2 October 2020

λ-calculus & categories 2

Functors & natural transformations

Functors and Natural Transformations

<u>Reminder</u>: every partial order $(X, \leq)$ defines a category whose objects are the elements of the ordered set, and the maps are defined in this way:

$$\mathrm{Hom}(x,y) = \begin{cases} \text{singleton when } x \leq y \\ \text{empty otherwise} \end{cases}$$

what this means is that there exists a map

$$x \longrightarrow y$$

precisely when $x \leq y$. Moreover this map is unique.

[2]

$$\boxed{\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & y \arrow[r, "g"] & z \end{tikzcd}}$$

transitivity of the order

$$\boxed{x \xrightarrow{id_x} x}$$

reflectivity of the order

[3] associativity of composition and neutrality of the identity map

$$\boxed{(h\circ g)\circ f = h\circ(g\circ f)}$$

$$\boxed{f\circ id_A = f = id_B \circ f}$$

at most one map between two objects

$$\begin{tikzcd} & B \arrow[r, "g"] & C \arrow[dr, "h"] & \\ A \arrow[ur, "f"] \arrow[rrr] & & & D \end{tikzcd}$$

$(h\circ g)\circ f = h\circ(g\circ f)$

$$\begin{tikzcd} & A \arrow[r, "f"] & B \arrow[dr, "id_B"] & \\ A \arrow[ur, "id_A"] \arrow[rrr] & & & B \end{tikzcd}$$

$f\circ id_A = f = id_B\circ f$

<u>Example</u>: the notion of "infimum" in order theory greatest lower bound "glb" $a \wedge b$

def: the glb $a \wedge b$ of a and b (when it exists) in an ordered set $(X, \leq)$ is an element:

① $a \wedge b$ is smaller than a and than b

$$a \wedge b \leq a \qquad a \wedge b \leq b$$

② for all $x \in X$,

$$(x \leq a \text{ and } x \leq b) \Rightarrow x \leq a \wedge b.$$

whenever
$$\begin{tikzcd}
a & & b \\
& a \wedge b \arrow[ul] \arrow[ur] & \\
& x \arrow[uul, bend left] \arrow[uur, bend right] &
\end{tikzcd}$$
then
$$\begin{tikzcd}
a & & b \\
& a \wedge b \arrow[ul] \arrow[ur] & \\
& x \arrow[uul, bend left] \arrow[uur, bend right] \arrow[u, dashed] &
\end{tikzcd}$$

So we see that the definition of cartesian product generalizes to <u>every category</u> the definition of glb coming from <u>order theory</u>.

(in particular a glb is the same thing as a cartesian product) in the special case of an order category

① Functors.

ordered sets

def. monotone function

$$(X, \le_X) \xrightarrow{f} (Y, \le_Y)$$

$$\boxed{\forall x, x' \in X, \quad x \le_X x' \implies f(x) \le_Y f(x')}$$

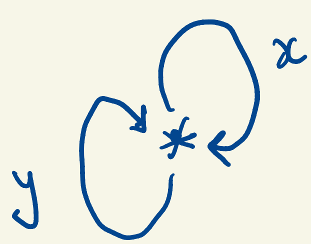
monoid

$$\underset{(M, \cdot_M, e_M)}{M} \xrightarrow{f} \underset{(N, \cdot_N, e_N)}{N}$$

def a homomorphism (of monoid) is a function $f: M \to N$ such that:

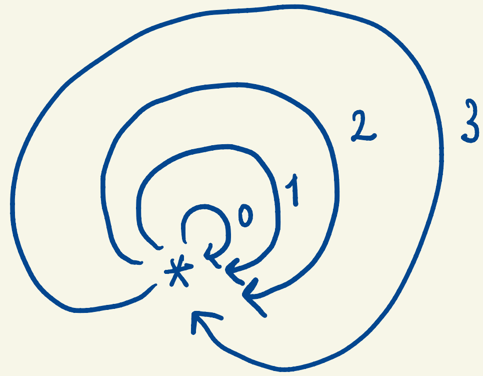
$$\forall x, y \in M \quad f(x \cdot_M y) = f(x) \cdot_N f(y)$$
$$f(e_M) = e_N$$

reminder a category with one object $*$ is the same thing as $\boxed{\text{a monoid.}}$



