$*$-autonomous categories

The structure of the category of coherence spaces (3)

$A, \perp$ $\perp$ bottom (think of it as "false")

A general observation

Every pair of objects $A, \perp$ in a smcc $(\mathcal{C}, \otimes, \mathbf{1})$ comes with an identity

$$id_{A \multimap \perp} : A \multimap \perp \longrightarrow A \multimap \perp$$

which is transported by the bijection $\phi^{-1}_{A \multimap \perp, A, \perp}$ to the morphism

$$\mathrm{eval}_{A,\perp} : A \otimes (A \multimap \perp) \longrightarrow \perp$$

which becomes by precomposing with symmetry:

$$(A \multimap \perp) \otimes A \longrightarrow \perp$$

which is then transported by the bijection $\phi_{A \multimap \perp, A, \perp}$ to the morphism

$$\boxed{A \longrightarrow (A \multimap \perp) \multimap \perp}$$

general principle of logic that $A \vdash \neg\neg A$

A implies its double negation.

$*$-autonomous categories

Definition

An object $\perp$ is called dualizing when the canonical morphism

$$\eta_A \quad : \quad A \longrightarrow (A \multimap \perp) \multimap \perp$$

is an isomorphism for every object A.

Definition

A $*$-autonomous category is a smcc with a dualizing object.

$$\frac{A \times B \xrightarrow{f} C}{B \xrightarrow{g} A \Rightarrow C} \phi$$

$$b \longmapsto \left(a \longmapsto f(a, b)\right)$$

The category $\mathbf{Coh}$ is $*$-autonomous

$\perp = 1^\perp$ is the coherence space with singleton web $|\perp| = \{*\}$.

$$
\begin{array}{lll}
e = id_{A \multimap \perp} & A \multimap \perp \longrightarrow A \multimap \perp & \{((a,*),(a,*)) \mid a \in |A|\} \\
f = \phi^{-1}_{A \multimap \perp, A, \perp}(e) & A \otimes (A \multimap \perp) \longrightarrow \perp & \{((a,(a,*)),*) \mid a \in |A|\} \\
g = f \circ \gamma_{A, A \multimap \perp} & (A \multimap \perp) \otimes A \longrightarrow \perp & \{(((a,*),a),*) \mid a \in |A|\} \\
h = \phi_{A \multimap \perp, A, \perp}(g) & A \longrightarrow (A \multimap \perp) \multimap \perp & \{(a,((a,*),*)) \mid a \in |A|\}
\end{array}
$$

The morphism h is an isomorphism with inverse the clique

$$h^{-1} = \{(((a,*),*),a) \mid a \in |A|\}$$

The category Coh is

symmetrical monoidal closed.

We have a natural bijection

$$\boxed{\phi_{A,B,C}} : \mathrm{Coh}(A \otimes B, C) \cong \mathrm{Coh}(B, A \multimap C)$$

which comes from the $\boxed{\text{graph}}$ isomorphism

$$\boxed{(A \otimes B) \multimap C \;\cong\; B \multimap (A \multimap C)}$$

$$\begin{array}{ccc}
((a,b),c) & \longmapsto & (b,(a,c)) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
|(A \otimes B) \multimap C| & & |B \multimap (A \multimap C)|
\end{array}$$

hence, there is a bijection $\boxed{\phi_{A,B,C}}$

between the cliques of

$$(A \otimes B) \multimap C$$

and the cliques of $B \multimap (A \multimap C)$

Evaluation map :

$$\mathrm{Id}_{A\multimap B} = \left\{ \left((a,b),(a,b)\right), \begin{matrix} a \in |A| \\ b \in |B| \end{matrix} \right\}$$

$$\mathrm{Id}_{A\multimap B} : (A\multimap B) \longrightarrow (A\multimap B)$$

when we apply $\phi^{-1}_{A,A\multimap B,B}$ we get

$$\mathrm{eval}_{A,B} : A \otimes (A\multimap B) \longrightarrow B$$

$$\mathrm{eval}_{A,B} = \left\{ \left(\underbrace{(a,(a,b))}_{|A\otimes(A\multimap B)|}, \underbrace{b}_{|B|}\right) \;\middle|\; \begin{matrix} a \in |A| \\ b \in |B| \end{matrix} \right\}$$

Multiplicative linear logic (MLL)

$$\vdash A_1, \ldots, A_n$$

$\parr$

disjunction

$$A, B ::= A \otimes B \mid \mathbf{1} \mid A \parr B \mid \bot \mid \alpha$$

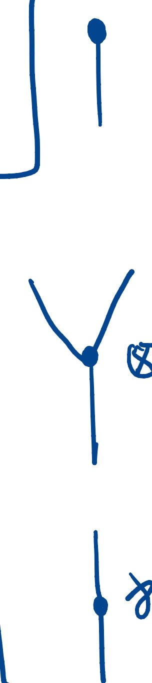
Axiom $$\frac{}{\vdash A^{\bot}, A}$$

$\otimes$ $$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

$\parr$ $$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$$

1 $$\frac{}{\vdash 1}$$

$\bot$ $$\frac{\vdash \Gamma}{\vdash \Gamma, \bot}$$



nodes of the derivation trees

MLL may be interpreted in every $*$-autonomous category

$$A_1, \ldots, A_n \vdash B \rightsquigarrow A_1 \otimes \ldots \otimes A_n \longrightarrow B$$

$$\vdash A_1, \ldots, A_n \rightsquigarrow 1 \longrightarrow A_1 \parr \ldots \parr A_n$$

Intuitionistic LL

Classical LL

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \quad \rightsquigarrow \quad \frac{\vdash \Gamma^\perp, A \qquad \vdash \Delta^\perp, B}{\vdash \Gamma^\perp, \Delta^\perp, A \otimes B}$$

$$\frac{\Delta \vdash A \qquad \Gamma, B \vdash C}{\Gamma, \Delta, \underbrace{A \multimap B} \vdash C} \quad \rightsquigarrow \quad \frac{\vdash \Delta^\perp, A \qquad \vdash \Gamma^\perp, B^\perp, C}{\vdash \Delta^\perp, \Gamma^\perp, \underbrace{A \otimes B^\perp}_{(A \multimap B)^\perp}, C}$$

$$A \multimap B = A^\perp \parr B$$

input output

$$= \left(\underbrace{A \otimes B^\perp}\right)^\perp$$

$$A \otimes B^\perp$$

what the environment gives

what the environment expects

Interpretation of Classical Linear Logic

In a $*$-autonomous category

every derivation tree of sequent

$$\boxed{\vdash A_1, \ldots, A_n}$$

is translated as

a morphism

$$\boxed{1 \longrightarrow A_1 \parr \ldots \parr A_n}$$

where

$$A_1 \parr \ldots \parr A_n = \neg(\neg A_1 \otimes \ldots \otimes \neg A_n)$$

in any $*$-autonomous category:

$$A \parr B = \neg(\neg A \otimes \neg B)$$

where $\neg A = (A \multimap \bot)$

<u>Prop</u> $\parr$ defines a monoidal structure on $\mathcal{C}$

Hence every $*$-autonomous category $\mathcal{C}$ comes with two monoidal structures:

$$\otimes \quad 1$$

$$\parr \quad \bot$$

$$A \parr B = \neg(\neg A \otimes \neg B)$$

$$\bot \cong \neg 1 = 1 \multimap \bot$$

In a general symmetric monoidal closed category $\mathcal{C}$ $\otimes$ $\multimap$

where we fix an object $\perp$

we can define $\neg A = A \multimap \perp$.

and

$$\boxed{A_1 \parr \dots \parr A_n = \neg(\neg A_1 \otimes \dots \otimes \neg A_n)}$$

then we observe that we have canonical maps:

$$A \parr (B \parr C) := \neg(\neg A \otimes \neg \underbrace{\neg(\neg B \otimes \neg C)}_{B \parr C})$$

$$A \parr (B \parr C) \longrightarrow (A \parr B \parr C)$$

not an isomorphism in general

$$\begin{tikzcd}[column sep=small]
A \parr (B \parr C) \arrow[dr, "\bar{\alpha}"'] & & (A \parr B) \parr C \arrow[dl, "\bar{\alpha}"] \\
& A \parr B \parr C &
\end{tikzcd}$$

in a $*$-autonomous category these two maps are isomorphisms

$$\begin{tikzcd}[column sep=small]
A \parr (B \parr C) \arrow[rr, "\bar{\alpha}^{-1}_{(AB)C} \circ \bar{\alpha}_{A(BC)}"] \arrow[dr, "\bar{\alpha}_{A(BC)}"'] & & (A \parr B) \parr C \arrow[dl, "\bar{\alpha}_{(AB)C}"] \\
& A \parr B \parr C &
\end{tikzcd}$$

the associativity isomorphism of the $\parr$ rule.

Multiplicative additive linear logic (MALL)

$$A, B ::= \boxed{A \oplus B} \mid A \otimes B \mid \boxed{0} \mid \mathbf{1} \mid \boxed{A \& B} \mid A \parr B \mid \boxed{\top} \mid \bot \mid \alpha$$

disjunction zero conjunction top

MLL+

disjunction

$\oplus$ left	$\dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$
$\oplus$ right	$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}$
$\&$	$\boxed{\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B}}$
$\boxed{0}$	no rule
$\top$	$\dfrac{}{\vdash \Gamma, \top}$

same context Γ

MALL may be interpreted in every $\left\{ \begin{array}{l} \text{*-autonomous} \\ \text{and cartesian} \end{array} \right\}$ category.

The exponential modality

The structure of the category of coherence spaces (4)

The new ingredient: the exponential

The l'exponential $!A$ of a coherence space A is the graph

- whose web $|!A|$ is the set of finite cliques of A,
- $u \coh_{!A} v$ iff the union $u \cup v$ is a finite clique of A.

The coherence space $?A$ is defined as

$$?A \quad = \quad (!A^{\perp})^{\perp}$$

The exponential alchimy

The exponential transmutes the additives into multiplicatives !

The terminology "exponential" is justified by the isomorphisms:

$$\boxed{!(A \& B) \;\cong\; !A \otimes !B} \qquad\qquad \boxed{!\top \;\cong\; 1}$$

Reminiscent of $\wp(A+B) \cong \wp(A) \times \wp(B)$ in Set.

