IOQM ASSIGNMENTS: BASIC NUMBER THEORY

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Exercise 1. Find all primes p such that $p+2$ and p^2+2p-8 are primes.
(Albania, 2012)

Exercise 2. Let p and q be primes such that the numbers $p+q$ and $p+7q$ are both perfect squares. Find the value of p.
(HMMT-Algebra, 2002)

Exercise 3. Prove that the equation $x^2 = 2008! + y^2$ has no solutions in integers.
(Iberoamerican, 2008)

Exercise 4. Find all pairs of positive integers (n, k) such that $n! + 8 = 2^k$.
(Australia, 2018)

Exercise 5. If a, b and c are natural numbers such that $10! = a! \cdot 8! \cdot c!$, then find the number of solutions to the above equation.
(South Africa, Junior Maths Olympiad - 2014)

Exercise 6. Let p, q, r be distinct primes. Prove that $p+q+r+pqr$ is composite.
(Berkeley Maths Circle, 2009)

Exercise 7. Find all primes p and q such that $p+q = (p-q)^3$.
(Russia, 2003)

Exercise 8. Find all primes p, q, r such that $15p + 7pq + qr = pqr$.
(Slovenia, 2010)

Exercise 9. Find all integers x for which $2x^2 - x - 36$ is the square of a prime number.
(Croatia, 2001)

Exercise 10. Let P be a fixed prime number. Find all pairs (x, y) of positive integers satisfying $P(x-3) = xy$.
(Estonia, 1996)

Exercise 11. The prime numbers p, q, r satisfy the equations $pq+pr = 80$ and $pq+qr = 425$. Find the value of $p+q+r$.
(Australia, Intermediate-2013)

Exercise 12. Find all triples (p, q, r) of primes such that $pq = r+1$ and $2(p^2+q^2) = r^2+1$.
(RMO-2013)

Exercise 13. Find all prime numbers p such that $p^2 + 2007p - 1$ is also prime.
(Berkeley Maths Circle - 2008)

Exercise 14. Find the triples of primes (a, b, c) such that $a-b-8$ is prime.
(Hitotsubashi University, 2014)

Exercise 15. Find all positive integers x, y, z such that $2^x + 2^y + 2^z = 2336$.
(Vietnam, 1982)

Exercise 16. Suppose that n is a positive integer and let $d_1 < d_2 < d_3 < d_4$ be the four smallest positive divisors of n. Find all integers n satisfying this condition.
(Balkan-1989; Iran-1999)

Exercise 17. Find all primes p such that $p^2 + 11$ has exactly six different divisors (including 1 and itself).
(Russia, 1995)

Exercise 18. The positive integer n is divisible by 24. Show that the sum of all the positive divisors of $n - 1$ is also divisible by 24.
(Putnam -1969; Bosnia & Herzegovina - 2011)

Exercise 19. The digits of n strictly increase from left to right. Find the sum of the digits of $9n$.
(Russia, 1999)

Exercise 20. Find the primes p, q, r, given that one of the numbers pqr and $p + q + r$ is 101 times the other.
(Nordic, 2015)

Exercise 21. Let p, q be two consecutive odd prime numbers. Prove that $p + q$ is a product of at least 3 natural numbers > 1 (not necessarily distinct).
(Baltic Way, 1992)

Exercise 22. Find all prime numbers p, q, r satisfying $p^4 + 2p + q^4 + q^2 = r^2 + 4q^3 + 1$.
(Turkey Junior National Olympiad, 2013)

Exercise 23. Find all prime numbers p, q, r such that $p > q > r$ and the numbers $p - q$, $p - r$, and $q - r$ are also prime.
(Slovenia, 2010)

Exercise 24. Knowing that the numbers p, $3p + 2$, $5p + 4$, $7p + 6$, $9p + 8$, and $11p + 10$ are all primes, prove that $6p + 11$ is a composite number.
(Czech and Slovak Olympiad, 2009)

