A Further Study of Methods of Constructing Life Tables when Certain Causes of Death are Eliminated

Author(s): M. Noel Karn
Source: *Biometrika*, Vol. 25, No. 1/2 (May, 1933), pp. 91-101
Published by: Oxford University Press on behalf of Biometrika Trust
Stable URL: https://www.jstor.org/stable/2332264
Accessed: 22-09-2021 14:04 UTC

A FURTHER STUDY OF METHODS OF CONSTRUCTING LIFE TABLES WHEN CERTAIN CAUSES OF DEATH ARE ELIMINATED.

By M. NOEL KARN, M.A.

In a recent paper entitled "An Inquiry into Various Death-rates and the Comparative Influence of certain Diseases on the Duration of Life"* D'Alembert's method was applied to the construction of life tables for a population from which cancer and tuberculosis were supposed to be eliminated as causes of death in order to estimate the effect of these diseases in shortening the duration of life. In that paper no reference was made to the work on this problem by Dr Farr towards the end of last century, nor to Louis I. Dublin's work in recent years. In the present paper a comparison has been made of the results of the several methods when applied to the same data in order to determine what, if any, are the practical advantages of employing D'Alembert's formula over others which have been used. As to the theoretical advantages of D'Alembert's formula, I think there can be no question.

Comparison between Results of Farr's Method and D'Alembert's Formula.

Dr Farr's work was published in the *Supplement to the Thirty-Fifth Annual Report of the Registrar-General*, 1875, and entitled "Effect of the Extinction of any single Disease on the Duration of Life†." In this he made passing reference to the previous work on the same kind of problem in connection with the controversy over inoculation, mentioning Daniel Bernoulli and D'Alembert, giving Duvillard's value for the increase in the mean life time which would result from the extinction of small-pox. He then referred to the short method which he had described in the Appendix to the R.-G.'s Fifth Annual Report‡, using it for this particular purpose as being sufficiently exact, and thus saving the labour of constructing and graduating full life tables.

In his short method Farr made use of quinquennial age-groups. The number of deaths in age-group x to $x + 5$ years divided by the population of that age-group gave the probability p of living one year in the middle of the period. The fifth power was therefore taken in order to obtain the number of survivors at the end of the period. Thus the chance of living for five years at age x was p^5.

* *Annals of Eugenics*, Vol. IV. pp. 279—326, 1931.

† P. xxxviii. § 21.

‡ P. 362, "A short method of constructing Life Tables."

The method was only applicable for ages after 5 years. The survivors at age 5 would have to be obtained from the known deaths and populations year by year for the ages before 5.

Farr attempted the problem of eliminating a particular disease from the life table population, taking, among other diseases, as examples, cancer and phthisis. To do this in the case of cancer he constructed a new life table on the basis of the mortality from all diseases except cancer, viz. $m_x - c_x$*, using the short method above described. This he assumed gave a life table population as it would be if there were no cancer mortality.

I have first tried to ascertain how closely the numbers of survivors and expectations of life at different ages obtained by this method approximate to the more exact results obtained by D'Alembert's full formula.

The amount of error due to the approximations used in the short method of constructing an ordinary life table was investigated by Farr and is given in Table 58 of the *Supplement*† for the case of the English Life Table for Males. The result is an excess in the short method in the expectation of life, to the amount of ·37 to ·53 of a year in the mid-age groups up to 65, followed by a rapid increase in the excess at the older ages.

Table 60 of the same *Supplement*† gives survivors of a life table based on the deaths for the years 1861—70, and also the survivors for a life table with cancer excluded. I have worked out, and show in Table I, the expectation of life for both these tables for comparison with the results tabulated in Table VI of my previous paper‡.

TABLE I.

Life Tables for Males (calculated from the Facts recorded during 1861—70).