We will study the categorical properties of the exponential:

- every $!A$ defines a comonoid $(!A, d_A, e_A)$ in **Coh**,
- the exponential defines a comonad $(!, \delta, \varepsilon)$ in **Coh**,
- the cartesian diagonal

$$\boxed{A \longrightarrow A \& A}$$

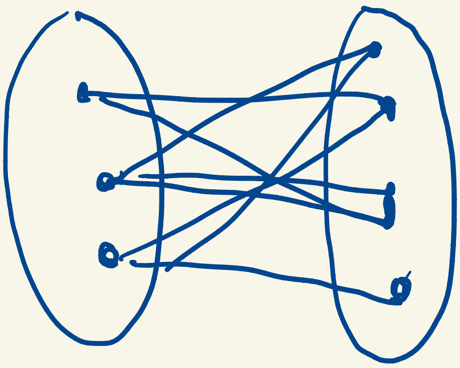
is transported to the comonoidal diagonal

$$\boxed{!A \longrightarrow !A \otimes !A.}$$

$$\boxed{e^{x+y} = e^x e^y}$$

$$!(A \& B) \cong \,!A \otimes !B$$

at the level of linear logic



A $\qquad$ B

$\underbrace{\qquad\qquad}_{A \& B}$

$$\boxed{D_{A\&B} \cong D_A \times D_B}$$

$$!\underbrace{A\&B}_{\text{additive}} \cong \,!A \otimes !B$$

multiplicati

every finite clique u of $A\&B$

is a pair (u_A, u_B)

consisting of a clique u_A of A

u_B of B

$$\boxed{!A\&B \cong !A \otimes !B}$$

$$u \longmapsto (u_A, u_B)$$

$$u = u_A \uplus u_B \longleftarrow (u_A, u_B)$$

What about the coherence relations?

$$u \frown_{!A\&B} v$$

precisely when

by definition of coherence in $!X$

$$u_A \frown_{!A} v_A \quad \text{and} \quad u_B \frown_{!B} v_B$$

hence precisely when

by def. of coherence in $X \otimes Y$

$$(u_A, u_B) \frown_{!A \otimes !B} (v_A, v_B)$$

Linear logic (LL)

$$A, B ::= A \oplus B \mid A \otimes B \mid !A \mid 0 \mid \mathbf{1} \mid A \& B \mid A \parr B \mid ?A \mid \top \mid \bot \mid \alpha$$

MALL+

contraction	$\dfrac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$
weakening	$\dfrac{\vdash \Gamma}{\vdash \Gamma, ?A}$

comonoid structure of !A

dereliction	$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?A}$
digging	$\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}$

Comonad structure of !

Monoids

A monoid in a monoidal category $(\mathcal{C}, \otimes, 1)$ is an object A equipped with two morphisms

$$1 \xrightarrow{\;u\;} A \xleftarrow{\;m\;} A \otimes A$$

such that the following diagrams commute:

$(x,(y,z)) \longmapsto ((x,y),z) \longmapsto (m(x,y),z) \longmapsto m(m(x,y),z)$

$$\begin{tikzcd}
A \otimes (A \otimes A) \arrow[r, "\alpha"] \arrow[d, "A\otimes m"'] & (A \otimes A) \otimes A \arrow[r, "m\otimes A"] \arrow[d, phantom, "*"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[rr, "m"] & & A
\end{tikzcd}$$

$(x, m(y,z)) \longmapsto m(x, m(y,z)) = m(m(x,y),z)$

$(*,x) \longmapsto (u,x) \longmapsto x = m(u,x)$

$$\begin{tikzcd}
1 \otimes A \arrow[r, "u\otimes A"] \arrow[d, "\lambda"'] \arrow[dr, phantom, "**"] & A \otimes A \arrow[d, "m"] \arrow[dr, phantom, "**"] & A \otimes 1 \arrow[l, "A\otimes u"'] \arrow[d, "\rho"] \\
A \arrow[r, equal] & A \arrow[r, equal] & A
\end{tikzcd}$$

$(*,x) \longmapsto x$

In Set

In the category Set

$$1 \xrightarrow{u} A \xleftarrow{m} A \times A$$

$u \in A$ neutral element

binary product multiplication

$$\{*\} \xrightarrow{u} A$$

$$* \longmapsto u$$

$\boxed{\text{associativity}}$

$$m(x, m(y,z)) \overset{(*)}{=} m(m(x,y), z)$$

$\boxed{\text{neutrality}}$

$$m(x, u) \overset{(**)}{=} x \overset{(**)}{=} m(u, x)$$

Comonoids

Dually, a comonoid is an object A equipped with two morphisms

$$1 \xleftarrow{e} A \xrightarrow{d} A \otimes A$$

such that the following diagrams commute:

$$\begin{tikzcd}
A \arrow[rr, "d"] \arrow[d, "d"'] & & A \otimes A \arrow[d, "d \otimes A"] \\
A \otimes A \arrow[r, "A \otimes d"] & A \otimes (A \otimes A) \arrow[r, "\alpha"] & (A \otimes A) \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes A \arrow[d, "\lambda"'] & A \otimes A \arrow[l, "e \otimes A"'] \arrow[r, "A \otimes \eta"] & A \otimes 1 \arrow[d, "\rho"] \\
A \arrow[r, equal] & A \arrow[u, "d"'] \arrow[r, equal] & A
\end{tikzcd}$$

$\boxed{\text{example}}$ of a <u>comonoid</u> in Set:

$$1 \xleftarrow{e} A \xrightarrow{d} A \times A$$

d diagonal function

$$\begin{aligned} A &\longrightarrow A \times A \\ a &\longmapsto (a, a) \end{aligned}$$

e is the unique canonical function

$$\begin{aligned} A &\longrightarrow \{*\} \\ a &\longmapsto * \end{aligned}$$

every set A is a comonoid!

same in any cartesian category:

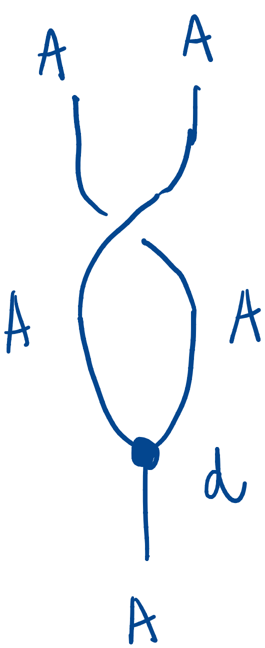
every object A defines a comonoid.

Commutative comonoid

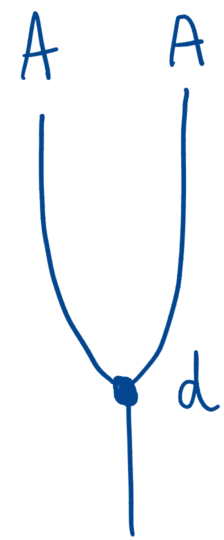
A comonoid

$$(A, d, e)$$

in a symmetric monoidal category is commutative when



$$\underbrace{A \xrightarrow{d} A \otimes A} \xrightarrow{\gamma_{A,A}} A \otimes A \quad = \quad A \xrightarrow{d} A \otimes A$$



Every object $!A$ is a comonoid in $\mathbf{Coh}$

The coherence space $!A$ equipped with the following cliques:

- a diagonal map or co-multiplication

$$\boxed{!A \xrightarrow{d_A} !A \otimes !A}$$

defined as

$$\boxed{\{(\underbrace{u, (v, w)}) \in |!A \multimap !A \otimes !A| \mid u = v \cup w\}}$$

- a weakening map or co-unit

$$\boxed{!A \xrightarrow{e_A} 1}$$

defined as the singleton

$$\boxed{\{(\emptyset, *)\}}$$

remark: a partial function from the output to the input.

Associativity of $!A \xrightarrow{d_A} !A \otimes !A$

w

$$\begin{tikzcd}
!A \arrow[r, "d_A"] \arrow[d, "d_A"'] & !A \otimes !A \arrow[d, "d_A \otimes !A"] \\
!A \otimes !A \arrow[r, "!A \otimes d_A"'] & !A \otimes !A \otimes !A
\end{tikzcd}$$

(u_1, u_2, u_3)

I want to show that given four finite cliques w, u_1, u_2, u_3 of A

$$(w, (u_1, u_2, u_3)) \in (d_A \otimes !A) \circ d_A$$

$$\Leftrightarrow$$

$$(w, (u_1, u_2, u_3)) \in (!A \otimes d_A) \circ d_A$$

the proof boils down to the fact that union of finite cliques is a partial function which is <u>associative</u> and has a <u>neutral element</u> (the empty clique)

the comultiplication

$$!A \longrightarrow !A \otimes !A$$

is also commutative because the partial function on finite cliques of A

$$u, v \longmapsto u \cup v$$

is commutative

Next lecture

- interpretation of LL in the category of coherence spaces,
- construction of the Kleisli category of a comonad.

Teaser... in the web of $\;!(A \multimap A) \multimap (A \multimap A)$

$$\lambda f : !(A \multimap A).\ \ \lambda x : A.\ \ f x$$

is interpreted as the set of the form $(\{(a,b)\}, a, b)$

$$\lambda f : !(A \longrightarrow A).\ \ \lambda x : A.\ \ x$$

is interpreted as the set of the form $(\varnothing, a, a)$

$$\lambda f : !(A \multimap A).\ \ \lambda x : A.\ \ f(f x)$$

is interpreted as the set of the form $(\{(a,b),(b,c)\}, a, c)$

Stable functions between coherence spaces

Domain associated to a coherence space

Every coherence space A induces a domain D_A

- ▷ whose elements are the cliques u of A,
- ▷ with partial order defined as set-theoretic inclusion:

$$x \leq_A y \iff x \subset y$$

Property:
$(\mathrm{D}_A, \leq_A)$ is a domain: it has a least element and all filtered limits.

Stability

A function

$$f \quad : \quad \mathrm{D}_A \longrightarrow \mathrm{D}_B$$

is stable when:

- f is monotone: for every pair of elements x and y,

$$x \leq y \implies f(x) \leq f(y)$$

- f is continuous: for all filter $\mathcal{F}$ of elements of D_A:

$$f(\bigvee \mathcal{F}) \quad = \quad \bigvee f(\mathcal{F})$$

- f preserves the intersections of compatible cliques:

$$\forall x, y \in \mathrm{D}_A, \qquad x \uparrow y \implies f(x \wedge y) = f(x) \wedge f(y)$$

Stability (alternative formulation)

Observation. Every monotone function

$$f \quad : \quad D_A \quad \longrightarrow \quad D_B$$

induces a family of functions

$$f_z \quad : \quad D_z \quad \longrightarrow \quad D_{f(z)}$$

A continuous function f is stable when every f_z has a left adjoint

$$g_z \quad : \quad D_{f(z)} \quad \longrightarrow \quad D_z$$

This means that

$$\forall x \leq z, \forall y \leq f(z) \qquad y \leq f(x) \leq f(z) \quad \Longleftrightarrow \quad g_z(y) \leq x \leq z$$

Decomposition as step functions

For every stable function

$$f : \mathrm{D}_A \longrightarrow \mathrm{D}_B$$

the trace $\mathrm{Tr}(f)$ is defined as the set of pairs

$$(x, b) \quad \in \quad \mathrm{D}_A \times |B|$$

such that for every pair of elements y and z satisfying

$$x \leq z \quad \text{and} \quad y \leq z$$

one has

$$x \leq y \iff b \leq f(y).$$

Note that $b \leq f(x)$ in that case.

Decomposition as step functions

Property. Every stable function is entirely described by its trace.

More precisely, the image of an element u of D_A is given by the set

$$f(x) \;=\; \left\{ \; b \in |B| \;\; \middle| \;\; \exists y \leq x, \; (y, b) \in \mathrm{Tr}(f) \; \right\}$$

Proof.

