Group Theory and Origami Modular Design

Presented by
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The talk is part of Quantum Formalism's Group Theory Crash Course.

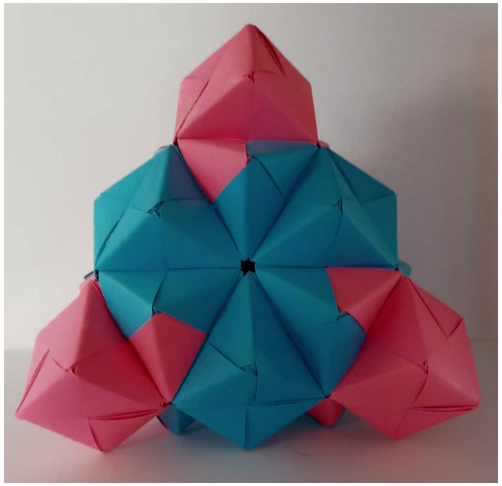
Justification

In our Quantum Formalism Group Theory Course we have learned about group theory in the abstract. This talk discusses how to use group theory to build origami objects that can be held in your hand. We go from abstraction to getting our hands dirty.

Much of the knowledge in this talk comes from mineralogy and crystallography. Besides origami, it has applications in chemistry, metallurgy, materials science, and condensed matter physics.

What I Do

+ **Group Theory** =

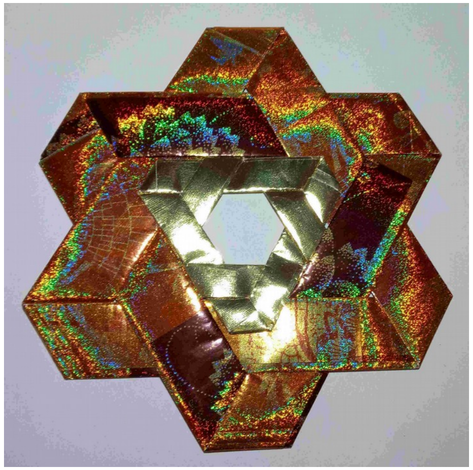


or

or

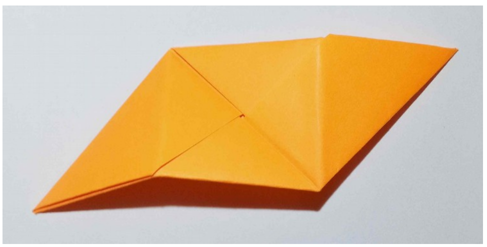
Modular Origami

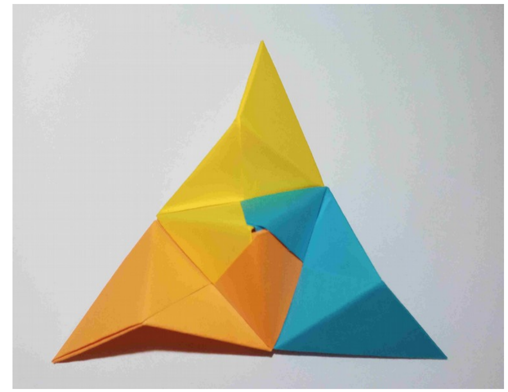
Modular origami is the craft of folding paper into individual units, the units are then combined into a finished model. The units lock together via flaps and pockets, requiring no adhesive to create the finished model.



