Modeling the Motion of an Electric Train - An Experimental and Numerical Method

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Abstract

This paper details the process of experimentation and analysis of the motion of an electric train. After controlled data was taken at 12 V, 10 V, 8 V, 6 V, and 4 V, the results were used to create a numerical/theoretical model of the train's velocity versus time in relation to the voltage run through the track. At each voltage, five separate trials were taken to ensure a good data set, and to create a reliable average of the data. Graphs to compare the velocity versus time and steady state velocity versus voltage were generated and analyzed to acquire constants. The collected data was manipulated to obtain constants values of B_S, B_I and A of 0.0478, 0.1189, and 0.350727, respectively. The results show that the train had increased velocity and displacement as voltage and time increased. A numerical model was generated to predict the velocity, position or voltage of the electric train, provided the other necessary values were given. The challenge problems tested the accuracy of predictions with the numerical model. Five challenge problems proved the accuracy of the numerical model of the train and compared the data and model accuracy of the train to those of other electric trains.

Introduction

In this experiment, an HO scale electric train was used to create a model of its motion in relation to the voltage supplied to the track/train system. The train was placed on an electric track and run at different voltages. Flags were attached to the side of the train and photogates were used to track the train's progress across a 2 meter stretch of straight, level track. After data was taken, the results were used to create a numerical/theoretical model of the train's velocity in relation to the voltage run through the track. Five different voltages were utilized to collect data:

12 V, 10 V, 8 V, 6 V, and 4 V. At each voltage, five separate trials were run to increase accuracy and ensure a good data set. The five datasets gathered for each voltage were averaged for analysis and utilized to test the accuracy of the numerical model. The data was manipulated to produce constant values for the slope of the best fit line of the velocity steady state versus the voltage (B_S), the intercept of the best fit line of the velocity steady state versus the voltage (B_I) and the cross-sectional area of the coil (A). B_S, B_I, and A values were discovered to be 0.0478, 0.1189, and 0.350727, respectively. The experimental and theoretical data were utilized to generate a model to predict voltage, position, and velocity curves of the electric train. Collected arbitrary track data was utilized to test the accuracy of the velocity and position models in challenge problems.

<u>Theoretical</u>

To interpret and model the behavior of our toy train, we used the basic principle of Newton's 2nd Law for rotations and translations, and mechanical and electrical descriptions and derivations of the inner workings of our train. We used ***Equation 1.6*** to describe the forces acting on the model. We then used ***Equation 1.7*** to find the forces acting on the motor of the train, including the torque, or rotational force, of the motor and friction. Then we used ***Equation 1.8*** to find the forces acting on the wheel of the train. We combined ***Equation 1.7*** and ***1.8*** and used the fact that $P_1 = P_2$. P_1 and P_2 are both contact forces between the gearbox and the flywheel, and due to Newton's 3rd Law, they are equal. We then related α, the rotational acceleration, to γ, the gear ratio, using ***Equation 1.9*** and ***Equation 1.10***. To derive the mechanical model, we combined Newton's 2nd Law equations and ***Equations 1.7, 1.8, 1.9***, and

1.10. The combined mechanical model equation is ***Equation 1.3***. We then divided the whole equation by k, which is a constant, and set the terms inside the parentheses on the right side equal to I_{EQ}, and the result is ***Equation 1.4***.

The electrical model of the train is about how a voltage produces the torque on the motor of the train. The equation for the torque of the motor is ***Equation 1.12***, and 2rl is equal to A, the area of the coil, and ABsinθ is equal to a constant, C. Then using Ohm's Law (***Equation 1.11***) and the resulting torque equation of $\tau_m = Ci$, the resulting equation is $\tau_m = \frac{Cv}{R}$. ***Equation 1.5*** is the initial combination of the electrical and mechanical model. The term $\frac{dV}{dt}$ is the rotational acceleration. Then we use ***Equation 1.2*** to get the velocity steady state, which is the velocity the train stays at after acceleration. Then we combine ***Equation 1.2*** and ***Equation 1.5*** to get ***Equation 1.1***. ***Equation 1.1*** is the final equation that includes the mechanical and electrical parts of the train.

The operational parameters of the train are used to model its behavior in graphs. The three operational parameters are B_s, B_i, and A. The three previous terms are constants that represent several variables measured from our train and experiments. ***Equation 1.13***, uses all three variables. B_s represents the slope of the graph of V_{ss} vs. Voltage. The variable B_i represents the y-intercept of the same graph, and A is the cross sectional area of the coil in the train's motor. The graph V_{ss} vs Voltage was created by finding the average of the velocity data points (ignoring outliers) after the train stopped accelerating exponentially and plotting these values against their corresponding Voltage values (4, 6, 8, 10, 12 V). The A value is an average of each Voltage's A value, excluding the outlier of 4V.. Each individual A is the slope of the graph of ln(1-v(t)/Vss) vs t, time, during the exponential growth period of the velocity.

We created a numerical model to predict how our train would perform with a time-dependent voltage profile. To predict how our train would behave, we numerically approximated voltages over small time steps. The numerical model created, had the B_s, B_i, and A values that were put into equations. There were columns that took the three variables into account, and calculated voltage, velocity, and position. The time column started at 0, and went up by a hundredth of a second. For the voltage column, a certain voltage is put into the column, and used to find velocity and position. To find the velocity, we used ***Equation 1.13***, and substituted $V_N - V_{N-1}$ for the change in velocity, and Δt for the change in time to get ***Equation 1.14***. We then multiplied the whole equation by Δt, divided everything by A, and added V_{N-1} to the other side of the equation to get ***Equation 1.15.*** Then we used kinematics to find the position of the train at any time.

