

Financial Econometrics I: Course Project

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This report summarizes results for the Financial Econometrics I course project. The project studies the characteristics of financial time series data, focusing on typical features such as volatility clustering, heavy tails, skewness, and general non-normality of returns. The analyses were conducted using Python and relevant econometric libraries are listed in the appendix.

The rest of this report is organized as follows. Section 1 focuses on exploratory data analysis of the return data. Section 2 focuses on AR(1) modeling with GARCH(1,1) volatility. Finally, Section 3 concludes.

1 Characteristics of Financial Time Series

This section uses stock (S&PCOMP(RI)), commodity (RJEFCRT(TR)), and government bond (SPUTBIX(RI)) return data to analyze typical characteristics of financial time series at daily and weekly frequencies.

1.1 Exploratory Analysis of Index Returns

This section analyzes daily and weekly log returns of the selected indices. The log returns are computed as

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right) = \log(P_t) - \log(P_{t-1}). \quad (1)$$

As we are mostly focused on weekly and daily frequencies, the daily returns are computed using consecutive trading days, while the weekly returns are computed using Sunday-to-Sunday periods.

1.1.1 Main Crashes and Booms on Index Returns

Table 1 presents the five largest positive returns (booms) and five largest negative returns (crashes) for the S&P 500 stock index at both daily and weekly frequencies. Notable periods of extreme volatility include the 2008 Global Financial Crisis, the 2020 COVID-19 pandemic, and various other major economic events. Daily returns are calculated as log returns of consecutive trading days, while weekly returns aggregate returns over Sunday-to-Sunday periods.

The table shows that during financial market distress, price movements are typically very extreme. The next section examines whether these extreme moves can be explained under the assumption that the data generating process is normal.

1.1.2 Normality of Extreme Moves

To assess how likely extreme events such as those in Table 1 are under an assumed normal distribution, I calculate probabilities of events as extreme or more extreme as

Table 1: Top 5 Booms and Crashes for S&P 500, Government Bonds, and Commodities (Daily and Weekly)

Daily Returns

Date	Return (%)	Explanation
Top 5 Crashes		
2020-03-16	-12.76	COVID-19 lockdowns and economic shut-down fears
2020-03-12	-9.97	Pandemic uncertainty and market panic
2008-10-15	-9.46	Lehman Brothers collapse aftermath
2008-12-01	-9.35	Global Financial Crisis deepening
2008-09-29	-9.20	Initial banking crisis panic
Top 5 Booms		
2008-10-13	10.96	Government intervention announcements during GFC
2008-10-28	10.25	Relief rally on bank rescue measures
2025-04-09	9.09	Trump tariff pause relief
2020-03-24	8.98	COVID-19 stimulus package hopes
2020-03-13	8.91	Emergency Fed measures announced

Weekly Returns

Date	Return (%)	Explanation
Top 5 Crashes		
2008-10-12	-20.02	Peak panic during GFC; bank failures and credit freezes
2020-03-22	-16.20	COVID-19 lockdowns and economic shut-down fears
2001-09-23	-12.29	September 11 terrorist attacks aftermath
2020-03-01	-12.15	Initial COVID-19 global spread shock
2000-04-16	-11.12	Dot-com bubble collapse
Top 5 Booms		
2020-04-12	11.46	COVID-19 crash rebound via fiscal stimulus
2008-11-30	11.41	Government rescue measures during GFC
2009-03-15	10.25	Financial crisis bottom rally; stimulus anticipation
2008-11-02	10.01	Policy intervention expectations during GFC
2020-03-29	9.78	Emergency rate cuts and pandemic relief

$$P(X \geq x) = 1 - \Phi\left(\frac{x - \mu}{\sigma}\right) \quad \text{for booms,} \quad (2)$$

and

$$P(X \leq x) = \Phi\left(\frac{x - \mu}{\sigma}\right) \quad \text{for crashes.}$$

In equation (2), Φ is the cumulative distribution function of the standard normal distribution, and μ and σ are the sample mean and standard deviation of the return series, respectively. The results are summarized in Table 2.

Table 2 shows that the probabilities of observing such extreme returns under the normality assumption are extremely low, often in the range of 10^{-5} to 10^{-20} or lower. This indicates that extreme events observed in financial markets are extremely unlikely to occur if returns were normally distributed.

These observations further support the well-known fact that financial returns exhibit fat tails and are not well-modeled by a normal distribution. In other words, the magnitude of these crashes is inconsistent with the normality assumption.

Having established that extreme returns are inconsistent with normality, I next examine more general characteristics of the return distributions and test whether they are aligned with normality using the Jarque-Bera test.

1.1.3 Test of Normality using Jarque-Bera Test

Table 3 displays sample skewness, kurtosis, Jarque-Bera (JB) statistics, and corresponding p-values for daily and weekly returns of the S&P 500 stock index, government bonds, and commodities. The JB test assesses whether the sample data deviate from a normal distribution by examining skewness and kurtosis.

The statistics in Table 3 are computed as

$$S = \frac{1}{n} \sum_{i=1}^n \left(\frac{r_i - \bar{r}}{\sigma} \right)^3 \quad (\text{Sample skewness}) \quad (3)$$

$$K = \frac{1}{n} \sum_{i=1}^n \left(\frac{r_i - \bar{r}}{\sigma} \right)^4 \quad (\text{Sample kurtosis}) \quad (4)$$

$$JB = \frac{n}{6} \left(S^2 + \frac{(K - 3)^2}{4} \right), \quad (5)$$

where n is the sample size, S is the sample skewness, and K is the sample kurtosis. Under the null hypothesis of normality, the JB statistic follows a chi-squared distribution with 2 degrees of freedom, which I use to compute the p-values of the statistic.

The results in Table 3 show that the JB statistics are significantly large across all asset classes and frequencies, with p-values effectively equal to zero. Thus, I reject the null hypothesis of normality for all return series.

This further supports the earlier observations that asset returns are not normally distributed, and that sample skewness and kurtosis values are inconsistent with normality. The only exception is the skewness of government bond weekly returns, which is close to zero.

1.1.4 Autocorrelation and the Ljung-Box test

Having established that returns are not normally distributed, I next examine whether there is significant autocorrelation in the returns or squared returns.

Table 2: Probabilities of Observed Extreme Returns Under Normality Assumption
Panel A: Weekly Returns

Type	Return (%)	Z-score	Probability
<i>S&P 500</i>			
Crash	-20.02	-8.12	2.43e-16
Crash	-16.20	-6.58	2.40e-11
Crash	-12.29	-5.00	2.80e-07
Crash	-12.15	-4.95	3.70e-07
Crash	-11.12	-4.54	2.86e-06
Boom	11.46	4.55	2.65e-06
Boom	11.41	4.53	2.90e-06
Boom	10.25	4.06	2.42e-05
Boom	10.01	3.97	3.62e-05
Boom	9.78	3.88	5.28e-05
<i>Government Bonds</i>			
Crash	-1.96	-3.79	7.68e-05
Crash	-1.93	-3.73	9.47e-05
Crash	-1.92	-3.71	1.03e-04
Crash	-1.81	-3.51	2.26e-04
Crash	-1.79	-3.47	2.60e-04
Boom	3.09	5.66	7.55e-09
Boom	2.44	4.44	4.46e-06
Boom	2.39	4.35	6.89e-06
Boom	2.18	3.96	3.76e-05
Boom	1.86	3.35	3.99e-04
<i>Commodities</i>			
Crash	-16.06	-6.75	7.48e-12
Crash	-12.82	-5.39	3.44e-08
Crash	-11.88	-5.00	2.82e-07
Crash	-11.00	-4.63	1.79e-06
Crash	-10.11	-4.26	1.01e-05
Boom	12.61	5.24	8.25e-08
Boom	8.44	3.49	2.40e-04
Boom	8.31	3.44	2.95e-04
Boom	6.87	2.84	0.0023
Boom	6.32	2.61	0.0046

Type	Return (%)	Z-score	Probability
<i>S&P 500</i>			
Crash	-12.76	-10.65	8.66e-27
Crash	-9.97	-8.33	4.07e-17
Crash	-9.46	-7.90	1.37e-15
Crash	-9.35	-7.81	2.90e-15
Crash	-9.20	-7.68	7.82e-15
Boom	10.96	9.10	0.00e+00
Boom	10.25	8.51	0.00e+00
Boom	9.09	7.55	2.26e-14
Boom	8.98	7.45	4.62e-14
Boom	8.91	7.40	7.03e-14
<i>Government Bonds</i>			
Crash	-1.69	-6.77	6.22e-12
Crash	-1.67	-6.71	9.80e-12
Crash	-1.42	-5.69	6.22e-09
Crash	-1.16	-4.66	1.58e-06
Crash	-1.15	-4.61	1.97e-06
Boom	1.79	7.07	7.59e-13
Boom	1.74	6.89	2.79e-12
Boom	1.55	6.12	4.72e-10
Boom	1.38	5.46	2.41e-08
Boom	1.26	4.96	3.51e-07
<i>Commodities</i>			
Crash	-11.09	-10.32	2.92e-25
Crash	-7.94	-7.39	7.27e-14
Crash	-7.33	-6.83	4.33e-12
Crash	-6.88	-6.40	7.62e-11
Crash	-6.02	-5.61	1.02e-08
Boom	5.88	5.45	2.53e-08
Boom	5.75	5.33	5.05e-08
Boom	5.53	5.12	1.54e-07
Boom	5.22	4.84	6.64e-07
Boom	5.21	4.82	7.01e-07

Table 3: Skewness, Kurtosis and Jarque-Bera Test for Normality of Returns on a daily and weekly frequencies for S&P 500, Government Bonds, and Commodities indices.

