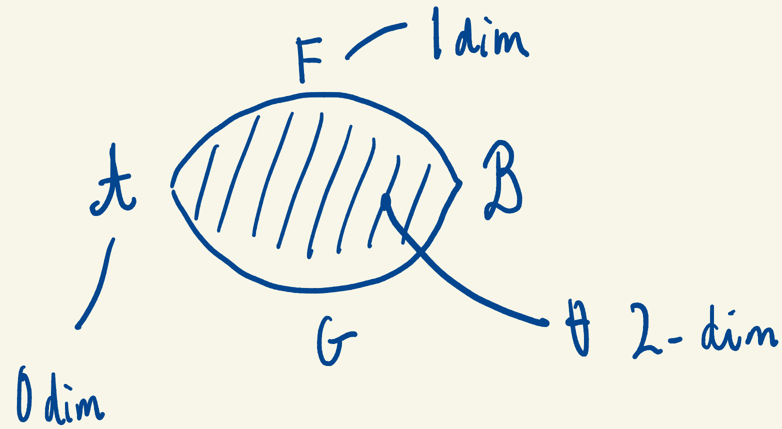
October the 9th

The 2-category of functors & natural transformations

λ-calculus & categories

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{B}
\arrow[from=U, to=D, Rightarrow, "\theta"]
\end{tikzcd}
$$

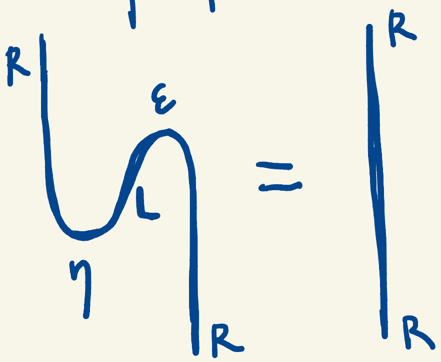
θ natural transformation between the functors F and G



this will define $\boxed{\text{a } \underline{\text{2-category}}}$ $\underline{\mathrm{Cat}}$

whose objects are categories
whose morphisms are functors
whose 2-cells are natural transformations.

<u>Next purpose</u>: introduce <u>string diagrams</u>



(in French diagrammes de <u>cordes</u>)

a graphical calculus of functors & nat. transformations

Reminder

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathscr{D} \end{tikzcd}$$

a transformation $\theta : F \Rightarrow G$

is a family

$$\left(\theta_A : FA \longrightarrow GA\right)_{A \in \mathrm{ob}\,\mathscr{C}}$$

of morphisms in $\mathscr{D}$.

a transformation $\theta : F \Rightarrow G : \mathscr{C} \longrightarrow \mathscr{D}$

is called natural when the diagram

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\theta_A"'] & FB \arrow[d, "\theta_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

$$\boxed{\begin{array}{c} \theta_B \circ Ff \\ = \\ Gf \circ \theta_A \end{array}}$$

commutes for every map $f : A \to B$ in $\mathscr{C}$.

1 vertical composition

every pair θ_1, θ_2

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\theta_1"] \arrow[r, phantom, shift right=3ex, "\Downarrow\theta_2"] & \mathcal{D}
\end{tikzcd}$$

of transformations.

