λ-calculus & categories 5

23 october 2020

the free cartesian category

The free cartesian category

a set $A \longmapsto A^*$ the free monoid

a category $\mathcal{A} \longmapsto \mathrm{Fam}(\mathcal{A})$ the free cartesian category

$\boxed{\begin{array}{c} \mathcal{M} \\ \text{a cartesian category} \end{array}}$

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{M} & \longrightarrow & \mathcal{M} \\ (A, B) & \longmapsto & A \times B \end{array}$$

cartesian product

$$\begin{array}{ccc} \mathbb{1} & \longrightarrow & \mathcal{M} \\ * & \longmapsto & 1 \end{array}$$

terminal object of $\mathcal{M}$

in a cartesian category

$$A \times 1 \cong A \cong 1 \times A$$

"$\cong$" isomorphic to

hence 1 is "some kind of" neutral element for $\times$.

Something remarkable to remember:

the free cartesian category is only defined "up to equivalence"

① the notion of equivalence between categories.

1.1 def. an equivalence between two categories $\mathcal{A}$ and $\mathcal{B}$ is a pair of functors

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \qquad \mathcal{B} \xrightarrow{G} \mathcal{A}$$

and a pair of natural isomorphisms

$$\varphi : \mathrm{Id}_{\mathcal{A}} \overset{\sim}{\Longrightarrow} G \circ F \qquad \psi : \mathrm{Id}_{\mathcal{B}} \overset{\sim}{\Longrightarrow} F \circ G$$

$$\boxed{\begin{tikzcd} \mathcal{A} \arrow[loop left, "G\circ F"] \arrow[r, bend left, "F"] & \mathcal{B} \arrow[l, bend left, "G"] \arrow[loop right, "F\circ G"] \end{tikzcd}}$$

Reminder:

a natural isomorphism $\theta : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ is a natural transformation such that $\theta_A : FA \overset{\sim}{\Rightarrow} GA$ is an isomorphism for all A object in $\mathcal{C}$.

1.2 examples.

ⓐ the category $\boxed{\underline{\mathrm{FinSet}}}$ of finite sets and functions.

the category $\boxed{\mathbb{F}}$ whose objects are natural numbers

$n \in \mathbb{N}$ $\quad$ $[n] = \{0, \dots, n-1\}$ the finite set with n elements.

$$[0] = \emptyset$$

whose morphisms

$$[p] \xrightarrow{f} [q]$$

are the <u>functions</u> f from $\{0, \dots, p-1\}$ to $\{0, \dots, q-1\}$.

Clearly, $\mathbb{F}$ is a subcategory of FinSet.

In particular, we have an inclusion functor

$$\mathbb{F} \xrightarrow{\mathrm{incl}} \mathrm{FinSet}.$$

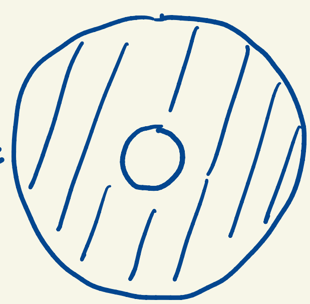
<u>Exercise</u>: construct a $\boxed{\text{functor}}$ $G : \mathrm{FinSet} \longrightarrow \mathbb{F}$ defining an equivalence between $\mathbb{F}$ and Finset

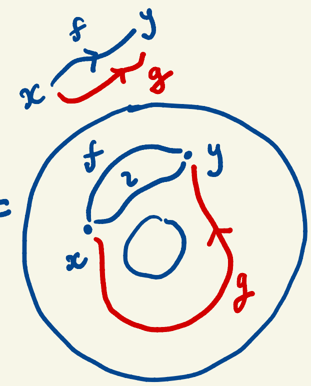
$$G \circ F = \mathrm{Id} \qquad \begin{tikzcd} \mathbb{F} \arrow[r, bend left, "F = \mathrm{incl}"] & \mathrm{FinSet} \arrow[l, bend left, "G = \#"] \end{tikzcd} \qquad F \circ G \cong \mathrm{Id}$$

