QF Group Theory CC2022
By
Zaiku Group

Lecture 13

Delivered by Bambordé Baldé

Friday, 26/8/2022

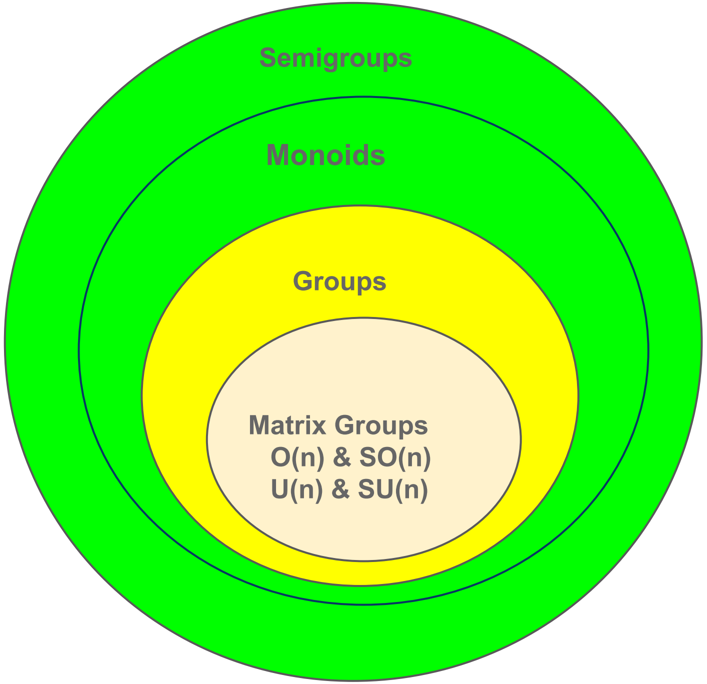
Learning Journey Timeline

Topology & Differential Geometry
9

Group Theory
13/20

Lie Groups & Rep.
10 - 16?

Completed | Ongoing | TBC (summer) | n is the number of live lectures |



Course Approach Overview

Completed!

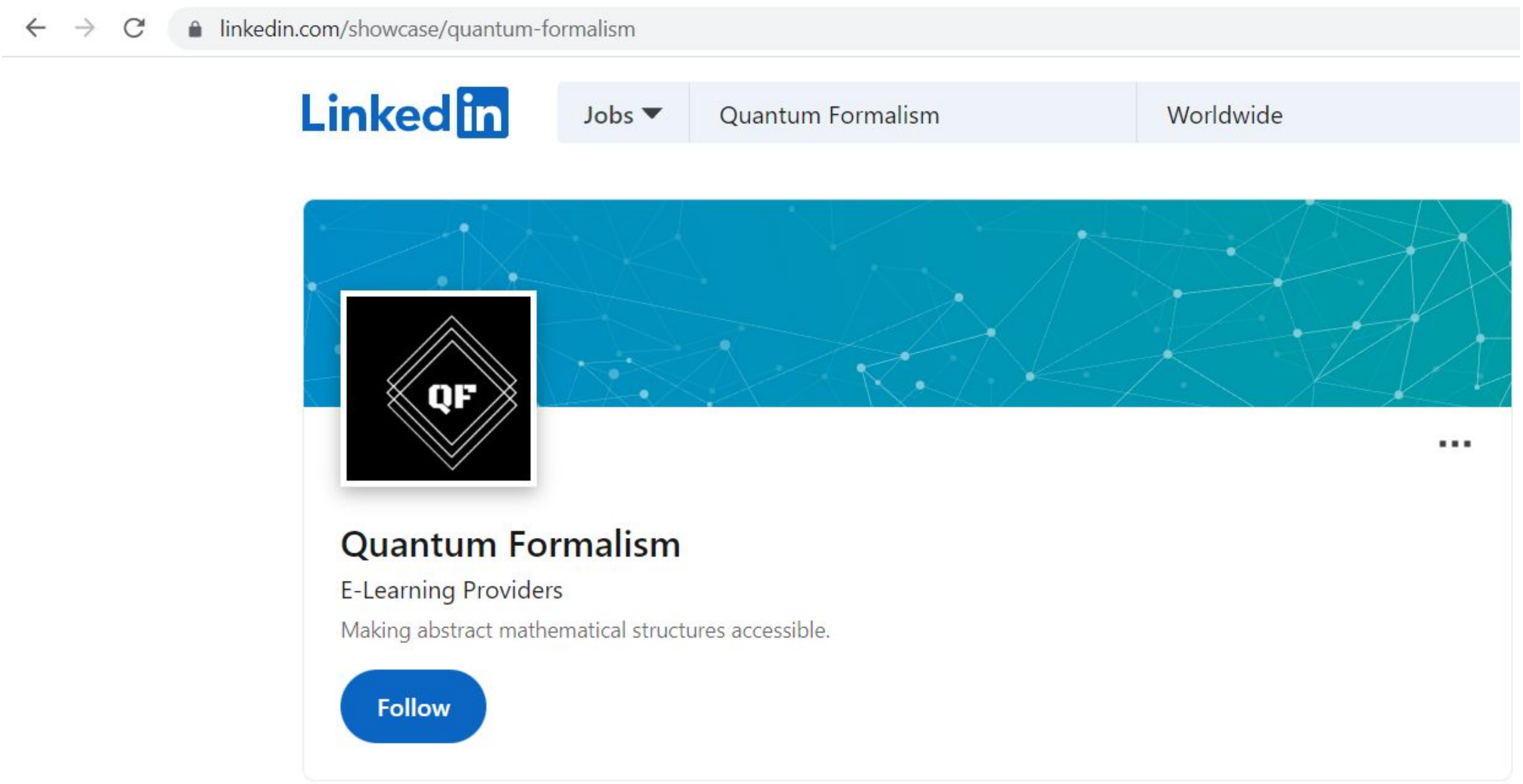
We're here!