Age x	To die of all Diseases		Cancer excluded		Increase in $\mathring{e}_x$
	l_x	$\mathring{e}_x$	l_x	$\mathring{e}_x^{\bullet}$	
0	510 622	40·55	510 622	40·75	·20
5	367 817	50·32	367 835	50·65	·33
10	353 129	47·31	353 165	47·65	·34
15	345 341	43·32	345 393	43·61	·29
20	334 867	39·60	334 951	39·90	·30
25	321 013	36·20	321 126	36·51	·31
35	290 755	29·45	291 032	29·76	·31
45	254 138	22·97	254 890	23·27	·30
55	209 825	16·77	211 563	17·02	·25
65	150 844	11·37	153 945	11·52	·15
75	77 409	7·41	80 501	7·46	·05
85	17 826	5·45	18 970	5·44	·01
95	789	5·11	823	5·11	·00
105	9	—	9	—	—

* See Table 59, p. clxix. *Supplement to Thirty-Fifth Annual Report of R.-G.*, 1861—70.
† *Ibid.* ‡ *Annals of Eugenics*, Vol. iv. p. 309.

The order of the differences in the last column of this table is never greater than about a third of a year for the period 1861—70. Whatever method is used, the results show that these differences have increased to nearly one and a half years in mid-life for the period 1919—23.

I turned next to the modern data and have found the effect of the elimination of cancer mortality from Life Table 9, applying Farr's short method instead of the formula of D'Alembert.

In Table II are the results obtained by Farr's short method from the data to which the full formula of D'Alembert was applied in my original paper.

TABLE II.

Comparison of the Increases in Expectation of Life due to the Elimination of Cancer Mortality, calculated by two Methods.

Age	$\mathring{e}_x$ for Life Table 9	$\mathring{e}_x$ for Life Table with Cancer eliminated by use of D'Alembert's formula	Increase in $\mathring{e}_x$ by D'Alembert's formula	$\mathring{e}_x$ for Life Table with Cancer eliminated by use of Farr's short method	Increase in $\mathring{e}_x$ by Farr's short method	Excess by Farr's method	Percentage excess on increase in expectation of life due to using Farr's method
0	55·62	56·89	1·27	56·97	1·35	·08	6
5	58·81	60·25	1·44	60·35	1·54	·10	7
10	54·64	56·10	1·46	56·19	1·55	·09	6
15	50·12	51·59	1·47	51·68	1·56	·09	6
20	45·78	47·26	1·48	47·37	1·59	·11	7
25	41·60	43·09	1·49	43·21	1·61	·12	8
30	37·40	38·91	1·51	39·03	1·63	·12	8
35	33·25	34·77	1·52	34·88	1·63	·11	7
40	29·19	30·71	1·52	30·84	1·65	·13	9
45	25·22	26·72	1·50	26·85	1·63	·13	9
50	21·36	22·79	1·43	22·93	1·57	·14	10
55	17·73	19·02	1·29	19·17	1·44	·15	12
60	14·36	15·45	1·09	15·63	1·27	·18	17
65	11·36	12·20	·84	12·38	1·02	·18	21
70	8·75	9·34	·59	9·53	·78	·19	32
75	6·59	6·96	·37	7·20	·61	·24	65
80	4·93	5·14	·21	5·40	·47	·26	124

The difference due to using the short method ranges from ·08 to ·26 of a year in excess, and would increase the additional expectation of life by the elimination of cancer from a maximum value of 1·52 years at 40 to 1·65 years, that is, by 9 per cent.

An estimate of the amount of error permissible by using a short method is given by Dr Snow * as ·08 of a year, and if this is to be taken as a criterion the evaluation of the increase in the expectation of life in this special problem would not be

* "An Elementary Rapid Method of Constructing an Abridged Life Table," E. C. Snow. *Supplement to the Seventy-Fifth Annual Report of the R.-G.*, Part II.

TABLE III.

Comparison of the Increases in Expectation of Life due to the Elimination of Pulmonary Tuberculosis Mortality, calculated by two Methods.

