Homework 9

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Let $\{f_n\}$ be a sequence of measurable functions with respect to μ.

   a) Prove that if $1 \le p < \infty$ and $a > 0$ then

   $$\mu(\{x : |f(x)| \ge a\}) \le \frac{\int |f|^p \, d\mu}{a^p}.$$

   This is known as Chebyshev's inequality.

   b) If f_n converges to f in L^p, then it converges in measure.

   Solution:

   (a) Let $A = \{x : |f(x)| \ge a\}$, and note that

   $$\int |f|^p \, d\mu \ge \int_A |f|^p \, d\mu \ge \mu(A) a^p,$$

   which we can rearrage to be the desired inequality.

   (b) Let $\epsilon > 0$ be arbitrary. We now apply Chebyshev's inequality to get that

   $$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \le \epsilon^{-p} \int |f_n - f|^p \to 0.$$

2. Suppose $|f_n| \le g \in L^1$ and $f_n \to f$ in measure. It can be shown than if $f_n \ge 0$ and $f_n \to f$ in measure, that

   $$\int f \le \liminf \int f_n.$$

   Use this variant of Fatou's lemma to prove the following.

   a) Show that

   $$\int f = \lim_{n\to\infty} \int f_n$$

   b) Show that $f_n \to f$ in L^1.

Solution:

(a) Using the variant of Fatou's Lemma for convergence in measure, we just repeat the processes in the proof of the Dominated convergence theorem. In essence, what we have here is a variant of DCT for when functions converge in measure!

(b) Note that $||f_n - f|| \leq |g| + |f| \in L^1$, and $|f_n - f|$ converges to 0 is meansure, so by part (a) we have

$$\int |f - f_n| \to 0,$$

which is the definition of convergence in L^1.

3. Suppose that f_n and f are measureable functions such that for each $\epsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \infty.$$

Prove that $f_n \to f$ a.e.

Solution: In class we proved that under these conditions we can find a subsequence $f_{n_j} \to f$ a.e. We can recycle that argument. Let $A_{n.k} = \{x : |f_n - f| > 1/k\}$. By the given condition, for each fixed k, $\sum \mu(A_{n,k})$ is finite, so $\mu(A_{n,k}) \to 0$, so there is some N_k such that $\mu(A_{m,k})$ is as small as we want for all $m \geq N_k$. We consider the set

$$A = \cap_{k=1}^{\infty} \left(\cup_{n=1}^{\infty} A_{n,k}\right),$$

and proceed as we did in the proof in class. That is, we note that

$$\mu(A) = \lim \mu(\cup A_{n,k}) \leq \lim_{k \to \infty} \sum_{n=k}^{\infty} \mu(A_{n,k}) = 0,$$

where the last equality comes from the fact that the tail of a converging series must go to zero. However now for any $k > 0$, if $x \notin A$, there is some N_k such that $|f_m(x) - f(x)| < 1/k$ for all $m \geq N_k$, that is, $f_n \to f$ a.e.