In one direction.

$$f(x) \quad \supseteq \quad \left\{ \; b \in |B| \;\; \middle| \;\; \exists y \leq x, \; (y, b) \in \mathrm{Tr}(f) \; \right\}$$

because if $(y, b) \in \mathrm{Tr}(f)$ for a clique $y \leq x$, then

$$b \leq f(y) \leq f(x).$$

Decomposition as step functions

In the other direction

$$f(x) \subseteq \left\{ \, b \in |B| \;\middle|\; \exists y \leq x, \; (y, b) \in \mathrm{Tr}(f) \, \right\}$$

because if b is an element of the clique $f(x)$, one defines

$$y = \bigwedge \left\{ \, z \leq x \;\middle|\; b \leq f(z) \, \right\}.$$

The clique y is the mininum clique included in x such that:

$$b \leq f(y).$$

One applies the stability hypothesis to show that y satisfies

$$(y, b) \in \mathrm{Tr}(f)$$

Decomposition as step functions

Property. The trace of a stable function f is a clique of $!A \multimap B$.

It is sufficient to check that two elements (x_1, b_1) and (x_2, b_2) of the trace are compatible in the coherence space $!A \multimap B$:

$$(x_1, b_1) \frown_{!A \multimap B} (x_2, b_2).$$

To that purpose, one must establish that

$$x_1 \frown_{!A} x_2 \quad \Longrightarrow \quad b_1 \frown_{B} b_2$$

and

$$b_1 \asymp_{B} b_2 \quad \Longrightarrow \quad x_1 \asymp_{!A} x_2.$$

Forward direction

Suppose that

$$x_1 \frown_{!A} x_2.$$

In that case, the elements x_1 and x_2 are bounded by a clique $z \in \mathrm{D}_A$. Thus,

$$f(x_1) \leq f(z) \quad \text{and} \quad f(x_2) \leq f(z).$$

But we know that

$$b_1 \in f(x_1) \quad \text{et} \quad b_2 \in f(x_2).$$

Hence b_1 and b_2 are elements of the same clique $f(z)$.

This establishes that

$$b_1 \frown_B b_2.$$

Backward direction

Suppose that

$$b_1 \asymp_B b_2$$

We treat the two possible cases in turn:

(1) suppose that

$$b_1 \smile_B b_2.$$

In that case, we deduce from the “forward” part that

$$x_1 \smile_{!A} x_2.$$

Backward direction

(2) suppose that

$$b_1 \quad = \quad b_2 \quad = \quad b$$

Suppose that the two cliques x_1 and x_2 are compatible in $!A$:

$$x_1 \supset\!\!\!\smile_{!A} x_2$$

The stability of the function f implies that

$$f(x_1) \cap f(x_2) = f(x_1 \cap x_2)$$

this implying in turn that

$$b \in f(x_1 \cap x_2).$$

We deduce from the definition of a trace that $x_1 = x_2$.

This establishes that

$$b_1 = b_2 \quad \Longrightarrow \quad x_1 \asymp_{!A} x_2.$$

Decomposition as step functions

Property.
Conversely, every clique u of $!A \multimap B$ defines a stable function

$$f(x) \;=\; \left\{\; b \in |B| \;\middle|\; \exists y \leq x,\; (y, b) \in u \;\right\}$$

whose trace coincides with the clique u.

One shows that two stable functions f and g are ordered by the stable ordering:

$$f \leq_s g$$

if and only if

$$\mathrm{Tr}(f) \subseteq \mathrm{Tr}(g).$$

Hence, the idea of deducing the category of coherence spaces and stable functions from the category of coherence spaces and cliques.

$$D_{A \Rightarrow B} \cong D_{!A \multimap B}$$

the domain of stable functions from D_A to D_B

in the TD.

Kleisli construction

how we will reconstruct the cartesian closed category of stable functions from the category of coherence spaces.

Idea: construct a ccc from a smcc

Starting point: the category $\mathbf{Coh}$ of coherence spaces.

At the same time:

- ▷ a cartesian category $\boxed{(\mathbf{Coh}, \&, \top)}$
- ▷ a symmetric monoidal closed category $\boxed{(\mathbf{Coh}, \otimes, 1, \multimap)}$

The $\boxed{\text{exponential}}$ will connect the cartesian and the monoidal worlds.

Comonad

$\boxed{\text{A comonad} \quad (T, \delta, \varepsilon)}$ in a category $\mathcal{C}$ is the data

▷ of a functor $\boxed{T : \mathcal{C} \longrightarrow \mathcal{C}}$

▷ of two natural transformations

$$\boxed{\delta : T \longrightarrow T \circ T} \qquad \boxed{\varepsilon : T \longrightarrow Id_{\mathcal{C}}}$$

such that the following diagrams commute:

$$\begin{tikzcd}
T \arrow[r, "\delta"] \arrow[d, "\delta"'] & T \circ T \arrow[d, "T \circ \delta"] \\
T \circ T \arrow[r, "\delta \circ T"] & T \circ T \circ T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, equal] \arrow[d, equal] & T \arrow[r, equal] \arrow[d, "\delta"'] & T \arrow[d, equal] \\
T & T \circ T \arrow[l, "\varepsilon \circ T"'] \arrow[r, "T \circ \varepsilon"] & T
\end{tikzcd}$$

families of maps:

$$\begin{cases} TA \xrightarrow{\delta_A} TTA \\ TA \xrightarrow{\varepsilon_A} A \end{cases}$$

making the diagrams below commute:

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] \arrow[dr, phantom, "\circledast"] & TTA \arrow[d, "\delta_{TA}"] \\
TTA \arrow[r, "T\delta_A"'] & TTTA
\end{tikzcd}$$

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[dr, "id_{TA}"'] & TTA \arrow[d, "T\varepsilon_A"] \arrow[dl, phantom, "(**)"] \\
& TA
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[dr, "id_{TA}"'] & TTA \arrow[d, "\varepsilon_{TA}"] \arrow[dl, phantom, "(**)"] \\
& TA
\end{tikzcd}$$

Kleisli category

The Kleisli category $\mathcal{C}_T$ of a comonad (T, δ, ε) is defined as

- ▷ the category with the same objects as $\mathcal{C}$
- ▷ with morphisms $A \dashrightarrow B$ the morphisms $TA \longrightarrow B$ of the category $\mathcal{C}$

The identity $A \dashrightarrow A$ is given by the morphism $\varepsilon_A : TA \longrightarrow A$.

Composition of two morphisms $f : A \dashrightarrow B$ and $g : B \dashrightarrow C$ is defined as

$$\begin{tikzcd}
TTA \arrow[dr, "Tf"] & & \\
TA \arrow[u, "\delta_A"] \arrow[dr, "f"] & TB \arrow[dr, "g"] & \\
& B & C
\end{tikzcd}$$

Show the identities are identities and that composition is associative.

$$
\begin{tikzcd}
TTTA \arrow[dr, "TTf"] & & \\
TTA \arrow[u, "T\delta_A"] \arrow[dr, "Tf"] & TTB \arrow[dr, "Tg"] & \\
TA \arrow[u, "\delta_A"] \arrow[dr, "f"] & TB \arrow[dr, "g"] & TC \arrow[dr, "h"] & \\
& B & C & D
\end{tikzcd}
$$

$h \circ_K (g \circ_K f)$

$$
\boxed{h \circ_K (g \circ_K f) \stackrel{?}{=} (h \circ_K g) \circ_K f}
$$

$(h \circ_K g) \circ_K f$

$$
\begin{tikzcd}
& TTB \arrow[dr, "Tg"] & & \\
TTA \arrow[ur, phantom] \arrow[r, "Tf"] & TB \arrow[u, "\delta_B"] \arrow[dr, "g"] & TC \arrow[dr, "h"] & \\
TA \arrow[u, "\delta_A"] \arrow[r, "f"] & B & C & D
\end{tikzcd}
$$

the proof of associativity needs that

$$
\begin{tikzcd}
& TTA \arrow[dr, "T\delta_A"] & \\
TA \arrow[ur, "\delta_A"] \arrow[dr, "\delta_A"'] \arrow[rr, phantom, "\circlearrowright"] & & TTTA \\
& TTA \arrow[ur, "\delta_{TA}"'] &
\end{tikzcd}
$$

commutes

what is the identity map
of the Kleisli category:

$$TA \xrightarrow{\varepsilon_A} A$$

<u>exercise</u>:

show that ε_A defines
an identity map
in the Kleisli map

on the left
and
on the right.

Cartesian products in $\mathcal{C}_T$

Theorem.
Let (T, δ, ε) be a comonad in a cartesian category $(\mathcal{C}, \&, \top)$.
Then, the Kleisli category $(\mathcal{C}_T, \&, \top)$ is cartesian too.

Proof.

- the object $\top$ is terminal in the category $\mathcal{C}_T$
- $A\&B$ is the product of A and B equipped with the projections

$$\pi_1^\bullet \quad : \quad A\&B \dashrightarrow A \quad = \quad T(A\&B) \xrightarrow{\varepsilon_{A\&B}} A\&B \xrightarrow{\pi_1} A$$
$$\pi_2^\bullet \quad : \quad A\&B \dashrightarrow B \quad = \quad T(A\&B) \xrightarrow{\varepsilon_{A\&B}} A\&B \xrightarrow{\pi_2} B$$

Indeed, every pair of morphisms

$$f : X \dashrightarrow A \quad = \quad T(X) \longrightarrow A \qquad\qquad g : X \dashrightarrow B \quad = \quad T(X) \longrightarrow B$$

defines a morphism

$$(f, g) \quad : \quad X \dashrightarrow A\&B \quad = \quad T(X) \longrightarrow A\&B$$

Cartesian products in $\mathcal{C}_T$ (2)

This morphism

$$(f, g) \; : \; X \dashrightarrow A\&B$$

is solution to the universal problem

$$X \overset{(f,g)}{\dashrightarrow} A\&B \dashrightarrow A \quad = \quad \begin{tikzcd} T(T(X)) \arrow[dr, "T(f{,}g)"] & & \\ T(X) \arrow[u, "\delta_X"] \arrow[dr, "(f{,}g)"'] & T(A\&B) \arrow[d, "\varepsilon_{A\&B}"] \arrow[dr, "\pi_1^{\bullet}"] & \\ & A\&B \arrow[r, "\pi_1"] & A \end{tikzcd} \quad = \quad X \overset{f}{\dashrightarrow} A$$

Cartesian products in $\mathcal{C}_T$ (3)

In order to establish uniqueness, suppose that $h : X \dashrightarrow A\&B$ satisfies

$$X \overset{h}{\dashrightarrow} A\&B \overset{\pi_1^\bullet}{\dashrightarrow} A \quad = \quad X \overset{f}{\dashrightarrow} A \qquad\qquad X \overset{h}{\dashrightarrow} A\&B \overset{\pi_2^\bullet}{\dashrightarrow} B \quad = \quad X \overset{g}{\dashrightarrow} B$$

The universality of $A \overset{\pi_1}{\leftarrow} A\&B \overset{\pi_2}{\rightarrow} B$, and the following diagrams

$$X \overset{h}{\dashrightarrow} A\&B \dashrightarrow A \quad = \quad \begin{tikzcd} T(T(X)) \arrow[drr, "Th"] & & & & \\ T(X) \arrow[u, "\delta_X"] \arrow[drr, "h"] & & T(A\&B) \arrow[d, "\varepsilon_{A\&B}"] \arrow[drr, "\pi_1^\bullet"] & & \\ & & A\&B \arrow[rr, "\pi_1"] & & A \end{tikzcd} \quad = \quad X \overset{f}{\dashrightarrow} A$$

establish that

$$h \quad = \quad (f, g) \quad : \quad T(X) \longrightarrow A\&B$$

in the category $\mathcal{C}$.