Age	$\mathring{e}_x$ for Life Table 9	$\mathring{e}_x$ for Life Table with Pulmonary Tuberculosis eliminated by use of D'Alembert's formula	Increase in $\mathring{e}_x$ by D'Alembert's formula	$\mathring{e}_x$ for Life Table with Pulmonary Tuberculosis eliminated by use of Farr's short method	Increase in $\mathring{e}_x$ by Farr's short method	Excess by Farr's method	Percentage excess on increase in expectation of life due to using Farr's method
0	55·62	57·40	1·78	57·48	1·86	·08	5
5	58·81	60·79	1·98	60·89	2·08	·10	5
10	54·64	56·63	1·99	56·72	2·08	·09	5
15	50·12	52·07	1·95	52·17	2·05	·10	5
20	45·78	47·58	1·80	47·70	1·92	·12	7
25	41·60	43·13	1·53	43·24	1·64	·11	7
30	37·40	38·67	1·27	38·79	1·39	·12	9
35	33·25	34·28	1·03	34·41	1·16	·13	13
40	29·19	29·98	·79	30·12	·93	·14	18
45	25·22	25·80	·58	25·94	·72	·14	24
50	21·36	21·76	·40	21·91	·55	·15	38
55	17·73	17·99	·26	18·15	·42	·16	62
60	14·36	14·51	·15	14·71	·35	·20	133
65	11·36	11·44	·08	11·64	·28	·20	—
70	8·75	8·78	·03	9·01	·26	·23	—
75	6·59	6·60	·01	6·91	·32	·31	—
80	4·93	4·93	·00	5·31	·38	·38	—

TABLE IV.

Comparison of the Increases in Expectation of Life due to the Elimination of Heart Diseases Mortality, calculated by two Methods.

Age	$\mathring{e}_x$ for Life Table 9	$\mathring{e}_x$ for Life Table with Heart Diseases eliminated by use of D'Alembert's formula	Increase in $\mathring{e}_x$ by D'Alembert's formula	$\mathring{e}_x$ for Life Table with Heart Diseases eliminated by use of Farr's short method	Increase in $\mathring{e}_x$ by Farr's short method	Excess by Farr's method	Percentage excess on increase in expectation of life due to using Farr's method
0	55·62	57·32	1·70	57·41	1·79	·09	5
5	58·81	60·74	1·93	60·85	2·04	·11	6
10	54·64	56·58	1·94	56·67	2·03	·09	5
15	50·12	52·03	1·91	52·12	2·00	·09	5
20	45·78	47·67	1·89	47·78	2·00	·11	6
25	41·60	43·45	1·85	43·57	1·97	·12	7
30	37·40	39·20	1·80	39·32	1·92	·12	7
35	33·25	35·01	1·76	35·11	1·86	·10	6
40	29·19	30·90	1·71	31·02	1·83	12	7
45	25·22	26·89	1·67	27·00	1·78	·11	7
50	21·36	22·97	1·61	23·10	1·74	·13	8
55	17·73	19·26	1·53	19·39	1·66	·13	9
60	14·36	15·77	1·41	15·94	1·58	·17	12
65	11·36	12·61	1·25	12·78	1·42	·17	14
70	8·75	9·77	1·02	9·96	1·21	·19	19
75	6·59	7·39	·80	7·62	1·03	·23	29
80	4·93	5·50	·57	5·76	·83	·26	46

sufficiently accurately worked by the short method of Farr, especially when a comparison is to be made of the alteration in the expectation of life due to the elimination of cancer mortality for two or three past decades.

Farr's short method applied for the elimination of Pulmonary Tuberculosis and Heart Diseases separately from the ordinary life table shows differences in expectation of life ranging again from ·08 of a year at birth to ·38 in the case of Pulmonary Tuberculosis, and from ·09 to ·26 in the case of Heart Diseases when compared with the fuller method, as shown in Tables III and IV.

These differences become large in the later ages when regarded as percentage excess on the increase in expectation of life, especially in the Pulmonary Tuberculosis investigation. This shows that the method of Farr is not accurate enough for this purpose.

Re-calculation in five-yearly Groups of the Data on which Life Table 9 *is based.*

In estimating the additional expectation of life resulting from eliminating a cause of death by the shorter method, the calculation should perhaps be made not from the standard life table, but from one based on the same data but calculated also by the same method. The comparative values of the normal expectation of life calculated on this basis are set out in Table V.

TABLE V.

