Monoidal categories

The first step towards a categorical account of linear logic

Intuition

Redo everything as in a cartesian closed category...
but replace the cartesian product $\times$ by an arbitrary bifunctor $\otimes$.

1. replace the **universal properties** of $\times$ by **coherence diagrams** of $\otimes$.

One obtains in this way a **symmetric monoidal category.**

2. replace the adjunction

$$\frac{\boxed{A \times} B \longrightarrow C}{B \longrightarrow \boxed{A \Rightarrow} C} \qquad A \times - \dashv A \Rightarrow -$$

by an adjunction

$$\frac{\boxed{A \otimes} B \longrightarrow C}{B \longrightarrow \boxed{A \multimap} C} \qquad A \otimes - \dashv A \multimap -$$

One obtains a **symmetric monoidal closed category (smcc**)
where **intuitionistic linear logic** (linear λ-calculus) may be interpreted.

Bénabou

Monoidal Categories

A monoidal category is a category $\mathcal{C}$ equipped with a functor:

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

an object:

$$I$$

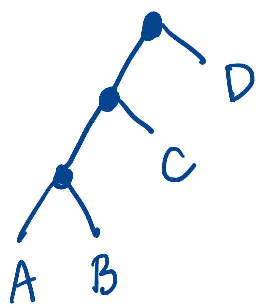
and three natural isomorphisms:

$$(A \otimes B) \otimes C \xrightarrow{\alpha} A \otimes (B \otimes C)$$

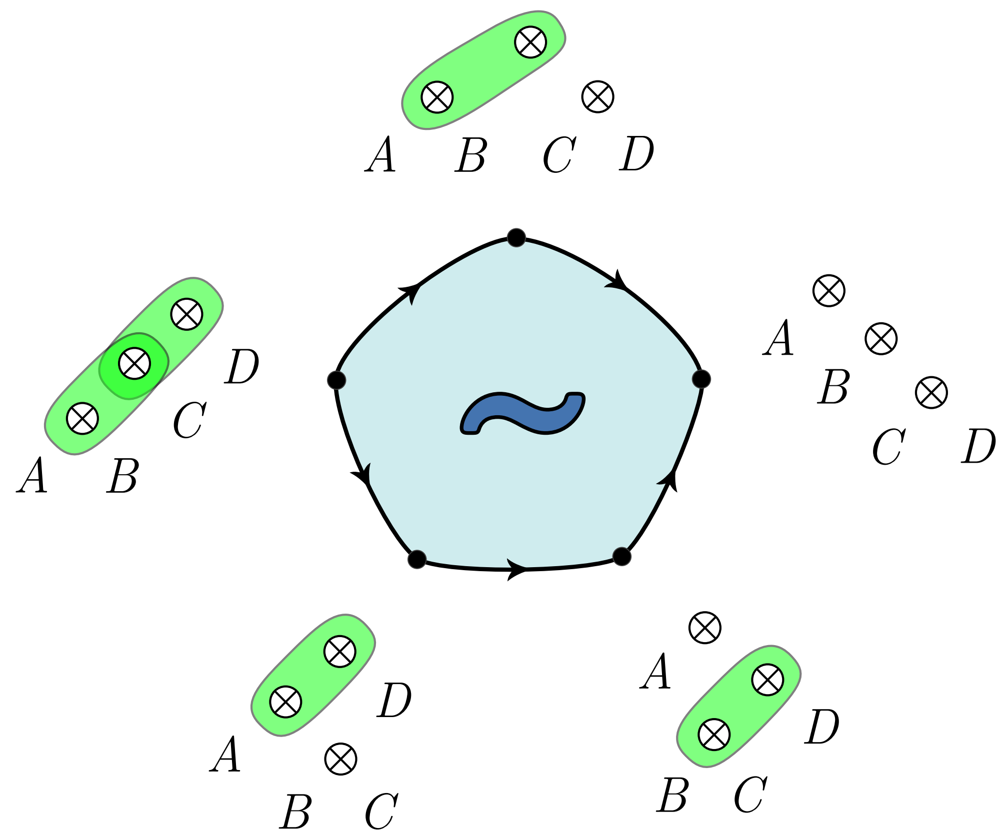
$$I \otimes A \xrightarrow{\lambda} A \qquad\qquad A \otimes I \xrightarrow{\rho} A$$

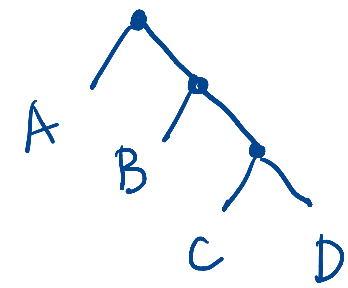
satisfying two coherence properties.

MacLane's pentagon



"left comb"

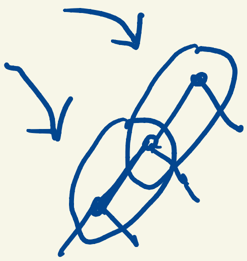


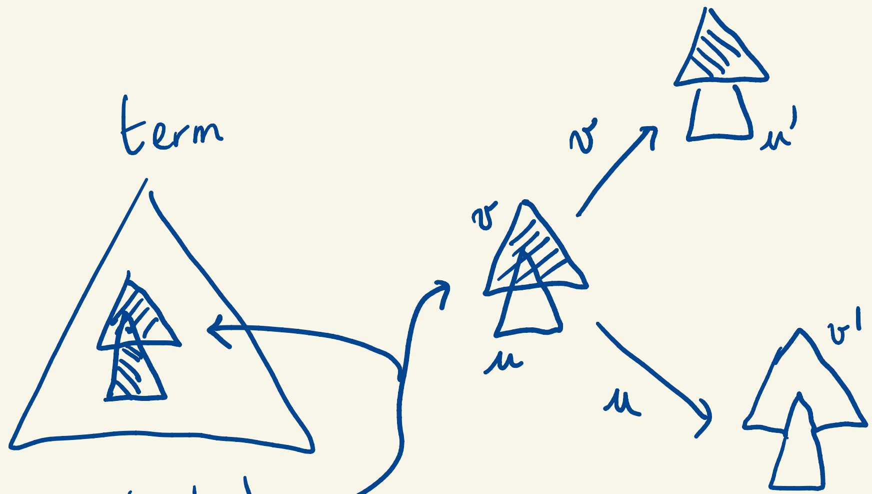


"right comb"

The associativity patterns are indicated in green

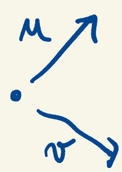
critical pair



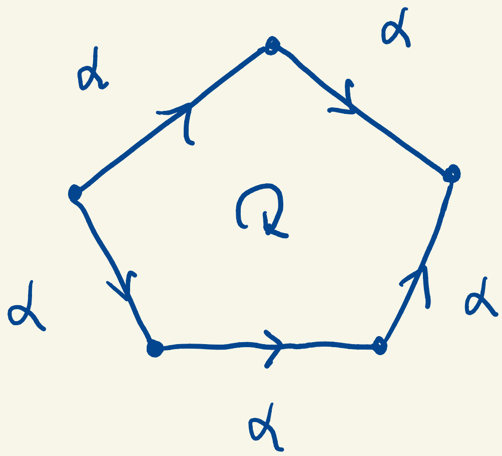


critical pair in rewriting theory

usually produce "non determinism" in rewriting

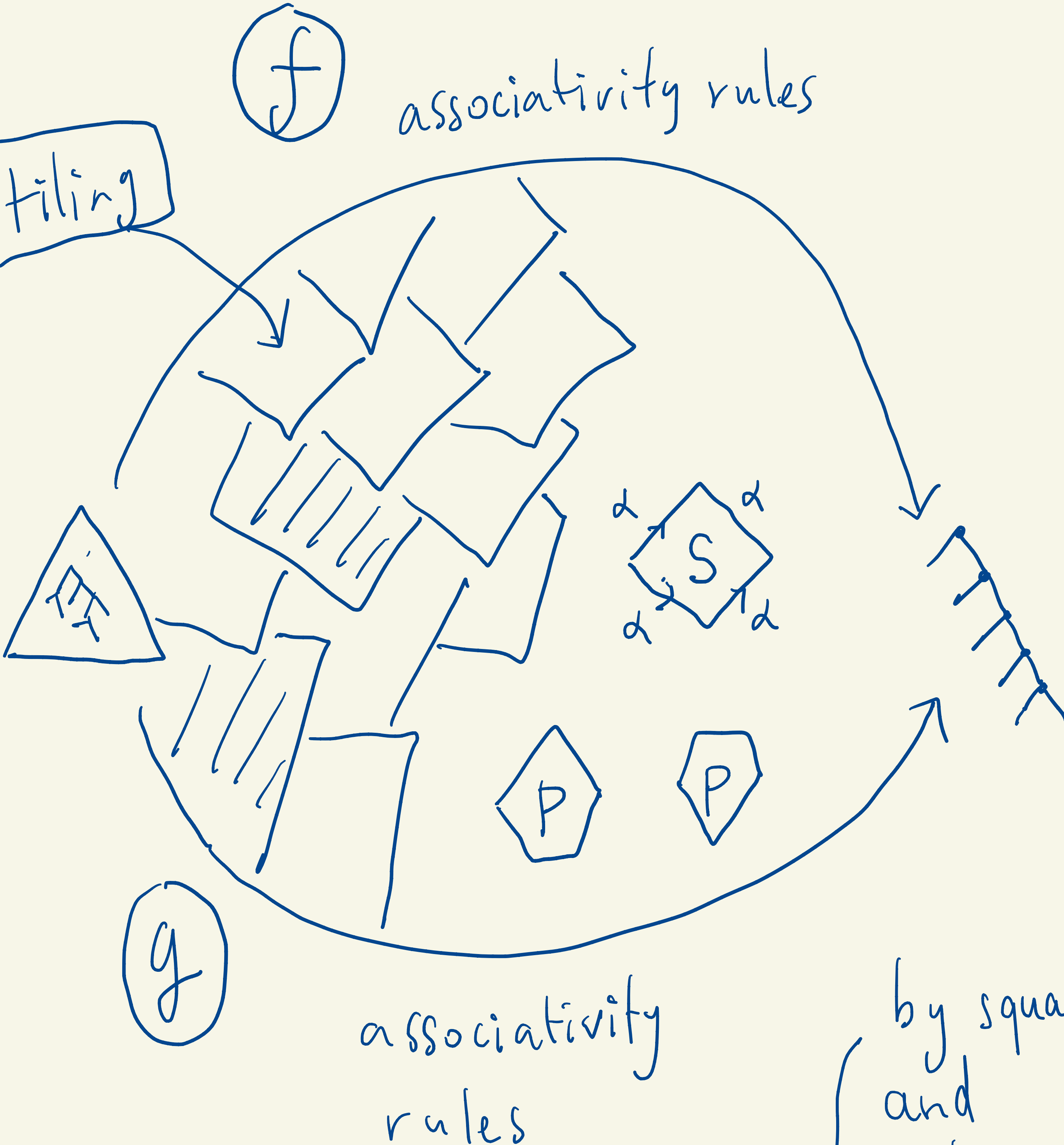


no confluence



confluence diagram

coherence property : the diagram induces the same composite morphism in $\mathcal{C}$



