QF Group Theory CC2022
By
Zaiku Group

Lecture 04

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Friday, 8/04/2022

Session Agenda

1. Learning Journey Timeline
2. Course Approach Overview
3. Mini Schools Series Idea

Pre-session Comments

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1. Inverse Semigroup
2. Identity Element
3. Identity Element Extension
4. Monoid Structure
5. Monoid Homomorphism Kernel

Main Session

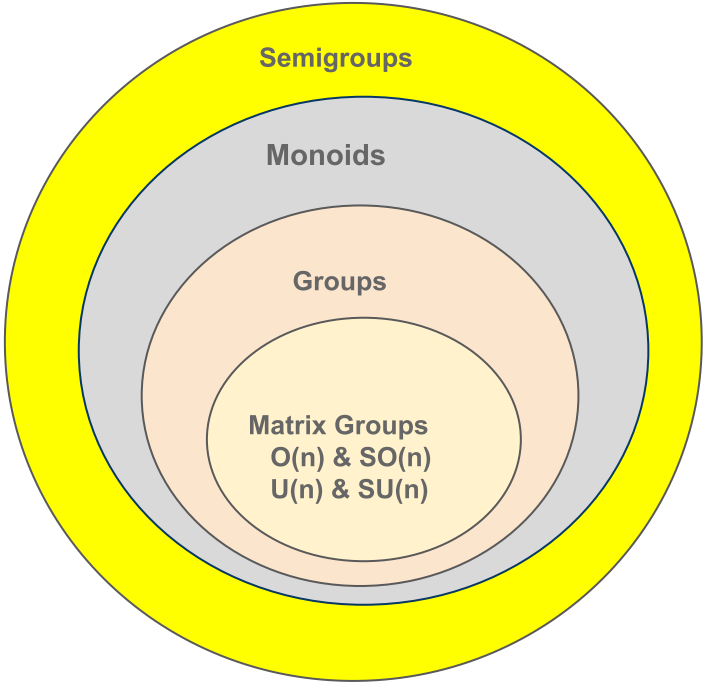
Learning Journey Timeline

Topology & Differential Geometry
9

Group Theory
4/14?

Lie Groups & Rep.
10 - 16?