$\theta_1 : F \Rightarrow G$

$\theta_2 : G \Rightarrow H.$

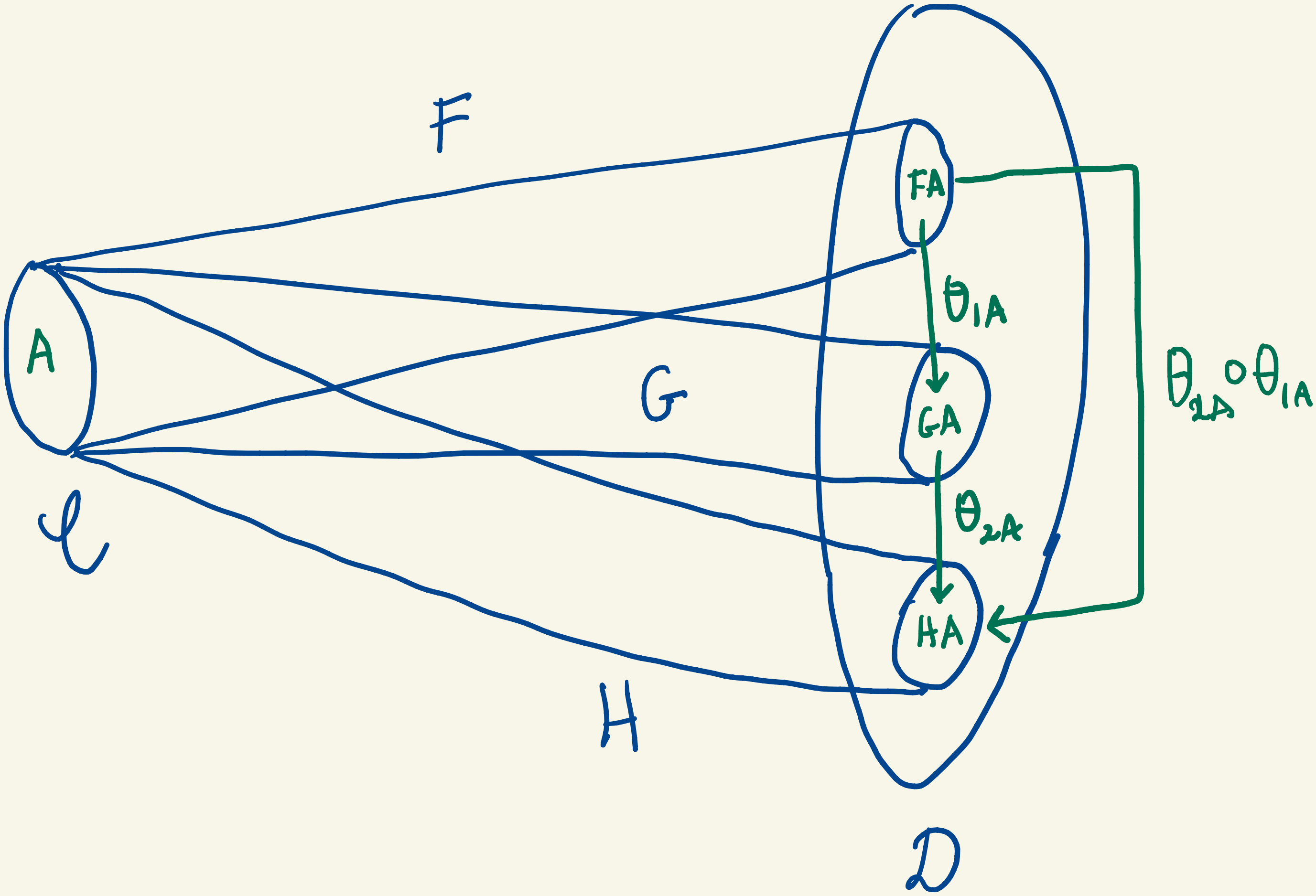
induces a transformation noted

$$\theta_2 * \theta_1 : F \Longrightarrow H : \mathcal{C} \longrightarrow \mathcal{D}$$

defined as the family of morphism

$$(\theta_2 * \theta_1)_A := \theta_{2A} \circ \theta_{1A} : FA \longrightarrow HA$$

where A is an object of $\mathcal{C}$.



$$
\begin{tikzcd}
FA \arrow[r, "\theta_{1A}"] \arrow[rr, bend right, "(\theta_2 * \theta_1)_A"'] & GA \arrow[r, "\theta_{2A}"] & HA
\end{tikzcd}
\quad \text{in } \mathcal{D}
$$

<u>Remark</u>: the transformation $(\theta_2 * \theta_1)_{A \in \mathrm{Ob}\,\mathcal{C}}$ is natural

when the transformations θ_1 and θ_2 are natural.

$A \xrightarrow{f} B$ in $\mathcal{C}$

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\theta_{1A}"'] & FB \arrow[d, "\theta_{1B}"] \\
GA \arrow[d, "\theta_{2A}"'] & GB \arrow[d, "\theta_{2B}"] \\
HA \arrow[r, "Hf"'] & HB
\end{tikzcd}$$

does this diagram commute

for $f : A \to B$ in $\mathcal{C}$?

diagram chase.

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\theta_{1A}"'] & FB \arrow[d, "\theta_{1B}"] \\
GA \arrow[r, "Gf"] \arrow[d, "\theta_{2A}"'] & GB \arrow[d, "\theta_{2B}"] \\
HA \arrow[r, "Hf"'] & HB
\end{tikzcd}$$

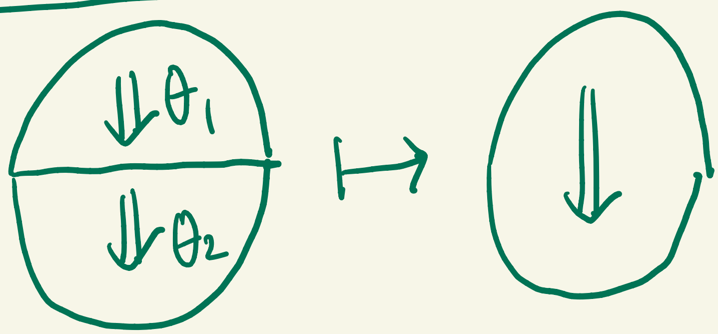
θ_1 is natural:

$$\theta_{1B} \circ Ff = Gf \circ \theta_{1A} \quad (1)$$

θ_2 is natural:

$$\theta_{2B} \circ Gf = Hf \circ \theta_{2A} \quad (2)$$

$$\theta_{2B} \circ \theta_{1B} \circ Ff \overset{①}{=} \theta_{2B} \circ Gf \circ \theta_{1A} \overset{②}{=} Hf \circ \theta_{2A} \circ \theta_{1A}$$

