IOQM ASSIGNMENTS: LCM AND HCF

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Exercise 1. Find the least positive integer n for which

$$\frac{n-10}{9n+11}$$

is a non-zero reducible fraction. *(Singapore, 2009)*

Exercise 2. Find the smallest positive integer n such that the 73 fractions

$$\frac{19}{n+21}, \quad \frac{20}{n+22}, \quad \dots, \quad \frac{91}{n+93}$$

are all irreducible. *(ConoSur, 1991)*

Exercise 3. Let p and q be positive integers such that

$$\text{LCM}(p,q) + \gcd(p,q) = p + q.$$

Prove that one of them is divisible by the other. *(Russia, 1995)*

Exercise 4. Prove that the fraction

$$\frac{21n+4}{14n+3}$$

is irreducible for every natural number n. *(IMO, 1959)*

Exercise 5. For any positive integer n, show that n^2+n-1 and n^2+2n have no common factors greater than 1. *(Mexico, 1987)*

Exercise 6. Prove that for any two positive integers a and b, the equation

$$\text{LCM}(a, a+5) = \text{LCM}(b, b+5)$$

implies $a = b$. *(Tournament of Towns, 1998)*

Exercise 7. Evaluate:

$$\gcd(2002+2,\ 2002^2+2,\ 2002^3+2,\ \dots)$$

(HMMT, 2002)

Exercise 8. Show that there are no two positive integers a and b such that

$$ab + \gcd(a,b) + \text{LCM}(a,b) = 2014.$$

(South Africa, 2014)

Exercise 9. Determine all possible values of $M+N$ where M and N are positive integers satisfying

$$\text{LCM}(M,N) - \gcd(M,N) = 103.$$

(CMIMC Number Theory, 2017)

Exercise 10. Let x and y be positive integers such that

$$2(x+y) = \gcd(x,y) + \mathrm{LCM}(x,y).$$

Find the numerical value of

$$\frac{\mathrm{LCM}(x,y)}{\gcd(x,y)}.$$

Exercise 11. The numbers in the sequence $101, 104, 109, 116, \dots$ are of the form $a_n = 100 + n^2$ where $n = 1, 2, 3, \dots$. For each n, let $d_n = \gcd(a_n, a_{n+1})$. Let k be the maximum value of d_n as n ranges over the positive integers. Find k.

Exercise 12. Solve the equation

$$MN = (\gcd(M,N))^2 + \mathrm{LCM}(M,N)$$

in positive integers. *(Kyiv Mathematical Festival, 2011)*

Exercise 13. Find the number of ordered triples (a,b,c) of positive integers such that

$$\mathrm{LCM}(a,b) = 1000, \quad \mathrm{LCM}(b,c) = 2000, \quad \mathrm{LCM}(c,a) = 2000.$$

(AIME, 1987)

Exercise 14. Are there such natural numbers x and y such that

$$\gcd(x,y) + \mathrm{LCM}(x,y) + x + y = 20162017?$$

Exercise 15. Find all positive integer solutions (a,b) to the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{n}{lcm(a,b)} = \frac{1}{\gcd(a,b)}$$

for (i) $n = 2007$, (ii) $n = 2010$. *(New Zealand Camp Selection Problems, 2010 Junior)*

Exercise 16. Find the minimum value of n such that

$$\mathrm{LCM}(n,2) + \mathrm{LCM}(n,3) + \mathrm{LCM}(n,4) + \mathrm{LCM}(n,5) \geq 2021.$$

Exercise 17. Find the number of unordered pairs of natural numbers x and y such that

$$\mathrm{LCM}(x,y) = 360.$$

Exercise 18. Two positive integers M, N satisfy the two equations

$$M^2 + N^2 = 3789, \quad \gcd(M,N) + \mathrm{LCM}(M,N) = 633.$$

Compute $M + N$. *(Thailand Maths Olympiad, 2008)*

Exercise 19. Given positive integers a, b, c, find all possible triples (a,b,c) satisfying

$$\gcd(a,20) = b, \quad \gcd(b,15) = c, \quad \gcd(a,c) = 5.$$

Exercise 20. What is the largest possible value of the expression

$$\gcd(n^2+3,\ (n+1)^2+3)$$

for all natural numbers n? *(Latvia – TST, 2016)*