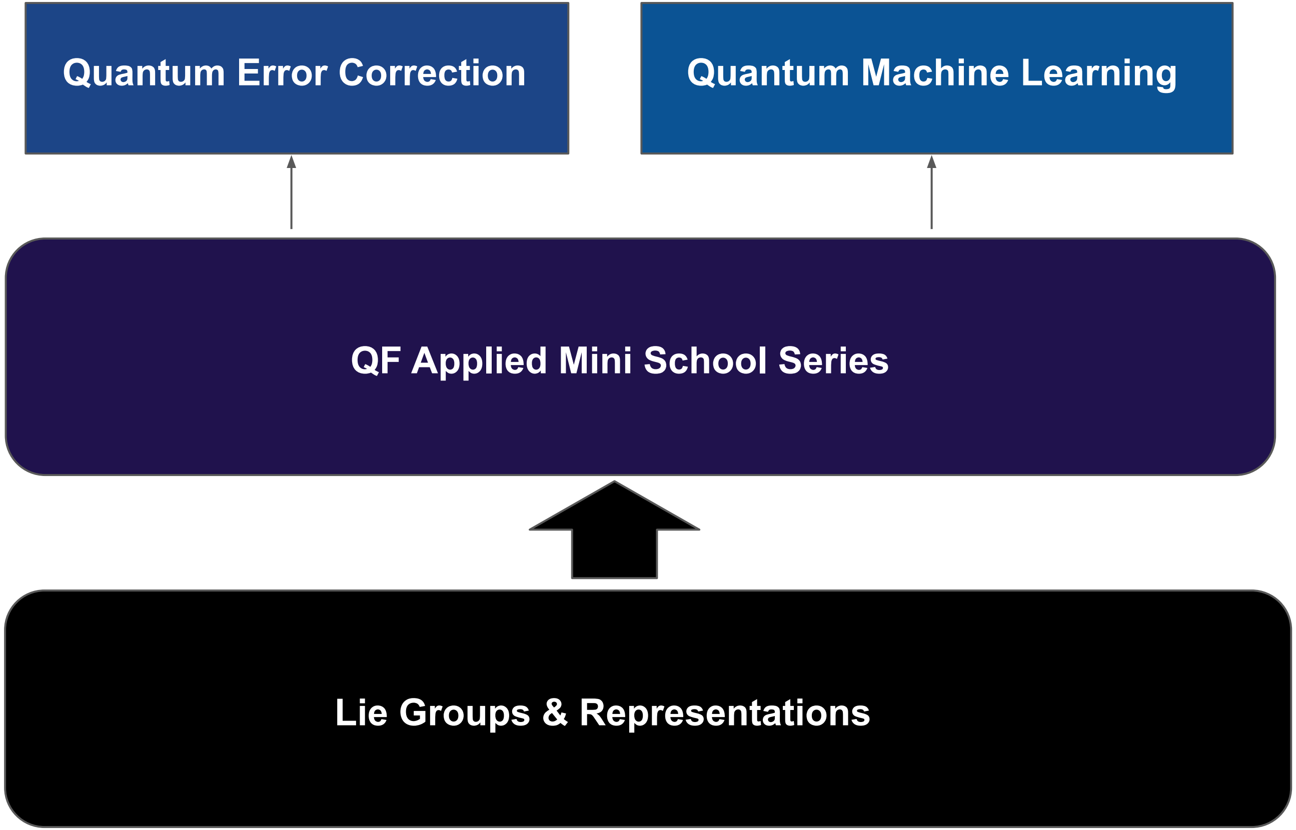
Completed | Ongoing | TBC (summer) | n is the number of live lectures |



Course Approach Overview

We’re here!

Applied QF Initiatives



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Inverse Semigroup

Definition 1.0

Let $(S, *)$ be a semigroup and $x \in S$. Then x is said to be invertible if there exists some $\tilde{x} \in S$ such that $x * \tilde{x} * x = x$ and $\tilde{x} * x * \tilde{x} = \tilde{x}$.

- The element $\tilde{x}$ is as you can guess is called an inverse for x!

Definition 1.1

A semigroup $(S, *)$ is called 'inverse semigroup' if for every $x \in S$ there is a unique $\tilde{x} \in S$ such that $x * \tilde{x} * x = x$ and $\tilde{x} * x * \tilde{x} = \tilde{x}$.

- When dealing with inverse semigroups, the notation x^{-1} is used to denote the inverse of $x \in S$ instead of $\tilde{x}$!

Homework Challenge 1

Let $(S_1, *_1)$ and $(S_2, *_2)$ be inverse semigroups. Now suppose that a map $f : S_1 \longrightarrow S_2$ is a homomorphism.

- Is it true that $f(x^{-1}) \in S_2$ is the inverse of $f(x) \in S_2$ for all $x \in S_1$?

Semigroup Identity Element

Definition 1.2

Let $(S, *)$ be a semigroup. An element $\boldsymbol{e} \in S$ is called:

1. A left identity if $\boldsymbol{e} * x = x$ for all $x \in S$.
2. A right identity if $x * \boldsymbol{e} = x$ for all $x \in S$.
3. A two sided identity if $\boldsymbol{e} * x = x * \boldsymbol{e} = x$ for all $x \in S$.

- For our purposes, we are only interested in semigroups with two sided identity elements!

Spoiler alert: A semigroup with a two sided identity is called a monoid!

Homework Challenge 2

Let $(S, *)$ be a semigroup with a two sided identity element $\boldsymbol{e} \in S$.

- Is it true that $\boldsymbol{e}$ is unique i.e. if $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are two sided elements then $\boldsymbol{e}_1 = \boldsymbol{e}_2$?

