Lett Mat Int (2015) 2:185–192
DOI 10.1007/s40329-015-0073-5

The smallpox vaccine: the dispute between Bernoulli and d'Alembert and the calculus of probabilities

Camilla Colombo · Mirko Diamanti

Published online: 27 January 2015

Abstract The urgent and dramatic need to introduce and promote a vaccine against smallpox, a scourge for society at the end of the 1700s, provided the occasion for a lively debate between Daniel Bernoulli and Jean Le Rond d'Alembert. This article discusses the motivations and arguments of the dispute, illustrating the probabilistic model proposed by Bernoulli to justify the greater "reasonableness" of the campaign in favour of the vaccine, and the objections raised by d'Alembert. The aim of this analysis is, beginning with a reconstruction of the contributions of the two authors, to show how the newly founded "art of conjecture" was the object of divergent interpretations, from the characterisation of its theoretical principles, such as the concept of expectation, to the question of its legitimacy in applications as a "guide to practical living".

Keywords Daniel Bernoulli · Jean Baptiste Le Rond d'Alembert · Smallpox · Vaccinations · Calculus of probabilities · Art of conjecture · Logic · Mathematical models

1 Introduction

From the first half of the seventeenth century, the problem of smallpox epidemics was considered to be urgent and dramatic. In cities with a high level of population density, such as London and Paris, it was estimated that some ten percent of all deaths were due to smallpox. Children and young people were above all vulnerable to the disease, which was caused by an airborne virus; though not always fatal, it often left its victims with heavy scarring or permanent blindness. Because of its widespread contagion and the percentage of the population affected, smallpox was considered a social scourge, and was a symbol of medicine's impotence. Methods for its cure and prevention thus attracted the focused attention of scientists and the public alike.

Although there were no known effective treatments, Asia had for some time practiced a rudimentary technique of vaccination, known as variolation, which consisted in the inoculation of the virus by means of insufflation of infected matter. This practice was introduced in England in 1718 by Lady Montague, wife of the British ambassador to the Ottoman Empire. The risks connected to the treatments were however significant, since often those who were inoculated contracted the disease and died within 2 months. Further, the vaccination caused low fevers as well as a low contagiousness: patients thus had to remain in isolation for several weeks following the inoculation, making it practical except among the wealthiest social classes.

In France, in spite of the high rates of incidence of the disease in all levels of society, the vaccine was regarded with suspicion. It was in fact feared that this technique would contribute to the spread of the contagion, and bring only doubtful benefits. Voltaire's summary of the situation in the *Lettres Philosophiques* was emblematic of the general diffidence:

> It is inadvertently affirmed in the Christian countries of Europe that the English are fools and madmen.

C. Colombo (✉)
43 Bronwen Court, Grove End Road, London NW8 9RX, UK
e-mail: camillafcolombo@gmail.com

M. Diamanti
Strada di Tuillo 3/o, 05100 Terni, TR, Italy
e-mail: mirko.diama@gmail.com

> Fools, because they give their children the small-pox to prevent their catching it; and madmen, because they wantonly communicate a certain and dreadful distemper to their children, merely to prevent an uncertain evil ([7], Letter XI).[1]

It was thus in a context that was very vivacious and of pressing interest that the intellectuals undertook a cultural campaign to promote the vaccination, a campaign that often assumed the tenor of a genuine ideological battle in the name of progress ([2], pp. 83–89). From the seats of institutions to newspapers, the arguments in favour or against vaccination were the object of frequent debates between 1750 and 1770, with the impassioned contributions of many French intellectuals. Voltaire, who survived smallpox in 1723, after having observed firsthand some of the vaccinations in England, enthusiastically declared himself in favour of it in his *Lettres Philosophiques* of 1743. La Condamine made the question the subject of a talk to the Académie des Sciences in Paris in 1754. The objective sought by those in favour was for the authorities to render vaccination obligatory, or at least publicly encourage it, thus overcoming the resistance of obscurantists and enemies of progress.

