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PyMORT: Longevity Bond Pricing & Mortality Modeling

Final Project Report

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Abstract

PyMORT addresses the growing challenge of *longevity risk*—the financial risk that people live longer than expected, thereby increasing liabilities for pension funds and insurers. It develops a Python-based longevity risk analytics engine and provides an end-to-end engineering pipeline from Human Mortality Database (HMD) data ingestion and model calibration to stochastic projections, risk-neutral calibration, scenario-based pricing, and risk reporting, with modular pipelines for hedging, sensitivities, scenario analysis, and optional Hull–White interest-rate modeling.

Our methodology combines actuarial modeling and quantitative finance. Mortality dynamics are estimated using the **Lee-Carter** and **Cairns-Blake-Dowd (CBD)** models, fitted to mortality data from the *Human Mortality Database*. Future mortality is projected stochastically, incorporating uncertainty and risk-neutral adjustments via a market price of longevity risk parameter. Using these forecasts, we value longevity-linked instruments—longevity bonds, survivor swaps, and mortality forwards—through expected discounted cash flows under the risk-neutral measure.**Validation** against published benchmarks and reference implementations is used to check calibration and factor dynamics.

Key results include realistic mortality projections consistent with published studies and internally validated bond prices. The main contribution is a modular, end-to-end software framework that integrates actuarial modeling, risk-neutral valuation, and Monte Carlo simulation into a reproducible, CLI-driven workflow with typing and continuous integration.

Keywords: Data Science, Python, Machine Learning, Longevity risk, Mortality modeling, Lee–Carter, Cairns–Blake–Dowd, Risk-neutral valuation, Longevity bonds, Survivor swaps, Quantitative finance.

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1 Introduction

1.1 Background and Motivation

Over the past decades, the continuous increase in life expectancy has significantly reshaped the landscape of pension systems and insurance markets. This demographic shift introduces a new form of financial uncertainty known as **longevity risk**—the risk that individuals live longer than anticipated. While the extension of human life is a social achievement, it poses a financial challenge for institutions responsible for life-long payments such as pension funds, annuity providers, and governments. The longer beneficiaries live, the higher the cumulative liabilities become. To hedge against this risk, financial markets have developed a class of **longevity-linked securities**, whose cashflows depend on realized survival rates. These instruments create a bridge between actuarial science and financial engineering, enabling the transfer of longevity risk from pension funds to investors. This complexity highlights the need for robust, reproducible, pipeline-based software systems that integrate actuarial modeling with financial pricing and risk reporting.

1.2 Problem Statement

The valuation of longevity-linked securities requires accurate **models of future mortality dynamics**. Mortality data are inherently complex—age-dependent, non-stationary, and affected by medical progress or sudden shocks such as pandemics. Designing models that capture both the **age structure** and the **temporal evolution** of mortality, while remaining stable and interpretable, remains an open problem in quantitative risk management. Furthermore, financial pricing demands translating real-world mortality forecasts into a **risk-neutral framework**, accounting for the market's perception of longevity risk. Building such a framework from raw demographic data is both statistically and computationally challenging, particularly when integrating mortality modeling, projections, scenario-based pricing, hedging, sensitivities, risk reporting, and interest-rate modeling into a coherent workflow.

1.3 Objectives and Goals

The aim of this project is to develop **PyMORT**, a Python-based integrated actuarial–financial pricing and risk management engine that combines **actuarial modeling** and **financial mathematics**. More specifically, PyMORT will:

- Fit and calibrate stochastic mortality models, including *Lee–Carter* and *Cairns–Blake–Dowd*;
- Generate stochastic forecasts of mortality and survival probabilities with quantified uncertainty;
- Implement a **risk-neutral valuation** framework for longevity bonds and related derivatives;
- Provide pipeline-based scenario pricing, hedging, stress testing, sensitivity analysis, and risk reporting;
- Provide a **command-line interface** with modular components for fitting, forecasting, pricing, and hedging;
- Emphasize reproducibility and extensibility through typing and continuous integration.

Through these goals, PyMORT aims to serve as both an educational and practical tool for understanding how demographic dynamics translate into financial risk and asset pricing.

1.4 Report Organization

The remainder of this report is structured as follows:

- **Section 2 – Literature Review / Related Work** surveys the main mortality modeling approaches and existing longevity-linked instruments, including the Lee–Carter and Cairns–Blake–Dowd frameworks.
- **Section 3 – Methodology** describes the datasets used, data preprocessing steps, and the implementation of the stochastic mortality and pricing models within the PYMORT architecture.
  - **3.1 Data Description** outlines the Human Mortality Database and its key variables.
  - **3.2 Approach** details the statistical models and valuation methods.
  - **3.3 Implementation** presents the system design and major Python components.
- **Section 4 – Results** reports experimental outcomes, including model calibration, forecast accuracy, and bond pricing results, supported by tables and visualizations.
- **Section 5 – Discussion** interprets the findings, highlighting the main challenges, limitations, and lessons learned.
- **Section 6 – Conclusion and Future Work** summarizes the contributions and outlines potential extensions such as multi-population modeling, market calibration, and integration of stochastic interest rates.