Panel A: Weekly Returns

Asset	Skewness	Kurtosis	JB-stat	p-value
S&P 500	-0.867	7.084	2998.44	0.0000
Government Bonds	0.001	2.171	265.67	0.0000
Commodities	-0.737	4.061	1052.19	0.0000

Panel B: Daily Returns

Asset	Skewness	Kurtosis	JB-stat	p-value
S&P 500	-0.350	11.099	34837.71	0.0000
Government Bonds	-0.015	3.273	3017.02	0.0000
Commodities	-0.536	5.339	8352.10	0.0000

This is done using the Ljung-Box test, which tests the null hypothesis of no autocorrelation up to a specified lag h . The test statistic is computed as

$$Q = n(n+2) \sum_{k=1}^h \frac{\hat{\rho}_k^2}{n-k}, \quad (6)$$

where n is the sample size, $\hat{\rho}_k$ is the sample autocorrelation at lag k , and h is the number of lags being tested. Under the null hypothesis, the Q statistic follows a chi-squared distribution with h degrees of freedom. Table 4 shows the computed Ljung-Box statistics.

Figure 1 present the autocorrelation function (ACF) plots for daily and weekly returns as well as for squared daily and weekly returns of the selected assets.

Figure 1 clearly shows that there is only very small autocorrelation in the returns themselves, as most ACF values lie within the 95% confidence intervals. However, as expected, the squared returns in Figure 2 are highly autocorrelated, indicating volatility clustering and predictability in the squared returns.

The Ljung-Box statistics in Table 4 confirm that the null hypothesis of no autocorrelation in 10 lags is rejected at the 5% significance level for all asset classes and frequencies except commodities at the daily frequency.

To conclude this section, there is a significant autocorrelated component in the return series. The next section examines whether these results are robust to different frequencies.

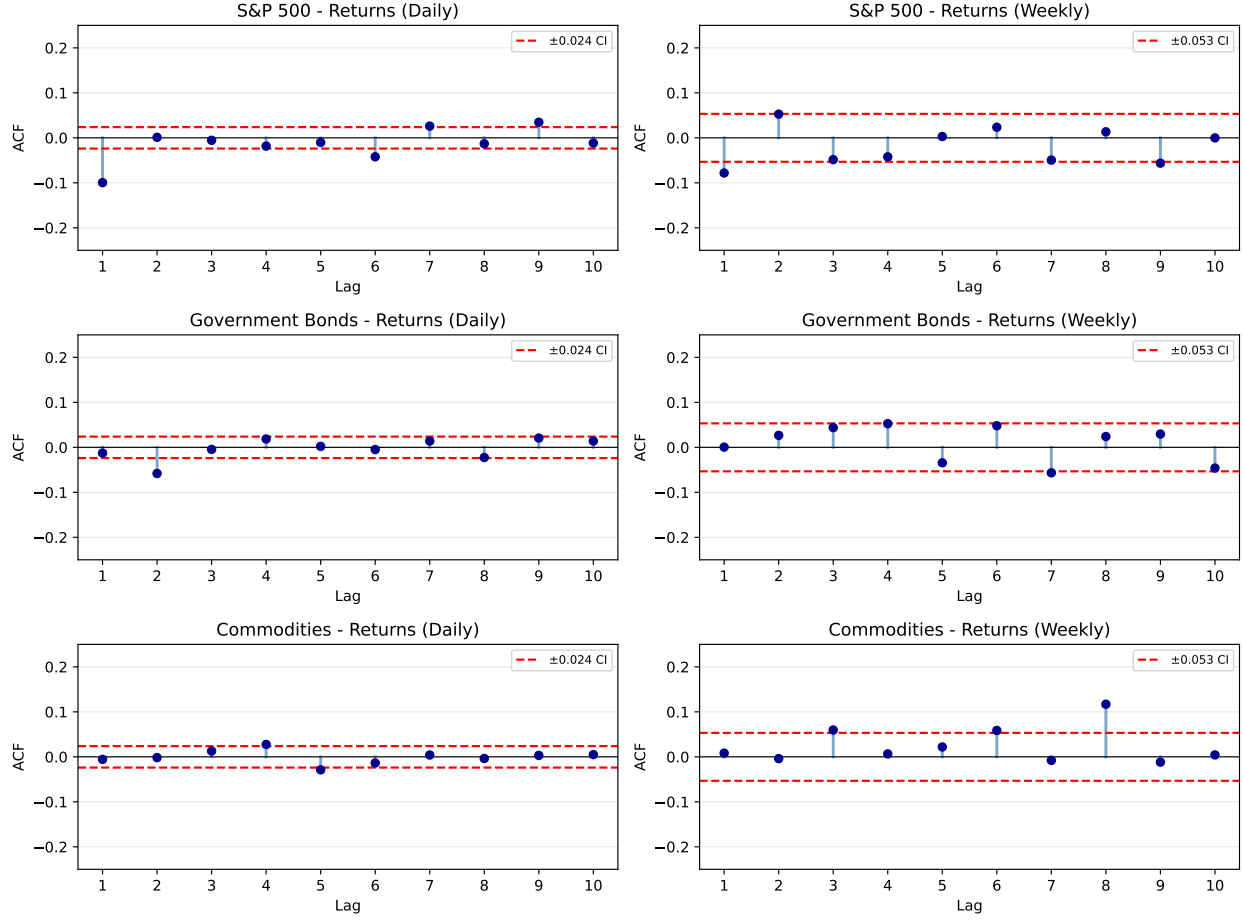


Figure 1: Autocorrelation Function (ACF) and Barlett's formula for individual autocorrelations. The plots show the ACF and the computed 95% confidence intervals for Daily Returns of S&P 500, Government Bonds, and Commodities. The top row shows the ACF plots up to lag 10. Horizontal dashed lines in the ACF plots represent the 95% confidence intervals. The CIs are calculated as $ci = 1.96/\sqrt{n}$.

Table 4: Ljung-Box Test Statistics and p-values for Daily and Weekly Returns and Squared Returns of S&P 500, Government Bonds, and Commodities indices.

Panel A: Returns				Panel B: Squared Returns		
Asset	Frequency	Q-stat	p-value	Frequency	Q-stat	p-value
S&P 500	Daily	97.01	0.0000	Daily	6025.09	0.0000
	Weekly	26.32	0.0033	Weekly	386.75	0.0000
Government Bonds	Daily	35.53	0.0001	Daily	2012.88	0.0000
	Weekly	21.32	0.0190	Weekly	117.68	0.0000
Commodities	Daily	13.86	0.1794	Daily	1845.51	0.0000
	Weekly	29.26	0.0011	Weekly	394.60	0.0000