In this excited climate of crusade, Daniele Bernoulli and Jean Baptiste Le Rond d'Alembert found themselves on opposite sides in weighing the risks and benefits of vaccination. In 1760 an article by Bernoulli was published in the *Mércure de France*, in which he set out the results obtained by means of the application of the calculus of probabilities to the question of the vaccine. Such a quantitative analysis, in the author's opinion, would prove to any "reasonable man" the necessity of promoting this practice. That same year, Bernoulli presented the details of his calculation to the Académie des Sciences, but the paper, entitled "Essai d'une nouvelle analyse de la mortalité causée par la petite vérole et des avantages de l'inoculation pour la prévenir", was not published until 1766. D'Alembert's reply to Bernoulli's intervention, contained in the "Onzième mémoire, Sur l'application du calcul des probabilités à l'inoculation de la petite vérole" in his *Opuscules mathématiques* was instead published 5 years earlier, in 1761: Bernoulli's proposal was harshly criticised before the publication of the final text, leading to polemics and complaints about his colleague's lack of respect.

Beyond the polemical turns taken by the debate, d'Alembert's assertions are interesting because they move the question to the more general level of the interpretation of the concept of probability itself, and of the possible criteria and relative drawbacks of the application of the new calculus. It was precisely the implications contained in the criticism of Bernoulli's work that made deeper examination from the theoretical point of view proposed by d'Alembert opportune. In attempting to give an account of the position that the encyclopaedist took on probability in the system of human knowledge, we intend here to show that the epistemological arguments he used ended up circumscribing the field of applications of probability within a context that was exclusively abstract, denying them any possibility to contribute to the management of uncertainty in real life.

2 Daniel Bernoulli's proposal

In 1759 Maupertuis, an enthusiastic supporter of the vaccination, convinced Daniel Bernoulli, his colleague in Basel, to devote himself to a mathematical analysis of the question of the vaccine. The solution to the problem, according to Bernoulli, had to take the form of an answer to the following question: Was the government to promote vaccination for all individuals at birth? The purpose of the calculation is, given the probability of being ill with and dying from smallpox due to vaccination, compare the risks and benefits of the two possible strategies. Once the difficulties posed by the lack of data and its reliability had been overcome, Bernoulli believed that the rigorous analysis would determine which alternative the government should reasonably follow. The author's conclusion was a positive response to the initial question: the vaccination should be encouraged because it leads to an increase of the average life expectancy.

We can summarise Bernoulli's argument in three steps:

1. The construction of a population model, that is, a curve that describes the variation of the population over time (mortality);
2. The calculation of the variation in life expectancy with respect to the initial condition, once smallpox has been eradicated (that is, if all individuals were vaccinated at birth);
3. The calculation of the variation of life expectancy taking account of the risk of dying from the vaccination.

Briefly, Bernoulli proposed comparing two "states of humanity", distinguished only by the endemic presence of smallpox. The criterion for comparison of the curves of population that described the two states was given by the subtended integral of the curve, that is, by average life expectancy. The necessary condition to carry out this

[1] *On dit doucement, dans l'Europe chrétienne, que les Anglais sont des fous et des enragés: des fous, parce qu'ils donnent la petite vérole à leurs enfants, pour les empêcher de l'avoir, des enragés, parce qu'ils communiquent de gaieté de coeur à ces enfants une maladie certaine et affreuse, dans la vue de un mal incertain* (Voltaire, *Lettres Philosophiques*, Onzième Lettre. Sur l'insertion de la petite vérole).

operation of comparison is the isolation of the component of mortality due to smallpox from total mortality.

3 The model

Bernoulli's argument is presented explicitly as a mathematical model based on hypotheses that the author acknowledges to be simplifying and imprecise, and on defective data. As we will see further on, the possibility of applying the calculus of probabilities beginning from uncertain and incomplete data constitutes one of the prime motives behind the criticism in d'Alembert's *Essai*. Bernoulli's initial assumptions are the following:

- The individuals infected with smallpox for the first time die with a probability p and survive with a probability $1 - p$;
- Each individual has the probability q of being infected each year. In an infinitesimal interval of time dx, the probability of being infected between age x and age $x + dx$ (with $dx = 1$ for the sake of simplicity) is qdx.
- The individuals who survive smallpox are immunized for the remainder of their lives.