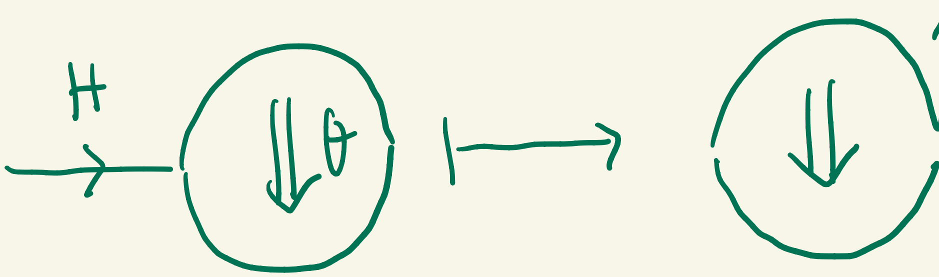


vertical composition

$\theta_2 * \theta_1$

left action of H on θ

$H \circ_L \theta$



right action of H on θ

$\theta \circ_R H$

② $\boxed{\text{left action}}$ "image"

given a transformation θ and a functor H

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \arrow[r, "H"] & \mathcal{E} \end{tikzcd}$$

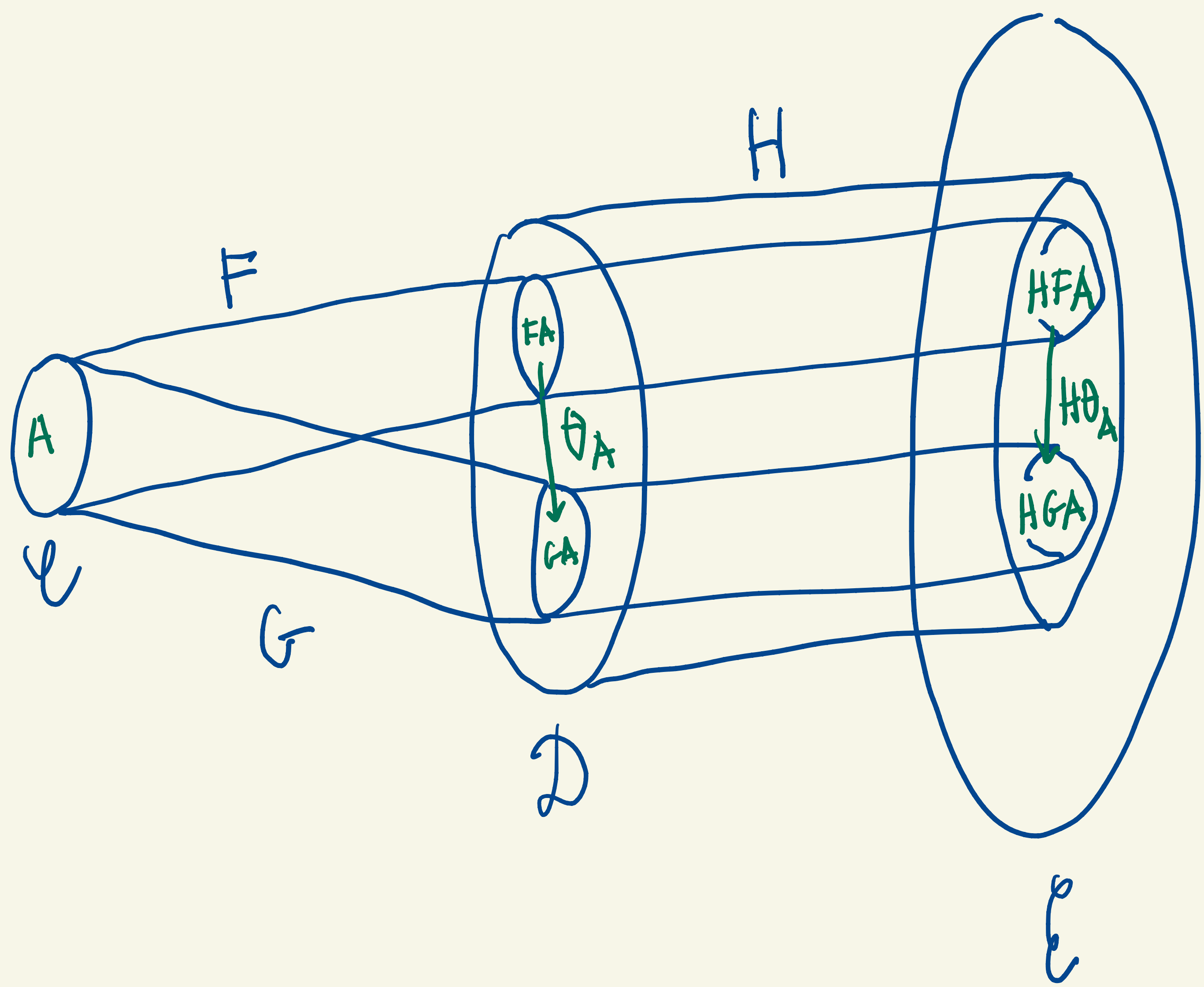
we define the transformation

$$H \circ_L \theta : H \circ F \Rightarrow H \circ G : \mathcal{C} \to \mathcal{E}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H \circ F"] \arrow[r, bend right, "H \circ G"'] \arrow[r, phantom, "\Downarrow H \circ \theta"] & \mathcal{E} \end{tikzcd}$$

for every object A of $\mathcal{C}$

$$H \circ F(A) \xrightarrow{H(\theta_A)} H \circ F(B)$$



$H \circ_L \theta$ is defined as the "image" of θ along the functor H

(2) the right action "substitution"

given a transformation $\theta : F \Rightarrow G$

and a functor H

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] & \mathcal{D} \arrow[r, bend left=60, "F", ""{name=U, below}] \arrow[r, bend right=60, "G"', ""{name=D, above}] & \mathcal{E} \arrow[Rightarrow, from=U, to=D, "\theta"] \end{tikzcd}$$

