Category Theory:

a crash course

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§1 Categories, functors, & Naturality

"To understand a structure, it is necessary to understand the morphisms that preserve it" – goguen

	objects	morphisms
Sets	sets	functions
Grp	groups	group homomorphisms
Vect_k	vector spaces	linear maps
Top	topological spaces	continuous f'ns
	posets	order-preserving f'ns
	⋮	⋮
Cat	categories	functors
	⋮	⋮
	?	?

Def: A **category** is (loosely) a collection of **objects** & a collection of **morphisms** between objects

$\mathcal{C}$

(i) a collection of objects $Ob(\mathcal{C})$

(ii) for any X & Y in $Ob(\mathcal{C})$, a set $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) :=$ set of morphisms from X to Y

$\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{R}}}(\mathbb{R}^n, \mathbb{R}^m) = \mathcal{M}_{m \times n}(\mathbb{R})$ space of matrices

(sometimes called "arrows")

$X \xrightarrow{f} Y$

Compatibility

(iii) for any $X, Y, Z \in Ob(\mathcal{C})$, $\underbrace{\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z)}_{\text{sets}} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$(X \xrightarrow{f} Y, Y \xrightarrow{g} Z) \longmapsto X \xrightarrow{g \circ f} Z$

$g \circ f$ = composition of f'ns

Compatibility

(iv) function composition is associative

$$h\circ(g\circ f) = (h\circ g)\circ f$$

$$X\xrightarrow{f} Y,\quad Y\xrightarrow{g} Z,\quad Z\xrightarrow{h} W$$

(v) identity morphisms $\mathbb{1}_X = id_X$

for any $X \in Ob(C)$, $\exists\ id_X \in Hom_C(X,X)$

for any $g \in G$ $\quad e\cdot g = g\cdot e = g$

Equivalent

for any $f: X\to Y$,
$g: Z\to X$,

$$\begin{cases} f\circ id_X = f \\ g = id_X\circ g \end{cases}$$

in terms of "commutative diagrams"

- $$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$
  "this diagram commutes"

- $$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\ & Z \arrow[r, "h"'] & W \end{tikzcd}$$
  $(h\circ g)\circ f = h\circ(g\circ f)$

- $$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "id_X\circ g"'] & X \arrow[d, "id_X"] \arrow[dr, "f\circ id_X"] & \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

when are two objects "the same"?

Def: two objects X & Y of $\mathcal{C}$ are isomorphic if $\exists$ a pair of morphisms

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

such that $\begin{cases} g\circ f = id_X \\ f\circ g = id_Y \end{cases}$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \end{tikzcd}$$

$g = f^{-1}$, etc.

Examples: Set, Grp, Ab, Top, $Vect_k$, $Vect_k^{f.d.}$, Mon, Met, ...

Banach Analytic Manifolds

Ban Ana Man

X is an initial object in $\mathcal{C}$ $\overset{?}{\text{iff}}$ X is a terminal object in $\mathcal{C}^{op}$

Ex: Let $\mathcal{C}$ be a category, its opposite category, $\mathcal{C}^{op}$, is given by:

(i) $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$

(ii) $Hom_{\mathcal{C}^{op}}(X,Y) = Hom_{\mathcal{C}}(Y,X)$

$$X \xrightarrow{f} Y \quad \text{in } \mathcal{C}$$

$$X \longleftarrow Y \quad \text{in } \mathcal{C}^{op}$$

Let R be a ring w/ unity,

R-mod $=$ category of left R-modules

The category of right R-modules is $(R\text{-mod})^{op}$

{ in $\mathcal{D}_X$-modules
sheaf of linear diff operators
w/ holomorphic coefficients
on a complex manifold }

Example $\Rightarrow$ $\mathcal{C} = Set$, $\emptyset =$ emptyset $\in Ob(Set)$

for any set X, $\emptyset \subseteq X$.

Categorically: for any X, $\exists!\ \emptyset \xrightarrow{\iota} X$

if Y satisfies this property for any X, then $Y \cong \emptyset$ (isomorphic)

"inclusion" $\hookrightarrow$ \hookrightarrow

$Hom_{Set}(\emptyset, X) = \{\emptyset \to X\}$

Exercise 1: show that $\emptyset$ is the only set with this property (up to isomorphism)

$\boxed{\text{initial object}}$

suppose a set A has the same property as $\emptyset$

$\Rightarrow$ for every $X \in Ob(Set)$, $\exists$ $A \to X$

$A = B$ iff $A \subseteq B$ & $B \subseteq A$

- by assumption, $\emptyset \subseteq A$ $\quad \emptyset \xrightarrow{i_1} A$

but also, know $\exists$ $A \xrightarrow{i_2} \emptyset$

$Hom(A, \emptyset) \neq \{i_2 : A \to \emptyset\}$.