With $m(x)$ representing mortality due to causes other than smallpox, and the related value $m(x)dx$, Bernoulli constructs a suitable population model that makes it possible to isolate the mortality due to the disease. Let P_O be a group of individuals born the same year: $S(x)$ is the number of those, at risk of becoming ill, who live and are never infected by the age of x; $R(x)$ is the number of those who have survived smallpox at age x; thus $P(x) = S(x) + R(x)$ is the number of those who are alive at age x. Bernoulli then set $x = 0$ as the birth, where $S(0) = P(0) = P_0$ and $R(0) = 0$. Between x and $x + dx$ each individual at risk has the probability qdx of being infected, and $m(x)dx$ of dying due to other causes. The variation of those at risk is thus $dS = -Sqdx - Sm(x)dx$, which leads to the differential equation:

$$\frac{dS}{dx} = -qS - m(x)S \tag{1}$$

where dS is the derivative of the function $S(x)$. During the same period of time, the number of individuals who die from smallpox is $pSqdx$ and the number of those who survive smallpox is $(1 - p)Sqdx$. The number is then added of those who die from other causes $Rm(x)dx$. Hence:

$$\frac{dR}{dx} = q(1 - p)S - m(x)R \tag{2}$$

Summing Eqs. (1) and (2), we have:

$$\frac{dP}{dx} = -pqS - m(x)P. \tag{3}$$

At this point Bernoulli can deduce the fraction of individuals at risk in the population for a certain age x, using the methods known from the calculus of differential equations:

$$\frac{S(x)}{P(x)} = \frac{1}{(1 - P)e^{qx} + p}. \tag{4}$$

4 The data of the problem

After having isolated within the population the group $R(x)$ of individuals at risk, Bernoulli possesses all of the elements to calculate the variation in life expectancy. The model must however be "filled" with the data relative to the mortality between x and $x + 1$ and the respective rates of incidence and mortality of smallpox p and q. Those data were not easy to obtain: in the majority of the registries of baptisms and funerals of the time no ages at death were given, making it impossible to determine the mortality rate with respect to an age x. This obstacle notwithstanding, beginning in the 1660s the spread of life insurance contributed to the promotion of techniques for calculating mortality rates. The solution adopted consisted in using the data relative to a stable population, in which the number of births and deaths each year were equal, in order to be able to construct a population curve. In 1693 Halley published an analysis of the population of the city of Breslau, in the Hapsburg Empire [6], which responded to these requirements. The result of the study, known as Halley's Life Table, was a table showing, among a group of 1,300 individuals born in year 0, how many would still be alive at age x.

In keeping with the observations of the day, Bernoulli then chose $p = 1/8$ and $q = 1/8$, which actually would turn out to be rather accurate. Using Halley's Table and his population model, it is possible to calculate the number $S(x)$ of the individuals at risk at age x and the number $R(x) = P(x) - S(x)$ of those at age x who survived smallpox. The number of deaths due to smallpox between age x and age $x + 1$ should be the integral $pq \int_x^{x+1} S(t)dt$, but the formula used by Bernoulli, $pq[S(x) + S(x + 1)]/2$, is a good approximation.

5 The comparison between life expectancies

At this point Bernoulli considers the situation in which the vaccine is inoculated into each individual at birth (his proposal in the *Mércure de France* is that obligatory vaccinations begin in orphanages), without causing any deaths. What is thus involved is calculating the increase in life expectancy once smallpox is eradicated. Beginning with the same number of births P_0, let us indicate with $P^*(x)$ the number of individuals of age x when smallpox disappears: $\frac{dP^*}{dx} = -m(x)P^*$.