The reference frame we used to determine the behavior of the train was based off of the track. The x range was along the rail of the track, and the y range was directly above and below the track, although we did not use the y range for any calculations. For the inner workings of the train, we considered that the torque of the motor, and the frictional torque, were in opposite directions. We also considered the forces between the flywheel and the gearbox of the train. We assumed the contact forces between both parts were equal, and that there was a frictional and gravitational force between the wheel of the train and the rail of the track. Overall, all equations, operational parameters, and considerations were used to accurately predict the behavior of an electric train on a track.

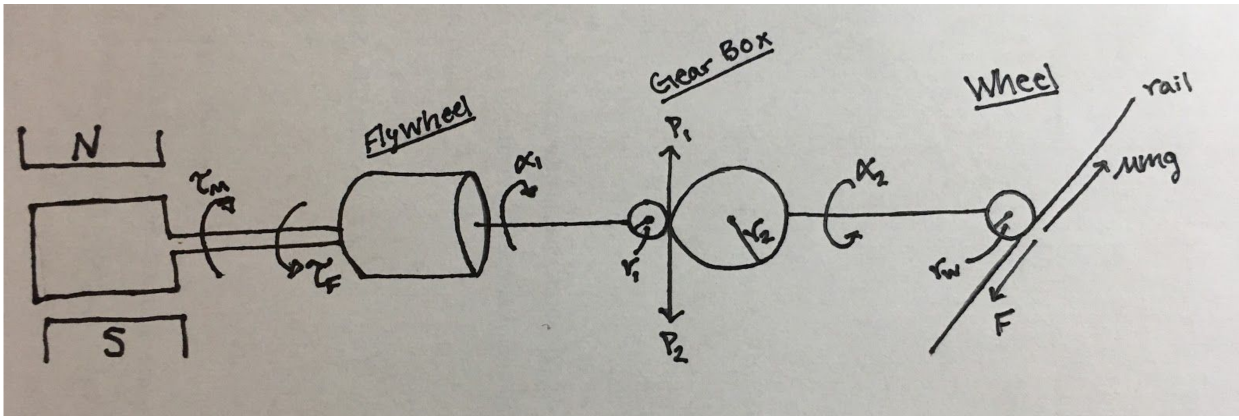


Figure 1: Schematic Diagram of Electric Train

Equations

$$V_{ss} - V = A\frac{dV}{dt} \tag{1.1}$$

$$V_{ss} = \frac{C}{Rk}\nu - \frac{\mu_k mgr_w}{\gamma} \tag{1.2}$$

$$\tau_m - kV - \left(\frac{\mu_k mgr_w}{\gamma}\right) = \left(\gamma I_1 + \frac{I_2}{\gamma} + \frac{mr_w^2}{\gamma}\right)\frac{a}{r_w} \tag{1.3}$$

$$\left(\frac{\tau_m}{k} - \frac{\mu_k mgr_w}{k\gamma}\right) - V = \left(\frac{I_{EQ}}{r_w k}\right)a \tag{1.4}$$

$$\left(\frac{C}{Rk}\nu - \frac{\mu_k mgr_w}{\gamma}\right) - V = A\frac{dV}{dt} \tag{1.5}$$

$$\sum F = ma \tag{1.6}$$

$$\tau_m - \tau_F - r_1 p_1 = I_1\alpha_1 \tag{1.7}$$

$$r_2 p_2 - r_w F = I_2 \alpha_2 \tag{1.8}$$

$$\frac{r_2}{r_1} = \gamma \tag{1.9}$$

$$\alpha_1 = \gamma \alpha_2 \tag{1.10}$$

$$\nu = IR \tag{1.11}$$

$$\tau_m = 2rilBsin(\theta) \tag{1.12}$$

$$B_s \nu - B_i - V = A\frac{dV}{dt} \tag{1.13}$$

$$B_s \nu - B_i - V_{N-1} = A\left(\frac{V_N - V_{N-1}}{\Delta t}\right) \tag{1.14}$$

$$V_N = \frac{\Delta t(B_S \nu - B_I - V_{N-1})}{A} + V_{N-1} \tag{1.15}$$

Theoretical Appendix:

V_{ss} = Velocity steady state
V = Velocity
ν = Voltage
A = Area of the coil
$\frac{dV}{dt}$ = Acceleration

Experimental

The equipment used in this experiment was: an HO scale model train, a length of electric train track, twenty photogates, an analog to digital converter used to transfer data to the computer and control the voltage supplied to the track, and a Capstone program that recorded photogate data. The first step was choosing the model train that would be used. The team selected a dark green Burlington train dubbed "Champ." Originally the train was not equipped with the flags needed to trigger the photogates used to determine velocity later in the experiment. A strip of 10 cm long plastic marked with ten black bands at 1 cm intervals was attached to the train and positioned to trigger the photogates.

