QF Group Theory CC2022
By
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Lecture 01 SOLUTIONS

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Binary Operation on a Set

Definition 1.0

Let S be a nonempty set. Informally, a binary operation $*$ on S is a rule that takes any two elements $a, b \in S$ to generate another element $a * b \in S$.

- More formally, a binary operation $*$ on S is a map $* : S \times S \longrightarrow S$.
- Hence, given $(a, b) \in S \times S$, $a * b$ is just an abbreviation for $*((a, b))$ i.e. $a * b$ is an abuse of notation!
- It is possible to equip a set S with more than one binary operation! For example, the algebraic structures of rings and fields are obtained that way.

Definition 1.1

Let S be a nonempty set. A binary operation $*$ on S is said to be commutative (or abelian) if $a * b = b * a$ for any pairs $a, b \in S$. Otherwise, whenever we have $a * b \neq b * a$ for some $a, b \in S$, we say that $*$ is a noncommutative (or non-abelian) binary operation on S.

Binary Operation Examples (Part A)

Example 1

Let S be the set of natural numbers $\mathbb{N}$ and let the operation $*$ be the ordinary addition of natural numbers $+$.

- $+$ defines a binary operation on $\mathbb{N}$ right? Yes cause $\forall a, b \in S$ also $a+b \in S$

Example 2

Let us consider $S = \{a \in \mathbb{N} \mid a \text{ is odd }\}$ and $*$ be the ordinary multiplication of natural numbers $\times$.

- Does $\times$ define a binary operation on S? $(2n+1)(2m+1) = 4nm+2n+2m+1$ $\underbrace{4nm+2n+2m}_{}$ = even $+1$ = odd
  So odd $\times$ odd = odd Yes, $*$ on S is a Bin. Oper.

Example 3

Let consider again $S = \{a \in \mathbb{N} \mid a \text{ is odd }\}$ and let now $*$ be the ordinary addition of natural numbers $+$.

- Does $+$ also define a binary operation on S?
  $(2n+1)+(2m+1) = \underbrace{2n+2m+2}_{\text{even!}}$. No, $*$ on S is not a Binary Operation

Binary Operation Examples (Part B)

Example 1

Let A be a non-empty set and let $S = \{f : A \longrightarrow A \mid f \text{ is a bijection } \}$. Now suppose that $*$ is the composition $\circ$ of maps in S.

- Is $\circ$ a binary operation on S? If yes, is it abelian or non-abelian?

Example 2

Let S be the set $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries and let the operation $*$ be the ordinary matrix multiplication.

- Is $*$ also a binary operation on $M_n(\mathbb{C})$? Is it abelian or non-abelian?

Example 3

Let S be the set denote $GL(n, \mathbb{C})$ of invertible $n \times n$ matrices with complex entries and let the operation $*$ be still the ordinary matrix multiplication.

- Is $*$ also a binary operation on $GL(n, \mathbb{C})$? Is it abelian or non-abelian?
- What if $*$ is now the ordinary addition of matrices?

Ex 1 $f: A \to A$ f is bijective Bijective = injective and surjective = | Injective: $\forall a,b \in A$ $f(a) = f(b)$ iff $a = b$

Surjective $\forall b \in B\ \exists a \in A$ s.t. $f(a) = b$

$\circ = f \circ f^{-1} (= g)$, $f: A \to A$, $g: A \to A$ $f \circ g : A \to A$

Since f and g are bijective also $(f \circ g)$ is bijective

Yes, $\circ$ is a binary operation on S. $(f \circ g : A \to A)$. It is NOT abelian cause $f \circ g \neq g \circ f$

Ex 2

$A, B = M_n(\mathbb{C})$ $A, B \in S$ $* = A \times B$ $A_{n \times n} \times B_{n \times n} = C_{n \times n}$ $C_n \in S$ -

Yes, $*$ is a binary operation on S. It is not abelian cause $AB \neq BA$

Ex 3

$GL(n, \mathbb{C})$ General Linear group of invertible matrix over $\mathbb{C}$

As per Ex 2 $*$ is a binary operation.