it is possible to reorganise by square and pentagon permutations

f in order to obtain g

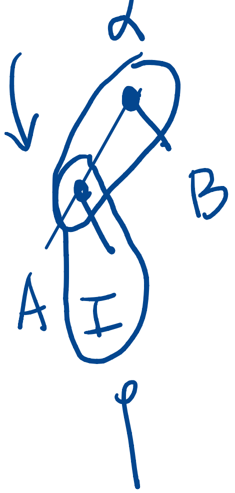
what the coherence property ensures

is that the two sequences f, g

of associativity rules

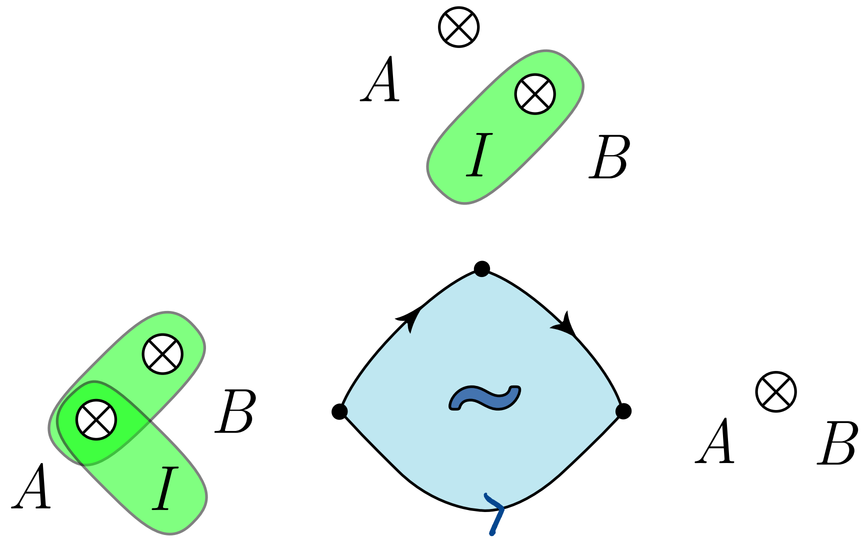
define the <u>same</u> morphism

in the underlying monoidal category.



MacLane's triangle

$$\begin{tikzcd}
 & A\otimes(I\otimes B) \arrow[dr, "A\otimes\lambda_B"] & \\
(A\otimes I)\otimes B \arrow[ur, "\text{assoc}"] \arrow[rr, bend right, "\rho_A\otimes B"'] & & A\otimes B
\end{tikzcd}$$



The associativity patterns are indicated in green

Coherence theorem

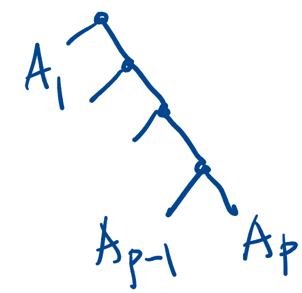
Idea: benefit from one consequence of the universality property... but without the universality property.

Given a sequence $A_1, ..., A_p$ of objects of a monoidal category $\mathcal{C}$, consider the words w over $A_1, \ldots, A_p$ defined as

- ▷ the object $\boxed{I}$ when $p = 0$,
- ▷ the object $\boxed{u \otimes v}$ where $\begin{cases} u & \text{is a word on } (A_1, ..., A_m) \\ v & \text{is a word on } (A_{m+1}, ..., A_p) \end{cases}$
  for some $1 \leq m \leq p$.

Among these words, one finds the **canonical** word

$$\boxed{(\cdots(A_1 \otimes A_2) \otimes \cdots A_p)}$$



Coherence theorem.
There exists a unique structural morphism built from "α, λ, ρ" from any word on $(A_1, ..., A_p)$ to the canonical word.

String diagrams

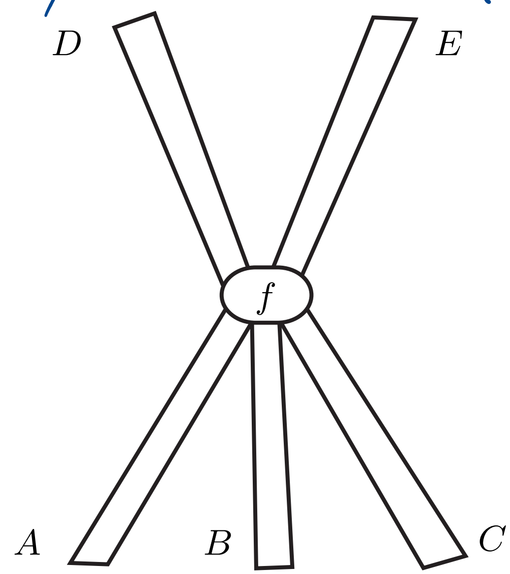
A graphical notation for monoidal categories

Penrose — Joyal-Street

String Diagrams

A morphism $f : A \otimes B \otimes C \longrightarrow D \otimes E$ is depicted as:

2 outputs

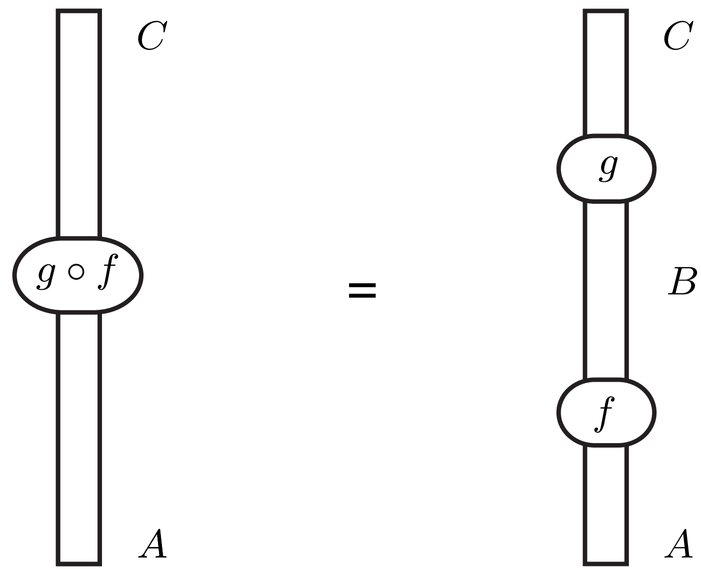


3 inputs

Composition

The morphism $A \xrightarrow{f} B \xrightarrow{g} C$ is depicted as

Sequential
Composition

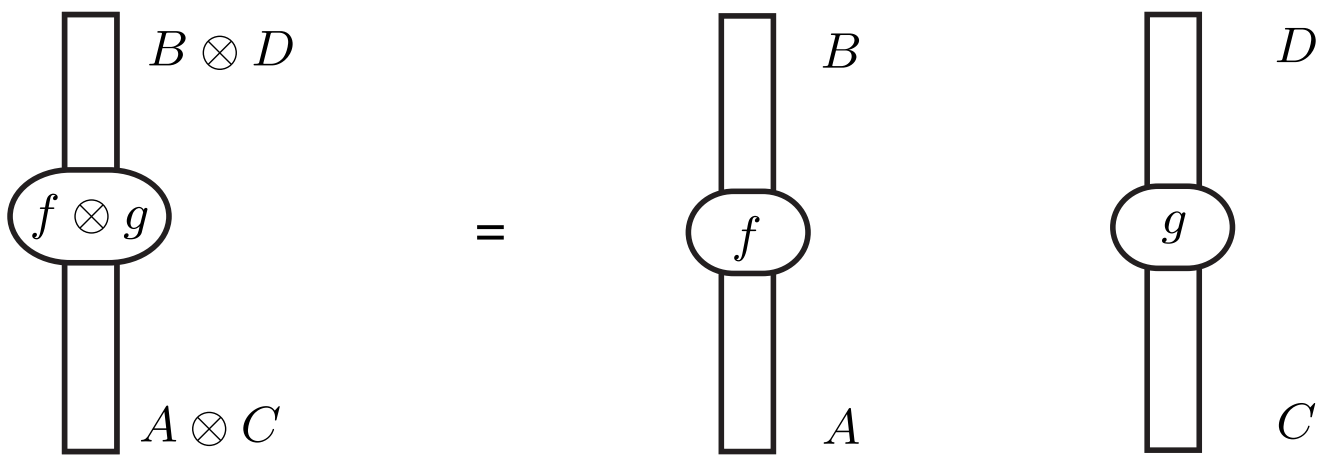


Vertical composition

Tensor product

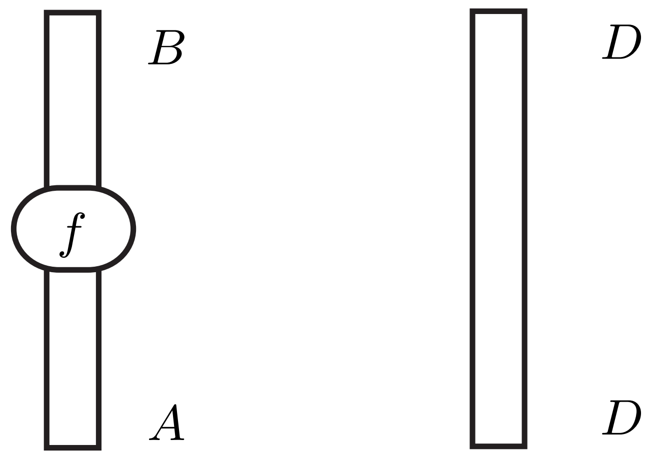
The morphism $(A \xrightarrow{f} B) \otimes (C \xrightarrow{g} D)$ is depicted as

parallel composition



Horizontal tensor product

Example



$f \otimes id_D$

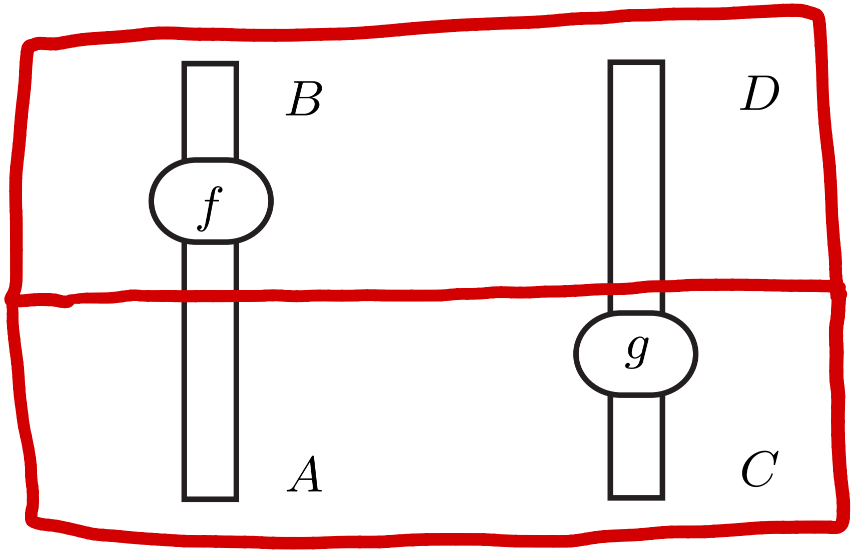
$f : A \longrightarrow B$ $\qquad$ $g : C \longrightarrow D$

