Monday 30 November 2020

λ-calculus cartesian & closed categories

Purpose of this session:

we will study / explain how every

$\boxed{\lambda\text{-term } M}$ of type:

$$x_1 : A_1 , \ldots , x_k : A_k \vdash M : B$$

can be turned into

$\boxed{\text{a morphism}}$

$$[A_1 \times \ldots \times A_k] \xrightarrow{[M]} [B]$$

in any $\boxed{\boxed{\text{cartesian closed}}}$ category $\mathscr{C}$.

here $A_1, \ldots, A_k$ are the interpretations of the type $A_1, \ldots, A_k$ in the category $\mathscr{C}$

Starting point: the notion of cartesian-closed category $\mathcal{C}$.

def: a cartesian-closed category $\mathcal{C}$ is a cartesian category $(\mathcal{C}, \times, 1)$ equipped with a family of functors

$$\boxed{A \Rightarrow - : \mathcal{C} \longrightarrow \mathcal{C}}$$

parametrized by objects A of the category $\mathcal{C}$ together with an adjunction

$$\boxed{A \times - \dashv A \Rightarrow -}$$

which says that

$A \Rightarrow -$ is right adjoint to $A \times -$

Here remember that every object A in a cartesian category $\mathcal{C}$ defines a functor

$$A \times - : \mathcal{C} \longrightarrow \mathcal{C} \qquad B \longmapsto A \times B$$

From this definition, one can deduce a functor:

$$\boxed{\begin{array}{rl} - \Rightarrow - \; : \; \mathcal{C}^{op} \times \mathcal{C} & \longrightarrow \mathcal{C} \\ (A \;\; , \; B) & \longmapsto A \Rightarrow B \end{array}}$$

$$\begin{array}{ccccc}
A & & B & & A \\
h_A \big\downarrow & & \big\downarrow h_B & & \big\uparrow h_A \\
A' & & B' & & A' \\
\text{in } \mathcal{C}^{op} & & \text{in } \mathcal{C} & & \text{in } \mathcal{C}
\end{array}
\qquad
\boxed{\begin{array}{c} A \Rightarrow B \\ \big\downarrow h_A \Rightarrow h_B \\ A' \Rightarrow B' \end{array}}$$

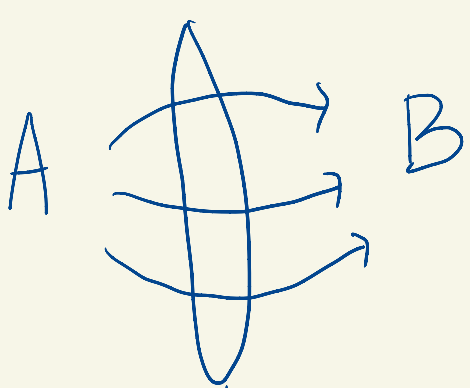
the construction of the functor $\Rightarrow$ uses the parameter theorem (see MacLane's book Categories for the Working Mathematician)

Example: the category Set of sets and functions.

① cartesian $\boxed{A, B \mapsto A \times B}$

② cartesian closed

$A \Rightarrow B$ = set of functions from A to B



this defines a set!

