Second moment of $GL(3)$ L-functions in the depth aspect

ADAMANT 2025, ISI Kolkata

Sampurna Pal

February 10, 2025

Introduction

Let

- F be a Hecke-Maass cusp form for $SL(3, \mathbb{Z})$,
- $A(m, n)$ be the normalized Fourier coefficients of F,
- χ be a primitive Dirichlet character modulo q.

Then, the L-function associated with the twisted form $F \times \chi$ is given by the Dirichlet series

$$L(s, F \times \chi) = \sum_{n=1}^{\infty} \frac{A(1, n)\chi(n)}{n^s} \qquad \text{for } \Re(s) > 1$$

and $L(s, F \times \chi)$ is a degree three Automorphic L-function of level q^3.

Introduction

Let p be an odd prime and ℓ is a positive integer. We consider

$$\sum_{\substack{\chi \bmod p^\ell \\ \chi \text{ primitive}}} |L(1/2, F \times \chi)|^2 .$$

Applying the approximate functional equation, we can trivially bound the second moment by

$$\sum_{\chi \bmod p^\ell}^{*} |L(1/2, F \times \chi)|^2 \ll_{F,\epsilon} p^{\frac{3\ell}{2}+\epsilon}.$$

Our goal is to obtain a non-trivial bound for the second moment in the depth aspect, i.e.

$$\sum_{\chi \bmod p^\ell}^{*} |L(1/2, F \times \chi)|^2 \ll_{F,\epsilon} p^{\frac{3\ell(1-\delta)}{2}+\epsilon} \quad \text{for some } \delta > 0 \text{ as } \ell \to \infty$$

Main result

Theorem ((upcoming) Dasharatharaman, Kumar, Leung, P.)

Let F be a Hecke-Maass cusp form for $SL(3,\mathbb{Z})$ and let p be an odd prime. Then,

$$\sum_{\chi \bmod p^{\ell}}^{*} |L(1/2, F \times \chi)|^2 \ll_{F,\epsilon} p^{\frac{4\ell}{3}+\epsilon}.$$

There should be some lower bound of ℓ. Specifically, we cannot take $\ell = 1$.

Second moment in t-aspect

This problem (second moment in the depth aspect) can be thought of as a p-adic analog of the problem of obtaining a non-trivial bound for the second moment of the $GL(3)$ L-functions in the t-aspect:

$$\int_T^{2T} |L(1/2+it, F)|^2 dt \ll_{F,\epsilon} T^{3/2-\delta+\epsilon} \quad \text{for some} \quad \delta > 0$$

Results:

- P. (2023) $\delta = \frac{3}{88}$
- Dasgupta-Leung-Young (2024) $\delta = \frac{1}{6}$

Trivial Bound

By the approximate functional equation:

$$\sum_{\chi \bmod p^\ell}^{*} |L(1/2, F \times \chi)|^2 \ll \frac{1}{p^{3\ell/2}} \sum_{n \sim p^{3\ell/2}} \sum_{m \sim p^{3\ell/2}} A(n)\bar{A}(m) \sum_{\chi \bmod p^\ell} \chi(n\bar{m})$$

By the orthogonality of multiplicative characters:

$$\frac{1}{p^{\ell/2}} \sum_{n \sim p^{3\ell/2}} \sum_{m \sim p^{3\ell/2}} A(n)\bar{A}(m)\delta(n \equiv m \bmod p^\ell).$$

By Cauchy-Schwartz inequality and Ramanujan type bound on average, i.e., $\sum_{m^2 n \ll N} |A(m,n)|^2 \ll N^\epsilon$, we can bound the above expression by

$$\frac{1}{p^{\ell/2}} \cdot p^{3\ell/2+\epsilon} \cdot p^{\ell/2} \ll p^{3\ell/2+\epsilon}.$$

Proof

We start with

$$\sum_{n\sim p^{3\ell/2}} \sum_{m\sim p^{3\ell/2}} A(n)\bar{A}(m)\delta(n \equiv m \bmod p^{\ell})$$

We write $m = n + hp^{\ell}$ and apply the DFI delta method with the conductor lowering mechanism.

$$\sum_{h\sim p^{\ell/2}} \sum_{n\sim p^{3\ell/2}} \sum_{m\sim p^{3\ell/2}} A(n)\bar{A}(m)\delta(n \equiv m \bmod p^{r})\delta\left(\frac{m-n-hp^{\ell}}{p^{r}}\right)$$

$\delta(n) = 1$ when $n = 0$ and 0 otherwise.

Proof

The congruence relation can be realized by the orthogonality of additive characters

$$\delta(n \equiv m \bmod p^r) = \frac{1}{p^r} \sum_{a \bmod p^r} e\left(\frac{a(n-m)}{p^r}\right)$$

and for $\delta(n)$ we will use the delta symbol expansion of DFI

$$\delta(n) \asymp \frac{1}{Q} \sum_{q \leq Q} \frac{1}{q} \sideset{}{^*}\sum_{a \bmod q} \int_{x \ll Q^{\epsilon}} e\left(\frac{an}{q}\right) e\left(\frac{nx}{qQ}\right)$$

So, we arrive at

$$\sum_{q \sim Q} \sideset{}{^*}\sum_{a \bmod qp^r} \sum_{h \sim p^{\ell/2}} e\left(\frac{ahp^{\ell-r}}{q}\right) \sum_{n \sim p^{3\ell/2}} A(n) e\left(\frac{an}{qp^r}\right) \sum_{m \sim p^{3\ell/2}} \bar{A}(m) e\left(\frac{-am}{qp^r}\right)$$

Proof

We apply the Voronoi summation formula on the m and n sum and the Poisson summation formula on the h sum. With some simplification in the character sum, we get

$$\sum_{h\sim\frac{Q}{H}}\sum_{q\sim Q}\sum_{\chi \bmod p^r}\left|\sum_{n\sim N'} A(n)S(p^{\ell-2r}\overline{h},n;q)\chi(n)\right|^2$$

Now, we apply the duality principle of the large sieve to get

$$\sup_{||\alpha||_2=1}\sum_{h\sim\frac{Q}{H}}\sum_{n\sim N'}\left|\sum_{q\sim Q}\sum_{\chi \bmod p^r}\alpha(q,\chi)S(p^{\ell-2r}\overline{h},n;q)\chi(n)\right|^2$$

where $||\alpha||_2 = 1$ means $\sum_{q\sim Q}\sum_{\chi \bmod p^r}|\alpha(q,\chi)|^2 = 1$.

Proof

Now, we open up the absolute square and apply the Poisson summation formula on the n sum :

$$\sum_{n \sim N'} S(p^{\ell-2r}\overline{h}, n; q_1) S(p^{\ell-2r}\overline{h}, n; q_2) \chi_1 \bar{\chi}_2(n).$$

After some careful analysis of the character sum, along with the use of reciprocity and multiplicative Fourier expansion, we arrive at (roughly)

$$\sum_{\psi \bmod nh} \sum_{\chi \bmod p^r} \left| \sum_{q \sim Q} \alpha(q, \chi) \chi\psi(q) \right|^2$$

where we use the large sieve inequality to obtain the desired bound.

Thank You