Thus:

$$P^*(x) = \frac{P(x)}{1 - p + pe^{-qx}}$$

where $P(x)$ is the population at age x before the eradication of smallpox. Comparing $P(x)$ and P^*x means estimating life expectancy at birth, that is, the integral

$$1/P_0 \int_0^\infty P(x)dx$$

[with smallpox, or without smallpox substituting with $P^*(x)$]. Bernoulli uses the approximate formula: $\left[\frac{1}{2}P(0) + P(1) + P(2) + \cdots\right]/P_0$. The results obtained are:

- life expectancy with smallpox $= E = \left[\frac{1}{2}1300 + 00 + \cdots + 20\right]/1300 \cong 26.57$ years[2];
- life expectancy without smallpox $= E^* = \left[\frac{1}{2}1300 + 1015 + \cdots + 23\right]/1300 \cong 29.65$ years.

Vaccination at birth therefore guarantees an increase of more than 3 years in life expectancy.

However, vaccination against smallpox is not a safe procedure, and thus Bernoulli considers p^l, the probability of dying from the vaccine, with $p^l < p$; life expectancy is then $(1 - p^l)E^*$ if all individuals are vaccinated at birth.

To render vaccination reasonable, the author observes that the following inequality must be satisfied:

$$p^l < 1 - E/E^*,$$

that is, $p^l = 11$ %. Even in the absence of accurate data, Bernoulli estimates that mortality after vaccination will be less than 1 %. Vaccination must therefore be actively promoted by the government: "I simply hope that, in a question that so closely regards the wellbeing of the human race, no decision will be taken without considering all the information that a modest analysis and calculation can provide" [1].

6 D'Alembert's criticisms

Reprising d'Alembert's position in a coherent manner by articulating it through a series of counter-arguments to Bernoulli's model is not an easy task. Set forth in the *Onzième Mémoire* and successively enlarged in later texts, the criticisms are wide-ranging, going from political opportunity to the proposal to use different data. Further, at the end of the memoir d'Alembert declares himself to be in favour of vaccination, apparently in contrast with the tone he maintained throughout the entire text. The confusion that derives from this can be imputed to personal rivalry towards Bernoulli, together with the desire to not show himself openly in dispute with a position (that in favour of the vaccine) supported unanimously by men of science and culture of the day. Given the complex scenario of the dispute, it is useful to concentrate on two conceptually relevant elements of d'Alembert's criticisms that cannot be traced to a merely "opportunistic" interpretation of his position.

6.1 The perspective of the individual versus that of society: criticism of the criterion of average life expectancy

The first point of divergence between the two authors regards the perspective from which the reasonableness of the vaccine is evaluated. As shown in the previous section, Bernoulli poses the question of whether or not it is rational for the government to promote vaccination; as a consequence, the opportune criterion of maximisation is average life expectance, an increase in which determines for the state a greater number of able-bodied individuals who can work and be "useful". This position is clearly supported by Bernoulli's observation on the loss suffered by society as a whole caused by the death of a young person, who had been nourished and raised without having contributed in any way to its collective prosperity. In other words, Bernoulli addresses the question as a topic of public health, concentrating on the social consequences of a policy in favour of the vaccine.

In contrast, d'Alembert interprets the same problem as a matter of individual choice, asking himself what is rational for an individual who must choose between being vaccinated and not (or of whether or not to vaccinate his child). In this perspective, the author observes, the maximisation of the average life expectancy is not a sensible criterion for deciding. First of all, not all years of life are equal: for a man in his prime, the addition of 2 years of old age are not worth the risk of dying within 2 months of vaccination. In the second place, the analysis proposed by Bernoulli is not adequate to take into account the "moral experience" that one has in taking a risk. By means of the mental experiment of a hypothetical lottery that assigns a probability of ½ to death and the same probability of ½ to a healthy life expectancy of 100 years, d'Alembert shows that the psychology of risk cannot be described in terms that are purely quantitative. D'Alembert admits that while the average life expectancy is an index of

[2] The value 1300 refers to the data of Halley's Table that Bernoulli was working with.

maximisation for the government, it is not for the individual: for this reason, the government has no right to impose vaccination on individual, for whom, for example, it makes sense to use a different criterion of maximisation.

