IOQM ASSIGNMENTS: CUBIC IDENTITIES

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Exercise 1. Prove that 1280000401 is a composite number. (International Intellectual Marathon, 1993)

Exercise 2. It is given that p is a prime number such that $x^2 + y^2 - 3xy = p - 1$ for some positive integers x and y. Determine the largest possible value of p. (Singapore, Senior Section, 2011)

Exercise 3. Find all prime numbers p such that $16p + 1$ is a perfect cube. (New Zealand Camp, Selection Problem, 2017)

Exercise 4. Consider any three consecutive natural numbers. Prove that the cube of the largest cannot be the sum of the cubes of the other two. (Eötvös Competition, 1909)

Exercise 5. Given two odd integers a and b, prove that $a^n + b^n$ is divisible by $2n$, where n is any natural number. (Eötvös Competition, 1908)

Exercise 6. Represent the number $989001 \times 1007 + 320$ as a product of primes. (Leningrad, 1987)

Exercise 7. Prove that the number $512^3 + 675^3 + 720^3$ is composite. (Leningrad, 1991)

Exercise 8. Find all triples (x, y, z) of integers satisfying the equation $x^3 + y^3 + z^3 - 3xyz = 2003$. (Nordic, 2003)

Exercise 9. Find all integer pairs (x, y) such that $x^2 + y^2 = (x + y)^3$. (Turkey Junior National Olympiad, 2005)

Exercise 10. Determine all pairs of integers (a, b) that satisfy the equation $a^3 + b^3 = (a+b)^2$. (Italy, 2005)

Exercise 11. Find all pairs (p, q) of positive integers such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2013}$. (Hitotsubashi University, Japan 2013)

Exercise 12. Show that if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$. (Australian Polish, 1985; Lithuania, 2006)

Exercise 13. (i) Suppose a, b are positive real numbers such that $a^2 + b^2 = 1$ and $a + b = \sqrt{2}$. Find $a^4 + b^4$. (Pre-RMO, 2017)

(ii) Determine all pairs of positive real numbers (a, b) with $a > b$ that satisfy the equations: $a^2 + b^2 = 7$ and $a + b = 3$.

Exercise 14. Solve the system of equations in real numbers:

$$\begin{cases} x^2 + y^2 + z^2 = 6 \\ x + y + z = 0 \\ xyz = -6 \end{cases}$$

(Baltic Way, 2002)

Exercise 15. Determine all natural numbers n for which $2^n + 1$ is prime. (Austrian Polish, 2004)

Exercise 16. Show that the equation $x^6 + x + 1 = 0$ has no real roots. (IMO Longlist, 1970)

Exercise 17. Is the number $\sqrt[3]{17 + 12\sqrt{2}} + \sqrt[3]{17 - 12\sqrt{2}}$ rational or irrational? (IMO Longlist, 1973)

Exercise 18. Prove that the equation $x^4 - 2y^2 = 7$ has no integral solutions. (IMO Longlist, 1969)

Exercise 19. Let a, b, c be integers with $a^2 + b^2 + c^2$ divisible by 18. Prove that abc is divisible by 6. (Austria Beginner's Competition, 2015)

Exercise 20. Find all prime numbers a and b such that $a^3 + b^3 = a^2 + b^2 + ab + 1$. (Belarus TST, 2007)

Exercise 21. Let real numbers a, b, c, d satisfy $a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da$. Prove that the sum of some two of the numbers a, b, c, d is equal to zero. (Bosnia and Herzegovina, Regional – 2010)

Exercise 22. Solve for real x: $x^4 - 5x^2 + 6 = 0$. (RMO, 2002)

Exercise 23. Show that there do not exist any distinct natural numbers a, b, c, d such that $a^2 + b^2 + c^2 + d^2 = a + b + c + d$. (KVS-JMA, 2007)

Exercise 24. Let a, b, c be positive integers such that $a^2 + b^2 = c^2$ and $ab = c$. Prove that $a = b$. (Australia, 2016)

Exercise 25. Find all prime numbers p and q such that $p^2 + q^2 + 1 = 2pq$. (Russia, 2003)

Exercise 26. Let x and y be real numbers satisfying $x+y = 5$ and $xy = 6$. Evaluate x^3+y^3. (AIMC, 2015)

Exercise 27. The number 27,000,001 has exactly four prime factors. Find their sum. (HMMT Algebra, Feb 2005)

Exercise 28. Solve for integers x, y, z:

$$x^2 + y^2 + z^2 = 2xy + 2yz + 2zx$$

Exercise 29. What is the largest positive integer n for which n^3+100 is divisible by $n+10$? (AIME, 1986)

Exercise 30. Let m be the largest real solution to the equation $x^3 - 6x^2 + 11x - 6 = 0$. There are positive integers a, b, c such that $m = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$. Find $a + b + c$. (AIME, 2014)