$(A \subseteq \emptyset)$

$A = \emptyset$

$e_1, e_2 \in G$

$e_1 = e_1 \cdot e_2 = e_2$

X is an initial object in $\mathcal{C}$
if $\forall y \in Ob(\mathcal{C})$,
$\exists!\ X \to y$

$\rightsquigarrow$ this property uniquely defines X up to isomorphism.

Example: "the" singleton set $\{*\} \in Ob(\text{set})$.
"a set containing only one element"

are these different?

$A = \{a\} \simeq \{\text{chair}\} = B$

uniquely isomorphic! $A \xrightarrow{f} B$ $\quad$ $B \xrightarrow{g} A$

$a \mapsto \text{chair}$ $\quad$ $\text{chair} \mapsto a$

The singleton set is the unique (up to isomorphism) set $\{*\}$ satisfying:

For any set X, $\exists!\ X \xrightarrow{\alpha} \{*\}$

$\text{Hom}_{\text{set}}(X, \{*\}) = \{X \xrightarrow{\alpha} \{*\}\}$

$x \mapsto *$

terminal object

Exercise 2: formulate this property in terms an arbitrary category $\mathcal{C}$, & prove it uniquely defines this object.

Z is terminal in $\mathcal{C}$
if for every $y \in Ob(\mathcal{C})$,
$\exists!\ y \to Z$

Exercise 3: for Vect_k, is there an initial object or a terminal object?

$\Rightarrow$ yes

$\boxed{\phi \notin \text{Vect}_k \ldots \text{and?}}$

any v.space V contains a zero element $0 \in V$

$\{0\} \hookrightarrow V$ is always linear

on the other hand, $\exists!\ V \to \{0\}$

$\vec{v} \mapsto 0$

$\{0\}$ is initial & terminal!
"zero object"

in Grp, $\{e\}$ is a zero object!

$R = (\mathbb{Z}, +, \cdot)$ is a Ring

$(\mathbb{Q}, +, \cdot)$ is a field

$\mathrm{Vect}_{\mathbb{Q}}$ is "nice"

R-mod is "nice"

partially ordered set

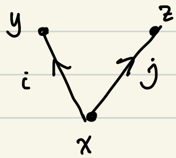
Example Let $(X, \leq)$ be a poset.

Consider as a category via:

$$\mathrm{Ob}(X, \leq) = X$$

for any two $x, y \in X$,

$$\mathrm{Hom}_{(X,\leq)}(x,y) = \begin{cases} \{x \to y\}, & \text{if } x \leq y \\ \varnothing, & \text{else} \end{cases}$$



$\mathrm{Hom}(x,y) = \{i\}$

$\mathrm{Hom}(y,z) = \varnothing$

$\mathrm{Hom}(x,z) = \{j\}$

Let X be any set.

$\mathcal{P}(X) = 2^X$ = set of subsets of X, partially ordered by inclusion.

power set of X

Complete Boolean algebra

New!

Exercise: Show, for any set X, that there is a bijection $\mathcal{P}(X) \simeq \mathrm{Hom}_{\mathrm{Set}}(X, \{0,1\})$

(power set of X)

$$\left\{ \begin{matrix} X \\ \cup\!| \\ A \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} X \xrightarrow{\chi_A} \{0,1\} \\ x \mapsto \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \end{matrix} \right\}$$

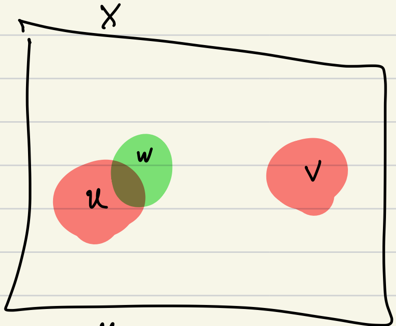
characteristic function of A

Let X be a topological space,

$\mathrm{Op}(X)$ = set of open subsets of X, partially ordered by inclusion.