Beyond several general observations on the psychology of risk, d'Alembert doesn't formulate an alternative to Bernoulli's model, nor does he propose an suitable index of individual maximisation. In his opinion, it is necessary to compare the risk of dying from today to a few weeks following vaccination with the risk of dying naturally from smallpox within the same interval of time. However, on the possibility of formalising this "moral experience" the author is very sceptical: How can we compare present risk to an unknown or remote benefit? With respect to this, the analysis of games of chance can tell us nothing" ([3], pp. 33–34). This profession of mistrust in the adequacy of the calculus of probabilities to treat the question of the smallpox vaccine leads our summary to the second point of disagreement with Bernoulli. In this phase, d'Alembert's criticism appears to be more generally directed against the attempt to apply methods of probability to a topic of practical relevance. This hypothesis requires a broader analysis of d'Alembert's position as a theoretician of probability.

6.2 The *art de conjecturer*

Volume IV of the *Mélanges de littérature, d'histoire et de philosophie* of 1759 contains d'Alembert's *Essai sur les éléments de philosophie*. After having presented the outline of the work, in which the author sets out to delineate the architecture of human knowledge, showing the principles that underlie the various areas of knowledge with their relative dependencies and correlations, d'Alembert speaks of logic in chapter V, criticising the claim of those philosophers who presume to use logic as a technique with general rules, the respect of which guarantees the correctness of reasoning:

> We have innumerable writings on logic, but does the science of reasoning need so many rules? To have success in it, it is as little necessary to have read all of these writings as it is to have read the great treatises on morals in order to be honest men. The geometers, without having fatigued themselves with acquiring the precepts of logic, and having only the natural sense as a guide, arrive by a pathway that is always sure to the most distant and abstract truths; while many *philosophes* [philosophers], or rather, so many writers of philosophy, seem to have placed at the beginning of their works the great treatises on the art of reasoning, only to lose themselves later with several methods; similar to these are those unlucky players who calculate at length and end up losing.[3]

D'Alembert also situates the art of conjecture within the area of logic, justifying this choice by noting that knowing how to conjecture well is an integral part of the capacity to reason:

> The mind that recognises truth only when it is directly affected, is far beneath the one who not only knows how to recognise it when it is close, but anticipates it and notices it in the distance in its fugitive character.[4]

Volume V of the *Mélanges* published in 1767, contained the *Eclairsissements sur les éléments de philosophie*, in which d'Alembert broadened and specified the considerations he had made in the *Essai*. Chapter VI is dedicated to the art of conjecture, and distinguishes three fields in which it can be applied. The first is that of "the analysis of the probabilities in games of chance … subject to known and certain rules, or at least seen as such by the mathematicians".[5] The second field is identified as the extension of this analysis to questions regarding daily life, such as the duration of a lifetime, pensions, maritime insurance, vaccination and other similar topics.

According to d'Alembert, these questions differ from the problems related to games of chance because

> …while in these [problems of games of chance] the rules of mathematical combinations suffice … to determine the number and ratio of possible cases, in those [problems of daily life], to the contrary, only experience and observation can instruct us as to the number of cases, and only instruct us approximately.[6]

[3] *Nous avons sur la logique des écrits sans nombre; mais la science du raisonnement a-t-elle besoin de tant de règles? Pour y réussir, il est aussi peu nécessaire d'avoir lu tous ces écrits, qu'il l'est d'avoir lu nos grands traités de morale pour être honnête homme. Les géomètres, sans s'épuiser en préceptes sur la logique, et n'ayant que le sens naturel pour guide, parviennent par une marche toujours sure aux vérités les plus détournées et les plus abstraites*; *tandis que tant de philosophes, ou plutôt d'écrivains en philosophie, paraissent n'avoir mis à la tête de leurs ouvrages de grands traités sur l'art du raisonnement, que pour s'égarer ensuite avec plus de méthode*; *semblables à ces joueurs malheureux qui calculent longtemps et finissent per perdre* ([5], I, p. 152).

[4] *L'esprit qui ne reconnait le vrai que lorsqu'il en est directement frappé, es bien au-dessous de celui qui sait non-seulement le reconnaitre de près, mais encore le pressentir et le remarquer dans le lointain à des caractères fugitifs* ([5], I, p. 154).