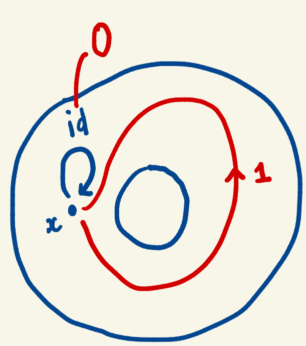
$$\Psi : \mathrm{Id} \overset{\sim}{\Longrightarrow} F \circ G$$

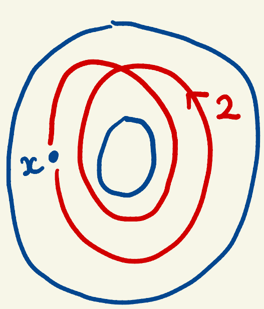
(b) every topological space Ω defines a category $\mathcal{C}_\Omega$ whose objects are the elements of the topological space Ω

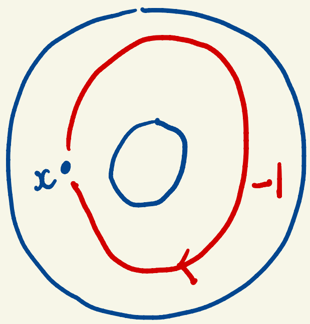
whose maps $x \xrightarrow{f} y$ are the paths from x to y in Ω up to homotopy.

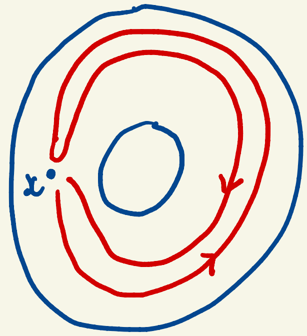
$\Omega =$ 

$\mathcal{C}_\Omega =$ 

$\mathcal{C}_\Omega$ 







$1+(-1)=0$

$$\mathrm{Hom}_{\mathcal{C}_\Omega}(x,x) = \mathbb{Z}$$

the fundamental group of Ω is $\underbrace{(\mathbb{Z}, +, 0)}_{\text{a group}}$

Poincaré group

hence a <u>monoid</u>

hence a category with one object $\Sigma\mathbb{Z}$

$\Sigma\mathbb{Z}$

$$\begin{tikzcd} * \arrow[loop right, "m \in \mathbb{Z}"] \end{tikzcd}$$

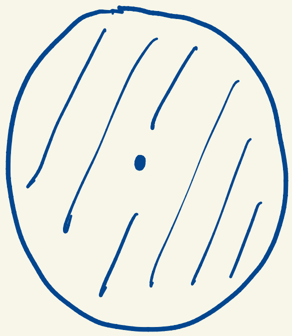
Poincaré group

topological space seen as a category

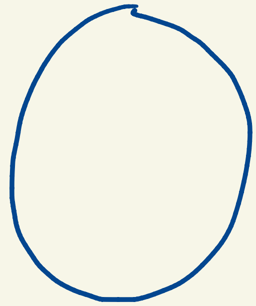
$$\begin{tikzcd} \Sigma\mathbb{Z} \arrow[r, bend left, "\text{incl}"] & \mathcal{C}_\Omega \arrow[l, bend left, "\text{poincaré}"] \end{tikzcd}$$

which defines $\boxed{\text{an equivalence of categories}}$

(more generally true for any <u>pointed</u> topological space)



$\simeq$

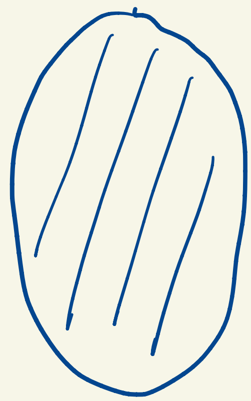


equivalent up to homotopy equivalence.

