AN INQUIRY INTO VARIOUS DEATH-RATES AND THE COMPARATIVE INFLUENCE OF CERTAIN DISEASES ON THE DURATION OF LIFE

By M. NOEL KARN, M.A.

PART I. INTRODUCTION. HISTORICAL, STATISTICAL AND MATHEMATICAL.

CONTENTS.

A. *Introduction.* The method used in this paper for determining the influence of the death-rates from some particular diseases on the duration of life is based on suggestions which were made in the first place by Daniel Bernoulli in a memoir read on April 30th, 1760, before the French Academy of Sciences. This ultimately appeared in the *Mémoires de l'Académie Royale des Sciences* under the title of "De la mortalité causée par la petite vérole et des avantages de l'inoculation pour la prévenir." Though it was not published till 1765, the reading of it had made it known before then, and it had aroused considerable interest. The reading of Bernoulli's memoir was followed, a few months later, namely in November, 1760, by a memoir by D'Alembert, which was published in his *Opuscules*, T. II, in 1761, under the title "Sur l'application du calcul des probabilités à l'inoculation de la petite vérole." In this paper D'Alembert criticised Bernoulli's work, and worked out another formula which would be easier to use*.

* Also *Opuscules*, T. IV, Paris, 1768, p. 98, "Sur un mémoire de M. Bernoulli concernant inoculation," and p. 310, "Sur les calculs relatifs à l'inoculation."

Bernoulli's method of obtaining a measure of the influence of a special disease on the duration of life was a comparison of two life-tables; one, Halley's life-table, the other, a hypothetical life-table showing the number of survivors there would be at each age, if the mortality from small-pox were entirely eliminated from the population.

During the greater part of the eighteenth century a controversy was carried on by doctors and mathematicians regarding the merits of inoculation. Bernoulli and D'Alembert joined in the fray, writing several polemical papers, besides those already mentioned which form the basis of the method used in the present memoir.

The History of Inoculation. In order to follow the steps by which the calculus of probabilities came to be applied to problems connected with death-rates, it is necessary to study something of the history of inoculation. The practice of inducing a mild kind of small-pox by inoculation is, according to the slight evidence available, very old, but the acute controversy only began with the eighteenth century.

Inoculation in England. Reports of the practice and some knowledge of the method of inoculation were conveyed to England early in the eighteenth century by those who had seen inoculation in use among the Turks and Greeks of Constantinople and Smyrna.

Emanuel Timoni, a Doctor of Medicine of Padua, who practised in Constantinople, wrote* concerning inoculation in 1713 in a letter to Dr Woodward, Secretary of the Royal Society, from his observations and inquiries during eight years of residence in that city.

Peter Kennedy, a Scot, who had spent several years in different medical schools in Europe, and had also visited Constantinople, wrote† in 1715 *An Essay on External Remedies* which contains some information as to the practice of inoculation in Constantinople.

Inquiries were made by Sir Hans Sloane, Secretary of the Royal Society, 1693–1712, with reference to the facts as given in Timoni's account, and in answer to these a treatise‡ on inoculation as practised among the Greeks was written in 1715 by Pylarini, a Greek doctor of medicine who had had much experience in practice in Smyrna.

In 1717, Lady Mary Wortley Montague wrote§ from Adrianople to a friend recommending the practice of inoculation. She was so much impressed by the benefits which it seemed to bring that she caused her son Edward aged four to be inoculated by Mr Maitland, surgeon to the British Embassy. "The small-pox," she writes, "so fatal and so general amongst us is here entirely harmless, by the invention of engrafting."

The Method of Inoculation. The details as to the method of inoculation employed in Constantinople are given by Timoni and Pylarini. Other writers|| give the methods used elsewhere; and the accounts have much in common. I condense them here.

* "An Account of the Procuring of the Small Pox by Incision, or Inoculation; as it has for some time been practised at Constantinople. Being an extract of a letter from Emanuel Timonius, Oxon. and Patav., M.D., S.R.S., dated Constantinople, December, 1713." Communicated by John Woodward, M.D. and S.R.S. *Phil. Trans.* Vol. XXIX, 1714, No. 339, p. 72.

† *An Essay on External Remedies*, p. 153. London, 1715.

‡ "A New and Safe Method of communicating the Small-pox by Inoculation, lately invented and brought into use. By Jacob Pylarini, M.D., formerly Venetian Consul at Smyrna." *Phil. Trans.* Vol. XXIX, 1716, No. 347, p. 393. Same also published in Venice in 1715.

§ *Letters of the Right Honourable Lady Mary Wortley Montague*, Vol. II, Letter XXXI, 3rd ed. London, 1763.

|| "The Method of Inoculating the Small-Pox in New England," communicated by Henry Newman, Esq., of the Middle Temple. *Phil. Trans.* Vol. XXXII, 1722, No. 370, p. 33. "Concerning the Inoculation of the Small-Pox at Halifax in Yorkshire," by Dr Nettleton. *Phil. Trans.* Vol. XXXII, 1722, No. 370, pp. 35, 49. "On the Method of procuring the Small-Pox in South Wales," by Perrot Williams, M.D. *Phil. Trans.* Vol. XXXII, 1723, No. 375, p. 262. "A further

The purpose of inoculation was of course to communicate a mild form of small-pox to one person from another who was already suffering from it.

It was necessary for the former to be in a good state of health. As the latter a patient, of strong constitution, suffering from small-pox, was chosen, and on the 12th or 13th day of the disease, some of the pustules were pricked, and the matter from them allowed to drop into a glass vessel held for the purpose. This was then securely closed, and carried to the room of the person to be inoculated. Incisions or needle pricks were made in the muscles of the upper arm or the forearm, or other fleshy part of the body, so that blood would flow from the incision. Then the small-pox matter was rubbed into the wound, and thus the infection was communicated to the patient. The wound was then covered with a half walnut shell or other small concave article, and bound over for protection. The operation by preference was performed either at the beginning of winter or in the spring.

The small-pox would begin to appear sooner in some than in others, but usually on the 7th day, and the number and disposition of the pustules which arose would vary. They would generally break out all over the body, just as in small-pox taken in the ordinary way. In some the pustules formed only at the places of incision and yet this was sufficient to protect the inoculated from taking small-pox afterwards, though they lived with persons suffering from it. It was said that no pitting or scars were left as in small-pox, except at the places of incision.

The inoculated pox was scarcely ever used for a new inoculation. It was observed that the inoculation did not take successfully on persons who had already had small-pox, and that incisions made on them and infected with the matter presently dried up and healed. Timoni considered that his observations undoubtedly established the facts that inoculation induces a mild form of small-pox, that, with proper care in choosing the patient, it does not cause death, and that it protects the patient, for a time at any rate, from an attack of natural small-pox.

The First Inoculations in England. The topic became a source of controversy in London early in the eighteenth century, but the first person to have the operation of inoculation performed in England was Lady Mary Wortley Montague*. She had returned home with her family in 1718, and in 1721 she again called in Mr Maitland to inoculate her younger child Mary, who was at that time three years old.

This became known to several of Lady Mary's friends, among whom was the Princess of Wales, later Queen Caroline, whose eldest daughter, the Princess Anne, had lately suffered severely from small-pox. Thinking that she might save her two younger daughters, Amelia and Caroline, by inoculation from the danger of taking small-pox, the Princess of Wales begged the King to reprieve six Newgate criminals condemned to death that they might be submitted to the experiment of inoculation. This was done on August 9th, 1721†, with the result that the patients became ill and recovered.

Those who advocated inoculation said that they had taken small-pox satisfactorily, but there were others† who considered that they had not taken small-pox at all through the inoculation, and that the test was not a fair one.

Account of procuring the Small-Pox," by the same author. *Phil. Trans.* Vol. XXXII, 1723, No. 375, p. 264. "A further Account of 'buying' the Small-Pox," by Mr Richard Wright, Surgeon at Haverford West. *Phil. Trans.* Vol. XXXII, 1723, No. 375, p. 267.

* "An Account of Inoculation given to Mr Ranby to be published, anno 1736," by Sir Hans Sloane. *Phil. Trans.* Vol. XLIX, 1755, p. 516. *De Variolis et Morbillis Liber.* Richard Mead. London, 1747.

† Maitland's *Account of Inoculating the Small Pox.* London, 1722.

Dr Wagstaffe, Fellow of the Royal Society, and physician at St Bartholomew's Hospital, wrote* in June, 1722 objecting to the ignorant experimentation in inoculation, with special reference to this attempt to inoculate the prisoners.

The Princess then asked that six charity children might also be inoculated as a test, and after further assurance by Sir Hans Sloane the younger princesses were inoculated in 1722 at the ages of 11 and 9.

Some people of noble birth followed the example of the royal family in introducing the operation of inoculation into their households, but the news of the death of the Earl of Sunderland's son through inoculation and of Lord Bathurst's footman spread and helped to prevent any rapid adoption of the practice.

