IOQM ASSIGNMENTS: SQUARE IDENTITIES

PANKAJ AGARWAL (PAMATHSFAC@GMAIL.COM)

Exercise 1. Prove that the cube of any positive integer greater than 1 can be represented as a difference of the squares of two positive integers. (Bosnia and Herzegovina, 2004)

Exercise 2. Let a, b, c be positive integers. Prove that it is impossible for all three of the numbers $a^2 + b + c$, $a + b^2 + c$, $a + b + c^2$ to be perfect squares. (APMO, 2011)

Exercise 3. Show that there is no natural number N for which $\frac{10^N+1}{121}$ is an integer. (Canada, 1971)

Exercise 4. The real numbers a, b, c satisfy $a^2 + b^2 + c^2 = ab + bc + ca$. Show that $a + b + c$ is an integer. (Nordic, 2012)

Exercise 5. Find all integer solutions (M, N) to the equation $M^2 + N^2 + MN = 1999$. (RMO-1999, Belarus – 1996)

Exercise 6. Let a, b, c be real numbers not equal to zero and satisfying:

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}$$

Prove that $(a - b)(b - c)(c - a) = 1$. (Poland, 2018)

Exercise 7. Prove that there do not exist integers a, b, c such that $a^2+b^2+c^2 = 2(ab+bc+ca)$. (Canada, 1969)

Exercise 8. Find all real numbers x such that $\sqrt{x + \sqrt{2x - 1}} = 5$. (Ukraine, 1997)

Exercise 9. Evaluate: (i) $\sqrt{(3 + \sqrt{5})(3 - \sqrt{5})}$ (British Maths Olympiad, 2007)
(ii) $\sqrt{9 + 4\sqrt{5}} + \sqrt{9 - 4\sqrt{5}}$ (Hong Kong – Preliminary 2015)

Exercise 10. Prove that 1 plus the product of any four consecutive integers is a perfect square. (Moscow, 1941)

Exercise 11. Evaluate the product: (i) $\prod_{k=1}^{4}\left(1 + \frac{1}{k(k+1)}\right)$ (AIME-1986)
(ii) $\prod_{k=1}^{3}\left(1 - \frac{1}{(2k)^2}\right)$ (Australia – Intermediate, 2000)

Exercise 12. Compute: $\sum_{k=1}^{9} \frac{1}{k(k+1)}$ (AIME – 1989)

Exercise 13. If $n + 20$ and $n - 21$ are both perfect squares and n is a natural number, find n.

Exercise 14. Evaluate: $\frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{2}+1}$ (British Maths Olympiad, 2013)

Exercise 15. Find all positive integers a and b such that $a^2 + b^2 + ab = 3ab$. (UK – 2013)

Exercise 16. (i) Determine the set of integers n such that $n^2 + 19n + 92$ is the square of an integer. (RMO, 1992)
(ii) Find all natural numbers n such that $n^2 - 19n + 91$ is a perfect square. (China, 1991)
(iii) Find the sum of all positive integers n for which $n^2 - 19n + 99$ is a perfect square. (AIME-1999)

Exercise 17. There exist unique positive integers x and y such that $x^2 + 84x + 2008 = y$. Find $x + y$. (AIME – 2008)

Exercise 18. Find all positive integers n such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square. (China Western Maths Olympiad, 2002)

Exercise 19. The non-zero real numbers a, b satisfy $a^b = b^a$. Prove that a and b are not both rational. (Russia-2000; Kosova-2016)

Exercise 20. Let a, b be real numbers such that $(a + b)^2 = a^2 + b^2 + 2ab$. Find the value of $2ab$. (Online Maths Open Problems, 2012)

Exercise 21. Determine the unique pair of real numbers (x, y) that satisfy the equation $x + \frac{1}{y} = y + \frac{1}{x}$. (USA-MTS, 1998-99)

Exercise 22. It is given that $x^3 + y^3 + z^3 = 3xyz$. Prove that $x + y + z = 0$. (Baltic Way, 1992)

Exercise 23. Show that if p and q are integers such that $p^3 + q^3$ is divisible by 9, then p and q are both divisible by 3. (Eotvos Competition, 1958)

Exercise 24. How many of the numbers $1^2, 2^2, 3^2, \ldots, 100^2$ have odd numbers as their tens digit? (International Mathematics Competition, 1999)

Exercise 25. Determine all possible ordered triplets (x, y, z) such that $x + y + z = xyz$. (Canada Open Maths Challenge, 2005)

Exercise 26. Calculate the value of $\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}$. (Japan Maths Olympiad – Preliminary, 2016)

Exercise 27. Let a, b, c be non-zero integers with $a \neq c$ such that $a^2 + b^2 + c^2 = ac + 2b^2$. Factorise $a^3 - c^3$. (Romania, 1999)

Exercise 28. Prove that 2005^2 can be written in at least 4 ways as the sum of 2 perfect (non-zero) squares. (Flanders, Junior – 2005)

Exercise 29. Find all real pairs (a, b) that satisfy the system:

$$a + b = 4, \quad ab = 3$$

Exercise 30. Determine the pair of positive integers p and q that satisfy $p^2 = q^2 + p + q + 2018$. (Lusophon, 2018)

Exercise 31. Find all positive integers x such that $2x + 1$ is a perfect square but none of the integers $2x + 2, 2x + 3, \ldots, 3x + 3$ are perfect squares. (Spain, 2018)

Exercise 32. Prove that $5^n + 4$ is not the square of any natural number. (Tuymada, 2019)

Exercise 33. Show that the number $n^2 - 22014 \cdot 2014^n + 4 \cdot 2013 \cdot (2014^2 - 1)$ is not prime, where n is a positive integer. (Conosur, 2014)

Exercise 34. Do there exist pairwise distinct rational numbers x, y, z such that $x^2+y^2+z^2 = x + y + z$? (Baltic Way, 2014)

Exercise 35. Prove that the equation $\sqrt{x + \sqrt{x + \sqrt{x}}} = 3$ has no rational solution. (IMO Longlist, 1970)

Exercise 36. If $x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}$, show that both fractions are equal to $x + y + z$. (Argentina Consur Olympiad, 2013)

Exercise 37. Solve for real x: $\sqrt{5 + \sqrt{24}} + \sqrt{5 - \sqrt{24}} = x$. (Bangladesh, 2018)

Exercise 38. Show that the system of equations $x^2 + y = 5, y^2 + z = 5, z^2 + x = 5$ has no real solutions. (Bosnia and Herzegovina – Regional, 2018)

Exercise 39. Find all natural numbers n for which $n^2 + n + 17$ is not a prime number. (Assam Maths Olympiad, 2015)

Exercise 40. Find all pairs of real numbers (x, y) satisfying $\sqrt{x^2 + y^2} = x + y$. (Croatia, 2003)

Exercise 41. Solve in real numbers:

$$\begin{cases} x^2 + y = 2 \\ y^2 + z = 2 \\ z^2 + x = 2 \end{cases}$$

(Metropolises, 2018)

Exercise 42. Find all real numbers x satisfying $\sqrt{x + \sqrt{x + \sqrt{x}}} = 3$. (Ukraine – 1997; Korea – 2000)

Exercise 43. Determine all real numbers $x > 1, y > 1, z > 1$ satisfying $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. (Nordic, 1992)

Exercise 44. Find all real numbers x, y such that $x^4 + y^4 + 2 = 4xy$. (RMO, 2014)

Exercise 45. Solve for real values x, y, z such that:

$$x + y + z = 3, \quad x^2 + y^2 + z^2 = 5, \quad x^3 + y^3 + z^3 = 7$$