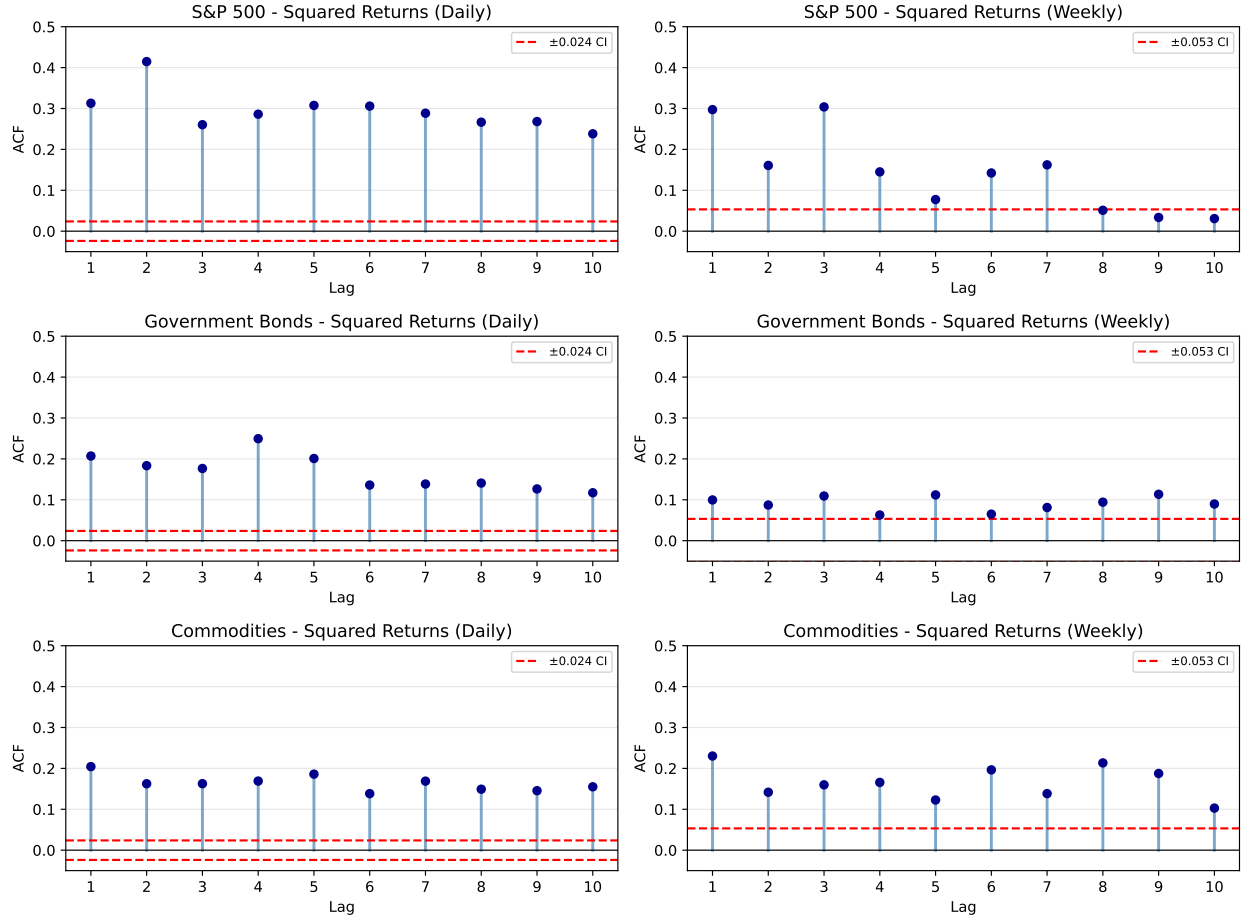


Figure 2: Autocorrelation Function (ACF) and Barlett's formula for individual autocorrelations. The plots show the ACF and the computed 95% confidence intervals for Squared Daily Returns of S&P 500, Government Bonds, and Commodities. The top row shows the ACF plots up to lag 10. Horizontal dashed lines in the ACF plots represent the 95% confidence intervals. The CIs are calculated as $ci = 1.96/\sqrt{n}$

Table 5: Effect of Temporal Aggregation on Normality of S&P 500 Returns

Frequency	N	Skewness	Kurtosis	JB-stat	JB p-val	Q-stat	LB p-val
Daily	6760	-0.350	11.099	34837.71	0.0000	97.01	0.0000
Weekly	1352	-0.867	7.077	2990.39	0.0000	26.36	0.0033
Monthly	311	-0.677	1.144	40.73	0.0000	15.56	0.1130
Yearly	26	-1.347	1.782	11.30	0.0035	25.06	0.0052

1.1.5 Robustness to change in Frequency

As sampling frequency decreases from daily to weekly to monthly, returns should tend toward normality if they are independent with finite second moments¹. By the CLT, returns should be closer to normality at longer sampling periods.

Table 2 and Table 3 confirm this pattern. The extreme moves are less extreme (in terms of Z-score) at weekly frequencies than at daily frequencies. Similarly, JB statistics tend to be smaller at lower frequencies.

For the S&P 500 index, Table 5 summarizes these results at daily, weekly, monthly, and yearly frequencies. Returns become closer to normal with smaller skewness and kurtosis values, lower JB statistics, and higher p-values as frequency decreases. Similarly, autocorrelations tend to be stronger at lower frequencies as indicated by the Ljung-Box statistics.

1.2 Diagnostics for a Portfolio

This section studies the characteristics of portfolios consisting of stocks, government bonds, and commodities. The portfolios are equal-weighted and the returns are computed as *simple* returns as they allow easy cross-sectional aggregation based on portfolio weights. The simple returns are computed as

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}. \quad (7)$$

The analysis uses daily and weekly simple returns of the selected indices.

1.2.1 Summary Statistics of Daily Portfolios

Table 6 presents summary statistics for daily simple returns of individual index series and an equal-weighted portfolio of these assets. The statistics include mean return, standard deviation, skewness, kurtosis, minimum and maximum returns, and the number of observations.

Comparing the portfolio to holding only the stock index, the portfolio has significantly lower standard deviation and less extreme values than the stock index alone.

However, when aggregating cross-sectionally, the skewness and kurtosis of the portfolio returns remain quite high, with skewness slightly elevated compared to the single equity index. This indicates non-normality and suggests that with such a small number of likely cross-sectionally dependent assets, diversification benefits are limited and the contemporaneous aggregation toward normality is weak and slow. In other words, more independent assets would be needed.

¹Or satisfy other conditions for the CLT to hold.

Table 6: Summary statistics for individual index series and an equal-weighted portfolio of these assets.

Asset	Mean (x100)	Std (x100)	Skewness	Kurtosis	Min (x100)	Max (x100)
S&P 500	0.037	1.199	-0.117	10.957	-11.98	11.58
Government Bonds	0.013	0.251	0.004	3.285	-1.67	1.80
Commodities	0.023	1.074	-0.424	4.896	-10.50	6.06
Portfolio	0.024	0.617	-0.707	9.832	-6.47	4.64

Table 7: Summary statistics for individual index series and an equal-weighted portfolio of these assets.

Asset	Mean (x100)	Std (x100)	Skewness	Kurtosis	Min (x100)	Max (x100)
S&P 500	0.180	2.466	-0.571	5.872	-18.14	12.15
Government Bonds	0.066	0.534	0.035	2.214	-1.94	3.14
Commodities	0.113	2.376	-0.544	3.516	-14.83	13.44
Portfolio	0.120	1.304	-1.016	7.266	-10.30	6.79

1.2.2 Summary Statistics of Weekly Portfolios

Table 7 presents summary statistics for weekly asset and portfolio returns, similar to the daily results in Table 6.

Again, the portfolio has lower standard deviation and less extreme values than the stock index alone. Skewness and kurtosis exhibit similar behavior with no strong convergence toward normality.

Regarding temporal aggregation, weekly returns naturally have higher means and standard deviations, which should scale approximately linearly and with the square root of time, respectively. However, convergence toward normality in the temporal dimension is weaker than with log returns, which is expected since simple returns aggregate multiplicatively rather than linearly over time.

1.2.3 Implications for Risk and Portfolio Management

From a risk and portfolio management perspective, the results are interesting. First, there are contemporaneous diversification benefits from aggregating across assets, which reduces portfolio standard deviation. This diversification is costly in terms of lower expected returns; however, the Sharpe ratio improves if the assets added to the portfolio are not perfectly correlated.

Second, the weak convergence toward normality implies that normal distributions should only be used for large portfolios with multiple independent assets. For small portfolios, non-normality is significant and should be accounted for in risk management. This is especially true during periods of market distress when extreme moves are more likely to occur.

2 Modeling Volatility and Non-Normality

This section studies volatility modeling using the GARCH(1,1) model for volatility of corporate bond and S&P 500 stock index excess log returns.

The returns are calculated as follows: First, I calculate the daily excess returns for corporate bonds and the stock index by subtracting the risk-free rate (proxied by government bond returns) using simple returns, giving the simple excess returns for stocks and bonds as

$$R_{i,t}^{excess} = R_{i,t} - R_{rf,t} \text{ for } i \in \{stocks, bonds\}. \quad (8)$$

Then, I convert the simple excess returns to log excess returns as

$$r_{i,t}^{excess} = \log(1 + R_{i,t}^{excess}) \text{ for } i \in \{stocks, bonds\}. \quad (9)$$

When building the GARCH model in this section, I effectively assume that the excess log return process has an autocorrelated component, and the residual process follows a GARCH(1,1) model with non-normal innovations. In other words, the model is specified as

$$r_{i,t}^{excess} = \mu + \phi r_{i,t-1}^{excess} + \epsilon_{i,t}, \quad (10)$$

$$\epsilon_{i,t} = \sigma_{i,t} z_{i,t}, \quad (11)$$

$$\sigma_{i,t}^2 = \omega + \alpha \epsilon_{i,t-1}^2 + \beta \sigma_{i,t-1}^2, \quad (12)$$

where $z_{i,t}$ are i.i.d. innovations following a Normal, Student-t, or Skewed Student-t distribution. Note that eventhought variables have the subscript i the analysis is done in time series setting. It is worth noting that the AR(1) specification used in this project contains the intercept term μ as well as the lagged return term with coefficient ϕ . I keep the intercept term because asset pricing theory predicts that expected excess returns should be positive on average under the physical measure due to risk premia associated with risky assets.