[5] *…l'analyse des probabilités dans les jeux de hasard … soumise à des règles connues et certaines, ou du moins regardées comme telles par les mathématiciens* … ([5], p. 157).

[6] *…que dans celles-ci, les règles des combinaisons mathématiques suffisent … pour déterminer le nombre et le rapport des cas possibles; au lieu que dans celles-là, l'expérience et l'observation seules peuvent nous instruire de nombre de ces cas, et ne nous en instruisent qu'à peu prés* ([5], p. 157).

Nevertheless, d'Alembert states that in this second branch of applications of the art of conjecture, recourse to mathematical calculation is still possible because "uncertainty, if there is any, only regards the facts that serve as principles; these facts supposed, the consequences are out of reach".[7]

The third type of art of conjecture regards the subjects where it is only rarely or never possible to arrive at proofs in which the art of conjecture is in any case necessary. These sciences are divided into speculative and practical. Speculative sciences include physics and history; practical sciences include medicine, jurisprudence, and the "science of the world", that is, the capacity of a human being to live in society, drawing from relationships with other humans the maximum benefit for themselves.

7 D'Alembert and the "ordinary theory of probabilities"

While in the context of the analysis of the structure of human knowledge d'Alembert appears to accept the possibility of a use of the calculus of probabilities in arenas in which the art of conjecture is necessary, the numerous essays on probability show in contrast a decided scepticism. The impression is that, in practice, the first two fields of the art of conjecture in which recourse to the calculus of probabilities is acceptable tend to collapse on the third.

We have already seen how the case of vaccination, explicitly cited in the *Eclairsissement* as an example of a subject to be included in the second field of the art of conjecture, was considered by d'Alembert to be substantially incapable of being treated with mathematical analysis. Instead, with regard to the case of the analysis of games of chance, it is interesting to note that d'Alembert appears to cast doubt on the possibility of applying mathematics to this type of question as well. D'Alembert devoted various essays to criticism of the foundations of what he defined as the "ordinary theory of probability".[8] The arguments presented by d'Alembert can be divided into two types, mathematical and empirical.[9] With mathematical arguments d'Alembert intends to cast doubt on the theoretical aspects of the calculus of probabilities, subjecting to criticism, among other things, the definition of the concept of expectation, the usual way of calculating the probability of events on the basis of the identification of equiprobable cases. The point of view that d'Alembert seems to assume in these criticisms is that of *philosophe* who unmasks sophisms of false geometers, as though the theory of probability, more than a subject of mathematical study, were a question that exclusive regarded logic. All else aside, it is the classification itself of the *Éléments* and the related *Eclairsissement* that suggest an intermediate position for this theory, suspended between mathematics and logic.

In this sense, one of d'Alembert's most interesting arguments of criticism is that with which he accuses the theorists of probability of appealing to a paralogism to prove their results. In the tenth memoire of the *Opuscules*, d'Alembert presents this reasoning as fallacious:

1. The probability of getting tails on the first toss is equal to that of getting heads on the first toss;
2. The probability of getting heads on the first toss is double that of getting heads on the first toss and tails on the second, or heads on the first and heads on the second;
3. Thus the probability of getting tails on the first toss is double that of getting heads on the first toss and tails on the second, or heads on the first and heads on the second.

Here is how d'Alembert criticises this reasoning:

> In these two propositions [1 and 2] the *mean term* is not the same, since in the first, the mean term is the probability of getting heads on the first toss before having done it, and in the second it is the probability of getting *heads* on the first toss in comparison to the probability of getting *heads* or *tails on the second*. Now, this last probability (that of getting *heads* or *tails* on the second toss) presupposes that the first toss

[7] …*l'incertitude, s'il y en a, ne tombe que sur les faits qui servent de principes; ces faits supposés, les conséquences sont hors d'atteinte* ([5], p. 157).

[8] In addition to the articles in the *Encyclopédie* devoted to this topic (*Avantage*, *Croix ou pile*, *Gageur*, *Dé*, *Loterie*, *Pari*), we also recall the essays contained in volume II (memoirs 10 and 11) and volume IV (memoir 23) of the *Opuscules mathématiques* and the essay entitled "Doutes et questions sur le calcul des probabilités" in the *Mélanges*.