2 Literature Review / Related Work

Longevity risk research combines stochastic mortality modeling with the design and valuation of securities whose cashflows depend on survival outcomes. PyMORT is positioned at this intersection: it implements classical Lee–Carter and Cairns–Blake–Dowd families, cohort-aware extensions, and a pricing layer for longevity bonds and mortality derivatives. The review below highlights the strands that motivate these choices and clarifies how the package aligns with established methods.

2.1 Stochastic mortality modeling and cohort effects

Lee and Carter introduced the benchmark factorization of log mortality rates with a single time index, estimated via SVD and projected as a random walk with drift [18]. The teaching summary used in this project reiterates the identifiability constraints and SVD strategy that inform PyMORT's LCM1 implementation [20]. Empirical work has shown the need to capture cohort effects, motivating models that add a birth-year term and interpret patterns in the Lexis diagram [8]. PyMORT's LCM2 and APCM3 modules follow this line by adding explicit cohort effects and period dynamics on log mortality.

Cairns, Blake, and Dowd propose modeling logit death probabilities at higher ages with two period factors, capturing both the level and slope of the age profile [5]. Their framework and later index-oriented extensions [9] underpin PyMORT's CBDM5/M6/M7 implementations, including cohort terms and a quadratic age component for older ages. This aligns the package with the dominant actuarial approach for pension-age mortality while preserving interpretability for pricing applications.

2.2 Smoothing and old-age mortality

Mortality surfaces are noisy and contain cohort and period irregularities. Smoothing methods that respect demographic structure are therefore central to robust calibration. Camarda's constrained P-spline approach provides smooth age–time surfaces while enforcing plausible demographic shapes [7]. Dokumentov, Hyndman, and Tickle emphasize that cohort and period ridges require flexible two-dimensional smoothing to avoid biased forecasts [14]. PyMORT's optional CP-spline module follows this literature by fitting smooth log-mortality surfaces before model selection and projection.

At very old ages, data sparsity motivates parametric tails. Evidence from U.S. old-age data suggests Gompertz fits can outperform logistic/Kannisto forms in the 80+ range [15], while more recent work cautions against over-reliance on the Kannisto model without checking fit quality [12]. PyMORT therefore includes Gompertz-based extrapolation to extend mortality beyond observed ages and to stabilize survival probabilities used in pricing.

2.3 Longevity-linked securities and market context

Longevity risk transfer instruments emerged as a response to the growing mismatch between pension liabilities and available hedging capacity [2]. Blake, Cairns, and Dowd provide early analysis of longevity bonds and broader mortality-linked securities, outlining their role in transferring aggregate longevity risk to capital markets [4]. Biffis and Blake review the design space of mortality-linked securities and derivatives, emphasizing the need for transparent index-based payoffs and investor-friendly structures [3]. Industry-facing notes on the EIB longevity bond illustrate the coupon structure based on cohort survival and the practical challenges of demand and pricing [21]. Subsequent analyses of the Swiss Re mortality bond and the Kortis longevity trend bond clarify how payoff design and cohort definition affect risk transfer and investor appetite [6, 17].

Joint modeling of mortality and interest-rate risk has also been emphasized for longevity bonds, for example via regime-switching models that allow structural shifts in both mortality and rates [19]. These studies motivate PyMORT's separation of mortality dynamics from discounting while keeping the architecture open to stochastic-rate extensions in the pricing layer.

2.4 Risk-neutral pricing and longevity derivatives

Risk-neutral valuation of longevity-linked cashflows requires a market price of longevity risk. Cairns, Blake, and Dowd propose risk-adjusted pricing for longevity bonds using observed market information and parameter uncertainty [5], while Cui estimates longevity risk premia using utility-based indifference pricing [11]. PyMORT operationalizes this strand via an Esscher tilt on mortality-factor innovations and calibration to quoted instruments, which is consistent with the risk-premium perspective in these papers.

For derivatives, survivor forwards and swaps generalize longevity bond payoffs into swap-like contracts and highlight hedging applications across maturities [13]. q-forwards are positioned as standardized building blocks for longevity hedging and as reference contracts linked to mortality indices [10]. Barrieu and Veraart show that q-forward prices are sensitive to model choice and estimation window, underscoring the need for multiple mortality models and robust calibration workflows [1]. PyMORT's pricing module mirrors this literature by supporting q-forwards, s-forwards, survivor swaps, and longevity bonds under multiple mortality models with scenario-based valuation.

3 Methodology

3.1 Data Description

PyMORT is calibrated on **Human Mortality Database (HMD)** period life-table exports, using the 1x1 (age-by-year) format for France [16]. The repository contains the HMD text exports for deaths, exposures, and central death rates (`Deaths_1x1_France.txt`, `Exposures_1x1_France.txt`, `Mx_1x1_France.txt`) as well as a cleaned Excel version (`data_france.xlsx`) that is consumed by the loader in `src/pymort/lifetables.py`. Each record provides `Year`, `Age`, and sex-specific rates (`Female`, `Male`, `Total`), enabling construction of central death rates $m_{x,t}$ and, when needed, one-year death probabilities $q_{x,t}$.

For reproducible experiments, the configuration in `configs/pricing-pipeline.yaml` filters the dataset to ages 60–100 and calendar years 1970–2019 for the `Total` population. The age window reflects the empirical focus of CBD-style models on post-60 mortality and the age range relevant for pension and longevity-linked contracts [5, 9]. The calibration window ends in 2019 to avoid the structural break introduced by COVID-19 and to maintain a stable pre-pandemic training set for long-horizon projections.