2.1 Estimation of GARCH Model

2.1.1 Non-Normality and Autocorrelation

Before estimating the GARCH model, I first examine the non-normality and autocorrelation characteristics of the excess log returns to confirm the model assumptions. Table 8 presents the Jarque-Bera and Ljung-Box statistics for daily excess log returns of corporate bonds and the S&P 500 stock index.

First, Table 8 displays very large Jarque-Bera statistics for both corporate bond and S&P 500 excess log returns, with p-values effectively equal to zero. This indicates that I reject the null hypothesis of normality for both return series, supporting the evidence of non-normality in financial returns.

Second, the Ljung-Box statistics for both return series are also significant at the 1% level. The autocorrelation effect tends to be even stronger in corporate bond excess log returns, as indicated by the very high Ljung-Box statistic and p-value effectively equal to zero. These results indicate significant autocorrelation in the excess log returns that must be filtered out before estimating the GARCH model.

Table 8: Jarque-Bera, and Ljung-Box statistics for Corporate Bond and Stock (S&P 500) excess log returns. Excess returns are computed by subtracting government bond returns from the respective simple asset returns.

Asset	JB-stat	JB p-value	Q-stat	LB p-value
Stocks	25446.38	0.0000	14.99	0.0047
Corp Bonds	44381.78	0.0000	788.89	0.0000

Table 9: Estimated AR(1) Model for excess corporate bond and stock returns. The table displays the estimated coefficients along with their standard errors (in parentheses), t-statistics, and p-values. The model is estimated using conditional maximum likelihood estimation (MLE) with Normal distribution. The standard errors are Heteroscedasticity and Autocorrelation Consistent (HAC) standard errors with 6 lags.

Parameter	Coefficient	Std. Error	p-value
Panel A: S&P 500			
Constant	0.000173	0.000142	0.2237
ϕ_1	-0.038332	0.026339	0.1456
Panel B: Corporate Bonds			
Constant	0.000080	0.000049	0.1045
ϕ_1	0.243030	0.033753	0.0000

2.1.2 AR(1) Model to Filter Autocorrelation

To filter the observed autocorrelation in the excess log returns, I use an AR(1) model as specified in equation (10). The model is estimated using conditional maximum likelihood estimation (MLE). The estimated coefficients are displayed in Table 9.

The coefficients in Table 9 show that the autocorrelation component is small in both series. However, the autocorrelation coefficient is larger for corporate bonds than stocks, and when standard errors are HAC adjusted, the coefficient is not statistically significant for stocks. Furthermore, the drift term is positive but, as expected, small in magnitude and significance, especially for corporate bonds.

Figure 3 displays the residuals from the estimated AR(1) models for both excess return series. There is significant volatility clustering in both residual series, indicating that a time-varying volatility model such as GARCH is appropriate for modeling the residuals.

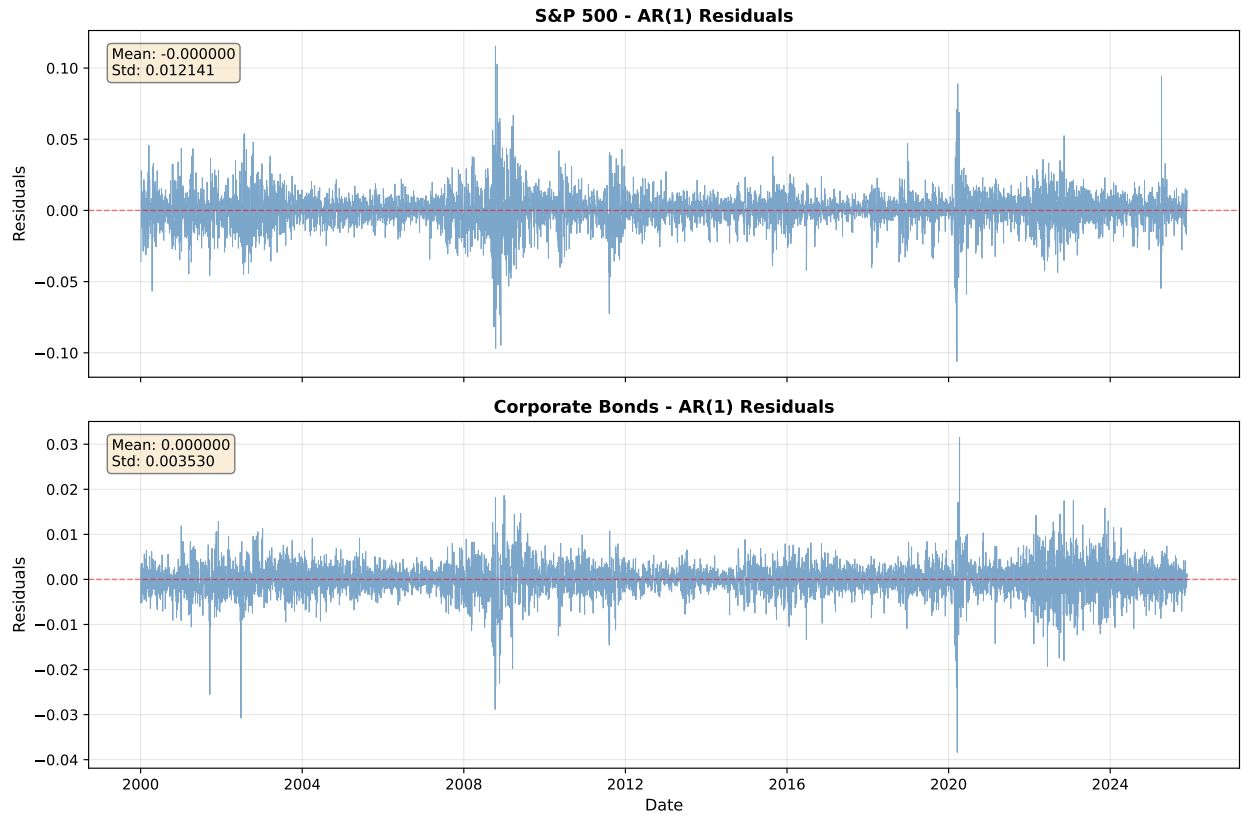


Figure 3: Residuals from the estimated AR(1) models for Corporate Bond and Stock (S&P 500) excess log returns. The plots show the time series of residuals, highlighting volatility clustering in both series.

2.1.3 ARCH Effects in AR(1) Residuals

Given the time-varying volatility in the estimated residuals, I first test for ARCH effects in the AR(1) residuals using Engle's ARCH test before proceeding to GARCH estimation. The results are summarized in Table 10.

The results in Table 10 show that the squared residuals from the AR(1) model exhibit sta-

Table 10: Engle’s ARCH Lagrange Multiplier Test for ARCH effects in AR(1) residuals of Corporate Bond and Stock (S&P 500) excess log returns. The table displays the test statistics along with their p-values. The null hypothesis of the test is that there are no ARCH effects up to lag 10. The test statistics tests the null hypothesis of $Var_t(\epsilon_{t+1}) = \sigma^2$, and under null the statistic follows a chi-squared distribution with degrees of freedom equal to the number of lags tested (here 4).

Asset	LM-stat	p-value	Reject H_0
S&P 500	1489.21	0.0000	Yes
Corporate Bonds	926.17	0.0000	Yes

Table 11: Estimated GARCH(1,1) Model for AR(1) residuals of Corporate Bond and Stock (S&P 500) excess log returns with Normal innovations. The table displays the estimated coefficients along with their standard errors (in parentheses), t-statistics, and p-values. The model is estimated using conditional maximum likelihood estimation (MLE) with Normal distribution. The parameters are estimated from the residuals of the AR(1) model fitted to excess returns and for reasons of numerical stability the residuals are scaled by 100 before estimation. Because of the scaling the estimated parameter ω is in units of 100^2 , but the α and β , and $\alpha + \beta$ are not scaled (as these are scale invariant).

Asset	ω	α	β	Persistence	Log-Lik	AIC
S&P 500	0.0233 (0.0046)	0.1113 (0.0101)	0.8712 (0.0109)	0.9825	-9390.35	18786.70
Corporate Bonds	0.0011 (0.0004)	0.0950 (0.0156)	0.8995 (0.0159)	0.9945	-1428.42	2862.85

tistically significant autocorrelation, as the ARCH test statistics are large and the p-values are effectively equal to zero for both series. Thus, I reject the null hypothesis of no ARCH effects in both series.

