λ-calculus
& categories 4
16 october 2020

String diagrams & adjunctions

String diagrams in 2-categories

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diagrammes de cordes

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{B}
\arrow[from=U, to=D, phantom, "\Downarrow\theta"]
\end{tikzcd}
\qquad \theta : F \Rightarrow G : \mathcal{A} \to \mathcal{B}
$$

$$
\begin{tikzcd}
& \mathcal{B} \arrow[dr, "G"] & \\
\mathcal{A} \arrow[ur, "F"] \arrow[rr, bend right, "H"'] \arrow[rr, phantom, "\Downarrow\theta"] & & \mathcal{C}
\end{tikzcd}
\qquad \theta : G \circ F \Rightarrow H : \mathcal{A} \to \mathcal{C}
$$

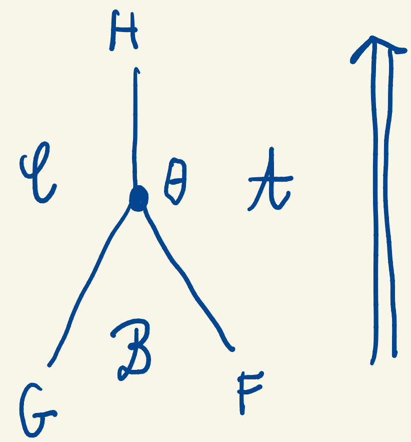
pasting diagrams

$$
\begin{tikzcd}
& \mathcal{B} \arrow[dr, "G"] \arrow[rr, bend left, "J"] \arrow[rr, phantom, "\Downarrow\varphi"] & & \mathcal{D} \\
\mathcal{A} \arrow[ur, "F"] \arrow[rr, bend right, "H"'] \arrow[rr, phantom, "\Downarrow\theta"] & & \mathcal{C} \arrow[ur, "K"'] &
\end{tikzcd}
$$

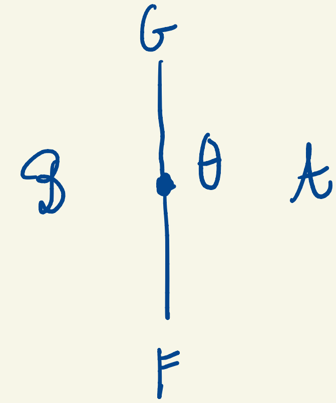
these diagrams can be represented in another way, based on the ideas of Penrose, Joyal and Street ___ very simple: "Poincaré duality"

2-cells are depicted as nodes (0-dimensional)
1-cells / morphisms are depicted as strings (1-dim)
0-cells / objects are depicted as "areas" (2-dimensional)

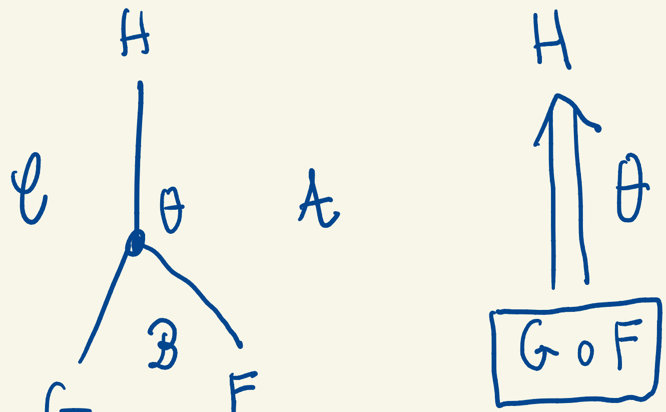
$$\begin{tikzcd} & \mathcal{B} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow\theta"] & \\ \mathcal{A} \arrow[ur, "F"] \arrow[rr, bend right, "H"'] & {} & \mathcal{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{C} & {} \arrow[d, phantom, "\Uparrow\theta"] & \mathcal{A} \arrow[ll, bend right, "H"'] \arrow[dl, "F"] \\ & \mathcal{B} \arrow[ul, "G"] & \end{tikzcd} \longmapsto$$



