Category Theory:

a crash course

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§1 Categories, functors, & Naturality

["To understand a structure, it is necessary to understand the morphisms that preserve it"] - goguen

	objects	morphisms
Sets	sets	functions
Grp	groups	group homomorphisms
Vect_k	vector spaces	linear maps
Top	topological spaces	continuous f'ns
	posets	order-preserving f'ns
	⋮	⋮
Cat	categories	functors
	⋮	⋮
	?	?

Def: A category is (loosely) a collection of objects & a collection of morphisms between objects

$\mathcal{C}$

(i) a collection of objects $\mathrm{Ob}(\mathcal{C})$

(ii) for any X & Y in $\mathrm{Ob}(\mathcal{C})$, a set
$f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) :=$ set of morphisms from X to Y

space of matrices
$\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{R}}}(\mathbb{R}^n, \mathbb{R}^m) = M_{m\times n}(\mathbb{R})$

(sometimes called "arrows")

$X \xrightarrow{f} Y$

Compatibility

(iii) for any $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$,

sets

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$(X \xrightarrow{f} Y,\ Y \xrightarrow{g} Z) \longmapsto X \xrightarrow{g\circ f} Z$

$g\circ f$ = composition of f'ns

Compatibility

(iv) function composition is associative

$$h\circ(g\circ f) = \underline{(h\circ g)}\circ f$$

$$X \xrightarrow{f} Y,\quad Y \xrightarrow{g} Z,\quad Z \xrightarrow{h} W$$

(v) identity morphisms $\mathbb{1}_X = id_X$

for any $X \in Ob(C)$, $\exists\ id_X \in Hom_C(X,X)$

for any $g \in G$ $\quad e\cdot g = g\cdot e = g$

Equivalent

for any $f: X \to Y$,
$g: Z \to X$,

$$\begin{cases} f\circ id_X = f \\ g = id_X \circ g \end{cases}$$

in terms of "commutative diagrams"

- "this diagram commutes"
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

- $$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\ & Z \arrow[r, "h"'] & W \end{tikzcd}$$

$(h\circ g)\circ f = h\circ(g\circ f)$

- $$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "id_X\circ g"'] & X \arrow[d, "id_X"] \arrow[dr, "f\circ id_X"] & \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

<u>when are two objects "the same"?</u>

Def: two objects X & Y of $\mathcal{C}$ are <u>isomorphic</u> if $\exists$ a pair of morphisms

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

such that $\begin{cases} g\circ f = id_X \\ f\circ g = id_Y \end{cases}$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \end{tikzcd}$$

$g = f^{-1}$, etc.

Examples: <u>Set</u>, <u>Grp</u>, <u>Ab</u>, <u>Top</u>, <u>Vect</u>$_k$, <u>Vect</u>$_k^{f.d.}$, <u>Mon</u>, <u>Met</u>, ...

Banach Analytic Manifolds

<u>BanAnaMan</u>

X is an initial object in $\mathcal{C}$
iff?
X is a terminal object in $\mathcal{C}^{op}$

Ex: Let $\mathcal{C}$ be a category, its opposite category, $\mathcal{C}^{op}$, is given by:

(i) $\mathrm{Ob}(\mathcal{C}^{op}) = \mathrm{Ob}(\mathcal{C})$

(ii) $\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$

$$X \xrightarrow{f} Y \quad \text{in } \mathcal{C}$$

$$X \longleftarrow Y \quad \text{in } \mathcal{C}^{op}$$

Let R be a ring w/ unity,

R-mod = category of left R-modules

The category of right R-modules is $(R\text{-mod})^{op}$

{ in $\mathcal{D}_X$-modules

sheaf of linear diff operators w/ holomorphic coefficients on a complex manifold }

Example $\Rightarrow$ $\mathcal{C} = \mathrm{Set}$, $\emptyset$ = emptyset $\in \mathrm{Ob}(\mathrm{Set})$

for any set X, $\emptyset \subseteq X$.

