Cartesian categories

Products

The product of two objects A and B in a category $\mathcal{C}$ is an object $A \times B$ equipped with two morphisms

$$\pi_1 : A \times B \longrightarrow A \qquad\qquad \pi_2 : A \times B \longrightarrow B$$

such that for every diagram

$$\begin{tikzcd}
 & & A \\
X \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & B
\end{tikzcd}$$

there exists a unique morphism $h : X \longrightarrow A \times B$ making the diagram

$$\begin{tikzcd}
 & & A \\
X \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] \arrow[r, "h"] & A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & B
\end{tikzcd}$$

commute.

Illustrations

1. The cartesian product in the category **Set**,

2. The lub $a \wedge b$ of two elements a and b in an ordered set $(X, \leq)$,

3. The cartesian product in the category **Dom**,

Terminal object

An object $\mathbf{1}$ is terminal in a category $\mathcal{C}$ when $\mathbf{Hom}(A, \mathbf{1})$ is a singleton for all objects A.

One may consider $\mathbf{1}$ as the nullary product in $\mathcal{C}$.

Example 1. the singleton $\{*\}$ in the categories $\mathbf{Set}$ and $\mathbf{Dom}$,

Example 2. the maximum of an ordered set $(X, \leq)$

Cartesian category

A cartesian category is a category $\mathcal{C}$ equipped with a product

$$A \times B$$

for all pairs A, B of objects, and of a terminal object

$$\mathbf{1}$$

Cartesian categories

In every cartesian category, one finds

- weakening maps $\epsilon_A : A \longrightarrow \mathbf{1}$,
- diagonal maps $\delta_A : A \longrightarrow A \times A$ obtained as

$$\begin{tikzcd} & & A \\ A \arrow[r, "\delta_A"] \arrow[urr, bend left, "id_A"] \arrow[drr, bend right, "id_A"'] & A \times A \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ & & A \end{tikzcd}$$

- symmetry maps $\gamma_{A,B} : A \times B \longrightarrow B \times A$ obtained as

$$\begin{tikzcd} & & B \\ A \times B \arrow[r, "\gamma_{A,B}"] \arrow[urr, bend left, "\pi_2"] \arrow[drr, bend right, "\pi_1"'] & B \times A \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ & & A \end{tikzcd}$$

Exercise: Show that $(- \times -)$ defines a bifunctor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$.

Cartesian closed categories

First definition

Ccc

A cartesian closed category is a cartesian category

$$\boxed{(\mathcal{C}, \times, 1)}$$

together with the following data for all objects A and B:

- of an object $\boxed{A \Rightarrow B}$
- of a morphism $\boxed{eval_{A,B} : A \times (A \Rightarrow B) \longrightarrow B}$

such that for every object X and morphism

$$f : A \times X \longrightarrow B$$

there exists a unique morphism $h : X \longrightarrow A \Rightarrow B$ making the diagram

$$\begin{tikzcd}
A \times (A \Rightarrow B) \arrow[r, "eval_{A,B}"] & B \\
A \times X \arrow[u, "A \times h"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commute.

$\mathcal{C} = Set$

$A \Rightarrow B$ the set of functions from A to B

eval :

$(a, f) \longmapsto f(a)$

Domains

$h : x \longmapsto (a \longmapsto f(a, x))$

a function

$X \longrightarrow (A \Rightarrow B)$

Cartesian closed category

Second definition

<u>Exercise</u>. Suppose that A and B are sets.

Show that a bijection between A and B is the same thing as a pair of functions:

$$A \xrightarrow{L} B \qquad\qquad B \xrightarrow{R} A$$

such that:

$$\forall a \in A \quad \forall b \in B \qquad \boxed{La =_B b \overset{(*)}{\Longleftrightarrow} a =_A Rb}$$

① Suppose that L and R define a bijective pair in the sense that $R = L^{-1}$.