Example

$(f \otimes id_D)$

$\circ$

$(id_A \otimes g)$



$B \otimes D$

$\uparrow$

$A \otimes C$

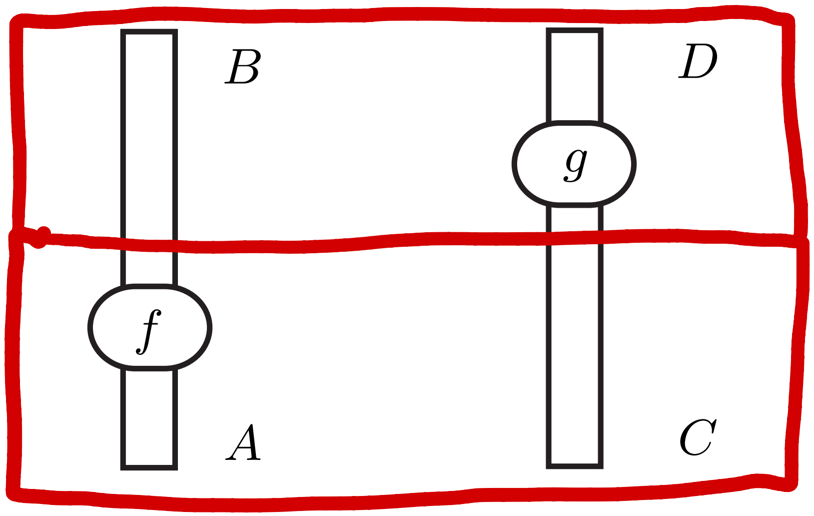
$$\boxed{(f \otimes id_D) \circ (id_A \otimes g)}$$

Example

$id_B \otimes g$

$\circ$

$f \otimes id_C$

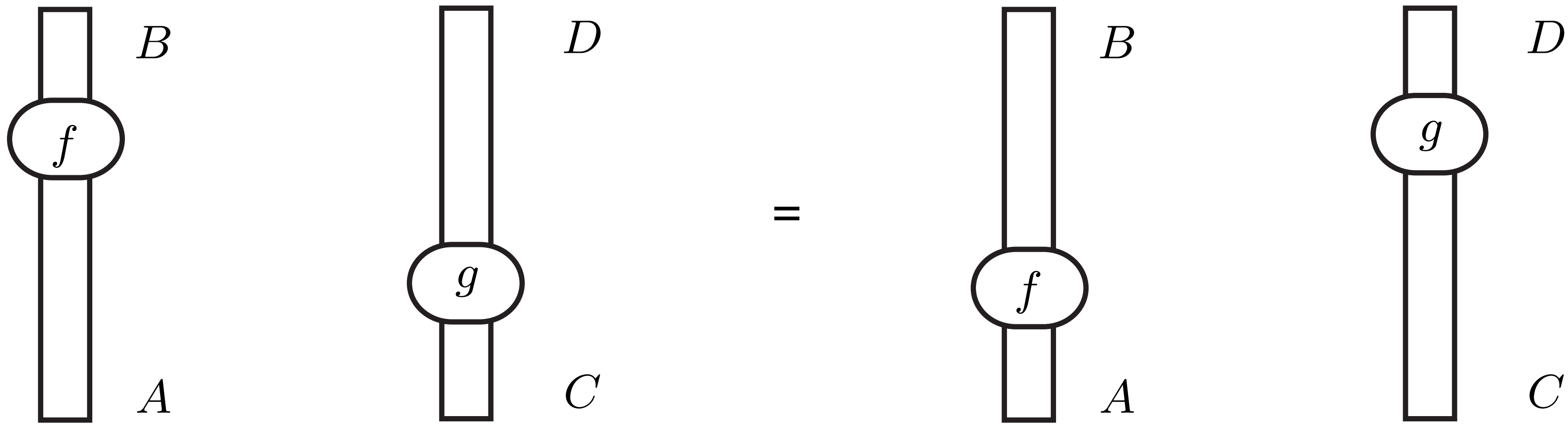


$B \otimes D$

$A \otimes C$

$\boxed{(id_B \otimes g) \circ (f \otimes id_C)}$

Meaning preserved by deformation



$$(f \otimes id_D) \circ (id_A \otimes g) \quad = \quad (id_B \otimes g) \circ (f \otimes id_C)$$

functoriality of $\otimes$ $\qquad\qquad$ functoriality of $\otimes$

$$(f \circ id_A) \otimes (id_D \circ g) \qquad (id_B \circ f) \otimes (g \circ id_C)$$

$$= f \otimes g =$$

Symmetric monoidal categories

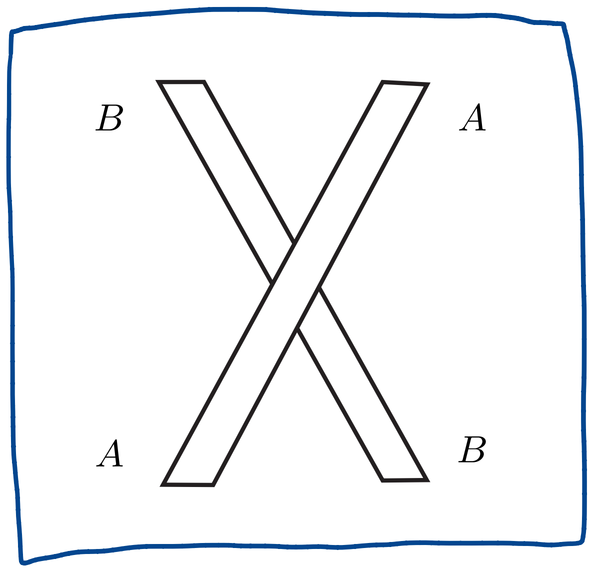
The structure of the category of coherence spaces (1)

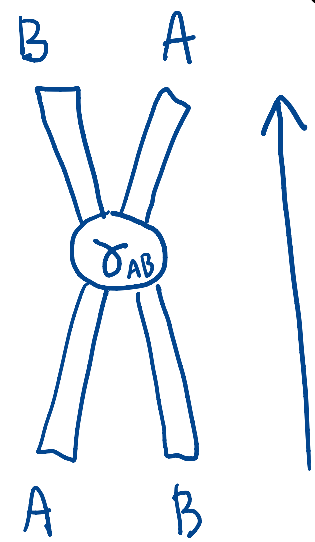
Braided categories

A monoidal category $\mathcal{C}$ equipped with a family of isomorphisms

$$\gamma_{A,B} \quad : \quad A \otimes B \quad \longrightarrow \quad B \otimes A$$

natural in A and B, represented pictorially as the positive braiding





Braided categories

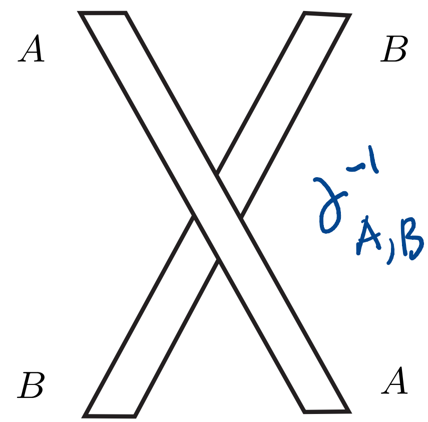
As expected, the inverse map

$$\gamma^{-1}_{A,B} \quad : \quad B \otimes A \quad \longrightarrow \quad A \otimes B$$

is represented pictorially as the negative braiding

$$\boxed{\gamma^{-1}_{A,B} \circ \gamma_{A,B} = id_{A \otimes B}}$$

$$\boxed{\gamma_{A,B} \circ \gamma^{-1}_{A,B} = id_{B \otimes A}}$$



1

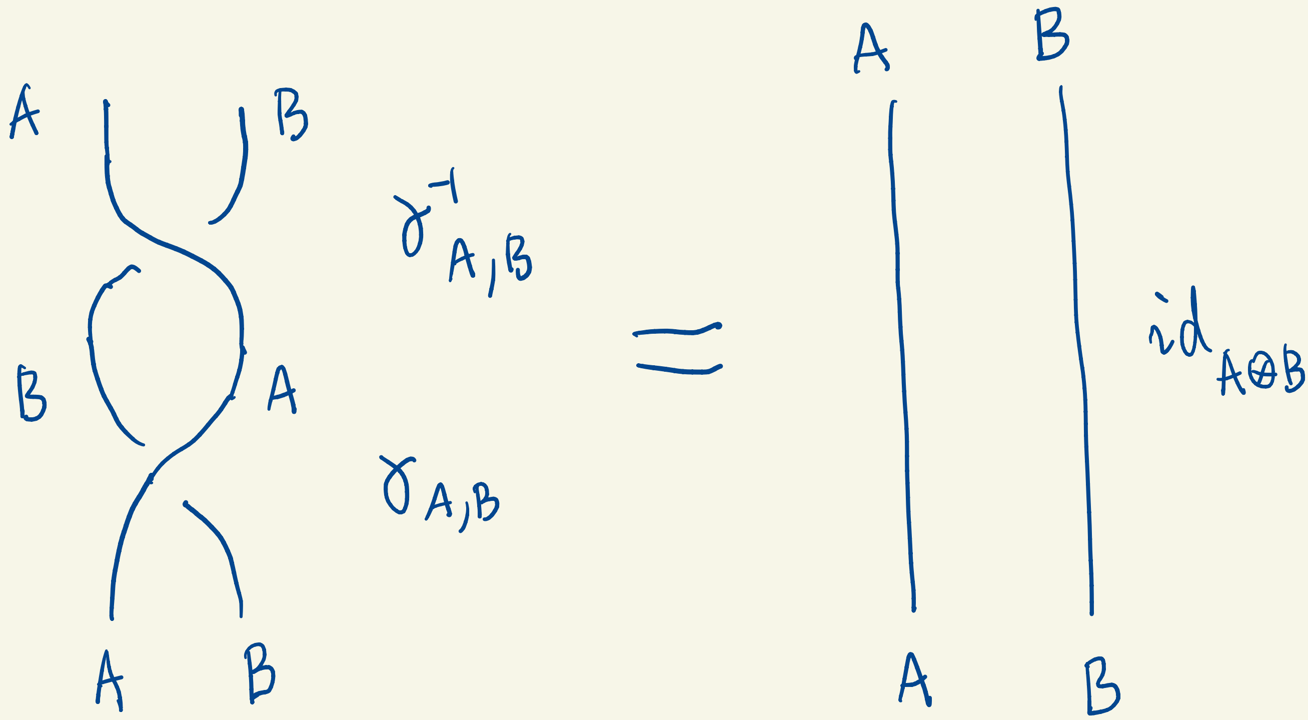
$$\boxed{\gamma^{-1}_{A,B}\circ\gamma_{A,B} = id_{A\otimes B}}$$