[9] Such a distinction, even though not made explicit by d'Alembert, emerges in the essay "Doutes et questions sur le calcul des probabilités" in the *Mélanges*, in which d'Alembert distinguishes topics of criticism that can be understood only by mathematicians and those instead that can be understood by readers who are less expert: *J'adopte donc, o plutôt j'admets pour bonne dans la rigueur mathématique, la théorie ordinaire des Probabilités; & je vais seulement examiner si les résultats de cette théorie, quand ils seroient hors d'atteinte dans l'abstraction géométrique, ne sont pas susceptibles de restriction, lorsqu'on applique ces résultats à la nature* ([4], p. 277). I adopt thus, or better, I accept, the ordinary theory of probabilities, and I limit myself to analysing whether the results of this theory, when they are beyond the reach of geometric abstraction, are liable to restrictions when applied to nature".

> has been done and that it came up *heads*; thus this last probability presupposes that the first (that of getting heads on the first toss) is no longer a *probability*, but a *certainty*.[10]

The terminology of logic used by d'Alembert in this passage clearly show the tone of his criticism. In passing, it is also interesting to note that the objection raised by the encyclopaedist at this point, while not entirely clear, appears to point to the concept of conditional probability, which would effectively emerge only at the end of the 1700s, with the work of Bayes.

The empirical arguments that d'Alembert uses as a criticism of the possibility of applying the calculus of probabilities to concrete situations in games of chance can be traced back to the definition of the categories of metaphysically possible and physically possible:

> It is necessary to distinguish that which is *metaphysically* possible from that which is possible *physically*. In the first class enter all the things whose existence has nothing of the absurd; in the second not only those whose existence has nothing of the absurd, but also nothing that is too extraordinary and out of the normal course of events. It is *metaphysically* possible to get two sixes with two dice, a hundred times in a row; but this is *physically* impossible, as this has never occurred, nor will it ever occur.[11]

The distinction between the metaphysically possible and the physically possible has serious consequences from the point of view of the analysis of the probability of real phenomena. An important example is that of the analysis of the coin tosses. According to d'Alembert, the need to establish what is physically possible renders it necessary to carry out experiments by means of which to establish the probability of the occurrence of a certain outcome. With regard to this, he also notes that the occurrence of a certain outcome many times often renders its occurrence less possible in the future. This, in its turn, implies that the series of tosses are not all equiprobable: a series of tosses constituted of almost the same number of heads and tails will be considered more physically possible than one in which the number of one of the outcomes is much smaller than the other.

8 Conclusion

As we have seen, among the arguments criticising the traditional theory of probability is the unmasking of a presumed fallacy, undertaken by d'Alembert borrowing the language of logicians that he harshly criticised. This strategy of explanation is neither intrinsic nor opportune: we have seen how, in the classifications of the *Eléments de philosophie*, d'Alembert had traced probabilistic reasoning to the context of the logic of conjecture, thus to one of the two branches of logic that he identified. Even though in the *Eclairsissements* he expressed himself in favour of the applicability of mathematics to the art of conjecture, the criticisms of probability analysed here show a decided distrust of the possibility of articulating this thesis concretely.

D'Alembert was reluctant to consider logic as an instrument that could be prepared to tell man how to reason in all spheres of knowledge. The calculus of probability, ambiguously collocated between the area of logic and that of mathematics, must be subject to the same doubts as logic. Logic, taken as the art of reason, must limit itself to furnishing a few rough instructions for how to reason:

> This consists in observing exactly their mutual dependence [of truths]; do not resort to a false genealogy to fill the gaps where filiation is lacking; finally, imitate those geographers who, meticulously giving the details of known regions, do not fear leaving empty spaces in correspondence to unknown lands.[12]

D'Alembert appears to imply that the attempt to apply probability to nature, is like the work of an inept geographer, not sufficiently scrupulous in marking unknown lands, those where uncertainly reigns and reasoning can gain little hold.

