IOQM ASSIGNMENTS: POLYNOMIALS

PANKAJ AGARWAL (PAMATHSFAC@GMAIL.COM)

Exercise 1. The product of two of the four roots of the equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Determine the value of k. *(USAMO, 1984)*

Exercise 2. Determine all the roots, real or complex, of the system of equations:

$$x + y + z = 3, \quad x^2 + y^2 + z^2 = 3, \quad x^3 + y^3 + z^3 = 3.$$

(USAMO, 1973)

Exercise 3. Let a, b, c be distinct integers and $P(x)$ be a polynomial with integer coefficients. Show that it is impossible that

$$P(a) = b, \quad P(b) = c, \quad P(c) = a.$$

(INMO, 1986; USAMO, 1974)

Exercise 4. Let $P(x)$ be a polynomial of degree n such that

$$P(k) = \frac{k}{k+1}, \quad \text{for } k = 0, 1, 2, \ldots, n.$$

Determine $P(n+1)$. *(Singapore MO, 1988; USAMO, 1974)*

Exercise 5. Without solving the cubic equation $x^3 - x + 1 = 0$, compute the sum of all roots of the equation. *(Vietnam, 1975)*

Exercise 6. Can the equation

$$z^3 - 2z^2 - 2z + m = 0$$

have three distinct rational roots? Justify your answer. *(Vietnam, 1980)*

Exercise 7. Find the polynomial of the lowest degree with integer coefficients such that one of its roots is

$$\sqrt{2} + \sqrt[3]{3}.$$

(Vietnam, 1984)

Exercise 8. Find all polynomials $p(x)$ of the lowest degree with rational coefficients such that

$$p(3 + \sqrt[3]{3}) = p(3 - \sqrt[3]{3}).$$

(Vietnam, 1997)

Exercise 9. Find all real values of a for which the equation

$$x^4 + 2ax^2 - 4x + a = 0$$

has all real roots. *(RMO, 2000)*

Exercise 10. Prove that the product of the first 200 positive even integers differs from the product of the first 200 positive odd integers by a multiple of 407. *(RMO, 2001)*

Exercise 11. Let $P_1(x) = ax^2 - bx - c$, $P_2(x) = bx^2 - cx - a$, $P_3(x) = cx^2 - ax - b$, where a, b, c are nonzero real numbers. Suppose there exists a real number α such that

$$P_1(\alpha) = P_2(\alpha) = P_3(\alpha).$$

Prove that $a = b = c$. *(RMO, 2011)*

Exercise 12. Let $P_1(x) = x^2 + a_1x + b_1$ and $P_2(x) = x^2 + a_2x + b_2$ be two quadratic polynomials with integer coefficients. Suppose $a_1 \neq a_2$ and there exist integers $m \neq n$ such that

$$P_1(m) = P_2(n), \quad P_2(m) = P_1(n).$$

Prove that $a_1 - a_2$ is even. *(RMO, 2015)*

Exercise 13. Suppose a, b are integers and $a + b$ is a root of the equation

$$x^2 + ax + b = 0.$$

What is the maximum possible value of b^2? *(Pre-RMO, 2018)*

Exercise 14. Suppose $1, 2, 3$ are the roots of the equation

$$x^4 + ax^2 + bx = c.$$

Find the value of c. *(Pre-RMO, 2017)*

Exercise 15. Suppose a and b are real numbers such that $ab < 1$ and the equations

$$120a^2 - 120a + 1 = 0, \quad b^2 - 120b + 120 = 0$$

hold. Find the value of

$$\frac{a + b + ab}{a}.$$

(Pre-RMO, 2016)

Exercise 16. Let x_1, x_2, x_3 be the roots of the equation

$$x^3 + 3x + 5 = 0.$$

Evaluate the expression

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right)?$$

(Pre-RMO, 2012)

Exercise 17. Determine the number of distinct real solutions of the equation

$$(x-1)(x-3)(x-5)\cdots(x-2015) = (x-2)(x-4)(x-6)\cdots(x-2014).$$

(Australian Maths Olympiad, 2015)

Exercise 18. Find a polynomial of degree 3 with real coefficients such that each of its roots is equal to the square of one root of the polynomial

$$P(x) = x^3 + 9x^2 + 9x + 9.$$

(Moldova, 1999)

