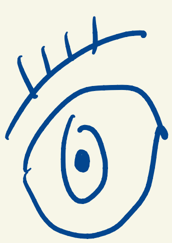
Lamba-Calculs & categories 11

14 décembre 2020

adjunctions in string diagrams

invariance par action
de groupe
de l'interprétation
d'un λ-terme clos
dans la catégorie des $\boxed{G\text{-ensembles}}$
est lié à la propriété
de paramétricité
de Reynolds

 Emmy Noether

invariance par action de groupe
des lois de la physique.

Adjunctions in string diagrams

pasting diagrams in 2-categories $\longleftrightarrow$ string diagrams

$$G \circ F \overset{\theta}{\Longrightarrow} H$$

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow\theta"] & \\ \mathcal{A} \arrow[ur, "F"] \arrow[rr, bend right, "H"'] & {} & \mathcal{C} \end{tikzcd}$$

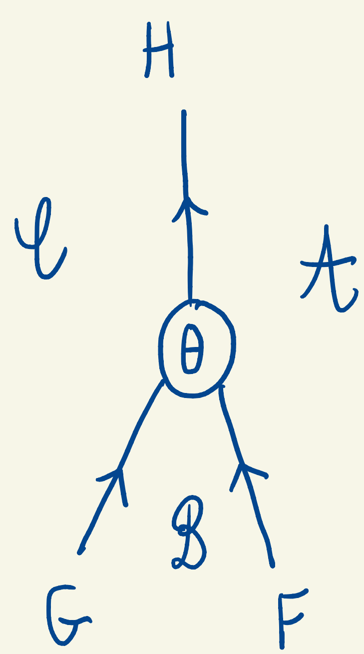
2-cell $\longmapsto$ 0-dim point

1-cell $\longmapsto$ 1-dim string

0-cell $\longmapsto$ 2-dim area zone

$\Big\downarrow$

$$\begin{tikzcd} \mathcal{C} & {} \arrow[d, phantom, "\Uparrow\theta"] & \mathcal{A} \arrow[ll, bend right, "H"'] \arrow[dl, "F"] \\ & \mathcal{B} \arrow[ul, "G"] & \end{tikzcd} \longmapsto$$



$$\begin{tikzcd} H \\ G \circ F \arrow[u, Rightarrow, "\theta"'] \end{tikzcd}$$

The plan today:

show that an adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

$L : \mathcal{A} \to \mathcal{B}$

$R : \mathcal{B} \to \mathcal{A}$

that is: a pair of functors L, R equipped with a family of bijections

$$\phi_{A,B} : \mathcal{A}(LA, B) \xrightarrow{\cong} \mathcal{B}(A, RB)$$

natural in A and B is

the same thing as a pair of functors $L : \mathcal{A} \to \mathcal{B}$

$R : \mathcal{B} \to \mathcal{A}$

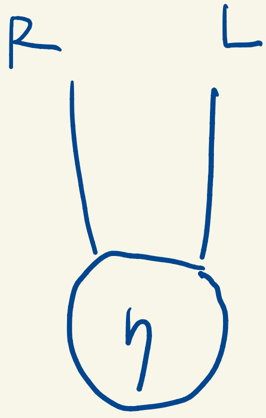
equipped with two natural transformations

$$\begin{cases} \eta : \mathrm{Id}_{\mathcal{A}} \Longrightarrow R \circ L & \boxed{\text{unit}} \\ \varepsilon : L \circ R \Longrightarrow \mathrm{Id}_{\mathcal{B}} & \boxed{\text{counit}} \end{cases}$$

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend left, "\mathrm{Id}_{\mathcal{A}}"] \arrow[dr, "L"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{A} \\ & \mathcal{B} \arrow[ur, "R"'] & \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{A} \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\ \mathcal{B} \arrow[ur, "R"] \arrow[rr, bend right, "\mathrm{Id}_{\mathcal{B}}"'] & \ & \mathcal{B} \end{tikzcd}$$

in string diagrams:

$$\begin{array}{c} R \circ L \\ \eta \Big\Uparrow \\ \mathrm{Id}_{\mathcal{A}} \end{array}$$



$$\begin{array}{c} \mathrm{Id}_{\mathcal{B}} \\ \varepsilon \Big\Uparrow \\ L \circ R \end{array}$$

satisfying the triangular laws:

$$\begin{tikzcd} A \arrow[rr, "Id_A"] \arrow[dr, "L"'] & & A \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\ & B \arrow[ur, "R"] \arrow[rr, "Id_B"'] \arrow[u, phantom, "\Downarrow\eta"] & & B \end{tikzcd} \overset{1}{=} \begin{tikzcd} A \arrow[r, bend left, "L"] \arrow[r, bend right, "L"'] \arrow[r, phantom, "\Downarrow Id_L"] & B \end{tikzcd}$$

$$\begin{tikzcd} & A \arrow[rr, "Id_A"] \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\varepsilon"] & & A \\ B \arrow[ur, "R"] \arrow[rr] & & B \arrow[ur, "R"'] \arrow[u, phantom, "\Downarrow\eta"] & \end{tikzcd} \overset{2}{=} \begin{tikzcd} B \arrow[r, bend left, "R"] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\Downarrow Id_R"] & A \end{tikzcd}$$

The triangular laws mean that the diagrams

(1)
$$\begin{tikzcd} LA \arrow[r, "L\eta_A"] \arrow[rr, bend right, "id_{LA}"'] & LRLA \arrow[r, "\varepsilon_{LA}"] & LA \end{tikzcd}$$

(2)
$$\begin{tikzcd} RB \arrow[r, "\eta_{RB}"] \arrow[rr, bend right, "id_{RB}"'] & RLRB \arrow[r, "R\varepsilon_B"] & RB \end{tikzcd}$$

commute for all objects

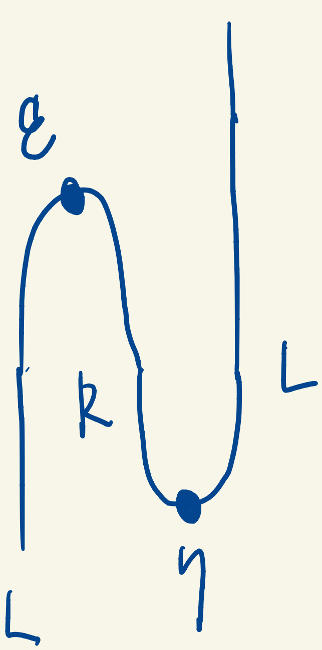
A of $\mathcal{A}$

B of $\mathcal{B}$

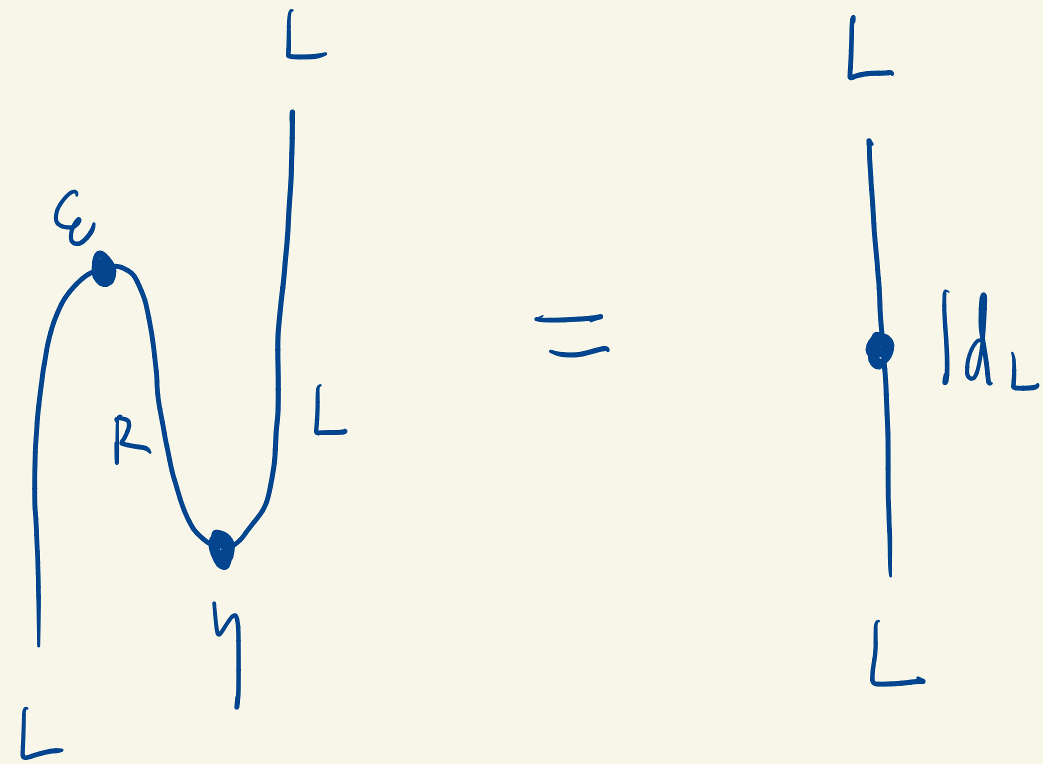
The equality between pasting diagrams:

$$
\begin{tikzcd}
A \arrow[rr, "\mathrm{Id}_A"] \arrow[dr, "L"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & A \arrow[dr, "L"] & \\
& B \arrow[ur, "R"] \arrow[rr, "\mathrm{Id}_B"'] & {} \arrow[u, phantom, "\Downarrow\varepsilon"] & B
\end{tikzcd}
\overset{1}{=}
\begin{tikzcd}
A \arrow[r, bend left=50, "L"] \arrow[r, bend right=50, "L"'] \arrow[r, phantom, "\Downarrow \mathrm{Id}_L"] & B
\end{tikzcd}
$$

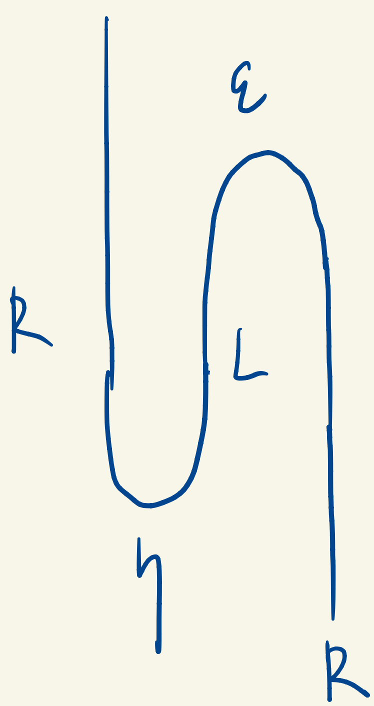
become in string diagrams



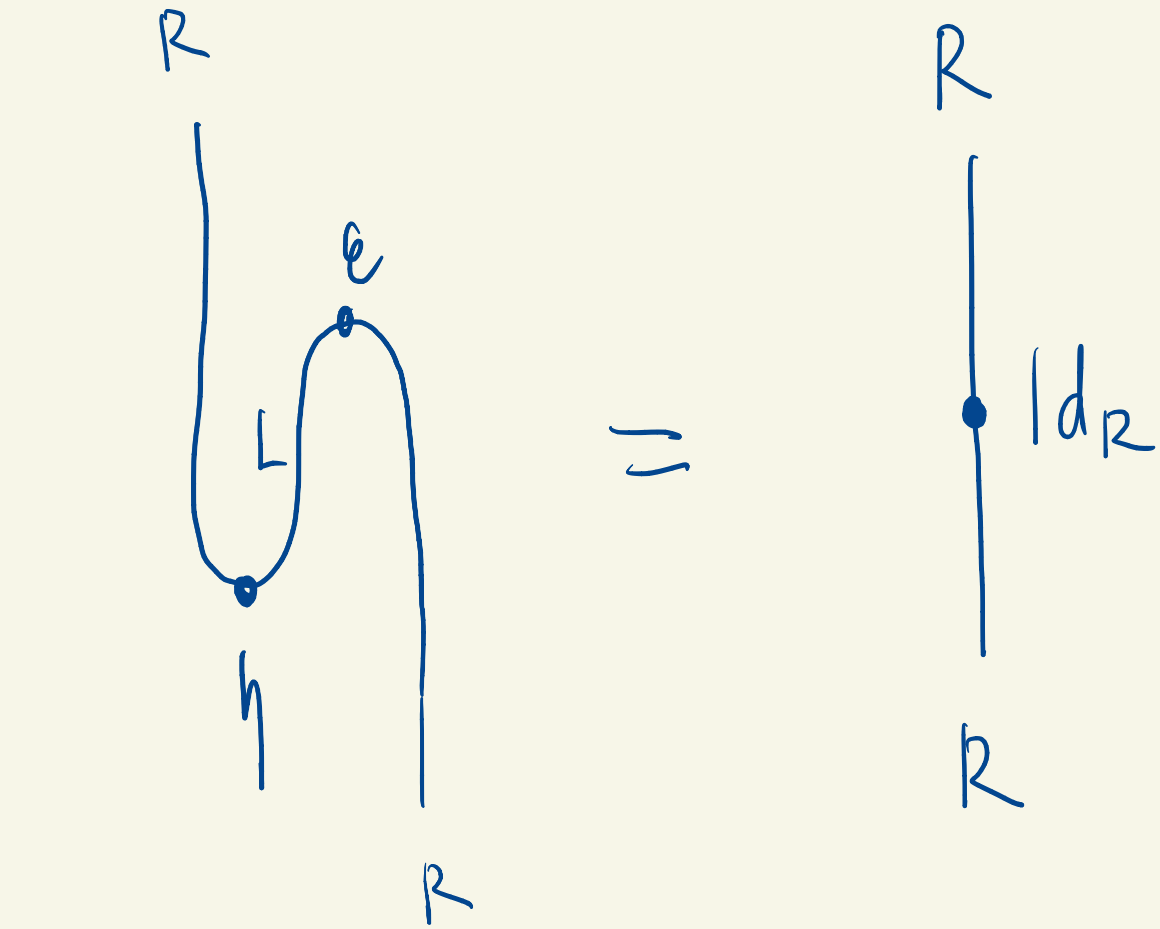
$$
\begin{array}{c}
L \\
\Uparrow \; \varepsilon \circ L \\
L \circ R \circ L \\
\Uparrow \; L \circ \eta \\
L
\end{array}
$$