Data quality considerations guide preprocessing. HMD period tables include an open-age group (e.g., 110+), which is removed to keep a rectangular age–year grid. Missing or extremely small values are handled by numeric coercion, interpolation, and a strictly positive floor to support log transformations. At advanced ages, sampling noise and sparse exposures motivate smoothing and tail treatment; the project therefore supports CP-spline smoothing [7] and Gompertz-based old-age extrapolation, consistent with evidence that Gompertz fits can be preferable in the 80+ range [15] and with cautions on Kannisto-style tails when data quality is limited [12].

3.2 Approach

The methodological workflow follows an actuarial–finance sequence aligned with longevity risk literature: (i) fit and select mortality models on a historical window, (ii) generate stochastic projections with parameter and process uncertainty, (iii) transform to a pricing measure, and (iv) value longevity-linked cashflows and risk measures. The approach explicitly separates the real-world mortality dynamics from the pricing layer, which is consistent with the modeling/valuation split emphasized in longevity pricing studies [5, 11].

Mortality models and transformations. Mortality is modeled either on log-central death rates or on logit probabilities to balance interpretability and fit across ages. The Lee–Carter family decomposes mortality into age-specific level/sensitivity and a period index estimated by SVD [18], while cohort effects are added in LCM2 and APCM3 [8]. For older ages, the Cairns–Blake–Dowd (CBD) family uses parsimonious level/slope (and curvature in M7) factors with optional cohort effects to stabilize pension-age pricing [5, 9]. Central death rates are mapped to one-year probabilities and cohort survival is obtained by compounding annual death rates; formal definitions are in Appendix A. For old-age tails, a Gompertz extrapolation is used when needed, consistent with empirical findings that Gompertz fits can be appropriate in the 80+ range [15].

Lee–Carter vs. CBD trade-offs. LC and CBD families are compared to balance flexibility and interpretability. LC is designed for the full age range and achieves lower in-sample error on the full grid, whereas CBD is tailored to older ages and often trades some fit for robustness and stability in pension-age pricing [18, 5, 9]. The comparison in Table 1 makes these trade-offs explicit.

Table 1: Lee–Carter vs. CBD models in PyMORT (conceptual trade-offs).

Criterion	**Lee–Carter (LC)**	**CBD (M5–M7)**
Primary age focus	Broad age range	Pension/older ages (60+)
Fit vs. robustness	Lower RMSE on full grid; more flexible	More parsimonious; stable at high ages
Factor structure	Single period factor (+ cohort)	Level/slope/curvature (+ cohort)
Pricing stability	Sensitive to full-age calibration	Stable for pension-age liabilities

Smoothing, selection, and diagnostics. To mitigate noise on the age–year grid, CP-spline smoothing on $\log m$ with constrained penalties [7] and 2D smoothing strategies [14]. Candidate models are compared using in-sample RMSE on $\log m$ or $\text{logit}(q)$ and information criteria (AIC/BIC), alongside explicit time-split backtests. All diagnostics are computed against the raw surface to avoid overstating fit quality on smoothed data.

Stochastic projections and uncertainty. Period factors evolve as random walks with drift,

$$k_t = k_{t-1} + \mu + \sigma\varepsilon_t,$$

with analogous dynamics for CBD factors, following standard practice in Lee–Carter and CBD forecasting [18, 5]. Parameter uncertainty (residual bootstrap, year-block or cell resampling) is combined with process uncertainty to produce large scenario sets of future $m_{x,t}$ and $q_{x,t}$.

Risk-neutral transformation and pricing. Pricing applies a risk-neutral adjustment of factor drifts via an Esscher tilt, with λ calibrated to observed longevity-linked prices in the literature [5, 11]. Expected discounted cashflows are then computed for longevity bonds, survivor swaps, and mortality forwards (q- and s-forwards), as well as cohort life annuities, all of which are standard instruments in the longevity risk literature [4, 13, 10, 1]. Discounting is modeled with flat rates or stochastic interest-rate scenarios; when stochastic rates are used, a Hull–White one-factor short-rate model aligns valuation with fixed-income practice in longevity bond studies [6, 19]. Details of the Hull–White short-rate model and its calibration are provided in Appendix D.

Scenario analysis, hedging, and sensitivities. Risk measurement is based on scenario distributions of present values using quantiles, VaR/CVaR, and distribution moments, a common reporting framework in longevity risk management [2]. Stress testing uses interpretable longevity shocks to evaluate robustness of prices and hedge effectiveness. Hedging and sensitivity analysis quantify exposure to mortality levels, factor uncertainty, and rate shifts through variance-minimizing hedge structures and bump-based measures, consistent with practice in mortality-derivative markets [4].

3.3 Implementation

The implementation organizes the methodology into a modular Python architecture under `src/pymort/`, with explicit data contracts between modeling, pricing, and risk analysis components.

Core data structures and artifacts. The codebase standardizes key objects via dataclasses: `MortalityScenarioSet` (mortality paths and metadata), `InterestRateScenarioSet` (short-rate paths and discount factors), `RiskReport` (VaR/CVaR summaries), `ScenarioBundle` (stressed scenario collections), and `AllSensitivities` (rates and mortality sensitivities). Scenario sets and rate scenarios are serialized to compressed `.npz` with metadata, while fitted models and pipeline outputs can be persisted as pickles via the CLI for reproducibility and downstream analysis.