Categorically: for any X, $\exists!\ \emptyset \xhookrightarrow{i} X$

if Y satisfies this property for any X, then $Y \cong \emptyset$ (isomorphic)

"inclusion" $\hookrightarrow$ \hookrightarrow

$\mathrm{Hom}_{\mathrm{Set}}(\emptyset, X) = \{\emptyset \to X\}$

Exercise 1: show that $\emptyset$ is the only set with this property (up to isomorphism)

initial object

suppose a set A has the same property as $\emptyset$

$\Rightarrow$ for every $X \in \mathrm{Ob}(\mathrm{Set})$, $\exists$ $A \to X$

$A = B$ iff $A \subseteq B$ & $B \subseteq A$

- by assumption, $\emptyset \subseteq A$ $\quad \emptyset \xrightarrow{i_1} A$

but also, know $\exists$ $A \xrightarrow{i_2} \emptyset$

$\mathrm{Hom}(A, \emptyset) \neq \{i_2 : A \to \emptyset\}$.

$(A \subseteq \emptyset)$

$A = \emptyset$

$e_1, e_2 \in G$

$e_1 = e_1 \cdot e_2 = e_2$

X is an initial object in $\mathcal{C}$
if $\forall Y \in Ob(\mathcal{C})$,
$\exists! \; X \to Y$

$\rightsquigarrow$ this property uniquely defines X up to isomorphism.

Example: "the" singleton set $\{*\} \in Ob(set)$.
"a set containing only one element"

are these different?

$A = \{a\} \simeq \{chair\} = B$

uniquely isomorphic! $A \xrightarrow{f} B$ $\quad$ $B \xrightarrow{g} A$

$a \mapsto chair$ $\quad$ $chair \mapsto a$

The singleton set is the unique (up to isomorphism) set $\{*\}$ satisfying:

For any set X, $\exists! \; X \xrightarrow{\alpha} \{*\}$

$Hom_{set}(X, \{*\}) = \{X \xrightarrow{\alpha} \{*\}\}$

$x \mapsto *$

terminal object

Exercise 2: formulate this property in terms an arbitrary category $\mathcal{C}$, & prove it uniquely defines this object.

Z is terminal in $\mathcal{C}$
if for every $Y \in Ob(\mathcal{C})$,
$\exists! \; Y \to Z$

Exercise 3: for $Vect_k$, is there an initial object or a terminal object?

$\Rightarrow$ yes

$\boxed{\phi \notin Vect_k \ldots \text{ and?}}$

any v.space V contains a zero element $0 \in V$

$\{0\} \hookrightarrow V$ is always linear

on the other hand, $\exists! \; V \to \{0\}$

$\vec{v} \mapsto 0$

$\{0\}$ is initial & terminal!
"zero object"

in Grp, $\{e\}$ is a zero object!

Example Let $(X, \leq)$ be a poset. (partially ordered set)

Consider as a category via:

$$Ob(X,\leq) = X$$

for any two $x, y \in X$,

$$Hom_{(X,\leq)}(x,y) = \begin{cases} \{x \to y\}, & \text{if } x \leq y \\ \emptyset, & \text{else} \end{cases}$$

$$\begin{tikzcd} y & & z \\ & x \arrow[ul, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

$Hom(x,y) = \{i\}$

$Hom(y,z) = \emptyset$

$Hom(x,z) = \{j\}$

$R = (\mathbb{Z}, +, \cdot)$ is a Ring

$(\mathbb{Q}, +, \cdot)$ is a field

$Vect_{\mathbb{Q}}$ is "nice"

R-mod is "nice"

Let X be any set.

$\mathcal{P}(X) = 2^X =$ set of subsets of X, partially ordered by inclusion.

power set of X

$\wr$

Complete Boolean algebra

New!

Exercise: Show, for any set X, that there is a bijection

$$\mathcal{P}(X) \simeq Hom_{Set}(X, \{0,1\})$$

(power set of X)

$$\left\{ \begin{matrix} X \\ \cup\!\mid \\ A \end{matrix} \right\} \xrightarrow{\sim} \left\{ X \xrightarrow{\chi_A} \{0,1\} \quad x \mapsto \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \right\}$$

characteristic function of A

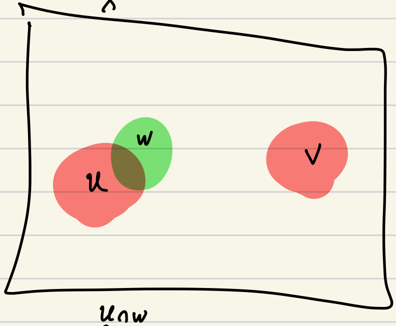
Let X be a topological space,

$Op(X) =$ set of open subsets of X, partially ordered by inclusion.