Similarly, the other triangular law means that:

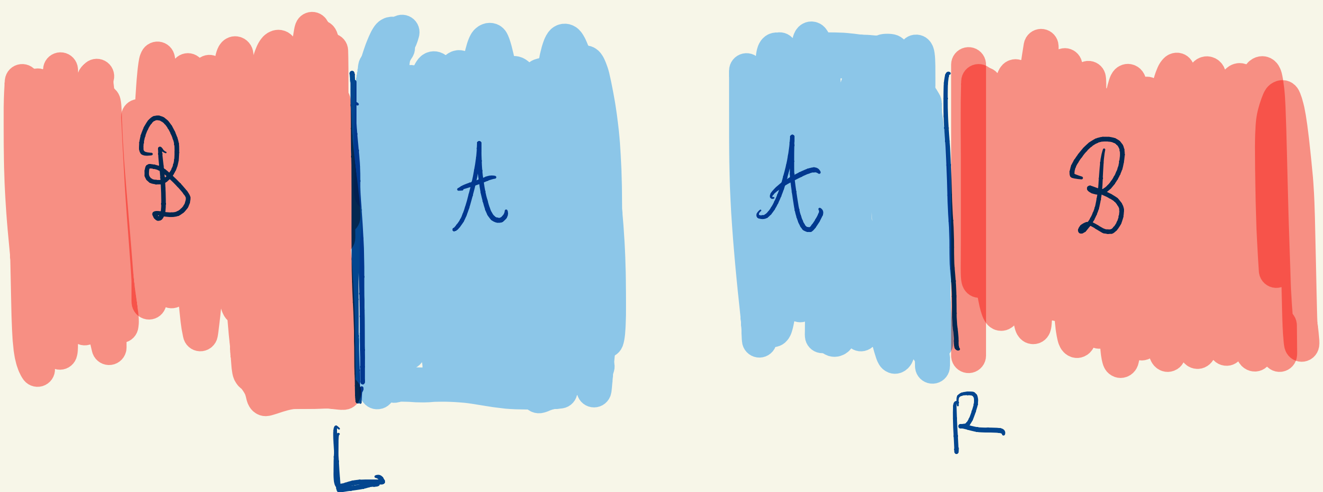


$$R \xRightarrow{\eta \circ R} R \circ L \circ R \xRightarrow{R \circ \varepsilon} R$$