Mr Maitland in Hertford and Dr Nettleton† in Halifax performed inoculations, and the practice continued for a time, but as the results were not altogether satisfactory, there was much opposition; the inoculated sometimes died of the operation, and sometimes they took natural small-pox later even though they had been inoculated, and there was much dispute as to whether genuine small-pox was conveyed through inoculation. So that by the year 1728 after about five or six years of trial the practice of inoculation decreased to almost nothing. The numbers of persons inoculated in these years are given in a table‡ on p. 287. For ten or twelve years more no rapid extension of the practice was made, but a few inoculations are mentioned by Thomas Frewen writing about an epidemic at Hastings§ in 1731.

Inoculation in the American Colonies. Simultaneously with the introduction of inoculation into England attempts were made to bring it into use in New England||. The Rev. Dr Cotton Mather, who had become acquainted with the writings of Timoni and Pylarini through Dr William Douglass¶ of Boston, introduced inoculation during the Boston epidemic of 1721, Dr Boylston** being the inoculator, having performed the operation first on his own child, and then on his negro servant. It was used also in the Boston epidemics of 1730 and 1752.

Opposition to and fear of inoculation became as great in America as in England, and especially it was believed that the practice intensified the epidemics, so that ultimately, by the middle of the 18th century, it was prohibited by the Assembly in the colony of New England except under very strict regulations.

Revival of the Practice of Inoculation in England. A revival of the practice of inoculation in England about 1740 was due to accounts of experience in the American colony of South Carolina. The method had been tried in the severe epidemic of 1738 in Charleston by J. Kilpatrick, who having lost one of his sons through small-pox hoped to save the other two by means of inoculation. After these first trials he performed many inoculations and believed that he saved many lives. He came to England in 1743, and his work on inoculation†† which had by that time been printed

* "A Letter to Dr Freind showing the Danger and Uncertainty of Inoculating the Small-Pox," 1722, by Dr Wagstaffe.

† "Concerning the Inoculation of the Small-Pox, and the Mortality of that Distemper in the natural way," by Dr Nettleton, Physician at Halifax. *Phil. Trans.* Vol. XXXII, 1722, No. 374, p. 209.

‡ *Account of the Success of inoculating Small-pox in the years* 1727 *and* 1728, by Dr Scheuchzer.

§ "An Account of the Condition of the Town of Hastings, after it had been visited by the Small-Pox," by Mr T. Frewen. *Phil. Trans.* Vol. XXXVII, 1731, No. 419, p. 108.

|| "A Comparison between the Danger of the Natural Small-Pox and of that given by Inoculation," by James Jurin, M.D. *Phil. Trans.* Vol. XXXII, 1722, No. 374, p. 213.

¶ William Douglass (1691–1752), physician of Boston.

** Zabdiel Boylston (1679–1766) published *An Historical Account of the Small-Pox Inoculated in New England upon all Sorts of Persons, Whites, Blacks, and of all Ages and Constitutions.* London, 1726.

†† *An Essay on Inoculation: occasioned by the Small-Pox being brought into S. Carolina in the year* 1738, by J. Kilpatrick. London, 1743. First printed in South Carolina.

in South Carolina was now published in London. He took his degree of M.D. and changed his name to Kirkpatrick, and became well known as an inoculator.

With the revival of inoculation a change in the usual method was introduced, and the inoculated matter from one person was now used for the inoculation of others.

In 1746 provision was made for the inoculation of poor people at the various hospitals which were established from that date in many parts of London for the care of the people in small-pox epidemics. Thus it was made possible for inoculation to become more general. The practice continued to increase in England without further check for about half a century till it was gradually superseded by the method of vaccination. The end of the history of inoculation came in 1840, when owing to the evils arising from the practice it was finally forbidden by Act of Parliament.

Inoculation in France. The first detailed history of the early practice of inoculation in France and in other European countries was written in 1754 by M. de la Condamine*, and published in the memoirs of the Academy of Sciences. This was followed by a second *mémoire* in 1758. The main facts of the history as stated by Condamine are given below.

In France the method of inoculation became known first through the same source as in England, that is, the Latin treatise on inoculation in Constantinople written by Timoni.

Detailed information explaining the advantages of inoculation and referring to the tables and calculations of Dr Jurin published in the *Philosophical Transactions* was given in 1723 in a letter from M. de la Coste to M. Dodard, both Doctors of Medicine.

So the controversy over inoculation in England became known by degrees in France, and the general population there was apparently influenced by the unfavourable accounts of inoculation in New England and by the writings of certain French medical men criticising it; thus it was not at once adopted.

About the same time, in 1723, nine doctors of the Sorbonne, after studying the account of Pylarini, agreed to try the experiment of inoculation, but public opinion was too hostile and no progress was made.

In 1724 the writings of Dr Jurin† were translated by M. Noguez, a doctor of Paris. After this for a period of thirty years there was no other French medical writer to recommend inoculation. The practice was established slowly only after the middle of the 18th century. It was first introduced in 1750 by M. Calendrini (1703–1758), Professor of Mathematics and Philosophy in the Academy of Geneva, later a counsellor of state and treasurer of the Republic of Geneva, who allowed his son to be inoculated. From that time the subject began to re-arouse interest, and in 1752 a reprint of Timoni's account of inoculation in Constantinople was published.

In 1755 M. Hosty, of Montpellier, was sent to England for the purpose of studying the methods and results of inoculation at the London Hospital. He returned to France, convinced in favour of inoculation and, under his direction, many inoculations were performed on people of noble birth, not only in Paris, but in other parts of France. In 1756 the Duke of Orleans, seeing the good results of inoculation, allowed his son, the Duke of Chartres, and his daughter to be submitted to it. The operations were performed with success by M. Tronchin, who had inoculated his own son.

In a collection of memoirs‡ published in 1786, but written much earlier, there are further

* "Mémoire sur l'inoculation de la petite vérole. Second Mémoire sur l'inoculation de la petite vérole." *Histoire de l'Académie Royale des Sciences.* 1754 and 1758.

† See p. 284.

‡ "Sur l'inoculation de la petite vérole, et principalement sur les variations de la méthode," p. 373. *Collection Académique contenant la suite de l'Histoire et des Mémoires de l'Académie Royale des Sciences de Paris.* Paris, 1786. This paper is dated as 1765 of the *Mémoires.*

references which show that the question of the usefulness of inoculation was keenly debated. The authorities were doubtful about it, and the people were afraid of it, so that in 1763 a decree was passed by the court of Paris by which provisionally it was forbidden to inoculate in the towns for fear of spreading the contagion of small-pox.

Inoculation in Germany. In Hanover the practice of inoculation was carried on for a short time after Prince Frederick had been inoculated by Maitland in 1723, but it was discontinued when the Prince left that country.

In other European countries the reports of inoculation in Constantinople do not seem to have been known as early as in England and France. The knowledge of inoculation was conveyed as a rule to neighbouring countries by the migration of some individual who had had experience of its supposed advantages; the practice of it was often established, after its first introduction, by the appointment of hospitals like those in London for the inoculation of poor people. Inoculation came to be generally practised in Europe during the latter half of the 18th century.

B. *The First Statistical Data of Inoculation.* From the first introduction of inoculation into England efforts were made to obtain statistical information regarding the results of it.

Accounts* were given by Dr Nettleton of his experience in inoculating in Yorkshire, during the years 1721 and 1722. These show that he inoculated 61 persons successfully before he had a fatal case. His purpose in writing of his experience was to show by figures collected with care that the risk of dying of inoculation was much less than the risk of dying of small-pox.

This kind of comparison is not of much use, for in calculating the chance of dying of small-pox it is necessary to consider how many persons take small-pox in the general population; and in calculating the chance of dying of inoculation, account ought to be taken as to how many of those inoculated would be likely to escape small-pox in the natural way, and further to pay accurate attention to the times of exposure to risk.

The following are the figures given by Dr Nettleton:

	Had Small-pox	Have Died of Small-pox	Proportion of Deaths
Halifax	276	43	$\frac{1}{6}$
Rochdale	177	38	$\frac{1}{5}$
Leeds	792	189	$\frac{1}{4}$

To these he added further results in his second paper making 636 deaths out of 3405 who took small-pox in the epidemics of 1721 and 1722, giving a proportion of 1 in 5. This mortality cannot be taken as an average rate, as it refers to times of epidemics. Hence a fair comparison cannot be made between the risk of 1 in 62 of death from inoculation and 1 in 5 as the risk of death from small-pox.

Dr James Jurin, Secretary of the Royal Society, writing about the year 1722†, followed Dr Nettleton in trying to make a convincing demonstration of the comparative risks of death from small-pox and from inoculation. He collected all the English data he could find from a variety of persons who had performed inoculation, the numbers inoculated being 182, of whom two had died. He took into account the records from Boston, New England, for the epidemic already mentioned on p. 282. The figures given by the Rev. Dr Cotton Mather, writing in 1721, were five or six

* *Phil. Trans.* Vol. XXXII, 1722, No. 374, p. 209. Referred to above, p. 282.