Superunits

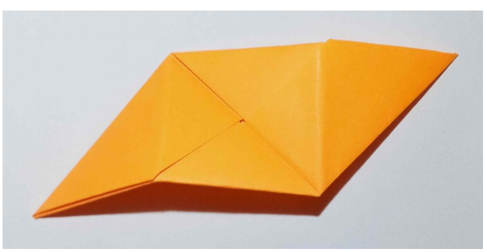
3 Sonobe units

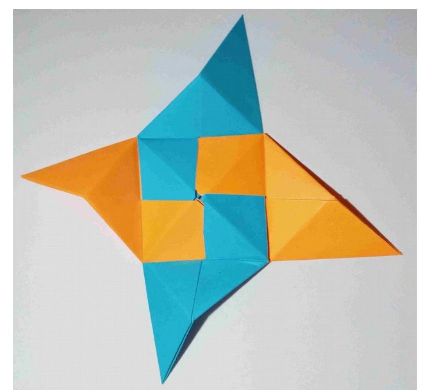




Triangular superunit has 3-fold symmetry

4 Sonobe units

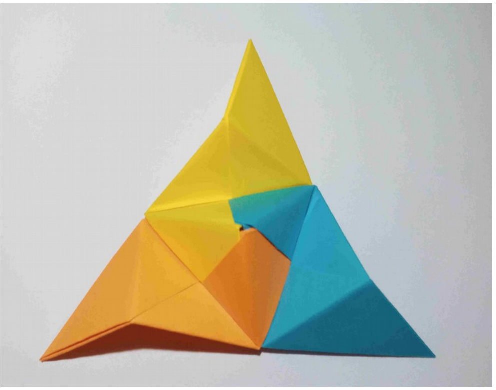


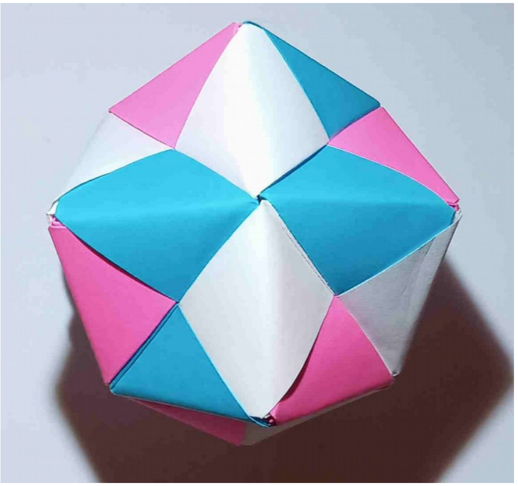


Square superunit has 4-fold symmetry

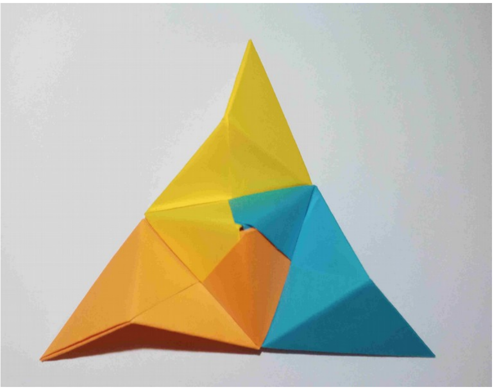
Combining Superunits

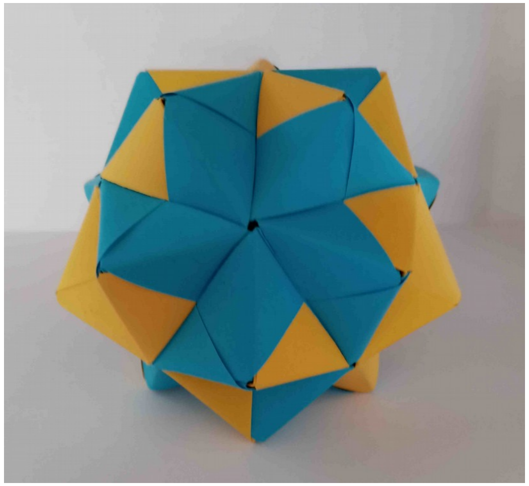
8 triangular superunits





20 triangular superunits

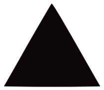




Symmetry Operators

For our discussions we will primarily deal with 3 types of symmetry operators:

- Rotation axis - rotation can be any integer > 1:

 2-fold axis  3-fold axis  4-fold axis

- Mirror plane - in 2d this operator is often called a line of reflection or a flip.

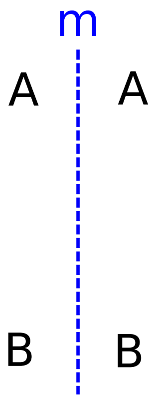
m -------------

- center of inversion - represented by the symbol i.

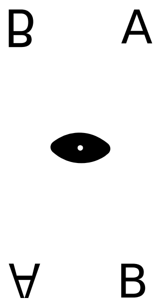
We will not be considering other symmetry elements such as translation, glide planes, screw axis, etc.

