Modèles des langages de programmation (MPRI 2.2)

Stable and linear functions between clique domains

— mardi 13 octobre 2020 —

In this problem, we call graph $A = (V, E)$ a set V of vertices equipped with a reflexive and symmetric relation $E \subseteq V \times V$ describing the edges. Recall that by a symmetric and reflexive relation, we mean that

$$\forall a \in V, \qquad (a, a) \in E$$

$$\forall a \in V, \forall a' \in V, \qquad (a, a') \in E \quad \Rightarrow \quad (a', a) \in E.$$

A clique of a graph A is defined as a subset $u \subseteq V$ such that

$$\forall a \in u, \forall a' \in u, \qquad (a, a') \subseteq E.$$

We recall that a continuous function is monotone by definition.

Question 1. Show that the set of cliques of A ordered by inclusion

$$u \leq_A v \qquad \overset{def}{\iff} \qquad u \subseteq v$$

defines a domain $(\mathrm{D}_A, \leq_A)$.

Question 2. Show that a continuous function

$$f \quad : \quad \mathrm{D}_A \quad \longrightarrow \quad \mathrm{D}_B$$

is entirely determined by its restriction

$$!A \quad \longrightarrow \quad \mathrm{D}_A \quad \xrightarrow{f} \quad \mathrm{D}_B$$

to the set (noted $!A$) of the finite cliques of the graph A.

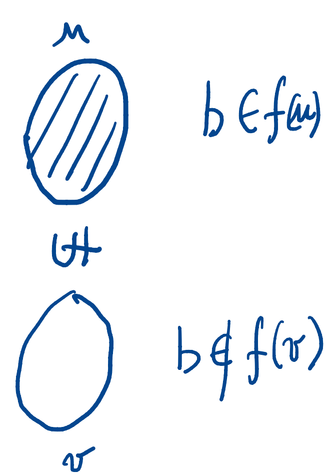
Question 3. From this, deduce the existence of a bijection between the set of continuous functions from D_A to D_B and the set of monotone functions from $!A$ to D_B — and describe how the bijection works.

Question 4. For every continuous function $f : \mathrm{D}_A \to \mathrm{D}_B$ one defines the set

$$\mathrm{Tr}(f) \quad \subseteq \quad !A \times B$$

of pairs (u, b) which satisfy the two properties below:

- $b \in f(u)$,
- $b \notin f(v)$ for every clique $v \in \mathrm{D}_A$ strictly included in u.



Show that the equality

$$f(u) \quad = \quad \{ \quad b \in B \quad | \quad \exists\, v \ \in !A, \quad v \leq_A u \quad \text{et} \quad (v, b) \in \mathrm{Tr}(f) \quad \}.$$

holds for every clique u of the graph A.

Question 1. A a graph

- D_A contains a minimum element $\perp$
  least

  defined as the empty clique

- suppose given a filter $\mathcal{F}$ of D_A.
  we want to show that $\bigvee \mathcal{F}$ exists (in D_A).
  the least upper bound of $\mathcal{F}$.

(a) consider the set $\bigcup \mathcal{F} = \{ a \mid \exists u \in \mathcal{F}, a \in u \}$

u is a clique

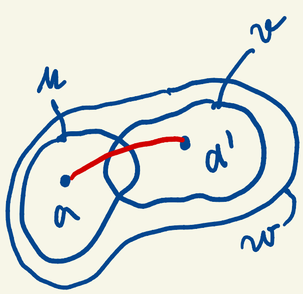
(b) we start by showing that $\bigcup \mathcal{F}$ is a clique.

for all $a \in \bigcup \mathcal{F}$ and $a' \in \bigcup \mathcal{F}$
we want to show that (a, a') is an edge of A.

by definition of $\bigcup \mathcal{F}$, there exists

- $u \in \mathcal{F}$ such that $a \in u$
- $v \in \mathcal{F}$ such that $a' \in v$

$\boxed{\mathcal{F} \text{ is a filter}}$, hence there exists $w \in \mathcal{F}$
such that $u \subseteq w$ and $v \subseteq w$.



w is an element of D_A, hence a clique of A.

$$\left.\begin{array}{ll} a \in u & \text{hence } a \in w \\ b \in v & \text{hence } b \in w \end{array}\right\} \text{ where } w \text{ is a clique.}$$

we conclude that (a, a') is an edge of A.

since the property holds for all $a, a' \in \bigcup \mathcal{F}$
we conclude that $\bigcup \mathcal{F}$ is a clique of A

(c) clearly $\bigcup \mathcal{F}$ is an element of D_A which defines an upper bound of $\mathcal{F}$ in D_A.

