# Abstract Mathematics 101 Bootcamp

## Lecture 7 (PART A)
## Introduction to Metric Spaces

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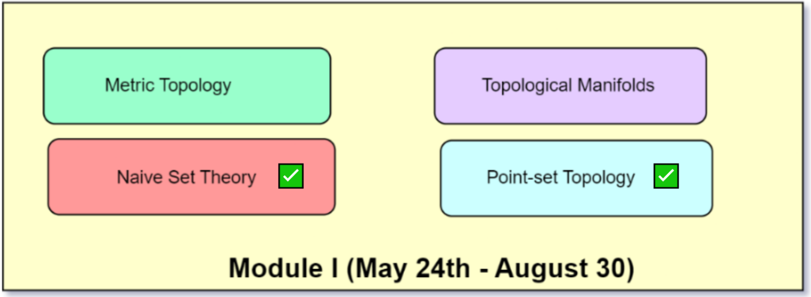
# Bootcamp Overview



# Metric Spaces

## Definition 1.0

Let $M$ be a nonempty set. A metric on $M$ is a map $d : M \times M \longrightarrow \mathbb{R}$ satisfying the following conditions:

1. $d(x, y) \geq 0$ for all $x, y \in M$.
2. $d(x, y) = 0$ iff $x = y$ for all $x, y \in M$.
3. $d(x, y) = d(y, x)$ for all $x, y \in M$.
4. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

- The pair $(M, d)$ is called a 'metric space'. Whenever the metric $d$ is understood from the context, we'll just write $M$.

# Concrete Example(i)

- For any nonempty set $M$, we can define the following metric:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

  It's not hard to verify that $d$ is a metric (discrete) on $M$.
- **Homework question:** Let $(M, d)$ be any metric space and let $\alpha \in \mathbb{R}^{+}$ (the set of all positive real numbers).
  Is $d_{\alpha}(x, y) = \alpha d(x, y)$ a metric on M?

# Concrete Example(ii)

- Let us consider $M = \mathbb{R}^n$. Then for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we can define the famous Euclidean metric (or distance) as follows:

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

In Machine Learning literature and other applied subjects, the Euclidean metric is often called $L2$-distance.

# Concrete Example(iii)

- Let us consider $M = \mathbb{R}^n$ again. Then for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we can define the famous Manhattan distance as follows:

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

In Machine Learning literature and other applied subjects, the Manhattan distance is also known as the $L1$-distance.

# Homework Exercises(i)

1. Let $(M_1, d_1), (M_2, d_2), \ldots, (M_k, d_k)$ be metric spaces. Construct a metric on $M_1 \times M_2 \times \cdots \times M_k$.
2. Let $\mathcal{H}$ be a finite dimensional complex Hilbert space (think $\mathbb{C}^n$). Construct a metric on $\mathcal{H}$ using the norm induced by the inner product.

# Metric Subspace

## Definition 1.1

Let $(M, d)$ be a metric space and $A \subseteq M$ be nonempty.We can construct $d_A : A \times A \longrightarrow \mathbb{R}$ as follows:

- $d_A(a_1, a_2) = d(a_1, a_2)$ for all $a_1, a_2 \in A$.

As you can guess, the pair $(A, d_A)$ is called metric subspace.

# Open Balls in Metric Spaces

## Definition 1.2

Let $(M, d)$ be a metric space. For each point $p \in M$ and $r \in \mathbb{R}^+$, we can define $B_r(p) = \{x \in M \mid d(p, x) < r\}$.

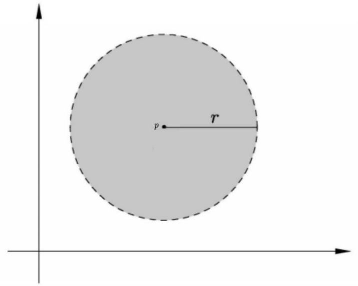
- $B_r(p)$ is called the open ball centred around the point $p$ with radius $r$.

## Proposition 1.0

Let $B_r(p)$ be an open ball in a metric space $(M, d)$. Then for any $p' \in B_r(p)$ there exists a $r' > 0$ such that $B_{r'}(p') \subseteq B_r(p)$.

# Concrete Example

- If $M = \mathbb{R}^2$ and $d$ is the Euclidean metric, then $B_r(p)$ is just the open disc of radius $r$ centred around $p$, i.e. the set of all points inside the circle of radius $r$ centred at point $p$:



# Homework Exercises (ii)

1. Let $(M, d)$ be a metric space where $d$ is the discrete metric. Given a point $p \in M$ and $r >$, what is $B_r(p)$?
2. Let $M = \mathbb{R}^2$ and $d$ be the Euclidean distance. If $p = (0, 0) \in \mathbb{R}^2$ and $r = 1$, what is $B_r(p)$?
3. Let $M = \mathbb{R}^2$ and $d$ now be the Manhattan distance. If $p = (0, 0) \in \mathbb{R}^2$ and $r = 1$, what is $B_r(p)$?
4. Let $M = \mathbb{R}^2$ and $d$ be the Manhattan distance again. If $p = (1, 0) \in \mathbb{R}^2$ and $r = 1$, what is $B_r(p)$? What if $p = (0, 1)$?

# Open Sets in Metric Spaces

## Definition 1.3

Let $(M, d)$ be a metric space. A subset $U \subseteq M$ is open in respect to the metric $d$ if for all $p \in U$ there exists an $r \in \mathbb{R}^+$ such that $B_r(p) \subseteq U$.

- As you have noticed, whether or not a subset $U$ is open really depends on the metric space that we are considering, i.e., being 'open' depends on the metric space structure being considered!

# Concrete Example

- Let $M = \mathbb{R}$ and $d$ the Euclidean metric. Then the open intervals (in the **Real Analysis** sense) $(a, b)$, $(-\infty, a)$, $(b, \infty)$ are also open in $M$.

**Natural homework question:** Is $\mathbb{R}$ itself open in respect to the above?

# Homework Exercises (iii)

1. Let $M = [a, b]$ and $d$ the Euclidean metric. Is it true that the intervals $[a, b]$ and $[a, b)$ are open?
2. Let $M = \mathbb{R}^2$ and $d$ the Euclidean metric. Is $\mathbb{R}$ viewed as subset of $\mathbb{R}^2$ open?
3. Let $M = \mathbb{R}^2$ and $d$ the Manhattan metric. Is $\mathbb{R}$ viewed as subset of $\mathbb{R}^2$ open?

# Properties of Open Sets in Metric Spaces

## Proposition 1.1

Given any metric space $(M, d)$, the following properties hold:

1. $\emptyset$ and $M$ are open.
2. If $U_1, U_2, \ldots, U_k$ are open subsets of $M$, then their intersection $U_1 \cap U_2 \cap \cdots \cap U_k$ is open.
3. If $\{U_i\}_{i \in I}$ is an arbitrary collection of open subsets of $M$, then the union over the collection $\bigcup_{i \in I} U_i$ is open.

$Proof$ : Homework!

- Anything familiar with the properties above?

# Congratulations for making it this far!