Comes up all the time in algebraic geometry, for Topos theory, Grothendieck topology ...

in **Set**, $\{0,1\}$ is called a subobject classifier

X



$U \cap W$

$U \hookrightarrow X$

$V \hookrightarrow$

Ex: Monoids.

like groups, but w/o inverses

$(\mathbb{Z}, +)$ vs. $(\mathbb{N}, +)$
group monoid

$M \times M \to M$

$(x, y) \mapsto x * y$

$e * x = x * e = x$

turn into a category: M

only one object, $Ob(M) = \{\bullet\}$

$Hom_{M}(M, M) = M$

monoid multiplication by x

$$\begin{tikzcd} M \arrow[loop above, "x"] \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "x"] \arrow[dr, "y \circ x := y * x"'] & M \arrow[d, "y"] \\ & M \end{tikzcd}$$

$x \in M \iff x : M \to M$

Exercise 4: prove the statement "a group is a category with one element, where every morphism is an isomorphism"

what is a "structure-preserving" morphism between categories?

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

?

"Functor"

F should define a function

(i) $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$
$X \mapsto F(X)$

(ii) $(X \xrightarrow{f} Y) \mapsto (F(X) \xrightarrow{F(f)} F(Y))$
in $\mathcal{C}$ in $\mathcal{D}$

$$Hom_{\mathcal{C}}(X,Y) \xrightarrow[F_*]{f \text{'n}} Hom_{\mathcal{D}}(F(X), F(Y))$$

respect composition

(iii) $F(g \circ f) = F(g) \circ F(f)$ covariant ☆

$F(g \circ f) = F(f) \circ F(g)$ contravariant

(iv) $F(id_X) = id_{F(X)}$

"Every sufficiently good analogy yearns to become a functor"
-Baez

Category theory := mathematics of analogies

→ Grothendieck topology on a category

$\underline{\mathbb{N}} = (\mathbb{N}, +)$ natural numbers under addition

$\mathbb{Z}_{\geq 0}$ $\quad \mathbb{N} \ni 0$

$m : \mathbb{N} \to \mathbb{N} \in Hom_{\underline{\mathbb{N}}}(\mathbb{N}, \mathbb{N})$
$n \mapsto n+m$

k & $m \in \mathbb{N}$
$k \circ m : \mathbb{N} \xrightarrow{m} \mathbb{N} \xrightarrow{k} \mathbb{N}$
$n \mapsto (n+m) \mapsto (n+m)+k$

$2 : \mathbb{N} \to \mathbb{N}$
$n \mapsto n+2$
$0, 1 \notin Im(2 : \mathbb{N} \to \mathbb{N})$

$\mathbb{Z} \xrightarrow{+2} \mathbb{Z}$ surjective & injective.
$n \mapsto n+2$

$g \in G$

$g : G \to G$
$h \mapsto h * g$

$g^{-1} : G \to G$
$h \mapsto h * g^{-1}$

g & g^{-1} are inverses

Ex: for any category $\mathcal{C}$, there's an identity functor: $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$

$$X \mapsto X$$
$$f \mapsto f$$

"forgetful functor"

Ex: $\mathcal{U}: \underline{\mathrm{Grp}} \to \underline{\mathrm{Set}}$

$(G, *) \mapsto G$

forget the binary operation!

$\mathcal{F}: \underline{\mathrm{Set}} \to \underline{\mathrm{Grp}}$

$S \mapsto \mathrm{Free}(S) = \{\text{free group on the set } S\}$

look this up, if hard, talk about it later.

$$\begin{tikzcd} \underline{\mathrm{Grp}} \arrow[r, bend left, "\mathcal{U}"] & \underline{\mathrm{Set}} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

generalized inverse functors to each other

they form an adjoint pair of functors.

