QF Group Theory CC2022
By
Zaiku Group

Lecture 01

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Crash Course Motivation

Lie Groups, Lie Algebras & Representations

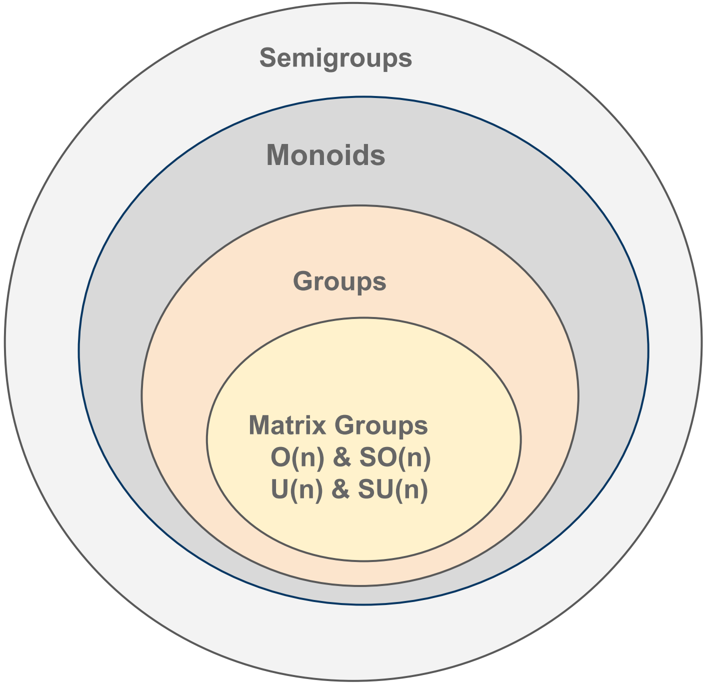
Module II (Summer 2022)

