Modèles des langages de programmation

Domaines, catégories, jeux

Programme de cette troisième séance:

Modèle cohérent de la logique linéaire ;

Catégories monoidales fermées

Synopsis

1 — coherence spaces

2 — monoidal categories

3 — string diagrams

4 — symmetric monoidal categories

5 — symmetric monoidal closed categories

6 — $*$-autonomous categories

Coherence spaces

Coherence spaces

This model is at the origin of linear logic (1986)

Linear decomposition of the category with

- ▷ dI-domains
- ▷ stable functions

Coherence spaces

Since then, several linearizations have been achieved:

Concrete data structures Berry-Curien 1985	$\Rightarrow$	Sequential games Lamarche 1992
dI-domains with coherence and strongly stable functions Bucciarelli-Ehrhard 1991	$\Rightarrow$	Hypercoherence spaces Ehrhard 1993
Bidomains Berry 1979	$\Rightarrow$	Bistructures Curien-Plotkin-Winskel 1996

Coherence spaces

A coherence space is a pair $A = (|A|, \mathrel{\overset{\frown}{\smile}}_A)$ consisting of

- a set $|A|$ called the web of A
- a reflexive and symmetric relation

$$\mathrel{\overset{\frown}{\smile}}_A \quad \subseteq \quad |A| \times |A|$$

called its coherence.

So, coherence space is a pedantic to say graph.

Notation: one writes

- $a \frown_A a'$ if $a \mathrel{\overset{\frown}{\smile}}_A a'$ and $a \neq a'$.
- $a \asymp_A a'$ if $\neg\,(a \mathrel{\overset{\frown}{\smile}}_A a')$ or $a = a'$.

Coherence spaces

Example 1. the coherence spaces $0 = \top$ of empty web and $1 = \bot$ of singleton web.

Example 2. for every set X, the **discrete** coherence space

$$(X, =)$$

In particular,

$$\begin{aligned} B &= (\{V, F\}, =) \\ N &= (\mathbb{N}, =) \end{aligned}$$

Interaction

A clique u in a graph A is a subset of $|A|$ such that

$$\forall (a, a') \in u, \qquad a \frown_A a'$$

An anticlique v in a graph A is a subset of $|A|$ such that

$$\forall (a, a') \in v, \qquad a \asymp_A a'$$

We are going to interpret

- the simple types of the λ-calculus as graphs,
- the programs u of type A as cliques of A,
- the counter-programs v of type A as anti-cliques of A
- the interaction between u and v as the intersection $u \cap v$.

Remark: $u \cap v$ contains at most one element (= the result)

Negation

Let A be a coherence space.

The negation $A^{\perp}$ is defined as its dual graph:

- $|A^{\perp}| = |A|$
- $a \frown_{A^{\perp}} a'$ if and only if $a \asymp_A a'$.

Remark: an anti-clique of A is a clique of $A^{\perp}$.

So, one makes a clique of A interact with a clique of $A^{\perp}$.

This reveals a fundamental duality between Player and Opponent:

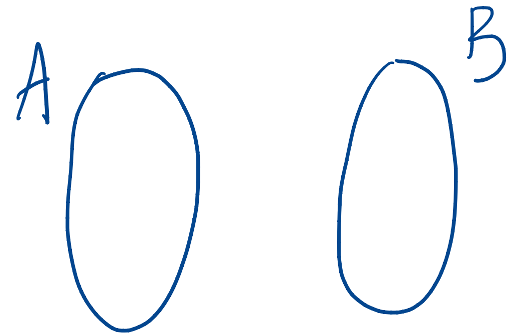
$$A = (A^{\perp})^{\perp}$$

The sum (plus)

The sum of two coherence spaces A and B

$$A \oplus B$$

is defined as their sum as graphs:

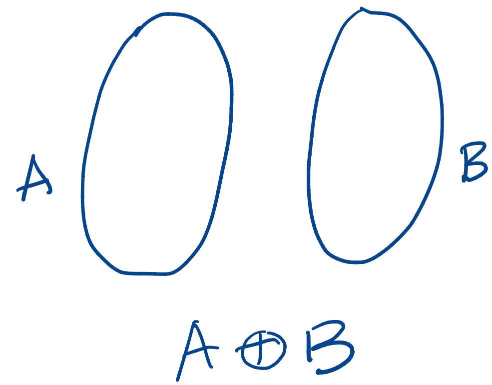


— $|A \oplus B| = |A| + |B|$

— $a \asymp_{A \oplus B} a'$ if and only if $a \asymp_A a'$,

— $b \asymp_{A \oplus B} b'$ if and only if $b \asymp_B b'$,

— $a \asymp_{A \oplus B} b$ never.