Expectation of Life.

Age	For Life Table 9	For Life Table calculated in five-yearly periods	Excess by Farr's method
0	55·62	55·71	·09
5	58·81	58·91	·10
10	54·64	54·74	·10
15	50·12	50·21	·09
20	45·78	45·89	·11
25	41·60	41·72	·12
30	37·40	37·52	·12
35	33·25	33·37	·12
40	29·19	29·32	·13
45	25·22	25·36	·14
50	21·36	21·51	·15
55	17·73	17·89	·16
60	14·36	14·56	·20
65	11·36	11·57	·21
70	8·75	8·98	·23
75	6·59	6·90	·31
80	4·93	5·31	·38
85	3·72	4·14	·42

In Table VI the increases in the expectation of life due to the elimination of the several diseases considered are shown, the results by Farr's method being compared with the standard life table re-calculated by the same method. The percentage difference on the increase is now diminished, as compared with the results in Tables II, III and IV, to an amount less than one, throughout the table in the case of Pulmonary Tuberculosis, and to age 60 in the table with Cancer eliminated.

TABLE VI.

Increase in Expectation of Life.

Age	With Cancer eliminated			With Pulmonary Tuberculosis eliminated			With Heart Diseases eliminated		
	By Farr's method compared with re-calculated Life Table	As in Table II	By D'Alembert's formula compared with Life Table 9	By Farr's method compared with re-calculated Life Table	As in Table III	By D'Alembert's formula compared with Life Table 9	By Farr's method compared with re-calculated Life Table	As in Table IV	By D'Alembert's formula compared with Life Table 9
0	1·26	1·35	1·27	1·77	1·86	1·78	1·70	1·79	1·70
5	1·44	1·54	1·44	1·98	2·08	1·98	1·94	2·04	1·93
10	1·45	1·55	1·46	1·98	2·08	1·99	1·93	2·03	1·94
15	1·47	1·56	1·47	1·96	2·05	1·95	1·91	2·00	1·91
20	1·48	1·59	1·48	1·81	1·92	1·80	1·89	2·00	1·89
25	1·49	1·61	1·49	1·52	1·64	1·53	1·85	1·97	1·85
30	1·51	1·63	1·51	1·27	1·39	1·27	1·80	1·92	1·80
35	1·51	1·63	1·52	1·04	1·16	1·03	1·74	1·86	1·76
40	1·52	1·65	1·52	·80	·93	·79	1·70	1·83	1·71
45	1·49	1·63	1·50	·58	·72	·58	1·64	1·78	1·67
50	1·42	1·57	1·43	·40	·55	·40	1·59	1·74	1·61
55	1·28	1·44	1·29	·26	·42	·26	1·50	1·66	1·53
60	1·07	1·27	1·09	·15	·35	·15	1·38	1·58	1·41
65	·81	1·02	·84	·07	·28	·08	1·21	1·42	1·25
70	·55	·78	·59	·03	·26	·03	·98	1·21	1·02
75	·30	·61	·37	·01	·32	·01	·72	1·03	·80
80	·09	·47	·21	·00	·38	·00	·45	·83	·57

Table VII shows the percentage difference for some of the later ages in the tables with Cancer or Heart Diseases eliminated, Farr's method now giving a defect.

TABLE VII.

Age	With Cancer eliminated		With Heart Diseases eliminated	
	Defect in increase in expectation of life due to Farr's method	Percentage defect	Defect in increase in expectation of life due to Farr's method	Percentage defect
45	·01	1	·03	2
50	·01	1	·02	1
55	·01	1	·03	2
60	·02	2	·03	2
65	·03	4	·04	3
70	·04	7	·04	4
75	·07	19	·08	10
80	·12	57	·12	21