$\forall u \in \mathcal{F}, \quad u \subseteq \bigcup \mathcal{F}$.

Moreover, if w is an upper bound of $\mathcal{F}$ then $\forall u \in \mathcal{F}, \quad u \leq_A w$ hence $u \subseteq w$

hence $\bigcup \mathcal{F} \subseteq w$

This establishes that $\boxed{\bigcup \mathcal{F} \leq_A w}$

We conclude that $\bigcup \mathcal{F}$ defines the least upper bound of $\mathcal{F}$ in the partial order D_A.

We conclude that D_A is a domain.

<u>Question 2</u>. Suppose given two continuous function

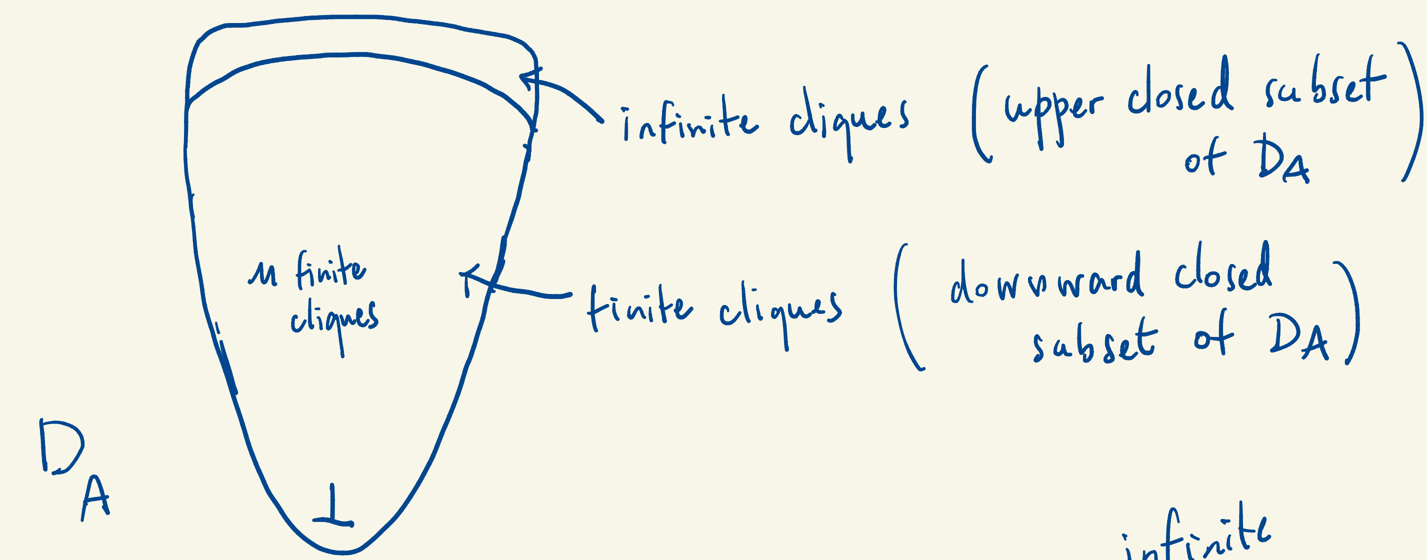
$$\begin{tikzcd} D_A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D_B \end{tikzcd}$$

