Monday 7 December 2020

the free cartesian closed category

The free cartesian closed category

Thm the interpretation of the λ-term M provides an invariant of the λ-term modulo β-reduction η-expansion

$$\boxed{M \simeq_{\beta\eta} N \text{ then } [M] = [N]}$$

proof by "surgery" on derivation trees.

in a given context with a given type

Example: the category of G-set is cartesian closed hence every simply-typed λ-term is interpreted in the category of G-sets as a function which preserves the group action (a homomorphism between G-sets)

<u>Reminder</u>: a G-set (A, act) is a set A equipped with a group action

$$act : G \times A \longrightarrow A$$

$$\begin{cases} act(g, act(g', a)) = act(g \cdot g', a) \\ act(e, a) = a \end{cases}$$

- the cartesian product of G-sets?

$$(A, act_A) \times (B, act_B) = (A \times B,\ act_{A\times B})$$

$$act_{A\times B}(g, (a,b)) := (act_A(g,a), act_B(g,b))$$

- the exponentiation $(A, act_A) \Rightarrow (B, act_B)$ of two G-sets?

$(A, act_A) \Rightarrow (B, act_B)$ is defined as the set of all function $A \Rightarrow B$ equipped with the action

$$G \times A \Rightarrow B \xrightarrow{act_{A\Rightarrow B}} A \Rightarrow B$$

$$\mathrm{act}_{A \Rightarrow B}(g, f) =$$

$$a \longmapsto \mathrm{act}_B(g, f(\mathrm{act}_A(g^{-1}, a)))$$

$$g \cdot f(g^{-1} \cdot a)$$

<u>Remark</u> that the terminal objet
in the category G-set of type A
is the singleton set 1
equipped with the obvious group action,
hence every closed simply typed
λ-term M is interpreted as
a homomorphism $1 \xrightarrow{[M]} [A]$

hence as an element of $[A]$

invariant under the group action!

$1 = \{*\}$

$$* \xrightarrow{[M]} [A]$$

$g \in G$

if I write $m = [M](*)$

$$g \cdot m = g \cdot [M](*) = [M](g \cdot *)$$

$act(g, m)$

$$= [M](*)$$

$$= m$$

$$\overset{g\,\circlearrowleft}{*} \xmapsto{\quad [M] \quad} \overset{g\,\circlearrowleft}{m}$$

So every $\boxed{\text{closed}}$ simply-typed λ-term is interpreted as an element invariant under the group action!

the intuition is that the λ-term has no information on the input — it just performs "pure" manipulations.

$$\lambda x.x \qquad \lambda f.\lambda x.\, f f (x)$$

What does it mean from a 2-categorical point of view?

behind this interpretation of simply-typed λ-terms in any cartesian closed category there is the construction of a free cartesian closed category generated by a category $\mathcal{C}$.

$$\text{free} - \text{ccc}(\mathcal{C})$$

similar to the free cartesian category —

Reminder: a cartesian functor between cartesian categories

$$F : \mathcal{A} \longrightarrow \mathcal{B}$$

is a functor such that every canonical morphism (in $\mathcal{B}$)

$$m_{A_1,A_2} : F(A_1 \times A_2) \longrightarrow FA_1 \times FA_2$$

$$m_1 : F(1) \longrightarrow 1$$

is $\boxed{\text{an isomorphism}}$ (for all A_1, A_2)

$$\begin{tikzcd}
 & A_1 \times A_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & & & F(A_1 \times A_2) \arrow[ddl, bend right, "F\pi_1"'] \arrow[ddr, bend left, "F\pi_2"] \arrow[d, "m_{A_1,A_2}"] & \\
A_1 & & A_2 & \longmapsto & & FA_1 \times FA_2 \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & & & FA_1 & & FA_2
\end{tikzcd}$$

Suppose that F is cartesian between cartesian closed categories.

Given two objects A_1, A_2 in $\mathcal{A}$ there is a morphism in $\mathcal{B}$

$$p_{A_1,A_2} : F(A_1 \Rightarrow A_2) \longrightarrow FA_1 \Rightarrow FA_2$$

defined as the morphism associated to the morphism

$$\begin{tikzcd}
FA_1 \times F(A_1 \Rightarrow A_2) \arrow[r, "m_{A_1, A_1 \Rightarrow A_2}"] \arrow[dr, "q_{A_1,A_2}"'] & F(A_1 \times (A_1 \Rightarrow A_2)) \arrow[d, "F(\mathrm{eval}_{A_1,A_2})"] \\
& F(A_2)
\end{tikzcd}$$

by the bijection

$$\phi_{F(A_1 \Rightarrow A_2), F(A_2)} : \mathcal{B}(FA_1 \times F(A_1 \Rightarrow A_2), FA_2) \xrightarrow{\cong} \mathcal{B}(F(A_1 \Rightarrow A_2), FA_1 \Rightarrow FA_2)$$

of the cartesian closed category $\mathcal{B}$.

