RPCA for Modal Decomposition of Corrupt Fluid Flows

Review of paper by *Scherl et. al.*

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November 10, 2022

Motivation

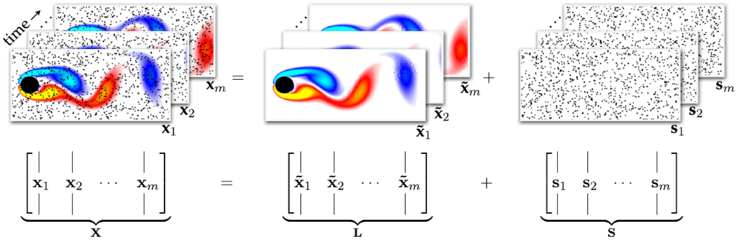

Figure: Schematic of RPCA filtering applied to corrupt flow field data. Corrupted snapshots are arranged as column vectors in the matrix **X**, which is decomposed into the sum of a low-rank matrix **L** and a sparse matrix of outliers **S**.

Outline

- Overview of standard POD and DMD
- Robust PCA
- Description of flow fields
- Results of RPCA filtering

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Proper orthogonal decomposition

There are several variants of POD , paper presents a variant of the *snapshot POD* of Sirovich that relies on the numerically stable SVD. POD modes are obtained by computing the SVD of $\mathbf{X} \in \mathbb{R}^{n \times m}$:

$$\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T, \tag{1}$$

where , $\mathbf{U} \in \mathbb{R}^{n \times n}$, $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times m}$ and $\mathbf{V} \in \mathbb{R}^{m \times m}$. The columns of $\mathbf{U}$ are *POD modes* with the same dimension as a flow field $\mathbf{x}$. POD modes are orthonormal so that $\mathbf{U}^T\mathbf{U} = \mathbf{I}$; similarly $\mathbf{V}^T\mathbf{V} = \mathbf{I}$. Moreover, the columns of $\mathbf{U}$ (resp. rows of $\mathbf{V}^T$) are arranged in order of their importance in describing the data.

Proper orthogonal decomposition

The matrix $\mathbf{X}$ will exhibit *low-rank structure*, so that it is well approximated by the first $r \ll m < n$ columns of $\mathbf{U}$ and $\mathbf{V}$:

$$\mathbf{X} \approx \mathbf{U}_r \boldsymbol{\Sigma}_r \mathbf{V}_r^T, \tag{2}$$

The Eckart-Young theorem states that this is the *optimal* rank-r approximation of the matrix $\mathbf{X}$ in a least-squares sense.

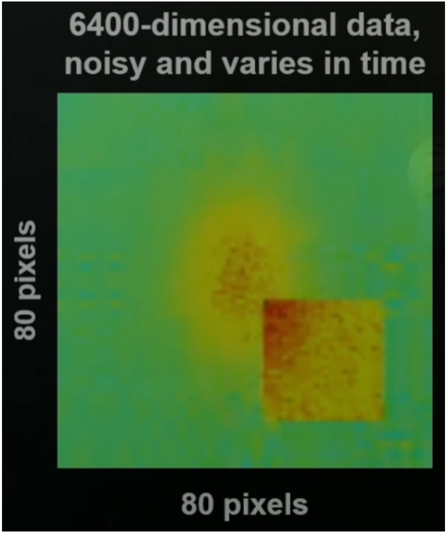
Dynamic mode decomposition

DMD is a modal decomposition technique that simultaneously identifies spatially coherent modes that are constrained to have the same linear behavior in time, given by oscillations at a fixed frequency with growth or decay

