updated angular velocities should equal:

$$\omega_1' = \omega_1 + \frac{-J}{I_1}$$

$$\omega_2' = \omega_2 + \frac{J}{I_2}$$

Substituting the above equations into Eq15:

$$\omega_2 + \frac{J}{I_2} - \omega_1 + \frac{J}{I_1} = -\omega_\mathrm{r}$$

Factor out the impulse:

$$J(\frac{1}{I_2} + \frac{1}{I_1}) + \omega_2 - \omega_1 = -\omega_\mathrm{r}$$

Rearranging:

$$J = \frac{\omega_1 - \omega_2 - \omega_\mathrm{r}}{\frac{1}{I_2} + \frac{1}{I_1}}$$