Abstract Mathematics 101 Bootcamp

Lecture 7 (PART C)
Introduction to Metric Topology

Bambordé Baldé

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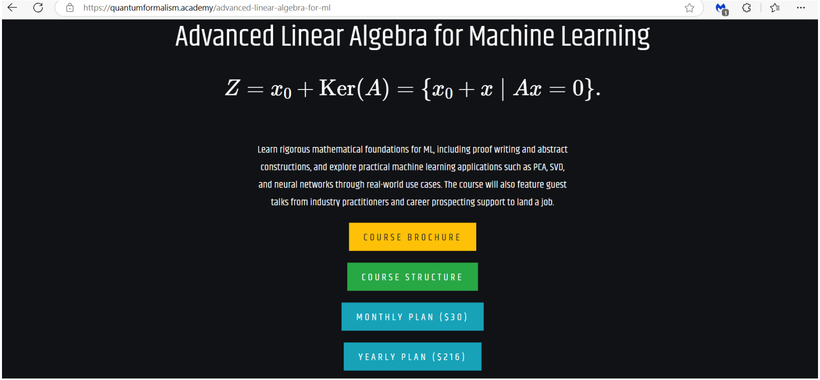

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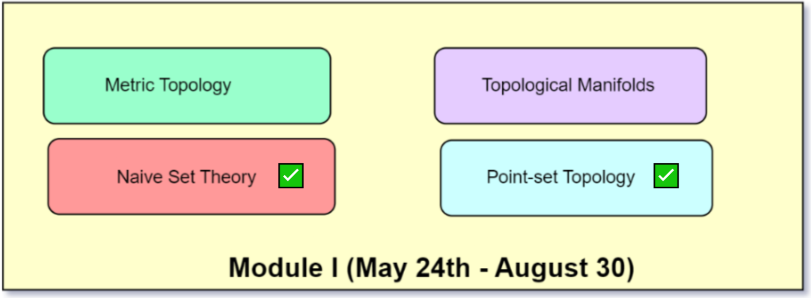
Bootcamp Overview



Lecture 7B Recap

- Closed Sets in Metric Spaces
- The Metric Topology
- The Standard Topology

Equivalent Metrics

Definition 1.0

Let d_1 and d_2 be two metrics on M with corresponding topologies $\mathcal{T}_{d_1}$ and $\mathcal{T}_{d_2}$. The metrics d_1 and d_2 are said to be equivalent if $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$, i.e., if the two metrics generate the same open sets.

- We write $d_1 \simeq d_2$ to indicate the equivalence between the two.

Curiosity question (homework): Let $M = \mathbb{R}^n$ and d the Euclidean metric with corresponding standard topology $\mathcal{T}_d$. Can you find other metrics defined in $\mathbb{R}^n$ that are equivalent to the Euclidean metric d?

The Metric Topology's Hausdorffness

Theorem 1.0

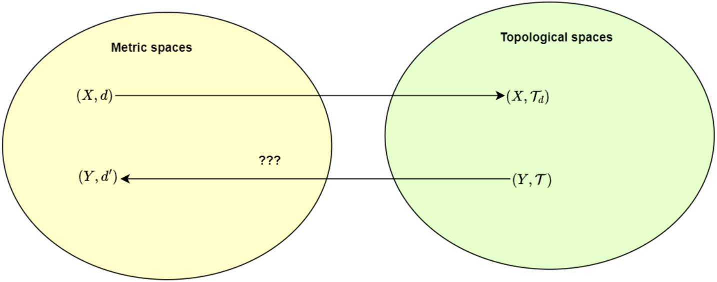
For any metric space (M, d), the induced topological space $(M, \mathcal{T}_d)$ is a Hausdorff space.

- Recall that $(M, \mathcal{T}_d)$ to be Hausdorff, it needs to satisfy the following: For all $p_1, p_2 \in M$ with $p_1 \neq p_2$, there exist some neighbourhoods N_{p_1} and N_{p_2} of the respective points such that $N_{p_1} \cap N_{p_2} = \emptyset$.

Question: Does the converse hold i.e. is every Hausdorff space a metric space? In other words, is being Hausdorff a minimum and necessary condition for a topological space to be metrisable?

The Metrisability Question

- Can we construct metric spaces from topological spaces? If yes, what are the minimum and necessary conditions for doing so?



Metrisable Topological Spaces

Definition 1.1

A topological space $(X, \mathcal{T})$ is *metrisable* if there exists a metric d on X such that $\mathcal{T} = \mathcal{T}_d$.

- A famous metrisation theorem by soviet mathematician Pavel Samuilovich Urysohn states that: Every second-countable, regular Hausdorff space X is metrisable.
- Nagata-Smirnov metrisation theorem: A topological space is metrisable if and only if it is regular, Hausdorff and has a countably locally finite (σ-locally finite) basis.

Homework Exercises (i)

- Let X be any set and $\mathcal{T} = \mathcal{P}(X)$. Is $(X, \mathcal{T})$ metrisable?
- What if we now have $\mathcal{T} = \{\emptyset, X\}$?
- Now consider $X = \{0, 1\}$ and $\mathcal{T} = \{\emptyset, X, \{1\}\}$ (Sierpinski topology). Is $(X, \mathcal{T})$ metrisable?

Natural curiosity (homework): Suppose now that $X = \mathbb{R}$, can you construct a non-trivial example of a non-metrisable topology on $\mathbb{R}$?

Congratulations for making it this far!

The Upcoming Lectures (8A,8B)

- Topological Bases
- Product Topology
- Projection Maps
- Topological Invariants
- Connected Spaces
- Connected Subsets