2.1.4 GARCH Model for AR(1) Residuals

Given the statistically significant ARCH effects in the AR(1) squared residuals, I now proceed to estimate the GARCH(1,1) model to capture the time-varying volatility, as specified in equations (11) and (12). Table 11 displays the estimated GARCH(1,1) model coefficients for both excess return series.

Table 11 shows that the coefficients for both series are statistically significant at the 1% level and economically large in magnitude.

Furthermore, the sum of the α and β coefficients, also known as the persistence parameter, is close to 1 for both series, indicating high persistence in volatility shocks. This suggests that volatility shocks have long-lasting effects and that spikes in volatility decay slowly over time.

2.2 MLE or QMLE

The previous section showed that there is significant persistence in volatility shocks for both excess return series. This section analyzes the adequacy of the Normal distribution for modeling the shocks in the GARCH innovation process.

The analysis in this section and the next applies the Diebold-Gunther-Tay (DGT) test for goodness-of-fit of the assumed distribution for the GARCH innovations. The test and its implementation are described as follows:

DGT Test Implementation Details Let a fitted GARCH model deliver conditional volatility $\hat{\sigma}_t$ and standardized innovations

$$\hat{z}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t},$$

together with an assumed innovation CDF F for z_t (e.g. Gaussian, Student- t , skew- t). The DGT test evaluates the adequacy of the conditional distribution by forming the probability integral transforms (PITs)

$$u_t = F(\hat{z}_t).$$

Under correct specification of both the volatility dynamics and the innovation distribution, the PITs then satisfy

$$u_t \stackrel{iid}{\sim} U(0, 1).$$

I implement the test in two parts. First, I check uniformity by partitioning $[0, 1]$ into N ($N = 40$) equal bins. Let F_n be the number of u_t in bin n , and compute Pearson's statistic

$$\text{DGT}(N) = \sum_{n=1}^N \frac{(F_n - T/N)^2}{T/N} \sim \chi^2(N - 1)$$

under H_0 . Second, to check independence and dynamics I define centered PITs $v_t = u_t - 1/2$ and apply a Ljung-Box test at lag 10 to v_t (and optionally higher moments), where the reported p-value corresponds to the joint null of no autocorrelation up to lag m . In the code, u_t is computed as $F(\hat{z}_t)$ using the chosen F , the chi-square test uses the binned PIT counts, and the Ljung-Box test is applied to v_t^k for selected moments k .

2.2.1 ML Estimates for GARCH with Normal Distribution

I start by estimating the GARCH(1,1) model for the residuals of the AR(1) model fitted in the earlier section using Maximum Likelihood Estimation (MLE), assuming that the innovations follow a Normal distribution. Table 12 summarizes the results of this estimation and tests the adequacy of the Normal distribution for modeling the standardized residuals from the GARCH model using the JB test. Panel A of Table 12 displays the estimated GARCH(1,1) model parameters assuming Normal innovations. The parameter estimates are the same as in Table 11 since both estimations use MLE with a Normal distribution. However, the standard errors differ: in the previous section they were computed using robust HAC standard errors, while here they are computed using the information matrix, which is valid under the assumption of correct model specification.

However, Panel B of Table 12 shows that these standard errors are likely too small, as the Jarque-Bera statistics for the standardized residuals from the GARCH model are very large with p-values effectively equal to zero.

Thus, the assumption that the innovations follow normality can be questioned, and estimators that weaken this normality assumption should be considered.

Table 12: Estimated GARCH(1,1) Model Parameters and Normality Test for Standardized Residuals, using the Maximum Likelihood Estimation (MLE) assuming Normal innovations. The first panel displays the estimated GARCH(1,1) model parameters along with their standard errors (in parentheses), t-statistics, and p-values. The second panel displays the Jarque-Bera statistics and p-values for testing normality of the standardized residuals from the GARCH model.

Panel A: GARCH(1,1) Parameter Estimates

Asset	ω	α	β	Persistence	AIC
S&P 500	0.0233 (0.0028)	0.1113 (0.0079)	0.8712 (0.0084)	0.9825	18786.70
Corporate Bonds	0.0011 (0.0002)	0.0950 (0.0081)	0.8995 (0.0084)	0.9945	2862.85

Panel B: Jarque-Bera Test for Normal Residuals

Asset	JB-stat	p-value	Skewness	Excess Kurtosis
S&P 500	1174.63	0.0000	-0.4830	1.7995
Corporate Bonds	591.88	0.0000	-0.2186	1.3823

2.2.2 QML Estimates for GARCH with Normal Distribution

To weaken the assumption of normality in the innovation process, I estimate the GARCH(1,1) model using Quasi-Maximum Likelihood Estimation (QMLE) assuming Normal innovations. Table 13 summarizes the results of this estimation.

The estimated parameters from QML estimation in Table 13 are the same as in the MLE with the normal model, as they should be. However, the standard errors are now computed using the robust sandwich estimator and are significantly larger than those from the MLE estimation. The MLE standard errors were computed as

$$SE_{MLE} = \sqrt{\text{diag}(H^{-1})}, \quad (13)$$

where H is the Hessian matrix of the log-likelihood function. For comparison, the sandwich standard errors for the QMLE estimation are computed as

$$SE_{QML} = \sqrt{\text{diag}(H^{-1}GH^{-1})}, \quad (14)$$

where $G = \sum_{t=1}^T s_t s_t'$ (outer product of score vectors), and s_t is the score vector at time t . This form does not make distributional assumptions as the classical MLE standard errors do.

Under correct specification, $G \approx H$, so $H^{-1}GH^{-1} \approx H^{-1}$ and both standard errors are similar. Thus, the large difference in standard errors indicates violations of the normality assumption.

2.2.3 Adequacy of Normal Distribution for Standardized Residuals

To further diagnose the normality problem in the GARCH innovation process, I now apply the Diebold-Gunther-Tay (DGT) test with $N = 40$ cells.

Table 13: Estimated GARCH(1,1) Model Parameters with QML and Normal Distribution. The table displays the estimated coefficients along with their distribution-robust standard. For a comparison, table also displays the MLE standard errors from Table 12, and the ratio of QML to MLE standard errors. Because of the scaling the estimated parameter ω is in units of 100^2 , but the α and β , and $\alpha + \beta$ are not scaled (as these are scale invariant).

Parameter	Estimate	SE(MLE)	SE(QML)	Ratio
Panel A: S&P 500				
ω	0.0233	0.0028	0.0046	1.662
α	0.1113	0.0079	0.0101	1.275
β	0.8712	0.0084	0.0109	1.303
Persistence	0.9825			
Panel B: Corporate Bonds				
ω	0.0011	0.0002	0.0004	1.528
α	0.0950	0.0081	0.0156	1.927
β	0.8995	0.0084	0.0159	1.904
Persistence	0.9945			

The test results are presented in Table 14. First, the uniformity results in Panel A show that the Normal distribution is not adequate for modeling the standardized residuals from the GARCH(1,1) model estimated with conditional MLE. In other words, the empirical distribution of the standardized residuals deviates significantly from the theoretical Normal distribution. Second, the Ljung-Box results in Panel B show that there is significant dependence in the standardized residuals, indicating that the model does not fully capture the dynamics of the process.

Overall, I reject the DGT null hypothesis that standardized innovations follow a Normal distribution. Next, I consider alternative non-normal distributions.

2.3 Estimation of GARCH with Student-t Distribution

As previous sections show, the innovations in the volatility process of the GARCH model behave in a non-normal manner. Thus, this section models the GARCH(1,1) process assuming the innovation z_t in equation (12) follows a Student-t distribution.

2.3.1 Fitting GARCH(1,1) with Student-t Distribution

Table 15 displays the estimated GARCH(1,1) model coefficients for both excess return series assuming Student-t innovations.

The parameter estimates are highly similar to those estimated earlier using the Normal distribution, which indicates that the volatility dynamics are robust to the distributional assumption. Furthermore, the Student-t specification clearly shows strong persistence in volatility.

However, the degrees of freedom parameter ν is estimated to be around 6 for stocks and around 8 for corporate bonds, implying fat tails in the innovation distribution. This further supports the evidence of non-normality in the innovation process.

Table 14: Diebold-Gunther-Tay Test for Adequacy of Normal Distribution for Standardized Residuals from GARCH(1,1) model estimated with QMLE. The table displays the DGT test statistics along with their p-values. The null hypothesis of the test is that the standardized residuals follow a Normal distribution. Under the null hypothesis, the DGT test statistic follows a chi-squared distribution with $N - 1$ degrees of freedom, where N is the number of cells (here 40).