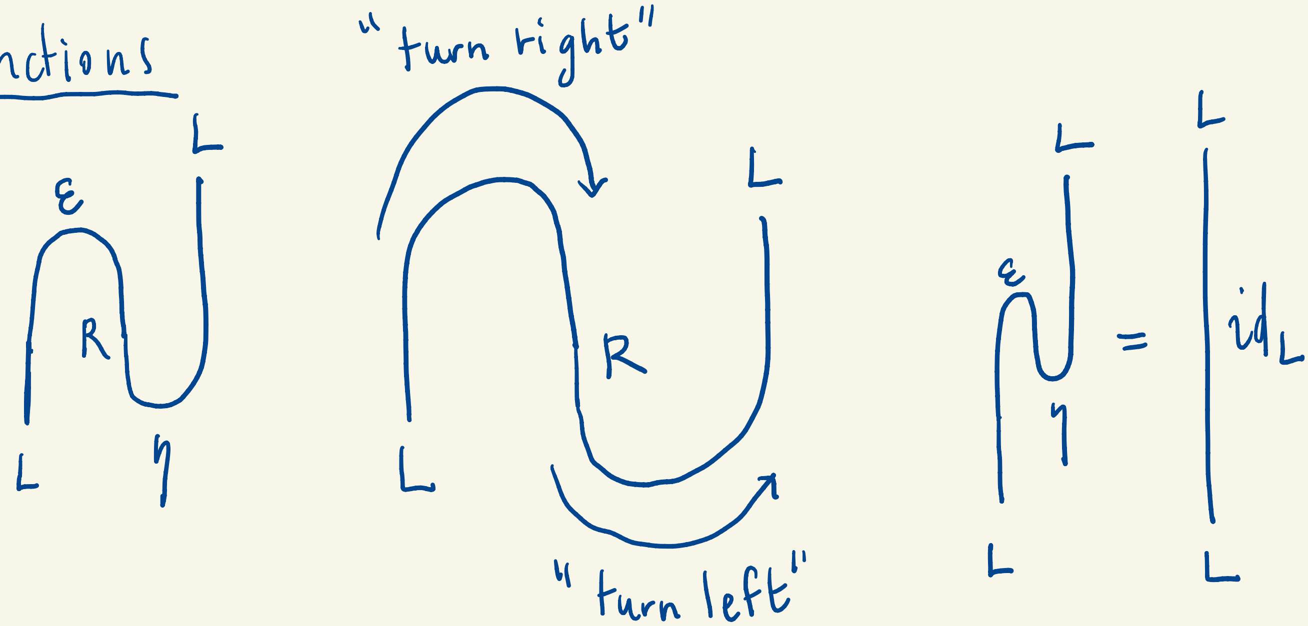
$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{B} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{B} & \mathcal{A} \arrow[l, bend right=50, "G"'] \arrow[l, bend left=50, "F"] \arrow[l, phantom, "\Uparrow\theta"] \end{tikzcd} \longmapsto$$