Exercise. Show that the graphs $A \oplus 0$ and A are isomorphic.

exercise: compare

$D_{A \oplus B}$

with D_A and D_B

exercise

$D_{A\&B} \cong D_A \times D_B$

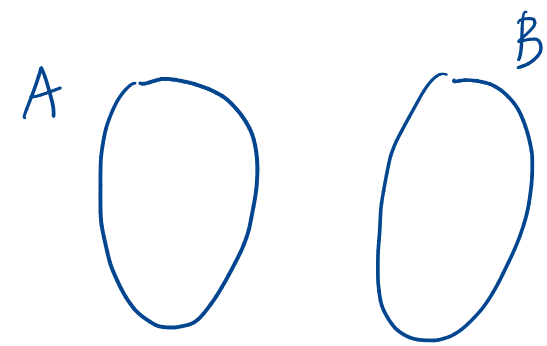
where D_A is the domain of cliques of A.

The product (with)

The product of two coherence spaces A and B

$$\boxed{A \,\&\, B}$$

is defined as an « alternative » sum of the two graphs:

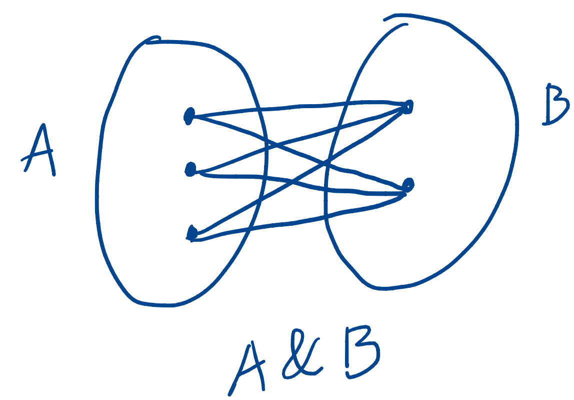


— $|A\&B| = |A| + |B|$

— $a \smile_{A\&B} a'$ if and only if $a \smile_A a'$,

— $b \smile_{A\&B} b'$ if and only if $b \smile_B b'$,

— $a \smile_{A\&B} b$ always.



Exercise. Show that

$$\boxed{A\&B \;=\; (A^{\perp} \oplus B^{\perp})^{\perp}}$$

de Morgan dual of $\oplus$

$A \oplus B$ disjunction

$A \& B$ conjunction

additive connectives

Tensor product

The tensor product of two coherence spaces A and B

$$\boxed{A \otimes B}$$

is defined as their product as graphs:

— $\boxed{|A \otimes B| = |A| \times |B|}$

— $\boxed{(a, b) \supset_{A \otimes B} (a', b')}$ if and only if $\boxed{a \supset_A a' \text{ and } b \supset_B b'.}$

Exercise. Show that the graphs $\boxed{A \otimes 1}$ and $\boxed{A}$ are isomorphic.

$$e^{x+y} = e^x e^y$$

$$!(A \& B) \cong !A \otimes !B$$

additive

multiplicative

Parallel product, or par

The parallel product of two coherence spaces A and B

$$A \parr B$$

is defined as an « alternative » product of the two graphs:

— $|A \parr B| = |A| \times |B|$

— $(a, b) \frown_{A \parr B} (a', b')$ if and only if $a \frown_A a'$ ou $b \frown_B b'$.

deMorgan dual of $\otimes$

Exercise. Show that

$$\boxed{A \parr B \ = \ (A^{\perp} \otimes B^{\perp})^{\perp}}$$

$\otimes$ conjunction

$\parr$ disjunction

multiplicative connectives.

Distributivity laws

$$A \otimes (B \oplus C) \;\cong\; (A \otimes B) \oplus (A \otimes C)$$

$$A \parr (B \& C) \;\cong\; (A \parr B) \& (A \parr C)$$

Reminiscent of

$$A \times (B + C) \cong (A \times B) + (A \times C)$$

in the category **Set**. Thus, one calls

— additives the connectives $\oplus$ and $\&$, and their units 0 and $\top$,

— multiplicatives the connectives $\otimes$ and $\parr$, and units 1 and $\bot$.

Remark: the sign $\cong$ means graph-isomorphism, or isomorphism in the category $\mathbf{Coh}$ constructed later.