Comes up all the time in algebraic geometry, for Topos theory, Grothendieck topology ...

in Set, $\{0,1\}$ is called a subobject classifier



$U \cap W$

$U \hookrightarrow X$

$V \hookrightarrow$

Ex: Monoids.

like groups, but w/o inverses

$(\mathbb{Z}, +)$ group vs. $(\mathbb{N}, +)$ monoid

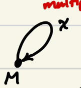
$$M \times M \to M$$
$$(x, y) \mapsto x * y$$
$$e * x = x * e = x$$

turn into a category: $\mathcal{M}$

only one object, $\mathrm{Ob}(\mathcal{M}) = \{\bullet\}$

$$\mathrm{Hom}_{\mathcal{M}}(\mathcal{M}, \mathcal{M}) = M$$

monoid multiplication by x



$$\begin{tikzcd} \mathcal{M} \arrow[r, "x"] \arrow[dr, "y \circ x"'] & \mathcal{M} \arrow[d, "y"] \\ & \mathcal{M} \end{tikzcd}$$

$y \circ x := y * x$

$$x \in M \Leftrightarrow x : \mathcal{M} \to \mathcal{M}$$

Exercise 4: prove the statement "a group is a category with one element, where every morphism is an isomorphism"

what is a "structure-preserving" morphism between categories?

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

?

"Functor"

F should define a function

(i) $F: \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$
$X \mapsto F(X)$

(ii) $(X \xrightarrow{f} Y) \mapsto (F(X) \xrightarrow{F(f)} F(Y))$
in $\mathcal{C}$ in $\mathcal{D}$

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow[F_*]{f \mapsto} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

respect composition

(iii) $F(g \circ f) = F(g) \circ F(f)$ covariant ☆

$F(g \circ f) = F(f) \circ F(g)$ contravariant

(iv) $F(id_X) = id_{F(X)}$

"Every sufficiently good analogy yearns to become a functor"
— Baez

Category theory := mathematics of analogies

→ Grothendieck topology on a category

$\underline{N} = (N, +)$ natural numbers under addition

$\mathbb{Z}_{\geq 0}$ $0 \in N$

$m: N \to N \in \mathrm{Hom}_{\underline{N}}(N, N)$
$n \mapsto n+m$

k & $m \in N$

$k \circ m : N \xrightarrow{m} N \xrightarrow{k} N$
$n \mapsto (n+m) \mapsto (n+m)+k$

$2 : N \to N$
$n \mapsto n+2$

$0, 1 \notin \mathrm{Im}(2: N \to N)$

$\mathbb{Z} \xrightarrow{+2} \mathbb{Z}$ surjective & injective.
$n \mapsto n+2$

$g \in G$

$g: G \to G$
$h \mapsto h * g$

$g^{-1}: G \to G$
$h \mapsto h * g^{-1}$

g & g^{-1} are inverses

Ex: for any category $\mathcal{C}$, there's an identity functor: $\quad id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$

$$X \mapsto X$$
$$f \mapsto f$$

"forgetful functor"

Ex: $\mathcal{U} : \underline{\mathrm{Grp}} \to \underline{\mathrm{Set}}$

$(G, *) \mapsto G$

forget the binary operation!

$\mathcal{F} : \underline{\mathrm{Set}} \to \underline{\mathrm{Grp}}$

$S \mapsto \mathrm{Free}(S) = \left\{ \begin{array}{l} \text{free group on the} \\ \text{set } S \end{array} \right\}$

look this up, if hard, talk about it later?

$$\begin{tikzcd} \underline{\mathrm{Grp}} \arrow[r, bend left, "\mathcal{U}"] & \underline{\mathrm{Set}} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

generalized inverse functors to each other

they form an adjoint pair of functors.

Ex: $\mathcal{C} = \underline{\mathrm{Vect}}_k$ dual vector space functor.

$D : \underline{\mathrm{Vect}}_k^{op} \to \underline{\mathrm{Vect}}_k$

$V \mapsto V^* = \mathrm{Hom}_k(V, k)$ $\quad \mathrm{Hom}_{\mathrm{Vect}_k}(V,k)$

$\mathrm{Hom}_k(W,k) \to \mathrm{Hom}_k(V,k)$

$(W \xrightarrow{\xi} k) \mapsto (V \to \xi)$

$(\xi \circ f) = f^*(\xi)$

first problem session next Sunday

Exercise: describe $D^2 : \underline{\mathrm{Vect}}_k \to \underline{\mathrm{Vect}}_k$ functor of "double dual"

$$D^2(V) = D(D(V)) = \mathrm{Hom}_k(\mathrm{Hom}_k(V,k), k)$$

& its relationship w/ the identity functor

$id_{\mathrm{Vect}_k} : \underline{\mathrm{Vect}}_k \to \underline{\mathrm{Vect}}_k$.