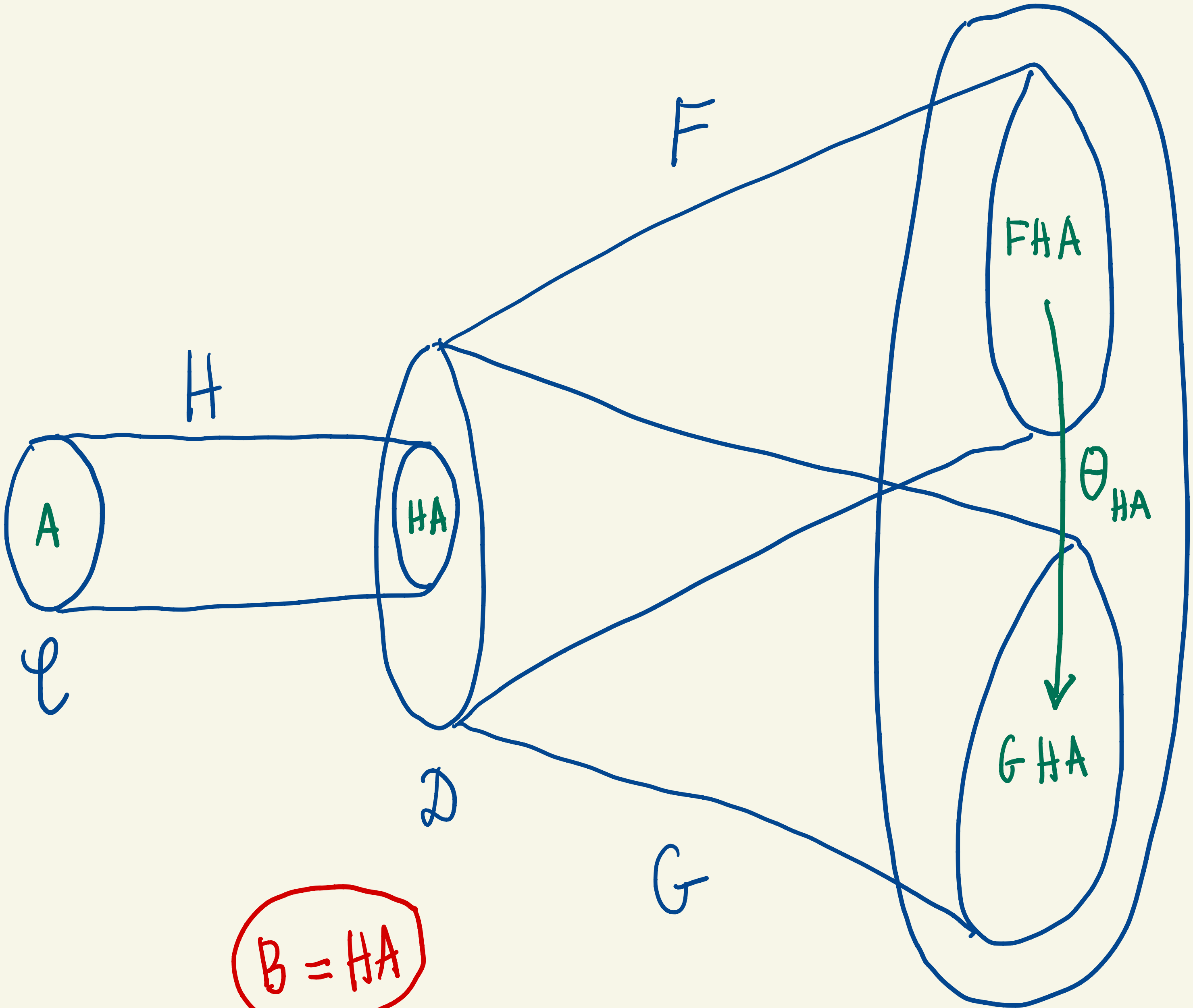
we define a transformation

$$\theta \circ_R H : F \circ H \Longrightarrow G \circ H : \mathcal{C} \longrightarrow \mathcal{E}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F \circ H", ""{name=U, below}] \arrow[r, bend right=60, "G \circ H"', ""{name=D, above}] & \mathcal{E} \arrow[Rightarrow, from=U, to=D, "\theta \circ_R H"] \end{tikzcd}$$

for every object A of the category $\mathcal{C}$

$$\Theta_{HA} : F \circ H(A) \longrightarrow G \circ H(A)$$



$B = HA$

$$\boxed{\Theta_B : FB \longrightarrow GB} \qquad B \in \mathrm{Ob}\,\mathcal{D}$$

substitute HA for the parameter B.