Symmetry Examples

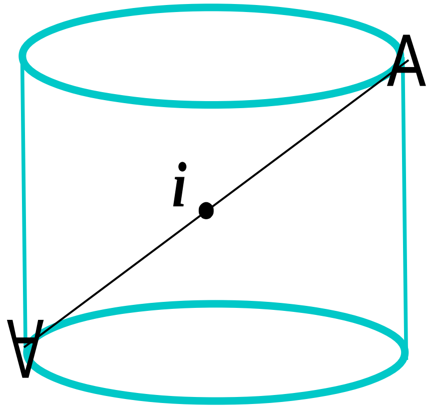
mirror



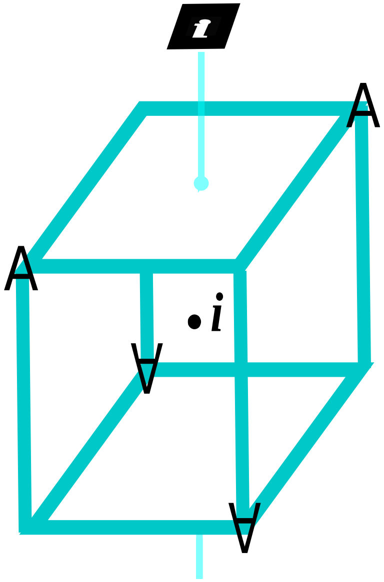
2-fold rotation



Center of Inversion



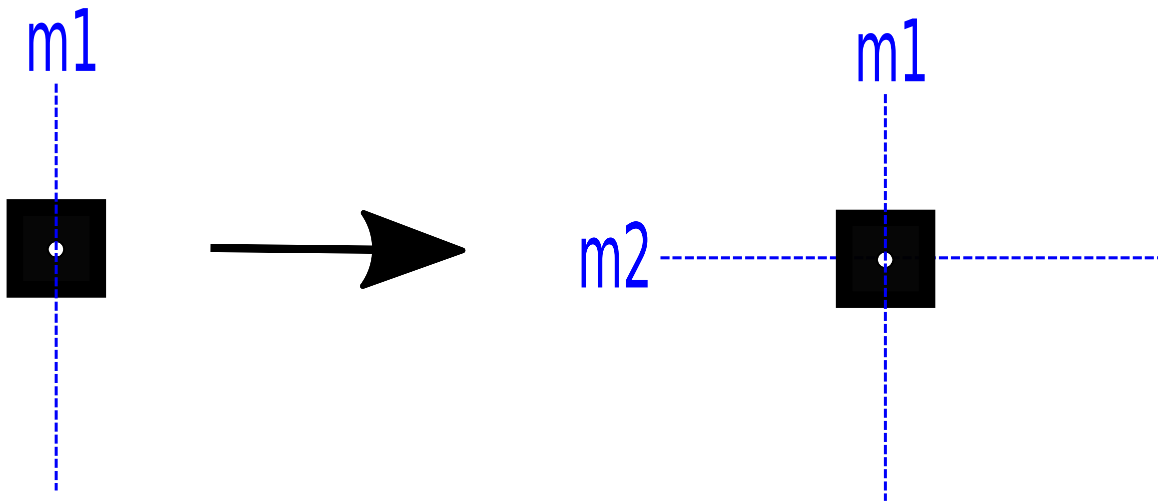
4-fold rotation + ***i***



Symmetry Combinations

Geometrically, not all symmetry operators are compatible Sometimes extra operators are needed to make a combination work!

For example if we have a 4 fold rotation axis and a mirror plane, another mirror plane is created/needed to make the combination feasible.

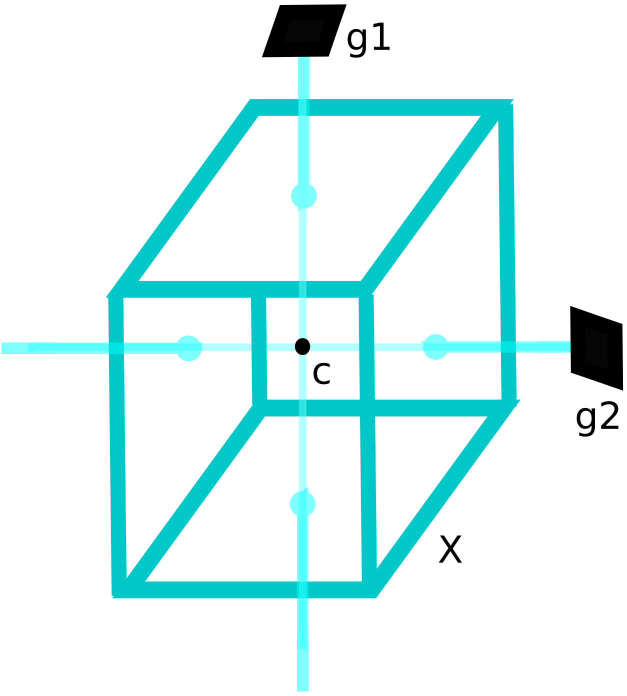


Note: m1 ○ rot(90°) = m2, this is the geometric equivalent of group closure. Or more broadly, **group closure is a requirement for geometric feasibility**. It provides a link between a group and a geometry.
The above is called 4mm, C_{4V} , or the dihedral group of order 8.

Definition of a Symmetry Group

For our purpose define a 3d symmetry group as {X, G, c }

- X = {set of all points in a closed polyhedral surface centered on point c}
- G = {set of symmetry operators operating on X}
- c is the center point of polyhedral X, c ∉ X



note: every mirror plane contains c, all rotation axis pass through c, and c is the center of any inversion ***i*** in G.

c is also invariant under any symmetry operation, i.e. for any g ∈ G g(c) = c.

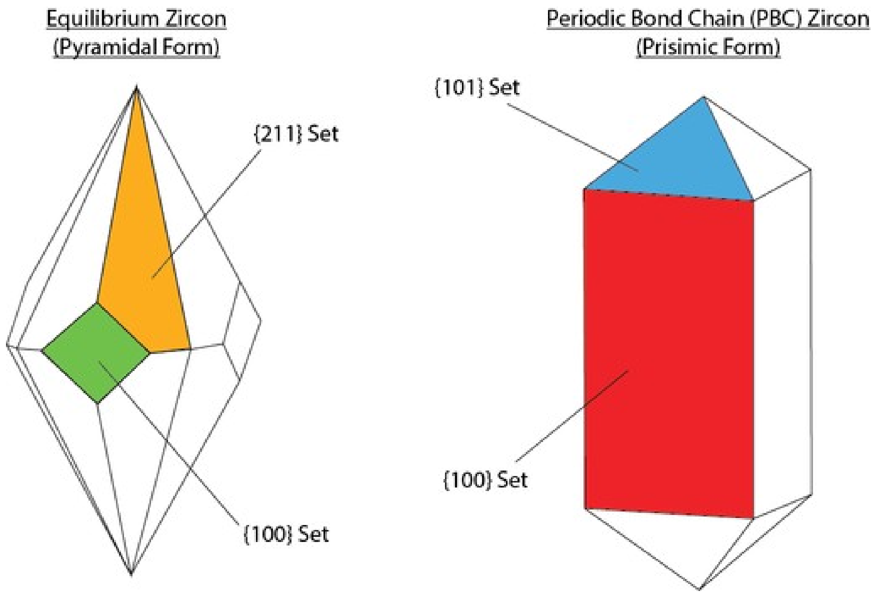
Method

My method for using groups to build origami modular models is as follows:

1) Pick a basic module. Study its properties, and determine what superunits can be made using it. Examine the symmetry elements of the module and superunits.
2) Pick a symmetry group which has symmetry operators similar to the superunits. Fold a polyhedron from the symmetry group. Use the group's symmetry to choose attractive color patterns.
3) Vary the polyhedron using dual and truncation techniques and come up with new polyhedra which have the same group symmetry.
4) Vary the polyhedron using modifications which change/break the group symmetry. You now have a new symmetry group to use as a guide.
5) Buy more paper and return to step 2 with your new group.