disk from which one removes a point

circle

topology

point $\bullet$ $\underset{\text{homotopy equivalence}}{\simeq}$  $=: \Omega$

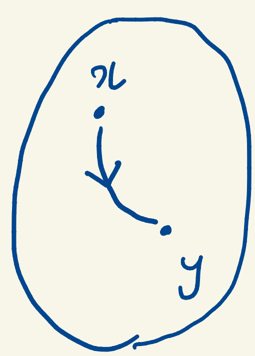
disk

$\mathcal{C}_\Omega(x,y)$ singleton

$\boxed{\text{Exercise}}$

category theory

$\mathbb{1} = * \circlearrowleft^{id}$ $\underset{\text{categorical equivalence}}{\simeq}$ $\mathcal{C}_\Omega$



② cartesian functor between cartesian categories

Observation. given $\boxed{\text{inf-lattice}}$ $(X, \leq)$ $(Y, \leq)$

(partial orders with a greatest lower bound

$$x, y \longmapsto x \wedge y \quad (\text{infimum})$$

for any pair of elements x and y

and with a maximum $\mathbb{1}$)

and a monotone function $f : (X, \leq) \longrightarrow (Y, \leq)$.

Then,

$$\boxed{\begin{array}{c} \forall x, x', \quad f(x \wedge x') \leq_Y f(x) \wedge f(x') \\ f(\mathbb{1}_X) \leq_Y \mathbb{1}_Y \end{array}}$$

Prf

$$x \wedge x' \leq_X x \quad \text{and} \quad x \wedge x' \leq_X x'$$

hence by monotonicity of f

$$f(x \wedge x') \leq_Y f(x) \qquad f(x \wedge x') \leq_Y f(x')$$

From this, we conclude that

$$f(x \wedge x') \leq_Y f(x) \wedge f(x')$$

since $f(x) \wedge f(x')$ is the greatest lower bound of $f(x)$ and $f(x')$.

$$\begin{array}{ccc} \text{partial order} & \qquad & \text{inf-lattice} \\ \Downarrow & & \Updownarrow \\ \text{category} & & \text{cartesian categories} \end{array}$$

Reminder:

a partial order $(X, \leq)$

is an inf-lattice

$\boxed{\text{precisely when}}$

the partial order $(X, \leq)$ seen as category

is a cartesian category.

Prop: given a pair $(\mathcal{A}, \times_{\mathcal{A}}, \mathbb{1}_{\mathcal{A}})$ and $(\mathcal{B}, \times_{\mathcal{B}}, \mathbb{1}_{\mathcal{B}})$ of cartesian categories

every functor $F : \mathcal{A} \longrightarrow \mathcal{B}$

comes equipped with a family of morphisms

$$m_{A_1, A_2} : F(A_1 \times_{\mathcal{A}} A_2) \longrightarrow FA_1 \times_{\mathcal{B}} FA_2$$

natural in A_1 and A_2, and with a morphism

$$m_{\mathbb{1}} : F(\mathbb{1}_{\mathcal{A}}) \longrightarrow \mathbb{1}_{\mathcal{B}} .$$

$$\boxed{\text{span}} \qquad \begin{tikzcd} A & & B \\ & S \arrow[ul] \arrow[ur] \end{tikzcd}$$

Construction of m_{A_1, A_2}

$$\begin{tikzcd} A_1 & & A_2 \\ & A_1 \times A_2 \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \end{tikzcd} \quad \overset{F}{\longmapsto} \quad (*) \quad \begin{tikzcd} FA_1 & & FA_2 \\ & FA_1 \times FA_2 \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] \\ & F(A_1 \times A_2) \arrow[u, dashed, "m_{A_1, A_2}"] \arrow[uul, bend left, "F\pi_1"] \arrow[uur, bend right, "F\pi_2"'] \end{tikzcd}$$

in the category $\mathcal{A}$ $\qquad\qquad$ in the category $\mathcal{B}$

m_{A_1,A_2} is $\boxed{\underline{\text{defined}}}$ as the unique morphism

$$F(A_1 \times A_2) \longrightarrow FA_1 \times FA_2$$

making the diagram commute.