Remark and exercise:

Show that $H \circ_L \theta$ and $\theta \circ_R H$ are <u>natural</u> transformations when the transformation θ is <u>natural</u>.

<u>Example of a natural transformation.</u>

1. consider the category $\mathbb{1}$ with one object $*$ and one morphism (= the identity morphism)

<u>Rem</u>: $\mathbb{1}$ is the terminal object in the category <u>Cat</u>

for any category $\mathcal{C}$ a unique functor

$$\mathcal{C} \xrightarrow[\text{unique functor}]{} \mathbb{1}$$

$$\begin{tikzcd} * \arrow[loop right, "id_*"] \end{tikzcd}$$

what is a functor ?

$$\mathbb{1} \xrightarrow{\ A\ } \mathcal{C}$$

it is the same thing as an object A
of the category $\mathcal{C}$.

what is a natural transformation ?

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left=60, "A"{name=U}] \arrow[r, bend right=60, "B"'{name=D}] & \mathcal{C} \arrow[from=U, to=D, phantom, "\Downarrow f"] \end{tikzcd}$$

it is the same thing as a morphism
$f : A \to B$ in the category $\mathcal{C}$.

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left=60, "A"{name=U}] \arrow[r, bend right=60, "B"'{name=D}] & Set \arrow[from=U, to=D, phantom, "\Downarrow f"] \end{tikzcd}$$

$$\boxed{A \xrightarrow[\ f\ ]{\text{functions}} B}$$

② in a cartesian category $\mathcal{C}$

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C} & \xrightarrow{\text{product}} & \mathcal{C} \\ (A,B) & \longmapsto & A\times B \end{array}$$

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C} & \xrightarrow{\text{first component}} & \mathcal{C} \\ (A,B) & \longmapsto & A \end{array}$$

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C} & \xrightarrow{\text{second component}} & \mathcal{C} \\ (A,B) & \longmapsto & B \end{array}$$

$$\boxed{\begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[r, bend left=50, "\text{product}"{name=U}] \arrow[r, bend right=50, "\text{first component}"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\pi_1"] & \mathcal{C} \\[-1em] (A,B) & \end{tikzcd}}$$

$$\begin{tikzcd} \text{product}(A,B) \arrow[d, phantom, "\parallel"] \\ \boxed{A\times B} \arrow[d, "\pi_1"] \\ \boxed{A} \end{tikzcd}$$

first component of (A,B)

$$\begin{tikzcd}
A \times B \arrow[r, "h_A \times h_B"] \arrow[d, "(\pi_1)_{(A,B)}"'] & A' \times B' \arrow[d, "(\pi_1)_{(A',B')}"] \\
A \arrow[r, "h_A"] & A'
\end{tikzcd}$$

commutes

for all maps

$$(A, B) \xrightarrow{(h_A, h_B)} (A', B') \quad \text{in } \mathcal{C} \times \mathcal{C}$$

hence for all pairs of maps

$$A \xrightarrow{h_A} A'$$
$$B \xrightarrow{h_B} B' \quad \text{in } \mathcal{C}.$$

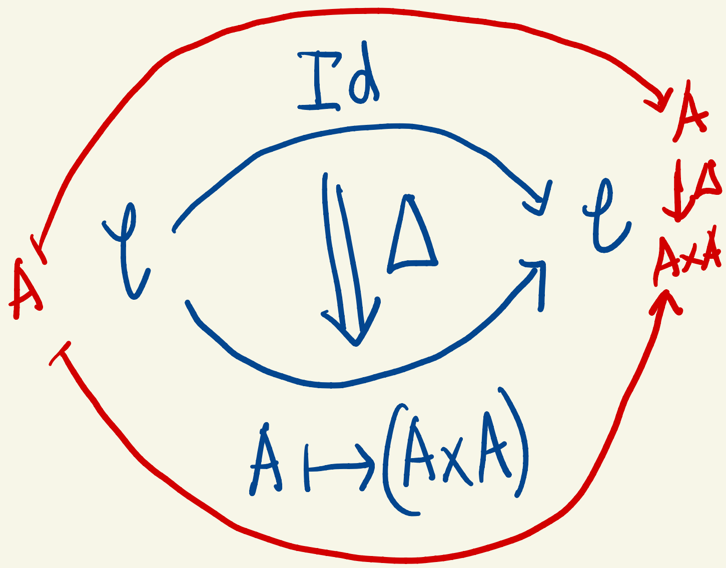
<u>Exercise</u>: Show that in any cartesian category $\mathcal{C}$, the diagonal