Method (continued)

One issue with the previous method is how to pick an appropriate symmetry group? What symmetry groups are available to choose from? Fortunately this problem was solved by mineralogists in the early 1800s!



Early Work in Symmetry Groups

Johann Friedrich Christian Hessel

(27 April 1796 – 3 June 1872)

In 1830 published the 32 crystal classes (symmetry groups) to which all crystals must belong. He discovered these naturally occurring groups using geometry and looking at a lot of rocks. Some of his crystal classes were not observed in minerals during his lifetime. He hypothesized their existence based on mathematics. He was right!

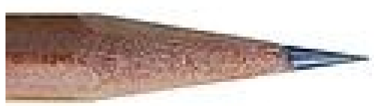
He published his work about the same time that Évariste Galois coined the term “group”. Hessel’s work was done independently of Galois.

Tools for Crystallography

Today crystallographers, mineralogists, and material scientists have all kinds of high tech tools. Computers, electron microscopes, x-ray diffraction equipment, etc.

In Hessel's day, high tech looked a little different...

High Tech Scientific devices in early 1800s



"I have seen a substance excellently adapted to the purpose of wiping from paper the mark of black-lead-pencil. ... It is sold by Mr. Nairne, Mathematical Instrument-Maker, opposite the Royal-Exchange." -- Joseph Priestley

32 Crystal Classes

Crystal Class	International Notation	Symmetry Elements	Schonflies
Triclinic System			
Pedial	1	none	C_1
Pinacoidal	$\bar{1}$	*i*	C_i (S_2)
Monoclinic System			
Sphenoidal	2	$1A_2$	C_2
Domatic	m	1m	C_s (C_{1h})
Prismatic	2/m	*i* $1A_2$ 1m	C_{2h}
Orthorhombic System			
Rhombic-disphenoidal	222	$3A_2$	D_2 (V)
Rhombic-pyramidal	2mm	$1A_2$ 2m	C_{2v}
Rhombic-dipyramidal	2/*m* 2/*m* 2/*m*	*i* $3A_2$ 3m	D_{2h} (V_h)

32 Crystal Classes (continued)

Crystal Class	International Notation	Symmetry Elements	Schonflies
Tetragonal System			
Tetragonal- pyramidal	4	$1A_4$	C_4
Tetragonal-disphenoidal	$\bar{4}$	$1\bar{A}_4$	S_4
Tetragonal-dipyramidal	4/m	i $1A_4$ m	C_{4h}
Tetragonal-trapezohedral	422	$1A_4$ $4A_2$	D_4
Ditetragonal-pyramidal	4mm	$1A_4$ 4m	C_{4V}
Tetragonal-scalenohedral	$\bar{4}2m$	$1\bar{A}_4$ $2A_2$ 2m	D_{2d} (V_d)
Ditetragonal-dipyramidal	4/m 2/m 2/m	i $1A_4$ $4A_2$ 5m	D_{4h}

32 Crystal Classes (continued)

Crystal Class	International Notation	Symmetry Elements	Schonflies
Isometric System			
Tetaroidal	23	$3A_2$ $4A_3$	T
Diploidal	$2/m\ \bar{3}$	$3A_2$ 3m $4\bar{A}_3$	T_h
Gyroidal	432	$3A_4$ $4A_3$ $6A_2$	O
Hextetrahedral	$\bar{4}3m$	$3\bar{A}_4$ $4A_3$ 6m	T_d
Hexoctahedral	$4/m\ \bar{3}\ 2/m$	$3A_4$ $4\bar{A}_3$ $6A_2$ 9m	O_h

32 Crystal Classes (continued)

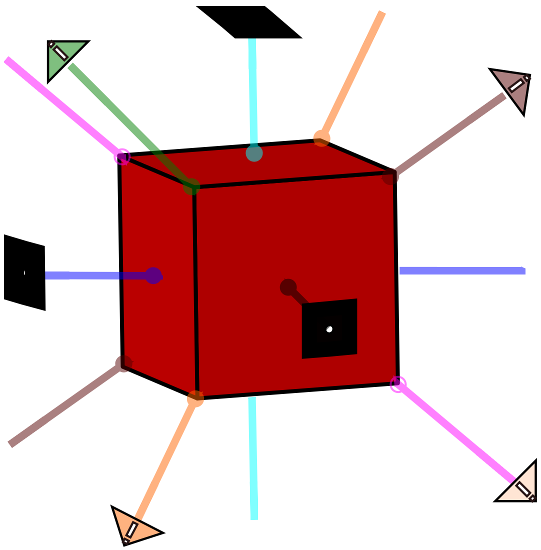
Crystal Class	International Notation	Symmetry Elements	Schonflies
Hexagonal System			
Trigonal subsystem			
Trigonal-pyramidal	3	$1A_3$	C_3
Rhombohedral	$\bar{3}$	$1\bar{A}_3$	C_{3i} (S6)
Trigonal-trapezohedral	32	$1A_3$ $3A_2$	D_3
Ditrigonal-pyramidal	3m	$1A_3$ 3m	C_{3v}
Hexagonal-scalenohedral	$\bar{3}$ 2/m	$1\bar{A}_3$ $3A_2$ 3m	D_{3d}