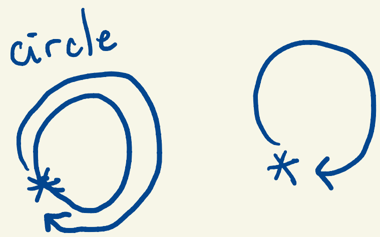
$x, y \mapsto x \circ y$

a multiplication associative with id as neutral element.



the monoid $\mathbb{N}$ with addition as composition

$$
\begin{tikzcd}
* \arrow[r, "m"] \arrow[rr, bend right, "n+m"'] & * \arrow[r, "n"] & *
\end{tikzcd}
$$



$\mathbb{Z}$

<u>def</u>. a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$

[0] the data of an object FA of $\mathcal{D}$ for every object A of $\mathcal{C}$.

[1] for every pair A, B of objects of $\mathcal{C}$

a function $F_{AB} : \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FA, FB)$

$$
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[rr, mapsto] & & FA \arrow[d, "Ff"] \\
B \text{ in } \mathcal{C} & & FB \text{ in } \mathcal{D}
\end{tikzcd}
$$

[2] F preserves composition and identities

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dd, bend left=60, "g\circ f"] \\ B \arrow[d, "g"'] \\ C \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} FA \arrow[d, "Ff"'] \arrow[dd, bend left=60, "{Fg \circ Ff = F(g\circ f)}"] \\ FB \arrow[d, "Fg"'] \\ FC \end{tikzcd}$$

in the category $\mathscr{C}$

$$\begin{tikzcd} A \arrow[d, "id_A"'] \\ A \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} FA \arrow[d, "Fid_A ="'] \arrow[d, bend left=60, "id_{FA}"] \\ FA \end{tikzcd}$$

<u>Rem</u>. • a functor $F : (X, \leq) \longrightarrow (Y, \leq)$ between ordered sets is the same thing as a monotone function.

• a functor $F : \Sigma(M, \cdot_M, e_M) \longrightarrow \Sigma(N, \cdot_N, e_N)$ between categories with one object (= monoids) is the same thing as a homomorphism $M \to N$.

$\Sigma(M, \cdot_M, e_M)$ "suspension" of the monoid

= category with one object

$$\begin{tikzcd} \ast \arrow[loop above, "m \in M"] \arrow[loop below, "m' \in M"] \end{tikzcd}$$

② Natural transformations

again, let us take partial orders $(X, \leq_X)$
$(Y, \leq_Y)$

$$\begin{tikzcd} (X, \leq_X) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\wedge\!\backslash"] & (Y, \leq_Y) \end{tikzcd}$$

f, g monotone functions

def.

$$f \leq g \iff \forall x \in X, \quad f(x) \leq_Y g(x)$$

<u>Similarly</u>:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \theta"] & \mathcal{D} \end{tikzcd}$$