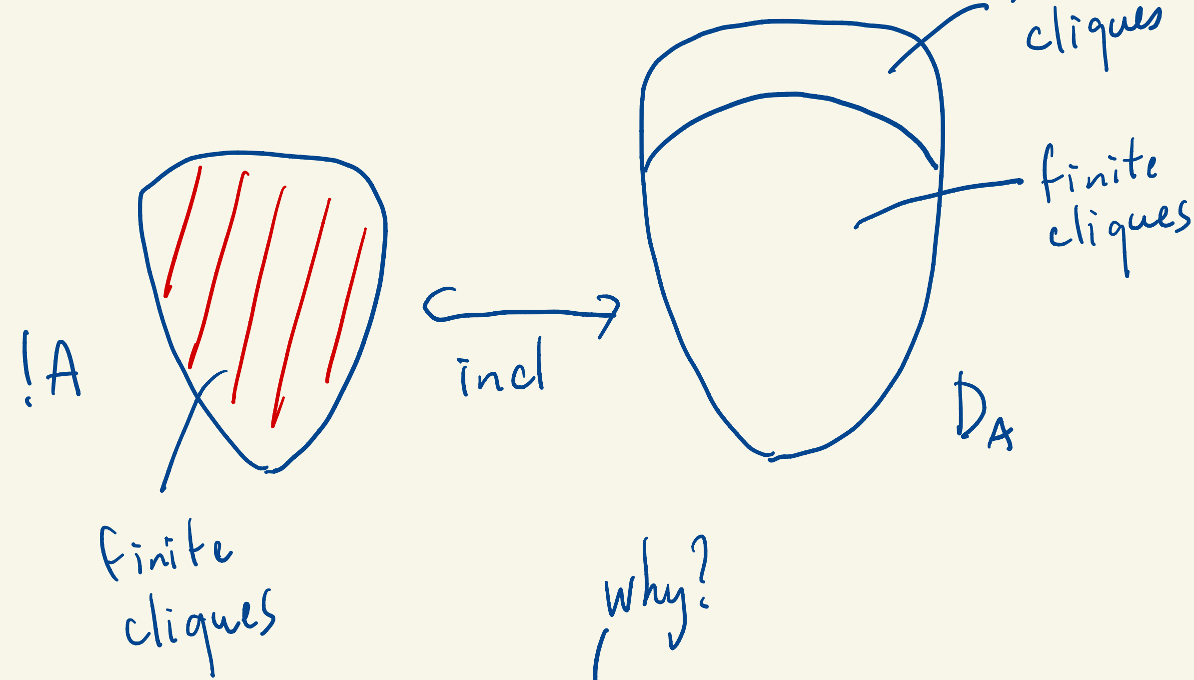
such that the composites

$$\begin{tikzcd} !A \arrow[r, "\mathrm{incl}"] & D_A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D_B \end{tikzcd}$$

are equal: $\quad f \circ \mathrm{incl} = g \circ \mathrm{incl} = h$

We want to show that $f = g$ in that case.





why?

$$\begin{tikzcd}
D_A \arrow[r, dashed, "f=g"] & D_B \\
!A \arrow[u, "\mathrm{incl}"] \arrow[ur, "h"'] &
\end{tikzcd}$$

Key observation: any clique w of A

induces a filter $\mathcal{F}_w$ of finite cliques:

$$\mathcal{F}_w = \{ u \text{ finite clique} \mid u \subseteq w \}$$

such that

$$\boxed{\boxed{w = \bigvee \mathcal{F}_w}}$$

<u>hence</u>: We know by continuity of $f : D_A \to D_B$

that $\quad f(w) = f\left(\bigvee \mathcal{F}_w\right) = \bigvee f(\mathcal{F}_w)$

From this we conclude that

$$f(w) = \bigvee h(\mathcal{F}_w) \underset{\text{by a similar argument.}}{=} g(w)$$

and thus that $f = g$.

<u>Question 3</u>. we want to deduce a bijection between ① the set of continuous functions $D_A \to D_B$

② the set of monotone functions $!A \to D_B$

We already defined a translation from 1 to 2 :

every continuous function $D_A \longrightarrow D_B$

induces a monotone function $!A \longrightarrow D_B$

by precomposing with the monotone function $!A \xrightarrow{\text{incl}} D_A$.