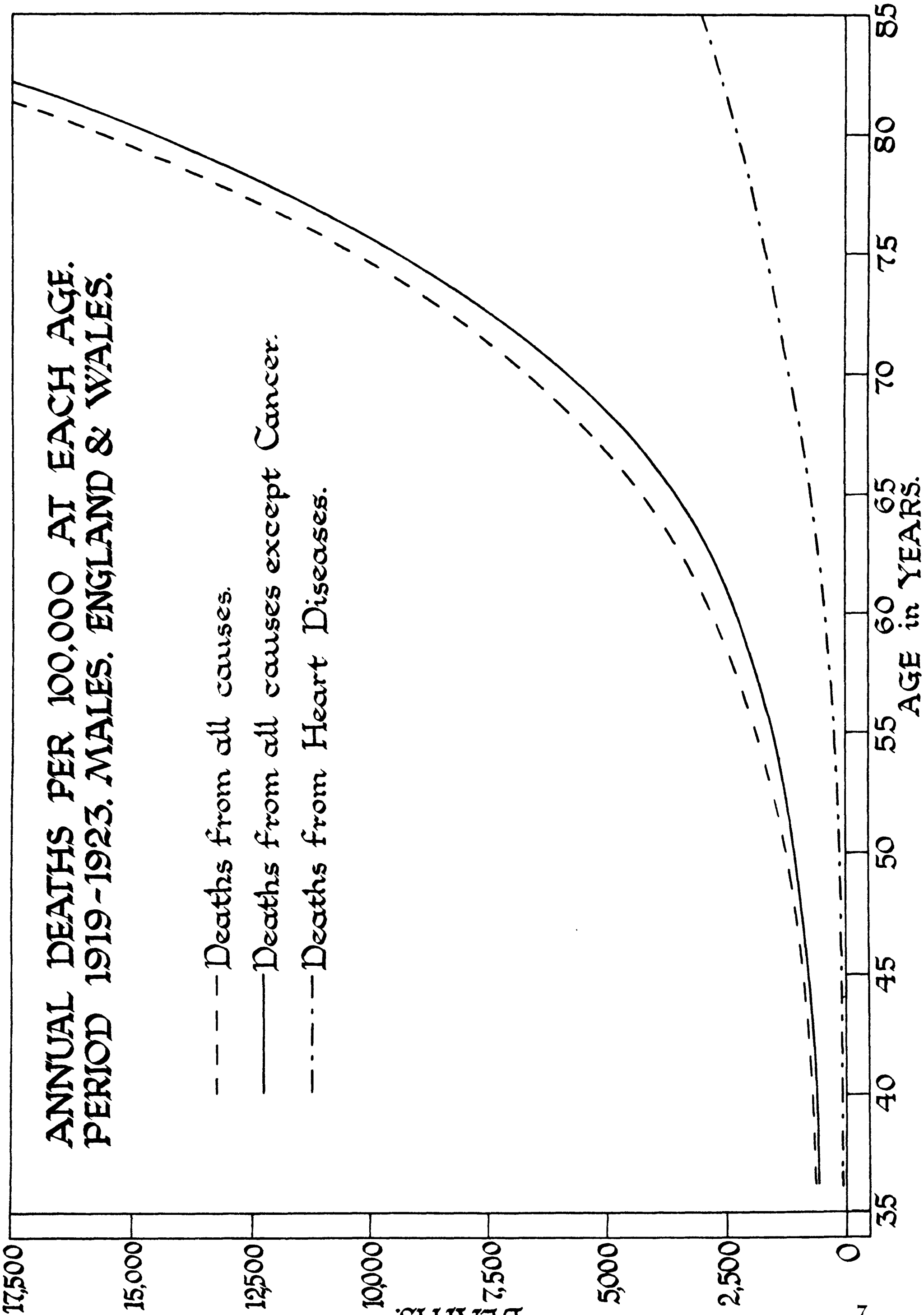
ANNUAL DEATHS PER 100,000 AT EACH AGE.
PERIOD 1919-1923. MALES. ENGLAND & WALES.
Deaths from all causes.
Deaths from all causes except Cancer.
Deaths from Heart Diseases.
DEATHS.
17,500
15,000
12,500
10,000
7,500
5,000
2,500
0
35
40
45
50
55
60
65
70
75
80
85
AGE in YEARS.

Fig. 1.