Lie group

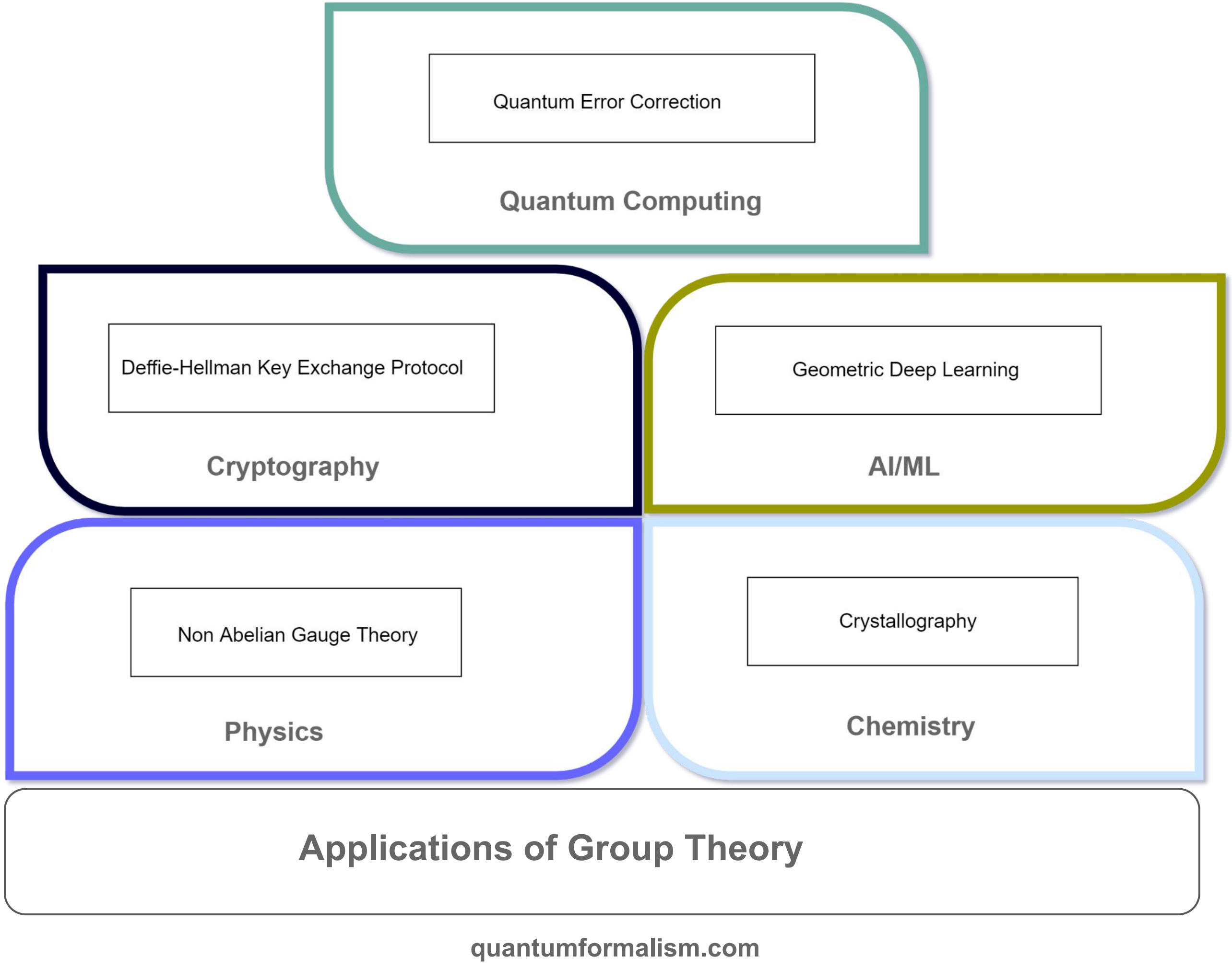
From Wikipedia, the free encyclopedia

Not to be confused with Group of Lie type.

In mathematics, a **Lie group** (pronounced /liː/ "Lee") is a group that is also a differentiable manifold. A manifold is a space that locally resembles Euclidean space, whereas groups define the abstract, generic concept of multiplication and the taking of inverses (division). Combining these two ideas, one obtains a continuous group where points can be multiplied together, and their inverse can be taken. If, in addition, the multiplication and taking of inverses are defined to be smooth (differentiable), one obtains a Lie group.



Course Approach Overview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Binary Operation on a Set

Definition 1.0

Let S be a nonempty set. Informally, a binary operation $*$ on S is a rule that takes any two elements $a, b \in S$ to generate another element $a * b \in S$.

- More formally, a binary operation $*$ on S is a map $* : S \times S \longrightarrow S$.
- Hence, given $(a, b) \in S \times S$, $a * b$ is just an abbreviation for $*((a, b))$ i.e. $a * b$ is an abuse of notation!
- It is possible to equip a set S with more than one binary operation! For example, the algebraic structures of rings and fields are obtained that way.

Definition 1.1

Let S be a nonempty set. A binary operation $*$ on S is said to be commutative (or abelian) if $a * b = b * a$ for any pairs $a, b \in S$. Otherwise, whenever we have $a * b \neq b * a$ for some $a, b \in S$, we say that $*$ is a noncommutative (or non-abelian) binary operation on S.

Binary Operation Examples (Part A)

Example 1

Let S be the set of natural numbers $\mathbb{N}$ and let the operation $*$ be the ordinary addition of natural numbers $+$.

- $+$ defines a binary operation on $\mathbb{N}$ right? Yes cause $\forall a, b \in S$ also $a+b \in S$

Example 2

Let us consider $S = \{a \in \mathbb{N} \mid a \text{ is odd }\}$ and $*$ be the ordinary multiplication of natural numbers $\times$.

- Does $\times$ define a binary operation on S? $(2n+1)(2m+1) = 4nm + 2n + 2m + 1$ $\quad \underbrace{4nm + 2n + 2m}_{} = \text{even}$, $+1 = \text{odd}$
  So odd $\times$ odd $=$ odd $\quad$ Yes, $*$ on S is a Bin. Oper.

Example 3

Let consider again $S = \{a \in \mathbb{N} \mid a \text{ is odd }\}$ and let now $*$ be the ordinary addition of natural numbers $+$.

- Does $+$ also define a binary operation on S?

$(2n+1)+(2m+1) = \underbrace{2n+2m+2}_{\text{even!}}$. $\quad$ No, $*$ on S is not a Binary Operation

Binary Operation Examples (Part B)

Example 1

Let A be a non-empty set and let $S = \{f : A \longrightarrow A \mid f \text{ is a bijection }\}$. Now suppose that $*$ is the composition $\circ$ of maps in S.

- Is $\circ$ a binary operation on S? If yes, is it abelian or non-abelian?

Example 2

Let S be the set $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries and let the operation $*$ be the ordinary matrix multiplication.

- Is $*$ also a binary operation on $M_n(\mathbb{C})$? Is it abelian or non-abelian?

Example 3

Let S be the set denote $GL(n, \mathbb{C})$ of invertible $n \times n$ matrices with complex entries and let the operation $*$ be still the ordinary matrix multiplication.

- Is $*$ also a binary operation on $GL(n, \mathbb{C})$? Is it abelian or non-abelian?
- What if $*$ is now the ordinary addition of matrices?