ⓘ $La = b \Longrightarrow RLa = Rb$

$a = RLa$ hence $a = Rb$

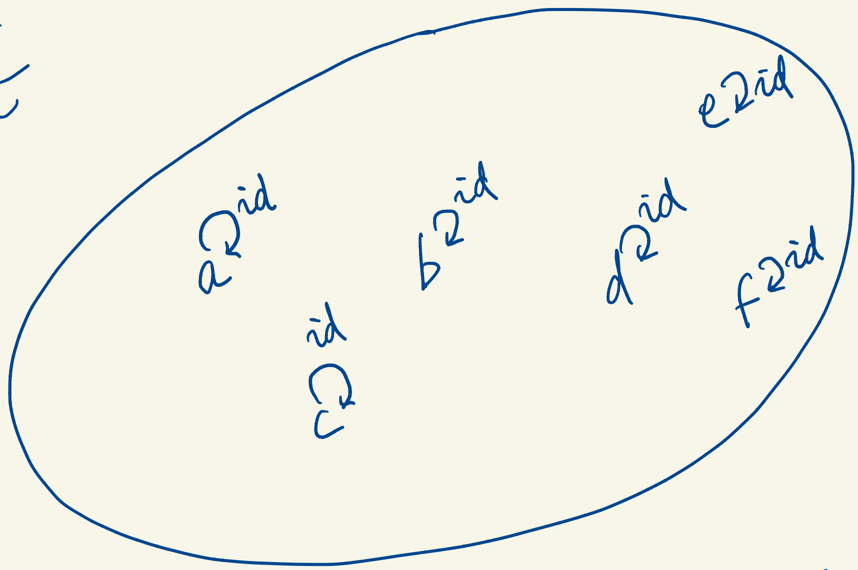
ⓘⓘ $a = Rb \Longrightarrow La = LRb = b$

② Suppose that the property $(*)$ holds.

$$\forall a \in A \quad \boxed{La =_B La} \overset{(*)}{\Longrightarrow} \boxed{a =_A RLa} \qquad b = La$$

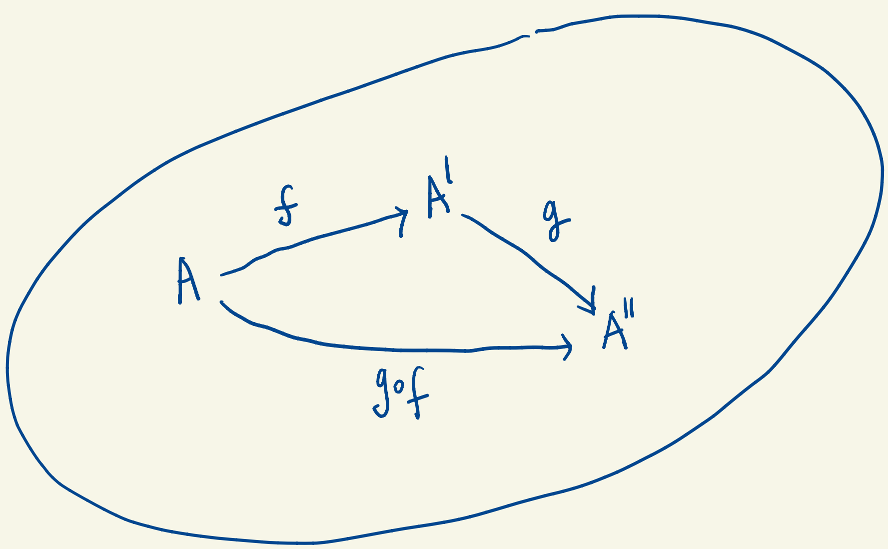
$$\forall b \in B \quad \boxed{Rb =_A Rb} \overset{(*)}{\Longrightarrow} \boxed{LRb =_B b} \qquad a = Rb$$

Set $\qquad\qquad\qquad\qquad A$



every set A can be seen as a category where the only maps are identity maps

Cat $\qquad\qquad\qquad\qquad \mathcal{C}$



every map $f : A \longrightarrow A'$ can be seen as some kind of "relaxed" equation

In a sense, the set of maps

$$\mathrm{Hom}(A, A') \text{ in a category } \mathcal{C}$$

replaces the identity relation

$$a = a' \text{ in a set } A.$$

The new thing is that there is an <u>orientation</u>!

$$\mathrm{Hom}(A, A') \qquad \mathrm{Hom}(A', A)$$

L left
R right

Adjunction

An adjunction is a triple consisting of two functors

$$L : \mathcal{A} \longrightarrow \mathcal{B} \qquad R : \mathcal{B} \longrightarrow \mathcal{A}$$

and a family of bijections

$$\boxed{\phi_{A,B} \quad : \quad \mathbf{Hom}_{\mathcal{B}}(LA, B) \ \cong \ \mathbf{Hom}_{\mathcal{A}}(A, RB)}$$

natural in A and B, for all pairs of objects A, B of $\mathcal{A}$ and $\mathcal{B}$.

$$\boxed{\frac{LA \xrightarrow{\mathcal{B}} B}{A \xrightarrow{\mathcal{A}} RB} \qquad \phi_{A,B}}$$

One writes $L \dashv R$ and one says that L is left adjoint to R.

