QF Group Theory CC2022
By
Zaiku Group

Lecture 16

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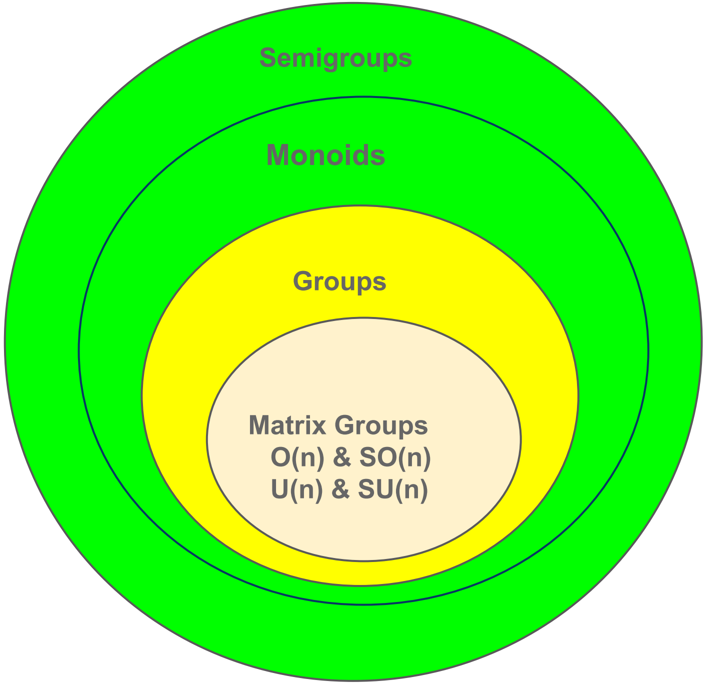
Learning Journey Timeline

Topology & Differential Geometry
9

Group Theory
16/20

Lie Groups & Rep.
10 - 16?

Completed | Ongoing | TBC (summer) | n is the number of live lectures |



Course Approach Overview

Completed!

We're here!

Direct product of groups

Definition 1.0

Let G_1 and G_2 be groups. The direct product of G_1 and G_2 is defined as $G_1 \times G_2 = \{(x, y) \in G_1 \times G_2 \mid x \in G_1, y \in G_2\}$ with the group operation on $G_1 \times G_2$ defined as $(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$ where obviously x_1x_2 is the group operation in G_1 and y_1y_2 is the group operation in G_2.

With the definition above, we can easily observe the following:

1. The pair $(1_{G_1}, 1_{G_2})$ is the group identity in $G_1 \times G_2$.
2. The group inverse of an element $(x, y) \in G_1 \times G_2$ is $(x, y)^{-1} = (x^{-1}, y^{-1})$.
3. The subsets $G_1' = \{(x, 1_{G_2}) \mid x \in G_1\} \subset G_1 \times G_2$ and $G_2' = \{(1_{G_1}, y) \mid y \in G_2\} \subset G_1 \times G_2$ are subgroups of $G_1 \times G_2$.
4. Also, we have two natural isomorphisms: $\phi_1 : G_1' \longrightarrow G_1$ that maps an element $(x, 1_{G_2}) \in G_1'$ to $x \in G_1$ and $\phi_2 : G_2' \longrightarrow G_2$ that maps an element $(1_{G_1}, y) \in G_2'$ to $y \in G_2$ i.e. $G_1' \simeq G_1$ and $G_2' \simeq G_2$

Toy examples

1. Let $G_1 = G_2 = \mathbb{Z}_2 = \{0, 1\}$ under *mod* 2 addition. Then $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ is a group.
2. Let $G_1 = \mathbb{Z}_2 = \{0, 1\}$ under *mod* 2 addition and $G_2 = \mathbb{Z}_3 = \{0, 1, 2\}$ under *mod* 3 addition. Then $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$ is also a group.

Direct product challenges

Challenge 1

Since $\mathbb{Z}_2$ and $\mathbb{Z}_3$ cyclic groups, are the direct products $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$ also cyclic groups?

- If $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$ are cyclic groups, then they must have at least a generator each!

Proposition 1.0

If G_1 and G_2 are abelian groups, then $G_1 \times G_2$ is also abelian.

Proof : Homework challenge!

- If either G_1 or G_2 is nonabelian, then $G_1 \times G_2$ is nonabelian right?
- What if G_1 and G_2 are nonabelian, is $G_1 \times G_2$ is nonabelian?

Challenge 2

Let $G_1 = \mathbb{Z}_2$ and $G_2 = S_3$. You're encouraged to construct the product $\mathbb{Z}_2 \times S_3$. Also, what is the order of $\mathbb{Z}_2 \times S_3$?

Interesting feature of direct products

Theorem 1.0

Let G be a group, H and K normal subgroups of G such that $H \cap K = 1$ and $HK = G$. Then $H \times K \simeq G$.

Proof: Homework challenge!

- What if we relax the condition above such that H is normal, but K is not? This is exactly what leads to the notion of semi-direct product that we'll define in the next session!