Panel A: DGT Uniformity Test

Asset	χ^2 Statistic	p-value
S&P 500	285.80	0.0000
Corporate Bonds	97.77	0.0000

Panel B: DGT Ljung-Box Test

Asset	Moment	Lag	LB Statistic	p-value
S&P 500	v^1	10	29.19	0.0012
	v^2	10	23.99	0.0076
	v^3	10	27.01	0.0026
	v^4	10	22.41	0.0132
Corporate Bonds	v^1	10	149.22	0.0000
	v^2	10	34.01	0.0002
	v^3	10	117.45	0.0000
	v^4	10	21.99	0.0152

Table 15: Estimated GARCH(1,1) Model for AR(1) residuals of Corporate Bond and Stock (S&P 500) excess log returns with Student-t innovations. The table displays the estimated coefficients along with their standard errors (in parentheses), t-statistics, and p-values. The model is estimated using conditional maximum likelihood estimation (MLE) with Student-t distribution. Because of the scaling the estimated parameter ω is in units of 100^2 , but the α and β , and $\alpha + \beta$ are not scaled (as these are scale invariant).

Asset	ω	α	β	ν	Persistence	AIC
S&P 500	0.0141	0.1107	0.8848	6.1215	0.9956	18480.30
	(0.0030)	(0.0097)	(0.0095)	(0.4617)		
Corporate Bonds	0.0009	0.0779	0.9161	8.3290	0.9940	2675.83
	(0.0003)	(0.0122)	(0.0131)	(0.7979)		

Table 16: Delta Method Test for $H_0 : 1/\nu = 0$ in GARCH(1,1) model with Student-t innovations for Corporate Bond and Stock (S&P 500) excess log returns. The table displays the estimated degrees of freedom parameter $\hat{\nu}$, its standard error, the computed test statistic, and the corresponding p-value for the test.

Asset	$\hat{\nu}$	$1/\hat{\nu}$	t-statistic	p-value
S&P 500	6.1215 (0.4617)	0.163359 (0.012320)	13.2600	0.0000
Corporate Bonds	8.3290 (0.7979)	0.120062 (0.011501)	10.4391	0.0000

Delta Method for Testing $H_0 : 1/\nu = 0$ Having estimated the GARCH(1,1) model with Student-t innovations, I apply the delta method to test the null hypothesis that $1/\nu = 0$, which corresponds to the case of infinite degrees of freedom, in which case the distribution is normal.

The delta method provides the asymptotic distribution of transformations of maximum likelihood estimators. For a differentiable function $g(\cdot)$ and MLE $\hat{\nu}$ with asymptotic normality $\sqrt{n}(\hat{\nu} - \nu) \xrightarrow{d} N(0, \sigma_\nu^2)$, we have $\sqrt{n}(g(\hat{\nu}) - g(\nu)) \xrightarrow{d} N(0, [g'(\nu)]^2 \sigma_\nu^2)$. Applying this to $g(\nu) = 1/\nu$ with derivative $g'(\nu) = -1/\nu^2$, the asymptotic standard error is $SE(1/\hat{\nu}) = |g'(\hat{\nu})| \times SE(\hat{\nu}) = SE(\hat{\nu})/\hat{\nu}^2$, yielding the test statistic

$$t = \frac{1/\hat{\nu} - 0}{SE(1/\hat{\nu})} = \frac{\hat{\nu}}{SE(\hat{\nu})} \stackrel{H_0}{\approx} N(0, 1). \quad (15)$$

The results for the delta method are summarized in Table 16, which shows that I reject the null hypothesis that $1/\nu = 0$ for both corporate bonds and stocks.

However, it is worth noting that *testing $H_0 : 1/\nu = 0$ with the delta method might be problematic* because this null corresponds to the boundary point $\nu = \infty$, which lies outside the parameter space $(0, \infty)$. The standard delta method assumes linearization around a finite interior point, but at $\nu = \infty$ the derivative $g'(\nu) \rightarrow 0$, causing the first-order approximation to be ill-defined. A more appropriate approach might be to test $H_0 : \nu \geq \nu^*$ for a large finite threshold ν^* , or to employ alternative tests for normality that avoid this boundary issue.

2.3.2 Test of Adequacy of Student-t Distribution for GARCH(1,1) Residuals

Having estimated the GARCH(1,1) model with Student-t innovations, I test the adequacy of the Student-t distribution for modeling the standardized residuals from the GARCH model using the DGT test, similar to the Normal distribution in the previous section.

Table 17 displays the results of the DGT test for adequacy of the Student-t distribution for standardized residuals from the GARCH(1,1) model estimated with Student-t innovations.

Looking first at the uniformity test in Panel A of Table 17, for stocks I reject the uniformity null hypothesis and note that the statistic values are very large. This implies that the Student-t GARCH density is not well calibrated for equities and the realized innovations fall too often in some regions relative to what the model predicts. For corporate bonds, I fail to reject the null hypothesis of uniformity, indicating that the Student-t distribution is well calibrated for corporate bonds.

However, the independence test in Panel B of Table 17 shows that for both stocks and corporate

Table 17: Diebold-Gunther-Tay Test for Adequacy of Student-t Distribution for Standardized Residuals from GARCH(1,1) model estimated with Student-t innovations.

Panel A: DGT Uniformity Test

Asset	χ^2 Statistic	p-value
S&P 500	127.68	0.0000
Corporate Bonds	40.18	0.4179

Panel B: DGT Ljung-Box Test

Asset	Moment	Lag	LB Statistic	p-value
S&P 500	v^1	10	28.92	0.0013
	v^2	10	18.67	0.0447
	v^3	10	28.17	0.0017
	v^4	10	16.56	0.0847
Corporate Bonds	v^1	10	148.13	0.0000
	v^2	10	40.88	0.0000
	v^3	10	125.68	0.0000
	v^4	10	31.79	0.0004

bonds, I reject the null hypothesis of independence. This means that the model does not capture all dynamics in the conditional distribution. Especially for corporate bonds, the test statistics are very large, indicating serial correlation or smoothing (stale prices, evaluated pricing), which is typical for non-traded securities.

To conclude, for both asset classes the DGT test rejects the null hypothesis that innovations from GARCH(1,1) follow a Student-t distribution.

2.4 Estimation of GARCH with Skewed Student-t Distribution

As observed in the previous section, using the Student-t distribution for the innovations in the GARCH(1,1) model provided a slight improvement over the Normal distribution but still failed to adequately capture the characteristics of the standardized residuals. This section estimates the GARCH(1,1) model assuming that the innovations follow a Skewed Student-t distribution, which allows for both fat tails and skewness in the distribution of innovations.

To fit the Skewed Student-t distribution, I use the parameterization proposed by Hansen [1994].

2.4.1 Fitting GARCH(1,1) with Skewed Student-t Distribution

Table 18 displays the estimated GARCH(1,1) model coefficients for both excess return residual series assuming Skewed Student-t innovations.

The estimation results yield very similar GARCH(1,1) estimates to the Normal distribution in Table 13 and the symmetric Student-t distribution in Table 15. Overall, the volatility dynamics appear robust to the distributional assumption, and the persistence in volatility shocks remains high across all specifications.

Table 18: Estimated GARCH(1,1) Model for AR(1) residuals of Corporate Bond and Stock (S&P 500) excess log returns with Skewed Student-t innovations. The table displays the estimated coefficients along with their standard errors (in parentheses), t-statistics, and p-values. The model is estimated using conditional maximum likelihood estimation (MLE) with Skewed Student-t distribution. Because of the scaling the estimated parameter ω is in units of 100^2 , but the α and β , and $\alpha + \beta$ are not scaled (as these are scale invariant).

Asset	ω	α	β	η	λ	Persistence	AIC
S&P 500	0.0139 (0.0030)	0.1108 (0.0094)	0.8847 (0.0092)	6.31 (0.49)	-0.1149 (0.0135)	0.9955 (0.0186)	18424.06
Corporate Bonds	0.0009 (0.0003)	0.0762 (0.0118)	0.9178 (0.0127)	8.41 (0.81)	-0.0366 (0.0158)	0.9940 (0.0246)	2672.87

Interestingly, Table 18 shows that the degrees of freedom parameter ν is estimated to be around 7 for stocks and around 9 for corporate bonds, similar to the symmetric Student-t case, indicating fat tails in the innovation distribution. Furthermore, the skewness parameter λ is estimated to be negative for both asset classes, suggesting that the innovation distribution is left-skewed, which is consistent with the common observation of negative skewness in financial return distributions.

The skewness parameter is also statistically significant at the 1% level for both asset classes, indicating that skewness is an important feature of the innovation distribution that should be accounted for in the model.