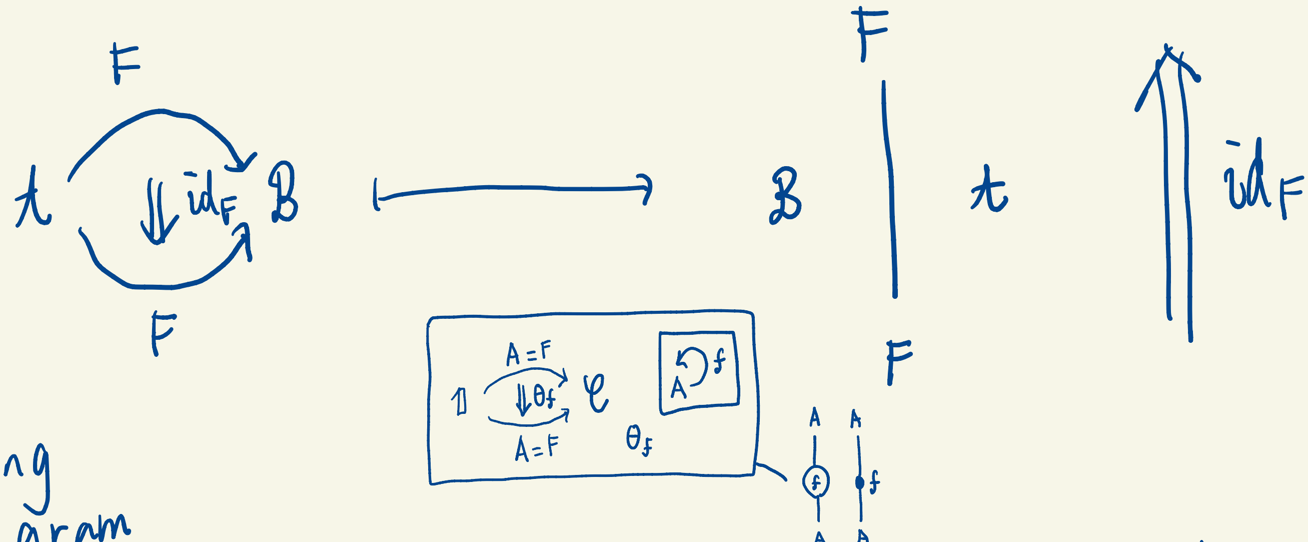


why the "rotation"