32 Crystal Classes (continued)

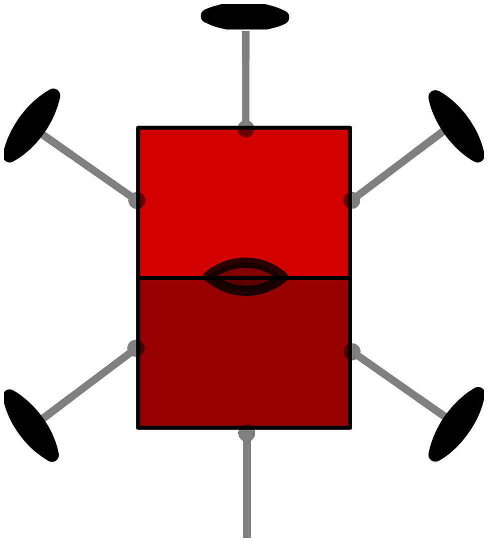
Crystal Class	International Notation	Symmetry Elements	Schonflies
Hexagonal System			
Hexagonal-pyramidal	6	$1A_6$	C_6
Trigonal-dipyramidal	$\bar{6}$	$1\bar{A}_6$	C_{3h}
Hexagonal-dipyramidal	6/m	$i\ 1A_6\ 1m$	C_{6h}
Hexagonal-trapezohedral	622	$1A_6\ 6A_2$	D_6
Dihexagonal-pyramidal	6mm	$1A_6\ 6m$	C_{6v}
Ditrigonal-dipyramidal	$\bar{6}m2$	$1\bar{A}_6\ 3A_2\ 3m$	D_{3h}
Dihexagonal-dipyramidal	6/m 2/m 2/m	$i\ 1A_6\ 6A_2\ 7m$	D_{6h}

Cube Example

As an example we will use the hexoctahedral group ($4/m\ \bar{3}\ 2/m$). A cube is a representative of this group. This group has symmetry elements $3A_4\ 4\bar{A}_3\ 6A_2\ 9m$.



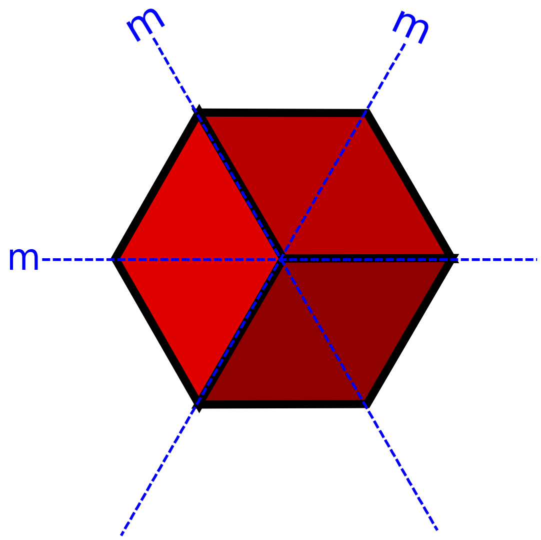
3-fold and 4-fold axis



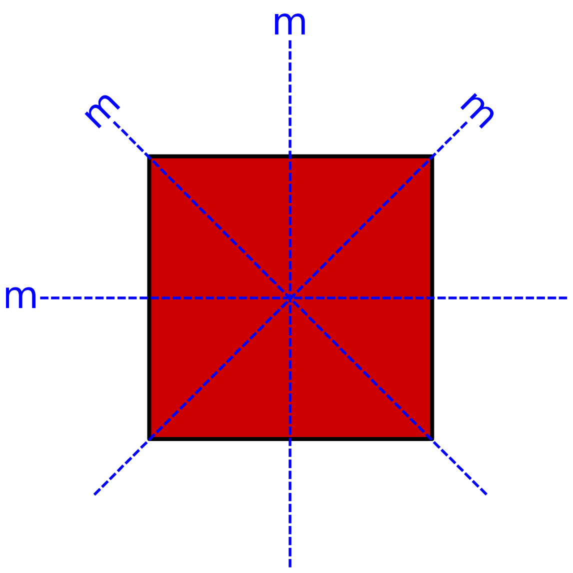
cube on edge with 2-fold axis

Cube Mirror Planes

Besides the rotation axis, the cube also has 9 mirror planes.



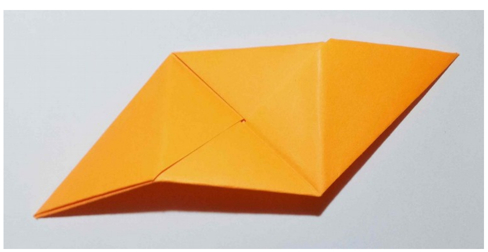
Mirror Planes passing through cube vertex.



Mirror Planes passing through cube face.

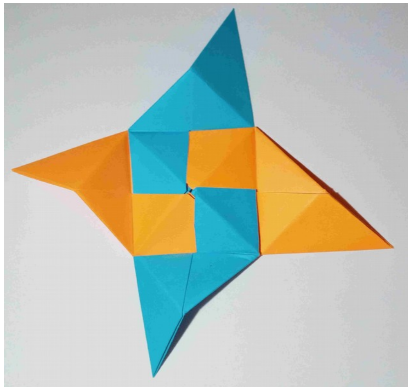
Build a Cube 3 Ways

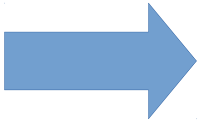
First we will build a cube using the 2-fold axis. Notice the edges of a cube all contain 2-fold axis. Also notice the individual module has 2-fold axis symmetry. If we use the module to build a cube along the 2-fold axis (or edges), we get



Build a Cube 3 Ways

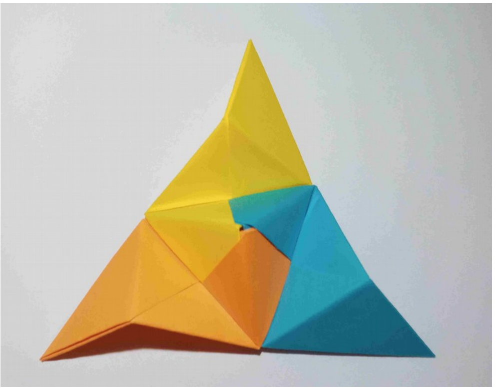
For our second cube, we will follow the 4-fold symmetry of a cube by building the six square faces from square superunits.