Measure Theory & Functional Analysis

$$\int_S f \,\mathrm{d}\mu.$$

Starts on August 29, 2022.

Upcoming Course!


linkedin.com/showcase/quantum-formalism
Linked in
Jobs
Quantum Formalism
Worldwide
QF
Quantum Formalism
E-Learning Providers
Making abstract mathematical structures accessible.
Follow

Orbits under Permutations

Definition 1.0

Let $\sigma \in S_n$ be an arbitrary permutation and $i \in \{1, \ldots, n\}$. The orbit of i under σ is defined as $Orb(i) = \{\sigma^r(i) | r \in \mathbb{Z}\}$.

- **Lecture 6 refresh:** Let G be a group under a binary operation $*$ and with identity e. Given a group element $g \in G$ and $r \in \mathbb{Z}$, the exponentiation operation is defined as:

$$g^r = \begin{cases} e \quad \text{if} \quad r = 0 \\ \underbrace{g * g * \ldots * g}_{\text{r times}} \quad \text{if} \quad r > 0 \\ \underbrace{g^{-1} * g^{-1} * \ldots * g^{-1}}_{\text{r times}} \quad \text{if} \quad r < 0 \end{cases}$$

- Recall that the exponentiation has the following properties:
  1. $g^{r_1} g^{r_2} = g^{(r_1 + r_2)}$ for all $g \in G$ and $r_1, r_2 \in \mathbb{Z}$.
  2. $(g^{r_1})^{r_2} = g^{r_1 r_2}$ for all $g \in G$ and $r_1, r_2 \in \mathbb{Z}$ (remember Diffie-Hellman?!!).

Facts about orbits under permutations (A)

Proposition 1.0

Let $\sigma \in S_n$ be an arbitrary permutation and $i \in \{1, \ldots, n\}$. Then there exists an integer $r \in \mathbb{Z}^+$ such that $\sigma^r(i) = i$.

Proof : Homework challenge? Try for at least $\sigma \in S_3$ and $i \in \{1, 2, 3\}$!

Proposition 1.1

Let $\sigma \in S_n$ be an arbitrary permutation and $i \in \{1, \ldots, n\}$. Now let $r \in \mathbb{Z}^+$ be the smallest integer such that $\sigma^r(i) = i$. Then $Orb(i) = \{i, \sigma(i), \ldots, \sigma^{r-1}(i)\}$ and its cardinality is $r - 1$ i.e. the elements of the orbit are all distinct.

Proof : Homework challenge? Try for at least $\sigma \in S_3$ and $i \in \{1, 2, 3\}$!

- The proposition above helps us computing orbits. Indeed, if we want to find $Orb(i)$ for a given σ, all we need to do is start listing $i, \sigma(i), \sigma^2(i), \ldots$ then we stop immediately when we get back to i. This list is guaranteed to be the complete orbit of i under σ.

Facts about orbits under permutations (B)

Definition 1.1

Let $\sigma \in S_n$ be an arbitrary permutation and $i, j \in X = \{1, \ldots, n\}.\ \}$. We can define a relation $\sim_\sigma$ on the set X as $i \sim_\sigma j$ if there exists an $r \in \mathbb{Z}$ such that $\sigma^r(i) = j$ i.e. if $j \in Orb(i)$.

- It's easy to see that $i \sim_\sigma i$ because $i \in Orb(i)$ right?
- Is $\sim_\sigma$ an equivalence relation on X?

Theorem 1.0

Let $\sigma \in S_n$ be an arbitrary permutation. Then σ can be written as product of disjoint cycles.

- The proof of the theorem above involves the use of orbits under σ. To express σ as a product of disjoint cycles, we can proceed as follows:
  1. Find all the orbits under σ.
  2. For each orbit, we get a cycle.
  3. Take the product of the cycles associated with each orbit, this should be the same as σ!

QUANTUM FORMALISM

GitHub: github.com/quantumformalism

YouTube: youtube.com/ZaikuGroup

Discord: discord.gg/SPcmcsXMD2

Twitter: twitter.com/ZaikuGroup

LinkedIn: linkedin.com/company/zaikugroup