$$\boxed{id_{A\otimes B} = id_A \otimes id_B}$$

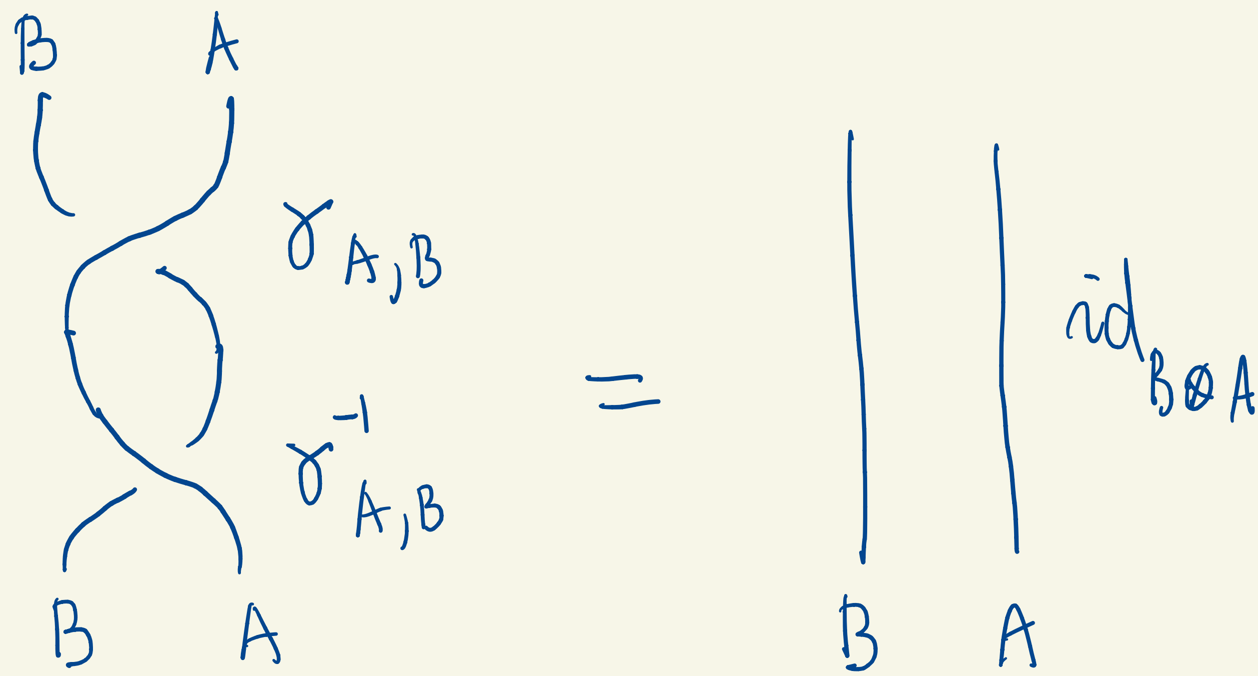
$$\boxed{\gamma_{A,B}\circ\gamma^{-1}_{A,B} = id_{B\otimes A}}$$

2

1



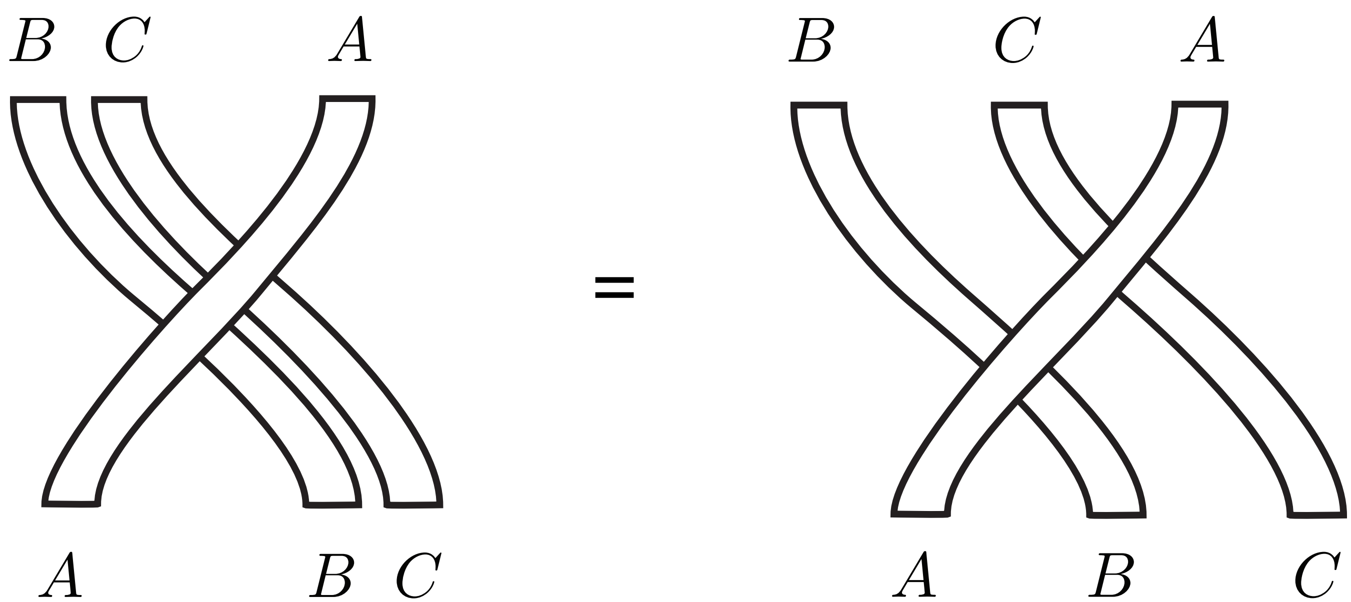
2



Coherence diagram for braids (1)

$$
\begin{tikzcd}
 & A \otimes (B \otimes C) \arrow[r, "\gamma"] & (B \otimes C) \otimes A \arrow[dr, "\alpha"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha"] \arrow[dr, "\gamma\otimes C"'] & & & B \otimes (C \otimes A) \\
 & (B \otimes A) \otimes C \arrow[r, "\alpha"] & B \otimes (A \otimes C) \arrow[ur, "B\otimes\gamma"'] &
\end{tikzcd}
$$

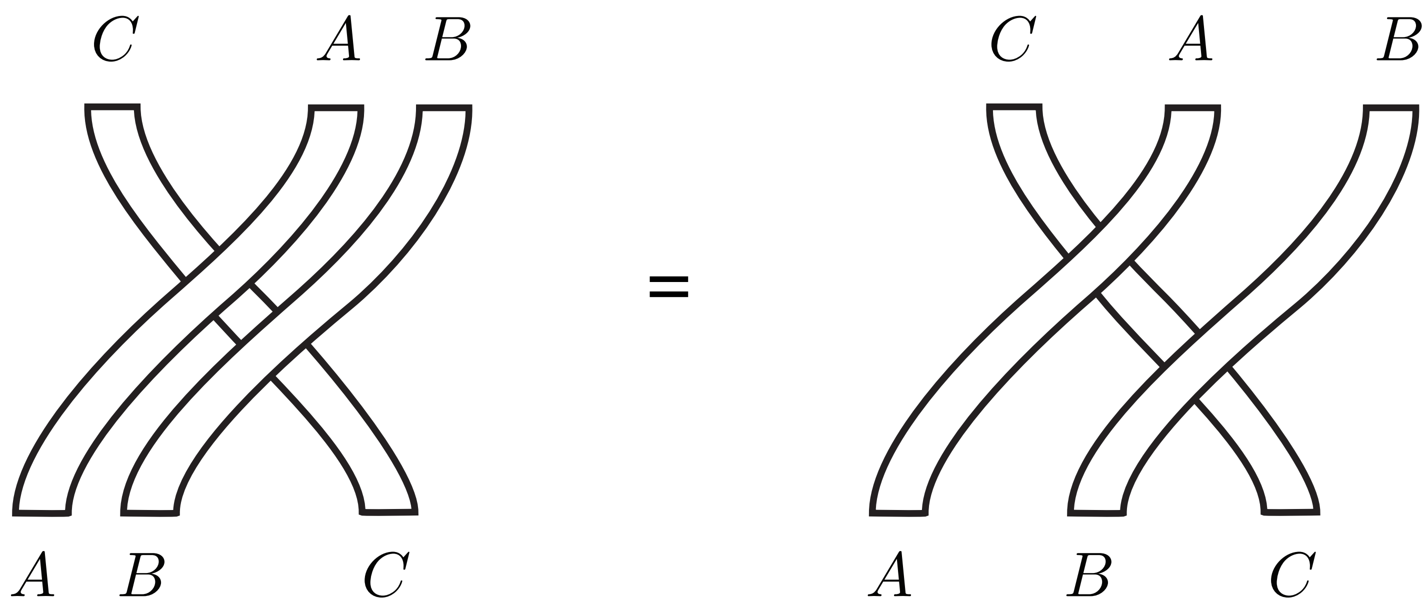
Same coherence diagram in string diagrams



Coherence diagram for braids (2)

$$\begin{tikzcd}
 & (A \otimes B) \otimes C \arrow[r, "\gamma"] & C \otimes (A \otimes B) \arrow[dr, bend left, "\alpha^{-1}"] & \\
A \otimes (B \otimes C) \arrow[ur, bend left, "\alpha^{-1}"] \arrow[dr, bend right, "A\otimes\gamma"'] & & & (C \otimes A) \otimes B \\
 & A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}"] & (A \otimes C) \otimes B \arrow[ur, bend right, "\gamma\otimes B"'] &
\end{tikzcd}$$