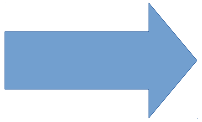




Build a Cube 3 Ways

Third, we will build a cube by replacing each of the vertices (3-fold axis) with triangular superunits.





Moving Beyond the Cubes

Now that we have used the group symmetry to build 3 cube models, what else can we build with this group?

Remembering our definition of a symmetry group: {X, G, c }

- X = {set of all points in a closed polyhedral surface centered on point c}
- G = {set of symmetry operators operating on X}
- c is the center point of polyhedral X, $c \notin X$

One approach is to transform the set X in a way that preserves the group symmetry G.

Taking the Dual

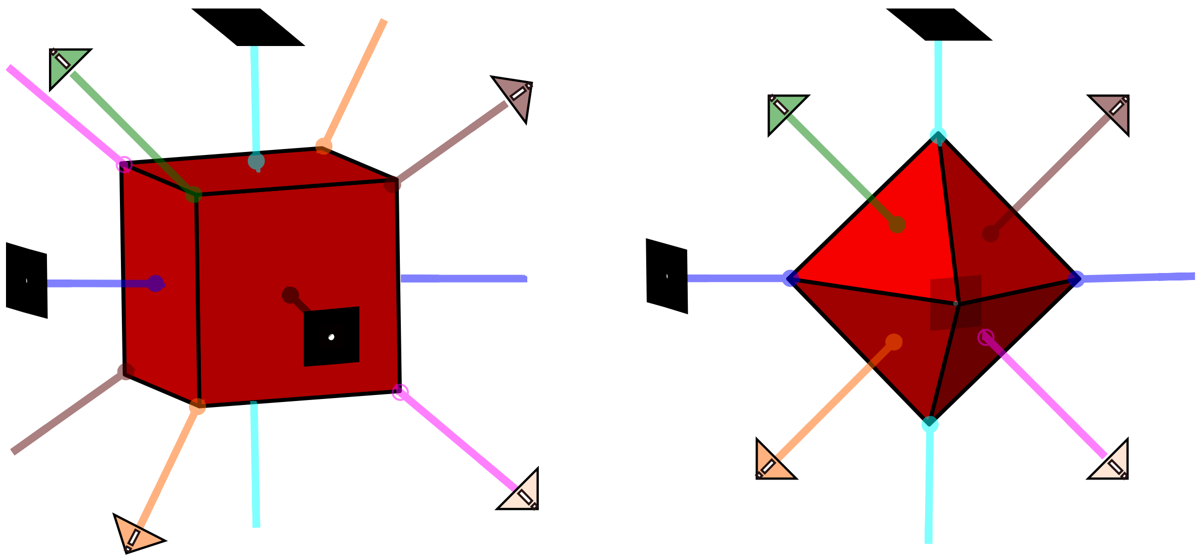
One way to modify set X while preserving the symmetry G is to take the dual of X. To take the dual of a polyhedron:

1) Replace every face of the polyhedron with a vertex. The number faces that intersect at the vertex should be the same as the number of sides on the faces being replaced.
2) Replace every vertex of the original polyhedra with a face having the same number of sides as faces intersecting at the vertex being replaced.

In the case of a cube every square face is replaced by a 4 way vertex, and every corner (3 way vertex) will be replaced by an equilateral triangle.

Dual of a Cube

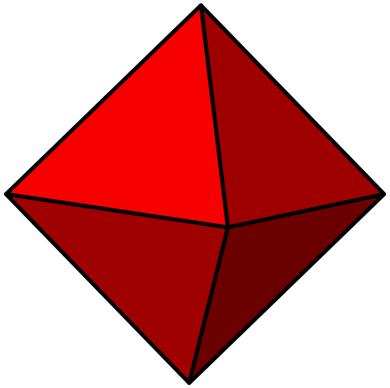
It turns out the dual of a cube is an octahedron.



The cube and the octahedron have the exact same symmetry!

Build an Octahedron

To build a model based on the octahedron, simply replace every triangular face on an octahederon with a triangular superunit.



