quantumformalism.com

Abstract Mathematics 101 Bootcamp

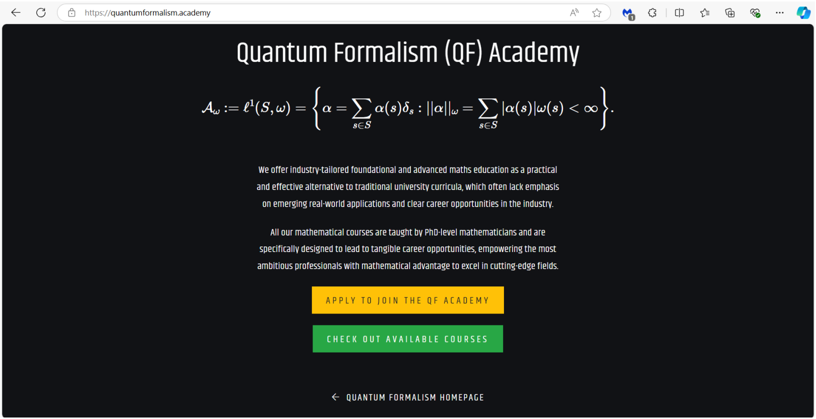
Lecture 7 (PART B)
Introduction to Metric Topology

Bambordé Baldé

Quantum Formalism (QF) Free Bootcamp
Brought to you by Zaiku Group.

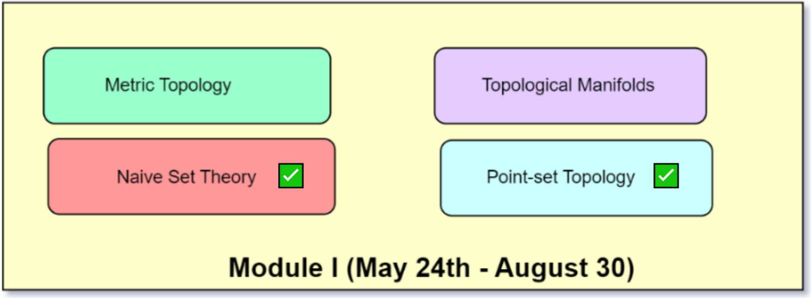
QF's CORE MISSION:
Make Abstract Mathematics Accessible.

QF Academy Program



quantumformalism.com

Bootcamp Overview



Lecture 7A Recap

- Metric spaces
- Open sets in metric spaces

Proposition (1.1 in Lecture 7A)

Given any metric space (M, d), the following properties hold:

1. $\emptyset$ and M are open.
2. If $U_1, U_2, \dots, U_k$ are open subsets of M, then their intersection $U_1 \cap U_2 \cap \cdots \cap U_k$ is open.
3. If $\{U_i\}_{i \in I}$ is an arbitrary collection of open subsets of M, then the union over the collection $\bigcup_{i \in I} U_i$ is open.

Closed Sets in Metric Spaces

Definition 1.0

Let (M, d) be a metric space. We say that a subset $A \subseteq M$ is closed in respect to the metric d if A^c is open.

Curiosity question (homework): Let $M = \mathbb{R}$ and d the Euclidean metric. Is it true that the intervals $[a, b]$ and $(-\infty, 0)$ are closed?

Closed Sets in Metric Spaces

Proposition 1.0

Given any metric space (M, d), the following properties hold:

1. $\emptyset$ and M are closed.
2. If $C_1, C_2, \ldots, C_k$ are closed subsets of M, then their union $C_1 \cup C_2 \cup \cdots \cup C_k$ is closed.
3. If $\{C_i\}_{i \in I}$ is an arbitrary collection of closed subsets of M, then the interesection over the collection $\bigcap_{i \in I} C_i$ is closed.

Question: The properties above are the same as the closed sets in point-set topology right?

The Metric Topology

Proposition 1.1

Given a metric space (M, d), the following collection of subsets is a topology on M:

$$\mathcal{T}_d = \{U \subseteq M \mid U \text{ is open with respect to } d\}.$$

Proof : Homework (trivial if you already proved the properties of open subsets)!

- $\mathcal{T}_d$ is called a metric topology on M, i.e. the topology induced by the metric d. Hence, we'll write $(M, \mathcal{T}_d)$ to denote the topological space induced by d.

The Standard Topology

Definition 1.1

When $M = \mathbb{R}^n$ and d is the Euclidean metric, then $\mathcal{T}_d$ is called the 'standard topology' on $\mathbb{R}^n$.

- Most topological spaces of interest in applied subjects such as physics will be subspaces of $(\mathbb{R}^n, \mathcal{T}_d)$.
- From now on, always assume $\mathcal{T}_d$ to be the standard topology whenever $M \subseteq \mathbb{R}^n$.
- When the metric is understood from the context, we'll also just write M for the underlying topology instead $(M, \mathcal{T}_d)$.

Homework Exercises (i)

- Consider the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Show that the subspace topology induced by the standard topology on $\mathbb{R}^2$ makes S^1 into a topological space.
- Show that the real line interval $[0, 1]$ equipped with the subspace topology from the standard topology on $\mathbb{R}$ is a topological space.

Important note: $[0, 1]$ is a very important topological space as it will help construct 'Path-Connected' topological spaces. Not to be mistaken with 'Connected' topological spaces!

Homework Exercises (ii)

1. Consider the topological spaces $X = (-1, 1)$ and $Y = (0, 5)$ constructed from the standard topology on $\mathbb{R}$. Let $f : X \longrightarrow Y$ be defined as $f(x) = \frac{5}{2}(x + 1)$. Is f a homeomorphism?
2. Consider now the topological spaces $X = (-1, 1)$ and $Y = \mathbb{R}$ again constructed from the standard topology on $\mathbb{R}$. Let $f : X \longrightarrow Y$ be defined as $f(x) = \tan\left(\frac{\pi x}{2}\right)$. Is f a homeomorphism?
3. Prove or disprove whether the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is homeomorphic to $X = [0, 1]$.

Congratulations for making it this far!