the solution before the query

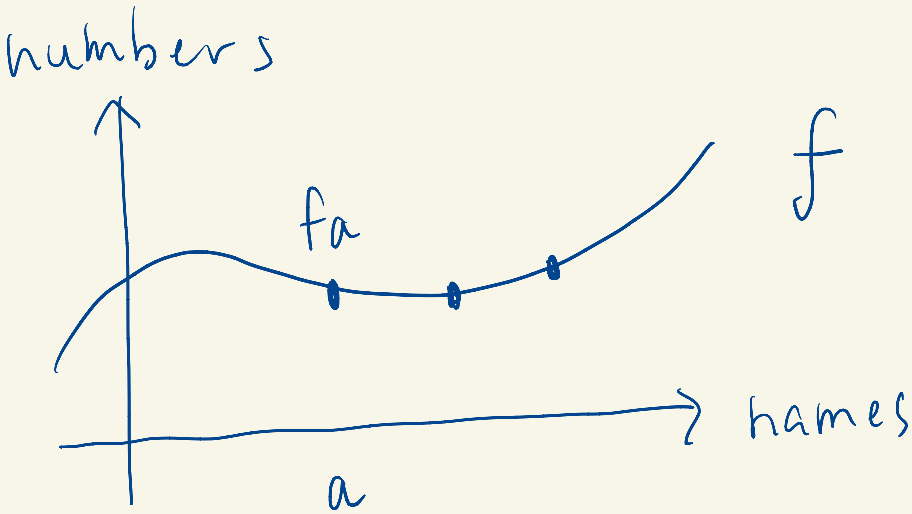
— Melliès — phone number

a <u>function</u>

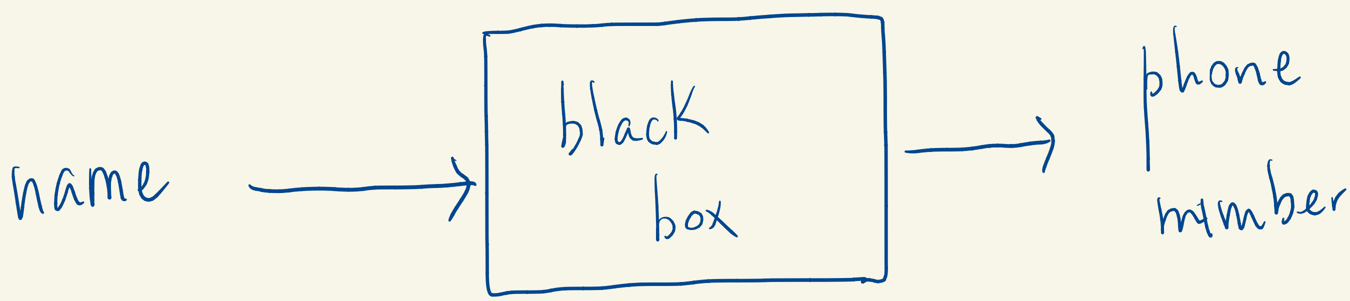
from the set of names

to the set of phone numbers

in set theory : $f : A \to B$ $\boxed{f \subseteq A \times B}$



we can also think of $f : A \to B$ as a means / a way / an algorithm which turns an input a into an output $b = f(a)$



an interesting question of temporality or dynamics.

Example : G - sets for G a group

a G - Set is a set A equipped with an $\boxed{\text{action}}$

$$\begin{array}{ccc} G \times A & \longrightarrow & A \\ (g, a) & \longmapsto & g \cdot a \end{array}$$

a particular case of a presheaf category

$$e \cdot a = a$$

$$g_1 \cdot (g_2 \cdot a) = (g_1 \cdot g_2) \cdot a$$

the action

in G

the action

G-Set is the category of G-sets and homomorphisms between them

a function $f : A \longrightarrow B$ such that $f(g \cdot a) = g \cdot f(a)$

for all g in G.

Example

interestingly, the category $\boxed{\text{Top}}$ of topological spaces and continuous functions is <u>not</u> cartesian closed.

however the subcategory of $\boxed{\text{compactly generated}}$ topological spaces is cartesian closed.

<u>def</u>. a topological space A is compactly generated when $\forall F \subseteq A$

$\left(\forall K \subseteq A \text{ compact } \left(F \cap K \underset{\text{set}}{\text{closed}} \right) \right) \Rightarrow F \text{ is } \underset{\text{set}}{\text{closed}}$

Claim :

① suppose given a cartesian closed category $\mathscr{C}$

and an object $[\alpha]$ in $\mathscr{C}$

for every type variable $\alpha \in TVar$.

then we can associate an object $[A]$

of the category $\mathscr{C}$ to any type A

of the simply-typed λ-calculus,

by structural induction on the type:

$[\alpha]$ has been already chosen

$$[A \times B] = [A] \times [B]$$

$$[A \Rightarrow B] = [A] \Rightarrow [B]$$

② to every λ-term M

of type

(*) $\boxed{x_1 : A_1, \ldots, x_k : A_k \vdash M : B}$

we can associate a morphism

$$\boxed{[A_1] \times \ldots \times [A_k] \xrightarrow{[M]} [B]}$$

in the cartesian-closed category.

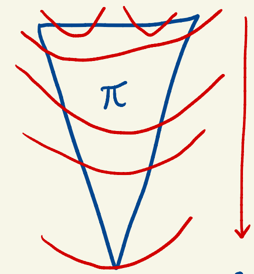
How do we do that?