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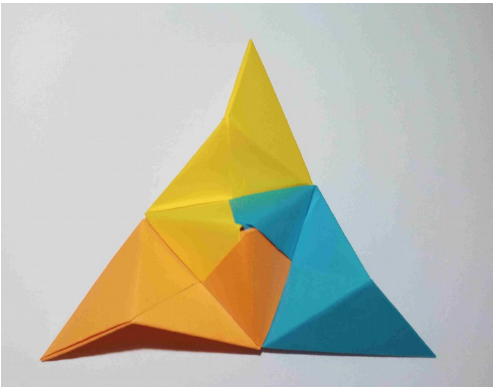
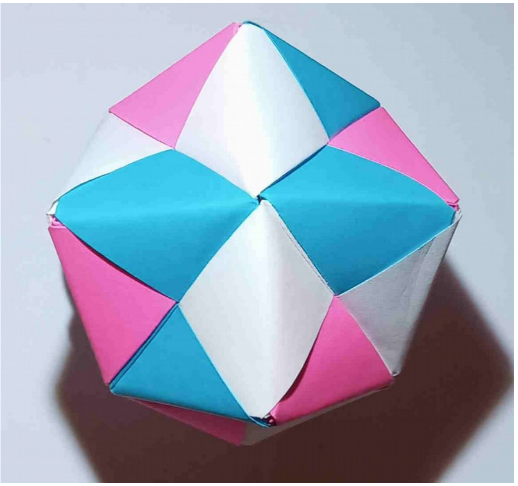
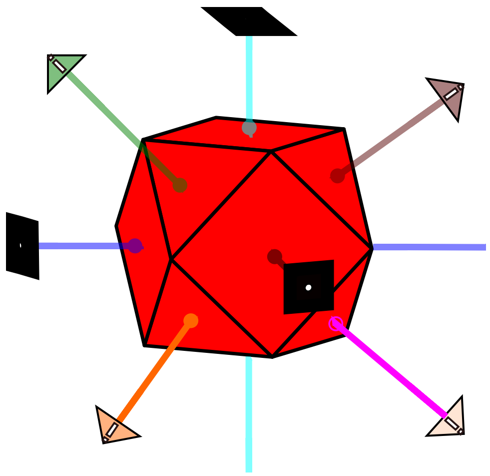


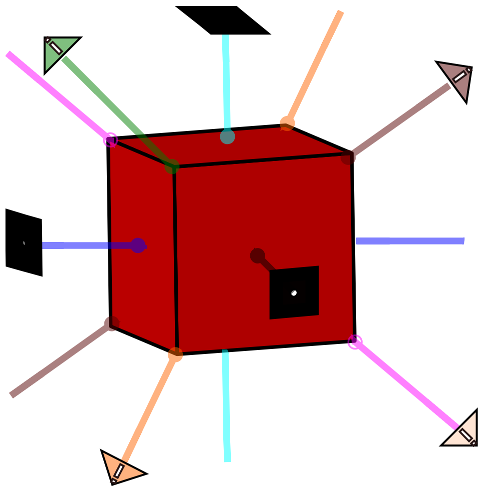


Truncation

Truncate a polyhedron’s vertex perpendicular to a symmetry axis is a symmetry preserving transformation. In the early 1600s Johannes Kepler pointed this out.

For our next shape we will truncate the corners of a cube. This gives a cuboctaheron.

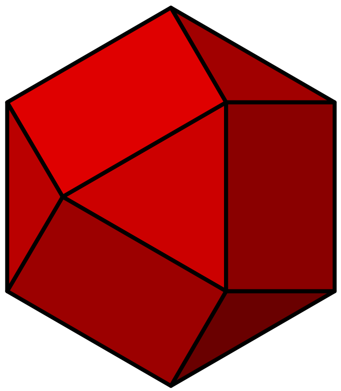




The cube and the cuboctahedron have the same symmetry!

Build a Cuboctahedron

Using the cuboctahedron as a template, add triangular and square superunits to create a new model.



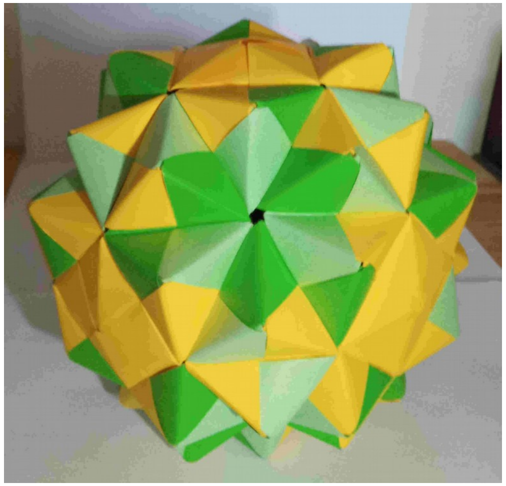
In this case I reversed the triangle superunit to point into the model.

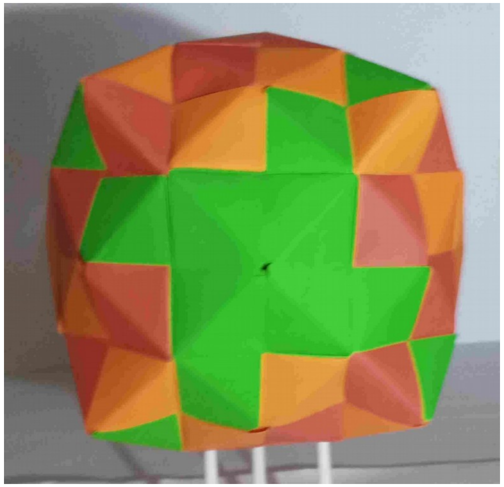
Transformations

The above symmetry preserving transformations provide templates for new models.

Other Truncations

These models were created from truncations of a cube shown in the previous video clip.



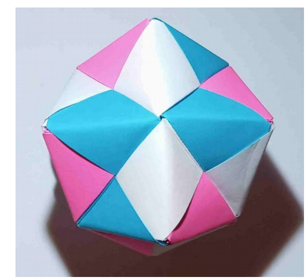


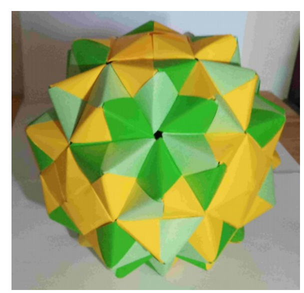
Both the above models have the same symmetry as a cube.