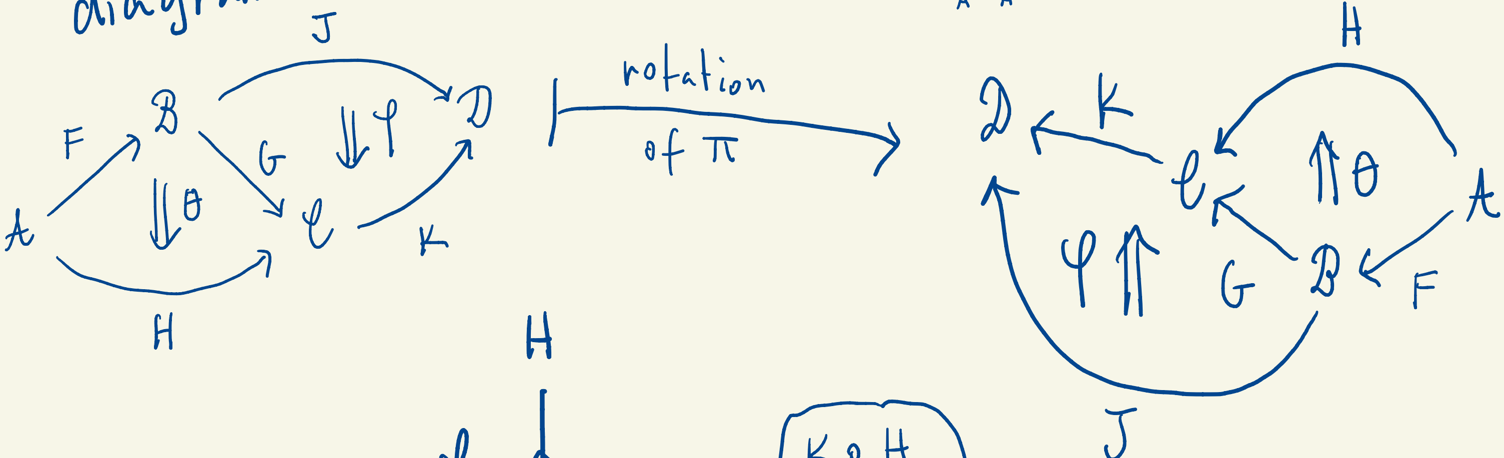


adjunctions

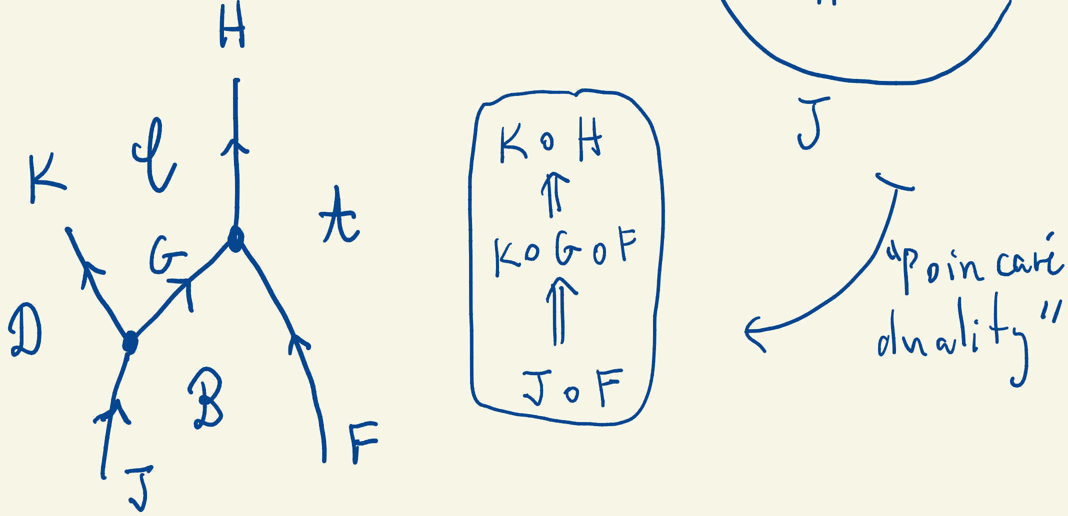




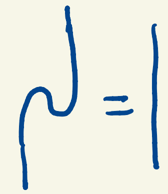
pasting diagram



string diagram



Adjunctions between categories



Exercise: given two sets A, B

given two functions $A \xrightarrow{L} B$

$A \xleftarrow{R} B$

the two following assertions are equivalent:

bijection

① the pair L and R defines an isomorphism pair in the sense that L and R are isomorphisms
bijections

and $\quad L = R^{-1} \quad R = L^{-1}$.

② for all $a \in A$, $b \in B$, we have that:

$$La =_B b \iff a =_A Rb$$

Prf: $\boxed{① \Rightarrow ②}$ is somewhat obvious.

$$La = b \Rightarrow RLa = Rb \overset{①}{\Rightarrow} a = Rb$$

because $a = RLa$.

similarly

$$a = Rb \Rightarrow La = LRb \Rightarrow La = b$$

because $LRb = b$.

② ⇒ ① (*) we want to show that for all $a \in A$

$$a = R \circ L(a)$$

and symmetrically that for all $b \in B$

$$L \circ R(b) = b$$

(*) we start from the equation:

$$La =_B La$$

then apply the equivalence ② and get

$$\boxed{a =_A RLa} \qquad \boxed{b = La}$$

② at instance.

symmetrically, starting from the equation:

$$Rb =_A Rb$$

we apply the equivalence ② and get

$$\boxed{LRb =_B b} \qquad \boxed{a = Rb}$$

② at instance

Definition: an adjunction $L \dashv R$ between two categories $\mathcal{A}$ and $\mathcal{B}$ is a pair of functors

$$\mathcal{A} \xrightarrow{L} \mathcal{B} \qquad \mathcal{B} \xrightarrow{R} \mathcal{A}$$

together with a family of bijections

$$\phi_{A,B} : \mathcal{B}(LA, B) \xrightarrow{\cong} \mathcal{A}(A, RB)$$

natural in A and B.

In that situation, we say that the functor L is <u>left adjoint</u> to R, the functor R is <u>right adjoint</u> to L.

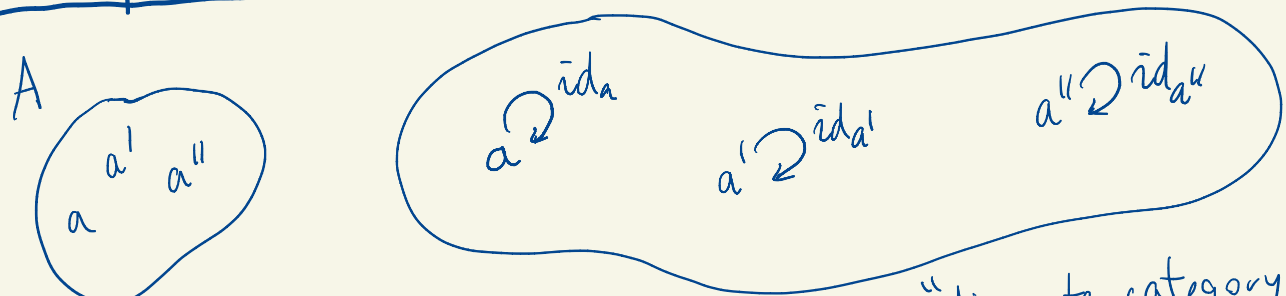
usual notation:

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "R"] \end{tikzcd} \qquad L \dashv R$$

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, bend right, "L"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, bend right, "L"'] \arrow[r, phantom, "\top"] & \mathcal{B} \arrow[l, bend right, "R"'] \end{tikzcd}$$

Remark: every set A can be seen as a category

$\boxed{\text{Example 0}}$ whose only morphisms are identity maps.



"discrete category"

what is an adjunction
$$\begin{tikzcd} A \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "R"] \end{tikzcd}$$
between two sets A, B ?

answer: it is just a pair of bijections L, R

such that $L = R^{-1}$ and $R = L^{-1}$.

First example

$$\begin{tikzcd} \mathcal{A} = \mathrm{Set} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{B} = \mathrm{Vect} \arrow[l, bend left, "R = U"] \end{tikzcd}$$

sets and functions

vector spaces and linear maps

$k = \mathbb{R}$

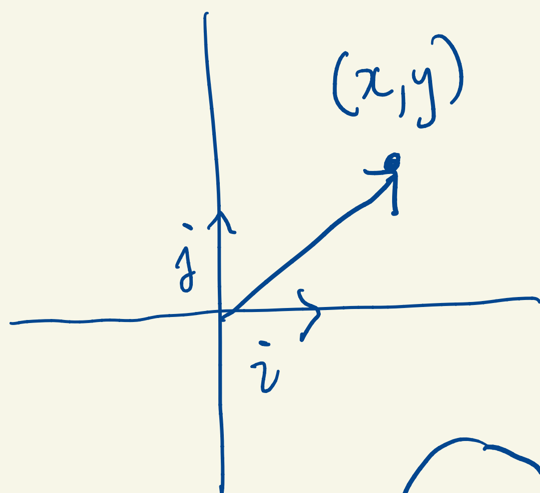
R "forgetful functor" U

$R : \mathrm{Vect} \longrightarrow \mathrm{Set}$

$$V \longmapsto \text{the underlying set}$$

V is a set
equipped with $+, -, 0,$
multiplication by scalars $\lambda \in \mathbb{R}$.

$\mathbb{R}^2$



$U(\mathbb{R}^2)$
the set of
pairs (x,y)
$x \in \mathbb{R}$ $y \in \mathbb{R}$

Vect **Set**

$$\begin{tikzcd}
V_1 \arrow[d, "f_1"'] & & UV_1 \arrow[d, "Uf_1"] \arrow[dd, bend left=60] \\
V_2 \arrow[d, "f_2"'] \arrow[r, phantom, "\longmapsto"] & & UV_2 \arrow[d, "Uf_2"] \\
V_2 & & UV_2
\end{tikzcd}$$

$$\boxed{U(f_2 \circ_{Vect} f_1) = Uf_2 \circ_{Set} Uf_1}$$

$L : \mathrm{Set} \longrightarrow \mathrm{Vect}$ "the free construction"

$X \longmapsto$ the vector space kX freely generated by X.