† "A Comparison between the Danger of the Natural Small-Pox and of that given by Inoculation," by James Jurin, M.D., R.S.Sec. *Phil. Trans.* Vol. XXXII, 1722, No. 373, p. 191; No. 374, p. 213.

deaths out of 300 inoculated, and 500 deaths out of 3000 who took small-pox, in a period of about six months, making the risk of death from inoculation about 1 in 60.

Dr Jurin made his comparison of the risk of inoculation and small-pox in a different way from Dr Nettleton. He did not, however, arrive at any useful or convincing comparison.

He consulted the yearly bills of mortality, as far back as the year 1667, and gave abstracts of these for 42 years, in two periods 1667–1686 and 1701–1722, showing the total number of burials, and the number of these deaths which were due to small-pox. Taking either period the result showed that 1 death in 14 was due to small-pox.

Table I A.

Years	Total No. of burials	Died of Small-pox		
		In all	In 1000	Proportion
1667–1686				
Total for 20 years	398,200	28,459	—	—
Average for 1 year	19,910	1,423	71½	$\frac{1}{14}$

Table I B.

1701–1722				
Total for 20 years	505,598	36,620	—	—
Average for 1 year	22,982	1,665	72	$\frac{1}{14}$
Average for 1 year on 42 years	21,519	1,550	72	$\frac{1}{14}$

As there was no reference to the populations in which these burials occurred, this did not lead to any very useful conclusion.

The Death-rate from Small-pox in London. It would have been interesting to have made a comparison between the death-rate among the inoculated and in the general population. This may be done roughly for London, for the population can be calculated by assuming with Sir William Petty that the deaths in London amounted in that period to $\frac{1}{30}$ of the population each year. Then, according to Jurin's figures, $\frac{1}{14}$ of all the deaths were due to small-pox, so that $\frac{1}{14} \times \frac{1}{30}$, that is $\frac{1}{420}$ of the population, died each year from small-pox. This is calculated on the average, and would vary from year to year according to the intensity of the epidemics. If Rickman's* estimate were used the fraction would be $\frac{1}{14} \times \frac{1}{25}$, that is $\frac{1}{350}$.

A table is given by Jurin showing the number of deaths from small-pox and the calculated population over a series of years, from which the risk of death from small-pox in the general population is obtained.

In order to eliminate the irregularities in the number of deaths caused by the variation in the small-pox epidemics from year to year, the population was calculated on the basis of the deaths from causes other than small-pox. These amounted to $\frac{13}{14} \times \frac{1}{30}$, i.e. $\frac{13}{420}$ of the population.

In using the figures of burials given in the Bills of Mortality it is necessary to make allowance for unrecorded burials. According to Dr Brackenbridge† these would be covered by the addition

* John Rickman (1771–1840), statistician, who prepared the first Census Act of 1800 and the census reports published in 1801, 1811, 1821 and 1831, calculated the death-rate of London in 1700 as 1 in 25, and in 1750 as 1 in 21 or 20. The population returns for these years, based on the Parish Registers, were 674,350 and 676,250 respectively. See *London Life in the Eighteenth Century*, by M. Dorothy George, 1925, pp. 24–25.

† "On the Number of Inhabitants within the London Bills of Mortality," by the Rev. W. Brackenbridge, D.D., F.R.S. *Phil. Trans.* Vol. XLVIII, p. 788.

of 2000 each year. So $\frac{1}{14} \times 2000$ of these would be due to small-pox, and $\frac{13}{14} \times 2000$, that is 1857, would be deaths from other causes.

The table gives the deaths from other causes, and thus the population is deduced by multiplying these by $\frac{420}{13}$ or 32·3077. Then, from the number of small-pox deaths and the population, the risk of death from small-pox is obtained for the given years. The results show that this had increased from about 1710, and had been generally above the average since then.

So each person in the population was under the average risk of 1 in 420 of death from small-pox in every year of life, while the risk of death from inoculation was about $\frac{1}{100}$, $\frac{1}{200}$ or $\frac{1}{300}$ according to different data* perhaps only once in a lifetime.

Table II.

Table showing the Risk of Death from Small-pox in London in a Series of Years.

	Burials	Deaths from Small-pox	Deaths from all Other Causes	Unrecorded 1857 added	Population Last col. multiplied by 32·3077	Small-pox deaths + 143 unrecorded	Risk of death from Small-pox 1 in
1701	20,471	1095	19,376	21,233	685,989	1238	554
1702	19,481	311	19,170	21,027	679,334	454	1496
1703	20,720	898	19,822	21,679	700,398	1041	673
1704	22,684	1501	21,183	23,040	744,369	1644	453
1705	22,097	1095	21,002	22,859	738,522	1238	597
1706	19,847	721	19,126	20,983	677,912	864	785
1707	21,600	1078	20,522	22,379	723,014	1221	592
1708	21,291	1687	19,604	21,461	693,355	1830	379
1709	21,800	1024	20,776	22,633	731,220	1167	627
1710	24,620	3138	21,482	23,339	754,029	3281	230
1711	19,833	915	18,918	20,775	671,192	1058	634
1712	21,198	1943	19,255	21,112	682,080	2086	327
1713	21,057	1614	19,443	21,300	688,154	1757	392
1714	26,569	2810	23,759	25,616	827,594	2953	280
1715	22,232	1057	21,175	23,032	744,111	1200	620
1716	24,436	2427	22,009	23,866	771,055	2570	300
1717	23,446	2211	21,235	23,092	746,049	2354	317
1718	26,523	1884	24,639	26,496	856,025	2027	422
1719	28,347	3229	25,118	26,975	871,500	3372	258
1720	25,454	1440	24,014	25,871	835,832	1583	528
1721	26,142	2375	23,767	25,624	827,852	2518	329
1722	25,750	2167	23,583	25,440	821,908	2310	356
Average for 1 year					748,704	1808	414

Some information regarding the population of London for this period, given by M. Dorothy George, is interesting for comparison with the results obtained here.

She quotes† that for the years 1728–1757, out of 100,000 living the number of small-pox deaths was 426 and the number of deaths from all causes 5200. This gives $\frac{426}{5200}$ or $\frac{1}{12}$ as the proportion of small-pox to other deaths against Jurin's $\frac{1}{14}$ for the earlier period, and $\frac{426}{100000}$ or 1 in 235 as the risk of death from small-pox in the general population.

Jurin concluded his memoir by recommending a house-to-house inquiry to find out how many people had had small-pox and how many had died of it in the preceding year.

* Figures quoted by Kirkpatrick in *The Analysis of Inoculation*, London, 1754:

The London Hospital	6 deaths in	1809	inoculations.
The Foundlings' Hospital	1 death in	186	,,
Frewen at Rye	1 death in	300	,,
At Salisbury	4 deaths in	422	,,

† *London Life in the Eighteenth Century*, Appendix D, p. 407. London, 1925.

An investigation of this kind had been made by Dr Nettleton in Yorkshire, Dr Whitaker in Chichester, and Dr Perrot Williams in South Wales. The following figures were the result:

	Sick of Small-pox	Died of Small-pox	Proportion died
Several towns in Yorkshire	3405	636	$\frac{1}{5}$
Chichester	994	168	$\frac{1}{6}$
South Wales	227	52	$\frac{1}{4}$
	4626	856	$\frac{1}{5}$

The following table was quoted by Thomas Frewen*, Surgeon at Rye in Sussex, taken from Dr Scheuchzer's *Account of the Success of inoculating Small-pox in the years* 1727 *and* 1728. It shows the death-rate among those who took small-pox, and among the inoculated, but not the death-rate from small-pox in the general population.

Table III.

A Table showing the Mortality of the Natural Small-pox from the year 1721 *to* 1728 *inclusive, compared with the State and Hazard of Inoculation during that time.*

		Natural Small-pox		Inoculated Small-pox				
		Died of Small-pox						
	Total number of Burials	In all	In proportion	Number of Persons Inoculated	Had Small-pox by Inoculation	Suspected to have died of Inoculation	Had an imperfect Small-pox	No effect
1721	26,142	2375	$\frac{1}{11}$					
1722	25,750	2167	$\frac{1}{12}$	483	449	9	—	—
1723	29,197	3271	$\frac{1}{9}$					
1724	25,952	1227	$\frac{1}{21}$	46	42	1	—	—
1725	25,523	3188	$\frac{1}{8}$	152	144	3	—	—
1726	29,647	1569	$\frac{1}{19}$	107	102	1	—	—
1727	28,418	2379	$\frac{1}{12}$	109	106	3	—	—
1728	27,810	2105	$\frac{1}{13}$					
Total	218,439	18281	$\frac{1}{12}$	897	843	17	13	24
At a medium	27,305	2285	$\frac{1}{12}$	—	—	—	—	—

The number of burials and the deaths from small-pox are taken from the Bills of Mortality for London.