Figure 2 - "Champ" pictured with flags

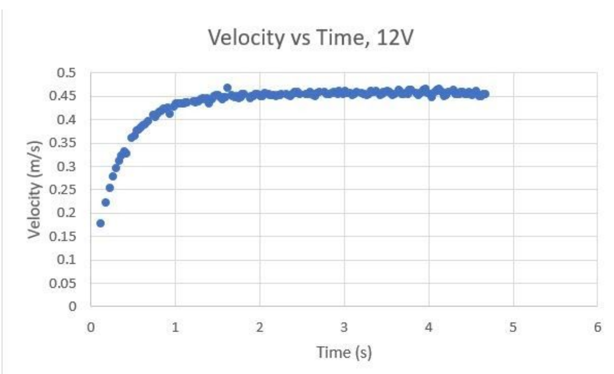
For the original multiple voltage trials, the train was positioned so that the flags aligned with the beginning of the photogates. The Capstone program was set to the first voltage being measured, 12V, and the train was set to run. Data was collected until the train had passed through all of the 20 photogates, and then stopped. The photogate data was copied and pasted into an excel sheet so it could be compiled and analyzed later. The voltages tested were 12, 10, 8, 6, and 4; five trials were completed for each voltage

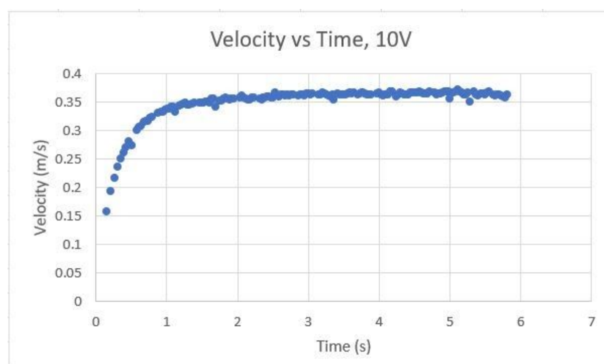
Figure 3 - Track with photogates setup

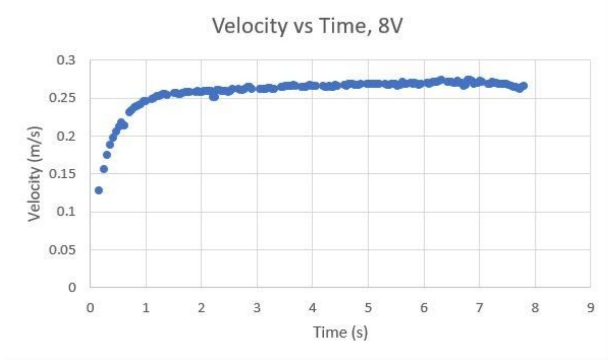
For the arbitrary voltage portion of the experiment, the procedure was similar to the phase before. The train was set back up on the track and again aligned with the first photogate. A different Capstone program was used that collected photogate and velocity data. The program utilized an arbitrary voltage and when the program was run the train was set to run for a total of 8 seconds, regardless of whether it had passed through all the photogates or not. This was repeated five times and data were recorded and stored in an excel sheet.

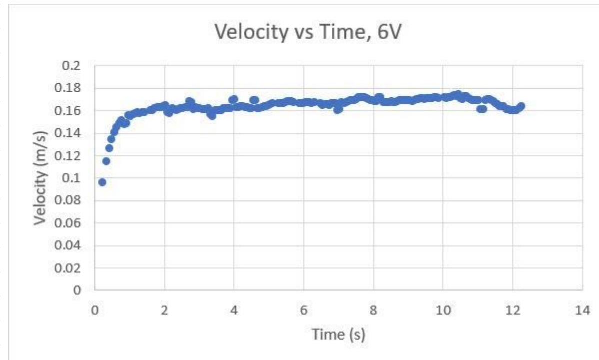
Results

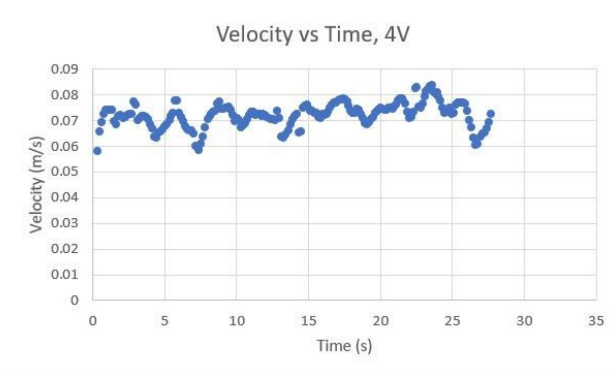
From these experiments, we gathered data to find the values of Bs, Bi, and A. The resulting raw data was in the form of timestamps from the flags on the train corresponding to voltage readings. These data points were then converted to droptimes with a binary checker and drop checker. This process was repeated for each set of raw data (5 trials for each of 5 voltages). The 5 data sets for each voltage were averaged vertically and horizontally. These averages were then used to calculate the near-instantaneous velocity of the train at the corresponding times. These velocities were plotted against time, ignoring significant outliers, which were commonly found at every 10th data point due to the nature of the experiment. For each voltage, the resulting graphs are shown below:











Figures 5-9, Velocity vs Time for Varying Voltages

For each voltage, the velocity values were averaged after the exponential growth ceased. This average was the Vss value (steady state velocity). Subsequently plotting the Vss values against the corresponding voltage values resulted in:

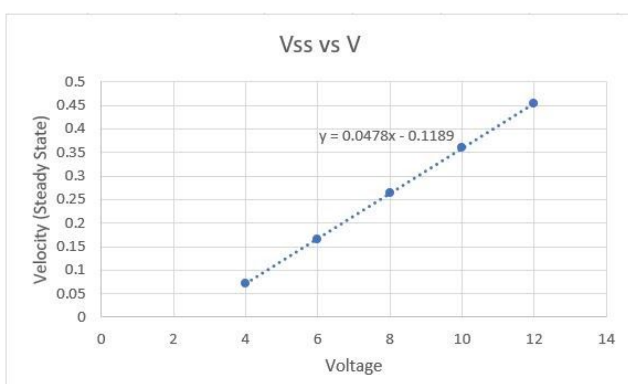
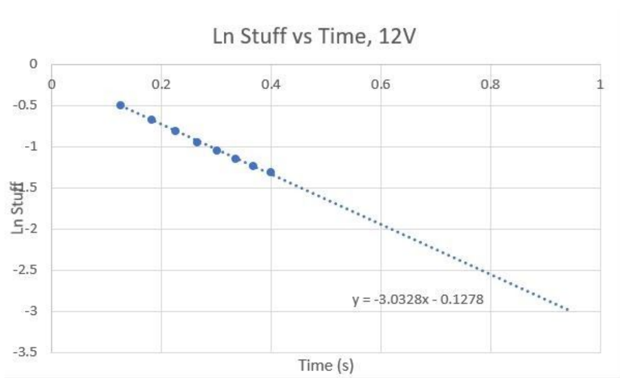
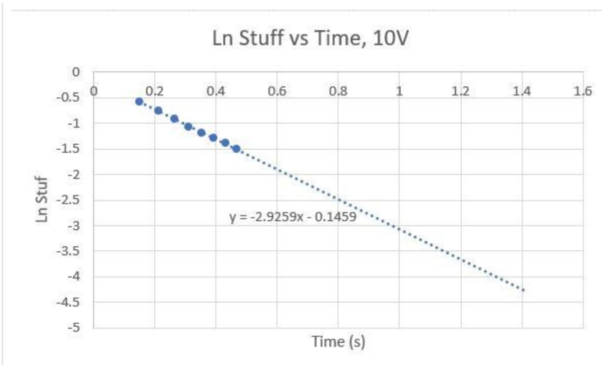


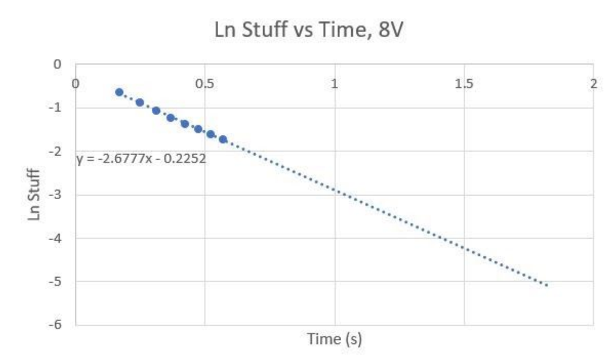
Figure 10 - Velocity Steady State vs Voltage

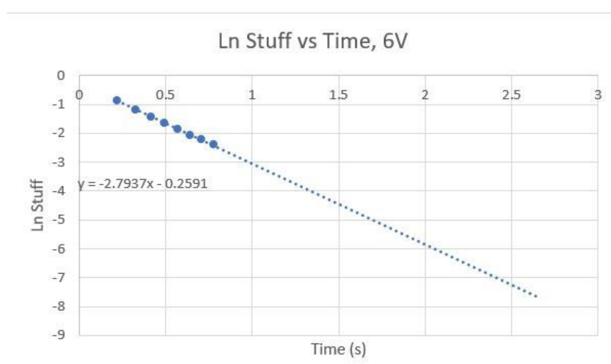
The values of Bs and Bi are the slope and y-intercept, respectively, of the trendline of this graph. Therefore, for our data, we found Bs = 0.0478 and Bi = 0.1189.

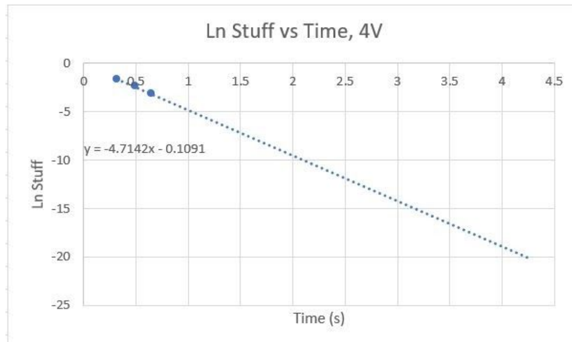
To find the value of A, we examined the exponential growth portion of the velocity vs time graph. These few velocity data points were converted to a useful measure, which was then plotted against time. The slope of the resulting linear set of data points gave the value of A by the equation A = -1/slope. The plot is ln(1-(v(t)/vss)), where v(t) is the velocity data point of exponential growth, and vss is a constant, the steady state velocity of the voltage in question. For each trial, we found:











Figures 11-15 - Plots to Find A

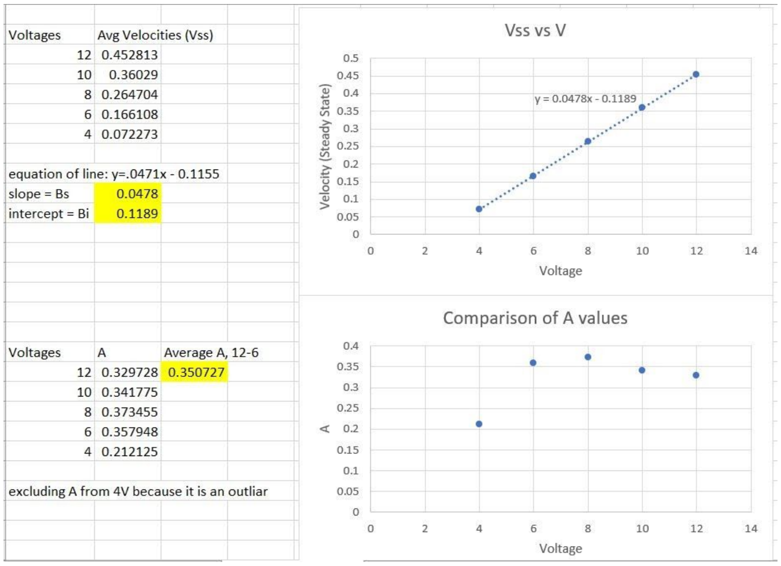
Therefore, the values of A can be represented by:

Voltage	A (-1/slope)
12	0.3297283
10	0.34177518
8	0.37345483
6	0.35794824
4	0.21212507

Figure 16 - Values of A

The A value used in our model is the average of these 5 data points excluding the outlier of the 4V point. This average is A = 0.350727.

These results as a whole can be summarized with our excel data:

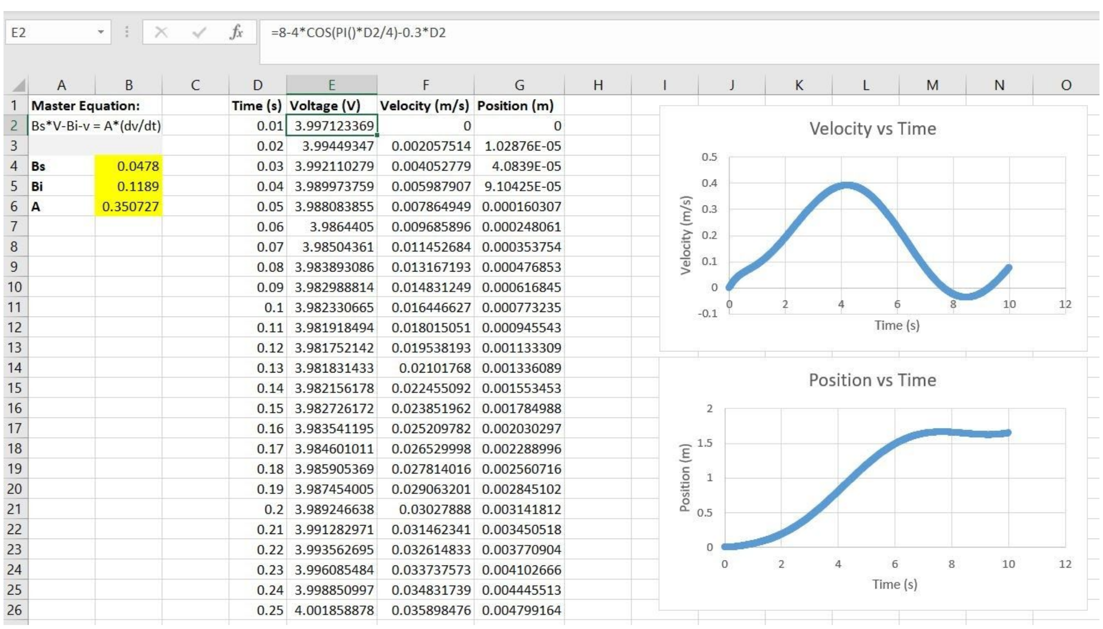


Voltages	Avg Velocities (Vss)	
12	0.452813	
10	0.36029	
8	0.264704	
6	0.166108	
4	0.072273	
equation of line: y=.0471x - 0.1155		
slope = Bs	0.0478	
intercept = Bi	0.1189	

Voltages	A	Average A, 12-6
12	0.329728	0.350727
10	0.341775	
8	0.373455	
6	0.357948	
4	0.212125	
excluding A from 4V because it is an outliar		

Figure 17 - Summary of Findings

We used these three highlighted values in our model to show data for an arbitrary voltage equation, V = 8 - 4cos(pi*time/4) - 0.3time. The model can be summarized with the following data and graphs:



E2 =8-4*COS(PI()*D2/4)-0.3*D2

	A	B	C	D	E	F	G
1	Master Equation:			Time (s)	Voltage (V)	Velocity (m/s)	Position (m)
2	Bs*V-Bi-v = A*(dv/dt)			0.01	3.997123369	0	0
3				0.02	3.99449347	0.002057514	1.02876E-05
4	Bs	0.0478		0.03	3.992110279	0.004052779	4.0839E-05
5	Bi	0.1189		0.04	3.989973759	0.005987907	9.10425E-05
6	A	0.350727		0.05	3.988083855	0.007864949	0.000160307
7				0.06	3.9864405	0.009685896	0.000248061
8				0.07	3.98504361	0.011452684	0.000353754
9				0.08	3.983893086	0.013167193	0.000476853
10				0.09	3.982988814	0.014831249	0.000616845
11				0.1	3.982330665	0.016446627	0.000773235
12				0.11	3.981918494	0.018015051	0.000945543
13				0.12	3.981752142	0.019538193	0.001133309
14				0.13	3.981831433	0.02101768	0.001336089
15				0.14	3.982156178	0.022455092	0.001553453
16				0.15	3.982726172	0.023851962	0.001784988
17				0.16	3.983541195	0.025209782	0.002030297
18				0.17	3.984601011	0.026529998	0.002288996
19				0.18	3.985905369	0.027814016	0.002560716
20				0.19	3.987454005	0.029063201	0.002845102
21				0.2	3.989246638	0.03027888	0.003141812
22				0.21	3.991282971	0.031462341	0.003450518
23				0.22	3.993562695	0.032614833	0.003770904
24				0.23	3.996085484	0.033737573	0.004102666
25				0.24	3.998850997	0.034831739	0.004445513
26				0.25	4.001858878	0.035898476	0.004799164