$\boxed{k = \mathbb{R}}$

$\boxed{\mathbb{R}X}$

$$\boxed{kX = \left\{ \text{functions } X \xrightarrow{f} \mathbb{R} \text{ with finite support} \right\}}$$

$\operatorname{supp} f =$ set of elements x of X such that $f(x) \neq 0$.

$$kX = \left\{ \text{formal sums } \sum_{x \in X} \lambda_x \boxed{e_x} \text{ where } \lambda_x \in \mathbb{R} \right\}$$

with finite support

an element of the basis of kX.

$$k\{\overset{1}{i}, \overset{2}{j}, \overset{3}{k}\} = \mathbb{R}^3$$

$$= \{ \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \}$$

we want a family of bijections

$$\phi_{A,B} : \mathcal{B}(LA, B) \xrightarrow{\cong} \mathcal{A}(A, RB)$$

$$(*) \quad \boxed{\mathrm{Vect}(\Bbbk X, V) \xrightarrow{\cong} \mathrm{Set}(X, UV)}$$

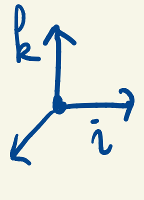
set of linear maps $\Bbbk X \longrightarrow V$ in Vect. $\rightleftharpoons$ set of functions $X \longrightarrow UV$ in Set.

$\boxed{X = \{i, j, k\}}$ $\quad V$ any vector space

a linear map $f : \mathbb{R}^3 \xrightarrow{\text{linear map}} V$

is entirely characterized

by the image of i, j, k vectors of the basis



$\xrightarrow{f} V$

$f(i)$
$f(j)$
$f(k)$

$$\lambda_1 i + \lambda_2 j + \lambda_3 k \longmapsto \lambda_1 f(i) + \lambda_2 f(j) + \lambda_3 f(k)$$

a general recipe to turn a function

$$X \xrightarrow{f} \boxed{V}$$

$U(V)$ the underlying set

into a linear map

$$kX \xrightarrow{f^+} V$$

where V a vector space:

$$f^+ : \sum_{x \in X} \lambda_x e_x \longmapsto \boxed{\sum_{x \in X} \lambda_x f(e_x)}$$

has finite support

this sum also has finite support

<u>Key observation</u>: this defines a bijection ⊛

between $\mathrm{Vect}(kX, V)$ and $\mathrm{Set}(X, UV)$

Second example

$$\begin{tikzcd} \mathbf{Set} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbf{Mon} \arrow[ll, bend left, "R = U"] \end{tikzcd}$$

sets and functions

monoids and homomorphisms

$\boxed{\text{a monoid } M}$

is a <u>set</u> equipped

with a binary operation $\bullet_M : M \times M \longrightarrow M$

$\bullet_M : M^2 \longrightarrow M$

binary operation

<u>associative</u>

and a neutral element $e_M : \mathbb{1} \longrightarrow M$

$\mathbb{1}$ = singleton set

$e_M : M^0 \longrightarrow M$

zero-ary operation

R = the forgetful functor which transports a monoid M to the underlying set UM

$L =$ the "free construction"

$$A \longmapsto A^*$$

set — A^* the monoid of <u>finite words</u> $[a_1 \dots a_n]$ on the alphabet A

A^* has <u>concatenation</u> as multiplication

$$[u] \cdot [v] = [uv]$$

and the empty word $[\,]$ as neutral element.

the fact that L is left adjoint to R can be expressed as the following property:

<u>Property</u>: given a set A and a monoid M there exists for every function $f : A \to M$ a <u>unique</u> homomorphism $f^+ : A^* \longrightarrow M$ such that the diagram below commutes:

$$[\,] \longmapsto e_M$$

$$[a_1 \dots a_n] \longmapsto f(a_1) \bullet_M \cdots \bullet_M f(a_n)$$

$$\begin{tikzcd}
{[a]} & A^* \arrow[r, dashed, "f^\dagger"] & M & \\
& & & f(a) \\
a \arrow[uu, mapsto] & A \arrow[uu, "\eta_A"] \arrow[uur, "f"'] & & \\
& a \arrow[uurr, mapsto] & &
\end{tikzcd}
\qquad
\boxed{f = f^\dagger \circ \eta_A}$$

where the function $\eta_A : A \longrightarrow A^*$ transport every element $a \in A$ to the word $[a]$ with a unique letter.