For any $V \in \underline{\mathrm{Vect}}_k$, define a function

$$V \xrightarrow{\delta_V} V^{**} = D^2(V)$$

$$\mathrm{Id}_{\mathrm{Vect}_k} \xrightarrow{\delta} D^2$$

"δ is a function between functors"

δ is a natural transformation

restricted to subcategory

$\mathrm{FinVect}_k$, δ is an isomorphism.

but not in general!

why?

$$V^* = \mathrm{Hom}_k(V, k)$$

$$\xi : V \to k$$

$$V \xrightarrow{\delta_V} V^{**} = \mathrm{Hom}_k(V^*, k) = \mathrm{Hom}_k(\mathrm{Hom}_k(V,k), k)$$

$$\vec{v} \longmapsto \left\{ \left\{ \xi : V \to k \right\} \longmapsto \xi(\vec{v}) \in k \right\}$$

$$\rotatebox{90}{$\subseteq$}$$

$$V^*$$

$$\delta_V(\vec{v}) = \text{“Evaluate the functional at } \vec{v}\text{”}$$

Lecture 2: Natural transformations & The Yoneda Lemma

given two categories $\mathcal{C}$ & $\mathcal{D}$,
a functor $F : \mathcal{C} \to \mathcal{D}$

is a structure-preserving morphism between categories

defines $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$

$$X \mapsto F(X)$$

$$Hom_{\mathcal{C}}(X,Y) \mapsto Hom_{\mathcal{D}}(F(X), F(Y))$$

$$\left(X \xrightarrow{f} Y \right) \longmapsto \left(F(X) \xrightarrow{F(f)} F(Y) \right)$$

→ a functor $F : \mathcal{C} \to \mathcal{D}$
is full if, for any $X, Y \in Ob(\mathcal{C})$,
the function

$$Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X), F(Y))$$

is surjective.

$F : \mathcal{C} \to \mathcal{D}$ is faithful if

$$Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X), F(Y))$$

is injective.

→ F is fully faithful if

"

is a bijection.

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \simeq \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$

Ex: (full) subcategories:

$$\begin{tikzcd} \underline{\text{FinSet}} \arrow[r, hook, "\iota"] & \underline{\text{Sets}} \end{tikzcd}$$

inclusion functor

$$X \longmapsto X$$

$$X \xrightarrow{f} Y \longmapsto X \xrightarrow{f} Y$$

$$\mathrm{Hom}_{\underline{\text{FinSets}}}(X,Y) \simeq \mathrm{Hom}_{\underline{\text{Set}}}(X,Y)$$

$\underline{\text{FinVect}}_k$ = cat. of finite dimensional vector spaces over a field k.

$$\begin{tikzcd} \underline{\text{FinVect}}_k \arrow[d, hook, "\iota"] \\ \underline{\text{Vect}}_k \end{tikzcd}$$

inclusion functor

$$\mathrm{Hom}_{\underline{\text{FinVect}}_k}(V,W) \simeq \mathrm{Hom}_{\underline{\text{Vect}}_k}(V,W).$$

Natural Transformations

given $F, G : \mathcal{C} \to \mathcal{D}$, what is a morphism $\eta : F \to G$?

associate, for $X \in \mathrm{Ob}(\mathcal{C})$, a morphism in $\mathcal{D}$: $\eta_X : F(X) \longrightarrow G(X)$

"the component of η over X"

for any morphism $X \xrightarrow{f} Y$, a commutative diagram:

$$G(f) \circ \eta_X = \eta_Y \circ F(f)$$

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

this diagram commutes

identity functor

$$\mathrm{id}_{\underline{\mathrm{Vect}}_k} = \mathbb{1}_k : \underline{\mathrm{Vect}}_k \to \underline{\mathrm{Vect}}_k$$

$\downarrow$

$$D = \underline{\mathrm{Vect}}_k^{op} \to \underline{\mathrm{Vect}}_k$$
$$V \mapsto \mathrm{Hom}_{\underline{\mathrm{Vect}}_k}(V, k) = D(V)$$

$\nexists$ a natural transformation $\mathbb{1}_k \to D$

but this does work for D^2, double dual.

$\exists$ $\mathbb{1}_k \to D^2$

$$\mathrm{id}_{\underline{\mathrm{FinVect}}_k} \xrightarrow{\sim} D^2|_{\underline{\mathrm{FinVect}}_k} \qquad \text{isomorphism}$$

$\to$ a finite-dim vector space is <u>naturally</u> isomorphic to its double-dual.