The comonad $(!, \delta, \varepsilon)$ in $\mathbf{Coh}$

▷ The modality $!$ defines a functor $\;! : \mathbf{Coh} \longrightarrow \mathbf{Coh}$.

Given a clique

$$f \quad : \quad A \quad \longrightarrow \quad B$$

the clique

$$!\,f \quad : \quad !\,A \quad \longrightarrow \quad !\,B$$

is defined as

$$!\,f \quad = \quad \left\{ \quad (u,v) \in |!A \multimap !B| \quad \middle| \quad \begin{array}{l} 1.\ \forall a \in u, \exists b \in v, (a,b) \in f \\ 2.\ \forall b \in v, \exists a \in u, (a,b) \in f \end{array} \quad \right\}$$

The comonad $(!, \delta, \varepsilon)$ in $\mathbf{Coh}$

- The modality ! defines a **comonad** $(!, \delta, \varepsilon)$.

The **promotion**

$$\delta_A \quad : \quad !A \quad \longrightarrow \quad !!A$$

is defined as

$$\{ \quad (u, v) \in |!A \multimap !!A| \quad | \quad v = \{u_1, \ldots, u_n\} \quad , \quad u = u_1 \cup \cdots \cup u_n \quad \}$$

The **dereliction**

$$\varepsilon_A \quad : \quad \underset{\{a\}}{!A} \quad \longrightarrow \quad \underset{a}{A}$$

singleton clique
$\{a\}$

is defined as

$$\{ \quad \cancel{(a, \{a\})} \quad | \quad a \in |A| \quad \}$$

$(\{a\}, a)$

The exponential alchemy from additives to multiplicatives

Key observation. a family of natural isomorphisms

$$!(A\&B) \cong\ !A \otimes\ !B \qquad\qquad !\top \cong 1$$

Theorem.

The category $(\mathbf{Coh}_!, \&, \top)$ is cartesian closed with exponentiation

$$B \Rightarrow A \quad = \quad !B \multimap A$$

Proof.

We have already seen that the category

$$(\mathbf{Coh}_!, \&, \top)$$

is cartesian.

The exponential alchemy
from additives to multiplicatives

Proof (continued).

The existence of the exponential

$$A \quad \mapsto \quad !\,B \multimap A$$

for every object B de $\mathbf{Coh}_!$ is established by the series of natural bijections

$\mathrm{Coh}_!(A \& B, C)$

$\cong$

$\mathrm{Coh}_!(A, !B \multimap C)$

$$\begin{array}{ccc}
A\&B & \dashrightarrow & C \\
\hline
!(A\&B) & \longrightarrow & C \\
\hline
!A \otimes !B & \longrightarrow & C \\
\hline
!A & \longrightarrow & !B \multimap C \\
\hline
A & \dashrightarrow & !B \multimap C
\end{array}$$

decomposition of the intuitionistic implication into a linear implication

$$B \Rightarrow C := !B \multimap C$$

we obtain a cartesian closed category

- whose objects are the coherence spaces
- whose morphisms

$Coh_!$

$$f : A \longrightarrow B$$

are the morphisms ("linear maps")

$$f : !A \longrightarrow B$$

in the category Coh.

Seen in the TD:

same as the stable functions

$$D_A \longrightarrow D_B.$$

hence $Coh_!$ is the category of coherence spaces and stable functions between them.

Why is it so enlightening?

$$g \circ_K f \overset{?}{\ni} c$$

$$\begin{tikzcd}
{!!A} \arrow[dr, "!f"] & & \\
{!A} \arrow[u, "\delta_A"] \arrow[dr, "f"'] & {!B} \arrow[dr, "\boxed{g}"] & \\
& B & C
\end{tikzcd}$$

$\{\{a_{11},\dots,a_{1j_1}\}, \{\dots\} \dots \{a_{n1}\dots a_{nj_n}\}\}$

$\{a_{11}\dots a_{nj_n}\} \mapsto \{\{a_{11},\dots,a_{1j_1}\}, \dots, \{a_{n1}\dots a_{nj_n}\}\}$

$\{a_{i1}\dots a_{ij_i}\} \mapsto b_i$

$\boxed{\{b_1,\dots,b_n\}} \subseteq |B|$

$\{b_1,\dots,b_n\} \mapsto \boxed{c \in |C|}$ output

$\{b_1,\dots,b_n\} \multimap c \in g$

$\{a_{i1},\dots,a_{ij_i}\} \multimap b_i \in f$

this "decomposes" / describes

the hidden combinatorics

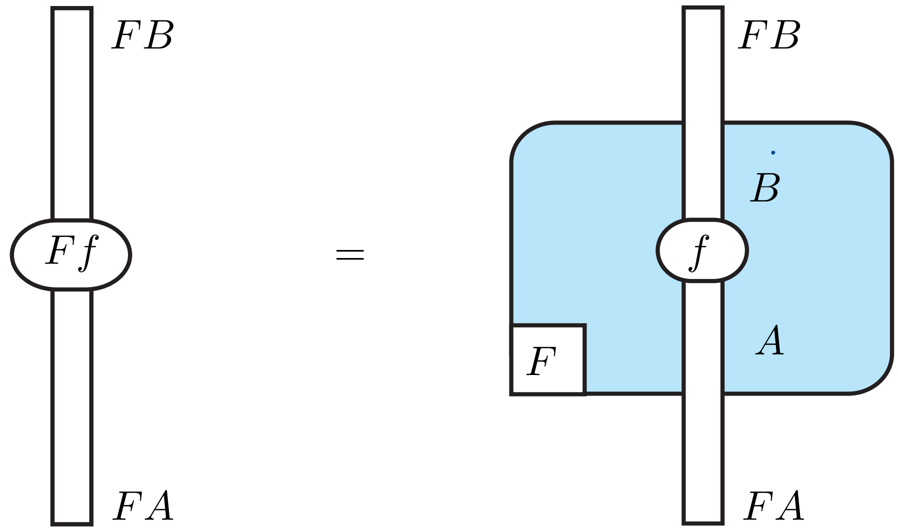
of composition of stable functions.

as functions

Functorial boxes

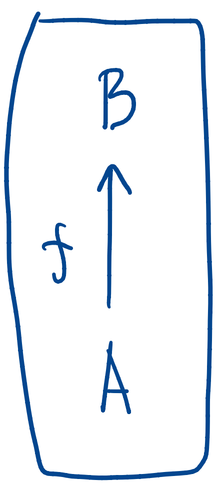
Functors in string diagrams

Functorial boxes

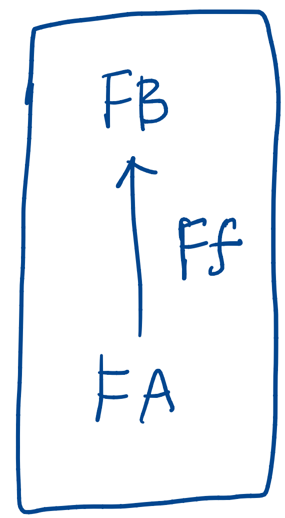


$$F \quad : \quad \mathcal{C} \longrightarrow \mathcal{D}$$

A window on the blue $\mathcal{C}$ inside the ambiant $\mathcal{D}$.

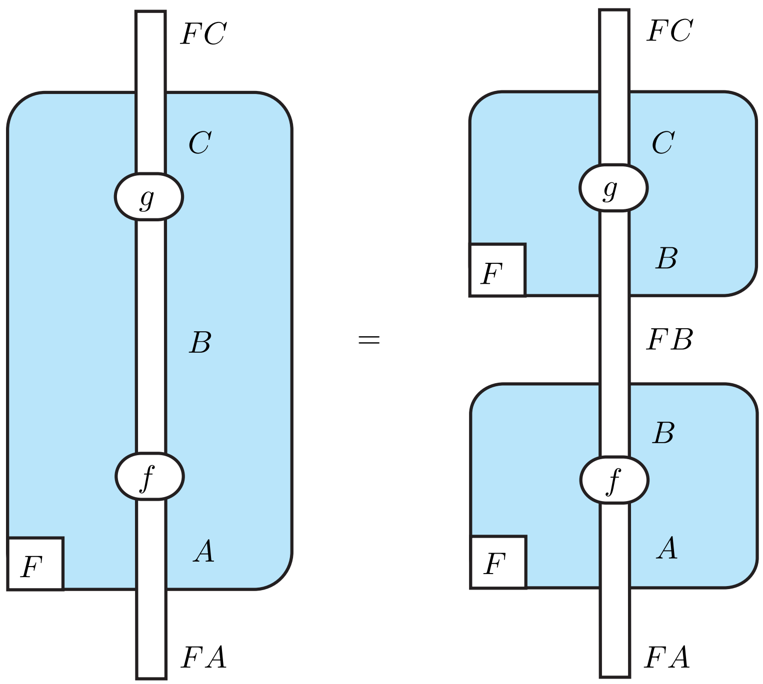


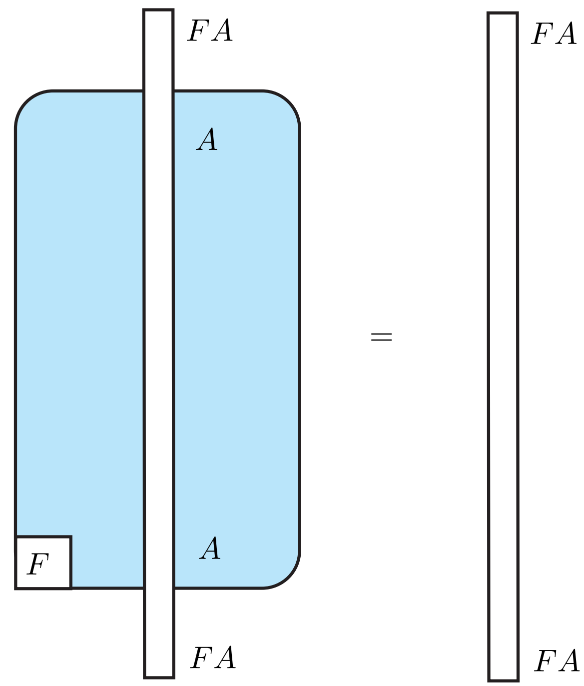
in $\mathcal{C}$



in $\mathcal{D}$

Functorial equalities





Lax monoidal functor

A lax monoidal functor is a functor

$$F \;:\; \mathcal{C} \;\longrightarrow\; \mathcal{D}$$

equipped with morphisms

$$m_{[A,B]} \;:\; FA \otimes FB \;\longrightarrow\; F(A \otimes B)$$

$$m_{[-]} \;:\; I \longrightarrow FI$$

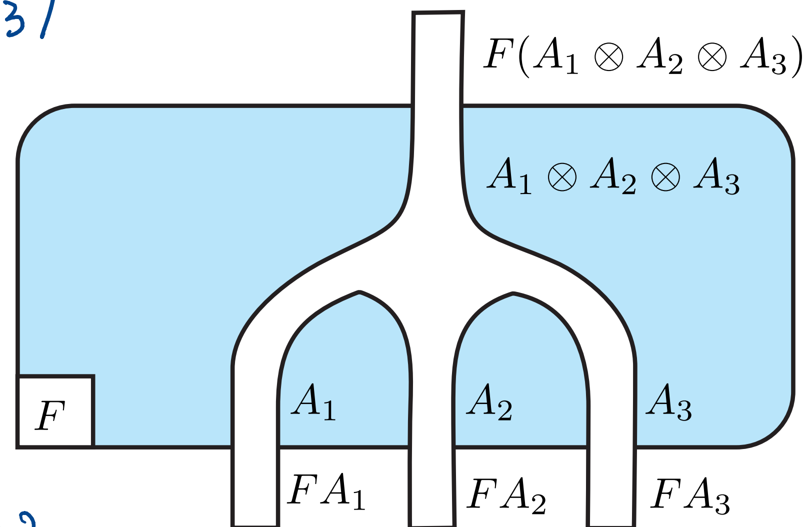
satisfying a series of coherence relations.