Ex: $\mathcal{C} = \underline{\mathrm{Vect}}_k$ dual vector space functor.

first problem session next Sunday

$D : \underline{\mathrm{Vect}}_k^{op} \to \underline{\mathrm{Vect}}_k$

$V \mapsto V^* = \mathrm{Hom}_k(V, k)$ $\quad \mathrm{Hom}_{\underline{\mathrm{Vect}}_k}(V,k)$

$\mathrm{Hom}_k(W,k) \to \mathrm{Hom}_k(V,k)$

$(W \xrightarrow{\xi} k) \mapsto (V \to \xi)$

$(\xi \circ f) = f^*(\xi)$

Exercise: describe $D^2 : \underline{\mathrm{Vect}}_k \to \underline{\mathrm{Vect}}_k$ functor of "double dual"

$$D^2(V) = D(D(V)) = \mathrm{Hom}_k(\mathrm{Hom}_k(V,k), k)$$

& its relationship w/ the identity functor $\mathrm{id}_{\underline{\mathrm{Vect}}_k} : \underline{\mathrm{Vect}}_k \to \underline{\mathrm{Vect}}_k$.

for any $V \in \underline{\mathrm{Vect}}_k$, define a function

$$V \xrightarrow{\delta_V} V^{**} = D^2(V)$$

$$\mathrm{Id}_{\mathrm{Vect}_k} \xrightarrow{\delta} D^2$$

"δ is a function between functors"

δ is a natural transformation

restricted to subcategory

$\mathrm{FinVect}_k$, δ is an isomorphism.

but not in general!

why?

$$V^* = \mathrm{Hom}_k(V, k)$$

$$\xi : V \to k$$

$$V \xrightarrow{\delta_V} V^{**} = \mathrm{Hom}_k(V^*, k) = \mathrm{Hom}_k(\mathrm{Hom}_k(V,k), k)$$

$$\vec{v} \longmapsto \left\{ \{ \xi : V \to k \} \longmapsto \xi(\vec{v}) \in k \right\}$$

$$\xi \in V^*$$

$\delta_V(\vec{v})$ = "Evaluate the functional at $\vec{v}$"

Lecture 2: Natural transformations & The Yoneda Lemma

given two categories $\mathcal{C}$ & $\mathcal{D}$,

a functor $F: \mathcal{C} \to \mathcal{D}$

is a structure-preserving morphism between categories

defines $F: Ob(\mathcal{C}) \to Ob(\mathcal{D})$

$$X \mapsto F(X)$$

$$Hom_{\mathcal{C}}(X,Y) \mapsto Hom_{\mathcal{D}}(F(X), F(Y))$$

$$\left(X \xrightarrow{f} Y \right) \longmapsto \left(F(X) \xrightarrow{F(f)} F(Y) \right)$$

→ a functor $F: \mathcal{C} \to \mathcal{D}$ is <u>full</u> if, for any $X, Y \in Ob(\mathcal{C})$, the function

$$Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X), F(Y))$$

is surjective.

$F: \mathcal{C} \to \mathcal{D}$ is <u>faithful</u> if

$$Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X), F(Y))$$

is injective.

→ F is <u>fully faithful</u> if

"

is a bijection.

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \simeq \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

Ex: (full) subcategories:

$$\begin{tikzcd} \underline{\mathrm{FinSet}} \arrow[r, hook, "\iota"] & \underline{\mathrm{Sets}} \\ X \arrow[r, mapsto] & X \\ X \xrightarrow{f} Y \arrow[r, mapsto] & X \xrightarrow{f} Y \end{tikzcd}$$

inclusion functor

$$\mathrm{Hom}_{\underline{\mathrm{FinSets}}}(X,Y) \simeq \mathrm{Hom}_{\underline{\mathrm{Set}}}(X,Y)$$

$\underline{\mathrm{FinVect}}_k$ = cat. of finite dimensional vector spaces over a field k.

$$\begin{tikzcd} \underline{\mathrm{FinVect}}_k \arrow[d, hook, "\iota"] \\ \underline{\mathrm{Vect}}_k \end{tikzcd}$$

inclusion functor

$$\mathrm{Hom}_{\underline{\mathrm{FinVect}}_k}(V,W) \simeq \mathrm{Hom}_{\underline{\mathrm{Vect}}_k}(V,W).$$

Natural Transformations

given $F, G : \mathcal{C} \to \mathcal{D}$, what is a morphism $\eta : F \to G$?