<u>Naturality</u> in A_1 and A_2 means that the diagram

$$\begin{tikzcd}
F(A_1 \times A_2) \arrow[r, "m_{A_1,A_2}"] \arrow[d, "F(h_1 \times h_2)"'] \arrow[dr, phantom, "(**)\quad G"] & FA_1 \times FA_2 \arrow[d, "Fh_1 \times Fh_2"] \\
F(A_1' \times A_2') \arrow[r, "m_{A_1',A_2'}"] & FA_1' \times FA_2'
\end{tikzcd}$$

commutes

for all maps $\begin{tikzcd} A_1 \arrow[d, "h_1"] \\ A_1' \end{tikzcd}$ $\begin{tikzcd} A_2 \arrow[d, "h_2"] \\ A_2' \end{tikzcd}$ in $\mathcal{C}$

Exercise: prove that the diagram $(**)$ commutes.

More conceptually, naturality means that the family $(m_{A_1,A_2})_{A_1\in \mathrm{ob}\,\mathcal{A},\, A_2\in \mathrm{ob}\,\mathcal{A}}$

defines a <u>natural</u> transformation

$$\begin{tikzcd}
 & & A_1\times A_2 \arrow[rrd, mapsto, bend left] & & \\
 & & \mathcal{A} \arrow[dr, "F"] \arrow[dd, phantom, "\Downarrow m"] & & F(A_1\times A_2) \arrow[dd, "m_{A_1,A_2}"] \\
(A_1,A_2)\in \arrow[uurr, mapsto, bend left] \arrow[ddrr, mapsto, bend right] & \mathcal{A}\times\mathcal{A} \arrow[ur, "\times_{\mathcal{A}}"] \arrow[dr, "F\times F"'] & & \mathcal{B} & \\
 & & \mathcal{B}\times\mathcal{B} \arrow[ur, "\times_{\mathcal{B}}"'] & & FA_1\times FA_2 \\
 & & (FA_1,FA_2) \arrow[rru, mapsto, bend right] & &
\end{tikzcd}$$

<u>Def</u>: a <u>cartesian</u> functor $F:\mathcal{A}\longrightarrow\mathcal{B}$ between cartesian categories is a functor such that

m_{A_1,A_2} and m_1 are isomorphisms for all A_1, A_2 in $\mathcal{A}$.

Note that a cartesian functor preserves cartesian products only « up to isomorphisms ».

Motivation for that definition:

Suppose that a category $\mathcal{A}$ is equipped with two cartesian structures:

$$(\mathcal{A}, \times, \mathbb{1}) \qquad\qquad (\mathcal{A}, \boxtimes, e)$$

$$\mathcal{A} \xrightarrow{\quad Id \quad} \mathcal{A}$$

the identity functor between

$$(\mathcal{A}, \times, 1) \quad \text{and} \quad (\mathcal{A}, \boxtimes, e)$$

is a cartesian functor!

Exercise: check that!

Illustration : $\mathcal{A} = \underline{Sets}$

$$A, B \longmapsto A \times B$$
$$A, B \longmapsto A \boxtimes B := B \times A$$

in that case

$$m_{A_1, A_2} : A_1 \times A_2 \xrightarrow{\cong} A_1 \boxtimes A_2$$

is defined as the symmetry

$$A_1 \times A_2 \xrightarrow{\sim} A_2 \times A_1$$
$$(a_1, a_2) \longmapsto (a_2, a_1)$$

strictly associative

$$(A \times B) \times C \;\boxed{=}\; A \times (B \times C)$$

Property. Suppose given a pair of functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathcal{B} \end{tikzcd}$$

between cartesian categories $\mathcal{A}$, $\mathcal{B}$ together with a natural isomorphism

$$\theta : F \Rightarrow G : \mathcal{A} \longrightarrow \mathcal{B}$$

$$\boxed{\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F", ""{name=U, below}] \arrow[r, bend right=50, "G"', ""{name=D, above}] & \mathcal{B} \arrow[from=U, to=D, Rightarrow, "\theta"] \end{tikzcd}}$$

In that situation,

F is a cartesian functor

implies that G is a cartesian functor

Proof left as exercise!