Exercise 25. Let p and q be prime numbers. Show that $p^2 + q^2 + 2020$ is composite.
(Kosovo, 2020)

Exercise 26. Let $1 = d_1 < d_2 < \cdots < d_n = n$ be the divisors of n. Find all values of n such that a given condition holds.
(Mexico, 2008)

Exercise 27. Find all pairs of prime numbers (p, q) for which $7pq^2 + p = q^3 + 43p^3 + 1$.
(Dutch, 2015)

Exercise 28. Solve $2a^2 + 3a - 44 = 3p^n$ in positive integers where p is a prime.
(Turkey - Junior, 2019)

Exercise 29. Find all the positive integers x and y that satisfy the equation $x(x-y) = 8y-7$.
(JBMO-Shortlist, 2008)

Exercise 30. Find all prime numbers p, q, r such that an additional condition holds.
(JBMO-Shortlist, 2008)

Exercise 31. Find all primes p, q such that $2p^3 - q^2 = 2(p+q)^2$.
(JBMO-Shortlist, 2011)

Exercise 32. Find all prime numbers p, q, r, k such that $pq + qr + rp = 12k + 1$.
(Iberoamerican, 2016)

Exercise 33. Prove that there aren't any positive integer numbers x, y such that $x^2 + y^2 = 3z^2$.
(Conosur, 1992)

Exercise 34. Find all positive prime numbers p, q, r, s so that $p^2 + 2019 = 26(q^2 + r^2 + s^2)$.
(Conosur, 2019)

Exercise 35. Find all positive prime numbers p, q, r such that p and q are primes and an additional condition holds.
(Centro American, 2011)

Exercise 36. Determine all triples (p, q, r) of positive integers, where p, q are primes, such that a condition holds.
(Centro American, 2018)

Exercise 37. Find all pairs of non-negative integers m and n that satisfy $3 \cdot 2^m + 1 = n^2$.
(New Zealand - Camp Selection Problems, Junior-2011)

Exercise 38. Find all positive integers n for which $4n + 2007$ is a perfect square.
(Greece, 2007)

Exercise 39. Does there exist any integers a, b, c such that $a^2bc + 2$, $ab^2c + 2$, $abc^2 + 2$ are all perfect squares?
(China Western, 2013)

Exercise 40. How many primes p are there such that $2p^4 - 7p^2 + 1$ is the square of an integer?
(Turkey National Olympiad, 2001 - Round 1)

Exercise 41. If p and $p^2 + 2$ are prime numbers, at most how many prime divisors can $p^3 + 3$ have?
(Turkey National Olympiad, 2006 – Round 1)

Exercise 42. Find all triples of primes in the form $(p, 2p+1, 4p+1)$.
(Moldova, 2002)

Exercise 43. Prove that the equation $p^4 + q^4 = r^4$ does not have solutions in the set of prime numbers.
(Bosnia & Herzegovina - Regional, 2008)

Exercise 44. Find all pairs (x, y) of positive integers that satisfy the equation $x^2 - xy + 2x - 3y = 2013$.
(Lusophon, 2013)

Exercise 45. Find all positive integer solutions to $2x^2 + 5y^2 = 11(xy - 11)$.
(Baltic Way, 1998)

Exercise 46. Let M and N be positive integers satisfying $MN^2 + 876 = 4MN + 217N$. Find the sum of all possible values of M.
(Singapore - Junior Section, 2012)

Exercise 47. Find all positive integer solutions of the equation $10(M + N) = MN$.
(Croatia, 1998)

Exercise 48. How many pairs of integers (x, y) are there such that $2x + 5y = xy - 1$?
(Turkey National Olympiad, 2003 - Round 1)

Exercise 49. Let M, N be natural numbers such that $M + N + 1$ is prime and divides $2(M^2 + N^2) - 1$. Prove that $M = N$.
(Switzerland, 2010)

Exercise 50. Two different positive integers a and b satisfy the equation $a^2 - b^2 = 2018 - 2a$. What is the value of $a + b$?
(Australia – Intermediate, 2018)