1-dim

2dim

0dim

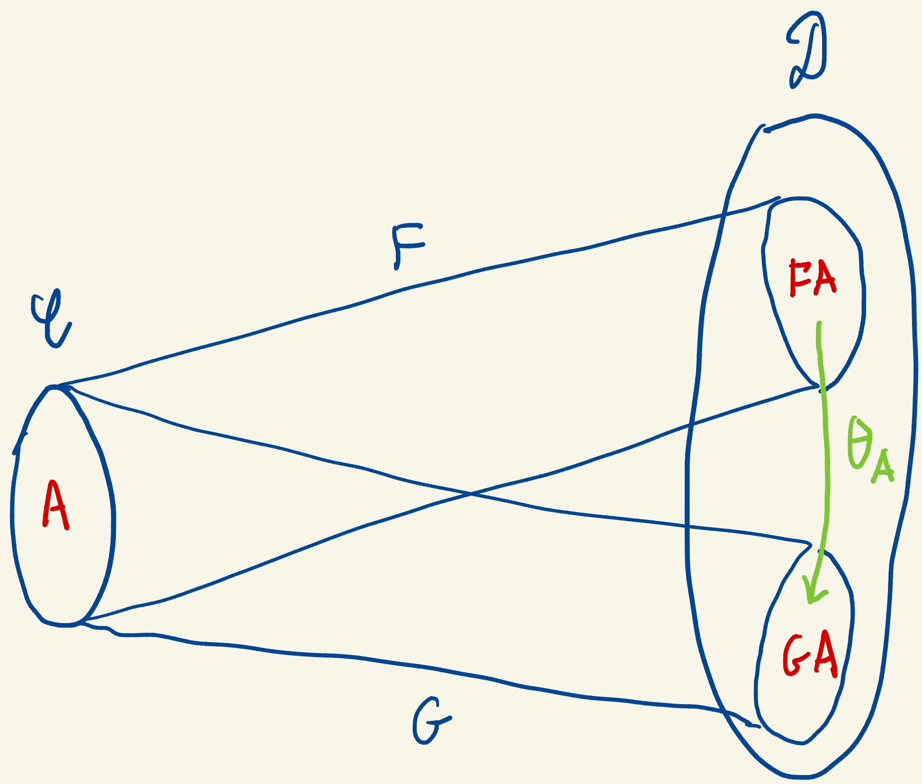
def. a transformation $\underbrace{\theta}_{\text{2dim}} : \underbrace{F}_{\text{1dim}} \Rightarrow \underbrace{G}_{\text{1dim}} : \underbrace{\mathcal{C}}_{\text{0dim}} \underbrace{\longrightarrow}_{\text{1dim}} \underbrace{\mathcal{D}}_{\text{0dim}}$

is a family $(\theta_A)_{A \in \mathrm{Obj}\, \mathcal{C}}$

of maps:

$$\theta_A : FA \longrightarrow GA \quad \text{in } \mathcal{D}$$

ie for every object A of $\mathcal{C}$
a map $\theta_A : FA \to GA$ of $\mathcal{D}$.



<u>Def</u>. a natural transformation

$$\theta : F \Rightarrow G : \mathcal{C} \longrightarrow \mathcal{D}$$

is a $\boxed{\text{transformation}}$ which satisfies the extra condition that for all maps $\boxed{f : A \to B}$ in $\mathcal{C}$

the diagram below commutes:

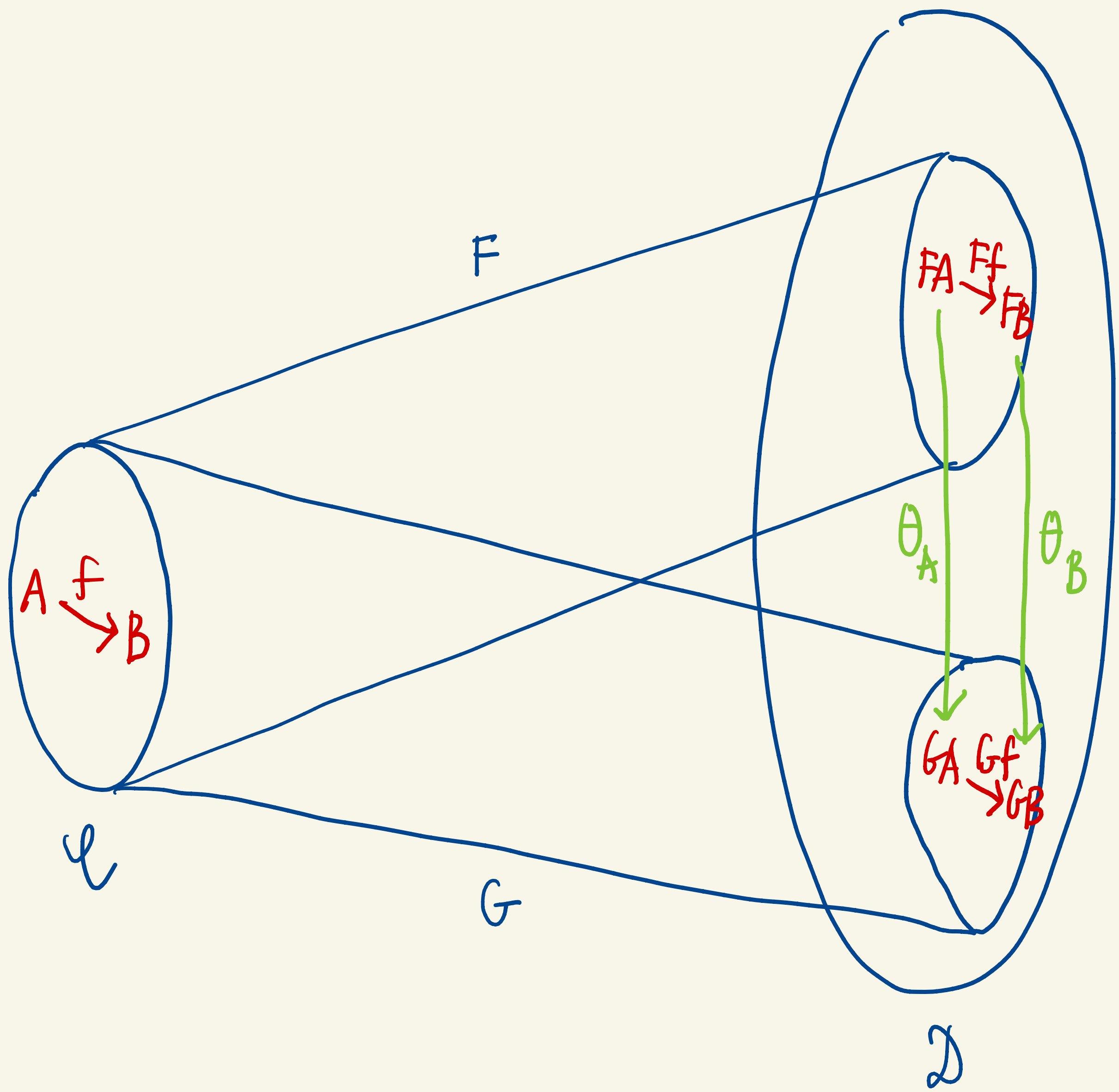
$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "(*)"] & FB \arrow[d, "\theta_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

$$\theta_B \circ Ff = Gf \circ \theta_A$$

Commutes.

① for every object A of $\mathcal{C}$, a map $FA \xrightarrow{\theta_A} GA$

② for every map $A \xrightarrow{f} B$, a commutative diagram $(*)$



<u>def</u>. a <u>cartesian category</u>
is a category $\mathcal{C}$ $\boxed{\text{equipped}}$

- for all pairs A, B of objects with a cartesian product $(A \times B, \pi_1, \pi_2)$ of A and B
- with a terminal object $\mathbb{1}$

Reminder. an object $\mathbb{1}$ in a category $\mathscr{C}$ is called terminal when for every object A of the category $\mathscr{C}$ there exists exactly one morphism $A \to \mathbb{1}$.

Rem

$$
\begin{tikzcd}
\mathbb{1} \arrow[r, "f"] \arrow[rr, bend right, "id"'] & \mathbb{1}' \arrow[r, "g"] & \mathbb{1}
\end{tikzcd}
$$

$g \circ f = id_{\mathbb{1}}$

$f \circ g = id_{\mathbb{1}'}$

hence

$\mathbb{1}$ and $\mathbb{1}'$ are isomorphic objects.

in a category $\mathscr{C}$

two terminal objects $\mathbb{1}$ and $\mathbb{1}'$ are isomorphic

"there exists only one terminal object up to isomorphism".

Exercise: (1) show that every cartesian category $\mathcal{C}$ comes with a $\boxed{\text{functor:}}$

$$\begin{aligned} \times : \mathcal{C} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ A, B &\longmapsto A \times B \end{aligned}$$

(2) From this, one obtains the functor

$$\begin{tikzcd} \mathcal{C} \arrow[r] & \mathcal{C} \times \mathcal{C} \arrow[r] & \mathcal{C} \\ A \arrow[r, mapsto] \arrow[rr, mapsto, bend right] & (A, A) \arrow[r, mapsto, "\times"] & A \times A \end{tikzcd}$$