The death-rate from inoculation is 17 in 897 or 1 in 53.

Data of Epidemic in Hastings. About the year 1731 an epidemic of small-pox occurred in the town of Hastings† and lasted for about a year and a half. Some figures were given for this epidemic by Mr Thomas Frewen (1704–1791).

Mr Frewen would be likely to be familiar with the conditions of the country in the south of Sussex, for he practised medicine at Rye, and later was physician at Lewes. He was known as one of the first doctors to inoculate for small-pox, and he published in 1749 in London *The Practice and Theory of Inoculation.*

He observed that many persons escaped small-pox, although they were exposed to the danger of taking it, but he did not remark as to whether they had been inoculated, or had previously had small-pox.

* *The Practice and Theory of Inoculation with an Account of its success in a letter to a Friend.* London, 1749.

† "An Account of the Condition of the Town of Hastings, after it had been visited by the Small-Pox," by Mr T. Frewen. *Phil. Trans.* Vol. XXXVII, 1731, No. 419, p. 108.

The details given for the Hastings epidemic were the number of inhabitants, the number who took small-pox, the number who died of it or of other diseases during the time of the epidemic, and the number of those who escaped it.

With regard to inoculation during the epidemic, Frewen said that nothing remarkable happened, but he did not give the number inoculated, except the four mentioned below. His figures are as follows:

Recovered of small-pox, including four inoculated	608
Died of small-pox	97
Escaped it	206
Died of other illnesses since the epidemic began...	50

He gave the whole number of inhabitants as 1636; 782 males, 854 females. This, according to the title of the memoir, should be the population at the end of the epidemic, not at the beginning.

Now the figures, as they stand, need much explanation because they give 608 + 206, i.e. 814, as the number alive at the end of the epidemic, whereas Frewen gives the population as 1636. There are thus 822 persons unaccounted for. It is possible that the number 206 of those who escaped small-pox means that 206 of those susceptible to it escaped it. In a place of less than 2000 inhabitants it would be possible to discover the number who had had small-pox and who had not. But if this be so, it would signify that more than half the inhabitants of Hastings had suffered before 1731 from the disease. Had this group been inoculated and escaped Frewen, as a keen inoculator, would certainly have mentioned it. Perhaps the 206 refer only to "contacts," or again the 822 persons may have been those who fled from Hastings, who had been inoculated, or who had previously had small-pox.

If we make the somewhat risky assumption that half the population had already had small-pox, the following information can be obtained from the figures:

(1) the proportion of deaths among those who took small-pox;
(2) the proportion of the population who took small-pox;
(3) the proportion who had already had small-pox;
(4) the chance of dying of small-pox in an epidemic.

It can be deduced:

(1) that the number of those who took small-pox was 608 + 97, that is 705;
(2) that the number susceptible was 705 + 206, i.e. 911;
(3) that the number who died of all diseases during the time of the epidemic was 97 + 50, i.e. 147;
(4) that the population at the beginning of the epidemic was 1636 + 147, i.e. 1783, no account being taken of births during the epidemic;
(5) that the number who had already had small-pox was 1783 − 911, i.e. 872.

Those who were born during the epidemic would be subject to the risk of taking and dying of small-pox, and so would be included in the above figures.

Proportion of deaths among those who took small-pox ... $\frac{97}{705} = \frac{1}{7 \cdot 3}$

Proportion of the population who took small-pox ... $\frac{705}{1783} = \frac{1}{2 \cdot 5}$

Proportion who had already had small-pox ... ... $\frac{872}{1783} = \frac{1}{2 \cdot 0}$

Chance of dying of small-pox in an epidemic of 18 months $\frac{97}{1783} = \frac{1}{18 \cdot 4}$

But the above numbers are of very doubtful validity. Frewen could have made in a couple of lines a statement which would have rendered his contribution of service; failing this statement his data are nearly worthless.

Inoculation Statistics in America. An account of the progress of inoculation in America was given by Benjamin Gale*, a physician in New England, writing about 1765. Small-pox had never been epidemic in the American colonies except in Boston, where it had occurred in 1649, 1666, 1678, 1689, 1702, 1721, 1730, 1752, 1764. Figures were given relating to the epidemics of 1721, 1730 and 1752†, as follows:

In 1721

		Dr Mather's rough estimate quoted above
Population, besides those who moved out to avoid the epidemic	10,567	
Attacked by small-pox	5,989	5000
Died of small-pox	844	900
Inoculated	286	300
Died from inoculation	6	5 or 6

Proportion which took small-pox ... ... $\frac{5989}{10567} = \frac{1}{1 \cdot 8}$

Proportion which died among those attacked ... ... $\frac{844}{5989} = \frac{1}{7 \cdot 1}$

Proportion which died of all population ... ... $\frac{844}{10567} = \frac{1}{12 \cdot 5}$

In 1730

Population, not given, would be about	12,000
Attacked by small-pox	4,000
Died of small-pox	500
Inoculated	400
Died from inoculation	12

This gives 1 in 8 as the proportion of deaths among those who took small-pox, and 1 in 24 as the proportion of the population which died of small-pox.

In 1752 a careful record was made on oath, by order of the magistrates of Boston, to obtain facts, in order to try and remove the prejudices against inoculation. Information was obtained from all who had had small-pox in the natural way, or by inoculation. The results were:

Population (including 1544 blacks)	15,734
Attacked by small-pox	5,544
Died of small-pox	514
Inoculated (including blacks) ...	2,113
Died from inoculation	30

As 1800 left the town to escape the epidemic, the remaining population was 13,934; then

Proportion who died of those attacked ... ... $\frac{514}{5544} = \frac{1}{10 \cdot 8}$

Proportion of the population who died of small-pox $\frac{514}{13934} = \frac{1}{27 \cdot 1}$

Proportion of the population who took small-pox... $\frac{5544}{13934} = \frac{1}{2 \cdot 5}$

* "Historical memoirs, relating to the Practice of inoculation for the Small-Pox, in the British American Provinces, particularly in New England," by Benjamin Gale. *Phil. Trans.* Vol. LV, 1765, p. 193.

† Dr Douglass' *Summary History*, Vol. II, p. 395.

Proportion who died from inoculation ... ... $\frac{30}{2113} = \frac{1}{70}$

Proportion who recovered from small-pox ... ... $\frac{5030}{5544} = \frac{1}{1 \cdot 10}$

The following table is given to show side by side the results of the different data examined:

Table IV.

Source	Risk of Death among those attacked by Small-pox	Risk of Death from Small-pox in the Population
Jurin's collection for epidemics	$\frac{1}{5}$	—
Frewen, epidemic	$\frac{1}{7}$	$\frac{1}{18}$
Douglass, epidemic	$\frac{1}{7}$	$\frac{1}{12}$
,, ,,	$\frac{1}{8}$	$\frac{1}{24}$
,, ,,	$\frac{1}{11}$	$\frac{1}{27}$
Annual average over many years:		
Bills of Mortality	$\frac{1}{14}$	$\frac{1}{420}$
Parish Registers of London	$\frac{1}{12}$	$\frac{1}{192}$

C. *Conclusions that may be drawn from the Controversy.* Discussion as to the chances of death from inoculation as compared with death from small-pox formed a great part of the 18th century controversy among both English and French writers. The question is hardly of any interest now, but there are some other points which are worth noting.

The Inoculated a Selected Population. By the middle of the 18th century much experience in inoculation had been gained, and the method had been perfected. It was acknowledged that inoculation should be performed only on those who were in a good state of health, so that the inoculated had the advantage of being a selected population from the point of view of the statistician, whereas the victims of natural small-pox would be chosen at random, and perhaps more particularly from among the weaker population. Hence the comparison between the death rates of natural small-pox and of inoculation is not a fair one. In addition to this we can only under very exceptional circumstances admit the use of a proposed preventive of disease which can occasionally be a cause of death.

The Methods used in inoculating. There was some discussion as to the best method of inoculating. The methods used were apparently very various. Some inoculators made a very mild operation of it, others cut a deep wound and ensured a very thorough infection with the small-pox matter. Later, inoculation was carried out with matter from an inoculated person who might have a third, fourth or later culture from the original small-pox matter. Some interesting questions soon arose, for example:

Did inoculation, by all the different methods, give a real attack of small-pox?

Did one inoculation ensure the patient against a subsequent attack of small-pox?

Was the inoculated person a source of infection to others?

Evidence was available on these points to show that the infection given by inoculation was not always recognisable as small-pox; that sometimes no result followed at all; that an inoculated person did sometimes take small-pox in after life; and that an inoculated person sometimes conveyed the disease to others.

With regard to the important problem as to whether one inoculation preserved the patient for the rest of his life from the danger of taking small-pox, the opinion that small-pox could be suffered only once in a lifetime, except in a very mild form, was stated by various writers to be based on the general observation of doctors.