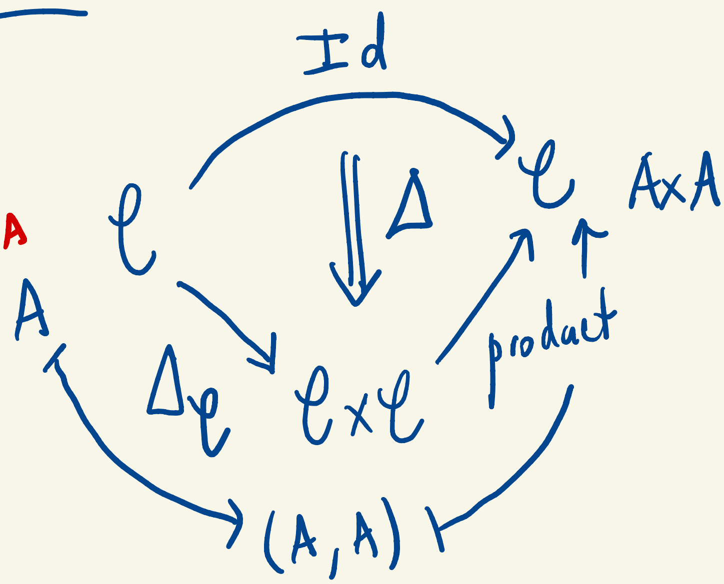
$$\Delta_A : A \longrightarrow A \times A$$

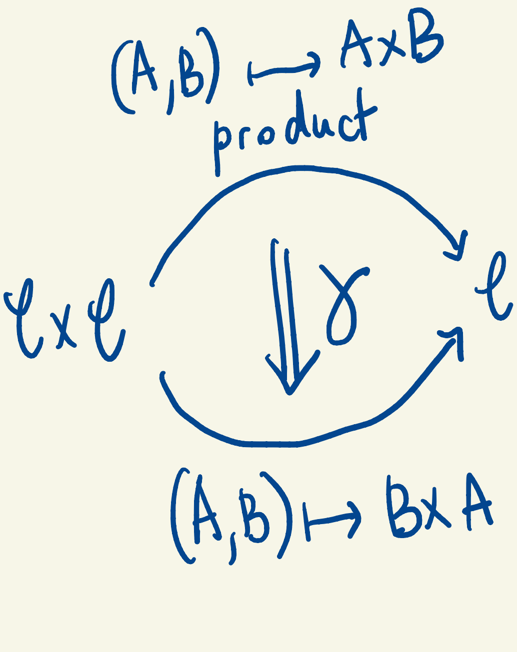
and the symmetry

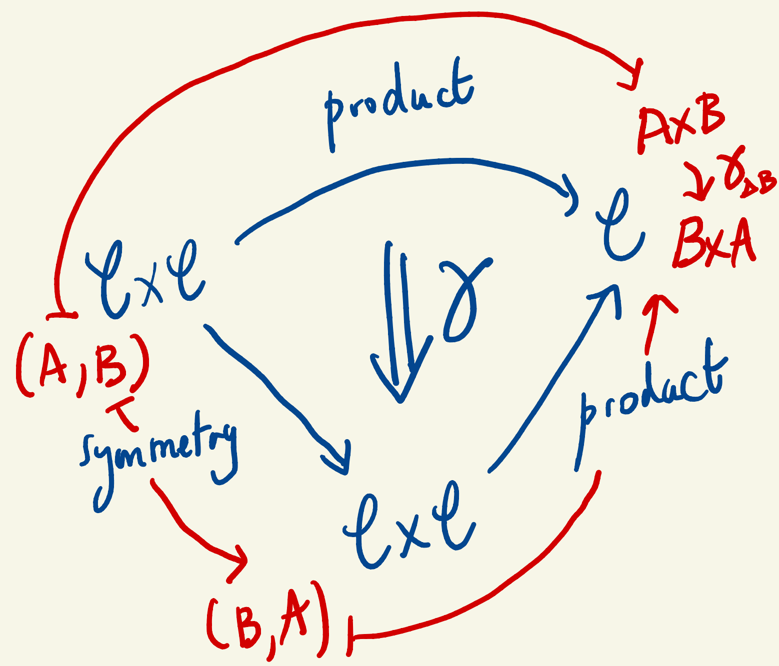
$$\gamma_{A,B} : A \times B \longrightarrow B \times A$$

are instances of <u>natural</u> transformations:









(3) Vect the category of vector spaces

$$
\begin{tikzcd}
\mathrm{Vect} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Bidual}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathrm{Vect}
\end{tikzcd}
$$

negation

$$V \longmapsto V^{**} = (V^*)^*$$

$$\varphi \in \boxed{V^* = \mathrm{Hom}(V, k)}$$

base field

$k = \mathbb{R}, \mathbb{C}$

$\boxed{\lambda k. k v}$

k continuation

the <u>natural</u> transformation:

$\boxed{\lambda \varphi. \varphi v}$

$$\eta_V : V \longrightarrow V^{**}$$

$$v \longmapsto \boxed{\varphi \mapsto \varphi(v)}$$

V^*

dual of V

$\neg V$

"negation" of V

V^{**}

bidual

$\neg(\neg V) = \neg\neg V$

double negation of V

$$V \xrightarrow{\eta_V} V^{**}$$

$$V \Longrightarrow \neg\neg V$$

general principle of logic

a formula A implies its double negation $\neg\neg A$.

