Homework 11

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Let A be some subset of a space X with topology τ. We say that x is a **cluster point** or **limit point** of A if for every open set U containing x, $U \cap A \backslash \{x\} \neq \emptyset$. We denote the set of all cluster points of A by A'. We denote the closure of A to be $\overline{A} = A \cup A'$.

   a) Show by example that A' need not be a subset of A.

   Solution: Consider $A = (0, 1]$ with standard topology on $\mathbb{R}$ Every open set containing 0 intersects A, so $0 \in A'$ but not in A.

   b) Suppose $\{a_n\}$ is a sequence of points such that for all n, $a_n \in A$ and $\{a_n\}$ converges to x. Prove that $x \in \overline{A}$.
   Hint: If $x \in A$, we're done. So suppose $x \notin A$, and show $x \in A'$.

   Solution: If $x \in A$ we're done, so suppose $x \notin A$. Let U be some arbitrary open set containing x. Since $a_n \to x$, there is some N such that for all $n \geq N$, $a_n \in U$. Since $a_n \in A$ and $x \notin A$, $a_n \neq x$. Thus $a_n \in U \cap A \backslash \{x\}$. Since U was arbitrary, this shows that x is a cluster point of A.

   c) Suppose we know that X is a metric space with τ a metric topology given by some metric d. Suppose $x \in \overline{A}$. Prove there exists a sequence of points in A that converges to x.

   Hint: If $x \in A$ the constant sequence $a_n = x$ works. So suppose $x \in A' \backslash A$, and use the fact that we have a metric to construct a sequence of points in A converging to x.

   Remark: If we have that X is metric space, parts (b) and (c) form an if and only if statement commonly referred to as **The Sequence Lemma**.

   Solution: Again, suppose $x \in A' \backslash A$, as otherwise we're done by taking a constant sequence. Since every open set containing x must intersect A at some point other than x (since $x \in A'$), we can pick $a_n \in B_{1/n}(x) \cap A \backslash \{x\}$. But now given any U containing x, there is some $B_\epsilon(x) \subseteq U$, and picking $N > 1/\epsilon$, it follows that for all $n \geq N$, $a_n \in B_\epsilon(x) \subseteq U$, which means $a_n \to x$.

2. We say as topological space X is Hausdorff if given any $x, y \in X$ with $x \neq y$, there exists open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

a) Show by example that not every space is a Hausdorff space.

Solution: The indiscrete topology on $\mathbb{R}$ is an easy example, as the only open set containing 1 and 2 is $\mathbb{R}$.

b) Prove every metric space is Hausdorff.

Solution: Given $x, y \in X$ with $x \neq y$, and a metric d, we know that $d(x, y) > 0$. We will let $\epsilon = d(x, y)/4$. Then $B_\epsilon(x) \cap B_\epsilon(y) = \emptyset$, and thus we found two disjoint open sets that contain x and y respectively.

To fully justify the intersection is empty, suppose not:, i.e., suppose there is some $z \in B_\epsilon(x) \cap B_\epsilon(y)$. But then by the triangle inequality, we have that

$$d(x, y) \leq d(x, z) + d(z, y) < \epsilon + \epsilon = d(x, y)/2,$$

an obvious contradiction since $d(x, y) > 0$.

Remark: The reverse implication is not true, a Hausdorff space need not be a metric space. It's above the scope of our crash course, but $\mathbb{R}_l$ can be shown to be Hausdorff and not metrizable (i.e., no metric can be defined on it).

c) Prove that limits are unique in Hausdorff spaces. That is, if X is a Hausdorff space, $\{x_n\}$ a sequence in X, $x_n \to y$ and $x_n \to z$, then $y = z$.

Hint: Suppose $y \neq z$, then using the definition of Hausdorff will lead to a contradiction.

Solution: If $y \neq z$, then we can find open sets U, V such that $y \in U$, $z \in V$, and $U \cap V = \emptyset$. Since $x_n \to y$, there is some N_1 such that for all $n \geq N_1$, $x_n \in U$. Since $x_n \to z$, there is some N_2 such that for all $n \geq N_2$, $x_n \in V$. Thus if $N = \max\{N_1, N_2\}$, for all $n \geq N$, $x_n \in U \cap V$, which is a contradiction since their intersection is empty. Thus it must be the case that $y = z$, thus the limit is unique.