A strong monoidal functor is lax monoidal with invertible coercions.

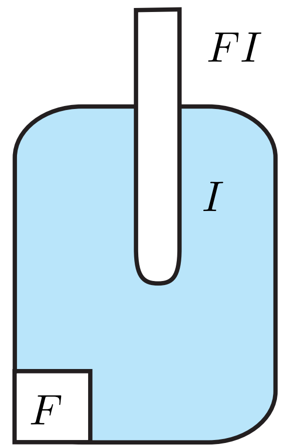
The purpose of coercions

$$FA_1 \otimes FA_2 \otimes FA_3 \longrightarrow F(A_1 \otimes A_2 \otimes A_3)$$

lax monoidal functor



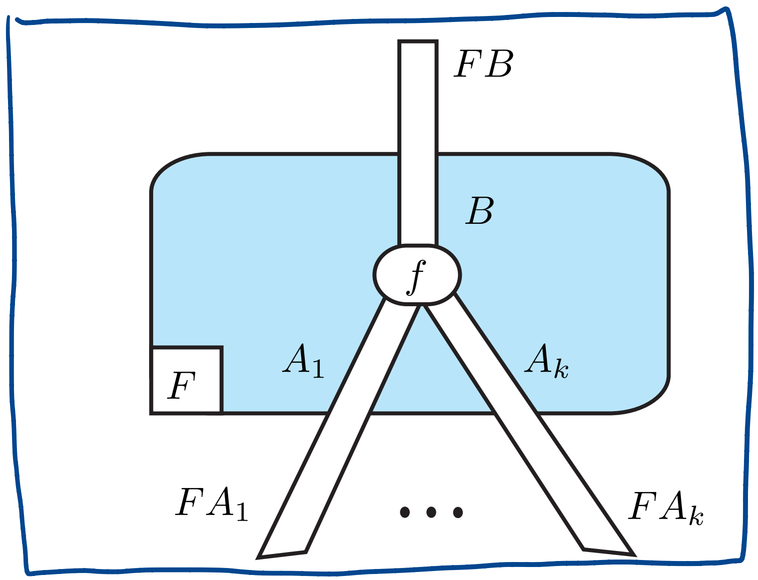
$m_{[A_1, A_2, A_3]}$



$m_{[-]}$

Lax monoidal functor

A lax monoidal functor is a box with many inputs - one output.

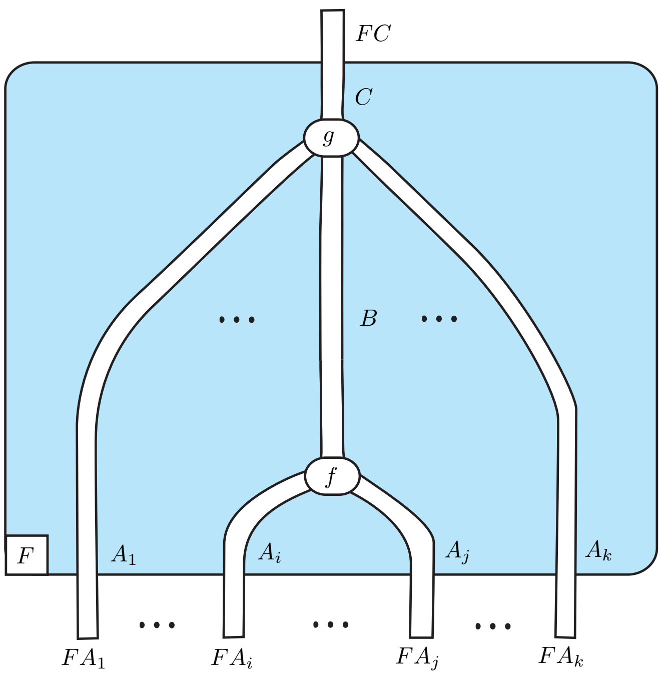


$f : A_1 \otimes \ldots \otimes A_k \to B$

$Ff : F(A_1 \otimes \ldots \otimes A_k) \to FB$

$$\boxed{F(f)} \circ \underbrace{m_{[A_1,\cdots,A_k]}}_{} \ : \ FA_1 \otimes \cdots \otimes FA_k \longrightarrow FB$$

Functorial equalities (on lax functors)



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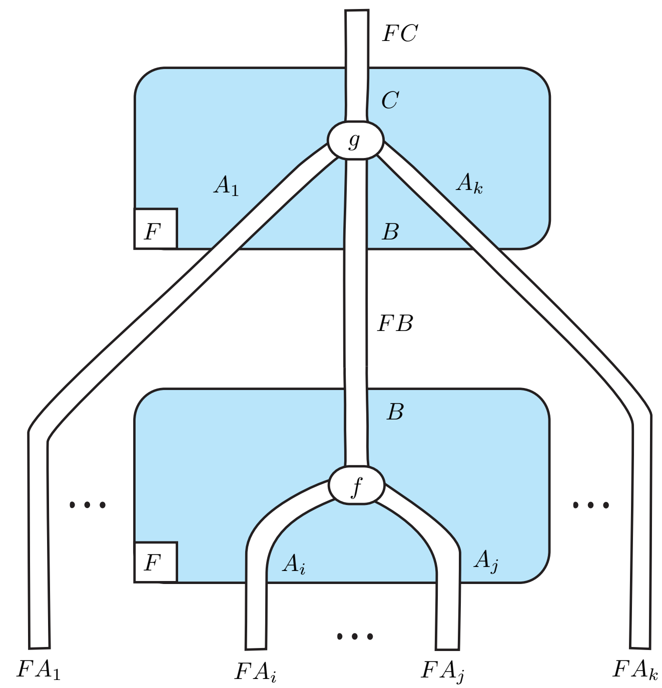
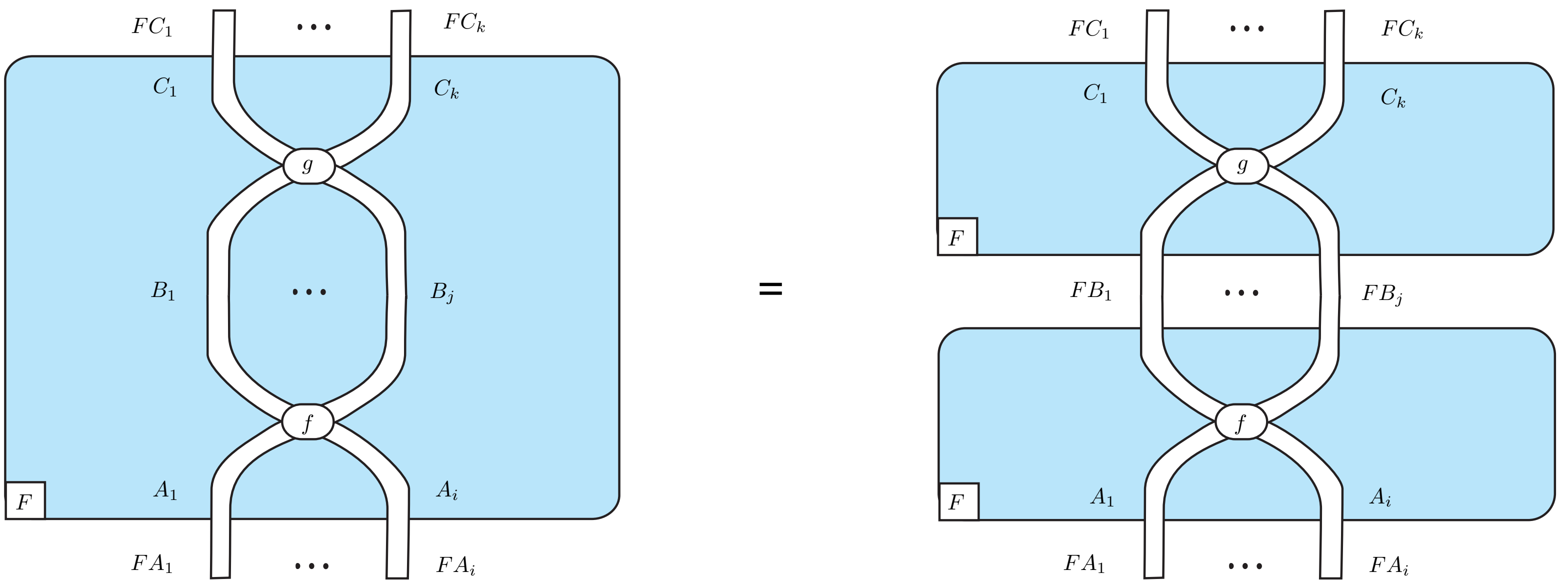


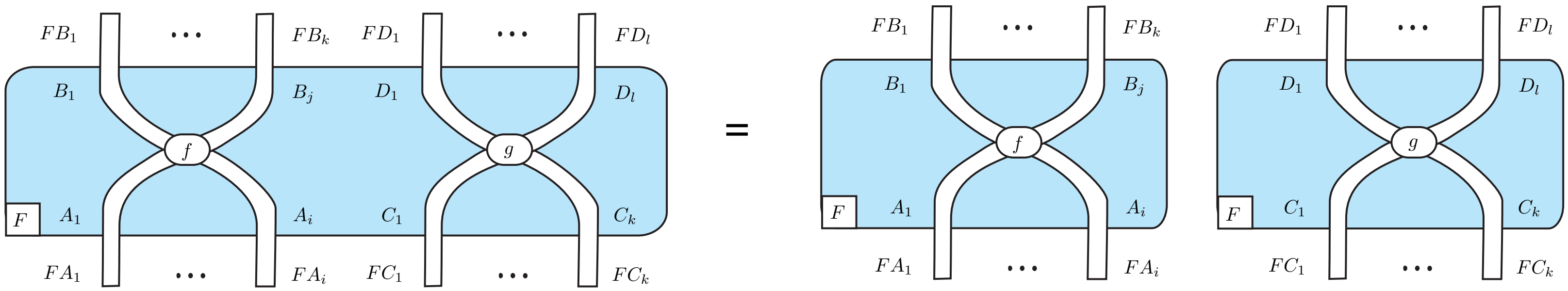
Strong monoidal functors

A strong monoidal functor is a box with many inputs - many outputs

Functorial equalities (on strong functors)



