Homework 9

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Let $\{f_n\}$ be a sequence of measurable functions with respect to μ.

   a) Prove that if $1 \leq p < \infty$ and $a > 0$ then

   $$\mu(\{x : |f(x)| \geq a\}) \leq \frac{\int |f|^p \, d\mu}{a^p}.$$

   This is known as Chebyshev's inequality.

   b) If f_n converges to f in L^p, then it converges in measure.

2. Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure. It can be shown than if $f_n \geq 0$ and $f_n \to f$ in measure, that

   $$\int f \leq \liminf \int f_n.$$

   Use this variant of Fatou's lemma to prove the following.

   a) Show that

   $$\int f = \lim_{n\to\infty} \int f_n$$

   b) Show that $f_n \to f$ in L^1.

3. Suppose that f_n and f are measureable functions such that for each $\epsilon > 0$, we have

   $$\sum_{n=1}^{\infty} \mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \infty.$$

   Prove that $f_n \to f$ a.e.
