From lasso to collaborative filtering and recommender systems

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Collaborative filtering: basics

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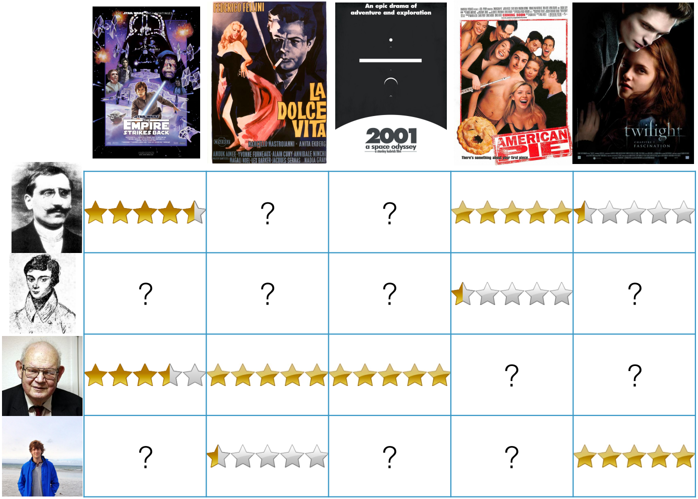
Collaborative filtering



Collaborative filtering

Problem: We partly observe a user-product matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$. Specifically, we only know the value of x_{ij} (e.g. the score of the movie) for a set of indices Ω.
We assume that there exist an (unknown) matrix $\mathbf{Z} \in \mathbb{R}^{m \times n}$, which is the "true user-product matrix", that we want to recover given our observations. We therefore have

$$\forall (i, j) \in \Omega,\ x_{ij} = z_{ij},$$

or more realistically

$$\forall (i, j) \in \Omega,\ x_{ij} \approx z_{ij}.$$

Exercise: How could we model $x_{ij} \approx z_{ij}$?

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Exercise: How could we model $x_{ij} \approx z_{ij}$? For example, we could say that $(x_{ij} - z_{ij})^2$ is minimised. A probabilistic approach would be

$$\forall (i,j) \in \Omega,\ x_{ij} = z_{ij} + \varepsilon_{ij},\ \text{with } \varepsilon_{ij} \sim \mathcal{N}(0, \sigma).$$

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This is clearly an ill-posed problem!(for now)

To be able to solve it, we need to assume that $\mathbf{Z}$ has some sort of simple structure. We need a notion of "simple structure" for matrices. Any idea?

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A first simple notion of simplicity is a constant matrix: all entries are identical. In a collaborative filtering context, it means all users have the exact same taste, and all products are equivalent... This is clearly overly simple!

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The notion of rank of a matrix generalises these simple heuristics!

What is the rank of a matrix?

The rank is a measure of complexity for matrices. A matrix $\mathbf{Z} \in \mathbb{R}^{n \times n}$ has rank $r \in \{1, ..., n\}$ if there exists two matrices $\mathbf{U} \in \mathbb{R}^{n \times r}$ and $\mathbf{V} \in \mathbb{R}^{n \times r}$ such that

$$\mathbf{Z} = \mathbf{U}\mathbf{V}^T.$$

Intuitively, it means that we can describe the matrix with only $2nr$ coefficients, rather than n^2. So when r is small there is a huge gain...

An equivalent definition is that the rank of $\mathbf{Z}$ is the number of nonzero eigenvalues of $\mathbf{Z}^T\mathbf{Z}$.

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which means that the columns of $\mathbf{Z}$ are proportional. Similarly, we can write

$$\mathbf{Z} = (u_1, ..., u_n)^T\mathbf{v} = (u_1\mathbf{v}, ..., u_n\mathbf{v})^T,$$

which means that the rows of $\mathbf{Z}$ are proportional.

Collaborative filtering with low-rank matrices

A way to have a well posed-problem: add the assumption that the "true matrix" should have low-rank k! This makes sense because it means that there are just a few factors of variation. In particular, we recover the naive solutions of the previous slides.

$$\hat{\boldsymbol{Z}} \in \operatorname{argmin}_{\boldsymbol{Z} \in \mathbb{R}^{m \times n}} \sum_{(i,j) \in \Omega} (z_{ij} - x_{ij})^2 + \lambda \operatorname{rank}(\boldsymbol{Z}).$$

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is unfortunately a super hard combinatorial problem to solve, because the rank is the number of non-zero singular values of Z.
Exercise:Can we replace this number of nonzeros by something smooth and convex?

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Yes! The ℓ_1 norm of the singular values, it's called the nuclear norm:

$$||Z||_* = \sum_{i=1}^{\min(n,m)} \lambda(Z)_i.$$

Like for the lasso, we replaced a number of nonzero elements by a ℓ_1 norm.

Collaborative filtering with the nucelar norm

The new problem is then:

$$\hat{\boldsymbol{Z}} \in \text{argmin}_{\boldsymbol{Z} \in \mathbb{R}^{m \times n}} \sum_{(i,j) \in \Omega} (z_{ij} - x_{ij})^2 + \lambda ||Z||_*.$$

It's convex and very fast algorithms exist!

We will use one called softimpute (Hastie, Mazumder, Lee, & Zadehin, JMLR 2015) in today's lab.