Functorial equalities (on strong functors)

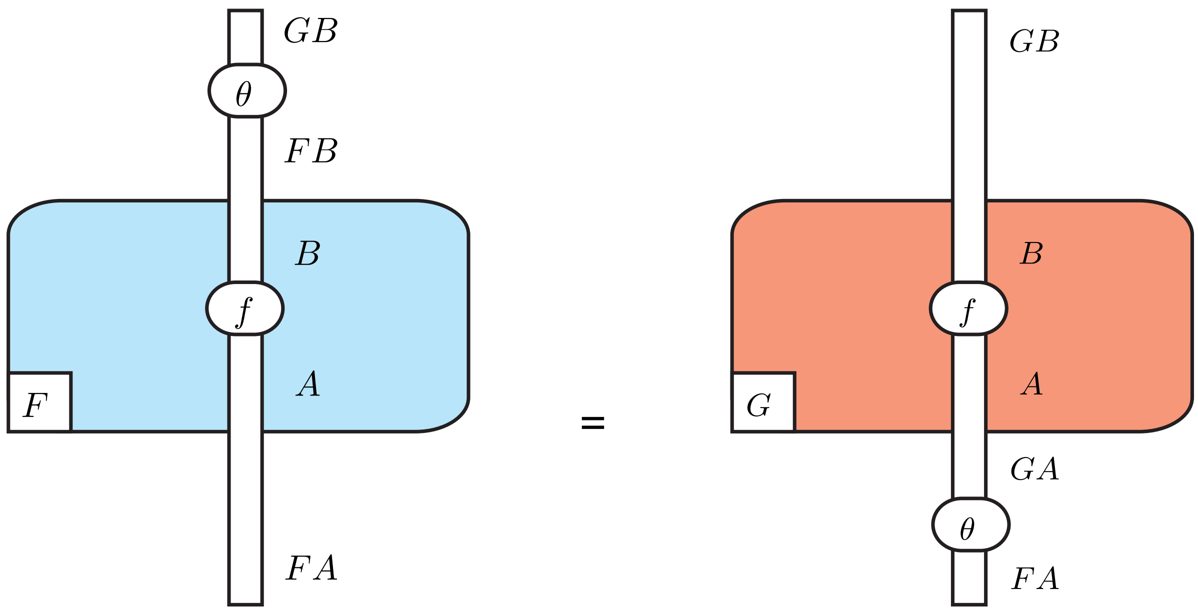


Natural transformations

A natural transformation

$$\theta \quad : \quad F \longrightarrow G \quad : \quad \mathcal{C} \longrightarrow \mathcal{D}$$

satisfies the pictorial equality:

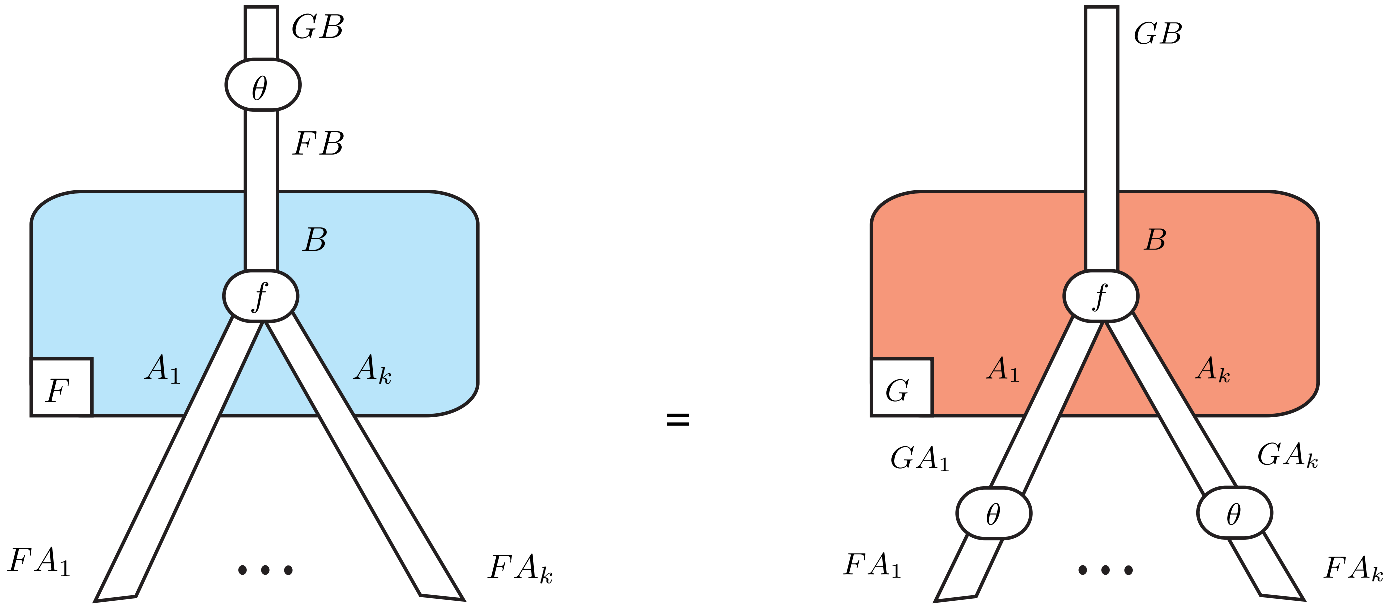


Monoidal natural transformations

A monoidal natural transformation

$$\theta \quad : \quad F \longrightarrow G \quad : \quad \mathcal{C} \longrightarrow \mathcal{D}$$

satisfies the pictorial equality:



Decomposing
the exponential box of linear logic

The categorical semantics of linear logic

A symmetric monoidal adjunction

Coh!
Stable maps

Linearizes

$$\begin{tikzcd} \mathcal{M} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathcal{L} \arrow[ll, bend left, "M"] \end{tikzcd}$$

$\times$ $\quad$ $\otimes$

Multiply

Coh
Coherence spaces and linear maps

$\boxed{\mathcal{M} \text{ cartesian}}$ $\qquad$ $\boxed{\mathcal{L} \text{ symmetric monoidal closed}}$

$$\boxed{\,! \quad = \quad L \circ M\,}$$

Equivalently: an adjunction whose left adjoint L is strong monoidal

$$L(A \times B) \xrightarrow{\cong} LA \otimes LB$$

in Coh! $\qquad$ $!(A \& B) \xrightarrow{\cong} !A \otimes !B$ $\qquad$ Seely isomorphism.

$$\begin{tikzcd}
\mathcal{M} \arrow[r, bend left, "\mathrm{Lin}"] \arrow[r, phantom, "\perp"] & \mathcal{L} \arrow[l, bend left, "\mathrm{Mult}"] \arrow[r, bend left, "\mathrm{neg}"] \arrow[r, phantom, "\perp"] \arrow[loop above, "!"] \arrow[loop below, "?"] & \mathcal{L}^{op} \arrow[l, bend left, "\mathrm{neg}"] \arrow[r, bend left, "\mathrm{Mult}^{op}"] \arrow[r, phantom, "\perp"] & \mathcal{M}^{op} \arrow[l, bend left, "\mathrm{Lin}^{op}"]
\end{tikzcd}$$

in a $*$-autonomous category

$$\begin{tikzcd}
\mathcal{L} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{L}^{op} \arrow[l, bend left]
\end{tikzcd}$$