Same coherence diagram in string diagrams



$$\boxed{\gamma^{-1}_{B,A} = \gamma_{A,B}}$$

equivalently

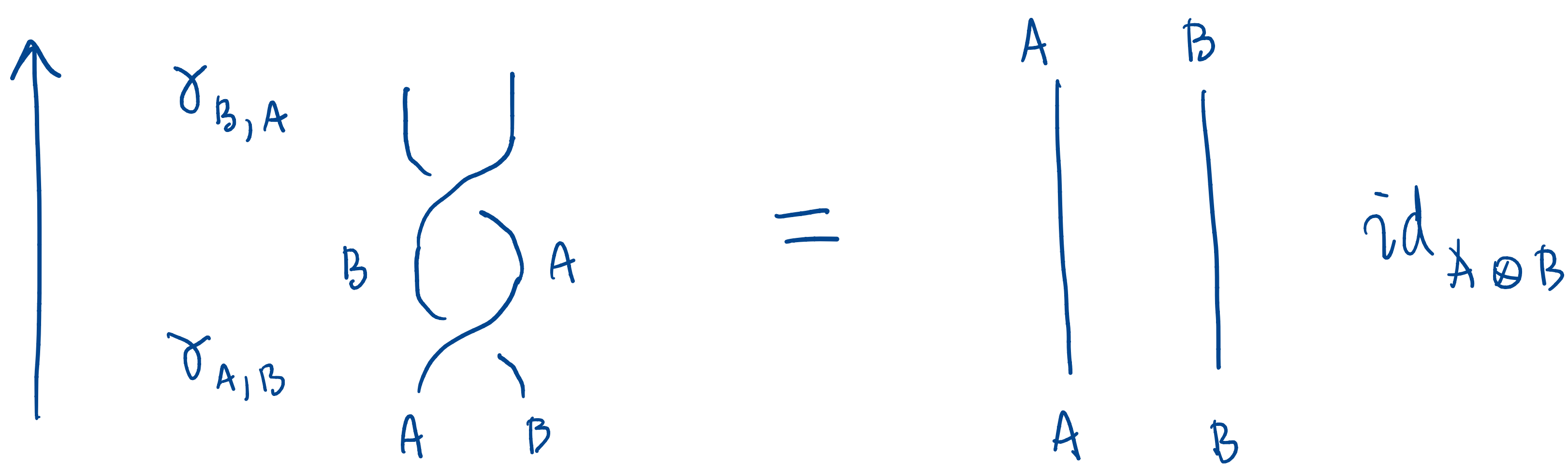
Symmetries

A symmetry in a monoidal category is a braiding

$$\gamma_{A,B} \;:\quad A\otimes B \longrightarrow B\otimes A$$

satisfying the additional equality

$$A\otimes B \xrightarrow{\gamma_{A,B}} B\otimes A \xrightarrow{\gamma_{B,A}} A\otimes B \quad = \quad A\otimes B \xrightarrow{id_{A\otimes B}} A\otimes B$$



Examples of monoidal categories

1

Monoidal:

▷ The category $End(\mathcal{C})$ of endofunctors of a category $\mathcal{C}$ with **endofunctors** as objects, **natural transformations** as morphisms, and **composition** as tensor product.

2

Braided:

▷ The category **Braid** with natural numbers as objects and braids as morphisms

Symmetric:

3

▷ The category **Perm** with natural numbers as objects and permutations as morphisms.

▷ Every **cartesian category** with the cartesian product as tensor.

▷ The opposite of a symmetric monoidal category.

▷ The category $\mathbf{Coh}$ with tensor product $\otimes$ and unit 1.

1 The monoidal category of endofunctors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{C} \arrow[from=U, to=D, Rightarrow, "\theta"] \end{tikzcd}$$

$\theta : F \Rightarrow G : \mathcal{C} \longrightarrow \mathcal{C}$

θ natural transformation from F to G.

$\mathrm{End}(\mathcal{C})$

objects = functors $F : \mathcal{C} \longrightarrow \mathcal{C}$

maps = natural transformations $\theta : F \Rightarrow G : \mathcal{C} \to \mathcal{C}$

tensor product = $F, G \longmapsto F \circ G$

It is not braided / symmetric in general

$F \circ G \not\cong G \circ F$ in general.

$F \overset{\theta}{\Longrightarrow} G \overset{\theta'}{\Longrightarrow} H$

$FA \xrightarrow{\theta_A} GA \xrightarrow{\theta'_A} HA$

$$\boxed{(\theta' \circ \theta)_A = \theta'_A \circ \theta_A}$$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "H"] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C} \arrow[r, "H'"] & \mathcal{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}
\end{tikzcd}
$$

$\theta \circ \theta'$ vertical composition

$F \circ F'$ $\otimes$ product.

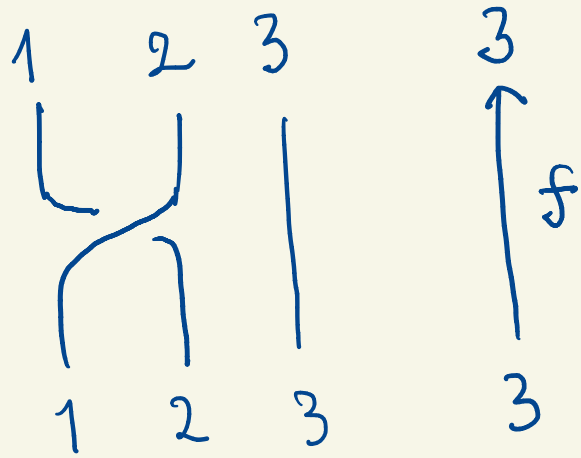
horizontal composition.

② the category Braid

with objects : natural numbers $n \in \mathbb{N}$

maps : $\mathrm{Braid}(n,n)$ is the set of topological braids on n strands

$f : 3 \longrightarrow 3$



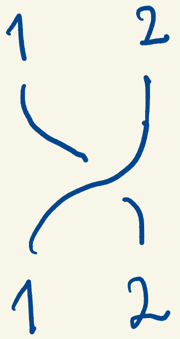
composition = concatenation of braids

$\mathrm{Braid}(p,q)$ is empty when $p \neq q$.

Note: the structure of the category Braid is entirely given by the family of groups $\mathrm{Braid}(n,n)$

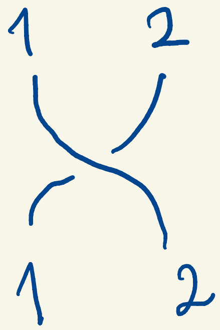
Observation: every map $f: n \to n$ is an isomorphism.

$$2 \xrightarrow{\sigma} 2$$



positive braiding

$$2 \xrightarrow{\sigma^{-1}} 2$$



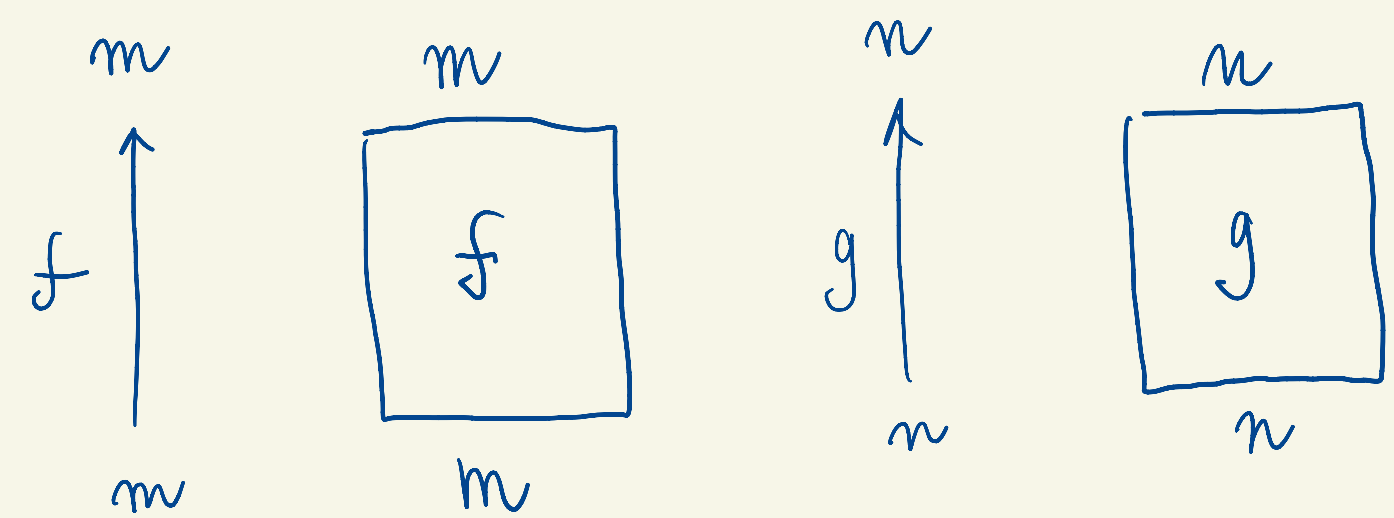
negative braiding

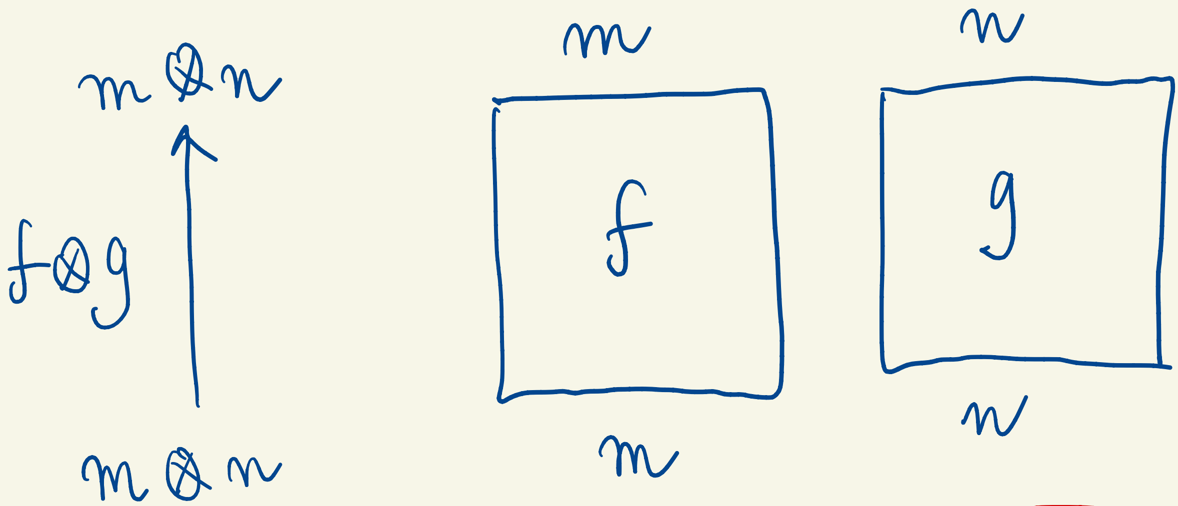
$$\sigma \circ \sigma^{-1} = id_2 = \sigma^{-1} \circ \sigma$$