<u>Conversely</u>: given a monotone function

$$!A \xrightarrow{\ h\ } D_B$$

we want to show that there exists a $\boxed{\underline{\text{unique}}}$

continuous function

$$D_A \xrightarrow{\ f\ } D_B$$

making the diagram below commute:

$$\begin{tikzcd} D_A \arrow[r, "f"] & D_B \\ !A \arrow[u, "\text{incl}"] \arrow[ur, "\boxed{h}"'] \arrow[u, phantom, shift right=4ex, "(*)"] & \end{tikzcd}$$

Given $w \in D_A$, we define $\boxed{f(w)}$ as:

$$\boxed{f(w)} = \bigvee \underbrace{h(\mathcal{F}w)}_{\text{properly defined + it is a filter in } D_B}$$

because $\mathcal{F}w \subseteq {!A}$

What do we need to do at this stage?

we need to check that our definition of f induces (indeed) a continuous function and that f makes the diagram (*) commute.

- Show that $\boxed{f}$ is montone.

$\boxed{v \leq_A w}$ implies $\mathcal{F}_v \subseteq \mathcal{F}_w$ hence $h(\mathcal{F}_v) \subseteq h(\mathcal{F}_w)$

from this follows that $\bigvee h(\mathcal{F}_v) \leq_B \bigvee h(\mathcal{F}_w)$

$$\boxed{f(v) \leq_B f(w)}$$

- Given a filter $\mathcal{F}$ of D_A, we want to check that

$$f(\bigvee \mathcal{F}) = \bigvee f(\mathcal{F})$$

$$\| \qquad\qquad\qquad \|$$

$$\boxed{\bigvee h(\mathcal{F}_{\bigvee \mathcal{F}})} \qquad \bigvee_{w \in \mathcal{F}} f(w) = \boxed{\bigvee_{w \in \mathcal{F}} h(\mathcal{F}_w)}$$

① remark: for each $w \in \mathcal{F}$, $\boxed{\mathcal{F}_w \subseteq \mathcal{F}_{\bigvee \mathcal{F}}}$ because $w \leq_A \bigvee \mathcal{F}$

$$h(\mathcal{F}_w) \subseteq h(\mathcal{F}_{\bigvee \mathcal{F}})$$

$$\bigvee_{w \in \mathcal{F}} h(\mathcal{F}_w) \leq \bigvee h(\mathcal{F}_{\bigvee \mathcal{F}})$$

$$\bigvee f(\mathcal{F}) \boxed{\leq} f(\bigvee \mathcal{F})$$

② how shall we prove

that $$\boxed{f(\bigvee \mathcal{F}) \leq \bigvee f(\mathcal{F})} \quad (**)$$

$$\bigvee h(\mathcal{F}_{\bigvee \mathcal{F}}) \boxed{\overset{?}{\leq}} \bigvee_{\boxed{w \in \mathcal{F}}} h(\mathcal{F}_w)$$

$\parallel$

$\leq$

for all $w \in \mathcal{F}$

$\mathcal{F}_w$ = the set of all finite cliques $u \subseteq w$

all finite cliques v u included in $\bigcup \mathcal{F}$ = the union of all cliques in $\mathcal{F}$.

$$\boxed{v \subseteq_{\text{finite}} \boxed{\bigcup \mathcal{F}}}$$

$u \subseteq_{\text{finite}} w$

$w \in \mathcal{F}$

$\boxed{\text{Claim}}$: any <u>finite clique</u> $v \subseteq_{finite} \bigcup \mathcal{F}$ is <u>included in an element</u> w of $\mathcal{F}$.

Suppose given an element a of v. (= a node a in the graph A)

by definition $a \in \bigcup \mathcal{F}$.

hence there exists $u_a \in \mathcal{F}$ such that $a \in u_a$

we use the fact that <u>$\mathcal{F}$ is a filter</u> to conclude that there exists an element $w \in \mathcal{F}$ such that $v \subseteq w$.