$\Rightarrow$ <u>Yoneda Lemma</u>:

for any $X \in \mathrm{Ob}(\mathcal{C})$, $\quad \mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ob}(\underline{\mathrm{Set}})$

$$h_X := \mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \underline{\mathrm{Set}}$$
$$Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

$$h_X(g) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$
$$(X \xrightarrow{f} Y) \mapsto (X \xrightarrow{g \circ f} Z)$$

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, bend left, "g\circ f"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

$h_X(g) = g_*$ "pushforward along g"

contravariant functor

$h^X := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$

$Y \longmapsto \mathrm{Hom}_{\mathcal{C}}(Y, X) = h^X(Y)$

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "f\circ g"'] & Z \arrow[d, "f"] \\ & X \end{tikzcd}$$

$h^X(Z) \xrightarrow{g^*} h^X(Y) = h^X(Y \xrightarrow{g} Z)$

$h^X(g) = g^*$ "pullback along g"

$\mathrm{Hom}_{\mathcal{C}}(Y, X) \longleftarrow \mathrm{Hom}_{\mathcal{C}}(Z, X)$

$$\begin{tikzcd} Y \arrow[r, "f\circ g"] \arrow[d, "g"'] & X \\ Z \arrow[ur, "f"'] & \end{tikzcd} \qquad Z \xrightarrow{f} X$$

$h^X(g)(f) := f \circ g$

$\Rightarrow$ what does a natural transformation

$\eta : h^X \longrightarrow G$

look like, for some $G : \mathcal{C}^{op} \to \underline{\mathrm{Set}}$?

This is context of yoneda lemma.

$\Rightarrow$ denote $[\mathcal{C}^{op}, \underline{\mathrm{Set}}]$ the category of contravariant functors from $\mathcal{C}$ to $\underline{\mathrm{Set}}$.

[yoneda]

$\mathrm{Hom}_{[\mathcal{C}^{op}, \underline{\mathrm{Set}}]}(h^X, G) \simeq G(X)$

$(\eta : \mathrm{Hom}_{\mathcal{C}}(-, X) \to G) \longmapsto \xi \in G(X)$

in particular, if $G = h^Y$, there is a bijection.

$\mathrm{Hom}_{[\mathcal{C}^{op}, \underline{\mathrm{Set}}]}(h^X, h^Y) \simeq h^Y(X)$

$= \mathrm{Hom}_{\mathcal{C}}(X, Y)$

Left as an exercise...

Stop now and prove it yourself!

Want a way to associate

$$(\eta : h^X \to G) \overset{\sim}{\longmapsto} \xi \in G(X)$$

$\exists$

$$\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X) = h^X(X) \xrightarrow{\eta_X} G(X)$$

$$\mathrm{id}_X \longleftrightarrow \eta_X(\mathrm{id}_X) =: \xi$$

eta

xi

Exercise 1: Show that $\xi := \eta_X(\mathrm{id}_X) \in G(X)$ uniquely determines the component

$$\eta_Y : h^X(Y) \to G(Y)$$

for any $Y \in \mathrm{ob}(\mathcal{C})$.

$$h^X \xrightarrow{\eta} G \longmapsto \eta_X(\mathrm{id}_X) \in G(X)$$

Exercise 2: Show this is a bijection!

$$\Longrightarrow \quad \mathrm{Hom}_{[\mathcal{C}^{op}, \underline{\mathrm{Set}}]}(h^X, h^Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

$$h^\wedge : \mathcal{C} \to [\mathcal{C}^{op}, \mathrm{Set}]$$

$$X \longmapsto h^X := \mathrm{Hom}_{\mathcal{C}}(-, X)$$

by yoneda, $h^\wedge$ is fully faithful (embedding)

$h^\wedge$ embeds $\mathcal{C}$ as a full subcategory of $[\mathcal{C}^{op}, \underline{\mathrm{set}}]$.

next class, we'll see $[\mathcal{C}^{op}, \underline{\mathrm{Set}}]$ is much nicer than $\mathcal{C}$

analogy w/ analysis,

$$\{\text{smooth } f\text{'ns}\} \hookrightarrow \{\text{Distributions}\}$$

$$f \longmapsto \left\{ \underset{\text{compact support}}{g} \mapsto \int_X fg \right\}$$

analogy w/

$$\mathcal{C} \xhookrightarrow{h^\wedge} [\mathcal{C}^{op}, \underline{\text{set}}]$$ ← can define a "topology" on this

$$X \longmapsto \{ Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X) \}$$

$\underline{\text{Yoneda}}$ gives a full subcategory of functors $\mathcal{C}^{op} \to \underline{\text{set}}$ isomorphic to h^X for some X.

→ if $G \simeq h^X$ for some X, call G a $\underline{\text{representable functor}}$.

→ In problem session 1;

for any set X,

$$\mathcal{P}(X) = \text{power set of } X$$

$$= \{\text{set of subsets of } X\}$$

by previous,

$$\mathcal{P}(X) \simeq \mathrm{Hom}_{\underline{\text{set}}}(X, \{0,1\})$$

in $\underline{\text{Set}}$, $\{0,1\}$ is a $\underline{\text{subobject classifier}}$

$$A \longmapsto \left\{ \chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases} \right\}$$

$$\mathcal{P} : \underline{\text{Set}}^{op} \to \underline{\text{Set}}$$

$X \xrightarrow{f} Y$ determines $\mathcal{P}(Y) \to \mathcal{P}(X)$, $B \mapsto f^{-1}(B)$

$\mathcal{P} \cong h^{\{0,1\}}$ is a representable functor!

