Abstract Mathematics 101 Bootcamp

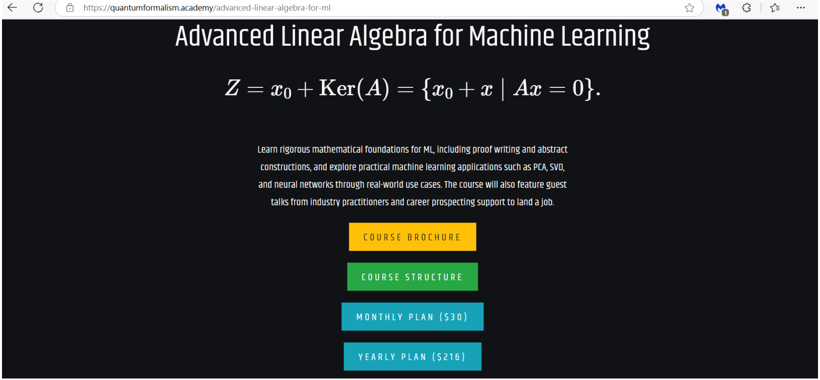
Lecture 8 (PART A)
Introduction to Metric Topology

Bambordé Baldé

Quantum Formalism (QF) Free Bootcamp
Brought to you by Zaiku Group.

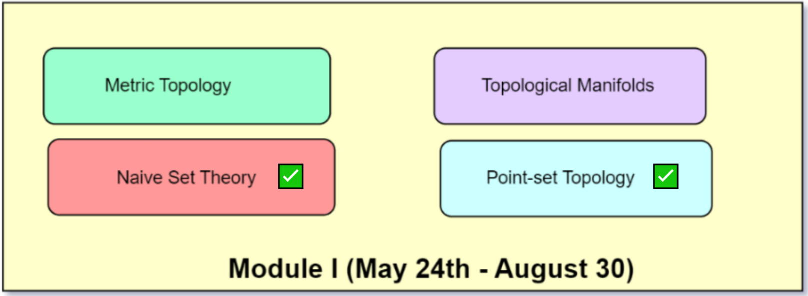
QF's CORE MISSION:
Make Abstract Mathematics Accessible.

QF Academy Career-focused Course



Visit quantumformalism.academy/courses to learn more.

Bootcamp Overview



Lecture 7C Recap

- Equivalent Metrics
- The Metric Topology's Hausdorffness
- Metrisable Topological Spaces

Topological Bases

Definition 1.0

Let $(X, \mathcal{T})$ be any topological space. A collection of open sets $\mathcal{B} \subseteq \mathcal{T}$ is called a **basis** (or just **base**) if for any open set $O \in \mathcal{T}$ there is a $\mathcal{B}' = \{B_i\}_{i \in I} \subseteq \mathcal{B}$ such that $O = \bigcup_{i \in I} B_i$.

- $\mathcal{T}$ itself is a basis, right? For any open set $O \in \mathcal{T}$, we can make $\mathcal{B}' = \{\emptyset, O\}$ and so $O = \emptyset \cup O$?
- If $X = \mathbb{R}^2$ with the standard topology $\mathcal{T}_d$ given by the Euclidean metric d, then $\mathcal{B} = \{B_r(p) \mid p \in \mathbb{R}^2,\ r > 0\}$ forms a basis.

The Metric Topology's Base

Proposition 1.0

If X is a topological space induced by a metric topology $\mathcal{T}_d$, then the collection of all the open balls $\mathcal{B} = \{B_r(p) \mid p \in X,\ r > 0\}$ forms a basis.

Proof : Homework!

- This shows the importance of open balls in metric spaces!

Natural curiosity: Can topological bases be uncountable? In the definition of basis, we didn't mention anything about cardinality right?

Second-Countable Spaces

Definition 1.1

A topological space $(X, \mathcal{T})$ is called 'second-countable' if it has a countable basis $\mathcal{B} \subseteq \mathcal{T}$.

- This notion of being second-countable is very important. Most practical topological spaces i.e. the ones used in applied topics such as theoretical physics, are second-countable.
- Indeed, the standard definition of a topological manifold requires the underlying topological to be second-countable and Hausdorff!

Homework Exercises

Exercise (i)

If $X = \{\beta_1, \beta_2, \beta_3\}$ and $\mathcal{T} = \{\emptyset, X, \{\beta_1\}, \{\beta_2\}, \{\beta_1, \beta_2\}\}$. Which of the following (if any) forms a basis:

1. $\mathcal{B}_1 = \{X, \{\beta_1\}, \{\beta_2\}\}$.
2. $\mathcal{B}_2 = \{\emptyset, X, \{\beta_1\}, \{\beta_2\}\}$.
3. $\mathcal{B}_3 = \{\emptyset, X, \{\beta_1\}\}$.
4. $\mathcal{B}_4 = \{\emptyset, X, \{\beta_2\}\}$.

Homework Exercises

Exercise (ii)

Which of the following statements are true (you need to prove):

1. The collection $\mathcal{B} = \{(a, b) \subseteq \mathbb{R} : a < b\}$ is a basis on $\mathbb{R}$ in respect to the standard topology.
2. The collection $\mathcal{B} = \{[a, b] \subseteq \mathbb{R} : a < b\}$ is a basis on $\mathbb{R}$ in respect to the standard topology.
3. The collection $\mathcal{B} = \{[a, b) \subseteq \mathbb{R} : a < b\}$ is a basis on $\mathbb{R}$ in respect to the standard topology.

Congratulations for making it this far!

The Upcoming Lectures (8B)

- Product Topology
- Projection Maps