This opinion, if correct, has a definite connection with the point just considered as to the nature of the illness following inoculation; for, assuming that the infection given by inoculation was small-pox, then one inoculation should be sufficient for a lifetime.

There is notice sometimes of the inoculated being purposely brought into close contact with severe cases of small-pox and yet taking no harm from it. But this test was made within a short time, perhaps within a few weeks only, of inoculation.

The objection to inoculation which lay in the danger of the spread of small-pox infection through lack of proper isolation of the inoculated came to be realised. It was largely from experience of this that inoculation was prohibited or restricted at various times as I have noticed previously in this paper*.

Dr Kirkpatrick† advised in 1754 that hospitals should be established outside large towns for the isolation of those inoculated. He considered that no one had a right to be inoculated if it meant exposing his neighbours to the danger of catching small-pox.

Attempts to obtain Statistical Information. There were some attempts to provide more details as to the results of inoculation than merely the number of deaths as shown in the tables already cited. An advertisement was printed, dated November 8th, 1728, requesting that all persons concerned in the practice of inoculation should keep a register of the "Name, Age, and Habitation of every person inoculated, the manner of the operation, the number of days of sickening and the duration of the eruption, the sort of small-pox that was produced, and the event. Where the true small-pox was not produced by inoculation it should be noticed whether the patient had any other kind of eruption, what symptoms, etc." In case any person died after inoculation a particular account should be given and attested. These accounts were to be sent for 1727 and 1728 to Dr Scheuchzer, M.D., F.R.S., at Sir Hans Sloane's house in Russell Square.

Another advertisement asked that any person who had opportunity of making any observation of the ill-consequences of inoculation should communicate the case with details to Dr Warren, of Bury St Edmund's.

The only tabulations with regard to age that I have been able to find are the following:

Table V.

A Table containing the several Ages of the Persons Inoculated in Great Britain from the year 1721 *to the end of the year* 1728, *together with the Success of the Operation.* (*Quoted by Kirkpatrick from Scheuchzer.*)

Age (years)	Persons Inoculated	Had Small-pox by Inoculation	Had an imperfect Small-pox	Had no effect	Suspected to have died of Inoculation
Under 1	24	24	—	—	2
1- 2	34	33	—	1	4
2- 3	65	65	—	—	1
3- 4	91	88	—	3	1
4- 5	65	63	—	2	1
5–10	257	249	3	5	3
10–15	140	131	1	8	1
15–20	104	95	3	6	2
20, etc.	110	91	6	13	2
Unknown	7	6	—	1	0
	897	845[1]	13	39[1]	17

[1] There are small discrepancies in Kirkpatrick's quotation of this table as compared with Frewen's.

* In Paris and in New England, and finally in England.

† *The Analysis of Inoculation*, London, 1754.

Table VI.

A Table given by Dr Boylston, March 10th, 1721. Inoculations in and near Boston, New England.*

Age (years)	Persons Inoculated	Had Small-pox by Inoculation	Had no effect	Suspected to have died of Inoculation
¾– 2	6	6	—	—
2– 5	14	14	—	—
5–10	16	16	—	—
10–15	29	29	—	—
15–20	51	51	—	1
20–30	62	60	2	1
30–40	44	42	2	1
40–50	8	7	1	—
50–60	7	6	1	2
60–67	7	7	—	1
Others	38	36	—	—
	282	274	6[1]	6

[1] It was proved subsequently that three of these had small-pox when very young.

As a side-issue of the main topic dealt with in this paper an outline has been given in the previous pages of a most remarkable piece of medical history. It shows the adoption of a suggested preventive of disease without the collection of any really reliable evidence as to its effect on the patient, as to the extent to which it might ensure against an attack of the disease it was designed to prevent, or as to its effect on the community as a whole. Finally, after a century of experimentation and argument from every point of view, it had to be suppressed by Act of Parliament on the ground that it was a means of propagating disease.

Have we to-day thoroughly learnt the lesson conveyed in this strange story? Is at the present time the collection of essential data, the provision of control series and the investigation of the whole material by independent statistically trained minds recognised as a *sine qua non* of the universal adoption of a new medical treatment? Or, does adoption follow very inadequate evidence, and only after long experimentation do we learn the advantages or disadvantages of the treatment?

D. *The Mathematical Writers.* After the middle of the 18th century there is more of interest for the purpose of this paper in the writings on the subject of inoculation of the mathematicians than in those of the medical men.

Bernoulli. A paper† was read in April, 1760, by Daniel Bernoulli, "De la mortalité causée par la petite vérole et des avantages de l'inoculation pour la prévenir," published in the *Mémoires de l'Académie Royale des Sciences.* In this memoir Bernoulli suggested a new idea which could indeed be illustrated only by means of accurate data. He wanted to compare the age distributions and the mean durations of life in two differently constituted populations: the one, the present population, suffering from a high mortality due to small-pox; the other, an imaginary population in which there would be no fatality from small-pox.

The same idea had been previously hinted at by Dr Kirkpatrick‡, who said that if all children were inoculated at five years of age the increase in the population would be very different.

Bernoulli was not so much interested in the question of the value of inoculation as in the mathematical theory of obtaining a life-table population freed from the mortality of a particular

* Published in *An Historical Account of the Small-Pox Inoculated in New England upon all Sorts of Persons, Whites, Blacks, and of all Ages and Constitutions,* by Zabdiel Boylston, Physician. London, 1726.

† See p. 279 above.

‡ See work cited on p. 291 in second footnote.

disease, but he referred to the question of the risk of death from inoculation, and for the purpose of his calculations took it to be 1 in 200. He assumed also that a person who had had small-pox would not suffer it a second time, and again that inoculation produced true small-pox.

The theory can well be applied in the cases of other diseases which affect in any marked degree the mortality of the race. It can be used in the case of cancer, for example, to find out the effect of the death-rate from that disease on the number of survivors at each age in the population at the present day. Bernoulli suggested that we should compare the two conditions by the amount of the mean duration of life in each case. For the first he was able to use Halley's life-table, and for the second he worked out a method of calculating the number of survivors at each age. As he had not enough data he could not obtain conclusions from his idea with much certainty of accuracy.

In order to know how many deaths from small-pox there were at each age, he had to make two hypotheses. These were, that at each age the same proportion of those living was attacked, and that, of those who were attacked the same proportion died at each age.

Bernoulli's Formula $s = \frac{m}{1 + (m-1)\, e^{x/n}}\, \xi$ *for the Number who have not had Small-pox at Age x.*

Bernoulli's results were given in two tables, referring respectively to the two conditions of the population. These and the formulae which he used require a little explanation.

Bernoulli's Table I* consists of:

Column 1. Age x in years.
2. Survivors ξ at the beginning of each age x, taken from Halley's life-table.
3. Number s, out of ξ survivors, who have not had small-pox at age x.
4. Number $\xi - s$ who have had small-pox.
5. Number who take small-pox during the year.
6. Number who die of small-pox during the year.
7. The sum u of those who have died of small-pox up to the beginning of age x.
8. The number who die of other diseases during the year.

For Column 3 the number s is given by the formula $s = \frac{m\xi}{(m-1)\, e^{x/n} + 1}$, which Bernoulli obtained in the following way. Let

$x =$ age in years,
$\xi =$ survivors at age x,
$s =$ number of those who have not had small-pox at age x,
1 in $n =$ proportion of those who take small-pox among those who have not had it,
1 in $m =$ proportion of those who die of small-pox among those who take it.

Bernoulli assumed values for m and n. He said that the evidence available showed that they varied with age and with the locality, but that there was not enough data to say what exactly they should be.

It is required to express generally the value of s. In a small time dx, the number of those who take small-pox is $-\,ds$, the sign being negative because s is decreasing.

But in one year, out of n persons, one takes small-pox, therefore in dx years, out of s persons, $dx \times \frac{s}{n}$ take small-pox. Of these, there will die $\frac{s\,dx}{mn}$ in time dx. The number of deaths from all diseases is $-\,d\xi$ in time dx, the sign being negative because ξ is diminishing; so the number of

* Extracts from these tables (pp. 44–45 of Bernoulli's memoir) are given on p. 294.

deaths from all diseases except small-pox is $-d\xi - \frac{s\,dx}{mn}$. This refers to a population ξ, so if applied to a population s, it must be multiplied by $\frac{s}{\xi}$.

Then an equation can be formed thus:

$$-ds = \frac{s\,dx}{n} - \frac{s}{\xi}\,d\xi - \frac{s^2}{\xi}\,\frac{dx}{mn};$$

hence
$$\frac{s\,d\xi}{\xi} - ds = \frac{s\,dx}{n} - \frac{s^2}{\xi}\,\frac{dx}{mn},$$

i.e.
$$\frac{s\,d\xi - \xi\,ds}{s^2} = \frac{\xi}{s}\,\frac{dx}{n} - \frac{dx}{mn}.$$

Put $\frac{\xi}{s} = q$, then $dq = \frac{s\,d\xi - \xi\,ds}{s^2}$.