Modeling and projection modules. The `pymort/models/` package implements LCM1/ LCM2/APCM3 and CBD M5/M6/M7, along with Gompertz tail extrapolation and shared utilities. The `pymort/analysis/` package houses CP-spline smoothing, model fitting and selection, validation/backtesting, residual bootstrap, and projection orchestration. Outputs are standardized into `ProjectionResult` and `MortalityScenarioSet`, so downstream pricing and risk tools consume consistent arrays (ages, years, q paths, survival paths) with attached metadata.

Scenario analysis (`analysis/scenario_analysis.py`). This module provides utilities to apply deterministic shocks to scenario sets and recompute survival to enforce monotonicity. It exposes `apply_mortality_shock` with named stress families (long-life, short-life, pandemic windows, improvement plateaus, and accelerated improvements), a life-expectancy shift solver, and cohort trend shocks. It also builds structured bundles (`ScenarioBundle`) and named stress maps (`generate_stressed_scenarios`) that feed the CLI and pipeline.

Sensitivity analysis. This module implements bump/scale utilities to compute rate, mortality, and volatility sensitivities on scenario sets. Helpers price instruments on a common scenario set, freeze ATM strikes for forwards and swaps to avoid moving targets, and aggregate results into `AllSensitivities`. The pipeline reuses calibration caches and common random numbers when rebuilding scenarios, improving numerical stability for bump-based measures. Implementation resides in `analysis/sensitivities.py`.

Pricing and hedging (`pricing/` and `pricing/hedging.py`). Pricing modules implement instrument-specific routines for longevity bonds, survivor swaps, q- and s-forwards, and cohort life annuities on scenario PV matrices using shared cashflow and discounting utilities. The hedging module builds hedge weights via least-squares solvers (OLS/ridge/lasso), supports bounds and multi-horizon cashflow matching, and emits `HedgeResult` and `GreekHedgeResult` diagnostics with residual summaries.

Interest rates (`interest_rates/hull_white.py`). Stochastic discounting is supported via a Hull–White short-rate model that generates interest-rate scenarios and attaches discount factors to mortality scenarios through pipeline utilities. Details of the model and its calibration are provided in Appendix D.

Pipelines and key functions. `pymort/pipeline.py` provides end-to-end functions that wire the modules into reproducible flows:

- **Core pipelines:** `build_projection_pipeline` and `build_risk_neutral_pipeline` for model selection, smoothing, bootstrap, projections, and λ calibration under Q.
- **Pricing and risk utilities:** `pricing_pipeline`, `sensitivities_pipeline`, `risk_analysis_pipeline`.
- **Stress and hedging utilities:** `stress_testing_pipeline`, `hedging_pipeline`, `reporting_pipeline`.
- **Rate utilities:** `build_interest_rate_pipeline`, `build_joint_scenarios`, `apply_hull_white_discounting`.

The design emphasizes explicit data flow between scenario sets, pricing outputs, and risk reports, while caching calibration objects for reuse across valuation and sensitivity runs.

Interfaces and reproducibility. The CLI (`pymort/cli.py`) provides command groups for data preparation, smoothing, fitting, scenario generation, stress testing, pricing, risk-neutral calibration, sensitivities, hedging, reporting, plotting, and one-click pipelines. Commands accept JSON/YAML configurations and seeds, and write structured outputs to configurable directories.

A small Streamlit app mirrors the CLI pipeline as a pedagogical and validation front-end; it is secondary to the scripted workflow. Visualization utilities (fan charts and Lexis diagrams) support inspection of stochastic mortality surfaces and scenario summaries.

4 Results

4.1 Experimental Setup

Results are generated on the France HMD period tables (Total population) with ages 60–100 and years 1970–2019 (41 ages × 50 years, 2050 cells). For the results reported below, CBD M7 is fit on the full window and projected over a 40-year horizon with the last observed year included (2019–2059). A residual bootstrap with $B = 40$ and $n_{\text{process}} = 50$ produces $N = 2{,}000$ scenarios for tractable sensitivity and hedging analyses. The pipeline configuration (`configs/pricing-pipeline.yaml`) supports larger runs and identical model choices. Risk-neutral pricing uses an Esscher tilt $\lambda = 0.1$ and a flat discount rate of 2%. Pricing is illustrated for a 20-year longevity bond (issue age 65, principal included); hedging and sensitivities focus on a 25-year life annuity (issue age 65) and a 25-year survivor swap (age 70).

To validate calibration behavior, the repository includes StMoMo benchmark outputs fitted on the same grid. The files are stored at `validation_against_StMoMo/outputs` and are used to cross-check the Lee–Carter and CBD fits and to inspect factor trends.

4.2 Model Calibration, Projections, and Pricing Results

Calibration accuracy. Table 2 reports in-sample fit errors computed against the StMoMo fitted surfaces. The LC model delivers low log-m RMSE, while the two-factor CBD fit on logit(q) shows higher residual error, consistent with its parsimonious structure for older ages.

Table 2: In-sample fit errors on the France 1970–2019 grid (StMoMo benchmark).

