# QF Group Theory CC2022 By Zaiku Group

Lecture 12

Delivered by Bambordé Baldé

Friday, 11/8/2022

# Learning Journey Timeline

Topology & Differential Geometry
9

Group Theory
12/20?

Lie Groups & Rep.
10 - 16?

Completed | Ongoing | TBC (summer) | n is the number of live lectures |

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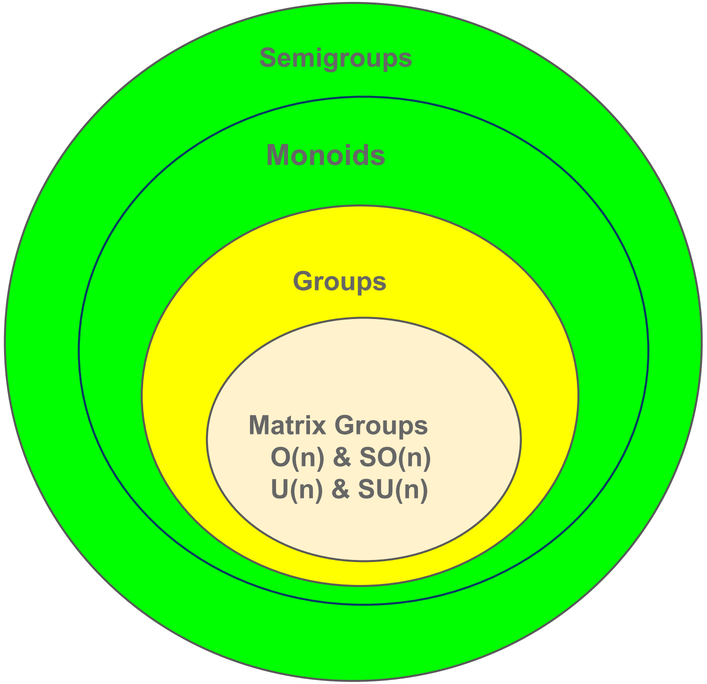


Course Approach Overview

Completed!

We’re here!

August 20, 4pm BST (11am EST)


Live Q&A: Measure Theory & Functional Analysis

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# Facts about cycles (A)

- It's clear that a $1-cycle$ $(a_1)$ corresponds to the identity for any $a_1$. This is why in the textbooks $1-cycles$ are omitted when considering cycle decomposition.
- A $2-cycle$ (aka transposition) $(a_1a_2)$ corresponds to a unique permutation $\sigma \in S_n$ which switches $a_1$ and $a_2$ but leaves all the other elements of $X = \{1, 2, \ldots, n\}$ unchanged. This means the cycle $(a_1a_2)$ is the same as $(a_2a_1)$ right?
- But $k \geq 3-cycles$ changing the order of the elements may result in a different cycle! For example, consider the $3-cycles$ $(123)$ and $(132)$. It's clear that $(132) \neq (123)$ right?

**Curiosity question 1:** Is there a way of changing the order of some elements of a $k-cycle$ $(a_1a_2a_3\ldots a_k)$ such that the underlying permutation stays the same?

# Facts about cycles (B)

**Proposition 1.0**

Let $(a_1 a_2 a_3 \ldots a_k)$ be a $k - cycle$. Then we have the following identities:

1. $(a_1 a_2 a_3 \ldots a_k) = (a_2 a_3 \ldots a_k a_1) = (a_3 a_4 \ldots a_k a_1 a_2) = (a_k a_1 \ldots a_{k-2} a_{k-1})$.
2. $(a_1 a_2 a_3 \ldots a_k)^{-1} = (a_k a_{k-1} \ldots a_k a_1)$.

**Some remarks:**

- The first identity above tells us that we can start a cycle at any point $a_i$. All we need is to then list the elements in order after that i.e. the next element must be $a_{i+1}$. Also, once we get to $a_k$ the next one is $a_1$ and so on.
- The second identity obviously means that the inverse element of a $k - cycle$ $(a_1 a_2 a_3 \ldots a_k)$ is the $k - cycle$ with the elements listed in

  the opposite order i.e $(a_k a_{k-1} \ldots a_k a_1)$.
- Hence, for a $2 - cycle$ (transposition) $(a_1 a_2)$, we have $(a_1, a_2)^{-1} = (a_2, a_1) = (a_1, a_2)$!

# Facts about cycles (C)

**Proposition 1.1**

Let $(a_1 a_2 a_3 \dots a_k)$ and $(b_1 b_2 b_3 \dots b_j)$ be disjoint i.e. $a_k \neq b_j$ for all $k$ and $j$. Then $(a_1 a_2 a_3 \dots a_k) \circ (b_1 b_2 b_3 \dots b_j) = (b_1 b_2 b_3 \dots b_j) \circ (a_1 a_2 a_3 \dots a_k)$.

- Hence, disjoint cycles commute with each other!

**Curiosity question 2 (challenge):** What if the cycles are non-disjoint? Do they necessarily not commute with each other?

**Proposition 1.2**

Let $\sigma \in S_n$ be an arbitrary permutation and $(a_1 a_2 a_3 \dots a_k)$ be a $k-cycle$. Then $\sigma \circ (a_1 a_2 a_3 \dots a_k) \circ \sigma^{-1} = (\sigma(a_1)\sigma(a_2)\sigma(a_3)\dots\sigma(a_k))$.

**Curiosity question 3 (challenge):** Is $\sigma \circ (a_1 a_2 a_3 \dots a_k) \circ \sigma^{-1}$ again a $k - cycle$?

# Even and Odd Permutations

**Theorem 1.0**

Let $\sigma \in S_n$ be an arbitrary permutation. Then $\sigma$ can be written as a product of $2-cycles$.

- For example, if we consider a $k-cycle$ $(a_1 a_2 a_3 \ldots a_k)$. Then $(a_1 a_2 a_3 \ldots a_k) = (a_1 a_n) \circ (a_1 a_{n-1)} \circ \ldots \circ (a_1 a_3) \circ (a_1 a_2)$.

**Definition 1.0**

A permutation $\sigma \in S_n$ is said to be even if it can be written as a product of an even number of $2-cycles$ and odd if it can be written as a product of an odd number of $2-cycles$.

- Hence, once you break down a permutation into a product of $2-cycle$ you'll know whether it's odd or even!

**Theorem 1.1**

Let $\sigma \in S_n$ be a permutation. Then $\sigma$ is either even or odd, not both!

**Theorem 1.2**

A $k - cycle$ is an even permutation if $k$ is odd and odd permutation if $k$ is even!

**Curiosity questions:**

1. Is the identity permutation $id \in S_n$ even or odd?
2. Are $2 - cycles$ even or odd permutations?

# The Sign of a Permutation

**Definition 1.1**

Let $\sigma \in S_n$ be a permutation. The the sign of $\sigma$ is defined as follows:

$$sign(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

- It's easy to see that $sign : S_n \longrightarrow \{1, -1\}$ is a group homomorphism i.e. $sign(\sigma_1 \circ \sigma_2) = sign(\sigma_1) sign(\sigma_2)$ for all $\sigma_1, \sigma_2 \in S_n$.
- The kernel of this homomorphism $Ker(sign) = \{\sigma \in S_n \mid sign(\sigma) = 1\}$ is a nontrivial subgroup of $S_n$ denoted $A_n$ (aka alternating group)!

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