In fact we do not interpret M "directly"

but the $\boxed{\text{derivation tree}}$

which establishes that (*)

The construction / interpretation works by *structural* induction on the derivation tree π ;



$x_1 : A_1 \ldots x_k : A_k \vdash M : B$

then show that all the derivation trees which establish that $(*)$ induce the same morphism $[M]$.

Derivation trees : $\boxed{6 \text{ rules}}$

(1) $$\boxed{\dfrac{}{x : A \vdash x : A}\ \text{Var}}$$

?

$$[A] \xrightarrow{id_{[A]}} [A]$$

② Lambda / Var rule

$$\frac{x : A,\ x_1 : A_1, \ldots, x_k : A_k \vdash M : B}{x_1 : A_1, \ldots, x_k : A_k \vdash \lambda x. M : A \Rightarrow B}$$

we suppose by induction that we have morphism

$$[A] \times [A_1] \times \ldots \times [A_k] \xrightarrow{[M]} [B]$$

we obtain a morphism

$$\begin{array}{ccc} [A_1] \times \ldots \times [A_k] & \xrightarrow{[\lambda x. M]} & [A \Rightarrow B] \\ & & \| \\ & & [A] \Rightarrow [B] \end{array}$$

defined using the adjunction

$$[A] \times \_ \dashv [A] \Rightarrow \_$$

in the category $\mathcal{C}$

$$[\lambda x. M] = \Phi_{[A],\,[B],\,[A_1]\times\dots\times[A_k]}\big([M]\big)$$

$$\Phi_{A,B,X} : \underbrace{\mathscr{C}(A \times X, B)}_{\text{a set of morphisms}} \xrightarrow{\cong} \underbrace{\mathscr{C}(X, A \Rightarrow B)}_{\text{a set of morphisms}}$$

a bijection

I stress here that the construction only uses the fact that

$$A \times - \dashv A \Rightarrow -$$

③ the application rule:

$$\boxed{\dfrac{\Gamma \vdash M : A \Rightarrow B \qquad \Delta \vdash P : A}{\Gamma, \Delta \vdash \mathrm{App}(M, P) : B}}$$

$\Gamma = x_1 : C_1, \ldots, x_p : C_p$

$\Delta = y_1 : D_1, \ldots, y_q : D_q$

$$\underbrace{[\Gamma]}_{} = [C_1] \times \ldots \times [C_p] \qquad \text{(notation)}$$

the interpretation of the context Γ.
in the cartesian closed category $\mathcal{C}$

By induction, we defined two morphisms

$$\boxed{\begin{array}{l} [\Gamma] \xrightarrow{\quad [M] \quad} [A \Rightarrow B] \\[1em] [\Delta] \xrightarrow{\quad [P] \quad} [A] \end{array}}$$

<u>Question</u>: how shall we interpret

$$App(M, P)$$

as a morphism

$$\underbrace{[\Gamma] \times [\Delta]}_{[\Gamma, \Delta]} \xrightarrow{\quad [App(M, P)] \quad} [B]$$

(**)

$$\boxed{[\Gamma] \times [\Delta] \xrightarrow{\quad [M] \times [N] \quad} [A \Rightarrow B] \times [A]}$$

here we use the fact that
$\times : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a functor

in any cartesian category $\mathcal{C}$

which means that given two maps

$$X \xrightarrow{f} Y \qquad U \xrightarrow{g} V$$

we can construct a map

$$X \times U \xrightarrow{f \times g} Y \times V.$$

We then post compose (**) with the map

$$\boxed{[A \Rightarrow B] \times [A] \xrightarrow{\mathrm{eval}_{[A],[B]}} [B]}$$

where

$$\mathrm{eval}_{X,Y} : (X \Rightarrow Y) \times X \longrightarrow Y$$

is the structural morphism of the cartesian closed category

For every pair X, Y of objects of $\mathcal{C}$:

$$\mathrm{id}_{X \Rightarrow Y} : X \Rightarrow Y \longrightarrow X \Rightarrow Y$$

in the cartesian closed category

we can apply

"decurrify"

$$\Phi^{-1}_{X, Y, X \Rightarrow Y} : \mathcal{C}(X \Rightarrow Y, X \Rightarrow Y) \longrightarrow \mathcal{C}(X \times (X \Rightarrow Y), Y)$$

to this map $\mathrm{id}_{X \Rightarrow Y}$

in order to a morphism:

$$X \times (X \Rightarrow Y) \longrightarrow Y$$

which we can then precompose with

permutation

$$(X \Rightarrow Y) \times X \longrightarrow X \times (X \Rightarrow Y)$$

From this, we obtain what we call the evaluation map

$$(X \Rightarrow Y) \times X \xrightarrow{\text{eval}_{X,Y}} Y$$

this enables us to interpret the derivation tree as

$$\begin{tikzcd}
{[\Gamma] \times [\Delta]} \arrow[r, "{[M] \times [P]}"] \arrow[dr, bend right, "{[App(M,P)]}"'] & {[A \Rightarrow B] \times [A]} \arrow[d, "{\text{eval}_{[A],[B]}}"] \\
& {[B]}
\end{tikzcd}$$

the other three rules:

weakening $\dfrac{\Gamma, \Delta \vdash M : B}{\Gamma, x : A, \Delta \vdash M : B}$

contraction $\dfrac{\Gamma, x : A, y : A, \Delta \vdash M : B}{\Gamma, z : A, \Delta \vdash M[x, y = z] : B}$

exchange $\dfrac{\Gamma, x : A, y : B, \Delta \vdash M : C}{\Gamma, y : B, x : A, \Delta \vdash M : C}$

will be interpreted by pre composition.

weakening:

Suppose by induction that we have a morphism:

$$[\Gamma] \times [\Delta] \xrightarrow{[M]} [B]$$

then we use the projection map

$$[\Gamma] \times [A] \times [\Delta] \xrightarrow{\pi_{13}} [\Gamma] \times [\Delta]$$

available in any cartesian category $\mathcal{C}$.

$$\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "\pi_1"'] \arrow[d, dashed, "\pi_{13}"] \arrow[dr, "\pi_3"] & \\
X & X \times Z \arrow[l] \arrow[r] & Z
\end{tikzcd}$$

another way to think about it

$$Y \xrightarrow[\text{unique map}]{} 1$$

$\boxed{1 \text{ is \underline{terminal}}}$

$$\begin{array}{ccc} X \times Y \times Z & \longrightarrow & X \times 1 \times Z \\ & & \wr\Vert \\ & & X \times Z \end{array}$$

<u>exercise</u> show that 1 is neutral for the cartesian in the sense that $A \times 1 \cong A$

<u>exercise</u>:

show that we obtain the same morphism $\pi_{1,3}$

$$X \times Y \times Z \longrightarrow X \times Z$$

we interpret the derivation tree

$$\left.\dfrac{\dfrac{\pi}{\Gamma, \Delta \vdash M : B}}{\Gamma, x : A, \Delta \vdash M : B}\right\} \text{derivation tree}$$

as the composite:

$$\begin{tikzcd}
{[\Gamma] \times [A] \times [\Delta]} \arrow[r, "\pi_{13}"] \arrow[dr, bend right] & {[\Gamma] \times [\Delta]} \arrow[d, "{[M]}"] \\
& {[B]}
\end{tikzcd}$$

the other two rules

work in the same way

Contraction uses the diagonal

$$\Delta_A : A \longrightarrow A \times A$$

available in any cartesian category

$$\begin{tikzcd}
{[\Gamma] \times [A] \times [\Delta]} \arrow[r, "\Delta_A"] \arrow[dr, bend right, "{[M[x,y=z]]}"'] & {[\Gamma] \times [A] \times [A] \times [\Delta]} \arrow[d, "{[M]}"] \\
& {[B]}
\end{tikzcd}$$

provides the interpretation

of the derivation tree

$$\dfrac{\dfrac{\pi}{\Gamma, x : A, y : A, \Delta \vdash M : B}}{\Gamma, z : A, \Delta \vdash M[x,y=z] : B}$$

Similarly for the exchange rule

$$A \times B \xrightarrow{\gamma_{A,B}} B \times A$$

available in any cartesian category.

Thm the interpretation of the λ-term M provides an invariant of the λ-term modulo β-reduction η-expansion

$$\boxed{M \simeq_{\beta\eta} N \text{ then } [M] = [N]}$$

in a given context with a given type

$$M \xrightarrow[\beta\eta]{} N \qquad \Gamma = x_1 : A_1, \dots, x_k : A_k$$

(1) if $\Gamma \vdash M : B$

then $\Gamma \vdash N : B$ (subject reduction)

and moreover

(2) M and N have the same interpretation in the sense that:

$$[A_1] \times \dots \times [A_n] \xrightarrow{[M] = [N]} [B]$$