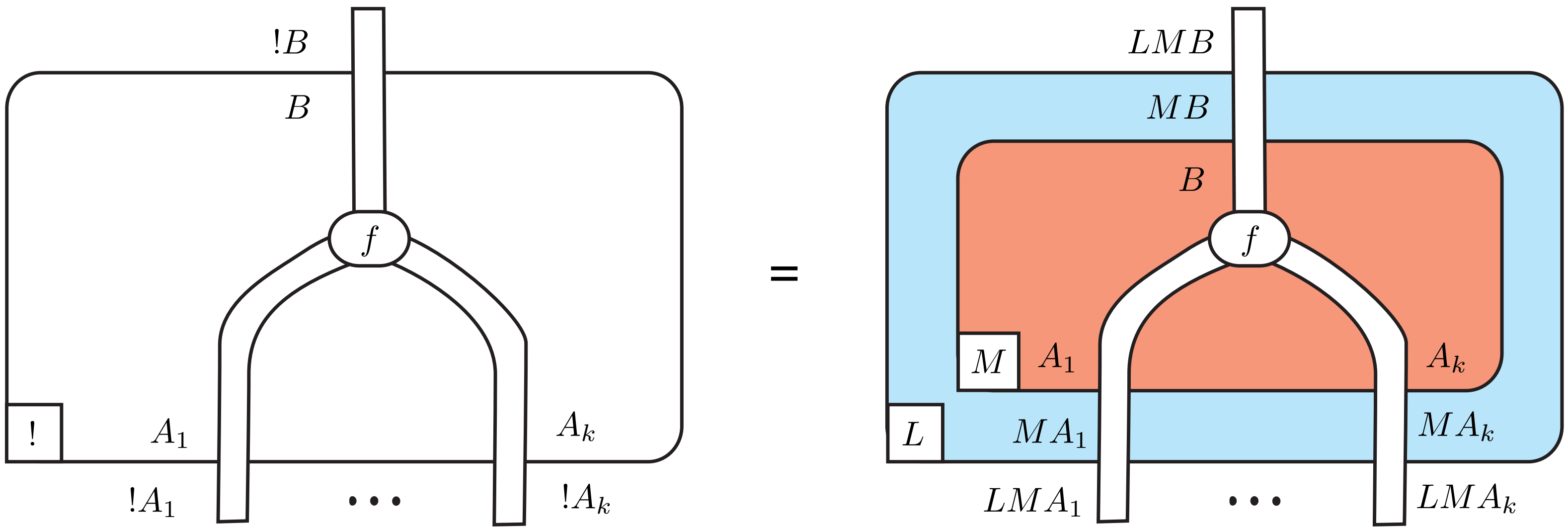
$$A \longmapsto A^{\perp}$$

$$A^{\perp} \longleftarrow\!\shortmid A$$

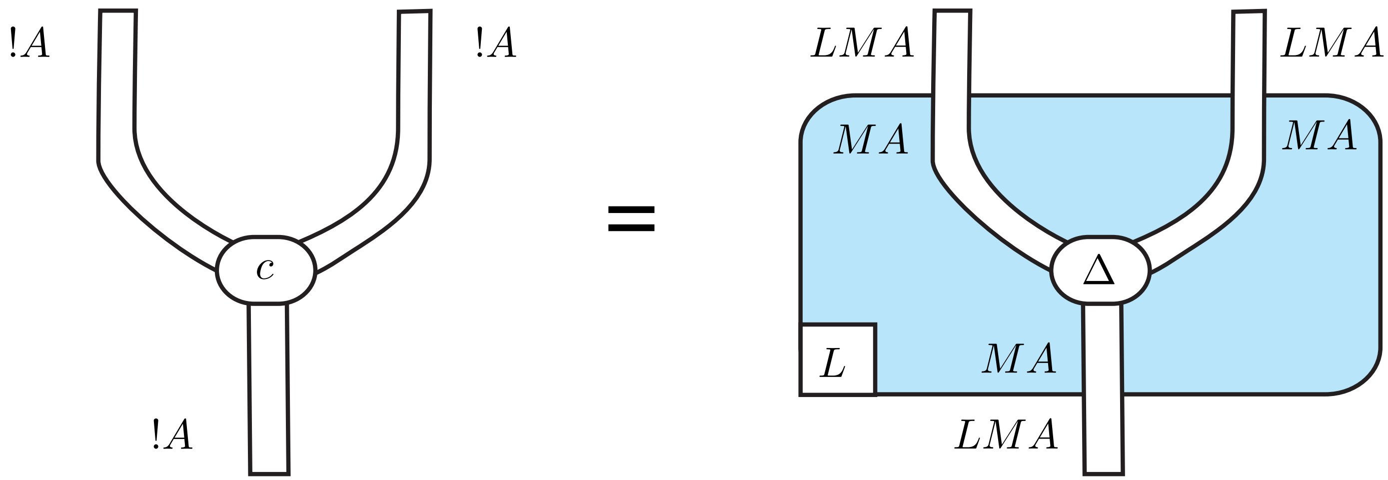
$$! = \mathrm{Lin} \circ \mathrm{Mult}$$

$$? = \mathrm{Lin}^{op} \circ \mathrm{Mult}^{op}$$

Decomposition of the exponential box



Decomposition of the contraction node

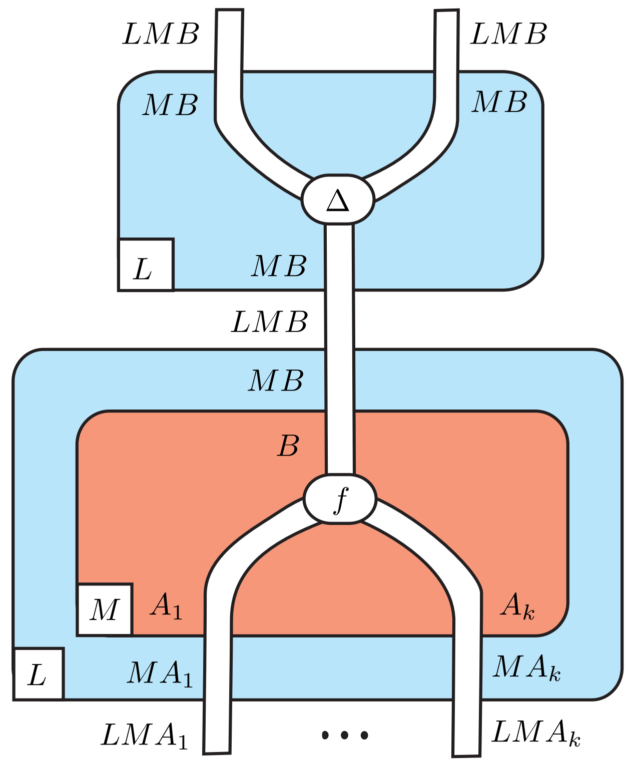


Coh!

||

$!A \to !A \otimes !A$ $\qquad$ $A \xrightarrow[\Delta_A]{\mathrm{diag}_A} A \& A$ in $\mathcal{M}$

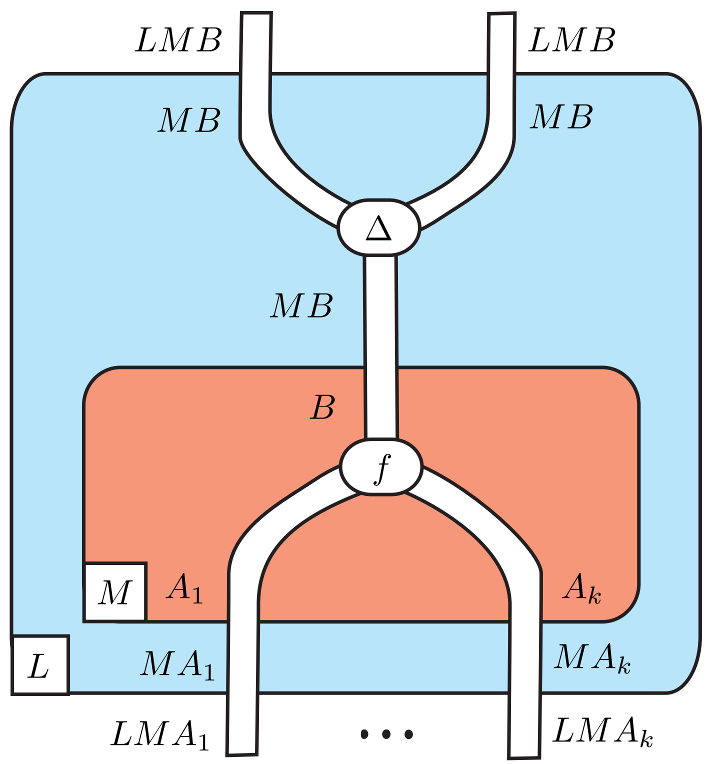
Illustration: duplication of the exponential box



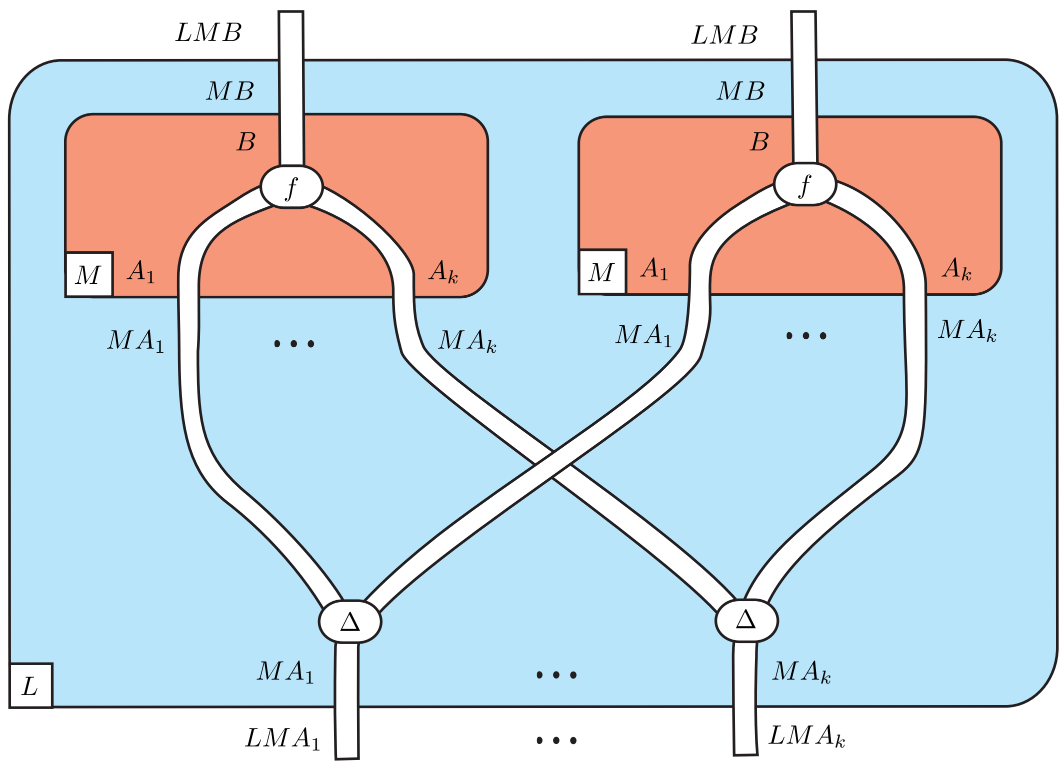
Duplication (step 1)