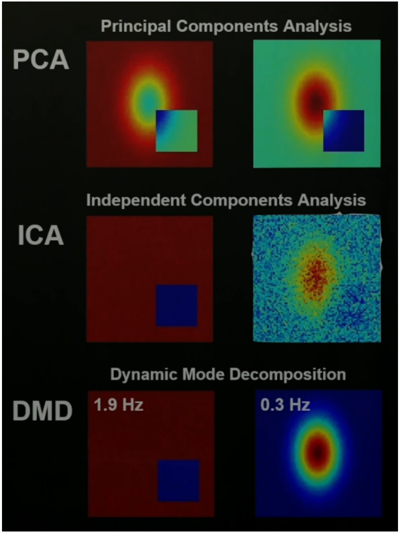
DMD seeks to identify the leading eigenvalues and eigenvectors of the best-fit linear operator $\mathbf{A}$ that evolves snapshots forward in time:

$$\mathbf{x}_{k+1} \approx \mathbf{A}\mathbf{x}_k. \tag{3}$$

Dynamic mode decomposition



Dynamic mode decomposition



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Robust PCA filtering

Mathematically, the goal is to find $\mathbf{L}$ and $\mathbf{S}$ that satisfy the following:

$$\min_{\mathbf{L},\mathbf{S}} \operatorname{rank}(\mathbf{L}) + \|\mathbf{S}\|_0 \text{ subject to } \mathbf{L} + \mathbf{S} = \mathbf{X}. \tag{4}$$

Robust PCA filtering

It is possible to solve for **L** and **S** with *high probability* using a convex relaxation of (4):

$$\min_{\mathbf{L},\mathbf{S}} \|\mathbf{L}\|_* + \lambda_0 \|\mathbf{S}\|_1 \ \text{ subject to } \ \mathbf{L} + \mathbf{S} = \mathbf{X}, \tag{5}$$

where $\|\cdot\|_*$ is the nuclear norm, given by the sum of singular values which is a proxy for the rank of the matrix and $\lambda_0 = \lambda/\sqrt{\max(n,m)}$ and $\|\cdot\|_1$ is the 1-norm of the matrix.

Robust PCA filtering

The convex problem in (5) is known as *principal component pursuit* (PCP), and may be solved using the augmented Lagrange multiplier (ALM) algorithm.

Robust PCA filtering

Specifically, an augmented Lagrangian may be constructed:

$$\mathcal{L}(\mathbf{L}, \mathbf{S}, \mathbf{Y}) = \|\mathbf{L}\|_* + \lambda_0 \|\mathbf{S}\|_1 + \langle \mathbf{Y}, \mathbf{X} - \mathbf{L} - \mathbf{S} \rangle + \frac{\upsilon}{2} \|\mathbf{X} - \mathbf{L} - \mathbf{S}\|_F^2. \quad (6)$$

Where Y is the matrix of Lagrange multipliers and υ is a hyperparameter. We then solve for $\mathbf{L}_k$ and $\mathbf{S}_k$ to minimize $\mathcal{L}$, update the Lagrange multipliers

$$\mathbf{Y}_{k+1} = \mathbf{Y}_k + \upsilon(\mathbf{X} - \mathbf{L}_k - \mathbf{S}_k),$$

and iterate until convergence.

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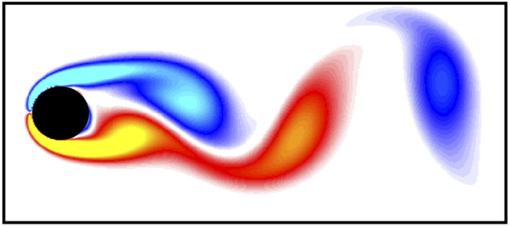
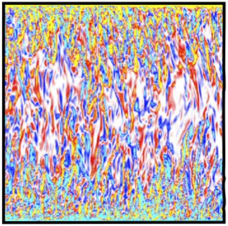
Flow fields

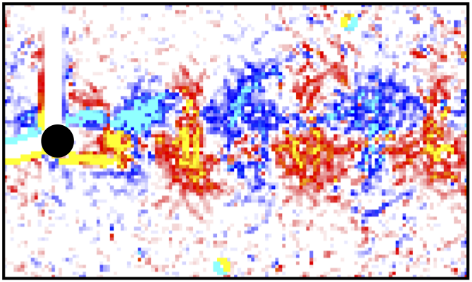
Flow past a cylinder, DNS



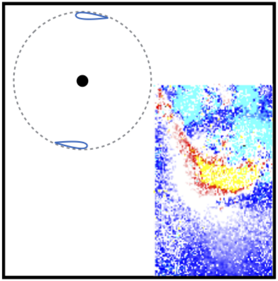
Channel flow, DNS



Flow past a cylinder, PIV



Turbine wake, PIV



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