Then the equation becomes
$$dq = q\,\frac{dx}{n} - \frac{dx}{mn},$$

or
$$mn\,dq = (mq - 1)\,dx;$$

thus
$$\frac{mn}{mq - 1}\,dq = dx.$$

Therefore integrating
$$n \log (mq - 1) = x + c,$$

i.e.
$$n \log \left(m\,\frac{\xi}{s} - 1\right) = x + c.$$

When $x = 0$, $\xi = s$, therefore $c = n \log (m - 1)$;

$$\therefore\; n \log \left(\frac{m\,\frac{\xi}{s} - 1}{m - 1}\right) = x;$$

$$\therefore\; m\,\frac{\xi}{s} - 1 = (m - 1)\,e^{x/n}.$$

Thus
$$m\,\frac{\xi}{s} = (m - 1)\,e^{x/n} + 1$$

and
$$s = \frac{m}{1 + (m - 1)\,e^{x/n}}\,\xi.$$

The following are extracts from Bernoulli's Tables I and II:

TABLE I.

Ages par années	Survivans selon M. Halley	N'ayant pas eu la petite vérole	Ayant eu la petite vérole	Prenant la petite vérole pendant chaque année	Morts de la petite vérole pendant chaque année	Somme des morts de la petite vérole	Morts par d'autres maladies pendant chaque année
0	1300	1300	0	—	—	—	—
1	1000	896	104	137	17·1	17·1	283
2	855	685	170	99	12·4	29·5	133
3	798	571	227	78	9·7	39·2	47
4	760	485	275	66	8·3	47·5	30

TABLE II.

Ages par années	État naturel et variolique	État non-variolique	Différences ou gains
0	1300	1300	0
1	1000	1017·1	17·1
2	855	881·8	26·8
3	798	833·3	35·3
4	760	802·0	42·0

Hence, substituting Bernoulli's constants $m = 8$ and $n = 8$,

$$s = \frac{8\xi}{1 + 7e^{x/8}}.$$

This is the formula from which Column 3 was calculated, and Column 4 follows at once. Column 5 was taken as $\frac{1}{8}$ of the value of s of Column 3, using the mean value of s at the beginning and end of the year. Column 6 is $\frac{1}{8}$ of the value of the previous column, and may be denoted by Δu, where u is the value of Column 7. Column 8 is the number $\Delta\xi - \Delta u$ of those who die of other diseases during the year.

Bernoulli's Table II was formed as follows:

Columns 1 and 2 as in his Table I.

Column 3. Number z of survivors in the population when freed from small-pox mortality.

This number z was obtained by Bernoulli by subtracting $\frac{z}{\xi}(\Delta\xi - \Delta u)$ from the value of z of the previous year, but is not quite correct for the required purpose, and makes the number of survivors too large.

For example, according to Column 2 the life-table populations at ages 0 and 1 respectively are 1300 and 1000, 17·1 dying of small-pox in the first year of life, and 283 dying of other diseases. Bernoulli deduced that the value of z, Column 3, would be 1017·1 at age 1, but, since the number 17·1 who were saved from death from small-pox would be exposed to the risk of dying of other diseases, they would be reduced by $17{\cdot}1 \times \frac{283}{1300}$, that is, by 3·7. The number of survivors at age 1 in Column 3 should therefore be 1013·4. This makes a difference in the number of survivors in the subsequent years. Column 4 shows the difference between Columns 3 and 2 at each age.

In taking a year as the unit of time instead of a small quantity dx, Bernoulli calculated these tables only approximately, but he worked out a formula which would give the results of Column 3 accurately, as follows:

Deduction of the Formula $z = \dfrac{me^{x/n}}{1 + (m-1)\, e^{x/n}}\, \xi$ *for the Survivors in the hypothetical Life-Table.* Using the symbols x, ξ, s, m, n, z with meanings already explained, the entire mortality during time dx is $-\,d\xi$. The mortality from small-pox is $s\,\dfrac{dx}{mn}$, so the whole mortality for the population free from small-pox deaths is $-\,d\xi - \dfrac{s\,dx}{mn}$. But this corresponds to the number ξ, so for population z it must be multiplied by $\dfrac{z}{\xi}$. Hence

$$-\frac{z}{\xi}\left(d\xi + \frac{s\,dx}{mn}\right) = -\,dz.$$

This is the equation in terms of differentials, while the rougher method gave for yearly increments

$$\frac{z}{\xi}(\Delta\xi - \Delta u) = \Delta z.$$

Taking the equation

$$-\frac{z}{\xi}\left(d\xi + \frac{s\,dx}{mn}\right) = -\,dz,$$

we have

$$\frac{dz}{z} - \frac{d\xi}{\xi} = \frac{s\,dx}{mn},$$

and substituting the value of s obtained before, this becomes

$$\frac{dz}{z} - \frac{d\xi}{\xi} = \frac{m\xi}{1 + (m-1)\, e^{x/n}} \frac{dx}{mn\xi};$$

thus

$$\frac{dz}{z} - \frac{d\xi}{\xi} = \frac{1}{n} \frac{dx}{1 + (m-1)\, e^{x/n}}$$

and

$$\frac{dz}{z} - \frac{d\xi}{\xi} = \frac{1}{n}\left[1 - \frac{(m-1)\, e^{x/n}}{1 + (m-1)\, e^{x/n}}\right] dx.$$

Integrating, this becomes

$$\log \frac{z}{\xi} = \frac{1}{n} x - \log\{1 + (m-1)\, e^{x/n}\} + c.$$

When $x = 0$, $z = \xi$, therefore $c = \log m$; accordingly:

$$\log \left\{\frac{z}{\xi} \times \frac{1 + (m-1)\, e^{x/n}}{m}\right\} = \frac{x}{n};$$

$$\frac{z}{\xi} \times \frac{1 + (m-1)\, e^{x/n}}{m} = e^{x/n};$$

and

$$z = \frac{me^{x/n}}{1 + (m-1)\, e^{x/n}}\, \xi.$$

The following life-tables are shown for comparison with one another. The second column gives Halley's life-table, up to the age of 21 years, the third is Bernoulli's table of survivors for a population from which small-pox fatality is eliminated, the fourth is the same table with Bernoulli's error corrected, while the last life-table in the fifth column gives the same population calculated from Bernoulli's formula for z.

Table VII.

x Age	ξ Halley's Table	Bernoulli's results for z, "non-variolique" population	Last column corrected (see p. 295)	z' calculated from Bernoulli's formula above[1]	Ratio $\frac{z'}{\xi}$
0	1300	1300	1300	1300	—
1	1000	1017·1	1013·4	1014·9	1·015
2	855	881·8	878·6	879·3	1·03
3	798	833·3	830·3	830·5	1·04
4	760	802·0	799·1	799·3	1·05
5	732	779·8	777·0	777·1	1·06
6	710	762·8	760·0	760·1	1·07
7	692	749·1	746·3	746·4	1·08
8	680	740·9	738·2	738·3	1·09
9	670	734·4	731·7	731·8	1·09
10	661	728·4	725·7	725·7	1·10
11	653	722·9	720·2	720·3	1·10
12	646	718·2	715·5	715·5	1·11
13	640	714·1	711·4	711·4	1·11
14	634	709·7	707·1	707·0	1·12
15	628	705·0	702·4	702·3	1·12
16	622	700·1	697·5	697·4	1·12
17	616	695·0	692·3	692·2	1·12
18	610	689·6	686·9	686·8	1·13
19	604	684·0	681·3	681·2	1·13
20	598	678·2	675·5	675·5	1·13
21	592	672·3	669·6	669·6	1·13

[1] Using his values of m and n.

The Mean Duration of Life calculated. The mean durations of life were calculated by Bernoulli for the ordinary life-table, and for that for the population free from small-pox. He took the sum of the survivors at each age, until all the 1300 who began life together had died, except that, for the first year of life, he took half the 1300 born as living at six months. Dividing the result by 1300, the mean life for Halley's life-table was 26 years 7 months. Bernoulli noticed that the ratio of z to ξ seemed to approach the value 8 to 7 (1·14), as x reached about 20 years and upwards.

Applying the same method to the "z" table, and for the years above 25 taking for the number of survivors $\frac{8}{7}$ of those of Halley's table at the corresponding ages, the mean life was 29 years 9 months. The small error in Bernoulli's work did not make any appreciable difference to this quantity.

If the death-rate from inoculation were taken into account and assumed to be 1 in 200, then the numbers of survivors in the z-population for the various groups would be slightly reduced, and would be 1012, 877, 829, etc.

Bernoulli calculated that the difference in the mean duration of life due to this mortality would be about 2 months, making the average duration 29 years 7 months.

The expectation of life could be found in the same way for the two tables, for each year of age, and by this means Bernoulli attempted to illustrate the advantages of eliminating small-pox as a cause of death.