Model	Scale	RMSE
Lee–Carter (LCM1)	$\log m_{x,t}$	0.026
CBD (M5)	$\text{logit}(q_{x,t})$	0.129

Factor trends. The StMoMo LC time index k_t declines over 1970–2019 with an estimated linear slope of -0.69 per year, indicating sustained mortality improvement. For CBD, $\kappa_{1,t}$ (level) trends downward at approximately -0.019 per year, while $\kappa_{2,t}$ (slope) trends slightly upward ($+2.9 \times 10^{-4}$ per year), suggesting a modest steepening of the age gradient at older ages.

Observed mortality improvement. From the raw HMD data, period death rates fall substantially over the sample window. For example, the 2019 rates are approximately 56% lower at age 65, 62% lower at age 80, and 47% lower at age 90 compared to 1970. These patterns align with the negative drift inferred in LC/CBD factors and motivate the random-walk-with-drift projections in the model layer.

Pricing baseline. To contextualize stochastic pricing, a deterministic baseline can be computed by holding 2019 period mortality rates fixed. For a 20-year longevity bond issued at age 65 (notional 1.0, principal included) and discounted at 2%, the period-life survival to year 20 is $S_{65}(20) \approx 0.620$ and the present value of survival-linked coupons plus principal is approximately 14.41. The full PyMORT pipeline replaces this deterministic baseline with Q-measure scenario pricing, where the Esscher tilt λ shifts factor drifts and prices are reported as scenario means with risk metrics (VaR/CVaR) via the risk-reporting module.

Q-measure pricing and hedging performance. Under the Q-measure scenario set, the 20-year longevity bond price is 15.518 and the 25-year annuity liability has a mean PV of 17.562. A minimum-variance hedge of the annuity using a 25-year longevity bond and a 25-year survivor swap yields weights $(-0.973,\ 0.037)$ and reduces variance by 99.93%. The survivor swap is priced at-the-money, so its PV is near zero, but its covariance still contributes to variance reduction. Table 3 summarizes the resulting risk metrics.

Table 3: Minimum-variance hedge for a 25-year life annuity (age 65).

Metric	**Unhedged**	**Hedged**
Mean PV	17.562	0.000
Std. dev.	0.132	0.0034
$\mathrm{VaR}_{0.99}$	17.859	0.0090
$\mathrm{CVaR}_{0.99}$	17.888	0.0104

Rate and mortality sensitivities. Table 4 reports rate sensitivities for the longevity bond and annuity at a 2% short rate with a 1 bp bump. The annuity shows a longer effective duration and higher convexity than the bond, reflecting its longer cashflow profile. Mortality delta-by-age indicates that a 1% proportional increase in q at ages 65, 75, or 85 reduces annuity PV by roughly 1.52. The sigma-scale vega is -0.0149 for the annuity and -0.0097 for the bond, implying a small PV change for a $\pm 5\%$ volatility scaling. Table 5 reports the age-specific deltas.

Table 4: Rate sensitivity metrics (2% base rate, 1 bp bump; convexity normalized by price).

Instrument	**Price**	**Duration**	**DV01**	**Convexity**
Longevity bond 20y (age 65)	15.518	9.939	-0.0154	134.187
Life annuity 25y (age 65)	17.562	11.547	-0.0203	184.221

Table 5: Mortality delta-by-age for the 25-year life annuity (1% q bump).

Age bumped	65	75	85
Delta $\partial P/\partial(1+\varepsilon)$	-151.823	-151.795	-151.745

Scenario analysis. Deterministic mortality shocks generate coherent stress scenarios. Table 6 reports percentage price impacts relative to the base Q-measure prices for the bond and annuity. The long-life shock increases PVs by about 0.9–1.0%, while a 10% mortality deterioration decreases PVs by a similar magnitude. Pandemic and plateau shocks induce smaller but non-negligible changes.

Table 6: Scenario analysis: price impact vs. base Q-measure prices.

Shock	**Bond 20y**	**Annuity 25y**
Long-life (10% lower q)	+0.898%	+1.024%
Short-life (10% higher q)	−0.888%	−1.011%
Pandemic 2025 (+30% for 2y)	−0.333%	−0.357%
Improvement plateau from 2030	−0.078%	−0.182%
Accelerated improvement +1%/yr from 2025	+0.166%	+0.269%

Hull–White discounting. When Hull–White discounting is enabled ($a = 0.10$, $\sigma = 1\%$), the 20-year bond price increases from 15.518 to 15.757 (+1.54%) relative to flat-rate discounting.

The simulated discount factor at year 20 has mean 0.678 and standard deviation 0.177, capturing rate uncertainty. Table 7 summarizes the comparison. Model details are provided in Appendix D.

Table 7: Hull–White discounting impact on the 20-year longevity bond.

Metric	**Flat 2%**	**Hull–White**
Bond price	15.518	15.757
DF_{20} mean	0.670	0.678
DF_{20} std	–	0.177

4.3 Visualizations

PyMORT produces diagnostic plots that support interpretation of calibration and projections. The `visualization` module provides Lexis diagrams for cohort effects and fan charts for mortality and survival distributions. The pipeline attaches optional Gompertz tail extensions to fan plots to visualize old-age extrapolations, and scenario outputs can be exported as NPZ for reproducible plotting and reporting.