<u>the tensor product</u> is defined by horizontal composition / juxtaposition

$$m \otimes n = m + n$$

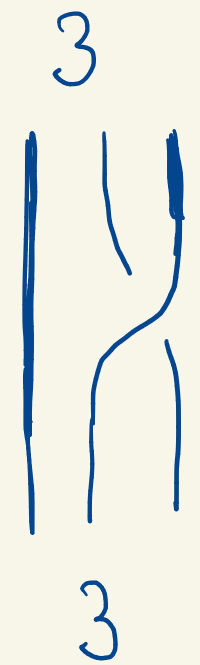
The tensor product (continued)



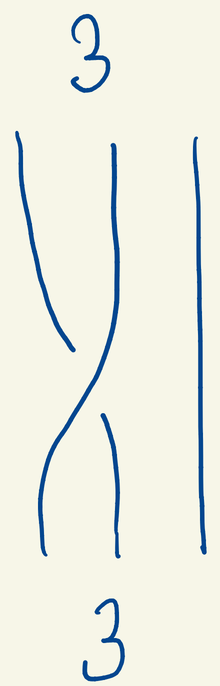


illustration

$1 \otimes \sigma =$



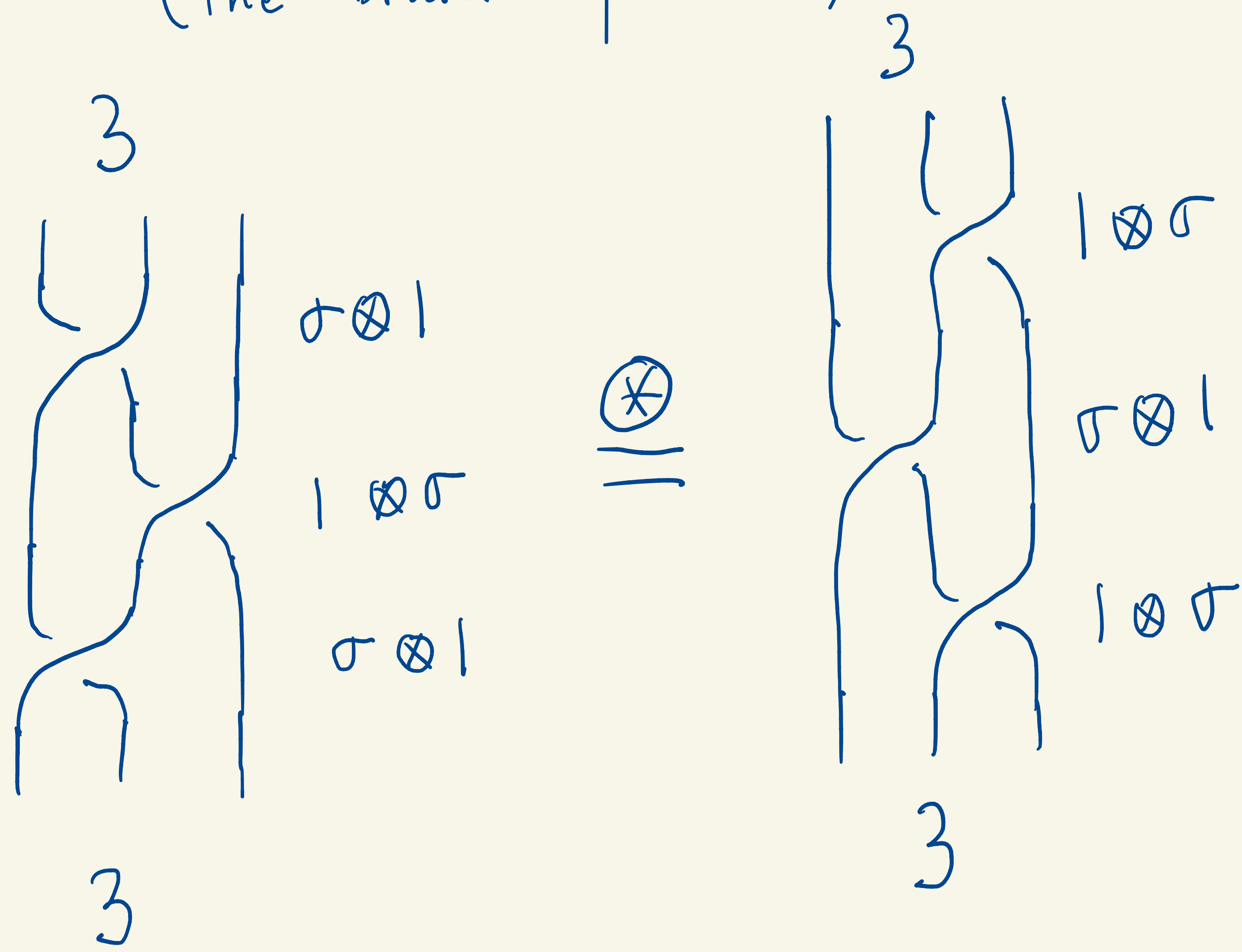
$\sigma \otimes | =$



$1 \otimes \sigma : 3 \longrightarrow 3$ $\qquad$ $\sigma \otimes | : 3 \longrightarrow 3$

Main equation of Braid:

(the braid equation)



the two maps $3 \longrightarrow 3$ are equal in the category Braid by definition of the maps of Braid as topological braids.

__Fact:__ __Braid__ is the monoidal category freely generated by

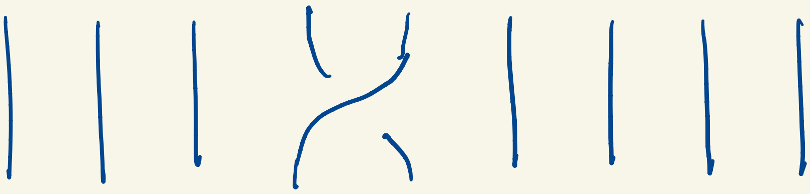
$$\boxed{\sigma} : 2 \to 2$$

σ isomorphism with an inverse

and the equation above $\circledast$
(not explained here)

typically, braiding two strands $p+1$ and $p+2$ among $p+2+q$ strands is performed by:

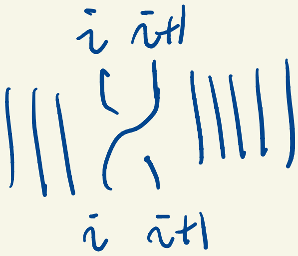
$$p \otimes 2 \otimes q$$



$$\underbrace{p \otimes 2 \otimes q}_{\sigma_{p+1} \,=\, p \otimes \sigma \otimes q}$$

$$p \otimes 2 \otimes q \xrightarrow{\ id_p \otimes \sigma \otimes id_q\ } p \otimes 2 \otimes q$$

This is related to the traditional fact that the group $\mathbb{B}_n$ of braids on n strands is presented by the generators

$\boxed{\sigma_i}$ $\quad 1 \leq i \leq n-1 \qquad \sigma_i =$ 

σ_i is the action of permuting the strands i, $i+1$

and the equations

$$\boxed{\sigma_i \, \sigma_j = \sigma_j \, \sigma_i} \qquad |j-i| \geq 2$$

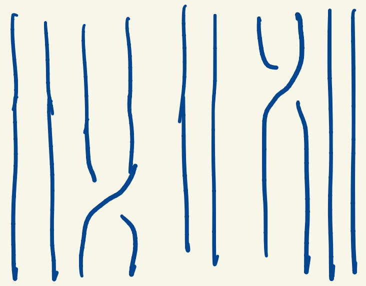
$$\boxed{\sigma_i \, \sigma_{i+1} \, \sigma_i \underset{(*)}{=} \sigma_{i+1} \, \sigma_i \, \sigma_{i+1}} \qquad 1 \leq i \leq n-1$$

what does it mean from the point of view of Braid?

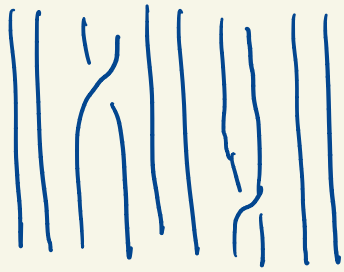
in the traditional picture:

σ_7

σ_3



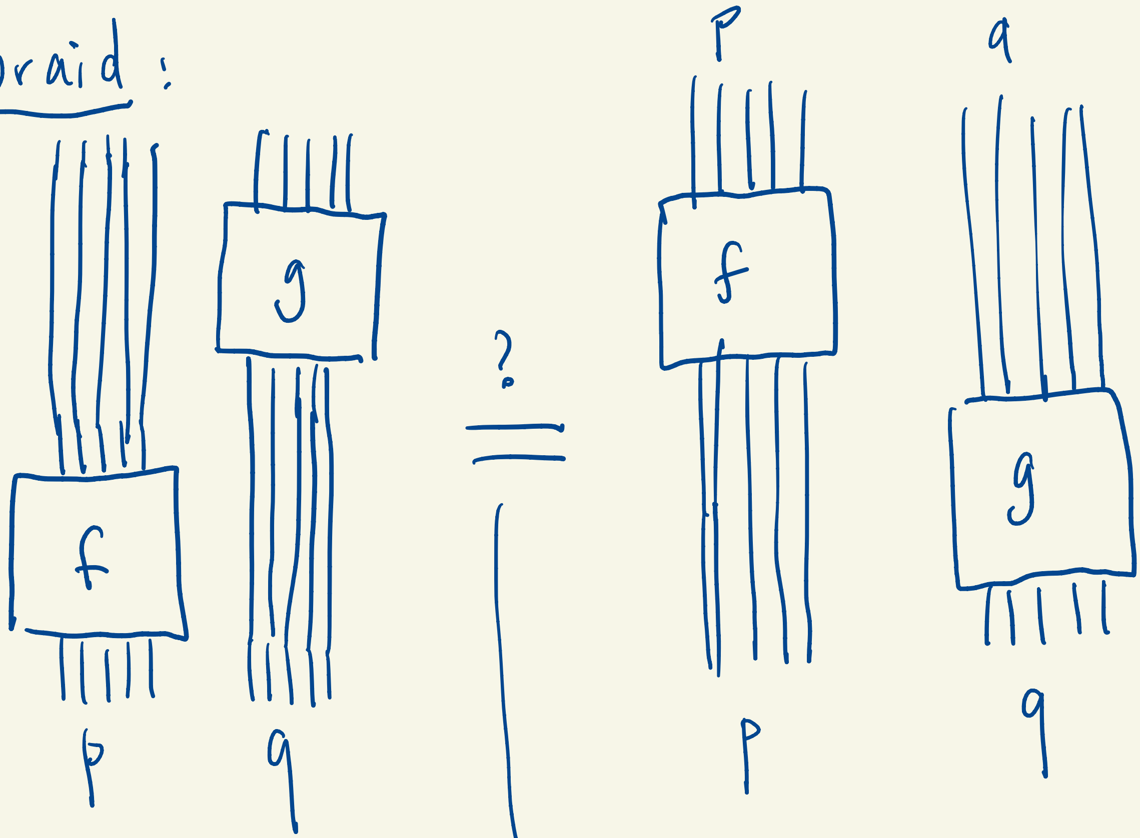
$=$ equation in B_{10}



σ_3

σ_7

in Braid:

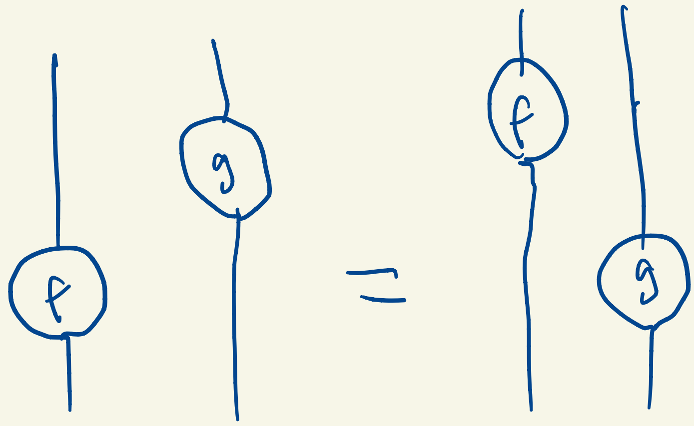


why is it equal?

$f \otimes g$

$//$

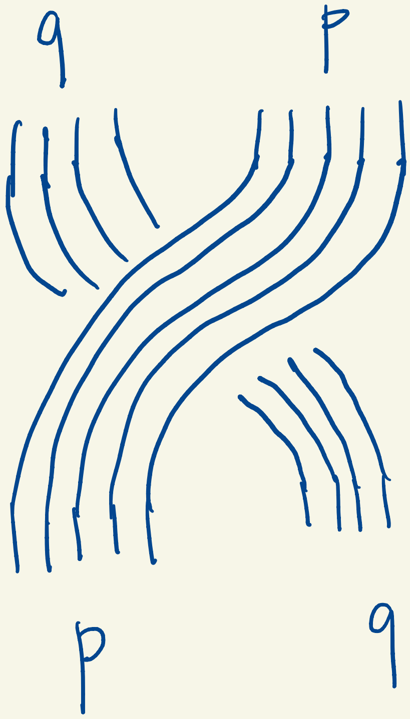
functoriality of the tensor product



what does it mean that the category Braid is <u>braided</u> monoidal?

$$\gamma_{p,q} : p \otimes q \longrightarrow q \otimes p.$$

$\gamma_{p,q}$



$$p \otimes q = p + q$$

what does it mean that γ is natural?

$$\begin{tikzcd}
\underset{(A,\,B)}{\mathcal{C}\times\mathcal{C}} \arrow[rr, bend left, "\otimes"] \arrow[dr, bend right, "Sym"'] & \arrow[d, phantom, "\Downarrow\gamma"] & \mathcal{C} \\
& \underset{(B,\,A)}{\mathcal{C}\times\mathcal{C}} \arrow[ur, bend right, "\otimes"'] &
\end{tikzcd}$$

$$A \otimes B \xrightarrow{\gamma_{A,B}} B \otimes A$$

Naturality of γ means that the diagram below

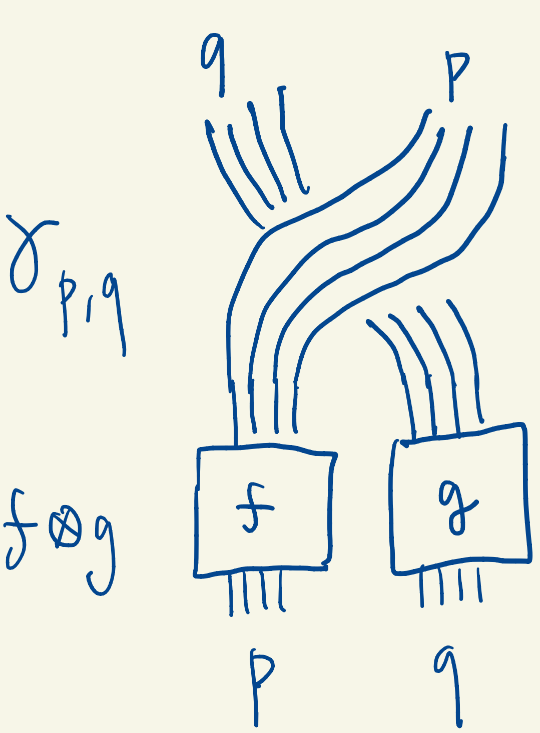
$$\begin{tikzcd}
A \otimes B \arrow[r, "\gamma_{A,B}"] \arrow[d, "h_A \otimes h_B"'] \arrow[dr, phantom, "(**)"] & B \otimes A \arrow[d, "h_B \otimes h_A"] \\
A' \otimes B' \arrow[r, "\gamma_{A',B'}"] & B' \otimes A'
\end{tikzcd}$$

commutes for all morphisms in $\mathcal{C}$:

$h_A : A \to A'$ $\qquad$ $h_B : B \to B'$

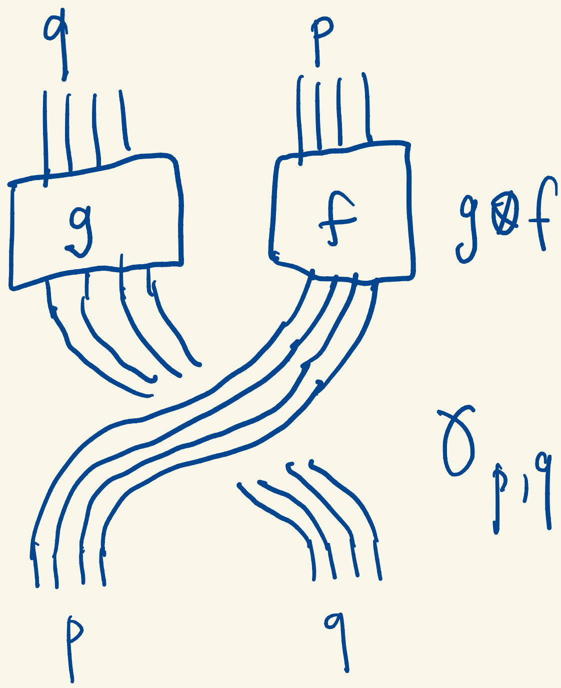
In the case of Braid:

$$\begin{tikzcd}
p \otimes q \arrow[r, "\gamma_{p,q}"] \arrow[d, "f \otimes g"'] \arrow[dr, phantom, "(**)"] & q \otimes p \arrow[d, "g \otimes f"] \\
p \otimes q \arrow[r, "\gamma_{p,q}"] & q \otimes p
\end{tikzcd}$$



$\overset{(**)}{=}$

naturality of γ



Symmetric monoidal $\boxed{\text{closed}}$ categories

The structure of the category of coherence spaces (2)

Symmetric monoidal $\boxed{\text{closed}}$ categories (smcc)

A smcc is a symmetric monoidal category

$$\boxed{(\mathcal{C}, \otimes, 1)}$$

together with the following data for all objects A and B:

- ▷ of an object $\boxed{A \multimap B}$
- ▷ of a morphism $\boxed{\mathrm{eval}_{A,B} : A \otimes (A \multimap B) \longrightarrow B}$

such that for every object X and morphism

$$\boxed{f : A \otimes X \longrightarrow B}$$

there exists a $\boxed{\text{unique}}$ morphism $h : X \longrightarrow A \multimap B$ making the diagram

$$\begin{tikzcd}
A \otimes (A \multimap B) \arrow[rr, "\mathrm{eval}_{A,B}"] & & B \\
& {} \arrow[u, phantom, "*"] & \\
A \otimes X \arrow[uu, "A \otimes h"] \arrow[uurr, "f"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes (A \multimap B) \arrow[r] & B \\
A \otimes X \arrow[u, dashed, "A \otimes h"] \arrow[ur] &
\end{tikzcd}$$

commute.

Monoidal exponentiation

Suppose given an object A of a symmetric monoidal category $\mathcal{C}$.

Definition.

A monoidal exponentiation of A is a pair consisting of a functor

$$\boxed{(A \multimap -) : \mathcal{C} \longrightarrow \mathcal{C}}$$

and a family of bijections

$$\boxed{\phi_{A,B,C} \quad : \quad \mathbf{Hom}(A \otimes B, C) \longrightarrow \mathbf{Hom}(B, A \multimap C)}$$

naturelle en B et C.

$\boxed{A \otimes \_ : \mathcal{C} \longrightarrow \mathcal{C}}$ $\qquad A \otimes \_ \dashv A \multimap \_ \qquad$ $\boxed{A \multimap \_ : \mathcal{C} \to \mathcal{C}}$

Alternative definition

A smcc is a symmetric monoidal category

$$(\mathcal{C}, \otimes, \mathbf{1})$$

equipped with a monoidal exponentiation

$$\boxed{\dfrac{A \otimes B \longrightarrow C}{B \longrightarrow A \multimap C} \quad \phi_{A,B,C}}$$

$$\left(A \multimap \_ : \mathcal{C} \longrightarrow \mathcal{C} \right)$$

for every object A of the category.

By the parameter theorem, this defines a unique bifunctor

$$\boxed{\multimap \quad : \quad \mathcal{C}^{op} \times \mathcal{C} \quad \longrightarrow \quad \mathcal{C}}$$

such that the family $(\phi_{A,B,C})_{A,B,C}$ is natural in A, B and C.

$$\begin{tikzcd} A & B \arrow[d, "h_B"] \\ A' \arrow[u, "h_A"] & B' \end{tikzcd} \longmapsto \begin{tikzcd} A \multimap B \arrow[d, "h_A \multimap h_B"] \\ A' \multimap B' \end{tikzcd} \quad \text{image of} \quad \begin{tikzcd} (A, B) \arrow[d] \\ (A', B') \end{tikzcd} \text{ in } \mathcal{C}^{op} \times \mathcal{C}$$

The evaluation morphism

In that formulation, the morphism

$$\boxed{\mathrm{eval}_{A,B} : A \otimes (A \multimap B) \longrightarrow B}$$

is defined in the following way:

$$\boxed{\dfrac{A \multimap B \xrightarrow{id} A \multimap B}{A \otimes (A \multimap B) \longrightarrow B} \quad \phi^{-1}_{A \multimap B, A, B}}$$

Exercise.