Exercise 19. Let $P(x)$ be a cubic polynomial with roots r_1, r_2, r_3. Suppose that

$$\frac{P(1/2) + P(-1/2)}{P(0)} = 1000.$$

Find the value of

$$\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1}.$$

(Australian Maths Olympiad, 1996)

Exercise 20. Find an integer x that satisfies the equation

$$x^5 - 101x^3 + 999x - 1009000 = 0.$$

(Singapore Maths Olympiad, 2005)

Exercise 21. Determine the value of $x^2 + y^2 + z^2 + w^2$ if the following equations hold:

$$\begin{aligned}
\frac{x^2}{2^2 - 1^2} + \frac{y^2}{2^2 - 3^2} + \frac{z^2}{2^2 - 5^2} + \frac{w^2}{2^2 - 7^2} &= 1, \\
\frac{x^2}{4^2 - 1^2} + \frac{y^2}{4^2 - 3^2} + \frac{z^2}{4^2 - 5^2} + \frac{w^2}{4^2 - 7^2} &= 1, \\
\frac{x^2}{6^2 - 1^2} + \frac{y^2}{6^2 - 3^2} + \frac{z^2}{6^2 - 5^2} + \frac{w^2}{6^2 - 7^2} &= 1, \\
\frac{x^2}{8^2 - 1^2} + \frac{y^2}{8^2 - 3^2} + \frac{z^2}{8^2 - 5^2} + \frac{w^2}{8^2 - 7^2} &= 1.
\end{aligned}$$

(AIME, 1984)

Exercise 22. For how many real numbers a does the quadratic equation

$$x^2 + ax + 6a = 0$$

have only integer roots for x? *(AIME, 1991)*

Exercise 23. Let

$$P_0(x) = x^3 + 313x^2 - 77x - 8.$$

For integers $n \geq 1$, define recursively

$$P_n(x) = P_{n-1}(x - n).$$

What is the coefficient of x in $P_{20}(x)$? *(AIME, 1993)*

Exercise 24. Suppose the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, c. Let the roots of $x^3 + rx^2 + sx + t = 0$ be $a + b, b + c, c + a$. Find t. *(AIME, 1996)*

Exercise 25. Consider the polynomials

$$P(x) = x^6 - x^5 - x^3 - x^2 - x, \quad Q(x) = x^3 - x^2 - x - 1.$$

Given that z_1, z_2, z_3, z_4 are the roots of $Q(x) = 0$, find the value of

$$P(z_1) + P(z_2) + P(z_3) + P(z_4).$$

(AIME, 2003)

Exercise 26. Let C be the coefficient of x^2 in the expansion of

$$(1-x)(1+2x)(1-3x)(1+4x)\cdots(1-15x).$$

Find $|C|$. *(AIME, 2004)*

Exercise 27. Let p be the product of the non-real roots of the equation

$$x^4 - 4x^3 + 6x^2 - 4x = 2005.$$

Find $\lfloor p \rfloor$. *(AIME, 2005)*

Exercise 28. The polynomial $P(x)$ is cubic. What is the largest value of k such that both polynomials

$$Q_1(x) = x^2 + (k-29)x - k, \quad Q_2(x) = 2x^2 + (2k-43)x + k$$

are factors of $P(x)$? *(AIME, 2007)*

Exercise 29. Let $P(x)$ be a quadratic polynomial with real coefficients satisfying

$$x^2 - 2x + 2 \le P(x) \le 2x^2 - 4x + 3 \quad \text{for all real } x,$$

and suppose $P(11) = 181$. Find $P(16)$. *(AIME, 2010)*

Exercise 30. Prove that if the coefficients of the quadratic equation $ax^2 + bx + c = 0$ are odd integers, then its roots cannot be rational. *(INMO, 1987)*

Exercise 31. Let α, β, γ be the roots of $x^3 - x - 1 = 0$. Compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

(Canada National Olympiad, 1996)

Exercise 32. For what values of b do the equations

$$1988x^2 + bx + 8891 = 0, \quad 8891x^2 + bx + 1988 = 0$$

have a common root? *(Canada Maths Olympiad, 1988)*