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Lecture 6 — Friday, September 13.

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Sep 24 & Oct 1 — 6pm UK Time.

❑ **Lecture 1:**

- **Objective:** Examine the main types of proof: Direct proof, Proof by Contradiction, Proof by Induction, Proof by Contrapositive, and Counterexamples.
- **Lecture highlights:** For each proof type, we work through an example and we discuss common mistakes.

❑ **Lecture 2:**

- **Objective:** Learn to write formally and clearly in a proof, and how to check for mistakes.
- **Lecture highlights:** We discuss the importance of clarity and precision. We talk about how to structure a proof, and common phrases and conventions to use. We examine some proofs, looking for mistakes. These mistakes may be logical, but may be due to ambiguities in writing style, motivating the necessity of proper, formal writing.

Recap of the basic notions covered during the last session:

(a) **Subspace topology**

(b) **Neighbourhoods**

(c) **Hausdorff spaces**

Questions:

(a) **Did anyone find it hard to grasp these basic concepts?**

(b) **Did anyone find it hard to solve the assignments?**

If the answer to the above is yes, then you should go back and review the set theoretic concepts already covered!

Definition 1.0

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A map $f : X \to Y$ is said to be continuous if for any open set $O_Y \in \mathcal{T}_2$, the pre-image $\text{Preim}_f(O_Y) \in \mathcal{T}_1$. Recall that $\text{Preim}_f(O_Y) = \{p \in X \mid f(p) \in O_Y\}$.

Remark 1:

(a) Hence, under a continuous map f, the pre-image of any open set in the target topological space $(Y, \mathcal{T}_2)$ is an open set in the domain space X in respect to the chosen topology $\mathcal{T}_1$.

(b) Be warned that some authors use $f^{-1}(O_Y)$ instead of $\text{Preim}_f(O_Y)$. This can be confusing because the notation f^{-1} is also used to denote the inverse of f! Hence, it may give the wrong impression that all continuous maps are invertible!

(c) For those with real analysis experience, do you agree this definition of continuity is far simpler than the one you encounter in analysis with ϵ and δ?!

Curiosity question 1: Can we alternatively define the notion of continuity at a point $p \in X$ first and then generalise to all X?

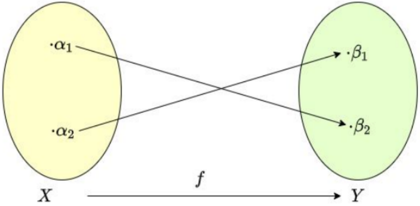
Exercise 1.0

(a) Let $(X, \mathcal{T})$ be a topological space. Is the identity map $id_X : X \to X$ continuous?

(b) Consider a topological space $(X, \mathcal{T}_1)$ with $\mathcal{T}_1 = \mathcal{P}(X)$ and $(Y, \mathcal{T}_2)$ be another topological space with $\mathcal{T}_2$ being any topology. Is it true that any map $f : X \to Y$ is continuous?

(c) Let $X = \{\alpha_1, \alpha_2\}$ and $Y = \{\beta_1, \beta_2\}$ with the respective topologies $\mathcal{T}_1 = \{\varnothing, X, \{\alpha_1\}, \{\alpha_2\}\}$ and $\mathcal{T}_2 = \{\varnothing, Y\}$.

Is the map defined by the diagram below continuous?

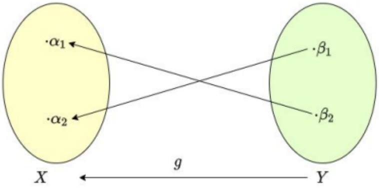


Exercise 1.1

(a) Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. Now let us consider $f : X \longrightarrow Y$ to be a continuous map. Is the pre-image of any closed set in Y also closed in X?

(b) Let again $X = \{\alpha_1, \alpha_2\}$ and $Y = \{\beta_1, \beta_2\}$ with the respective topologies $\mathcal{T}_1 = \{\varnothing, X, \{\alpha_1\}, \{\alpha_2\}\}$ and $\mathcal{T}_2 = \{\varnothing, Y\}$.

Is the map defined by the diagram below continuous?



Proposition 1.0

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A map $f : X \longrightarrow Y$ is continuous iff the pre-image of every closed set in Y is closed in X.

Proof : Homework!

Proposition 1.1

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. If $f : X \to Y$ is continuous and $A \subseteq X$, then the 'restriction' map $f|_A : A \to Y$ is also continuous.

Proof : Homework!

Remark 2: The proposition above is implicitly invoking the subspace topology of A.

Proposition 1.2

Let $(X, \mathcal{T}_1)$, $(Y, \mathcal{T}_2)$ and $(Z, \mathcal{T}_3)$ be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then the composition $g \circ f : X \longrightarrow Z$ is also continuous.

Proof : Homework!

Definition 1.1

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A continuous map $f : X \to Y$ is a homeomorphism if the following conditions hold:

(1) f is bijective i.e. f is injective and surjective.

(2) The inverse f^{-1} is also continuous.

Remark 3:

(a) We write $X \simeq Y$ and say the two topological spaces are homeomorphic if a homeomorphism between the two exists.

(b) Homeomorphisms are indeed the structure preserving maps in topology.

Remark 4 (digression): In Linear Algebra, isomorphisms are maps that preserve the structure of vector spaces. When two vector spaces, V_1 and V_2, are isomorphic, they must have the same dimension. Thus, dimension is an intrinsic property of a vector space, meaning it remains invariant under isomorphisms.

Curiosity question 2: Is there an analogous concept in topology? That is, are there one or more intrinsic topological properties such that if two topological spaces are homeomorphic, they must share these properties?

Proposition 1.3

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. If X is Hausdorff and $X \simeq Y$, then Y is also Hausdorff.

Proof : Homework!

Remark 5:

(a) Homeomorphisms are the reason why donuts and cups are the same thing in the eyes of a Topologist!

(b) In a nutshell, two spaces (think shapes) are homeomorphic if one can be deformed into the other without cutting or gluing.

Congratulations for making it this far!