If $* = +$ then $*$ is a binary operation and it is abellian cause $A_n, B_n \in GL$ $A_n + B_n = C_n$ $C_n \in GL$ and $A_n + B_n = B_n + A_n$

Semigroup Structure

Definition 1.2

A semigroup is a pair $(S, *)$ where S is a nonempty set and $*$ is a binary operation on S such that $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$.

- The condition $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$ is called the 'associativity law' and we say that the operation $*$ is associative.
- Whenever the operation $*$ is understood from the context and fixed, we just say S is a semigroup and we omit writing the pair $(S, *)$.
- A semigroup $(S, *)$ is said to be abelian or non-abelian if $*$ is a abelian or non-abelian binary operation respectively.

Definition 1.3

Let $(S, *)$ be a semigroup and $S' \subseteq S$. Then S' is said to be subsemigroup of $(S, *)$ if $(S', *)$ is also a semigroup.

- Obviously, $(S, *)$ is trivially a subsemigroup of itself!

Semigroup Examples

Example 1

Let A be a non-empty set and let $S = \{f : A \longrightarrow A \mid f \text{ is a bijection }\}$. Now suppose that $*$ is the composition $\circ$ of maps in S.

- Is S a semigroup under $\circ$? If yes, is it abelian or non-abelian?

1) Yes, it is a semigroup

$(f \circ g) \circ h = f \circ (g \circ h)$

No it is not abelian

$f \circ g \neq g \circ f$

eg: $A = \mathbb{N}$ $f = 2x+1$ $g = x^2$

Example 2

Let S be the set $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries and let the operation $*$ be the ordinary matrix multiplication.

- Is $M_n(\mathbb{C})$ a semigroup under matrix multiplication? Is it abelian or non-abelian? What about under matrix addition?

2) Yes it is a semigroup. under multiplication and addition.

$M_n(\mathbb{C})$ it is not abelian under multiplication ($A_n \times B_n \neq B_n \times A_n$) but it is abelian under addition ($A_n + B_n = B_n + A_n$)

Example 3

Let S be the set denote $GL(n, \mathbb{C})$ of invertible $n \times n$ matrices with complex entries and let the operation $*$ be still the ordinary matrix multiplication.

- Is $GL(n, \mathbb{C})$ a semigroup under matrix multiplication?

What about under matrix addition?

3) $GL(n, \mathbb{C})$ is a semigroup under multiplication and addition.

Semigroups Structure Challenge

1. Let $(S, *)$ be a semigroup and let $S' = \{a \in S \mid a * x = x * a \text{ for all } x \in S\}$. Is it true that $(S', *)$ is a subsemigroup of $(S, *)$? Yes.
2. Let $(S_1, *_1)$ and $(S_2, *_2)$ be two semigroups. Construct a semigroup structure on the Cartesian product $S_1 \times S_2$ using the respective semigroup structure. Can you generalise your construction to $(S_1, *_1), (S_2, *_2), \ldots, (S_n, *_n)$? $((S_1 \otimes S_2), (*_1 \circ *_2))$ $\otimes$ = Tensor Product
3. Assuming that $(S_1, *_1)$ is abelian and $(S_2, *_2)$ is non-abelian, is your constructed semigroup structure on $S_1 \times S_2$ abelian or non-abelian?
4. Identify at least a nontrivial subsemigroup structure for the constructed semigroup structure on $S_1 \times S_2$ above.
5. Let $\mathbb{Z}_2 = \{0, 1\}$, $\mathbb{Z}_3 = \{0, 1, 2\}$ and $\mathbb{Z}_4 = \{0, 1, 3\}$. Identify at least a semigroup structure for $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$.
6. Identify at least a subsemigroup structure (if any) from the identified semigroup structures on $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$ above.

QUANTUM FORMALISM

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