<u>Def</u>. a cartesian-closed functor

$$F : \mathcal{A} \longrightarrow \mathcal{B}$$

between cartesian-closed categories is a cartesian functor such that every canonical morphism P_{A_1, A_2} is an iso.

<u>Remark</u>: clearly we have isomorphisms

$$F(A_1 \times A_2) \xrightarrow{\cong} FA_1 \times FA_2$$

$$F(A_1 \Rightarrow A_2) \xrightarrow{\cong} FA_1 \Rightarrow FA_2$$

but it is more than that:

these isomorphisms are obtained by required that canonical morphisms are invertible.

<u>Thm</u>. for every category $\mathcal{C}$ there exists a free cartesian closed category

$$\text{free-ccc}(\mathcal{C})$$

together with a functor

$$\mathrm{incl} : \mathcal{C} \longrightarrow \text{free-ccc}(\mathcal{C})$$

(incl for inclusion).

By this we mean that

for every functor

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

where $\mathcal{D}$ is a cartesian-closed category

there exists a

cartesian-closed functor

$F^\dagger : \mathrm{freeccc}(\mathcal{C}) \longrightarrow \mathcal{D}$

making the diagram (*) commute

cartesian-closed functor

$$\begin{tikzcd} \text{free-ccc}(\mathcal{C}) \arrow[r, "F^\dagger"] & \mathcal{D} \\ \mathcal{C} \arrow[u, "\mathrm{incl}"] \arrow[ur, "F"'] & \end{tikzcd}$$

cartesian-closed category

$\times \Rightarrow$

similar to the free monoid

$$\begin{tikzcd} A^* \arrow[r, "f^\dagger", "\text{homomorphism}"'] & M \\ A \arrow[u, "\mathrm{incl}"] \arrow[ur, "f"'] & \end{tikzcd}$$

monoid

Moreover the functor F^+

is unique up to isomorphism

(in fact up to unique isomorphism)

note in particular that

whenever

$$\mathcal{A} \xrightarrow{F} \mathcal{B}$$

is cartesian closed

and $F \cong G$ natural isomorphism

then G is cartesian closed.

$$\begin{tikzcd}[row sep=huge]
X \Rightarrow \quad \mathrm{free\,ccc}(\mathcal{C}) \arrow[rr, bend left, "H"] \arrow[rr, "F^+"'] \arrow[rr, phantom, bend left=15, "\cong"] & & \mathcal{D} \\
\mathcal{C} \arrow[u, "\mathrm{incl}"] \arrow[urr, "F"'] & &
\end{tikzcd}$$

X, Y

$$\begin{aligned} H(X \times Y) &\cong HX \times HY \\ &= FX \times FY \end{aligned}$$

The free cartesian-closed category

$$\text{free-ccc}(\mathscr{C})$$

is constructed in this way:

$\boxed{\text{dim } 0}$ the objects of free-ccc$(\mathscr{C})$

are the contexts

$$x_1 : A_1, \ldots, x_p : A_p$$

where $A_1, \ldots, A_p$ are simple types

defined by the grammar:

$$A, B ::= 1 \mid A \times B \mid A \Rightarrow B \mid X$$

where X is an object of $\mathscr{C}$.

in other words

A, B are simple types

whose atoms are

the objects of $\mathscr{C}$.