→ For any category $\mathcal{C}$, & $X \in \mathrm{Ob}(\mathcal{C})$

$$\mathrm{Sub}_{\mathcal{C}}(X) = \{\text{subobjects of } X\} \in \underline{\text{Set}}$$

$$\mathrm{Sub}_{\mathcal{C}} : \mathcal{C}^{op} \to \underline{\text{Set}}$$

whether or not $\mathrm{Sub}_{\mathcal{C}}$ is representable is equivalent to $\mathcal{C}$ having a $\underline{\text{subobject classifier}}$ $\Omega \in \mathrm{Ob}(\mathcal{C})$

important in categorical logic, topos theory

$\underline{\text{Exercise 3:}}$ define & prove a covariant version of Yoneda lemma

$$h_X := \mathrm{Hom}_{\mathcal{C}}(X, -) \to F \quad \text{in } [\mathcal{C}, \underline{\text{set}}]$$

covariant functors

$$\mathrm{Hom}_{[\mathcal{C}, \underline{\text{set}}]}(h_X, F) \simeq F(X)$$

& analogue of Yoneda embedding:

$$h^\vee : \mathcal{C}^{op} \to [\mathcal{C}, \underline{\text{set}}]$$

3. Examples

The central point about examples of representable functors is:

Representable functors are ubiquitous .

To a fair extent, category theory is all about representable functors and the other universal constructions: Kan extensions, adjoint functors, limits, which are all special cases of representable functors – and representable functors are special cases of these.

Listing examples of representable functors in category theory is much like listing examples of integrals in analysis: one can and does fill books with these. (In fact, that analogy has more to it than meets the casual eye: see coend for more).

Keeping that in mind, we do list some special cases and special classes of examples that are useful to know. But any list is necessarily wildly incomplete.

let $\mathcal{C}$ be a Category
$X \in \mathrm{ob}(\mathcal{C})$

$$h^X := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \longrightarrow \underline{\mathrm{Set}}$$

$$h^X(Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

$$h_X = \mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \underline{\mathrm{Set}}$$

$$h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

pullback along f

$$h^X(Y \xrightarrow{f} Z) = h^X(Z) \xrightarrow{f^*} h^X(Y)$$

$$(Z \xrightarrow{g} X) \longmapsto \left(\begin{tikzcd} Y \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & & X \\ & Z \arrow[ur, "g"'] & \end{tikzcd} \right)$$

$$\begin{tikzcd} Y \arrow[r, "f"] & Z \arrow[d, "g"] \\ & X \end{tikzcd}$$

pushforward along f

$$h_X(Y \xrightarrow{f} Z) = h_X(Y) \xrightarrow{f_*} h_X(Z)$$

$$\begin{tikzcd} X \arrow[d, "g"'] \arrow[dr, "f \circ g"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd} \qquad (X \xrightarrow{g} Y) \longmapsto \left(\begin{tikzcd} X \arrow[rr] \arrow[dr, "g"'] & & Z \\ & Y \arrow[ur, "f"'] & \end{tikzcd} \right)$$

(Abelian category)

$\underline{D}: \underline{Vect}_k^{op} \to Vect_k$

$V \mapsto Hom_k(V, k) = h^k(V)$

$G: \mathcal{C}^{op} \to \underline{Set}$

$\boxed{\text{yoneda}}$

$$Hom_{[\mathcal{C}^{op}, \underline{Set}]}(h^X, G) \simeq G(X)$$

morphism between functors

G is representable if $\underline{G \simeq h^X}$

$\mathcal{C}$ is small if $Ob(\mathcal{C})$ is $\underline{\text{a set}}$

$\mathcal{C}$ is locally small if $Hom_{\mathcal{C}}(X, Y)$ is $\underline{\text{a set}}$

$G: \mathcal{C}^{op} \to \underline{\mathcal{D}}$

Sheaves of sets, etc? enriched over sets?

$$Hom_{[\mathcal{C}^{op}, \underline{Set}]}(\overrightarrow{h^X, h^Y}) \simeq Hom_{\mathcal{C}}(\overrightarrow{X, Y})$$

$\overset{\text{yoneda}}{\shortparallel}$

$h^Y(X) \underset{def}{:=} Hom_{\mathcal{C}}(X, Y)$

$$\boxed{\begin{aligned} h^{\wedge}: \mathcal{C} &\to [\mathcal{C}^{op}, Set] \\ X &\mapsto h^X \end{aligned}}$$

$$Hom_{[\mathcal{C}, Set]}(h_X, h_Y) \simeq h_Y(X) = Hom_{\mathcal{C}}(\overrightarrow{Y, X})$$

$$h^{\vee}: \mathcal{C}^{op} \to [\mathcal{C}, Set]$$

$$X \mapsto h_X$$

Lecture 3: Universal Properties

"A universal property is a property that uniquely characterizes (up to isomorphism) the result of some construction"

Initial objects
terminal objects } uniquely characterized by some property.