associate, for $X \in \mathrm{Ob}(\mathcal{C})$, a morphism in $\mathcal{D}$:

$$\eta_X : F(X) \longrightarrow G(X)$$

"the component of η over X"

for any morphism $X \xrightarrow{f} Y$, a commutative diagram:

$G(f) \circ \eta_X = \eta_Y \circ F(f)$

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

identity functor

$$id_{\underline{Vect}_k} = \mathbb{1}_k \; : \; \underline{Vect}_k \to \underline{Vect}_k$$

$$\downarrow$$

$$D = \underline{Vect}_k^{op} \longrightarrow \underline{Vect}_k$$

$$V \longmapsto \mathrm{Hom}_{\underline{Vect}_k}(V, k) = D(V)$$

$\nexists$ a natural transformation $\mathbb{1}_k \longrightarrow D$

but this does work for D^2, double dual.

$\exists \quad \mathbb{1}_k \to D^2$

$$id_{\underline{FinVect}_k} \xrightarrow{\sim} D^2|_{\underline{FinVect}_k} \qquad \text{isomorphism}$$

→ a finite-dim vector space is <u>naturally</u> isomorphic to its double-dual.

⇒ <u>Yoneda Lemma</u>:

For any $X \in Ob(\mathcal{C})$, $\quad \mathrm{Hom}_{\mathcal{C}}(X,Y) \in Ob(\underline{Set})$

$$h_X := \mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \underline{Set}$$

$$Y \longmapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

$$h_X(g) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

$$(X \xrightarrow{f} Y) \longmapsto (X \xrightarrow{g \circ f} Z)$$

$$\begin{tikzcd} X \arrow[dr, "g"'] & & Z \\ & Y \arrow[ur, "f"'] & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, bend left, "g \circ f"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

$h_X(g) = g_*$ "pushforward along g"

$$h^X := \mathrm{Hom}_{\mathcal{C}}(-,X) : \mathcal{C}^{op} \xrightarrow{\text{contravariant functor}} \mathrm{Set}$$

$$Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y,X) = h^X(Y)$$

$$h^X(Z) \xrightarrow{g^*} h^X(Y) = h^X(Y \xrightarrow{g} Z)$$

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "f\circ g"'] & Z \arrow[d, "f"] \\ & X \end{tikzcd}$$

$h^X(g) = g^*$ "pullback along g"

$$\mathrm{Hom}_{\mathcal{C}}(Y,X) \longleftarrow \mathrm{Hom}_{\mathcal{C}}(Z,X)$$

$$\begin{tikzcd} Y \arrow[rr, "f\circ g"] \arrow[dr, "g"'] & & X \\ & Z \arrow[ur, "f"'] & \end{tikzcd} \qquad Z \xrightarrow{f} X$$

$$h^X(g)(f) := f\circ g$$

$\Rightarrow$ what does a natural transformation

$\eta : h^X \longrightarrow G$

look like, for some $G : \mathcal{C}^{op} \to \mathrm{Set}$?

This is context of yoneda lemma.

$\Rightarrow$ denote $[\mathcal{C}^{op}, \mathrm{Set}]$ the category of contravariant functors from $\mathcal{C}$ to Set.

[yoneda]

$$\mathrm{Hom}_{[\mathcal{C}^{op},\mathrm{Set}]}(h^X, G) \simeq G(X)$$

$$(\eta : \mathrm{Hom}_{\mathcal{C}}(-,X) \to G) \mapsto \xi \in G(X)$$

in particular, if $G \simeq h^Y$, there is a bijection.

$$\mathrm{Hom}_{[\mathcal{C}^{op},\mathrm{Set}]}(h^X, h^Y) \simeq h^Y(X) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

Left as an exercise...

Stop now and prove it yourself!