Summary of Models

Using our symmetry group definition {X, G, c} we have created models using only the group G = {$3A_4$ $4\bar{A}_3$ $6A_2$ 9m}. We kept G constant and changed the set X. Here's the result.

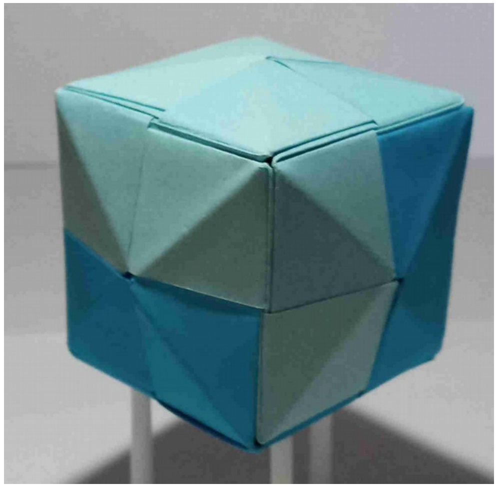


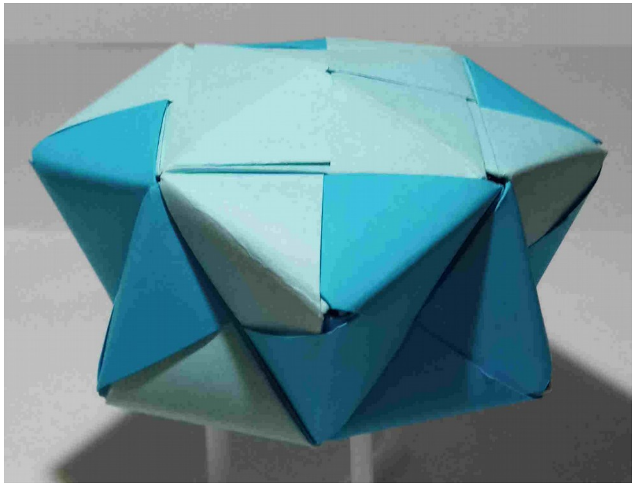



Transformations of Groups

Now let’s look at a transformation that changes the group symmetry (G) itself. Take a cube and twist the top square face a 1/8 turn around the vertical 4-fold axis.



Twist



The new shape retains it’s symmetry in the vertical direction, but all symmetry in the horizonal direction has been destroyed. We have transformed from $\{3A_4\ 4\bar{A}_3\ 6A_2\ 9m\}$ to $\{1A_4\ 4m\}$.

Snub Twist

The above video clip shows a snub twist on a cuboctahedron. All three 4 fold axis are twisted at once.

Snub Twist Model

The snub twist is another transformation which breaks group symmetry. In this transformation all 3 4-fold axis are given a 1/8 twist. The result keeps the 4-fold axis, 3-fold axis, and 2-fold axis, but eliminates all mirror planes.

Snub Twist

The symmetry goes from $\{3A_4\ 4\overline{A}_3\ 6A_2\ 9m\}$ to $\{3A_4\ 4A_3\ 6A_2\}$.

SIC Counting Theorems

One question which comes up is how many different color patters can I make in my modulars. This questions falls under the umbrella of counting Symmetry Independent Configurations (SIC). This problem is parallel to problems in chemistry, materials science, etc.

The solution is to use a series of theorems developed with group theory: Burnside’s lemma, Polya Enumeration Theorem, and deBruijns Theorem.

There is active research in condensed matter physics that uses these techniques, here’s an example

On the Use of Symmetry in Configurational Analysis for the Simulation of Disordered Solids by Mustapha, et. all. *Journal of Condensed Matter Physics*, Feb 2013.

Burnside's Lemma

Let G be a finite group that acts on the set X. Let X/G be the set of orbits of X (that is, each element of X/G is an orbit of X). For any element $g \in G$, let S(x) be the set of points of X which are fixed by g: $S(x) = \{x \in X \mid gx = x\}$. Then

$$|X/G| = (1/|G|) \sum_{g \in G} |S(x)|$$

where Orbit of $x = \{gx \mid g \in G\}$.

Note: Burnside's Lemma is also called the Cauchy-Frobenius-Burnside Lemma, and the Lemma Which Is Not Burnside's.

SIC Example

Let's look at the configuration of the 8 vertices of a cube. Let each vertex be either pointing in or out. Question how many Symmetry Independent Configurations (SIG) are possible? (To make it simplier, we will consider only rotations of a cube, no mirror planes, no inversions.)

If we let C be the set of all possible configurations of the 8 vertices, and c $\in$ C. Let G be group of all symmetry operators, G = $\{3A_4\ 4A_3\ 6A_2\}$. Since each SIC is an orbit, using Burnside

the number of SICs = $|C/G| = (1/|G|) \sum_{g \in G} |S(c)|$

Solution

Symmetry Operator	# of Operators	# of Stabilizers
e - 0 rotation	1	2^8
4-fold axis		
90	3	2^2
180	3	2^4
270	3	2^2
3-fold		
120	4	2^4
240	4	2^4
2-fold		
180	6	2^4
Total	24	552

Number of SICs = = 552/24 = 23

Thanks

QF QUANTUM FORMALISM

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