Homework Challenge 3

Let $(S_1, *_1)$ and $(S_2, *_2)$ be semigroups with two sided identity elements $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ respectively. Now suppose that a map $f : S_1 \longrightarrow S_2$ is a homomorphism.

- Is it true that $f(\boldsymbol{e}_1) = \boldsymbol{e}_2$?

Identity Element Examples

- Let $(S, *) = (\mathbb{R}, \times)$. Then 1 is an identity element right? Is it two sided identity?

- Let $\mathbb{R}^*$ denote the set of nonzero reals i.e $\mathbb{R}^*$ is the set of reals excluding zero. We can construct a binary operation $*$ on $\mathbb{R}^*$ as $a * b = |a|b$ for all $a, b \in \mathbb{R}^*$ where $|.|$ denotes the absolute value of reals.

    1. $(\mathbb{R}^*, *)$ forms a semigroup right?
    2. It's clear that 1 is a left identity? What about -1?
    3. Does $(\mathbb{R}^*, *)$ contain a right identity?

- Is $(\mathbb{R}^*, *)$ as constructed above an abelian semigroup?

Question: What if a semigroup doesn't have any identity? Can we invent one?!

Adding an Identity to a Semigroup

Definition 1.3

Let $(S, *)$ be a semigroup without an identity. We first define the set $S^1 = S \cup \{\mathbf{1}\}$. Then we can construct a binary operation $\hat{*}$ on S^1 as follows:

1. $a \hat{*} b = a * b$ for all $a, b \in S$.
2. $x \hat{*} \mathbf{1} = \mathbf{1} \hat{*} x = x$ for all $x \in S^1$.

- With $\hat{*}$ define above, $(S^1, \hat{*})$ forms a semigroup structure with a two-sided identity $\mathbf{1}$.

Monoid Structure

Definition 1.4

A monoid is a triple $(M, *, \boldsymbol{e})$ such that $(M, *)$ is a semigroup and $\boldsymbol{e} \in M$ is a two-sided identity in the semigroup $(M, *)$.

Question: Is $\boldsymbol{e}$ in a monoid unique i.e. if $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$ are two-sided identities then $\boldsymbol{e} = \tilde{\boldsymbol{e}}$?

Definition 1.5

Let $(M, *, \boldsymbol{e})$ be a monoid and $N \subseteq M$. If $(N, *, \boldsymbol{e}_N)$ is a monoid then we call it a submonoid.

- Is it true that we must have $\boldsymbol{e}_N = \boldsymbol{e}$?

Homework Challenge 4

Let $(M_1, *_1, \boldsymbol{e}_1)$ and $(M_2, *_2, \boldsymbol{e}_2)$ be monoids. Now let $\phi : M_1 \longrightarrow M_2$ be a homomorphism.

- Is it true that we must have $\phi(\boldsymbol{e}_1) = \boldsymbol{e}_2$?

Monoid Homomorphism Kernel

Definition 1.5

Let $(M_1, *_1, \boldsymbol{e}_1)$ and $(M_2, *_2, \boldsymbol{e}_2)$ be monoids. Now let $\phi : M_1 \longrightarrow M_2$ be a homomorphism. The set $ker(\phi) = \{x \in M_1 \mid \phi(x) = \boldsymbol{e}_2\}$ is called the kernel of the homomorphism ϕ.

- Obviously $ker(\phi)$ cannot be empty right?

Homework Challenge 5

Let $(M_1, *_1, \boldsymbol{e}_1)$ and $(M_2, *_2, \boldsymbol{e}_2)$ be monoids. Now let $\phi : M_1 \longrightarrow M_2$ be a homomorphism.

1. Is it true that $ker(\phi)$ is a submonoid of $(M_1, *_1, \boldsymbol{e}_1)$?
2. Is it true that ϕ is an isomorphism iff $ker(\phi) = \{\boldsymbol{e}_1\}$?

QUANTUM FORMALISM

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