③ the free cartesian category

Given a category $\mathcal{A}$, we construct a cartesian category $\mathrm{Fam}(\mathcal{A})$ as follows:

① the objects are finite sequences of objects of $\mathcal{A}$

$\underline{A} = [A_1, \ldots, A_n]$ where A_i are objects of $\mathcal{A}$

② the morphisms

$$[A_1, \ldots, A_p] \xrightarrow{f = (\varphi, [f_1, \ldots, f_q])} [B_1, \ldots, B_q]$$

are pairs consisting of

- a function $\varphi : \{1, \ldots, q\} \longrightarrow \{1, \ldots, p\}$

- a finite sequence $[f_1, \ldots, f_q]$ of morphisms

$$f_i : A_{\varphi(i)} \longrightarrow B_i$$

In a drawing:

$$
\begin{tikzcd}
A_1 \arrow[drr, "f_2"] & & B_1 \\
A_2 & & B_2 \\
A_3 \arrow[uurr, "f_1"'] & &
\end{tikzcd}
$$

$$[A_1, A_2, A_3] \xrightarrow{(\varphi, [f_1, f_2])} [B_1, B_2]$$

$$\varphi : \{1,2\} \longrightarrow \{1,2,3\}$$

$$1 \longmapsto 3$$

$$2 \longmapsto 1$$

$$f_1 : A_{\varphi(1)} \longrightarrow B_1$$

$$f_2 : A_{\varphi(2)} \longrightarrow B_2$$

Claim: the category $Fam(\mathcal{A})$ is a cartesian category.

Construction:

given two objects $\underline{A} = [A_1, \ldots, A_p]$

$\underline{B} = [B_1, \ldots, B_q]$

of the category $Fam(\mathcal{A})$

the cartesian product $\underline{A} \times \underline{B}$ is defined as the object

$[A_1, \ldots, A_p, B_1, \ldots, B_q]$ in $Fam(\mathcal{A})$

Moreover, note that the empty word $[\,]$ is the terminal object of $Fam(\mathcal{A})$.

What are the projection maps?

$$
\begin{tikzcd}
A_1 \arrow[rrd, "id_{A_1}"] & & \\
\vdots & & A_1 \\
A_p \arrow[rrd, "id_{A_p}"] & & \vdots \\
B_1 & & A_p \\
\vdots & & \\
B_q & &
\end{tikzcd}
$$

first projection in $Fam(\mathcal{A})$

$$\underline{A}\times\underline{B} \xrightarrow{\pi_1} \underline{A} \quad \text{in } Fam\,\mathcal{A}.$$

