IOQM ASSIGNMENTS: TELESCOPING METHOD

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Exercise 1. Evaluate:
$$\sum_{k=1}^{360} \frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}.$$
(Purple Comet, High School – 2005)

Exercise 2. Define
$$a_k = (k^2+1)k!, \quad b_k = a_1 + a_2 + \cdots + a_k.$$
Evaluate $\frac{a_{100}}{b_{100}}$. *(Purple Comet, 2003)*

Exercise 3. Let
$$a_n = \sqrt{1+\left(1-\frac{1}{n}\right)^2} + \sqrt{1+\left(1+\frac{1}{n}\right)^2}.$$
Evaluate:
$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{20}}.$$
(Purple Comet, 2003)

Exercise 4. Evaluate:
$$\left(1-\frac{1}{2^2-1}\right)\left(1-\frac{1}{2^3-1}\right)\left(1-\frac{1}{2^4-1}\right)\cdots\left(1-\frac{1}{2^{29}-1}\right).$$
(Purple Comet, Middle School, 2022)

Exercise 5. For integers $k \geq 1$, let
$$a_k = \frac{k}{4k^4+1}.$$
Find the least integer n such that
$$\sum_{k=1}^{n} a_k > \frac{505.45}{2022}.$$
(Purple Comet, High School, 2022)

Exercise 6. Evaluate the product:
$$\left(\frac{1+1}{1^2+1}+\frac{1}{4}\right)\left(\frac{2+1}{2^2+1}+\frac{1}{4}\right)\left(\frac{3+1}{3^2+1}+\frac{1}{4}\right)\left(\frac{2022+1}{2022^2+1}+\frac{1}{4}\right).$$
(Purple Comet, High School, 2022)

Exercise 7. Evaluate the product:

$$\left(\frac{1}{2^3-1}+\frac{1}{2}\right)\left(\frac{1}{3^3-1}+\frac{1}{2}\right)\left(\frac{1}{4^3-1}+\frac{1}{2}\right)\cdots\left(\frac{1}{100^3-1}+\frac{1}{2}\right).$$

(Purple Comet, High School, 2021)

Exercise 8. Let $T_k=\frac{k(k+1)}{2}$ be the k-th triangular number. Evaluate:

$$\sum_{k=4}^{\infty}\frac{1}{(T_{k-1}-1)(T_k-1)(T_{k+1}-1)}.$$

(Purple Comet, High School, 2017)

Exercise 9. Evaluate the product:

$$\left(1+\frac{1}{1+2^1}\right)\left(1+\frac{1}{1+2^2}\right)\left(1+\frac{1}{1+2^3}\right)\cdots\left(1+\frac{1}{1+2^{10}}\right)$$

(Purple Comet, High School, 2015)

Exercise 10. Determine the value of:

$$\sqrt{5+\sqrt{5^2-9}}-\sqrt{11+\sqrt{11^2-9}}+\sqrt{17+\sqrt{17^2-9}}-\sqrt{23+\sqrt{23^2-9}}$$
$$+\sqrt{29+\sqrt{29^2-9}}-\cdots+\sqrt{797+\sqrt{797^2-9}}.$$

(El Salvador, 2021)

Exercise 11. Evaluate:

$$\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+\cdots+\frac{99}{1+99^2+99^4}.$$

(NTSE Haryana, Stage-I, 2013–14)

Exercise 12. Evaluate:

$$\frac{(10^4+324)(22^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}.$$

(AIME-1987; NTSE AP, 2014–15)

Exercise 13. Find the value of:

$$\frac{(3^4+3^2+1)(5^4+5^2+1)(7^4+7^2+1)(9^4+9^2+1)(11^4+11^2+1)(13^4+13^2+1)}{(2^4+2^2+1)(4^4+4^2+1)(6^4+6^2+1)(8^4+8^2+1)(10^4+10^2+1)(12^4+12^2+1)}.$$

(Pre-RMO 2016, Delhi Region)

Exercise 14. Evaluate:

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}+\cdots+\sqrt{1+\frac{1}{2012^2}+\frac{1}{2013^2}}.$$

(Hong Kong IMO Preliminary, 2013)

Exercise 15. Evaluate:
$$\sum_{k=1}^{2002} \frac{k+2}{k!+(k+1)!+(k+2)!}.$$
(Hong Kong IMO Preliminary, 2004)

Exercise 16. If
$$A=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{2003\cdot 2004}, \quad B=\frac{1}{1003\cdot 2004}+\frac{1}{1004\cdot 2003}+\cdots+\frac{1}{2004\times 1003},$$
evaluate $\frac{A}{B}$. *(Hong Kong IMO Preliminary, 2004)*

Exercise 17. Evaluate:
$$\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+\cdots+\frac{100}{1+100^2+100^4}.$$
(Hong Kong IMO Preliminary, 2005)

Exercise 18. Let
$$f(n)=\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\cdots+\frac{1}{2017^n}$$
find
$$f(2)+f(3)+f(4)+\cdots.$$
(Hong Kong IMO Preliminary, 2017)

Exercise 19. Consider the following 50-term sums:
$$S=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{99\cdot 100}, \quad T=\frac{1}{51\cdot 100}+\frac{1}{52\cdot 99}+\frac{1}{53\cdot 98}+\cdots+\frac{1}{99\cdot 52}+\frac{1}{100\cdot 51}.$$
Express $\frac{S}{T}$ as an irreducible fraction. *(Argentina National Olympiad, 2014)*

Exercise 20. Decide whether S_n or T_n is larger, where
$$S_n=\sum_{k=1}^{n} \frac{k}{(2n-2k+1)(2n-k+1)}, \quad T_n=\sum_{k=1}^{n} \frac{1}{k}.$$
(Vietnam National Olympiad, 1983)

Exercise 21. Evaluate:
$$\sum_{k=1}^{40} \sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}.$$

Exercise 22. If
$$\sum_{k=1}^{N} \frac{2k+1}{(k^2+k)^2}=0.9999,$$
determine the value of N. *(IOQM, 2020–21)*

Exercise 23. Let
$$S=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{99}}+\frac{1}{\sqrt{100}}.$$
Find the greatest integer less than or equal to S. *(Pre-RMO 2016, West Bengal Region)*

Exercise 24. If

$$f(n) = \frac{2n+1+\sqrt{n(n+1)}}{\sqrt{n+1}+\sqrt{n}}$$

for all positive integers n, then evaluate the sum

$$\sum_{k=1}^{400} f(k).$$

Exercise 25. Find the consecutive integers bounded by the expression

$$\frac{1}{x_1+1} + \frac{1}{x_2+1} + \frac{1}{x_3+1} + \cdots + \frac{1}{x_{2001}+1} + \frac{1}{x_{2002}+1},$$

where

$$x_1 = 3, \quad x_{n+1} = x_n^2 + x_n.$$

Exercise 26. Prove that:

$$2010 < \frac{2^2+1}{2^2-1} + \frac{3^2+1}{3^2-1} + \cdots + \frac{2010^2+1}{2010^2-1} < 2010 + \frac{1}{2}.$$

Exercise 27. Evaluate:

$$\frac{1}{2\sqrt{1}+1\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + \frac{1}{4\sqrt{3}+3\sqrt{4}} + \cdots + \frac{1}{400\sqrt{399}+399\sqrt{400}}.$$

Exercise 28. Let

$$S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}}.$$

Find

$$\sum_{n=1}^{99} \frac{1}{S_n + S_{n-1}}.$$

(Pre-RMO 2013)

Exercise 29. Evaluate:

$$\sum_{n=1}^{9} \frac{1}{n(n+1)(n+2)}.$$

(Pre-RMO 2017)

Exercise 30. Find the greatest integer less than or equal to:

$$\sum_{n=1}^{1599} \frac{1}{n}.$$

(Pre-RMO 2019)

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