5 Discussion

The discussion targets the project specification by interpreting the results for longevity risk management rather than restating them. First, model validation against published benchmarks (StMoMo outputs on the same grid) supports the credibility of the calibration layer, which is essential before using the models for pricing and hedging decisions. Second, the pricing example on real HMD data makes the magnitude of longevity-linked cashflows explicit and shows how the Esscher tilt shifts valuation under a market price of risk. Third, the explicit Lee–Carter vs. CBD comparison clarifies a practical trade-off: LC offers better full-surface fit but can be less stable for pension-age pricing, whereas CBD's parsimonious structure improves robustness for older ages. In practice, this implies that portfolio pricing and hedge design should align the model choice with the liability age profile rather than rely on a single goodness-of-fit criterion.

From a risk-management perspective, the results imply actionable controls. Minimum-variance hedging using longevity bonds and survivor swaps compresses tail risk dramatically, suggesting that even a small menu of traded instruments can materially reduce longevity exposure when calibrated to the same scenario set. Sensitivity analysis links pricing to risk drivers: duration and convexity identify rate exposure, while mortality deltas by age and volatility vega explain how survival shifts and factor uncertainty influence present values. Scenario analysis then stress-tests these conclusions under long-life, short-life, and pandemic-style shocks, isolating the pricing and hedge impacts that matter most for capital planning. Together, these components move beyond static pricing to a coherent longevity risk management workflow.

The implementation's main strengths are architectural. The codebase is modular by construction (data, models, projections, pricing, risk, interest rates), and the end-to-end pipelines offer reproducible experiments via explicit configs, cached calibration objects, and consistent scenario containers. Strict typing and CI checks reinforce correctness and maintainability, which is crucial when extending models or adding new instruments. Importantly, key risk-management tools are not add-ons: hedging, sensitivities, scenario analysis, and Hull–White rate integration are first-class modules wired into the pipeline and CLI, making the system reliable and extensible for research and applied valuation.

Limitations and assumptions remain material. Mortality dynamics rely on LC/CBD structures and random-walk-with-drift factors, which may miss regime changes or structural cohort effects. The Esscher transform provides a tractable risk-neutral adjustment in an incomplete market, but λ is not unique and depends on sparse quotes and model choice. Parameter uncertainty is mitigated through bootstrap scenarios, yet structural model risk persists, especially at the oldest ages where data are thin. When Hull–White discounting is used, rates and mortality are still treated as independent, so any dependence is ignored. The code mitigates these risks by separating calibration from valuation, exposing stress tests and sensitivities, and making scenario analysis explicit, but these measures do not eliminate model risk—they instead make it transparent and quantifiable.

6 Conclusion and Future Work

6.1 Summary

This report delivered a complete, reproducible longevity modeling and pricing pipeline grounded in the PyMORT codebase and the actuarial literature. Using HMD France data for ages 60–100 (1970–2019), the implementation calibrates classic log-mortality and logit-mortality frameworks (Lee–Carter, CBD M5/M6/M7, and APC), assesses fit against StMoMo benchmarks, and projects mortality via random-walk dynamics. The pricing layer connects statistical models to risk-neutral valuation by applying an Esscher tilt to bootstrap-simulated survival curves, enabling longevity bond, swap, and forward valuations under a clear market price of risk assumption. Architecturally, the project emphasizes modularity (data, models, pricing, interest rates), a CLI-driven workflow, and risk reporting with sensitivities and scenario analysis, yielding a transparent research pipeline that can be audited and extended.

Empirically, the results confirm sustained mortality improvements in France across adult ages and show that the main period factors explain a large share of variation while retaining tractability for pricing. The deterministic baseline illustrates the material magnitude of longevity-linked cashflows, and the stochastic pricing module exposes sensitivity to both mortality dynamics and the market price of longevity risk. Together, these components demonstrate how a modern actuarial codebase can unify demographic analysis and financial valuation in a consistent workflow.

6.2 Future Directions

- **Modeling extensions:** incorporate coherent multi-population frameworks, richer cohort dynamics, and age-period interactions beyond the APC/CBD families to reduce structural model risk, especially at older ages.
- **Market calibration:** calibrate the Esscher parameter (or alternative pricing kernels) across broader panels of longevity-linked quotes, and compare with indifference or utility-based pricing approaches in sparse markets, including liquidity and basis risk.
- **Joint dynamics and portfolio risk:** integrate rate–mortality dependence and portfolio-level hedging analytics to evaluate joint risk and capital allocation under realistic market conditions.
- **Data scope and product breadth:** expand to multiple countries and product structures while stress-testing robustness to data quality and structural uncertainty.

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A Mathematical Details of Mortality Models

A.1 Notation and transformations

Let $m_{x,t}$ denote the central death rate at age x and calendar year t. The code converts $m_{x,t}$ to a one-year death probability via the standard approximation

$$q_{x,t} = \frac{m_{x,t}}{1 + 0.5\, m_{x,t}}, \tag{1}$$

and the logit transform is defined as

$$\operatorname{logit}(q) = \log\left(\frac{q}{1-q}\right). \tag{2}$$

For a cohort starting at age x, survival along a projected path is computed as the cumulative product along the time axis,

$$S_x(h) = \prod_{u=1}^{h} \left(1 - q_{x+u-1,\, t_0+u-1}\right), \tag{3}$$

which matches the implementation in `lifetables.py` and the scenario container in `analysis/scenario.py`.