$$\pi_1 = (\varphi, [id_{A_1} \dots id_{A_p}])$$

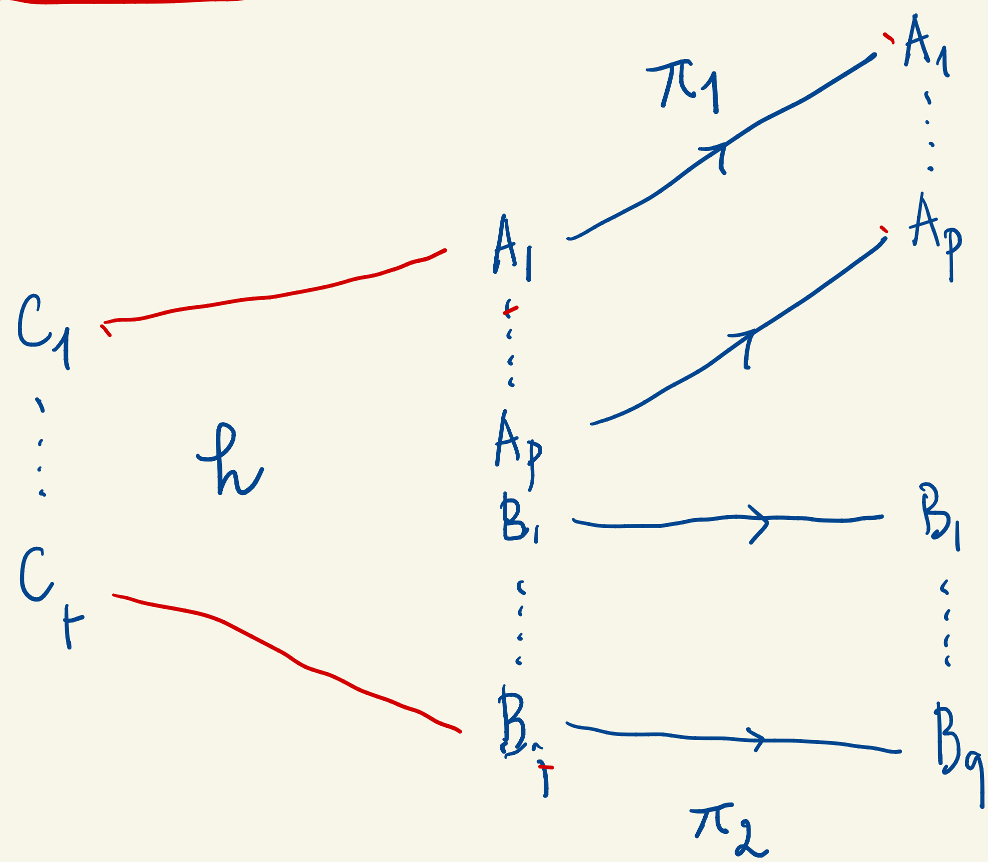
where $\varphi : j \longmapsto j$

Similarly for the second projection map

$$\underline{A}\times\underline{B} \xrightarrow{\pi_2} \underline{B} \qquad \varphi : j \mapsto p+j$$

Exercise: conclude the proof that $\mathrm{Fam}(\mathcal{C})$ is (indeed) a cartesian category.

$$\boxed{C_{\varphi i} \longrightarrow A_i}$$



$$\begin{tikzcd} \underline{C} \arrow[r, "h"] \arrow[rr, bend left, "f"] \arrow[rrd, bend right, "g"'] & \underline{A} \times \underline{B} \arrow[r] \arrow[rd] & \underline{A} \\ & & \underline{B} \end{tikzcd}$$

$$\boxed{C_{\psi j} \longrightarrow B_j}$$

h is (some kind of) concatenation of f and g

$$h = \left([\varphi, \psi], [f_1 \dots f_p \, g_1 \dots g_q] \right)$$

Main theorem:

given any cartesian category $\mathcal{M}$

$$(\mathcal{M}, \times, \mathbb{1})$$

and any <u>functor</u> $\mathcal{A} \xrightarrow{F} \mathcal{M}$

there exists a <u>cartesian functor</u>

$$\mathrm{Fam}(\mathcal{A}) \xrightarrow{F^+} \mathcal{M}$$

which makes the diagram below commute:

$$\begin{tikzcd}
\mathrm{Fam}(\mathcal{A}) \arrow[rr, dashed, "\boxed{F^+}"] & & \mathcal{M} \\
\mathcal{A} \arrow[u, "\text{inclusion}"] \arrow[urr, "F"'] & &
\end{tikzcd}$$

inclusion : $\mathcal{A} \longrightarrow \mathrm{Fam}\,\mathcal{A}$

$A \longmapsto [A]$

Moreover, the cartesian functor F^+ is unique up to unique isomorphism.