D'Alembert's Criticism of Bernoulli. D'Alembert, in his *Opuscules*, T. II, 1761*, criticised unfavourably the work of Bernoulli on the ground that he made two inadmissible hypotheses in regarding "m" and "n" as constant. D'Alembert worked out a formula for "z" which is the same as Bernoulli's as far as the differential equation

$$\frac{dz}{z} - \frac{d\xi}{\xi} = \frac{s}{mn}\frac{dx}{\xi}.$$

But he took du to represent the number who die of the given disease in time dx, and so avoided making any assumptions as regards the proportion of those who take the disease at any age, and of those who die of it.

D'Alembert's equation was expressed as

$$\frac{dz}{z} - \frac{dy}{y} = \frac{du}{y},$$

where y refers to the number of survivors in the ordinary life-table.

The integral of this is

$$z = Cye^{\int_0^x \frac{du}{y}}.$$

At time $x = 0$, $z = y$ and $\left(\int_0^x \frac{du}{y}\right)_{x=0} = 0$;

thus $$C = 1,$$

and $$z = ye^{\int_0^x \frac{du}{y}}.$$

Geometrical Method of obtaining the Formula $z = ye^{\int \frac{du}{y}}$. D'Alembert obtained this result by the following geometrical method:

Let the vertical AK represent y the number of persons born at the same time, and let equal intervals of time AB, BC, CD, ... be marked along the horizontal AO. Then the verticals BB', CC', DD', ... give the number of survivors at the end of each unit of time. Take RE, giving the y-population at the end of a period, AR, of n units. Then after time dx, represented by Rr, this is reduced to re. Draw $e\epsilon$ parallel to rR, to meet RE in point ϵ. Then $E\epsilon$ represents the number of deaths dy in time dx in the given population y.

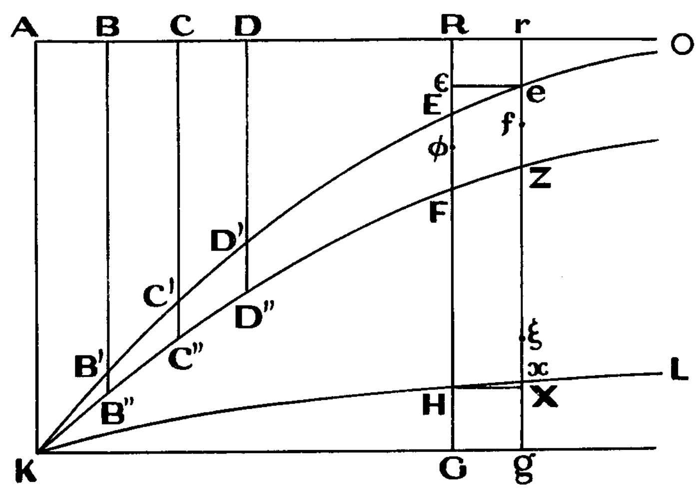
The object is to find how the population y would be affected if there were no deaths from a particular disease. Let the resulting population be z, and let the verticals BB'', CC'', ... RF, rZ

* See p. 279.

represent the z-population at the end of periods AB, AC, ... AR, Ar respectively. The number of deaths from all diseases in the z-population in time dx would be $E\epsilon \times \frac{RF}{RE}$. Let this be $F\phi$, marked off along FR.

Take next the curve $KHxL$ to represent the law of mortality u, from the particular disease which it is required to eliminate. At the end of periods AR, Ar the total deaths from that disease amount to HG, xg respectively. So Xx gives the number of deaths, du, in time dx, HX being drawn horizontally to meet rg in X. These are deaths in the population RE; hence, if the z-population RF were subject to death from the particular disease, then the number of deaths in time dx would be

$$X\xi = xX \times \frac{RF}{RE}.$$



Make $Zf = X\xi$; then the deaths from all diseases in the z-population, which is not subject to death from the particular disease, must be diminished by $X\xi$. Hence

$$RF - rZ = F\phi - X\xi,$$

or

$$dz = dy \times \frac{z}{y} \quad \left(- du \times \frac{z}{y}\right),$$

where z and y are decreasing while u is increasing;

$$\therefore \frac{dz}{z} = \frac{dy}{y} + \frac{du}{y}.$$

Integrating, this becomes $$z = Cy e^{\int \frac{du}{y}}.$$

When time $x = 0$, $z = y$ and $\int \frac{du}{y} = 0$, hence $C = 1$, and the formula becomes $z = y e^{\int \frac{du}{y}}$.

This is more direct than Bernoulli's formula, since the only information required is the number of deaths at each age for the particular disease, and not the number of those attacked. It can be adapted for use at the present day for any disease for which the records of deaths are given by the Registrar-General for various age groups.

A comparison of the results of the "z" life-table for different diseases would show the relative effect of those diseases on the number of survivors at each age in the population.

Tremblay. Another writer who was interested in Bernoulli's calculations was M. Jean Tremblay, who wrote in 1796 a paper entitled "Recherches sur la mortalité de la petite vérole," which was published in the *Mémoires de l'Académie Royale*, in Berlin*.

He referred in this to Bernoulli's work, and to D'Alembert's, and he considered that Bernoulli's formula was the more valuable, as being less complicated, and easier to apply. He pointed out the importance of having reliable data, if a law was to be deduced from observations.

He did not contribute anything to the mathematical problem of obtaining a life-table for the population when freed from the mortality of a special disease, but he suggested a method of obtaining Bernoulli's formula for the number in the population at a given time who had not had small-pox, by taking the intervals of time to be very small, but infinite in number, without using infinitesimal calculus. This he thought was an advantage.

He also tested Bernoulli's hypotheses that the "m" and the "n" in his formula might be taken to be constants, that is, that the proportion at each age who took small-pox among those who had not had it was constant, and that the proportion of those who died, among those who were attacked, was also constant at each age.

For this purpose he made use of the data of two mortality lists, one from The Hague, mentioned by M. Lambert†, giving details of the deaths from small-pox during 15 years, with age at death; the other table, which was apparently more reliable, was published in a German work by M. Möhsen, and contained the number of deaths from small-pox in Berlin during 12 years lying between 1758 and 1774, when the age at death was given.

In the Berlin table the figures were given year by year only for the first five years of life, and then in five-yearly periods, so that interpolation would be necessary as yearly deaths were required.

Having obtained Bernoulli's formula, without calculus, for the number of persons in the population who had not already had small-pox Tremblay endeavoured, by giving arbitrary values to "m," to find out the value of "n" by means of his formula, and with the help of the data. As a result he concluded that while "n" might be taken to be fairly constant, there was more variation in "m."

His tentative method seems unsatisfactory, but more data with accurate information would be required before a definite conclusion as to the variations in "m" and "n" for different age groups could be reached.

The following table shows the arbitrary values obtained by Tremblay:

Values of m and n obtained								
From Berlin data						From The Hague data		
Age	n	m	Age	n	m	Age	n	m
1	7	6	13	5·5	100	1	5·5	16·5
2	6	6	14	5·65	90	2	8	8·1
3	5·7	6	15	5·7	80	3	8	6·4
4	5·5	5·5	16	5·7	74	4	7·5	4·6
5	5·5	8	17	5·7	67	5	7	5·5
6	5·5	11	18	5·7	63	6	7	6·2
7	5·5	14	19	5·8	60	7	7	5·5
8	5·5	17	20	5·8	74	8	7	7
9	5·5	30	21	5·8	74	9	7	8
10	5·5	72	22	5·8	74	10	7	16
11	5·5	120	23	5·8	74	11	7	29
12	5·5	112				12	7	19

* "Recherches sur la mortalité de la petite vérole." *Mém. de l'Acad. Roy. des Sciences.* Berlin, 1796.

† *Beiträge zum Gebrauche der Mathematik*, B. III, S. 574, etc. Berlin, 1772.

Tremblay was concerned only with the problem of Bernoulli's first table, that is, of finding the number of persons at each age who had not had small-pox, so that he did not contribute anything to the question, which is of special interest in this paper, of finding the number of survivors at each age in a population which would be free from the mortality due to a particular disease.

He suggested that it would be of great value to obtain data for deaths from small-pox at each age for a period of fifty years before the introduction of inoculation, and for a like period after it had become general. A comparison of the results would show the effect of inoculation on the number who took small-pox at each age, and on the number who died of it, and to what extent inoculation might have contributed to perpetuating epidemics, and whether the number of persons who died without having had small-pox had increased since inoculation was introduced.

Duvillard. Duvillard, writing in his *Analyse et tableaux de l'influence de la petite vérole sur la mortalité à chaque âge*, Paris, 1806, suggested a great number of questions connected with mortality. He wanted to find out what would be the number of people living at each age out of a number born together in a "natural" state, that is, in a condition free from the ravages of disease, and what would be the mean life in that condition compared with what it was under the existing environment.

He proposed to discuss especially small-pox, the mortality from it at each age, the number at each age who take it, and the number who have not already had it.

