Homework 2

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. We call a collection of sets $\mathcal{R} \subset P(X)$ a ring if it is closed under finite unions and differences, that is, if $E, F \in \mathcal{R}$, then $E \backslash F \in \mathcal{R}$. If it is closed under countable unions, we call it a σ-ring.

   a) Let $\mathcal{R}$ be a ring such that $X \in \mathcal{R}$. Show that $\mathcal{R}$ is also an algebra. (The same holds for σ-rings/σ-algebras).

   b) Construct an example of a ring that is not an algebra.

   c) If $\mathcal{R}$ is a σ-ring, then

   $$\mathcal{A}_1 = \{E \subset X \mid E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$$

   and

   $$\mathcal{A}_2 = \{E \subset X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$$

   are both σ-algebras.

   Solution:

   (a) Since rings are closed under differences, and $X \in \mathcal{R}$, then for any $E \in \mathcal{R}$, we have that $E^c = X \backslash E \in \mathcal{R}$. Thus it is an algebra since we also get for free closure under finite unions.

   (b) We let $X = [0, 1]$, $\mathcal{R} = \{\emptyset, [0, 1/2)\}$. It is clear that $\mathcal{R}$ satisfies the definition of a ring. However it is not closed under compliments so it is not an algebra.

   (c) Note that since rings are closed under differences, this means that $\emptyset = E \backslash E$ is in every ring. Now since $X^c = \emptyset$, this means $X \in \mathcal{R}$, so by part (a), $\mathcal{A}_1$ is a σ-algebra. As for $\mathcal{A}_2$, observe that since $X \cap F = F$, by the definition of the set, $X \in \mathcal{R}$, thus it is a σ-algebra by part (a).

2. a) Let X be a non-empty set and $\mathcal{A}_1, \mathcal{A}_2, \ldots$ be a collection of σ-algebras on X. Verify that $\cap_{j=1}^{\infty} \mathcal{A}_j$ is a σ-algebra on X. (Recall we state this in lecture but never verified it).

   b) Provide an example to show that the analogous statement about a union of σ-algebras is false.

c) Suppose we add the condition that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ Is it now the case that $\cup_{j=1}^{\infty} \mathcal{A}_j$ is a σ-algebra? Prove it or provide a counter example.

Solution:

(a) We simply verify the definition of a σ-algebra. First let $E \in \cap_{j=1}^{\infty} \mathcal{A}_j$, and then since $E \in \mathcal{A}_j$ for all j, then since each $\mathcal{A}_j$ is a σ-algebra, $E^c \in \mathcal{A}_j$ for all j. But then that means $E^c \in \cap_{j=1}^{\infty} \mathcal{A}_j$, thus the intersection is closed under compliments. Now let $\{E_j\}_{j=1}^{\infty} \subset \cap_{j=1}^{\infty} \mathcal{A}_j$. Thus $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}_j$ for all j, and again since it is a σ-algebra, this means the union is in $\mathcal{A}_j$ for each j. So $\cup_{j=1}^{\infty} E_j \in \cap_{j=1}^{\infty} \mathcal{A}_j$. Thus it is also closed under countable unions, so it is indeed a σ-algebra.

(b) We will let $X = [0,1]$, $\mathcal{A}_1 = \{\emptyset, X, [0,1/2], (1/2,1]\}$, $\mathcal{A}_2 = \{\emptyset, X, [0,1/3), [1/3,1]\}$, and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. One can easily see that each $\mathcal{A}_j$ is a σ-algebra, however the union is not, as $E_1 = [0,1/3)$ and $E_2 = (1/2,1]$ are both in $\mathcal{A}$, but their union is not. So $\mathcal{A}$ isn't even an algebra.

(c) This still remains false. Let $X = [0,1]$, and let $\mathcal{E}_1 = \{\emptyset, X, [0,1/2), [1/2,1]\}$ and $\mathcal{A}_1$ the σ-algebra generated by $\mathcal{E}_1$ (in this case they will be the same). Let $\mathcal{E}_2 = \mathcal{E}_1 \cup \{[1/3,1]\}$ and $\mathcal{A}_2$ the σ-algebra generated by $\mathcal{E}_2$. In general we let $\mathcal{E}_n = \mathcal{E}_{n-1} \cup \{[1/(n+1),1]\}$ and $\mathcal{A}_n$ the σ-algebra it generates. Note that since $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots$, we have that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$, so our σ-algebras are nested as desired. Let $\mathcal{A} = \cup_{j=1}^{\infty} \mathcal{A}_j$, and we now pick our sets $E_n = [1/(n+1),1] \in \mathcal{A}$. However we note that $\cup_{j=1}^{\infty} E_j = (0,1]$ which is not in any $\mathcal{A}_j$, thus not in $\mathcal{A}$. So the union of the nested σ-algebras is still not a σ-algebra.

Remark: Interestingly enough, the union of the nested σ-algebras is closed under compliments, whereas the example in part (b) is not even closed under compliments.

3. An algebra $\mathcal{A}$ is a σ-algebra if and only if for any collection $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}$ with $E_1 \subset E_2 \subset \dots$, $\cup_{j=1}^{\infty} E_j \in \mathcal{A}$.

Solution: The forward direction has nothing to show, since if $\mathcal{A}$ is a σ-algebra, then it's closed under countable unions, and in particular closed under increasing countable unions. Suppose now that $\mathcal{A}$ is an algebra that is closed under countable increasing unions, and let $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}$. We will define $F_k = \cup_{j=1}^{k} E_j$. Since $\mathcal{A}$ is an algebra, it is closed under finite unions, so each $F_k \in \mathcal{A}$. Moreover, $F_1 \subset F_2 \subset F_3 \subset \dots$, so by our hypothesis, their union is in the algebra, that is, $\cup_{k=1}^{\infty} F_k \in \mathcal{A}$. However, note that $\cup_{k=1}^{\infty} F_k = \cup_{j=1}^{\infty} E_j$, so we have that $\cup_{j=1}^{\infty} E_j \in \mathcal{A}$, which shows that it is closed under countable unions. Thus $\mathcal{A}$ is indeed a σ-algebra.