this means that if

$$\begin{tikzcd}
\mathrm{Fam}\,\mathcal{A} \arrow[rr, "H \text{ cartesian}"] & & \mathcal{M} \\
\mathcal{A} \arrow[u, "\text{inclusion}"] \arrow[urr, "F"'] \arrow[ur, phantom, "\circlearrowright"] & &
\end{tikzcd}$$

commutes

$$\begin{tikzcd}
A^* \arrow[r, "f^+"] & M \\
A \arrow[u, "\eta_A"] \arrow[ur, "f"'] &
\end{tikzcd}$$

then there $\boxed{\text{exists}}$ a natural isomorphism

$$\begin{tikzcd}
\mathrm{Fam}\,\mathcal{A} \arrow[r, bend left=50, "F^+"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{M}
\end{tikzcd}$$

between F^+ and G such that

$$\begin{tikzcd}
\mathcal{A} \arrow[r] & \mathrm{Fam}\,\mathcal{A} \arrow[r, bend left=50, "F^+"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{M}
\end{tikzcd}$$

is the identity. Moreover θ is unique.

Final word:

suppose given an object A in a cartesian category $\mathcal{M}$.

the object A defines a functor

$\mathbb{1}$ terminal category

$$\begin{tikzcd} * \arrow[loop left, "id"] \end{tikzcd}$$

$$\mathbb{1} \xrightarrow{\quad A \quad} \mathcal{M}$$

hence a cartesian functor

$$\mathrm{Fam}(\mathbb{1}) \xrightarrow{\quad A^{+} \quad} \mathcal{M}$$

$\mathrm{Fam}(\mathbb{1})$ is the category with finite words of the form $\underbrace{[*, *, *, \ldots, *]}_{[n]}$

and maps

$$[p] \xrightarrow{\quad f = \varphi \quad} [q]$$

$$\varphi : \{1, \ldots, q\} \longrightarrow \{1, \ldots, p\}$$

What is $\mathrm{Fam}(\mathbb{1})$?

objects are natural numbers
maps are functions taken in the reverse direction.

$$\boxed{\mathrm{Fam}(\mathbb{1}) = \mathbb{F}^{op}}$$

$$\begin{tikzcd}
{[1]} & \mathbb{F}^{op} \arrow[r, "A^+"] & \mathcal{U} \\
{*} \arrow[u, mapsto] & \mathbb{1} \arrow[u, "\text{inclusion}"] \arrow[ur, "A"'] \arrow[u, phantom, shift left=20, "\circlearrowright"] &
\end{tikzcd}$$

$$A^+ : [1] \longmapsto A$$
$$[n] \longmapsto A^n$$

Every object A in a cartesian category $\mathcal{C}$ comes with a hierarchy of "powers"

$$A^0 = 1_{\mathcal{C}} \qquad A \qquad A^2 = A \times A \quad \dots \quad A^n \quad \dots$$

$$\begin{tikzcd}
{[3]} \arrow[d, shift right=4] \arrow[d, shift right=2, <-] \arrow[d] \arrow[d, shift left=2, <-] \arrow[d, shift left=4] \\
{[2]} \arrow[d, shift right=2] \arrow[d, <-] \arrow[d, shift left=2] \\
{[1]} \arrow[d] \\
{[0]}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A^3 \arrow[d, shift right=4, "\pi_{12}"'] \arrow[d, shift right=2, <-] \arrow[d, "\pi_{13}" description] \arrow[d, shift left=2, <-] \arrow[d, shift left=4, "\pi_{23}"] \\
A^2 \arrow[d, shift right=2, "\pi_1"'] \arrow[d, <-, "\Delta" description] \arrow[d, shift left=2, "\pi_2"] \\
A \arrow[d] \\
A^0
\end{tikzcd}$$

$$\boxed{\mathbb{F}^{op} = \mathrm{Fam}(\mathbb{1})}$$

describes the hierarchy of objects A^n in any cartesian category.