Ex 1 $f: A \to A$ f is bijective Bijective = injective and surjective = | Injective: $\forall a,b \in A$ $f(a)=f(b)$ iff $a=b$

$\circ = f \circ f^{'} (=g)$, $f: A \to A$, $g: A \to A$ $f \circ g : A \to A$

Since f and g are bijective also $(f \circ g)$ is bijective

| Surjective $\forall b \in B \ \exists a \in A$ s.t. $f(a)=b$

Yes, $\circ$ is a binary operation on S ($f \circ g : A \to A$). It is NOT abelian cause $f \circ g \neq g \circ f$

Ex 2

$A, B = M_n(\mathbb{C})$ $A, B \in S$ $* = A \times B$ $A_{n \times n} \times B_{n \times n} = C_{n \times n}$ $C_n \in S$ -

Yes, $*$ is a binary operation on S. It is not abelian cause $AB \neq BA$

Ex 3

$GL(n, \mathbb{C})$ General Linear group of invertible matrix over $\mathbb{C}$

As per Ex 2 $*$ is a binary operation.

If $* = +$ then $*$ is a binary operation and it is abellian cause $A_n, B_n \in GL$ $A_n + B_n = C_n$ $C_n \in GL$ and $A_n + B_n = B_n + A_n$

Semigroup Structure

Definition 1.2

A semigroup is a pair $(S, *)$ where S is a nonempty set and $*$ is a binary operation on S such that $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$.

- The condition $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$ is called the 'associativity law' and we say that the operation $*$ is associative.
- Whenever the operation $*$ is understood from the context and fixed, we just say S is a semigroup and we omit writing the pair $(S, *)$.
- A semigroup $(S, *)$ is said to be abelian or non-abelian if $*$ is a abelian or non-abelian binary operation respectively.

Definition 1.3

Let $(S, *)$ be a semigroup and $S' \subseteq S$. Then S' is said to be subsemigroup of $(S, *)$ if $(S', *)$ is also a semigroup.

- Obviously, $(S, *)$ is trivially a subsemigroup of itself!

Semigroup Examples

Example 1

Let A be a non-empty set and let $S = \{f : A \longrightarrow A \mid f \text{ is a bijection }\}$. Now suppose that $*$ is the composition $\circ$ of maps in S.

- Is S a semigroup under $\circ$? If yes, is it abelian or non-abelian?

1) Yes, it is a semigroup

$(f \circ g) \circ h = f \circ (g \circ h)$

No it is not abelian

$f \circ g \neq g \circ f$

eg: $A = \mathbb{N}$ $f = 2x+1$ $g = x^2$

Example 2

Let S be the set $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries and let the operation $*$ be the ordinary matrix multiplication.

- Is $M_n(\mathbb{C})$ a semigroup under matrix multiplication? Is it abelian or non-abelian? What about under matrix addition?

2) Yes it is a semigroup. under multiplication and addition.

$M_n(\mathbb{C})$ it is not abelian under multiplication ($A_n \times B_n \neq B_n \times A_n$) but it is abelian under addition ($A_n + B_n = B_n + A_n$)

Example 3

Let S be the set denote $GL(n, \mathbb{C})$ of invertible $n \times n$ matrices with complex entries and let the operation $*$ be still the ordinary matrix multiplication.

- Is $GL(n, \mathbb{C})$ a semigroup under matrix multiplication?

  What about under matrix addition?

3) $GL(n, \mathbb{C})$ is a semigroup under multiplication and addition.

Semigroups Structure Challenge

1. Let $(S, *)$ be a semigroup and let $S' = \{a \in S \mid a * x = x * a \text{ for all } x \in S\}$. Is it true that $(S', *)$ is a subsemigroup of $(S, *)$? Yes.
2. Let $(S_1, *_1)$ and $(S_2, *_2)$ be two semigroups. Construct a semigroup structure on the Cartesian product $S_1 \times S_2$ using the respective semigroup structure. Can you generalise your construction to $(S_1, *_1), (S_2, *_2), \ldots, (S_n, *_n)$? $((S_1 \otimes S_2), (*_1 \circ *_2))$ $\otimes$ = Tensor Product
3. Assuming that $(S_1, *_1)$ is abelian and $(S_2, *_2)$ is non-abelian, is your constructed semigroup structure on $S_1 \times S_2$ abelian or non-abelian?
4. Identify at least a nontrivial subsemigroup structure for the constructed semigroup structure on $S_1 \times S_2$ above.
5. Let $\mathbb{Z}_2 = \{0, 1\}$, $\mathbb{Z}_3 = \{0, 1, 2\}$ and $\mathbb{Z}_4 = \{0, 1, 3\}$. Identify at least a semigroup structure for $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$.
6. Identify at least a subsemigroup structure (if any) from the identified semigroup structures on $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$ above.

QF

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