$$\int_S f \, \mathrm{d}\mu.$$
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Homework 7

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Suppose $\{f_n\}$ is a sequence of non-negative decreasing integrable functions such that they converge to some f for every x. Prove that

$$\lim_{n\to\infty} \int f_n = \int f.$$

Solution: We will define a new sequence of functions $g_n := f_1 - f_n$, which are non-negative, and increasing to $g = f_1 - f$. Then we can simply apply Monotone Convergence Theorem to get that

$$\lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \int f_1 - f_n = \int f_1 - f = \int g.$$

However from the middle equality we get

$$\lim_{n\to\infty} \int f_1 - \lim_{n\to\infty} \int f_n = \int f_1 - \int f \implies \int f_1 - \lim_{n\to\infty} \int f_n = \int f_1 - \int f \implies$$
$$\lim_{n\to\infty} \int f_n = \int f,$$

as desired.

2. Suppose that f_n, g_n, f, and g are all integrable functions with $f_n \to f$, $g_n \to g$, $|f_n| \leq g_n$ for all n, and $\int g_n \to \int g$. Prove that $\int f_n \to \int f$.

Remark: This is often refered to as the "Generalized Dominated Convergence Theorem".

Solution: We follow along the same lines as the proof of the DCT version we proved in lecture. Note that since $|f_n| \leq g_n$ for all n, this means that $-f_n \leq g_n \implies f_n + g_n \geq 0$ for all n. We now have

$$\int f + \int g = \int f + g = \int \liminf (f_n + g_n) \leq \liminf \int f_n + g_n =$$
$$\liminf \int f_n + \liminf \int g_n = \liminf \int f_n + \int g$$

where we used Fatou's Lemma to pull the lim inf out of the integral. Thus subtracting the integral of g from each side we get

$$\int f \leq \liminf \int f_n.$$

Attacking from the other side using the fact that $g_n - f_n \geq 0$ for all n, we will get that

$$\int f \geq \limsup \int f_n \implies \limsup \int f_n \leq \int f \leq \liminf \int f_n \implies \lim \int f_n = \int f.$$

3. For the following integrals, prove the limit exists, then evaluate.

   a)
$$\lim_{n\to\infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n)\, dx.$$

   b) Given that $g(x)$ is a non-negative integrable function, and $f(x)$ is measurable, bounded, and continuous at 1,
$$\lim_{n\to\infty} \int_{-n}^{n} f\left(1 + \frac{x}{n^2}\right) g(x)\, dx.$$

Solution:

(a) We will use the generalized Dominated Convergence Theorem proven in problem (2) above. We note that

$$|f_n| = |(1 + (x/n))^{-n} \sin(x/n)| \leq (1 + (x/n))^{-n} = g_n.$$

And we see that $g_n \to e^{-x}$ as $n \to \infty$. Moreover a quick computation shows that

$$\int_0^\infty g_n\, dx = \int_0^\infty (1 + (x/n))^{-n}\, dx = \int_1^\infty nu^{-n}\, du = \frac{n}{n-1}.$$

and

$$\int_0^\infty g = \int_0^\infty e^{-x}\, dx = 1,$$

thus

$$\int g_n = \frac{n}{n-1} \to 1 = \int g.$$

Thus we can apply the result of problem (2), noting that $f_n \to e^{-x}\sin(0) = 0 = f$, thus

$$\int f_n = \int f = 0.$$

(b) Let $E_n = [-n, n]$ and we will rewrite the integral as follows,

$$\lim_{n\to\infty}\int_{\mathbb{R}} f\left(1+\frac{x}{n^2}\right) g(x)\chi_{E_n}\ dx.$$

Since f is bounded, $|f| \leq M$ for some $M > 0$. We set $f_n(x) = f\left(1+\frac{x}{n^2}\right) g(x)\chi_{E_n}$, then note that $|f_n(x)| \leq Mg(x)$ and $f_n \to f(1)g(x)\chi_{\mathbb{R}} = f(1)g(x)$ as $n \to \infty$ (this is where we invoke the continuity of f at 1). We can apply D.C.T. to get that

$$\lim_{n\to\infty}\int_{\mathbb{R}} f\left(1+\frac{x}{n^2}\right) g(x)\chi_{E_n}\ dx. = \int f(1)g(x)\ dx = f(1)\int g(x)\ dx.$$

4. Give an example of a sequence of non-negative functions f_n such that $f_n \to 0$ pointwise, $\int f_n \to 0$, but there is no integrable $g(x)$ such that $f_n \leq g$ for all n.

Solution: We will define f_n as follows

$$f_n(x) = \begin{cases} 4n^3x - 4n^4 & x \in [n, n+1/(2n^2)] \\ 4n^4 + 4n - 4n^3x & x \in (n+1/(2n^2), n+1/n^2] \\ 0 & x \notin [n, n1/n^2] \end{cases}$$

Notice that for all x, there is some $N > 0$ such that $x < N$, thus $f_N(x) = 0$ for all $n \geq N$, so $f_n \to 0$ pointwise. Moreover, the integral of f is just the area of the triangle, so we have

$$\lim_{n\to\infty}\int f_n = \lim_{n\to\infty}\frac{1}{n} = 0.$$

However, for any g such that $g \geq f_n$ for all n, then g must contain at least all of the triangles, meaning

$$\int g = \sum_{n=1}^{\infty}\frac{1}{n} = \infty,$$

so no such integrable g exists.