Joint Test for $H_0 : 1/\nu = 0$ AND $\lambda = 0$ This test is applied similarly to the delta method for testing normality in the Student-t case; however, here I consider the joint test for both fat tails and skewness. Taking derivatives of the transformation vector $\mathbf{g}(\boldsymbol{\theta}) = [1/\nu, \lambda]'$, I obtain the Jacobian $\mathbf{G} = \text{diag}(-1/\nu^2, 1)$. Furthermore, given the asymptotic covariance $\text{Var}(\mathbf{g}(\hat{\boldsymbol{\theta}})) = \mathbf{G} \text{Var}(\hat{\boldsymbol{\theta}}) \mathbf{G}'$, the test statistic is

$$W = \mathbf{g}(\hat{\boldsymbol{\theta}})' [\text{Var}(\mathbf{g}(\hat{\boldsymbol{\theta}}))]^{-1} \mathbf{g}(\hat{\boldsymbol{\theta}}) \stackrel{H_0}{\sim} \chi^2(2). \quad (16)$$

However, it is worth noting that testing $H_0 : 1/\nu = 0$ might be ill-defined, as discussed in the previous section for the standard Student-t case.

Table 19 summarizes the result of the delta method applied to the parameter vector. The table shows that I reject the null hypothesis that both $1/\nu = 0$ and $\lambda = 0$ for both corporate bonds and stocks.

2.4.2 Test of Adequacy of Skewed Student-t Distribution for GARCH(1,1) Residuals

I now test the adequacy of the Skewed Student-t distribution for modeling the standardized residuals from the GARCH model using the DGT test, similar to the normal and symmetric Student-t distributions in previous sections.

Similar to earlier sections, Table 20 rejects the null hypothesis that the standardized GARCH residuals in the GARCH(1,1) model (11) follow a Skewed Student-t distribution.

As earlier, Panel B shows that there is significant autocorrelation in the residuals for both asset classes, indicating that the model does not capture all dynamics in the conditional distribution.

Table 19: Delta method applied to the joint test for $H_0 : 1/\nu = 0$ AND $\lambda = 0$ in GARCH(1,1) model with Skewed Student-t innovations for Corporate Bond and Stock (S&P 500) excess log returns residuals. The table displays the computed test statistic along with its p-value for the test.

Asset	$\hat{\eta}$	$1/\hat{\eta}$	$\hat{\lambda}$	Joint Stat.	p-value
S&P 500	6.31 (0.49)	0.1586 (0.0122)	-0.1149 (0.0135)	247.49	0.0000
Corporate Bonds	8.41 (0.81)	0.1189 (0.0114)	-0.0366 (0.0158)	112.27	0.0000

Regarding uniformity in Panel A of Table 20, the conclusions are similar to those with the standard Student-t distribution: for stocks I reject the uniformity null hypothesis, while for corporate bonds I fail to reject it.

However, it is worth emphasizing that the uniformity test statistic for stocks is significantly smaller than with the standard Student-t distribution, indicating that the Skewed Student-t distribution provides a better fit for stocks and that the asymmetry in skewness is an important feature to capture in the innovation distribution.

2.4.3 Sample Moments of Normal ML GARCH vs. Theoretical Moments from Skewed Student-t

The results thus far illustrate that even though the Skewed Student-t distribution was rejected by the DGT test, it has been the best fitting distribution when compared to the Normal and standard Student-t distributions.

To further validate the distribution, I compare the theoretical moments from the Skewed Student-t distribution, computed based on the estimated parameters in Table 18, to the empirical moments calculated from the standardized residuals \hat{z}_t of the Normal model in Section 2.2. The comparison of the moments is presented in Table 21.

The theoretical moments that are used in the table are described in the Appendix A.

Overall, the results in the table show that the Skewed Student-t distribution moments match well with those in the residuals. In particular, the two parameters of the Skewed Student-t distribution that determine skewness and kurtosis are necessary to capture the sample skewness and kurtosis empirically observed in the Normal model.

2.4.4 Volatility Time Series Comparison

The previous results have shown that even though the Skewed Student-t distribution tends to be the best fitting distribution for the GARCH innovations, the parameter estimates for the GARCH(1,1) model have remained quite stable across different distributional assumptions in the MLE. This indicates that the estimated volatility dynamics are quite robust to the distributional assumption.

These observations are further confirmed by Figure 4, which plots the conditional volatilities for each of the three models.

Figure 4 shows that throughout the sample, forecasted volatilities have behaved similarly across all three distributional assumptions.

Table 20: Diebold-Gunther-Tay Test for Adequacy of Skewed Student-t Distribution for Standardized Residuals from GARCH(1,1) model estimated with Skewed Student-t innovations. The table displays the DGT test statistics along with their p-values. The null hypothesis of the test is that the standardized residuals follow a Skewed Student-t distribution. Under the null hypothesis, the DGT test statistic follows a chi-squared distribution with $N - 1$ degrees of freedom, where N is the number of cells (here 40).

Panel A: DGT Uniformity Test

Asset	χ^2 Statistic	p-value
S&P 500	65.37	0.0051
Corporate Bonds	34.31	0.6837

Panel B: DGT Ljung-Box Test

Asset	Moment (v^k)	LB Statistic	p-value
S&P 500	v^1	28.97	0.0013
	v^2	18.51	0.0469
	v^3	29.30	0.0011
	v^4	16.00	0.0997
Corporate Bonds	v^1	147.88	0.0000
	v^2	41.43	0.0000
	v^3	124.67	0.0000
	v^4	32.66	0.0003

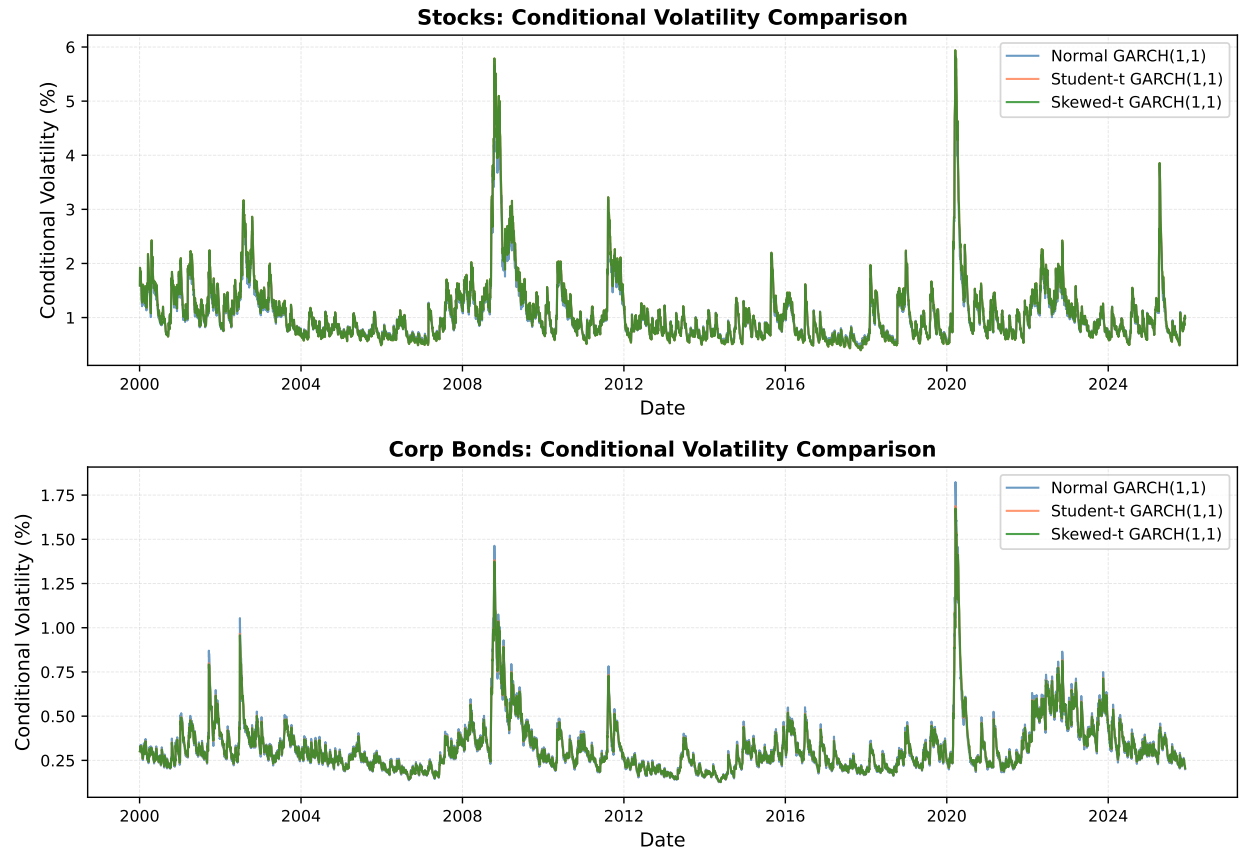


Figure 4: Comparison of Conditional Volatilities from GARCH(1,1) models estimated with Normal, Student-t, and Skewed Student-t innovations for Corporate Bond and Stock (S&P 500) excess log returns. The plots show the time series of conditional volatilities estimated from each model, highlighting the similarities and differences in volatility dynamics across different distributional assumptions.