In deciding whether the short method is sufficiently accurate when a particular disease is to be eliminated the form of the curve of mortality rates of the disease must be considered. In Fig. 1 on the previous page are given:

The curves of annual mortality rates per 100,000:

(1) for deaths from all causes,

(2) for deaths from heart diseases,

(3) for deaths from all causes except cancer in five-yearly groups are given for the period of the data under consideration.

Similar curves have already been given for cancer and for pulmonary tuberculosis in the former paper.

The curve for deaths from all causes rises rapidly towards the end of life, the curve for heart diseases rises in the same way but at a more gradual slope, and the curve for all deaths except cancer follows a course similar to that for all deaths.

The curve of mortality for pulmonary tuberculosis is different in form. The slope is generally gradual whether it is ascending or descending.

Provided then that one starts from a life table constructed in the same way, the short method of Farr is seen to lead to tolerably accurate differences in life expectation due to elimination of the diseases which have been under consideration in this memoir, except in the cases of cancer and heart diseases for ages after 60 when the inaccuracy increases. The method however involves the additional work of first re-computing the standard life table on the basis of five-yearly instead of yearly age-groups.

The error introduced in the short method arises from the fact that the numbers saved from the disease in a particular five-yearly period are regarded as exempt from risk of death from other diseases for the whole of the five years. This is not the case, for those saved from the particular disease will be subjected to the death-rate from other diseases from the moment in which they would have died of the special disease, which connotes on the average for half the period. This error does not occur when infinitesimal periods are used. In the use of D'Alembert's formula there is the additional advantage that a series of annual cancer (or other) mortality rates is obtained which is of interest and value in itself, and which proved to be of use in other problems dealt with in my former paper.

Some Work on the same Problem from Data of the Metropolitan Life Insurance Company of New York.

Some work on the same problem has been published in the *Statistical Bulletin** of the Metropolitan Life Insurance Company of New York, under the title "Effect of Cancer upon the Length of Life" and "Loss in Expectation of Life on account of organic Heart Diseases," for data of the Industrial Population 1911—1916. The tables for males, white, are reproduced as Table VIII.

* *Stat. Bull. M.L.I. Co.*, Oct. 1920, Vol. I. No. 10, p. 5; *Ibid.* Feb. 1921, Vol. II. No. 2, p. 6.

TABLE VIII.

Age	Number of years of Average After Lifetime Lost on Account of Cancer. (All Forms.) White, Males	Average Number of Years of After Lifetime Lost on Account of Organic Diseases of the Heart. White, Males
0	0·62	1·67
1	·70	1·86
2	·72	1·91
3	·73	1·94
4	·73	1·95
5	·72	1·95
15	·73	1·93
25	·75	1·89
35	·79	1·88
45	·80	1·87
55	·73	1·81
65	·53	1·56
75	·39	1·19
85	·30	·96
95	·17	·41

The column referring to cancer gives smaller figures than those for English data of about the same period, viz. 1909—1913, as given in Table VI of my former paper, but the results resemble those of English data of the last three decades in rising to a maximum loss of years between 40 and 50 years.

The results for heart diseases are very similar to those obtained from the English data 1919—1923, given in Table V *a* of my original paper.

The formula used to obtain the results given in Table VIII has been communicated to me by Messrs Dublin and Lotka as

$$q_x^{(-i)} = \frac{q_x - q_x^{(i)}}{1 - q_x^{(i)}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A$$

where q_x denotes the usual life table function when all causes of death are effective, $q_x^{(i)}$ denotes the corresponding life table function when only cause i of death is effective, and $q_x^{(-i)}$ denotes the corresponding life table function when all causes except i of death are effective.

The formula A may be obtained as follows:

The probability of living for one year at age x at risk of death from all causes is equal to the product of the probability of living for one year when only cause i of death is effective and the probability of living for one year when all causes except i are effective, that is

$$p_x = p_x^{(i)} \cdot p_x^{(-i)};$$

$$\therefore 1 - q_x = (1 - q_x^{(i)})(1 - q_x^{(-i)});$$

$$\therefore q_x^{(-i)} = \frac{q_x - q_x^{(i)}}{1 - q_x^{(i)}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A$$

p_x denoting the usual life table function when all causes of death are effective,

$p_x^{(i)}$ denoting the corresponding function when only cause i of death is effective, and

$p_x^{(-i)}$ denoting the corresponding function when all causes of death except i are effective.