He referred to the work of Bernoulli, D'Alembert, Tremblay and Lambert. He considered that, until the records of deaths were more reliable and more numerous, it would be necessary to make some hypotheses, such as Bernoulli had done, with regard to the proportions of those who at each age were attacked by small-pox and of those who died from it.

He tried, like Bernoulli, to form a life-table for the population as it would be if small-pox were eliminated as a cause of death. He assumed that it was characteristic of this disease that it did not attack a person a second time, so that the population at any moment might be divided into two classes, viz. those who have not had small-pox, but will eventually get it and die from it, and those who have already had it or will never have it, and so must necessarily die from other causes.

Duvillard worked algebraically and obtained the same result as D'Alembert, using the following method:

Let y be the number of survivors in the ordinary life-table at age x. Let v be the number of persons at age x who will die in the future of small-pox. Then the values of v would give the law of mortality of those who began life together who will die of small-pox, and the values of $y - v$ would give the law of mortality of those who die of other diseases.

Duvillard regarded the "v" population and the "$y - v$" population as existing under different conditions, the former as subject only to mortality from small-pox, and the latter from every other disease except small-pox. The mortality, then, of the individuals of the class "$y - v$" would be $\frac{\partial y - \partial v}{y - v}$, but if the mortality of small-pox were abolished entirely from the population, then the whole of the y-population would be exposed to the risk of death from other causes, and the rate of mortality would be $\frac{\partial y - \partial v}{y}$.

Let z be the required number of survivors in the life-table population at age x under these conditions. Then the difference equation

$$\frac{\partial z}{z} = \frac{\partial y - \partial v}{y}$$

has in the limit for integral $\log z = \log C + \log y + \int -\frac{dv}{y}$,

the limits for the integral being 0 and x.

When $x = 0$, $z = y$, and $\int -\frac{dv}{y}$ is zero, thus $\log C = 0$ and $C = 1$. Therefore

$$z = y e^{\int_0^x -\frac{dv}{y}},$$

where v is diminishing with x. This is the same as the result obtained by D'Alembert, that is,

$$z = y e^{\int_0^x \frac{du}{y}},$$

where u is increasing with x. For the actual calculation of z by means of this formula, Duvillard observed that

$$\int -\frac{dv}{y} \quad \text{or} \quad \int \left(-\frac{1}{y}\frac{dv}{dx}\right) dx$$

is the quadrature of the curve $-\frac{1}{y}\frac{dv}{dx}$ (or v say).

The series of values of y and v would be given in the Mortality Register and the differential $\frac{dv}{dx}$ might be obtained by means of finite differences at unit intervals.

The formula suggested by Duvillard for the evaluation of the quadrature $\int v dx$ in terms of differential coefficients of v was

$$\int v dx = C + S v - \frac{1}{2} v - \frac{1}{12}\frac{dv}{dx} + \frac{1}{720}\frac{d^3v}{dx^3} - \dots *,$$

the signs of $\frac{dv}{dx}$ and $\frac{d^3v}{dx^3}$ being negative because v diminishes with x.

The evaluation of an exact expression for z would always be complicated, but Duvillard thought it was desirable to have an approximate method easy for working. For this purpose he suggested using only first finite differences of y and of v.

A different quadrature formula which could be used without too much labour is employed for similar calculations in the second part of the present paper†.

Duvillard gave some approximations from which he calculated the tables at the end of his memoir.

It is necessary to warn the reader that the theory of the life-table with a given disease eliminated as developed by Bernoulli, D'Alembert, Tremblay and Duvillard supposes that the mortality from the given disease is non-selective, i.e. that the population after removal of disease A is as susceptible to diseases B, C, D, etc. as it was before the elimination of that disease. This may possibly be true of certain diseases, but if a disease like phthisis or small-pox be more fatal to the weaker than to the stronger members of the community, it may leave a population less subject to mortality from other diseases; and if phthisis or small-pox were eliminated the surviving population might be more subject to death from other diseases. The theory involves the random character as far as mortality is concerned of the disease eliminated. This point must be borne in mind in judging the results of the following Part. We assume the selective action of the eliminated diseases not to be of a very marked character.

* Boole's *Finite Differences*, p. 90 (7), 2nd ed. London, 1872.

† Page 304.

REFERENCES TO WORKS CONSULTED IN PART I.

D'ALEMBERT, J. DE R. "Sur l'application du calcul des probabilités à l'inoculation de la petite vérole." *Opuscules*, T. II. Paris, 1761. "Sur un mémoire de M. Bernoulli concernant inoculation," T. IV. Paris, 1768. "Sur les calculs relatifs à l'inoculation," T. IV. Paris, 1768.

BERNOULLI, D. "De la Mortalité causée par la petite vérole et des avantages de l'inoculation pour la prévenir." *Mém. de l'Acad. Roy. des Sciences*. Paris, 1760.

BOOLE, G. *Finite Differences*, 2nd ed. London, 1872.

BOYLSTON, Z. *An Historical Account of the Small-Pox Inoculated in New England upon all Sorts of Persons, Whites, Blacks, and of all Ages and Constitutions*. London, 1726.

BRACKENBRIDGE, W. *Phil. Trans.* Vol. XLVIII, 1754, No. 95, p. 788.

Census Reports. 1881–1921.

DE LA CONDAMINE, C. M. "Mémoire sur l'inoculation de la petite vérole. Second mémoire sur l'inoculation de la petite vérole." *Hist. de l'Acad. Roy. des Sciences*. Paris, 1754 and 1758.

CREIGHTON, C. *History of Epidemics in Britain*, Vol. II. Cambridge, 1894.

DOUGLASS, W. *A Practical Essay concerning the Small-Pox*. Boston, 1730.

DUVILLARD, E. É. *Analyse et tableaux de l'influence de la petite vérole sur la mortalité à chaque âge*. Paris, 1806.

FREWEN, T. *Phil. Trans.* Vol. XXXVII, 1731, No. 419, p. 108.

FREWEN, T. *The Practice and Theory of Inoculation with an Account of its success in a letter to a Friend*. London, 1749.

GALE, B. *Phil. Trans.* Vol. LV, 1765, p. 193.

GEORGE, DOROTHY. *London Life in the Eighteenth Century*. London, 1925.

HORNEMAN, S. *Phil. Trans.* Vol. XXXVIII, 1734, No. 432, p. 296.

HUXHAM. *Phil. Trans.* Vol. XXXIII, 1725, No. 390, p. 379.

JURIN, J. *Phil. Trans.* Vol. XXXII, 1722, No. 373, p. 191; Vol. XXXII, 1722, No. 374, p. 213.

KENNEDY, P. *An Essay on External Remedies*. London, 1715.

KILPATRICK (KIRKPATRICK), J. *An Essay on Inoculation*. London, 1743.

KIRKPATRICK, J. *The Analysis of Inoculation*. London, 1754.

LAMBERT, J. *Beiträge zum Gebrauche der Mathematik*, B. III, S. 574 et seq. Berlin, 1772.

MAITLAND, C. *Account of Inoculating the Small-Pox*. London, 1722.

MEAD, R. *De Variolis et Morbillis Liber*. London, 1747.

MÖHSEN, J. K. W. *Sammlung merkwürdiger Erfahrungen die den Werth und grossen Nutzen der Pockeninoculation näher bestimmen*. Berlin, 1775.

MONTAGUE, M. W. *Letters of Lady Mary Wortley*. Letter XXXI. 3rd ed. London, 1763.

NETTLETON, T. *Phil. Trans.* Vol. XXXII, 1722, No. 370, pp. 35, 49; Vol. XXXII, 1722, No. 374, p. 209.

NEWMAN, H. *Phil. Trans.* Vol. XXXII, 1722, No. 370, p. 33.

PYLARINI, J. *Phil. Trans.* Vol. XXIX, 1716, No. 347, p. 393.

REGISTRAR-GENERAL. *Statistical Reviews*, 1921–1927.

ROBIE, T. *Phil. Trans.* Vol. XXXIII, 1724, No. 382, p. 67.

RUSSELL, P. *Phil. Trans.* Vol. LVIII, 1768, p. 140.

SCHEUCHZER, J. G. *Account of the Success of inoculating Small-Pox in the years* 1727 *and* 1728.

SLOANE, HANS. *Phil. Trans.* Vol. XLIX, 1755, No. 71, p. 516.

TIMONI, E. *Phil. Trans.* Vol. XXIX, 1714, No. 339, p. 72.

TREMBLAY, J. "Recherches sur la mortalité de la petite vérole." *Mém. de l'Acad. Roy. des Sciences*. Berlin, 1796.

WAGSTAFFE, W. *A Letter to Dr Freind showing the Danger and Uncertainty of Inoculating the Small-Pox*. London, 1722.

WILLIAMS, P. *Phil. Trans.* Vol. XXXII, 1723, No. 375, p. 262 and p. 264.

WRIGHT, R. *Phil. Trans.* Vol. XXXII, 1723, No. 375, p. 267.