$\boxed{\text{dim } 1}$ the morphisms

$$\Gamma \xrightarrow{f} \Delta$$

$$\Gamma = x_1 : A_1, \ldots, x_p : A_p \qquad \Delta = y_1 : B_1, \ldots, y_q : B_q$$

are the sequences of

simply-typed λ-terms

$$(M_1, \ldots, M_q)$$

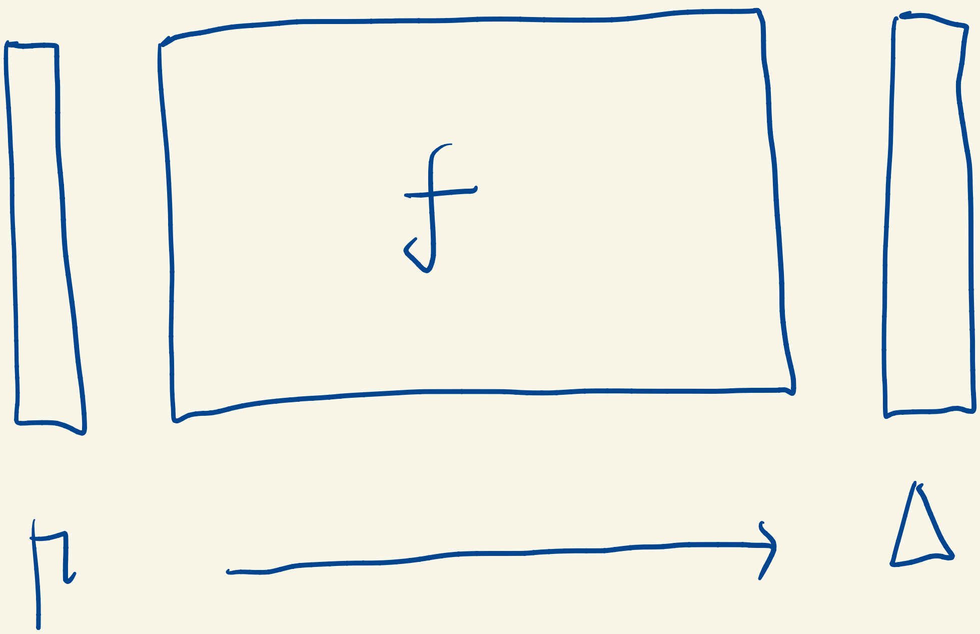
modulo

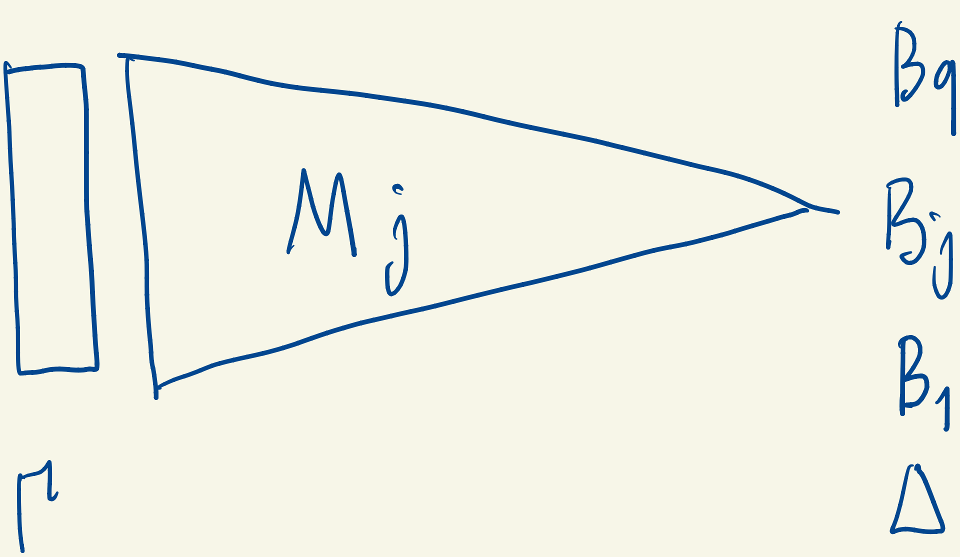
$\beta\eta$ conversion

of type

$$x_1 : A_1, \ldots, x_p : A_p \vdash M_j : B_j$$

for $1 \le j \le q$.





I should mention here that the original type theory is extended with the rules

$$\boxed{\dfrac{f : X \longrightarrow Y \text{ in } \mathcal{C}}{\vdash [f] : X \Rightarrow Y}} \qquad \boxed{f : X \to Y}$$

for every morphism

$$f : X \longrightarrow Y \text{ in } \mathcal{C}.$$

together with a number of equations.

with the equations $h = g \circ f$

in the category $\mathcal{C}$
reflected as equations
between the corresponding
λ-terms

$$\lambda x. [g]([f]x) = [h]$$

when $h = g \circ f$

in the category $\mathcal{C}$.