I have applied the formula A to the data in hand in yearly periods and find that it gives a very near approximation to the formula of D'Alembert, the survivors in the life table excluding a special disease being the same to within a few units in 100,000 starting life together, at all ages through life, whether the disease considered is Cancer, Pulmonary Tuberculosis, or Heart Diseases. The expectations of life are exactly the same for the two methods.

Comparison of Formula A with Farr's Method.

The formula which Farr used would be

$$m_x - m_x^{(i)} = m_x^{(-i)}, \qquad \text{B}$$

where $m_x = \dfrac{2q_x}{2 - q_x}$, and $m_x^{(i)}$, $m_x^{(-i)}$ have similar meanings.

Substituting these values for the m's Formula B gives on reduction

$$q_x^{(-i)} = \frac{q_x - q_x^{(i)}}{1 - q_x^{(i)} + \dfrac{q_x q_x^{(i)}}{4}},$$

which differs from formula A in the denominator of the right-hand side.

The amount of the difference in the expectation of life given by Formula B as compared with that given by D'Alembert's formula, or by Formula **A**, has already been shown.

It may be of interest to apply Formula A to the cancer data in five-yearly periods in order to compare the expectation of life with the results found in yearly periods. The results are given in Table IX.

This table shows that there is no sensible error involved in computing the additional expectation of life resulting from elimination of cancer as a cause of death by using five-yearly periods, so long as the life table with which comparison is made is computed in similar periods.

In conclusion, the difference in the methods used lies in the evaluation of q_x^i, D'Alembert's formula giving instantaneous values, Dublin's formula values at yearly intervals, and Farr's values at quinquennial periods.

For rapidity of calculation, combined with accuracy, the formula giving yearly values has some advantage over that giving instantaneous values.

In either case the results are arbitrary to some extent, as the original figures are usually only obtainable in quinquennial age-groups.

TABLE IX.

Expectation of Life and Increase in that Expectation over that of a Standard Life Table in a Population excluding Cancer.

Age	Calculated in yearly periods by formula A $\mathring{e}_x$	Increase in expectation of life over Life Table 9	Calculated in 5-yearly periods by formula A $\mathring{e}_x$	Standard Life Table (see Table V) $\mathring{e}_x$	Increase in expectation of life over Standard Life Table
0	56·89	1·27	56·99	55·71	1·28
5	60·25	1·44	60·37	58·91	1·46
10	56·10	1·46	56·21	54·74	1·47
15	51·59	1·47	51·69	50·21	1·48
20	47·26	1·48	47·39	45·89	1·50
25	43·09	1·49	43·23	41·72	1·51
30	38·91	1·51	39·04	37·52	1·52
35	34·77	1·52	34·90	33·37	1·53
40	30·71	1·52	30·86	29·32	1·54
45	26·72	1·50	26·87	25·36	1·51
50	22·79	1·43	22·96	21·51	1·45
55	19·02	1·29	19·20	17·89	1·31
60	15·45	1·09	15·66	14·56	1·10
65	12·20	·84	12·42	11·57	·85
70	9·34	·59	9·57	8·98	·59
75	6·96	·37	7·26	6·90	·36
80	5·14	·21	5·51	5·31	·20

BIBLIOGRAPHY.

(1) KARN, M. N. An Inquiry into Various Death-Rates and the Comparative Influence of Certain Diseases on the Duration of Life. *Annals of Eugenics*, Vol. IV. Parts III and IV, May, 1931.

(2) *Registrar-General's Fifth Annual Report.* Appendix.

(3) *Statistical Bulletin Metropolitan Life Insurance Company of New York.* Oct. 1920, Vol. I. No. 10. Feb. 1921, Vol. II. No. 2.

(4) *Supplement to the Registrar-General's 35th Report.*

(5) *Supplement to the Registrar-General's 75th Report*, Part II.