Want a way to associate

$$(\eta : h^X \to G) \overset{\sim}{\longmapsto} \xi \in G(X)$$

$\exists$

$$\begin{array}{ccc} id_X \in \mathrm{Hom}_{\mathcal{C}}(X,X) = h^X(X) & \xrightarrow{\eta_X} & G(X) \\ \Psi & & \Psi \\ id_X & \longmapsto & \eta_X(id_X) =: \xi \end{array}$$

eta

xi

Exercise 1: Show that $\xi := \eta_X(id_X) \in G(X)$ uniquely determines the component

$$\eta_Y : h^X(Y) \to G(Y)$$

for any $Y \in ob(\mathcal{C})$.

$$h^X \xrightarrow{\eta} G \longmapsto \eta_X(id_X) \in G(X)$$

Exercise 2: Show this is a bijection!

$\Longrightarrow$ $\mathrm{Hom}_{[\mathcal{C}^{op}, \underline{Set}]}(h^X, h^Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X,Y)$

$$h^\wedge : \mathcal{C} \to [\mathcal{C}^{op}, Set]$$

$$X \longmapsto h^X := \mathrm{Hom}_{\mathcal{C}}(-,X)$$

by yoneda, $h^\wedge$ is fully faithful (embedding)

$h^\wedge$ embeds $\mathcal{C}$ as a full subcategory of $[\mathcal{C}^{op}, \underline{Set}]$.

next class, we'll see $[\mathcal{C}^{op}, \underline{Set}]$ is much nicer than $\mathcal{C}$

analogy w/ analysis,

$$\{\text{smooth } f\text{'ns}\} \hookrightarrow \{\text{Distributions}\}$$

$$f \longmapsto \{ g \mapsto \int_X fg \}$$

compact support

analogy w/

$$\mathcal{C} \xhookrightarrow{h^\wedge} [\mathcal{C}^{op}, \underline{Set}]$$

← can define a "topology" on this

$$X \longmapsto \{ Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X) \}$$

Yoneda gives a full subcategory of functors $\mathcal{C}^{op} \to \underline{Set}$ isomorphic to h^X for some X.

⇝ if $G \simeq h^X$ for some X, call G a representable functor.

⇝ in problem session 1;

for any set X,

$$\mathcal{P}(X) = \text{power set of } X$$
$$= \{\text{set of subsets of } X\}$$

by previous,

in $\underline{Set}$, $\{0,1\}$ is a subobject classifier

$$\mathcal{P}(X) \simeq \mathrm{Hom}_{\underline{Set}}(X, \{0,1\})$$

$$A \longmapsto \left\{ \chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases} \right\}$$

$$\mathcal{P} : \underline{Set}^{op} \to \underline{Set}$$

$X \xrightarrow{f} Y$ determines $\mathcal{P}(Y) \to \mathcal{P}(X)$

$$B \mapsto f^{-1}(B)$$

$\mathcal{P} \cong h^{\{0,1\}}$ is a representable functor!

⇝ for any category $\mathcal{C}$,

& $X \in \mathrm{Ob}(\mathcal{C})$

$$\mathrm{Sub}_{\mathcal{C}}(X) = \{\text{subobjects of } X\} \in \underline{Set}$$

$$\mathrm{Sub}_{\mathcal{C}} : \mathcal{C}^{op} \to \underline{Set}$$

whether or not $\mathrm{Sub}_{\mathcal{C}}$ is representable is equivalent to $\mathcal{C}$ having a subobject classifier $\Omega \in \mathrm{Ob}(\mathcal{C})$

important in categorical logic, topos theory

Exercise 3: define & prove a covariant version of yoneda lemma

$h_X := \mathrm{Hom}_{\mathcal{C}}(X, -) \to F$ in $[\mathcal{C}, \underline{Set}]$

covariant functors

$$\mathrm{Hom}_{[\mathcal{C}, \underline{Set}]}(h_X, F) \simeq F(X)$$

& analogue of yoneda embedding:

$$h^\vee : \mathcal{C}^{op} \to [\mathcal{C}, \underline{Set}]$$

3. Examples

The central point about examples of representable functors is:

Representable functors are ubiquitous .

To a fair extent, category theory is all about representable functors and the other universal constructions: Kan extensions, adjoint functors, limits, which are all special cases of representable functors – and representable functors are special cases of these.

Listing examples of representable functors in category theory is much like listing examples of integrals in analysis: one can and does fill books with these. (In fact, that analogy has more to it than meets the casual eye: see coend for more).

Keeping that in mind, we do list some special cases and special classes of examples that are useful to know. But any list is necessarily wildly incomplete.

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