Translated from the Italian by Kim Williams

[10] … *je dirai que dans cet argument le* moyen terme *d'est pas le même dans les deux Propositions. Car le moyen terme dans la premiere Proposition, est la probabilité d'amener* pile *au premier coup,* avant d'avoir joué ce premier coup. *Dans la seconde Proposition, le moyen terme est la probabilité d'amener* pile *au premier coup,* comparée à la probabilité d'amener croix *ou* pile au second coup. *Or cette derniere probabilité (celle d'amener* croix *ou* pile *au second coup) suppose que le premier coup est joué, & qu'il a donné* pile; *ainsi cette dernier probabilité suppose que la premiere probabilité (celle d'amener* pile *au premier coup) n'est plus une* probabilité *mais une* certitude ([3], pp. 20–21).

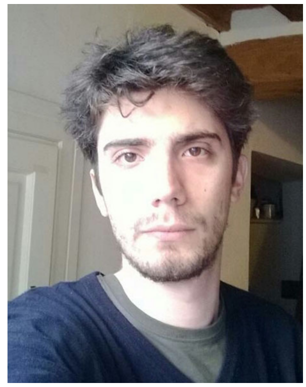
[11] *C'est qu'il faut distinguer entre ce qui est* métaphysiquement *possible, & ce qui est possible* physiquement. *Dans la premiere classe sont toutes les choses dont l'existence n'a rien d'absurde*; *dans la seconde sont toutes celles dont l'existence non-seulement n'a rien d'absurde*; *mais même rien de trop extraordinaire, & qui ne soit dans le cours journalier des événemens. Il est* métaphysiquement *possible, qu'on amene rafle de six avec deux dez, cent fois de suite*; *mais cela est impossible* physiquement, *parce que cela n'est jamais arrivé, & n'arrivera jamais* ([3], p. 10).

[12] *Elle consiste à observer exactement leur dépendance mutuelle; à ne point remplir par une fausse généalogie les endroits où la filiation manque; à imiter enfin ces géographes qui, en détaillant avec soin sur leurs cartes les régions connues, ne craignent point de laisser des espaces vides à la place des terres ignorées* ([5], vol. I, p. 152).

References

1. Bernoulli, D.: Essai d'une nouvelle analyse de la mortalité causée par la petite vérole et des avantages de l'inoculation pour le prévenir. *Histoire et Mémoires de l'Académie des Sciences*, 1760, parte 2, pp. 1-79 (1760)
2. Daston, L.: *Classical Probability in the Enlightenment*. Princeton University Press, Princeton, NJ (1988)
3. D'Alembert J.-B. L. R. *Opuscules mathématiques*, t. II. Chez David, Paris (1761)
4. D'Alembert J.-B. L. R. *Mélanges de litterature, d'histoire, et de philosophie*, t. V. Chatelain, Amsterdam (1767)
5. D'Alembert, J.-B.L.R.: *Oeuvres de d'Alembert*. A. Belin, Paris (1821)
6. Halley, E. An Estimate of the Degrees of the Mortality of Mankind, drawn from the curious Tables of the Births and the Funerals at the City of Breslau, with an Attempt to ascertain the Price of Annuities on Lives. *Philosophical Transactions of the Royal Society of London* **17**: 596–610 (1693)
7. Voltaire, F. M. A. de. *Letters on the English*. Vol. XXXIV, Part 2. The Harvard Classics. P.F. Collier & Son, New York (1909–14). Retrieved from www.bartleby.com/34/2/, 17 November 2014

Camilla Colombo is currently a Ph.D. candidate and graduate teacher assistant at the London School of Economics and Political Science, where she received a Master in "philosophy and public policy" in 2013. She is also continuing her studies at the Scuola Normale Superiore in Pisa and the University of Pisa, where she took a degree in philosophy in 2012. Her principal research interests regard decision theory and social choice theory, the philosophy of science and biology, bioethics and applied ethics.



Mirko Diamanti is a student at the Scuola Normale Superiore di Pisa, where he received a degree in philosophy in 2012 with a thesis on the subjective conception of probability. His principle research interests are in the history of the calculus of probabilities in the 1600s–1700s, and the logical and epistemological implications of probability theory.