Homework 6

Directions: Answer the following questions. You are encouraged to work together, join the discussion sessions, use discord, and ask me questions!

1. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and $f : X \to Y$. Suppose $X = A \cup B$ where $A, B \in \mathcal{M}$ So that f is measurable on X if and only if f is measurable on A and B.

   Solution: If f is measurable on X, then this means for all $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$. Moreover $f^{-1}(E) \cap A \in \mathcal{M}$, which means that f is measurable on A. Likewise f is measurable on B. Conversely, if f is measurable on both A and B, then note for any $E \in \mathbb{N}$, we have that $f^{-1}(E) \cap A \in \mathcal{M}$ and $f^{-1}(E) \cap B \in \mathcal{M}$, so $(f^{-1}(E) \cap A) \cup (f^{-1}(E) \cap B) = f^{-1}(E) \cap (A \cup B) = f^{-1}(E) \cap X = f^{-1}(E) \in \mathcal{M}$, thus f is measurable on X.

   Alternate Solution: One can observe that χ_A and χ_B are measurable functions, and as such, $f\chi_A$, $f\chi_B$ are measurable on A, B respectively if f measurable on X. If f is measurable on A and B then $f = f\chi_A + f\chi_B$ is measurable on X.

2. Suppose $f, g : X \to \mathbb{R}$ are measurable, and let $c \in \mathbb{R}$. Prove that fg and cf are both measurable.

   Hint: To show fg is measurable, first prove that f^2 is measurable, then note $2fg = (f + g)^2 - f^2 - g^2$.

   Solution: To see the cf, we will suppose w.l.o.g. that $c > 0$. Then note that since f is measurable, we have that for any$a \in \mathbb{R}$, $\{x \mid f(x) > a\} \in \mathcal{M}$, for the corresponding σ-algebra on X. In particular we note that

   $$\{x \mid cf(x) > a\} = \{x \mid f(x) > a/c\} \in \mathcal{M},$$

   thus cf is measurable. If $c < 0$ a similar argument holds. If $c = 0$ then it is vacuously true.

   To see the product fg, we note that $\{x \mid f^2(x) > a\} = \{x \mid f(x) > a\} \cup \{x \mid f(x) < -a\} \in \mathcal{M}$, so f^2 is measurable. Then just note since sums, squares, and constant multiples of measurable functions are still measurable, we have that $fg = 1/2((f + g)^2 - f^2 - g^2)$ is measurable.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

   Hint: Suppose f is increasing. Let $a \in \mathbb{R}$, $z = \sup\{y \mid f(y) \leq a\}$, and consider $f^{-1}((a, \infty))$.

   Solution: We will assume f is increasing. Given $a \in \mathbb{R}$, let $z := \sup\{y \mid f(y) \leq a\}$. Then if $f(z) \leq a$, then we have $f^{-1}((a, \infty)) = \{x \mid f(x) > a\} = (z, \infty)$, a Borel set. If $f(z) > a$

then $f^{-1}((a,\infty)) = \{x \mid f(x) > a\} = [z,\infty)$. So in either situation, $f^{-1}((a,\infty))$ is a Borel set, thus it is Borel measurable. If f is decreasing, then let $g = -f$, so g is increasing, and then we're done by our above argument.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable. Show that there exists a Borel measurable function g such that $f = g$ a.e.

   Hint: Consider a sequence of Lebesgue measurable simple functions that approach the function f, and then modify them to be Borel measurable.

   Solution: First we note that since the Lebesgue σ-algebra $\mathcal{L}$ is the completion of $\mathcal{B}_{\mathbb{R}}$, for any Lebesgue measurable characteristic function, χ_A, there is some Borel measurable set B such that $\chi_A = \chi_B$ almost everywhere, differing only by some Lebesgue null set. By the theorem from class we can find a sequence of simple functions φ_k that are Lebesgue measurable, such that $\varphi_k \to f$. For each φ_k there is some ψ_k which is Borel measurable, and $\varphi_k = \psi_k$ almost everywhere, say off of some null set N_k. In particular we have this for each k, and $m(\cup N_k) \leq \sum m(N_k) = 0$, so the union remains a null set. Thus there is some Borel measurable set N such that $N \supset \cup N_k$, and the Borel measure of N, $\mu(N) = 0$. Finally we now consider the functions $g_k := \psi_k \chi_{\mathbb{R}\setminus N}$, which is still Borel measurable, and equals φ_k a.e. Thus we have that $g_k \to f$ pointwise for all $x \in \mathbb{R}\setminus N$. In particular, $g := \lim g_k = f$ a.e. Since we proved in class that the limit of measurable functions remains measurable, we know that g is Borel measurable.