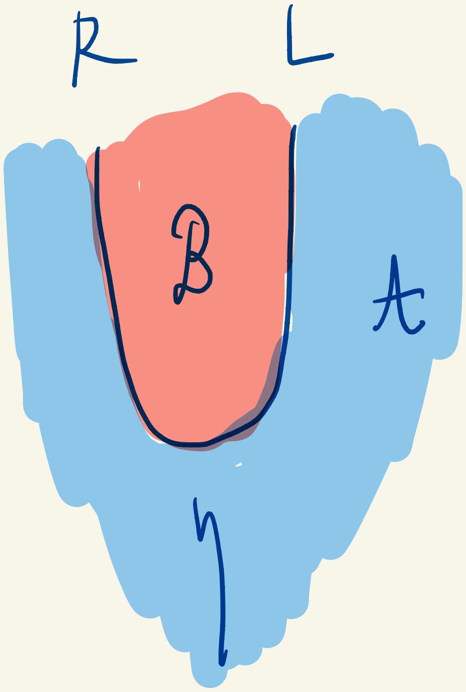


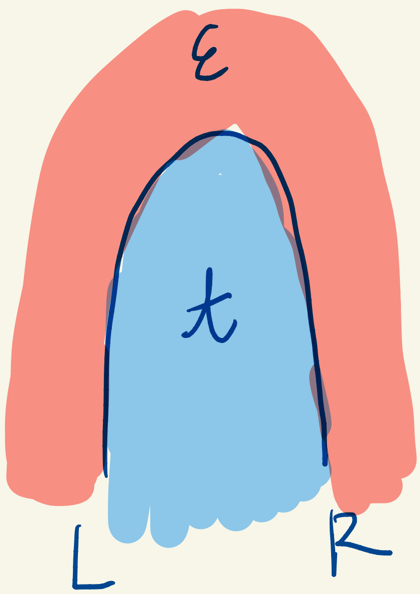
So an adjunction is just a pair of functors

$$L : \mathcal{A} \longrightarrow \mathcal{B}$$
$$R : \mathcal{B} \longrightarrow \mathcal{A}$$

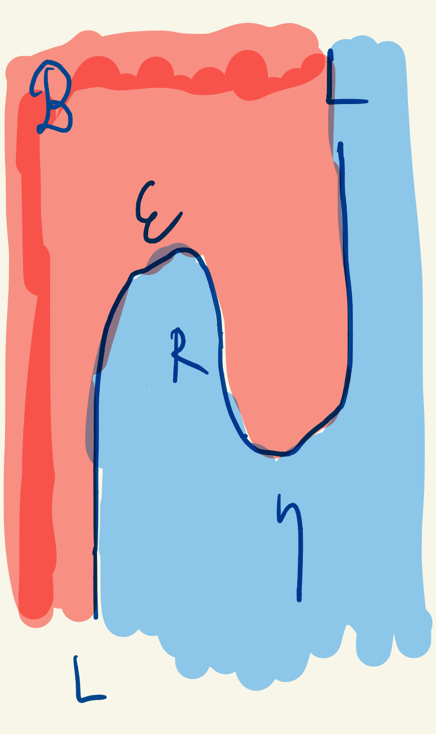


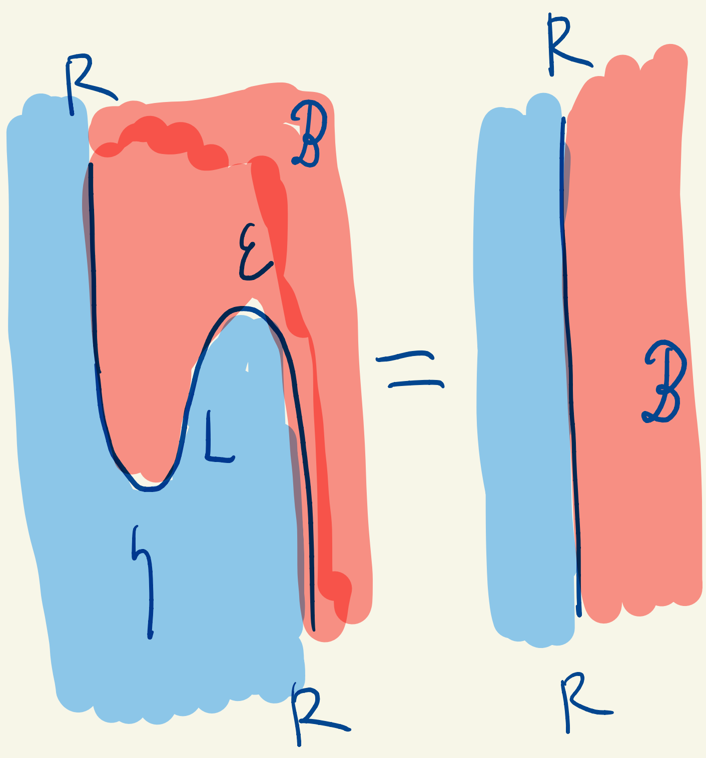
equipped with natural transformations





satisfying the zigzag equations:



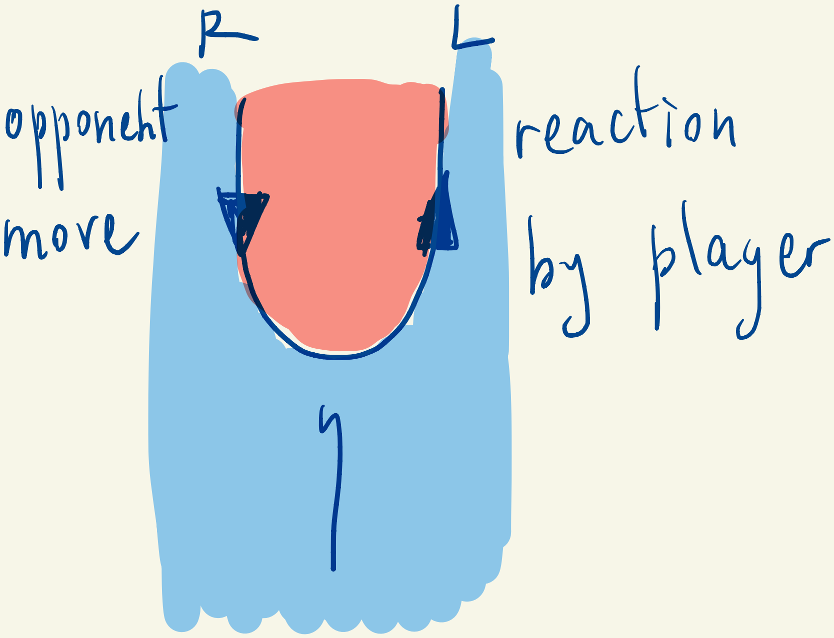


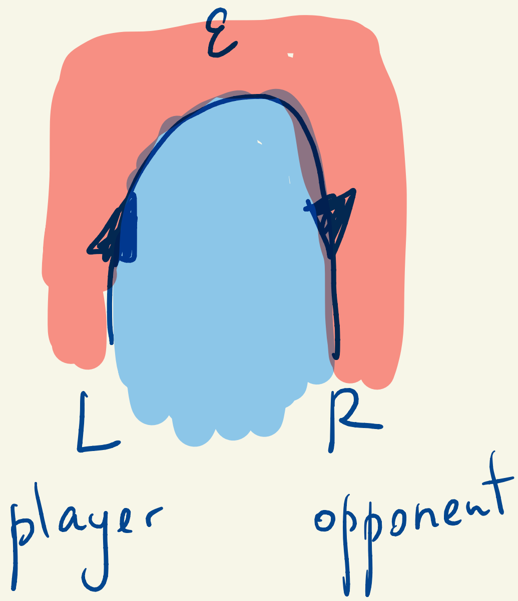
For instance can be used to give a "dynamic" and interactive description of proofs as strategies

<u>where</u>:

the right adjoint R is a move played by opponent

the left adjoint L is a move played by player





the functors L and R

seen as moves in a game

are induced from the adjunction

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{C}^{op} \arrow[l, bend left, "R"]
\end{tikzcd}$$

associated to $\boxed{\text{any negation } \neg}$

in a cartesian category

What is a negation?