I used the words "action"

and I want to justify that!

def

an action of a group G

on a set X

is a function $G \times X \longrightarrow X$

noted $(g, x) \longmapsto g \cdot x$

such that:

1. for all $g_1, g_2 \in G$, $x \in X$

$$g_1 \cdot (g_2 \cdot x) = (g_1 \circ_G g_2) \cdot x$$

2. for all $x \in X$

$$e \cdot x = x$$

neutral element of G

Key remark:

the left action of

$$H : \mathcal{D} \longrightarrow \mathcal{E}$$

induces a functor

$$\underline{\text{Trans}}(\mathcal{C}, \mathcal{D}) \xrightarrow{H \circ_L -} \underline{\text{Trans}}(\mathcal{C}, \mathcal{E})$$

Trans means the category of transformations

where $\boxed{\underline{\text{Trans}}(\mathcal{C}, \mathcal{D})}$ is the category

whose objects are the functors $F : \mathcal{C} \to \mathcal{D}$

whose maps are the transformations

between such functors.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, bend right=60, "G"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\theta"] \end{tikzcd}$$

$$\begin{tikzcd} F \arrow[d, "\theta"] \\ G \end{tikzcd} \quad \text{in } \underline{\text{Trans}}(\mathcal{C}, \mathcal{D})$$

Remark: the category Trans $(\mathcal{C}, \mathcal{D})$

contains as a subcategory

the category Nat $(\mathcal{C}, \mathcal{D})$

whose objects are functors $F : \mathcal{C} \to \mathcal{D}$

whose morphisms are

natural transformations

What this means:

Nat $(\mathcal{C}, \mathcal{D})$ is a subgraph of Trans $(\mathcal{C}, \mathcal{D})$

and inherits the composition law

and identity maps

from Trans $(\mathcal{C}, \mathcal{D})$.

Similarly, the right action of H

$$H : \mathcal{C} \longrightarrow \mathcal{D}$$

induces a functor

$$\mathrm{Trans}(\mathcal{D}, \mathcal{E}) \xrightarrow{- \circ_R H} \mathrm{Trans}(\mathcal{C}, \mathcal{E})$$

<u>what does this mean</u>?

$$\boxed{\begin{array}{l} H \circ_L - \\ \text{is a functor} \end{array}}$$

<u>the left action</u>:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, shift left=3, "\Downarrow \theta_1"] \arrow[r, phantom, shift right=3, "\Downarrow \theta_2"] & \mathcal{D} \arrow[r, "H"] & \mathcal{E}
\end{tikzcd}$$

$$H \circ_L (\theta_2 * \theta_1) \overset{①}{=} (H \circ_L \theta_2) * (H \circ_L \theta_1)$$

$$H \circ_L \mathrm{Id}_F \overset{②}{=} \mathrm{Id}_{H \circ F}$$

the right action:

$$\boxed{\begin{array}{c} - \circ_R H \\ \text{is a functor} \end{array}}$$

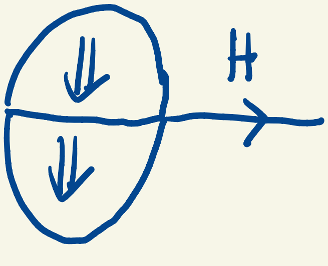
$$\begin{tikzcd}
\mathcal{C} \arrow[r, "H"] & \mathcal{D} \arrow[r, bend left=70, "F_1"{name=U}] \arrow[r, "F_2"{name=M, description}] \arrow[r, bend right=70, "F_3"'{name=L}] & \mathcal{E}
\arrow[Rightarrow, from=U, to=M, "\theta_1"] \arrow[Rightarrow, from=M, to=L, "\theta_2"]
\end{tikzcd}$$

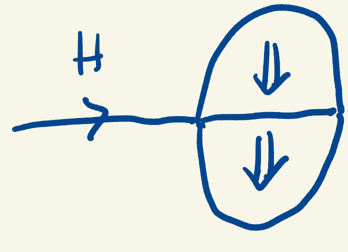
$$(\theta_2 * \theta_1) \circ_R H \overset{①}{=} (\theta_2 \circ_R H) * (\theta_1 \circ_R H)$$

$$\mathrm{Id}_F \circ_R H \overset{②}{=} \mathrm{Id}_{F \circ H}$$

What this means:

we can interpret univoquely diagrams such as:





this shows left/right actions "commute" with vertical composition

Left and Right actions

(1) Left actions: the terminology is justified by ⊛

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\theta\Downarrow"] & \mathcal{D} \arrow[r, "H_1"] & \mathcal{E} \arrow[r, "H_2"] & \mathcal{F} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Trans}(\mathcal{C},\mathcal{D}) \arrow[r, "H_1 \circ_L -"] \arrow[rr, bend right, "(H_2 \circ H_1)\circ_L -"'] & \mathrm{Trans}(\mathcal{C},\mathcal{E}) \arrow[r, "H_2 \circ_L -"] & \mathrm{Trans}(\mathcal{C},\mathcal{F}) \end{tikzcd}$$

$$\boxed{H_2 \circ_L (H_1 \circ_L \theta) \overset{⊛}{=} \underbrace{(H_2 \circ H_1)}_{\text{composition of functors}} \circ_L \theta}$$

$$g_2 \cdot (g_1 \cdot x) \overset{⊛}{=} (g_2 \circ g_1) \cdot x$$

$$e \cdot x \overset{⊛}{=} x$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \arrow[r, "\mathrm{Id}"] & \mathcal{D} \end{tikzcd}$$