Group Actions Recap

- In the previous session we have seen that given a group G and set X, an action of G on X is the same as having group homomorphisms $\phi : G \longrightarrow Sym(X)$. More precisely, for each $g \in G$, we get a homomorphism $\phi_g : G \longrightarrow Sym(X)$ such that the following conditions hold:
  1. $\phi_{1_G}(x) = x$ for all $x \in X$.
  2. $\phi_g \circ \phi_{g'} = \phi_{gg'}$ for all $g, g' \in G$.
- For example, let $X = \{1, \ldots, n\}$ and $G = S_n$. Then S_n acts on X naturally as follows: Given a permutation $\sigma \in S_n$, we set $\phi_\sigma(i) = \sigma(i)$ for all $i \in X$. With this, we can verify the following:
  1. $\phi_{1_X}(x) = x$ for all $x \in X$.
  2. For $\sigma, \sigma' \in S_n$, we have $\phi_\sigma \circ \phi_{\sigma'}(x) = \phi_\sigma(\sigma'(x)) = \sigma(\sigma'(x)) = (\sigma\sigma')(x) = \phi_{\sigma\sigma'}$ for all $x \in X$
- We can also consider G acting on itself i.e. $X = G$. For example, we can define $\phi_g : G \longrightarrow G$ as $\phi_g(x) = gx$ for all $g, x \in G$.
- An interesting action of G on itself is the action by 'conjugation' $\phi_g : G \longrightarrow G$ defined as $\phi_g(x) = gxg^{-1}$ for all $g, x \in G$.

Important Notation Alert

When we write an action of G on a set X as gx, always try to insert in your mind that what we really mean is the homomorphism $\phi_g(x)$!

Group Stabilisers

Definition 1.1

Let G be a group acting on a set X. For an element $x \in X$, the stabiliser of x in G is defined as the set $Stab_G(x) = \{g \in G \mid gx = x\}$.

- So $Stab_G(x)$ is the set of group elements of G that leave x untouched under the $G-$ action on X.

Proposition 1.1

Let G be a group acting on a set X. Then $Stab_G(x) = \{g \in G \mid gx = x\}$ is a subgroup of G.

Proof : Homework challenge!

Challenge 3

Let G be a group acting on a set X. Is $Stab_G(x) = \{g \in G \mid gx = x\}$ a normal subgroup of G?

Quantum Computing Alert

- For the quantum computing folks, this is the basic notion behind the famous 'Stabilizer Formalism' in Quantum Error Correction!
- In the Stabilizer Formalism, the group $G = U(n)$ (the unitary group) and $X = \mathbb{C}^n$ which carries a complex Hilbert space structure. Also, for QC purposes, $n = 2^k$ where k is the number of qubits under consideration.
- We'll comeback to all this in the planned 'Quantum Error Correction School' after the course on Lie Groups!

Stabiliser examples

1. Let $G = S_4$ and $X = \{1, 2, 3, 4\}$ under the natural group action defined in the recap section. Then $Stab_{S_4}(4) = \{1, (12), (13), (23), (123), (132)\}$ and so $Stab_{S_4}(4) \simeq S_3$ right?
2. Suppose that G acts on itself by conjugation as defined in the recap section and that $x \in Z(G)$ where $Z(G)$ is the center of G. Then it follows that $Stab_G(x) = G$!

Challenge 4

Consider again $G = S_4$ and $X = \{1, 2, 3, 4\}$ under the natural group action defined in the recap section. You're encouraged to compute the following the stabilisers:

1. $Stab_G(2)$.
2. $Stab_G(3)$.

- Also, is it true $Stab_G(2) \simeq S_3$?

Group Orbits

Definition 1.2

Let G be a group acting on a set X. For an element $x \in X$, the orbit of x in G is defined as the set $Orb_G(x) = \{gx \in G \mid g \in G\}$.

- Intuitively, the orbit of x is the set of points in X in which x can be moved to by the group action!
- It's obvious that $Orb_G(x)$ is just a subset of X because we are only assuming X to be a set. But if X is a group itself, for example $X = G$? Is $Orb_G(x)$ necessarily a subgroup of G? If not, under what circumstances $Orb_G(x)$ is a subgroup if $X = G$?

Proposition 1.2

Let G be a group acting on a set X. Then for all $x \in X$, $x \in Orb_G(x)$.

Proof : Homework challenge (trivial)!

Orbit examples

1. Suppose that G acts on itself by conjugation as defined in the recap section. Then for all $x \in G$, we have $Orb_G(x) = \{gxg^{-1} \mid g \in G\}$!
2. Let $G = S_4$ and $X = \{1, 2, 3, 4\}$ under the natural group action defined in the recap section. Then we have $Orb_G(2) = \{1, 2, 3, 4\}$.

Challenge 5

Consider again $G = S_4$ and $X = \{1, 2, 3, 4\}$ under the natural group action defined in the recap section. We know $Orb_G(2) = \{1, 2, 3, 4\}$, what are the orbits of the other elements?

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