$\boxed{\text{by induction on the size of } v}$

<u>Prop</u>. given a non empty finite set $\{u_1, \dots, u_n\} \subseteq \mathcal{F}$ there exists $w \in \mathcal{F}$ such that

$$\forall i \in [1, \dots, n], \quad u_i \subseteq w.$$

this establishes that (**) holds and thus that f is continuous.

$$\begin{tikzcd} D_A \arrow[r, dashed, "\exists! f \text{ continuous}"] & D_B \\ !A \arrow[u, "\text{incl.}"] \arrow[ur, "h \text{ monotone}"'] & \end{tikzcd}$$

introduced by Gérard Berry
analyzed by JY Girard.

Question 5. Two cliques u and v of the graph A are compatible (notation: $u \uparrow v$) when there exists a clique w which contains both of them:

$$u \uparrow v \quad \overset{def}{\Longleftrightarrow} \quad \exists w. \quad u \leq w \quad \text{and} \quad v \leq w.$$

A continuous function $f : \mathrm{D}_A \to \mathrm{D}_B$ is called *stable* when

$$\boxed{\forall u, v \in \mathrm{D}_A, \qquad u \uparrow v \quad \Rightarrow \quad f(u \cap v) = f(u) \cap f(v).}$$

Suppose that f is stable. Show that (u, b) is an element of $\mathrm{Tr}(f)$ if and only if, for every clique v compatible with u, the equivalence below holds:

$$u \leq v \quad \Longleftrightarrow \quad b \in f(v).$$

Question 6. A continuous function $f : \mathrm{D}_A \to \mathrm{D}_B$ is called *linear* when it is stable and satisfies the two properties below:

$$\begin{array}{lrcl} (1) & f(\emptyset) & = & \emptyset \\ (2) \quad \forall u, v \in \mathrm{D}_A, \quad u \uparrow v \ \Rightarrow & f(u \cup v) & = & f(u) \cup f(v). \end{array}$$

Show that a stable function $f : \mathrm{D}_A \to \mathrm{D}_B$ is linear if and only if every element (u, b) of the trace of f is of the form $(\{a\}, b)$.

Question 7. Show that the set of linear functions from D_A to D_B, ordered as follows:

$$f \leq g \quad \overset{def}{\Longleftrightarrow} \quad \forall u \in \mathrm{D}_A, \quad f(u) \leq_B g(u).$$

defines a domain. We write $\mathrm{D}_A \multimap \mathrm{D}_B$ for the domain of linear functions just defined.

Question 8. Define a graph $A \multimap B$ such that the equality holds:

$$\mathrm{D}_{A \multimap B} \quad = \quad \mathrm{D}_A \multimap \mathrm{D}_B$$

Question 9. Let 1 denote the graph with a unique vertex $*$. Show that the trace of a stable function

$$f \quad : \quad \mathrm{D}_A \quad \longrightarrow \quad \mathrm{D}_1 = \{\bot, \top\}$$

is of the form

$$\mathrm{Tr}(f) \quad = \quad \{ \quad (u, *) \quad | \quad u \in U \quad \}$$

where U is a set of finite and pairwise incompatible cliques of A.

Question 10. The ordered set $!A$ of finite cliques of the graph A defines a graph (also noted $!A$) where two finite cliques u and v of the graph A are connected by an edge precisely when $u \cup v$ is a clique of A. Construct a bijection between the set of stable functions from D_A to D_B and the set of linear functions from $\mathrm{D}_{!A}$ to D_B.

Question 11. Show that

$$\mathrm{D}_{(!A) \multimap B} \quad = \quad \mathrm{D}_{!A} \multimap \mathrm{D}_B \quad = \quad \mathrm{D}_A \Rightarrow \mathrm{D}_B$$

where $\mathrm{D}_A \Rightarrow \mathrm{D}_B$ denotes the set of stable functions from D_A to D_B, equipped with the ordering relation

$$f \leq_s g \quad \overset{def}{\Longleftrightarrow} \quad \forall u, v \in \mathrm{D}_A, \quad u \leq_A v \Rightarrow f(v) \cap g(u) = f(u).$$

Show in particular that

$$f \leq_s g \quad \Longleftrightarrow \quad \mathrm{Tr}(f) \subseteq \mathrm{Tr}(g).$$