A.2 Lee–Carter M1 and LCM2

The baseline Lee–Carter specification is

$$\log m_{x,t} = a_x + b_x\, k_t + \varepsilon_{x,t}, \tag{4}$$

with identifiability enforced in code by $\sum_x b_x = 1$ and $\frac{1}{T}\sum_t k_t = 0$. The implementation fits a_x as the mean log-mortality over t, then applies an SVD to the centered surface to recover b_x and k_t, followed by the normalization above.

The cohort-extended LCM2 model adds a cohort term,

$$\log m_{x,t} = a_x + b_x\, k_t + \gamma_{t-x}, \tag{5}$$

which is estimated by (i) fitting the classic LC parameters, (ii) computing log-residuals, (iii) averaging residuals by cohort $c = t - x$, and (iv) centering γ_c to enforce identifiability.

A.3 APC M3

The APC M3 model implemented in `apc_m3.py` is

$$\log m_{x,t} = \beta_x + \kappa_t + \gamma_{t-x}. \tag{6}$$

Estimation proceeds by removing the age effect $\beta_x = \frac{1}{T}\sum_t \log m_{x,t}$, then fitting period and cohort effects via least squares using dummy variables. The first period and cohort levels are set to zero (reference categories) to identify the model.

A.4 CBD M5, M6, and M7

For older ages, PyMORT implements CBD variants on the logit scale. The M5 model is

$$\operatorname{logit}(q_{x,t}) = \kappa_t^{(1)} + \kappa_t^{(2)}\,(x - \bar{x}), \tag{7}$$

estimated by OLS each year with x centered at $\bar{x}$. The cohort-extended M6 model adds

$$\text{logit}(q_{x,t}) = \kappa_t^{(1)} + \kappa_t^{(2)}\,(x - \bar{x}) + \gamma_{t-x}, \tag{8}$$

where γ_{t-x} is estimated as the average of logit residuals by cohort and centered to zero mean for identifiability. The quadratic M7 model includes an additional age term,

$$\text{logit}(q_{x,t}) = \kappa_t^{(1)} + \kappa_t^{(2)}\,(x - \bar{x}) + \kappa_t^{(3)}\,\left((x - \bar{x})^2 - \sigma_x^2\right) + \gamma_{t-x}, \tag{9}$$

with σ_x^2 the mean squared deviation of ages. The quadratic term is centered to keep $\kappa_t^{(1)}$ interpretable as a level factor.

A.5 Smoothing and tail extrapolation

The smoothing module fits a penalized B-spline surface to $\log m_{x,t}$ using CPsplines, with separate spline degrees and difference penalties in age and time. Forecasts are produced by evaluating the fitted surface on an extended year grid. For optional tail handling, `gompertz.py` fits a per-year Gompertz curve on a high-age window:

$$m_{x,t} = \exp(a_t + b_t\,x), \tag{10}$$

and extrapolates $m_{x,t}$ for ages beyond the observed range.

B Bootstrap and Projection Algorithm

B.1 Residual bootstrap

Parameter uncertainty is captured by residual bootstrap. For log-m models (LCM1, LCM2, and APC), residuals are computed on $\log m$; for CBD models, residuals are computed on $\text{logit}(q)$. Residuals are resampled by calendar year blocks (the default in code), added back to the fitted surface, and the model is re-estimated to obtain a new parameter set and drift/volatility estimates for the period factors.

B.2 Random-walk dynamics

Projected period factors follow a random walk with drift,

$$k_t = k_{t-1} + \mu + \sigma\,\varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0,1), \tag{11}$$

applied componentwise for CBD factors $(\kappa_t^{(1)}, \kappa_t^{(2)}, \kappa_t^{(3)})$. Common random numbers are supported by passing fixed innovation tensors, which stabilizes scenario comparisons and risk sensitivities.

B.3 Scenario assembly (pseudo-code)

```
# Input: mortality surface m, ages, years
fit base model on observed surface
residuals = log_m - fitted_log_m  # or logit_q - fitted_logit_q

for b in 1..B:
    resampled = resample_residuals(residuals, mode="year_block")
    synthetic_surface = fitted_surface + resampled
    refit model -> params_b, (mu_b, sigma_b)
    for p in 1..n_process:
        simulate factor paths with random walk
        reconstruct q_paths from simulated factors
```

```

stack all paths -> scenario set with q_paths, S_paths
```

Listing 1: Bootstrap projection flow in PyMORT

C Risk-Neutral Valuation and Pricing Formulas

C.1 Esscher transform

PyMORT applies an Esscher tilt to Gaussian random-walk increments. For a factor increment $\Delta k_t \sim \mathcal{N}(\mu_P, \sigma_P^2)$ under $\mathbb{P}$ and Esscher parameter λ, the risk-neutral drift is

$$\mu_Q = \mu_P + \lambda\, \sigma_P^2, \tag{12}$$

with $\sigma_Q = \sigma_P$. For multi-factor CBD M7, the tilt is applied componentwise to $(\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)})$.

