IOQM ASSIGNMENTS: ALGEBRAIC MANIPULATIONS

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Exercise 1 (Korea International Maths Competition - Team Contest, 2014)*.* Divide the 18 numbers: $1, 2, 3, \dots, 17, 18$ into nine pairs such that the sum of the two numbers in each pair is the square of an integer.

Exercise 2 (Romanian Mathematics Competition, 1996)*.* Find all real numbers x for which the following equality holds:

$$\sqrt{x+\sqrt{4x-3}} = \sqrt{x-\sqrt{4x-3}} + 1.$$

Exercise 3*.* Show that, for any integer n, the number $10^{3n} + 3 \cdot 10^{2n} + 3 \cdot 10^n + 1$ is not divisible by 81.
(Conosur Olympiad, 2015)

Exercise 4*.* Let S be the set of integers that can be written in the form $50M + 3N$ where M and N are non-negative integers. For example: 3, 50, 53 are all in S. Find the sum of all positive integers not in S.
(Singapore - Junior Section, 2009)

Exercise 5*.* There exists a block of 1000 consecutive integers containing no prime numbers, namely $1001! + 2, 1001! + 3, \dots, 1001! + 1001$. Does there exist a block of 1000 consecutive integers containing exactly 5 prime numbers?
(Tournament of Towns, Senior - O Level, 2001)

Exercise 6*.* Find the least natural number n such that $n^2(n-1)$ is a multiple of 2009.
(South Africa, 2009)

Exercise 7*.*

(1) If M and N are positive integers, show that 19 divides $11M + 2N$ if and only if it divides $18M + 5N$.
(Mexico, 1988)

(2) Let a, b, c be integers. If $4a + 5b - 3c$ is divisible by 19, prove that $6a - 2b + 5c$ is also divisible by 19.
(Croatia, First Round)

(3) Let M and N be positive integers. Prove that $25M + 3N$ is divisible by 83 if and only if $3M + 7N$ is divisible by 83.
(Baltic Way, 1970)

Exercise 8*.* For any positive integer n, let $S(n)$ denote the sum of the digits of n. If $n + S(n) = 2001$, then find the product of the digits of n.
(China, 2000)

Exercise 9. The number 888888 is written as the product of two 3-digit numbers. Find the larger.
(Australian, Intermediate - 2002)

Exercise 10. The numbers a, b, c are the digits of a three-digit number which satisfy $49a + 7b + c = 286$. What is the three-digit number $(100a + 10b + c)$?
(Canada Open Maths Challenge, 1996)

Exercise 11. Find all positive integers n such that n divides $3^n + 5^n$.
(St. Petersburg, 1996)

Exercise 12. Let p, q, r be real numbers satisfying $p + q + r = 0$ and $p^3 + q^3 + r^3 = 72$. Evaluate $p^2 + q^2 + r^2$.
(Online Maths Olympiad, 2012)

Exercise 13. Let a, b, c be positive real numbers for which $\frac{1}{1+a^3} + \frac{1}{1+b^3} + \frac{1}{1+c^3} = 1$. Evaluate $a + b + c$.
(Online Maths Olympiad, 2014)

Exercise 14. Find the smallest positive integer n such that $n(n + 1)(n + 2)$ is divisible by 247.
(Singapore - Junior Section, 2007)

Exercise 15. Let a, b, c be real numbers such that $a + b + c = 6$ and $ab + bc + ca = 9$. Find the value of $a^2 + b^2 + c^2$.
(Singapore - Junior Section, 2013)

Exercise 16. If $\sqrt{20 + 14\sqrt{2}} = \sqrt{a} + \sqrt{b}$, find the value of $a + b$.
(Asian Maths Olympiad, 2015)

Exercise 17. Find the least positive integer n for which $\frac{1}{n}$ is a non-zero reducible fraction.
(Singapore, 2009)

Exercise 18. Find all positive integers x, y, z such that $x^2 + y^2 + z^2 = 2xyz$.
(Vietnam, 1982)

Exercise 19. Let a, b, c be real numbers such that $abc(a + b)(b + c)(c + a) \neq 0$ and $\frac{a^3+b^3+c^3-3abc}{(a+b+c)^3} = \frac{1}{4}$. Prove that $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{a+b+c}{abc}$.
(Bosnia & Herzegovina Regional, 2017)

Exercise 20. Let a, b, c be real and positive parameters. Solve the equation $\sqrt{a^2 + b^2} + \sqrt{b^2 + c^2} = \sqrt{a^2 + c^2} + \sqrt{2}b$.
(Romania, 1974)

Exercise 21. Find a way to write all the digits from 1 to 9 in a sequence and without repetition, so that the numbers determined by any consecutive digits of the sequence are divisible by 7 or 13.
(Lusophon, 2019)

Exercise 22. Find the smallest positive integer n such that the 73 fractions $\frac{n+1}{n}, \frac{n+2}{n+1}, \ldots, \frac{n+73}{n+72}$ are all irreducible.
(Conosur, 1999)

Exercise 23. Does there exist a ten-digit number such that all its digits are different and after removing any six digits, we get a four-digit composite number?
(Tournament of Towns, 2013)

Exercise 24. For any natural number n, let $S(n)$ denote the sum of the digits of n. Find the number of three-digit numbers for which $S(S(n)) = 2$.
(RMO-2014, Region 2)

Exercise 25. Determine the odd prime number p such that the sum of digits of the number $p^4 - 5p^2 + 13$ is the smallest possible.
(SMO, Senior Section, 2010)

Exercise 26. If $x + \frac{1}{x} = 3$, find $x^7 + \frac{1}{x^7}$.
(Hong Kong Preliminary Selection Test, 2004)

Exercise 27. Show that there exists an integer divisible by 1996 such that the sum of its decimal digits is 1996.
(Nordic, 1996)

Exercise 28. Prove that the equation $a^3+b^3+c^3 = 3abc$ has infinitely many integer solutions $\{a, b, c\}$.
(Italy, 1996)

Exercise 29. Show that if 13 divides $n^2 + 3n + 51$, then 169 divides $21n^2 + 89n + 44$.
(RMO – 2012)

Exercise 30. Show that for any natural number n, the expression $n^2-3n-19$ is not divisible by 289.
(RMO-2009)

Exercise 31. A number consists of 3 different digits. The sum of the 5 other numbers formed with those digits is 2003. Find the number.
(Flanders, 2003)

Exercise 32. An integer consists of 7 different digits, and is a multiple of each of its digits. What digits are in this number?
(Flanders, 2000)

Exercise 33. Show that the number $111\ldots111$ (with 99 ones) is not prime.
(Flanders, 1991)

Exercise 34. N is a 50-digit number (in the decimal scale). All digits except the 26th digit (from the left) are 1. If N is divisible by 13, find the 26th digit.
(RMO-1990; Turkey National Olympiad - 2000, Round 1)

Exercise 35. Prove that the expressions $2x + 3y$ and $9x + 5y$ are divisible by 17 for the same set of integral values of x and y.
(Eötvös Competition, 1894)