it is a functor

$$\neg : \mathcal{C}^{op} \longrightarrow \mathcal{C}$$

$$\boxed{\dfrac{A \xrightarrow{f} B}{\neg B \xrightarrow{\neg f} \neg A}}$$

on a cartesian category $\mathcal{C}$

such that there is

a natural bijection

$$\mathcal{C}(A \times B, \neg C) \cong \mathcal{C}(B, \neg(C \times A))$$

Example:

every object $\perp$ of a

cartesian closed defines

such a negation as follows

$$A \longmapsto (A \Rightarrow \perp\!\!\!\perp)$$

<u>indeed</u>

there is a natural bijection

between

$$\mathscr{C}(A \times B, C \Rightarrow \perp\!\!\!\perp)$$

$\phi^{-1}_{A\times B, \perp\!\!\!\perp}$

and

$$\mathscr{C}(C \times A \times B, \perp\!\!\!\perp)$$

$\phi_{B, A\times C}$

and

$$\mathscr{C}(B, (A \times C) \Rightarrow \perp\!\!\!\perp)$$

So we have a natural bijection

$$\mathcal{C}(A \times B, C \Rightarrow \bot\!\!\!\bot) \cong \mathcal{C}(B, (C \times A) \Rightarrow \bot\!\!\!\bot)$$

<u>Fact</u>: every negation

$$\neg : \mathcal{C}^{op} \longrightarrow \mathcal{C}$$

is adjoint to "itself"

$$\neg : \mathcal{C} \longrightarrow \mathcal{C}^{op}$$

<u>Proof</u>:

there is a natural bijection between

$$\mathcal{C}(A, \neg B) \quad \text{and} \quad \mathcal{C}(B, \neg A)$$

because both are in bijection with

$$\mathcal{C}(A \times B, \perp\!\!\!\perp)$$

where $\perp\!\!\!\perp$ is defined as $\neg 1$.

(1 is the terminal object)

$$
\begin{aligned}
\mathcal{C}(A, \neg B) &\overset{\text{(neutrality)}}{\cong} \mathcal{C}(A, \neg(B \times 1)) \\
\text{(negation)} \quad &\cong \mathcal{C}(B \times A, \neg 1) \\
\text{(symmetry)} \quad &\cong \mathcal{C}(A \times B, \neg 1) \\
\text{(negation)} \quad &\cong \mathcal{C}(B, \neg(A \times 1)) \\
\text{(neutrality)} \quad &\cong \mathcal{C}(B, \neg A)
\end{aligned}
$$

so we have that:

$$\underbrace{\mathcal{C}(A, \neg B)}_{\mathcal{C}(A,\, RB)} \cong \underbrace{\mathcal{C}(B, \neg A)}_{\underbrace{\mathcal{C}^{op}(\neg A,\, B)}_{\mathcal{C}^{op}(LA,\, B)}}$$

$$\mathcal{C}(A, RB) \overset{\text{adjunction}}{=} \mathcal{C}^{op}(LA, B)$$

$$R = \neg : \mathcal{C}^{op} \longrightarrow \mathcal{C}$$
$$B \longmapsto \neg B$$

$$L = \neg : \mathcal{C} \longrightarrow \mathcal{C}^{op}$$
$$A \longmapsto \neg A$$

Now, an adjunction is the same thing as a pair of functors

$$L : \mathcal{C} \xrightarrow{\text{negation}} \mathcal{C}^{op}$$

$$R : \mathcal{C}^{op} \xrightarrow{\text{negation}} \mathcal{C}$$

equipped with two natural transformations:

$$\eta : Id_{\mathcal{C}} \Longrightarrow R \circ L$$

$$\eta_A : A \longrightarrow \neg\neg A$$

we recover the logical principle
that every formula A
implies its double negation $\neg\neg A$

$$\varepsilon : L \circ R \Longrightarrow Id_{\mathcal{C}^{op}}$$

same logical principle
reflected in $\mathcal{C}^{op}$

in this logical reading

$A \longrightarrow B$ in $\mathcal{C}$

means that A implies B

(the territory of the prover)

$A \longrightarrow B$ in $\mathcal{C}^{op}$

means that A refutes B

(the territory of the refutator)