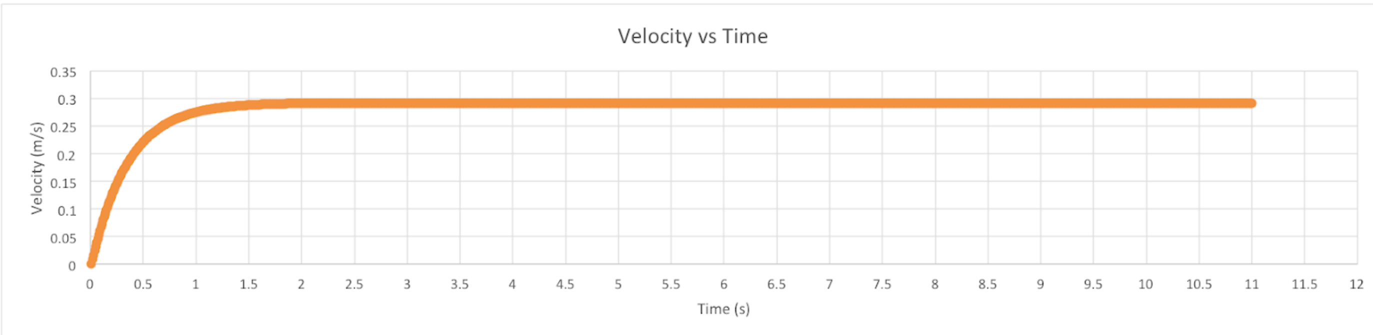
Figure 18 - Arbitrary Model with Graphs

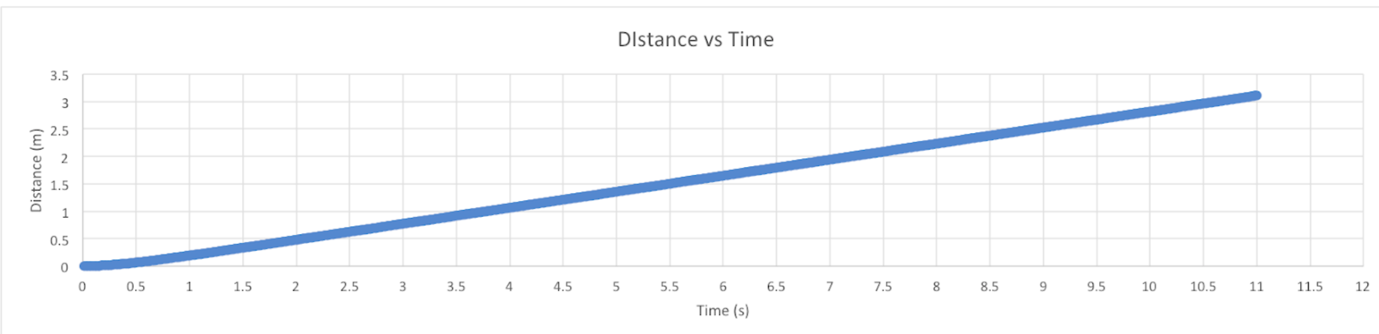
Conclusions

The theoretical model created to predict the behavior of an electric train only has certain predictive capabilities, and it has limitations. The model we created can use voltage to predict velocity and position of the train fairly accurately. Although, when we created the model, Microsoft Excel cannot integrate functions, so the position of the train is not as accurate as it can be because we had to integrate by hand, in the program, to find the position. Our model also only takes into account forces like gravity, contact, and friction and torques. The model lacks all of the variables that can impact the velocity of the train, this impacting the position. When we created the model, we assumed air resistance to be negligible, although it is not. The electric train can also take multiple trials to warm up to its full capacity, and the model does not take that variable into account either. Our theoretical model can only predict velocity, position, and voltage with the variables that it was given, so it is not as accurate as it could be. This model works well in a classroom, but to predict precise measurements, it would be highly ineffective. There was not enough information put into it to precisely determine velocity and position.