This induces a bijection:

$$\left[\ \phi_{A,M} : \mathrm{Mon}(A^*, M) \longrightarrow \mathrm{Set}(A, UM)\right.$$

$$\begin{cases} A^* \xrightarrow{h} M \longmapsto A \xrightarrow{\eta_A} A^* \xrightarrow{h} M \\ A^* \xrightarrow{f^\dagger} M \longleftarrow A \xrightarrow{f} M \end{cases}$$

which defines the adjunction $\underset{\text{Free}}{L} \dashv \underset{\text{Forgetful}}{R}$

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "L \text{ free}"] \arrow[r, phantom, "\perp"] & \mathrm{Magma} \arrow[l, bend left, "R \text{ forgetful}"] \end{tikzcd}$$

$$L : A \longmapsto \left\{ \begin{array}{c} \text{binary trees} \\ \text{with leaves} \\ \text{in } A \end{array} \right\}$$

What does <u>natural</u> mean in the definition of an adjunction?

a <u>natural</u> family of bijections

$$\phi_{A,B} : \mathcal{B}(LA, B) \xrightarrow{\cong} \mathcal{A}(A, RB)$$

<u>Key observation</u>: every category $\mathcal{C}$ induces a functor

$$\begin{aligned} \mathrm{Hom} : \mathcal{C}^{op} \times \mathcal{C} &\longrightarrow \mathrm{Set} \\ (A \;,\; B) &\longmapsto \mathcal{C}(A, B) \end{aligned}$$

why is this a functor?

what is a map in $\mathscr{C}^{op} \times \mathscr{C}$?

given categories $\mathscr{A}, \mathscr{B}$

- the objects of $\mathscr{A} \times \mathscr{B}$ are the pairs (A,B) consisting of an object A of $\mathscr{A}$, B of $\mathscr{B}$

- the maps of $\mathscr{A} \times \mathscr{B}$ are pairs

$$(A,B) \xrightarrow{(h_A, h_B)} (A', B')$$

consisting of a map

$$h_A : A \longrightarrow A' \text{ in } \mathscr{A}$$
$$h_B : B \longrightarrow B' \text{ in } \mathscr{B}.$$

in the case of $\boxed{\mathscr{C}^{op} \times \mathscr{C}}$, a map

$$\begin{array}{c} (A,B) \\ \Big\downarrow (h_A, h_B) \\ (A', B') \\ \text{in } \mathscr{C}^{op} \times \mathscr{C} \end{array}$$

is a pair of maps

$$\boxed{\begin{array}{c} A \\ \Big\downarrow h_A \\ A' \\ \text{in } \mathscr{C}^{op} \end{array}} \qquad \begin{array}{c} B \\ \Big\downarrow h_B \\ B' \\ \text{in } \mathscr{C} \end{array}$$

that is a pair of maps

$$\begin{tikzcd} A \\ A' \arrow[u, "h_A"'] \end{tikzcd} \text{ in } \mathcal{C} \qquad \begin{tikzcd} B \arrow[d, "h_B"] \\ B' \end{tikzcd} \text{ in } \mathcal{C}$$

note the orientation!

given such a map we want to construct a function:

a map in Set

$$\circledast \quad \boxed{\mathrm{Hom}(h_A, h_B) : \boxed{\mathrm{Hom}(A,B)} \longrightarrow \mathrm{Hom}(A',B')}$$

$$(h_A, h_B) : (A,B) \longrightarrow (A',B')$$

in the source category $\mathcal{C}^{op} \times \mathcal{C}$

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\mathrm{Hom}} \mathrm{Set}$$

$$(h_A, h_B) \longmapsto \mathrm{Hom}(h_A, h_B)$$

$$
\begin{tikzcd}
A & B \arrow[d, "h_B"] \\
A' \arrow[u, "h_A"] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[r, mapsto] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f \in \mathrm{Hom}(A,B)"] & B \arrow[d, "h_B"] \\
A' \arrow[u, "h_A"] \arrow[r, "h_B \circ f \circ h_A"'] & B'
\end{tikzcd}
$$

in $\mathcal{C}$ in $\mathcal{C}$

$h_B \circ f \circ h_A \in \mathrm{Hom}(A', B')$

very important!

