IOQM ASSIGNMENTS: FERMAT'S LITTLE THEOREM

PANKAJ AGARWAL (PAMATHSFAC@GMAIL.COM)

Exercise 1. Prove that for every positive integer n the following proposition holds:

$$3^n + n^3 \text{ is divisible by } 7 \iff 7 \mid 3^n \cdot n^3 + 1.$$

(Bulgaria, 1995)

Exercise 2. Find all triples of primes (p, q, r) satisfying

$$3p^4 - 5q^4 - 4r^2 = 26.$$

(JBMO-Shortlist, 2014)

Exercise 3. Find all prime numbers a, b, c and positive integers k which satisfy the equation

$$a^2 + b^2 + 16c^2 = 9k^2 + 1.$$

(JBMO, 2015)

Exercise 4. Show that the equation $x^4 + 131 = 3y^4$ has no solution in integers x and y. *(Australia, 1984)*

Exercise 5. Find the integral solutions of

$$19x^3 - 84y^2 = 1984.$$

(Moscow, 1984)

Exercise 6. Prove that for every integer x, the number

$$\frac{x^5}{5} + \frac{x^3}{3} + \frac{7}{15}x$$

is an integer. *(Australia, 1994)*

Exercise 7. Are there integers a, b, c, d which satisfy

$$a^4 + b^4 + c^4 + 2016 = 10d?$$

(Berkeley Maths Circle, 2015)

Exercise 8. Prove that if a is an integer relatively prime with 35, then

$$(a^4 - 1)(a^4 + 15a^2 + 1) \equiv 0 \pmod{35}.$$

Exercise 9. Let M and N be positive integers and

$$A(M, N) = M^{3^{4N}+6} - M^{3^{4N}+4} - M^5 + M^3.$$

Find all integers N such that $A(M, N)$ is divisible by 1992 for all integers M. *(Balkan, 1992)*

Exercise 10. Let $a_1, a_2, \ldots, a_9$ be integers. Prove that if 19 divides

$$a_1^9 + a_2^9 + \cdots + a_9^9,$$

then 19 divides $a_1 a_2 \cdots a_9$. *(Saudi Arabia Selection Test, 2013)*

Exercise 11. Prove that the equation

$$y^2 = x^5 - 4$$

has no integral solutions. *(Balkan, 1998)*

Exercise 12. Find all prime numbers p such that

$$2^p + p^2$$

is also a prime number. *(Albania TST, 2011)*

Exercise 13. Prove that there are no positive integers x, y, z such that

$$x^2 + y^2 = 3z^2.$$

(ConoSur, 1992)

Exercise 14. Prove that there are no integer numbers x, y such that

$$x^2 - 3y^2 = 17.$$

(Berkeley Math Circle)

Exercise 15. Find all prime numbers a and b such that

$$20a^3 - b^3 = 1.$$

(Belarus TST, 2017)

Exercise 16. Let x, y, z be integers with $z > 1$. Show that

$$(x+1)^2 + (x+2)^2 + \dot{s} + (x+99)^2 \neq y^z$$

(Hungary, 1998)

Exercise 17. Prove that the equation

$$\sqrt{x^3 + y^3 + z^3} = 1969$$

has no integral solutions. *(IMO-Longlist, 1969)*

Exercise 18. Given unequal integers x, y, z, prove that

$$(x-y)^5 + (y-z)^5 + (z-x)^5$$

is divisible by

$$5(x-y)(y-z)(z-x).$$

(All Soviet Union, 1962)

Exercise 19. Find all primes p such that

$$8p^4 - 3003$$

is a (positive) prime. *(Mexico, 1997)*

Exercise 20. Prove that 2014 divides

$$53n^{55} - 57n^{53} + 4n$$

for all integers n. *(Saudi Arabia Pre-Selection Test, 2014)*