$$\boxed{\mathrm{Id} \circ_L \theta \overset{⊛}{=} \theta}$$

(2) similarly for the right action:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "H_2"] & \mathcal{B} \arrow[r, "H_1"] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \end{tikzcd}$$

$$(\theta \circ_R H_1) \circ_R H_2 \overset{\circledast}{=} \theta \circ_R (\underbrace{H_1 \circ H_2}_{\text{composite functor}})$$

$$\theta \circ_R \mathrm{Id} \overset{\circledast}{=} \theta$$

③ compatibility relation between the left and right actions:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "H_1"] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \arrow[r, "H_2"] & \mathcal{E} \end{tikzcd}$$

$$\boxed{H_2 \circ_L (\theta \circ_R H_1) = (H_2 \circ_L \theta) \circ_R H_1}$$

What I have described:

- the left and right actions
- all the equations they satisfy

<u>defines</u> what is called

a $\underbrace{\text{sesqui}}_{1+1/2}$ - category

In particular, we have just described the <u>sesqui</u> category of categories

functors

<u>transformations</u>

what is missing in order to get a 2-category?

Suppose given two transformations:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, "F_1"{name=F1}] \arrow[r, bend right=50, "G_1"'{name=G1}] & \mathcal{B} \arrow[r, bend left=50, "F_2"{name=F2}] \arrow[r, bend right=50, "G_2"'{name=G2}] & \mathcal{C}
\arrow[from=F1, to=G1, phantom, "\Downarrow\theta_1"]
\arrow[from=F2, to=G2, phantom, "\Downarrow\theta_2"]
\end{tikzcd}$$

we would like to "compose" these two transformations together...

however, there are <u>two</u> different ways to do that —

1st "recipe" starting from θ_1 then applying θ_2

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & \mathcal{B} \arrow[r, bend left, "F_2"] & \mathcal{C}
\end{tikzcd}$$

$F_2 \circ_L \theta_1$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend right, "G_1"'] & \mathcal{B} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_2"] & \mathcal{C}
\end{tikzcd}$$

$\theta_2 \circ_R G_1$

$$\boxed{(\theta_2 \circ_R G_1) * (F_2 \circ_L \theta_1)}$$

defines a transformation

from $F_2 \circ F_1$ to $G_2 \circ G_1$

2nd "recipe" starting from θ_2 then applying θ_1

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F_1"] & \mathcal{B} \arrow[r, bend left, "F_2"{name=U}] \arrow[r, bend right, "G_2"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\theta_2"] & \mathcal{C}
\end{tikzcd} \qquad \theta_2 \circ_R F_1$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F_1"{name=U}] \arrow[r, bend right, "G_1"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\theta_1"] & \mathcal{B} \arrow[r, bend right, "G_2"'] & \mathcal{C}
\end{tikzcd} \qquad G_2 \circ_L \theta_1$$

$$\boxed{(G_2 \circ_L \theta_1) * (\theta_2 \circ_R F_1)}$$

defines a transformation from $F_2 \circ F_1$ to $G_2 \circ G_1$.

Def. we declare that θ_1 and θ_2 satisfy the "exchange rule" when the two transformations are equal.

Fact: the interchange rule is not always satisfied between two transformations.

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow f"] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \theta"] & \mathcal{D} \end{tikzcd}$$

Exchange rule:

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\theta_A"'] & FB \arrow[d, "\theta_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$ commutes

Exercise: show that a transformation θ_2 satisfies the exchange rule with every transformation θ_1 if and only if θ_2 is natural.

def: a 2-category is a sesqui-category in which the exchange rule is always satisfied.

What this means is that the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F_1"] \arrow[r, bend right=50, "G_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & \mathcal{B} \arrow[r, bend left=50, "F_2"] \arrow[r, bend right=50, "G_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & \mathcal{C} \end{tikzcd}$$

induces a 2-cell in a univoque way.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=60, "F_2 \circ F_1"] \arrow[r, bend right=60, "G_2 \circ G_1"'] \arrow[r, phantom, "\Downarrow \theta_2 \circ \theta_1"] & \mathcal{C} \end{tikzcd}$$

$\theta_2 \circ \theta_1$ is defined as this 2-cell

In our case,

$\boxed{\theta_2 \circ \theta_1}$ is the natural transformation

defined as

$$(\theta_2 \circ \theta_1)_{A \in \mathrm{Obj}\,\mathcal{A}} : F_2 \circ F_1(A) \longrightarrow G_2 \circ G_1(A)$$

$$\begin{tikzcd} & F_2 G_1 A \arrow[dr, "\theta_{2 G_1 A}"] & \\ F_2 F_1 A \arrow[ur, "F_2 \theta_{1A}"] \arrow[dr, "\theta_{2 F_1 A}"'] \arrow[rr, phantom, "\circlearrowright"] & & G_2 G_1 A \\ & G_2 F_1 A \arrow[ur, "G_2 \theta_{1A}"'] & \end{tikzcd}$$

This establishes that we have a 2-category **Cat**

with objects categories
with morphisms functors
with 2-cells natural transformations

Next time:

string diagrams for 2-categories.