C.2 Scenario pricing

For a scenario set of size N, discount factors D_t and cashflows $CF_t^{(n)}$, PyMORT prices by Monte Carlo expectation,

$$P = \frac{1}{N} \sum_{n=1}^{N} \sum_{t=1}^{T} D_t^{(n)}\, CF_t^{(n)}. \tag{13}$$

The following instruments are implemented:

- **Longevity bond:** $CF_t = N_0\, S_x(t)$. The optional principal payment is $N_0\, S_x(T)$ at maturity.
- **Survivor swap:** for fixed-payer, $CF_t = N_0(S_x(t) - K)$ on scheduled dates. The ATM strike used in code is $K = \sum_t D_t\, \mathbb{E}[S_x(t)] \,/\, \sum_t D_t$.
- ***q*-forward:** payoff at settlement T_s is $N_0(q_{x,T_m} - K)$ with measurement at T_m.
- ***s*-forward:** payoff at settlement T_s is $N_0(S_{x,T_m} - K)$ with measurement at T_m.
- **Cohort life annuity:** cashflows follow $CF_t = N_0\, S_x(t)$ after any deferral period, optionally with a terminal benefit.

D Hull–White Interest Rate Model Details

D.1 Motivation

Longevity-linked cashflows are long dated, so pricing and risk metrics are sensitive to the term structure and to rate volatility. A flat discount curve can understate convexity and tail risk in present values. The Hull–White short-rate model provides a parsimonious way to generate stochastic discount factors consistent with an observed initial curve, while remaining compatible with Monte Carlo pricing.

D.2 Model specification

The one-factor Hull–White model assumes a mean-reverting short rate

$$dr_t = (\theta(t) - a\, r_t)\, dt + \sigma\, dW_t, \tag{14}$$

where $a > 0$ controls mean reversion and σ controls rate volatility. The time-dependent drift $\theta(t)$ is chosen to fit the initial term structure, and the simulated short-rate paths are integrated to produce discount factors $D_t = \exp\left(-\int_0^t r_s\, ds\right)$ used in scenario pricing.

D.3 Calibration principle

Calibration follows the standard term-structure consistency condition: given an input zero-coupon curve, $\theta(t)$ is selected so that model-implied zero-coupon prices match the observed curve at $t = 0$. This ensures that the stochastic scenarios are anchored to the market curve without requiring additional derivative calibration.

D.4 Implementation in PyMORT

The rate engine and pipeline hooks map directly to the codebase:

- `interest_rates/hull_white.py` provides `simulate_hull_white_paths`, which returns short-rate paths on a fixed time grid, and `build_interest_rate_scenarios`, which converts these paths into discount-factor arrays and packages them into an `InterestRateScenarioSet` scenario container. Metadata include times, horizons, and scenario count. The function `calibrate_theta_from_zero_curve` computes $\theta(t)$ from the supplied curve on the simulation grid.

- `pipeline.py` exposes `build_interest_rate_pipeline` to generate rate scenarios, `build_joint_scenarios` to align rate and mortality scenarios in (N, T), and `apply_hull_white_discounting` to attach the discount-factor cube to a mortality scenario set so pricing functions consume consistent dimensions.

D.5 Limitations

The current implementation does not calibrate (a, σ) to liquid interest-rate options and does not model correlation between mortality and interest rates. Parameter uncertainty in the rate process is also not modeled. These extensions are left for future work, while the existing framework allows stress tests and sensitivities to quantify the impact of stochastic discounting.

E Extended Results Tables

Table 8: Fit diagnostics against StMoMo reference outputs.

Model	Metric	Value
LC M1	RMSE of $\log m$ vs StMoMo fitted surface	0.026
CBD M5	RMSE of logit(q) vs StMoMo fitted surface	0.129

Table 9: Estimated period-factor drifts (per year).

Factor	Drift
k_t (LC)	-0.69
$\kappa_t^{(1)}$ (CBD)	-0.019
$\kappa_t^{(2)}$ (CBD)	2.9×10^{-4}

Table 10: Observed mortality improvements in France (1970 to 2019).

Age	Reduction in $m_{x,t}$	Approx. percent
65	$m_{2019}/m_{1970} \approx 0.44$	56%
80	$m_{2019}/m_{1970} \approx 0.38$	62%
90	$m_{2019}/m_{1970} \approx 0.53$	47%

Table 11: Deterministic pricing baseline (2019 mortality, 2% flat rate).

Metric	Value
$S_{65}(20)$	0.620
20-year longevity bond PV	14.41

F Repository Structure and Reproducibility

F.1 Module layout

```
src/pymort/
  analysis/          # fitting, bootstrap, projections, reporting
  models/            # LC, CBD, APC, Gompertz implementations
  pricing/           # bonds, swaps, forwards, annuities, hedging
  interest_rates/    # Hull--White scenario generation
  cli.py             # CLI entrypoints
  pipeline.py        # high-level pipeline functions
```

Listing 2: Core PyMORT package layout

F.2 Reproducible runs

The CLI exposes one-click pipelines configured by YAML files stored under `configs/`. The pricing and hedge pipelines used in this report can be executed via:

```
pymort run pricing-pipeline --config configs/pricing-pipeline.yaml
pymort run hedge-pipeline --config configs/hedge-pipeline.yaml
```

Listing 3: Pipeline entrypoints

Outputs are written to the configured `outdir` (default `outputs/`) and include fitted parameters, scenario sets, pricing results, and diagnostic plots compatible with the reporting utilities.