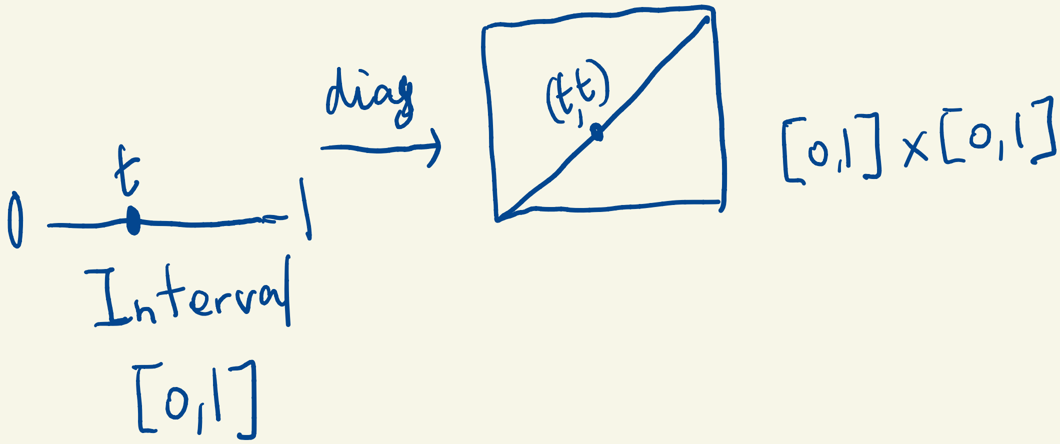
Show that this is a functor $\mathcal{C} \xrightarrow{\Delta} \mathcal{C}$

(3) Construct a natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\Delta"'] \arrow[r, phantom, "\Downarrow \mathrm{diag}"] & \mathcal{C} \end{tikzcd} \qquad \mathrm{diag}_A : A \longrightarrow A \times A$$

Remark: the category Set is cartesian

$$\begin{aligned} \mathrm{diag}_A : A &\longrightarrow A \times A \\ a &\longmapsto (a, a) \end{aligned}$$



$$\begin{tikzcd}
A & & A \\
& A \times A \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& A \arrow[uul, bend left, "id_A"] \arrow[uur, bend right, "id_A"'] \arrow[u, "\mathrm{diag}_A"] &
\end{tikzcd}$$

diag_A = the unique map making the diagram commute

For every object A such a map $A \longrightarrow A \times A$

Given two categories $\mathcal{A}, \mathcal{B}$

one defines the category $\mathcal{A} \times \mathcal{B}$

as the category whose $\boxed{\text{objects}}$

are the pairs (A, B)

of an object A of $\mathcal{A}$

and an object B of $\mathcal{B}$

whose $\boxed{\text{maps}}$ are the pairs

$$\begin{tikzcd} (A_1, B_1) \arrow[d, "{(f,g)}"] \\ (A_2, B_2) \end{tikzcd}$$

where

$$\begin{tikzcd} A_1 \arrow[d, "f"] \\ A_2 \end{tikzcd}$$ a map in $\mathcal{A}$

$$\begin{tikzcd} B_1 \arrow[d, "g"] \\ B_2 \end{tikzcd}$$ a map in $\mathcal{B}$

$$
\begin{tikzcd}
(A_1, B_1) \arrow[d, "{(f_1, g_1)}"] \arrow[dd, bend left=60] \\
(A_2, B_2) \arrow[d, "{(f_2, g_2)}"] \\
(A_3, B_3)
\end{tikzcd}
\qquad
\begin{aligned}
&(f_2, g_2) \circ (f_1, g_1) \\
&:= (f_2 \circ_{\mathcal{C}} f_1,\ g_2 \circ_{\mathcal{D}} g_1)
\end{aligned}
$$

$$
\begin{tikzcd}
(A, B) \arrow[d, "\mathrm{id}_{(A,B)}"] \\
(A, B)
\end{tikzcd}
:= (\mathrm{id}_A, \mathrm{id}_B)
$$

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr] & & \mathcal{C} \times \mathcal{C} \\
A \arrow[rr, mapsto] \arrow[d, "f"'] & & (A, A) \arrow[d, "{(f, f)}"] \\
B & & (B, B)
\end{tikzcd}
$$