Homework 10

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Let $D = [0, 1] \times [0, 1]$, and consider the following variant of Thomae's function:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ or } y \text{ is irrational} \\ \frac{1}{q} & \text{if } x \text{ and } y \text{ both rational and } y = p/q \text{ in reduced form} \end{cases}$$

   Show the following (note all the following are standard Riemann integrals):

   a) $\int_0^1 f(x, y)\, dy = 0$ for any $x \in [0, 1]$.

   b) $\iint_D f(x, y)\, dA = 0$.

   c) $\int_0^1 f(x, y)\, dx = 0$ for any irrational y but does not exist for rational y.

   d) Explain why this doesn't contradict Fubini's Theorem.

   e) What if instead of using the standard Riemann integral, we replaced dx and dy with $d\lambda(x)$ and $d\lambda(y)$, the Lebsegue integral?

2. For $i = 1, 2, 3, \ldots$, let $\varphi_i : \mathbb{R} \to \mathbb{R}$ be continuous real valued functions with support in $(1/(i + 1), 1/i)$ such that $\int_0^1 \varphi_i = 1$. Define

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y).$$

   a) Prove the following (note the following are both Riemann integrals):

$$\int_0^1 \int_0^1 f(x, y)\, dx\, dy = 0 \qquad \int_0^1 \int_0^1 f(x, y)\, dy\, dx = 1.$$

   b) Explain why the findings above do not contradict Fubini's Theorem.

   c) What if instead of using the standard Riemann integral, we replaced dx and dy with $d\lambda(x)$ and $d\lambda(y)$, the Lebsegue integral?