$$
\begin{tikzcd}
(A, B) \arrow[d, "{(h_A, h_B)}"'] & & \mathrm{Hom}(A,B) \arrow[d, "{\mathrm{Hom}(h_A, h_B)}"] \\
(A', B') & \text{in } \mathcal{C}^{op} \times \mathcal{C} \arrow[r, mapsto] & \mathrm{Hom}(A', B')
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{C} \arrow[r] & \mathrm{Set} \\
(A, B) \arrow[r, mapsto] & \mathrm{Hom}(A,B)
\end{tikzcd}
$$

Similarly, every functor

$$\boxed{L : \mathcal{A} \longrightarrow \mathcal{B}}$$

induces a functor

$$(*) \qquad \begin{aligned} \mathcal{A}^{op} \times \mathcal{B} &\xrightarrow{\mathcal{B}(L-,-)} \mathbf{Set} \\ (A, B) &\longmapsto \mathcal{B}(LA, B) \end{aligned}$$

$$\boxed{\begin{tikzcd} A & B \arrow[d, "h_B"'] \\ A' \arrow[u, "h_A"] & B' \\ \text{in } \mathcal{A} & \text{in } \mathcal{B} \end{tikzcd}} \longmapsto \begin{tikzcd} LA \arrow[r, "f"] & B \arrow[d, "h_B"] \\ LA' \arrow[u, "Lh_A"] \arrow[r, dashed, "h_B \circ f \circ Lh_A"'] & B' \end{tikzcd}$$

$$\longmapsto \begin{tikzcd} \mathcal{B}(LA, B) \arrow[d, "{\mathcal{B}(Lh_A, h_B)}"] \\ \mathcal{B}(LA', B') \end{tikzcd}$$

Symmetrically,

every functor $R : \mathcal{B} \longrightarrow \mathcal{A}$

induces a functor

$$\begin{aligned} \mathcal{A}^{op} \times \mathcal{B} &\xrightarrow{\mathcal{A}(-,R-)} \mathbf{Set} \\ (A, B) &\longmapsto \mathcal{A}(A, RB) \end{aligned}$$

So, given a pair of functors:

$L : \mathcal{A} \longrightarrow \mathcal{B}$ $\qquad$ $R : \mathcal{B} \longrightarrow \mathcal{A}$

we get two functors:

$$\begin{tikzcd} \mathcal{A}^{op} \times \mathcal{B} \arrow[r, bend left, "\mathcal{B}(L-,-)"] \arrow[r, bend right, "\mathcal{A}(-,R-)"'] & \mathbf{Set} \end{tikzcd}$$

a family $\phi_{A,B}$ of bijections

$$\phi_{A,B} : \mathcal{B}(LA, B) \xrightarrow{\cong} \mathcal{A}(A, RB)$$

is called $\boxed{\text{natural}}$ when it defines

a <u>natural transformation</u>

$$\begin{tikzcd} \mathcal{A}^{op} \times \mathcal{B} \arrow[r, bend left, "\mathcal{B}(L-,-)"] \arrow[r, bend right, "\mathcal{A}(-,R-)"'] \arrow[r, phantom, "\Downarrow \phi"] & Set \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B}(LA, B) \arrow[r, "\mathcal{B}(Lh_A, h_B)"] \arrow[d, "\phi_{AB}"'] \arrow[dr, phantom, "G"] & \mathcal{B}(LA', B') \arrow[d, "\phi_{A'B'}"] \\ \mathcal{A}(A, RB) \arrow[r, "\mathcal{A}(h_A, Rh_B)"] & \mathcal{A}(A', RB') \end{tikzcd}$$

<u>commutes</u> for all maps $\begin{tikzcd} A \\ A' \arrow[u, "h_A"] \end{tikzcd}$ in $\mathcal{A}$ $\begin{tikzcd} B \arrow[d, "h_B"] \\ B' \end{tikzcd}$ in $\mathcal{B}$

Exercise Show that the naturality condition holds in our two examples of adjunctions.

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathrm{Vect} \arrow[l, bend left, "R"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathrm{Mon} \arrow[l, bend left, "R"] \end{tikzcd}$$

Next time

explain the connection to the λ-calculus

and to string diagrams.