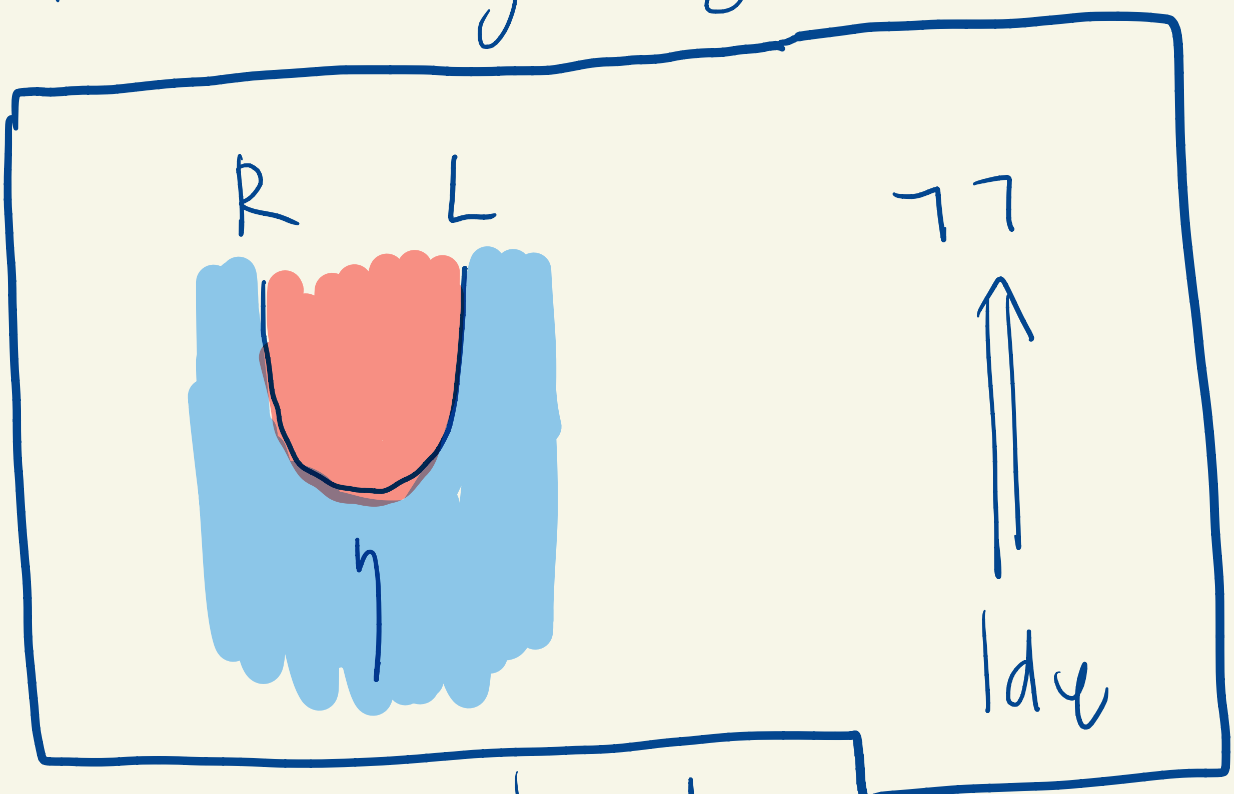
"territory" = the category $\mathcal{C}$
for prover
the category $\mathcal{C}^{op}$
for refutator

in that reading, the proof

$$A \vdash \neg\neg A$$

is interpreted as

the string diagram



R is understood

as an opponent move

L is understood

as a Player move.

this has to do with the notion of <u>continuations</u>

in programming:

$$a : A \vdash \lambda k. k a \ : \neg\neg A$$

$$\underbrace{(A \Rightarrow \perp\!\!\!\perp) \Rightarrow \perp\!\!\!\perp}$$

k has type

$$A \Rightarrow \perp\!\!\!\perp$$

k is called a continuation

$\perp\!\!\!\perp$ is called a pole.

$$\lambda k. \ \mathrm{App}(k, a)$$

In a cartesian closed category $\mathscr{C}$ we have an adjunction

$$A \times - \dashv A \Rightarrow -$$

for every object A of $\mathscr{C}$.

The $\boxed{\text{counit}}$ of the adjunction:

$$LRB = \underbrace{A \times \underbrace{(A \Rightarrow B)}_{R}}_{L}$$

$$\varepsilon_B : A \times (A \Rightarrow B) \longrightarrow B$$

evaluation map which interprets the λ-term:

$$a : A,\ f : A \Rightarrow B \vdash \mathrm{App}(f, a) : B$$

the unit of the adjunction

$$RLB = A \Rightarrow (A \times B)$$

$$\eta_B : B \longrightarrow A \Rightarrow (A \times B)$$

which interprets the λ-term

$$b : B \vdash \lambda a. \underbrace{(\langle a, b \rangle)}_{\text{a pair of } \lambda\text{-terms.}} : A \Rightarrow (A \times B)$$

<u>Exercise</u>: interpret the triangular laws of any cartesian closed categories as equations in the λ-calculus.