How composition is defined in the category free-ccc$(\mathcal{C})$?

$$\Gamma \xrightarrow{(M_1 \cdots M_q)} \Delta \xrightarrow{(N_1, \ldots, N_r)} \Xi$$

$$x_1 : A_1, \ldots, x_p : A_p \qquad y_1 : B_1, \ldots, y_q : B_q \qquad z_1 : C_1, \ldots, z_r : C_r$$

$$\Gamma \xrightarrow{(N_1, \ldots, N_r) \circ (M_1, \ldots, M_q)} \Xi$$

is defined as the sequence of $\underline{\underline{r}}$ λ-terms:

$$\left(N_1[y_1 := M_1, \ldots, y_q := M_q], \ldots, N_r[y_1 := M_1, \ldots, y_q := N_q]\right)$$

"composition by substitution".

<u>Example</u>: a morphism

from Γ to Δ where

$$\Delta = (y : B)$$

is a context with a <u>unique</u> type B

is the same thing as

a λ-term M of type

$$\Gamma \vdash M : B.$$ modulo $\beta\eta$ conversion.

$\boxed{\text{Cartesian Product}}$

The <u>cartesian product</u> of two contexts Γ and Δ is simply the context Γ, Δ

obtained by concatenation (taking care of the fact that the variables should be all different).

Note in particular that a map

$$\Xi \xrightarrow{h} \Gamma, \Delta$$

is the same thing as a pair of maps

$$\Xi \xrightarrow{f} \Gamma$$

$$\Xi \xrightarrow{g} \Delta$$

In fact, the projection maps

$$\begin{tikzcd} & \Gamma, \Delta \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Gamma & & \Delta \end{tikzcd}$$

are defined by the sequences

$$\Gamma = x_1 : A_1, \ldots, x_p : A_q$$

$$\Delta = y_1 : B_1, \ldots, y_q : B_q$$

$$\pi_1 = (x_1, \ldots, x_p)$$ a sequence of p λ-terms

$$\pi_2 = (y_1, \ldots, y_q)$$ a sequence of q λ-terms

$$
\begin{tikzcd}
& \Xi \arrow[dl, bend right, "{\boxed{\begin{matrix}(M_1,\dots,M_p)\\ \shortparallel \\ f\end{matrix}}}"'] \arrow[dr, bend left, "{\boxed{\begin{matrix}(N_1,\dots,N_q)\\ g =\end{matrix}}}"] \arrow[d, dashed, "h"] & \\
\Gamma & \Gamma,\Delta \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \Delta
\end{tikzcd}
$$

$$\boxed{h = (M_1 \dots M_p, N_1 \dots N_q)}$$

there exists a unique map h making the diagram commute.

h is defined by "concatenating" the sequences f and g of λ-terms.

Exponentiation

<u>Remark</u> : in any cartesian-closed category we have an isomorphism

$$A \Rightarrow (B \times C) \xrightarrow{\cong} (A \Rightarrow B) \times (A \Rightarrow C)$$

<u>Exercise</u> : construct the morphism and establish the claim that it is an isomorphism.

<u>Definition</u> of exponentiation (in two steps)

① $\Gamma = (x_1 : A_1, \ldots, x_p : A_p)$

$\Delta = (y : B)$ a context with a unique type

$$\Gamma \Rightarrow \Delta : (z : A_1 \Rightarrow \dots \Rightarrow A_p \Rightarrow B)$$

a context with a unique type.

2 more generally, when

$$\Delta = (y_1 : B_1, \dots, y_q : B_q)$$

then

$$\Gamma \Rightarrow \Delta = \left(\begin{array}{l} z_1 : A_1 \Rightarrow \dots \Rightarrow A_p \Rightarrow B_1, \\ z_2 : A_1 \Rightarrow \dots \Rightarrow A_p \Rightarrow B_2, \\ \dots \\ z_q : A_1 \Rightarrow \dots \Rightarrow A_p \Rightarrow B_q \end{array} \right)$$

Claim: this defines the free cartesian-closed category.

generated by the category $\mathcal{C}$.

$$\begin{tikzcd}
\text{free-ccc}(\mathcal{C}) \arrow[rr, "F^{+}"] & & \mathcal{D} \\
\mathcal{C} \arrow[u, "\text{incl}"] \arrow[urr, "F"'] & &
\end{tikzcd}$$

[Jim Lambek]

F^{+} is defined by interpreting

every type A as an object $[A]$
of the category $\mathcal{D}$

every λ-term (up to $\beta\eta$ conversion)
as a morphism of $\mathcal{D}$.