Show that every cartesian closed category is a smcc.

sequent calculus

Multiplicative intuitionistic $\boxed{\text{linear}}$ logic

$\boxed{A_1, \ldots, A_n \vdash B}$

$$\boxed{A, B \quad ::= \quad 1 \mid A \otimes B \mid A \multimap B \mid \alpha}$$

α type variable

only have Introduction rules + cut

Axiom $\boxed{\dfrac{}{A \vdash A}}$

$\multimap$ left $\boxed{\dfrac{\Delta \vdash A \qquad \Gamma, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}}$

$\multimap$ right $\boxed{\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}}$

$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$

$\dfrac{A, \Gamma \vdash B}{\Gamma \vdash B \multimap A}$

$\otimes$ left $\boxed{\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}}$

$\otimes$ right $\boxed{\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}}$

$\boxed{\dfrac{\Gamma \vdash A}{\Gamma, 1 \vdash A}}$

1 left $\dfrac{\Gamma, \cancel{A} \vdash A}{1, \Gamma \vdash A}$

1 right $\boxed{\dfrac{}{\vdash \mathbf{1}}}$

$\boxed{\text{Cut}}$ $\dfrac{\Delta \vdash \boxed{A} \qquad \Gamma, \boxed{A} \vdash B}{\Gamma, \Delta \vdash B}$

$\boxed{\text{Exchange}}$ $\dfrac{\Gamma, A_1, A_2, \Delta \vdash B}{\Gamma, A_2, A_1, \Delta \vdash B}$

Interpretation of the logic

- ▷ Axiom: $A \xrightarrow{id_A} A$

- ▷ Left $\multimap$

$\boxed{\Delta \xrightarrow{f} A}$ and $\boxed{\Gamma \otimes B \xrightarrow{g} C}$

become

$$\boxed{\Gamma \otimes \Delta \otimes (A \multimap B)} \xrightarrow{\Gamma \otimes f \otimes A \multimap B} \Gamma \otimes A \otimes (A \multimap B) \xrightarrow{\Gamma \otimes \mathrm{eval}_{A,B}} \Gamma \otimes B \xrightarrow{g} \boxed{C}$$

- ▷ Right $\multimap$

$$A \otimes \Gamma \xrightarrow{sym} \Gamma \otimes A \xrightarrow{f} B$$

becomes

fosym

$$\Gamma \xrightarrow{\phi_{\Gamma,A,B}(f)} A \multimap B$$

in the λ-calculus

Interpretation of the logic

▷ Left $\boxed{\otimes}$

$$\Gamma \otimes A \otimes B \xrightarrow{f} C$$

remains the same.

▷ Right $\boxed{\otimes}$

$$\Gamma \xrightarrow{f} A \quad \text{and} \quad \Delta \xrightarrow{g} B$$

become $\boxed{\Gamma \otimes \Delta \xrightarrow{f \otimes g} A \otimes B}$ using the fact that $\otimes$ is a functor.

$$\frac{\pi}{A_1, \ldots, A_n \vdash B} \qquad A_1 \otimes \ldots \otimes A_n \xrightarrow{[\pi]} B$$

Interpretation of the logic

morph.

- Cut

$$\boxed{\Delta \xrightarrow{f} A} \quad \text{and} \quad \boxed{\Gamma \otimes A \xrightarrow{g} B}$$

become

$$\boxed{\Gamma \otimes \Delta} \xrightarrow{\Gamma \otimes f} \boxed{\Gamma \otimes A} \xrightarrow{g} \boxed{B}$$

$$\frac{\Delta \vdash A \quad \Gamma, A \vdash B}{\Gamma, \Delta \vdash B}\text{cut}$$

- Exchange

$$\Gamma \otimes \boxed{A_1 \otimes A_2} \otimes \Delta \xrightarrow{f} B$$

becomes

$$\boxed{\Gamma \otimes A_2 \otimes A_1 \otimes \Delta} \xrightarrow{\Gamma \otimes \gamma_{A_2, A_1} \otimes \Delta} \Gamma \otimes A_1 \otimes A_2 \otimes \Delta \xrightarrow{f} \boxed{B}$$

Remark: in order to simplify notations, we suppose that the monoidal category is strict, this meaning that the structural morphisms α, λ and ρ are identities.

classical linear logic.

$*$-autonomous categories

The structure of the category of coherence spaces (3)

$A, \perp$ $\quad$ $\perp$ bottom (think of it as "false")

A general observation

Every pair of objects $A, \perp$ in a smcc $(\mathcal{C}, \otimes, \mathbf{1})$ comes with an identity

$$id_{A \multimap \perp} : A \multimap \perp \longrightarrow A \multimap \perp$$

which is transported by the bijection $\phi^{-1}_{A \multimap \perp, A, \perp}$ to the morphism

$$\mathrm{eval}_{A,\perp} : A \otimes (A \multimap \perp) \longrightarrow \perp$$

which becomes by precomposing with symmetry:

$$(A \multimap \perp) \otimes A \longrightarrow \perp$$

which is then transported by the bijection $\phi_{A \multimap \perp, A, \perp}$ to the morphism

$$\boxed{A \longrightarrow (A \multimap \perp) \multimap \perp}$$

general principle of logic that $A \vdash \neg\neg A$

A implies its double negation.

$*$-autonomous categories

Definition

An object $\perp$ is called dualizing when the canonical morphism

$$\eta_A \quad : \quad A \longrightarrow (A \multimap \perp) \multimap \perp$$

is an isomorphism for every object A.

Definition

A $*$-autonomous category is a smcc with a dualizing object.

$(a,*) \in |A \multimap \bot|$

$((a,*),(a,*)) \in |(A \multimap \bot) \multimap (A \multimap \bot)|$

The category Coh is $*$-autonomous

$\bot = 1^{\bot}$ is the coherence space with singleton web $|\bot| = \{*\}$.

$$e = id_{A \multimap \bot} \qquad A \multimap \bot \longrightarrow A \multimap \bot \qquad \{((a,*),(a,*)) \mid a \in |A|\}$$

$$f = \phi^{-1}_{A \multimap \bot, A, \bot}(e) \qquad \boxed{A \otimes (A \multimap \bot) \longrightarrow \bot} \qquad \{((a,(a,*)),*) \mid a \in |A|\}$$

$$g = f \circ \gamma_{A, A \multimap \bot} \qquad \boxed{(A \multimap \bot) \otimes A \longrightarrow \bot} \qquad \{(((a,*),a),*) \mid a \in |A|\}$$

$$h = \phi_{A \multimap \bot, A, \bot}(g) \qquad \boxed{A \longrightarrow (A \multimap \bot) \multimap \bot} \qquad \{(a,((a,*),*)) \mid a \in |A|\}$$

The morphism h is an isomorphism with inverse the clique

$$\boxed{h^{-1} = \{(((a,*),*),a) \mid a \in |A|\}}$$

$$A \xrightarrow{\cong} (A \multimap \bot) \multimap \bot$$

The category Coh is symmetrical monoidal closed.

We have a natural bijection

$$\boxed{\phi_{A,B,C}} : \mathrm{Coh}(A \otimes B, C) \cong \mathrm{Coh}(B, A \multimap C)$$

which comes from the $\boxed{\text{graph}}$ isomorphism

$$\boxed{(A \otimes B) \multimap C \cong B \multimap (A \multimap C)}$$

$$\begin{array}{ccc} ((a,b),c) & \longmapsto & (b,(a,c)) \\ \cap & & \cap \\ |(A \otimes B) \multimap C| & & |B \multimap (A \multimap C)| \end{array}$$

hence, there is a bijection $\boxed{\phi_{A,B,C}}$ between the cliques of

$$(A \otimes B) \multimap C$$

and the cliques of $B \multimap (A \multimap C)$

Evaluation map :

$$\mathrm{Id}_{A \multimap B} = \left\{ \left((a,b), (a,b) \right), \begin{array}{l} a \in |A| \\ b \in |B| \end{array} \right\}$$

$$\mathrm{Id}_{A \multimap B} : (A \multimap B) \longrightarrow (A \multimap B)$$

when we apply $\phi^{-1}_{A, A \multimap B, B}$ we get

$$\mathrm{eval}_{A,B} : \quad A \otimes (A \multimap B) \longrightarrow B$$

$$\mathrm{eval}_{A,B} = \left\{ \left(\underbrace{(a, (a,b))}_{|A \otimes (A \multimap B)|}, \underbrace{b}_{|B|} \right) \;\middle|\; \begin{array}{l} a \in |A| \\ b \in |B| \end{array} \right\}$$

Multiplicative linear logic (MLL)

$$A, B ::= A \otimes B \mid \mathbf{1} \mid A \parr B \mid \bot \mid \alpha$$

$A_1, \ldots, A_n \vdash B$

$\vdash A_1, \ldots, A_n$

$\vdash A^\perp, A$

$\boxed{A^\perp \parr A}$

Axiom	$\boxed{\dfrac{}{\vdash A^\perp, A}}$
$\otimes$	$\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$
$\parr$	$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$
1	$\dfrac{}{\vdash 1}$
$\bot$	$\dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}$

next time: $\parr$

MLL may be interpreted in every $*$-autonomous category

$A_1, \ldots, A_n \vdash B \quad \rightsquigarrow \quad A_1 \otimes \ldots \otimes A_n \longrightarrow B$

$\vdash A_1, \ldots, A_n \quad \rightsquigarrow \quad 1 \longrightarrow A_1 \parr \ldots \parr A_n$

in any $*$-autonomous category:

$$A \parr B = \neg(\neg A \otimes \neg B)$$

where $\neg A = (A \multimap \bot)$

<u>Prop</u> $\parr$ defines a monoidal structure on $\mathcal{C}$

Hence every $*$-autonomous category $\mathcal{C}$ comes with two monoidal structures:

$$\otimes \quad 1$$

$$\parr \quad \bot$$

$$A \parr B = \neg(\neg A \otimes \neg B)$$

$$\bot \cong \neg 1 = 1 \multimap \bot$$

Multiplicative additive linear logic (MALL)

$$A, B ::= A \oplus B \mid A \otimes B \mid 0 \mid \mathbf{1} \mid A \& B \mid A \parr B \mid \top \mid \bot \mid \alpha$$

MLL+

$\oplus$ left	$\dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$
$\oplus$ right	$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}$
$\&$	$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B}$
0	no rule
$\top$	$\dfrac{}{\vdash \Gamma, \top}$

MALL may be interpreted in every $\left\{ \begin{array}{l} *\text{-autonomous} \\ \text{and cartesian} \end{array} \right\}$ category.