- in **Set**, $\emptyset$ (the empty set) is "the" initial object.

  $\emptyset \subseteq X$ for any set X.

  $\emptyset \xrightarrow{\exists!} X \leftrightarrow$ "is a subset of every set"

- "the" singleton set $\{*\}$ is terminal in cat. of sets.

  $\{a\}$ vs. $\{\text{Chaw}\}$

  for any set X, $\exists!\ X \to \{*\}$
  $x \mapsto *$

  "admits a map from any set"
  $\Leftrightarrow$ $X \to \{a\}$.

Examples:

- free groups / Free vector spaces
- quotient groups / quotient vector spaces / quotient topological spaces
- (co)kernels of group hom's / (co)kernels of linear maps between v. spaces
- direct products, direct sums ...
- Completion of a metric space (e.g. $\mathbb{R}$ from $\mathbb{Q}$, $\mathbb{Z}_p$ (p-adics) from $\mathbb{Q}$)
- tensor products
- fiber products (pullbacks)

} ubiquitous in math!

Exercise 1: find more

Example: the Kernel of a group homomorphism

$$G \xrightarrow{\varphi} H \quad \text{a grp hom.}$$

[φ is a function between the underlying sets of G & H
s.t. $\forall g_1, g_2 \in G$,
$$\varphi(g_1 \underset{G}{*} g_2) = \varphi(g_1) \underset{H}{*} \varphi(g_2)$$]

$$V \xrightarrow{\varphi} W \quad \text{k-linear map of v. spaces}/k.$$

the Kernel of φ, $\operatorname{Ker}\varphi$, is the (normal) subgroup of G defined by

usual defn.

$$\operatorname{Ker}\varphi := \{ g \in G \mid \varphi(g) = 0_H \}.$$

$$\mathbb{Z} \xrightarrow{q} \mathbb{Z}/n\mathbb{Z}$$

$$m \longmapsto m + n\mathbb{Z}$$

$$\operatorname{Ker} q = n\mathbb{Z} := \{ nm \in \mathbb{Z} \mid m \in \mathbb{Z} \}.$$

$\rightsquigarrow$ "$\operatorname{Ker}\varphi$ is the largest (normal) subgroup of G that is annihilated by φ"

commutative diagram

$$\begin{tikzcd}
G \arrow[r, "\varphi"] & H \\
\operatorname{Ker}\varphi \arrow[u, "i_\varphi"] \arrow[ur, "0 = \varphi \circ i_\varphi"'] &
\end{tikzcd}$$

$\iota_K : K \hookrightarrow G$

means: "if $K \trianglelefteq G$ is any other normal subgroup such that $\varphi_{|K} = 0$ (i.e., $\varphi \circ i_K = 0$),

then $\boxed{K \leq \operatorname{Ker}\varphi}$."

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ K \arrow[u, "i_K"] \arrow[ur, "0 = \varphi \circ i_K"'] & \end{tikzcd} \quad \overset{\exists !}{\Longrightarrow} \quad \boxed{K \hookrightarrow \operatorname{Ker}\varphi}$$

$$\begin{tikzcd} & G \arrow[r, "\varphi"] & H \\ & \operatorname{Ker}\varphi \arrow[u, "i_\varphi"] \arrow[ur, "0"'] & \\ & K \arrow[u, dashed, "\exists !"] \arrow[uur, "0"'] \arrow[uu, bend left=60, "i_K"] & \end{tikzcd}$$

Define a category of diagrams

$$\left\{ \begin{tikzcd} G \arrow[r, "\varphi"] & H \\ K \arrow[u, "i_K"] \arrow[ur, "0"'] & \end{tikzcd} \right\} \leftarrow$$ consists of objects which are diagrams of this form (for some grp K w/ hom $K \xrightarrow{i_K} G$

a morphism of such diagrams is given by

$$\left(\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ K_1 \arrow[u, "\iota_1"] \arrow[ur, "0"'] & \end{tikzcd} \right) \xrightarrow{\psi} \left(\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ K_2 \arrow[u, "\iota_2"] \arrow[ur, "0"'] & \end{tikzcd} \right)$$

