$\int_S f \, d\mu.$ QUANTUM FORMALISM

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Homework 11

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**Directions:** Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Let $A$ be some subset of a space $X$ with topology $\tau$. We say that $x$ is a **cluster point** or **limit point** of $A$ if for every open set $U$ containing $x$, $U \cap A \backslash \{x\} \neq \emptyset$. We denote the set of all cluster points of $A$ by $A'$. We denote the closure of $A$ to be $\overline{A} = A \cup A'$.

   a) Show by example that $A'$ need not be a subset of $A$.

   b) Suppose $\{a_n\}$ is a sequence of points such that for all $n$, $a_n \in A$ and $\{a_n\}$ converges to $x$. Prove that $x \in \overline{A}$.
   **Hint:** If $x \in A$, we're done. So suppose $x \notin A$, and show $x \in A'$.

   c) Suppose we know that $X$ is a metric space with $\tau$ a metric topology given by some metric $d$. Suppose $x \in \overline{A}$. Prove there exists a sequence of points in $A$ that converges to $x$.

   **Hint:** If $x \in A$ the constant sequence $a_n = x$ works. So suppose $x \in A' \backslash A$, and use the fact that we have a metric to construct a sequence of points in $A$ converging to $x$.

   **Remark:** If we have that $X$ is metric space, parts (b) and (c) form an if and only if statement commonly referred to as **The Sequence Lemma**.

2. We say as topological space $X$ is Hausdorff if given any $x, y \in X$ with $x \neq y$, there exists open sets $U$ and $V$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

   a) Show by example that not every space is a Hausdorff space.

   b) Prove every metric space is Hausdorff.

   **Remark:** The reverse implication is not true, a Hausdorff space need not be a metric space. It's above the scope of our crash course, but $\mathbb{R}_l$ can be shown to be Hausdorff and not metrizable (i.e., no metric can be defined on it).

   c) Prove that limits are unique in Hausdorff spaces. That is, if $X$ is a Hausdorff space, $\{x_n\}$ a sequence in $X$, $x_n \to y$ and $x_n \to z$, then $y = z$.

   **Hint:** Suppose $y \neq z$, then using the definition of Hausdorff will lead to a contradiction.

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