$\Delta : \mathrm{Id} \to \mathrm{Id} \times \mathrm{Id}$

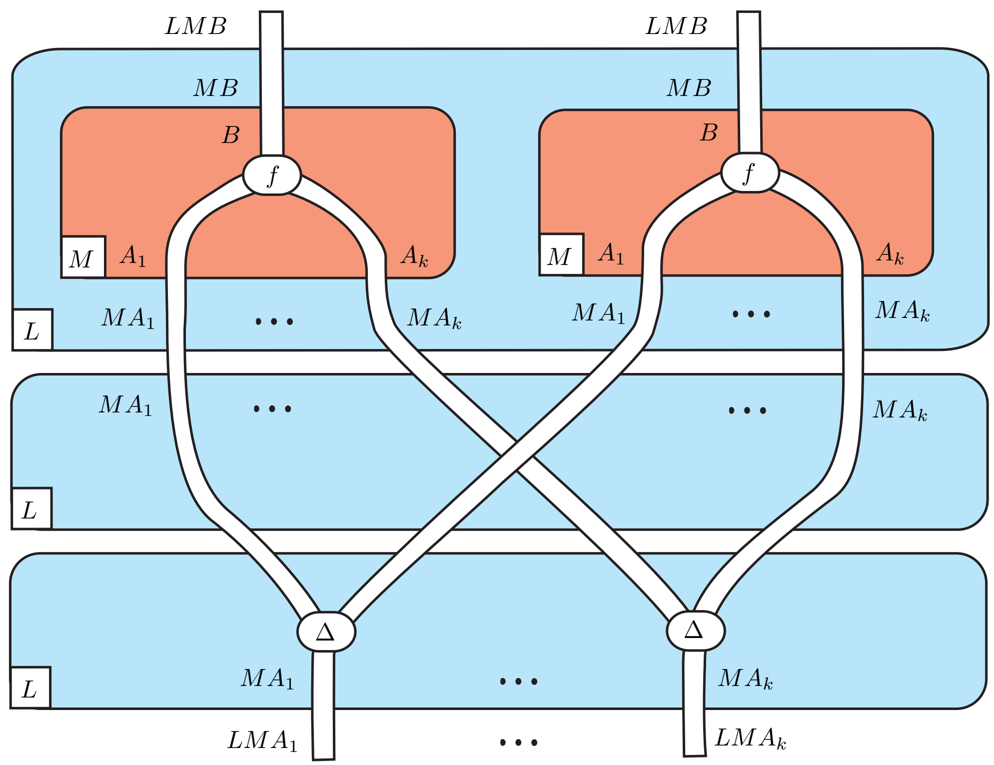
$\Delta_A : A \to A \times A$



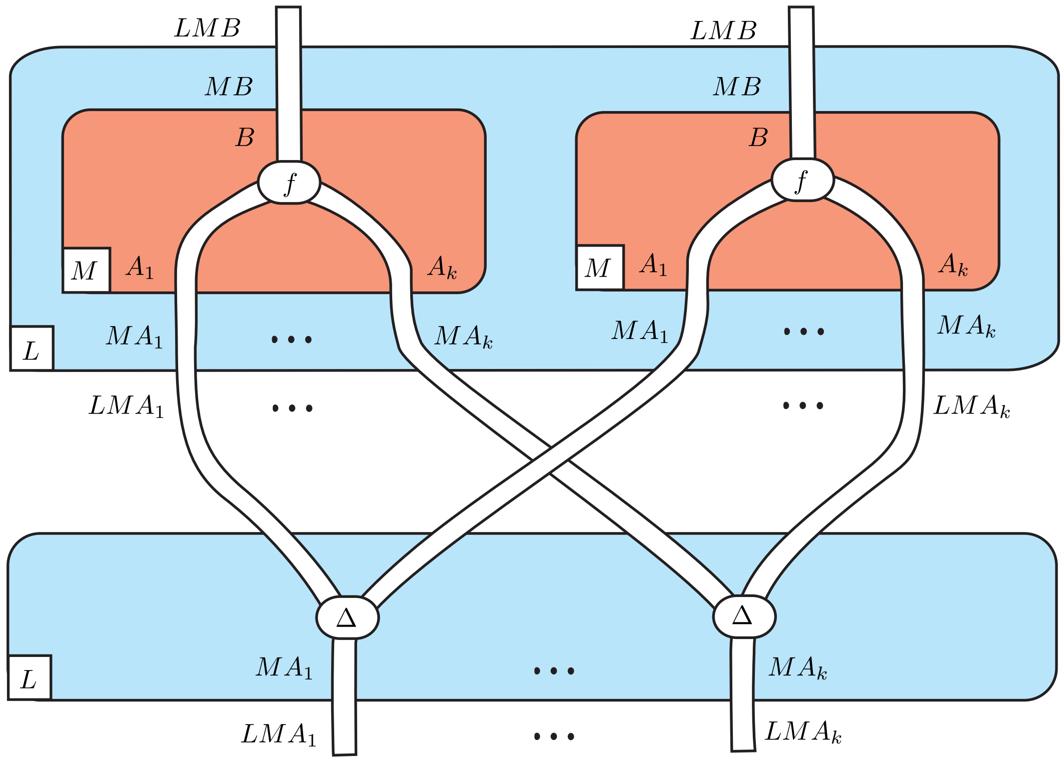
Duplication (step 2)



Duplication (step 3)

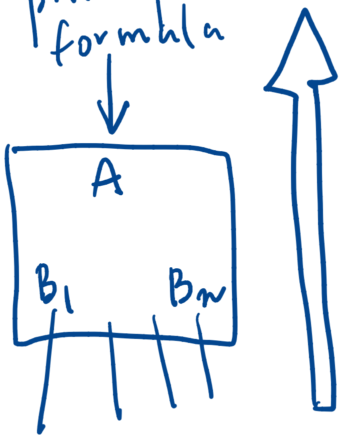


Duplication (step 4)

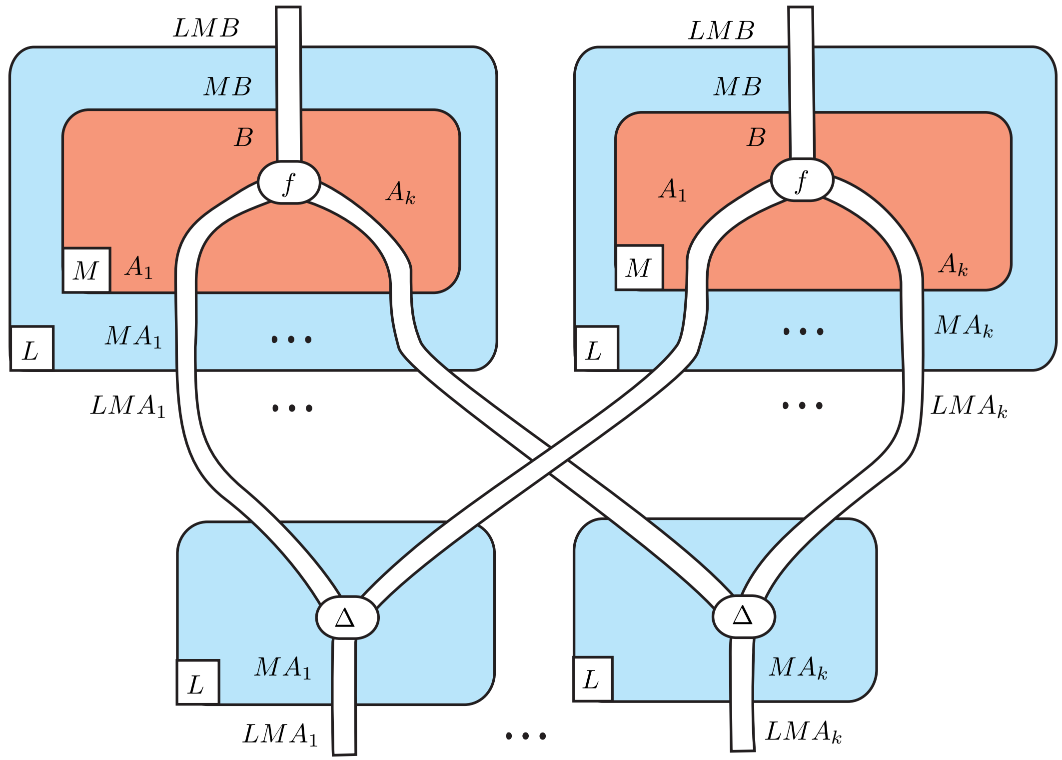


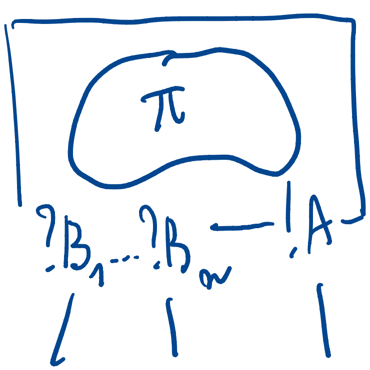
string diagrams

principal formula



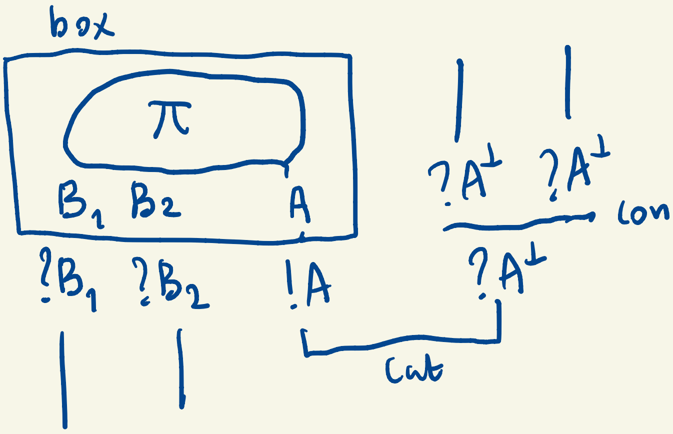
Duplication (step 5)

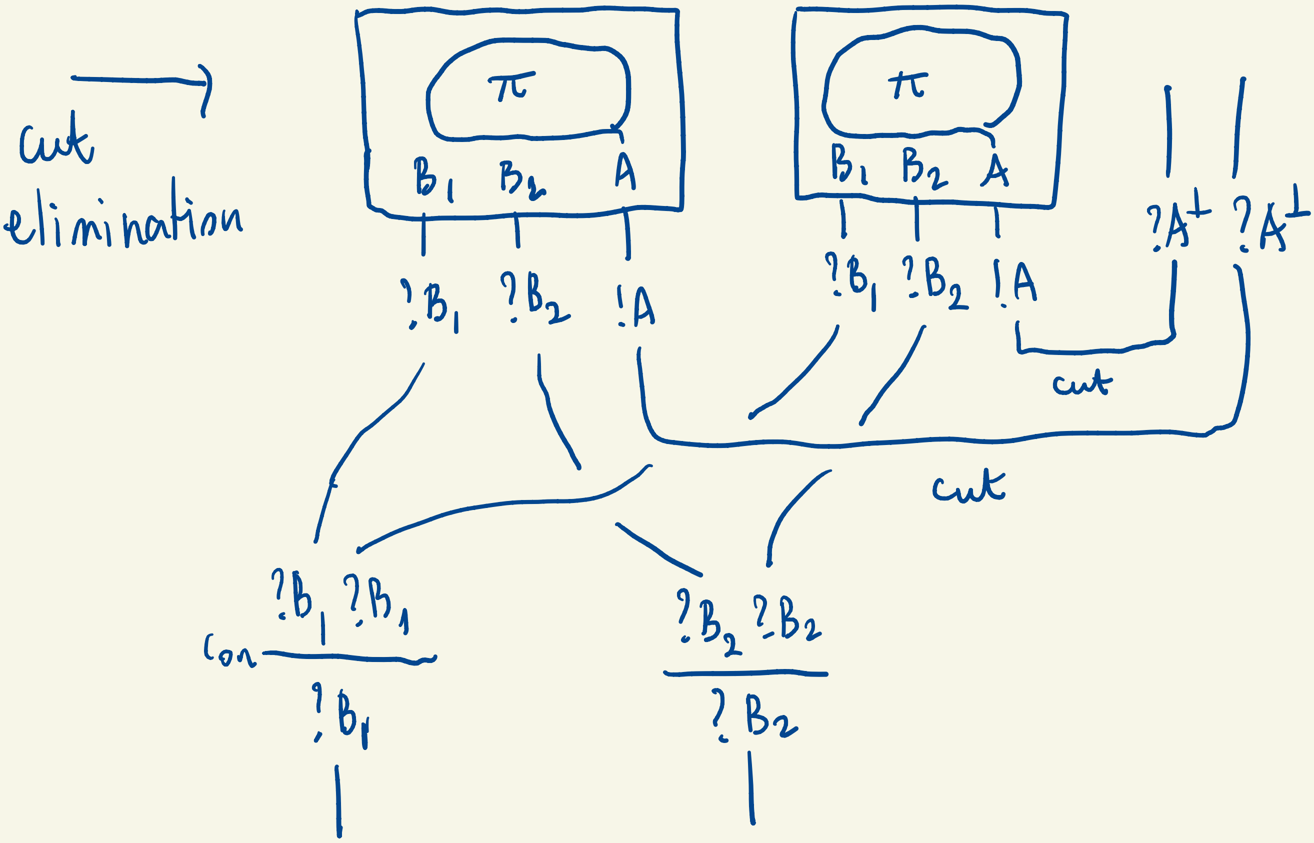




proof nets

In linear logic (proof nets)





Five steps instead of one!

Follows faithfully the categorical proof of soundness.