Exercise 2: check this is actually a category! $\mathcal{D}_\varphi$

given by a group hom

$$\psi : K_1 \to K_2$$

such that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ K_2 \arrow[u, "i_2"] \arrow[ur, "0"] & \\ K_1 \arrow[u, "\psi"] \arrow[uu, bend left, "i_1"] \arrow[uur, "0"'] & \end{tikzcd}$$

punchline:
$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ \ker\varphi \arrow[u, "\iota_\varphi"] \arrow[ur, "0"'] & \end{tikzcd}$$
is the terminal object in the category of diagrams $\mathcal{D}_\varphi$

Exercise 3: (part I)

for $K \xrightarrow{\iota_K} G$, didn't require that K is a subgroup / normal subgrp of G.

Q: why does this still work?

hint: think about Image (φ).

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ K \arrow[u, "\iota_K"] \arrow[ur, "0"'] & \end{tikzcd} \quad \text{vs} \quad \begin{tikzcd} G \arrow[r, "\varphi"] & H \\ \mathrm{Im}(\varphi) \arrow[u] \arrow[ur, "0"'] & \end{tikzcd} \quad ?$$

Exercise 3. (part II)

what is (if it exists) the property defining $\mathrm{Im}(\varphi)$?

Similarly, for cokernels:

$$G \xrightarrow{\varphi} H$$

$$\text{cokernel}(\varphi) = \text{Coker}(\varphi) := H/\text{Im}(\varphi)$$

Coker φ is the smallest quotient group of H s.t.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr, "0"'] & H \arrow[d, "j_\varphi"] \\ & \text{Coker}\varphi = H/\text{Im}(\varphi). \end{tikzcd}$$

$\Leftarrow$ this is the initial object in $\mathcal{D}^\varphi$.

next lecture, we will axiomatize this idea

cat. of diagrams: $\mathcal{D}^\varphi$

$$\left\{ \begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr, "0"'] & H \arrow[d, "j_L"] \\ & L \end{tikzcd} \right\}$$

Example: from topology

form category **Top** of topological spaces & continuous f'ns between them.

Let X_1 & X_2 be top. sp. Can form the product space $X_1 \times X_2$

as a set, this is cartesian product.

with topology given as follows:

Explicitly gives you the space $X_1 \times X_2$

$$
\begin{tikzcd}
 & X_1 \times X_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X_1 & & X_2
\end{tikzcd}
$$

the discrete topology on $X_1 \times X_2$ makes these maps cont. as well!

$$p_1(x_1, x_2) = x_1$$
$$p_2(x_1, x_2) = x_2$$

the topology on $X_1 \times X_2$ is the coarsest topology for which p_1 & p_2 are continuous.

if $U_i \subset X_i$ is open, then $p_i^{-1}(U_i)$ is open in $X_1 \times X_2$

universal properties identify objects that are the "best"/"most optimal" at that property

→ e.g. biggest object such that, smallest / finest / coarsest, etc..

define category of diagrams of top. spaces

top. space

$$\begin{tikzcd}
& Y \arrow[dl, "f_1"' , "\text{cont.}"] \arrow[dr, "f_2", "\text{cont.}"'] & \\
X_1 & & X_2
\end{tikzcd}$$

$$\begin{tikzcd}
& Y \arrow[d, "\psi"] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\
& Z \arrow[dl, "g_1"] \arrow[dr, "g_2"'] & \\
X_1 & & X_2
\end{tikzcd}$$

$Y \xrightarrow{\psi} Z$

cont.

s.t. the diagram commutes.

Then,

$$\begin{tikzcd}
& X_1 \times X_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X_1 & & X_2
\end{tikzcd}$$

is the terminal object in this category.

the whole diagram matters, not just $X_1 \times X_2$.

$$\begin{tikzcd}
& Y \arrow[d, dashed, "\exists! f"] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\
& X_1 \times X_2 \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X_1 & & X_2
\end{tikzcd}$$

$$p_i \circ f = f_i$$

Exercise 4: if I reverse some arrows

what am I defining?

$$\begin{tikzcd}
X_1 \arrow[dr, "g_1"] \arrow[ddr, bend right, "f_1"'] & & X_2 \arrow[dl, "g_2"'] \arrow[ddl, bend left, "f_2"] \\
& Z \arrow[d, "\psi"] & \\
& Y &
\end{tikzcd}$$

Exercise J: quotient vector spaces
quotient groups
quotient topological spaces
seem similar...
identify the universal property,
find the cat of diagrams,
& compute initial object/etc.