Appendix

Challenge 1



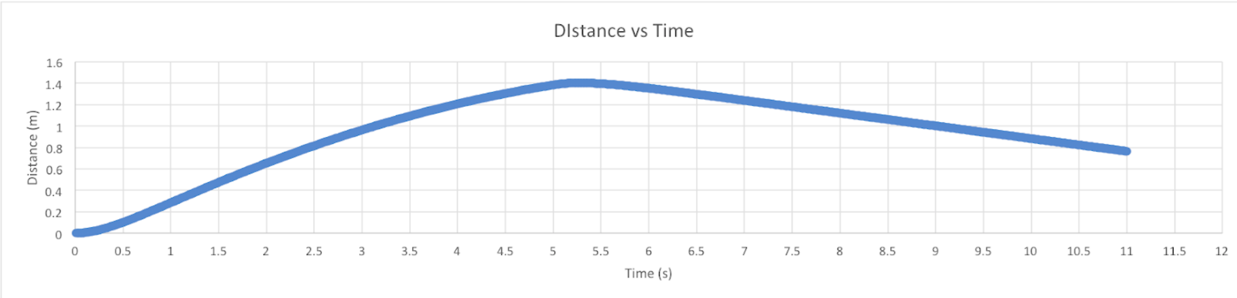


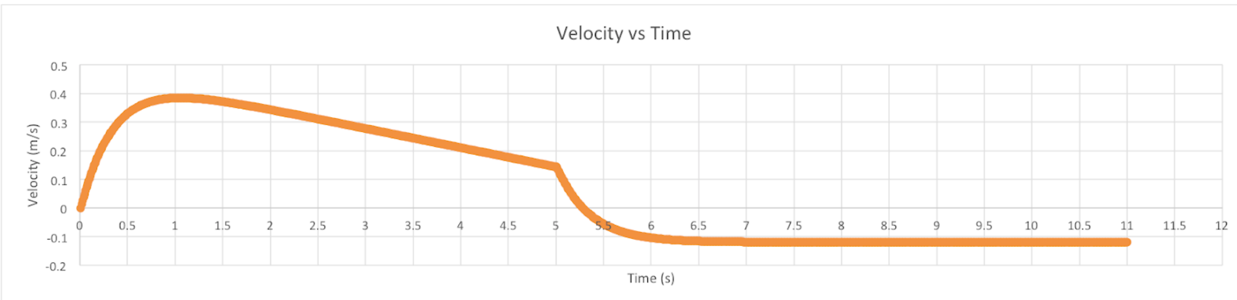
The first challenge asked us to pick a constant applied voltage that allowed our train to reach **Here** and **There** for the given times. We also had to pick the starting position of our train from **Here** in order to reach the selected spots at the given time. Our model gave us the following values:

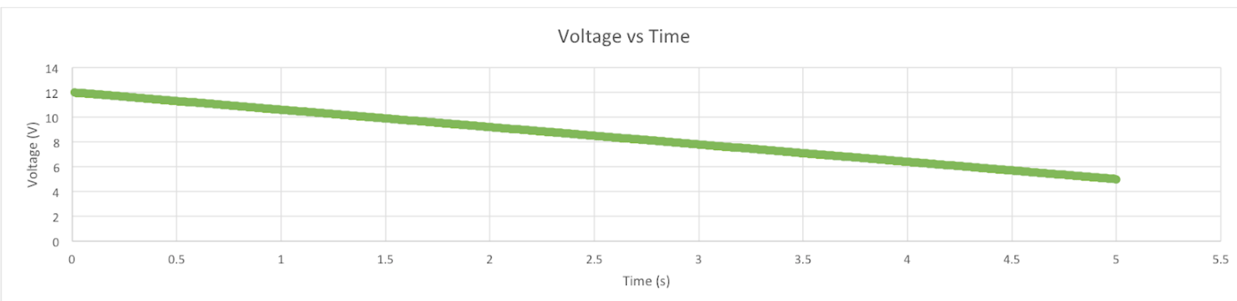
Voltage = 8.6 Volts

X start = 1.34 meters

Challenge 2







Challenge 2 wanted us to use the given voltage profile to get our train to stop as close to a point as possible without going over. To get our train to stop, we had to determine the start position and the stop time. Our model predicted the following values:

X start = 1.4 meters

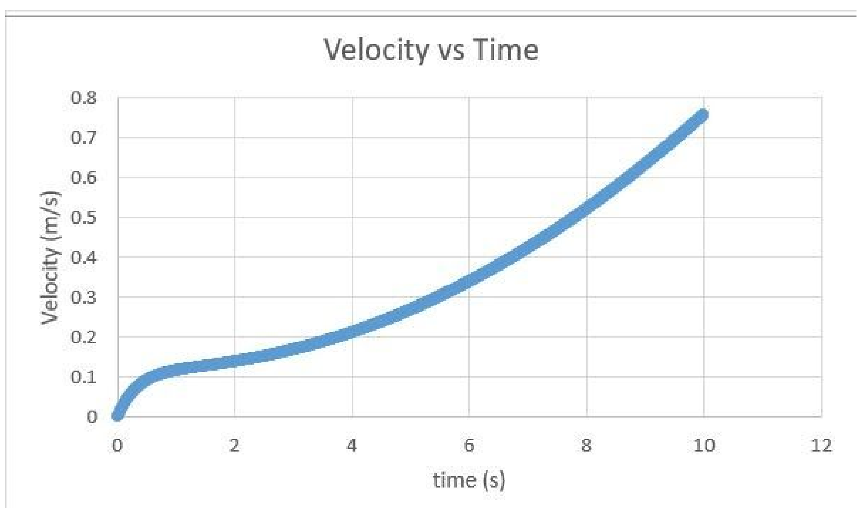
Stop Time = 5.25 seconds

Challenge 3

Challenge 3 asked us to find the position of the train at specified times (3.5 and 7 seconds) with a given voltage profile ($V(t) = 5 + (1/7)t^2$). Our model predicted the following values:

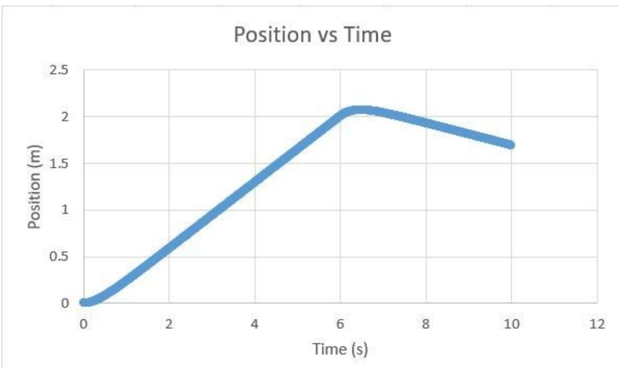
$X(3.5) = 0.451106$ m

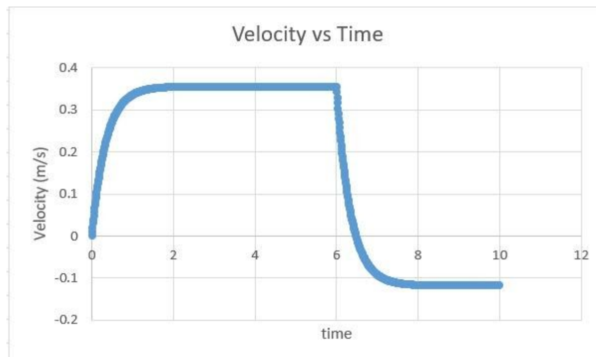
$X(7) = 1.472378$ m



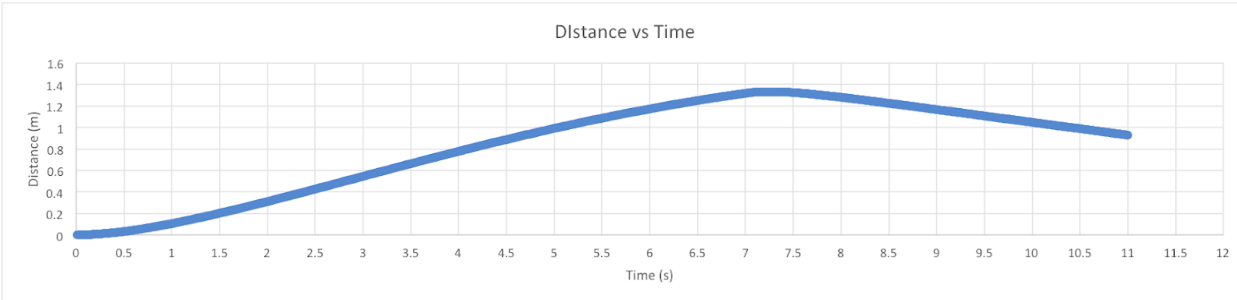
Challenge 4

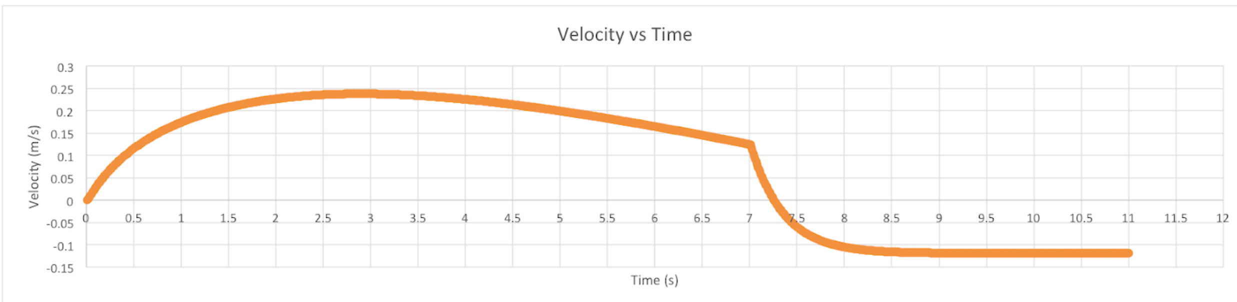
Challenge 4 asked us to find the optimal constant voltage value for 6 seconds such that the train would travel 170 cm then stop within a directly following 40 cm range. Our goal was to have the train stop at 200 cm total, giving us 10 cm for error. We found this voltage value through trial and error in our model, looking at the position of the train where velocity = 0. We determined: With voltage = 9.9V, X("v(t) = 0") = 2.069134 m.

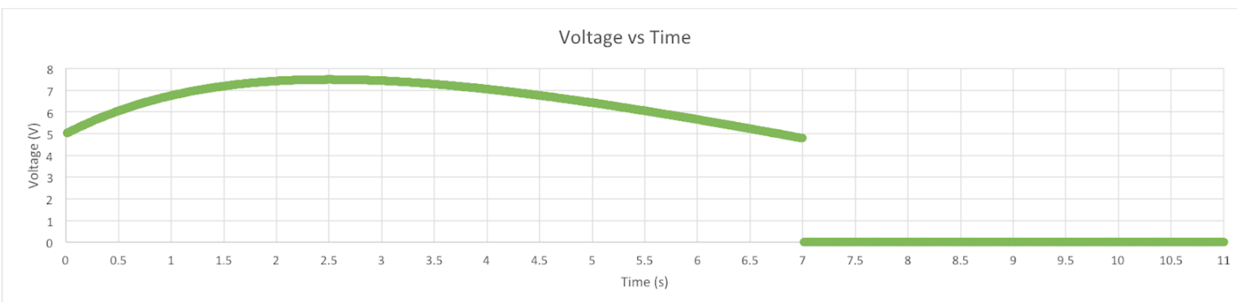




Challenge 5







Challenge 5 asked us to pick the correct goat and boat positions with the given voltage profile of $V(t) = 5 - t + 7(1 - e^{-t/2})$ in the range t < 7 s, and V = 0 when t > 7 s. It was also given that the train must reach the goat at 4.0 seconds, and it must reach the boat at 7.0 seconds.

X(4) = 0.778154 meters

X(7) = 1.318572 meters