Table 21: Comparison of Theoretical Moments from Skewed Student-t Distribution vs. Sample Moments from Standardized Residuals of Normal GARCH(1,1) Model for Corporate Bond and Stock (S&P 500). The table displays the theoretical moments (mean, variance, skewness, kurtosis) computed from the Skewed Student-t distribution using the estimated parameters from the GARCH(1,1) model with Skewed Student-t innovations, alongside the empirical moments calculated from the standardized residuals of the Normal GARCH(1,1) model.

Asset	Moment	Sample	Theoretical	Difference
S&P 500	Mean	-0.0011	-0.0000	-0.0011
	Variance	0.9941	1.0000	-0.0059
	Skewness	-0.5244	-0.3786	-0.1458
	Kurtosis	5.1071	5.7851	-0.6781
	Excess Kurtosis	2.1071	2.7851	-0.6781
Corporate Bonds	Mean	0.0063	0.0000	0.0063
	Variance	1.0014	1.0000	0.0014
	Skewness	-0.2465	-0.0976	-0.1489
	Kurtosis	4.4987	4.3702	0.1285
	Excess Kurtosis	1.4987	1.3702	0.1285

To further analyze the differences between the models, Figure 5 plots all pairwise differences in the conditional volatilities from the three models. For stocks, the Normal model tends to underestimate volatility relative to the other two models, especially during high volatility periods. Furthermore, the standard Student-t and Skewed Student-t models produce very similar volatility estimates (note that the scale of the difference is very small), but the Skewed Student-t model produces systematically larger volatility estimates than the standard Student-t model.

For corporate bonds, the main differences arise during high volatility periods, where the Normal model actually tends to overestimate volatility relative to the other two models.

3 Conclusion

This report first analyzed the characteristics of typical financial time series, and then proceeded to model the volatility dynamics of corporate bond and stock excess log returns using GARCH(1,1) models with different innovation distributions.

From the comparisons between Normal, Student-t, and Skewed Student-t distributions for the standardized GARCH innovations, I find that the best fitting distribution is the Skewed Student-t distribution, which captures both the fat tails and skewness observed in the financial return series.

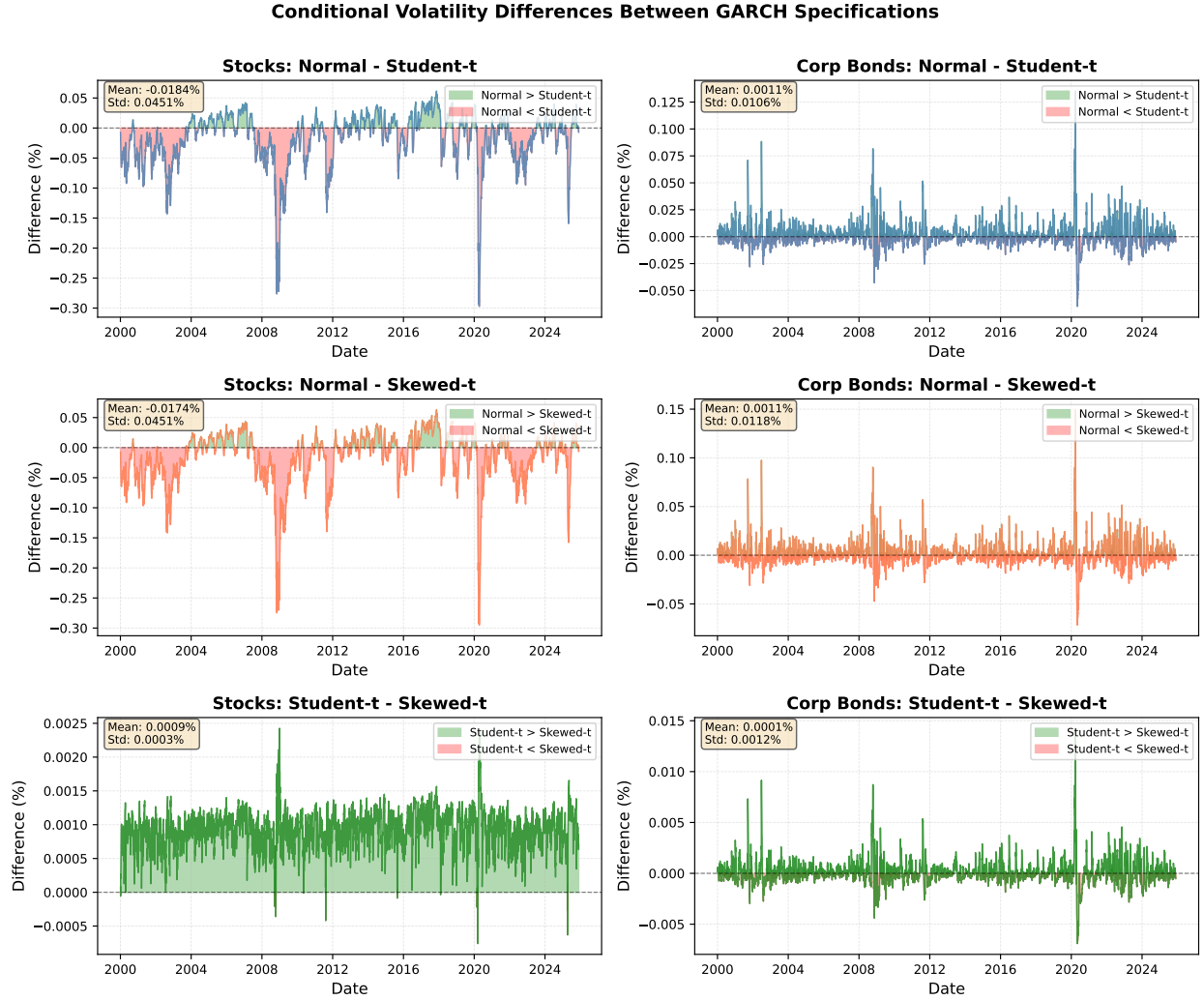


Figure 5: Difference in Conditional Volatilities between GARCH(1,1) models estimated with Normal, Student-t, and Skewed Student-t innovations for Corporate Bond and Stock (S&P 500) excess log returns. The plots show the time series of differences in conditional volatilities between each pair of models, highlighting the magnitude and patterns of differences in volatility estimates across different distributional assumptions.

References

Bruce E Hansen. Autoregressive conditional density estimation. *International Economic Review*, pages 705–730, 1994.

A Skewed-t Distribution Moments

The following definitions follow Hansen [1994]'s definition of the Skewed Student-t distribution. Let $Z \sim \text{SkewStudent}(\nu, \lambda)$ with $\nu > 2$, $\lambda \in (-1, 1)$, and define the following constants and variables:

$$c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)},$$

$$a = 4\lambda c \frac{\nu-2}{\nu-1}, \quad b = \sqrt{1+3\lambda^2-a^2}.$$

$$Y := a + bZ.$$

For Y we have obtain

$$\mathbb{E}[Y] = a, \quad \mathbb{E}[Y^2] = 1 + 3\lambda^2,$$

$$\mathbb{E}[Y^3] = 16\lambda(1+\lambda^2)c \frac{(\nu-2)^2}{(\nu-1)(\nu-3)} \quad (\text{for } \nu > 3),$$

and

$$\mathbb{E}[Y^4] = \frac{3(\nu-2)}{\nu-4} (1 + 10\lambda^2 + 5\lambda^4) \quad (\text{for } \nu > 4).$$

Lastly using the calculated moments for Y and then solving for the moments of Z we obtain the four following moments that are used in the report

$$m_1 := \mathbb{E}[Z] = 0, \quad m_2 := \mathbb{E}[Z^2] = 1.$$

$$\begin{aligned} m_3 := \mathbb{E}[Z^3] &= \frac{\mathbb{E}[(Y-a)^3]}{b^3} \\ &= \frac{\mathbb{E}[Y^3] - 3a\mathbb{E}[Y^2] + 2a^3}{b^3} \\ &= \frac{\mathbb{E}[Y^3] - 3a(1+3\lambda^2) + 2a^3}{b^3} \quad (\text{for } \nu > 3). \end{aligned}$$

$$\begin{aligned} m_4 := \mathbb{E}[Z^4] &= \frac{\mathbb{E}[(Y-a)^4]}{b^4} \\ &= \frac{\mathbb{E}[Y^4] - 4a\mathbb{E}[Y^3] + 6a^2\mathbb{E}[Y^2] - 3a^4}{b^4} \\ &= \frac{\mathbb{E}[Y^4] - 4a\mathbb{E}[Y^3] + 6a^2(1+3\lambda^2) - 3a^4}{b^4} \quad (\text{for } \nu > 4). \end{aligned}$$

B Python Libraries

- Main library for ARCH type modelling is `arch`, version 5.3.1.
- For some timeseries tests I used `statsmodels`, version 0.14.0.
- Many of the tests used in this project are coded by myself, and the code is available at the start of the notebook