Example: the free vector space

$$\begin{tikzcd} UV \arrow[d, "Uf"'] & \boxed{\mathbf{Set}} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \boxed{\mathbf{Vect}} \arrow[l, bend left, "R"] & V \arrow[d, "f"] \\ UW & & & W \end{tikzcd}$$

where

$\mathcal{A} = \mathbf{Set}$: the category of sets and functions
$\mathcal{B} = \mathbf{Vect}$: the category of vector spaces on a field k

R : the « forgetful » functor $V \mapsto U(V)$
L : the « free vector space » functor $X \mapsto kX$

$$kX := \left\{ \sum_{x \in X} \lambda_x \, x \;\middle|\; \lambda_x \in k \text{ null almost everywhere.} \right\}$$

$X = \{i, j, k\}$ $\quad k = \mathbb{R}$ $\quad \mathbb{R}X = \mathbb{R}^3$

$$\{i, j, k\} \xrightarrow[\text{in Set}]{\text{function } f} U(V)$$

3 elements set

forgetful functor

in $\underbrace{\text{Set}}_{A}$

it is the choice of three vectors

$$\left.\begin{matrix} f(i) \\ f(j) \\ f(k) \end{matrix}\right) \quad \text{in the vector space } V.$$

$$\mathbb{R}^3 \xrightarrow{\text{linear map}} V$$

$$\lambda_1 i + \lambda_2 j + \lambda_3 k \longmapsto \lambda_1 f(i) + \lambda_2 f(j) + \lambda_3 f(k)$$

$$(\lambda_1, \lambda_2, \lambda_3)$$

in $\underbrace{\text{Vect}}_{B}$

Illustration: the free monoid

$$
\begin{tikzcd}
UM \arrow[d, "Uf"'] & & & M \arrow[d, "f"] \\
UN & \boxed{\mathbf{Set}} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \boxed{\mathbf{Mon}} \arrow[l, bend left, "R"] & N
\end{tikzcd}
$$

homomorphism

where

$\mathcal{A} = \mathbf{Set}$: the category of sets and functions
$\mathcal{B} = \mathbf{Mon}$: the category of monoids and homomorphisms,

R : the « forgetful » functor $M \mapsto U(M)$.
L : the « free monoid » functor $A \mapsto A^*$.

$$A \;:=\; \coprod_{n \in \mathbb{N}} A^n = \{\text{finite words on the alphabet } A\}$$

$$\underline{\mathrm{Set}}(A, UM) \cong \underline{\mathrm{Mon}}(A^*, M)$$

$(M, \cdot_M, e_M)$

M monoid
A set

the monoid A^* has concatenation as product
empty word as neutral element.

$f^\dagger$ homomorphism (unique!)

$$\begin{tikzcd} A^* \arrow[r, dashed] & M \\ A \arrow[u, "\eta_A"] \arrow[ur, "f"'] \arrow[u, phantom, shift left=20, "\circlearrowright"] & \end{tikzcd}$$

f function

$$\boxed{f^\dagger \circ \eta_A = f}$$

$f^\dagger([a]) = f(a)$

η_A the function
$$A \longrightarrow A^*$$
$$a \longmapsto [a]$$
the word with one letter a

$$f^\dagger : [a_1 \ldots a_n] \longmapsto f(a_1) \cdot_M \cdots \cdot_M f(a_n)$$
$$[\,] \longmapsto e_M$$

in Set

$$\begin{tikzcd} A \arrow[r, "f \text{ function}"] \arrow[d, bend right, "f \mapsto f^\dagger"'] & M \\ A^* \arrow[r, "h \text{ homomorphism}"] & M \arrow[u, bend right, "\mathrm{hom}_A"'] \end{tikzcd}$$

$U(M)$ the set underlying M.

in Mon

$$\mathrm{Mon}(A^*, M) \cong \mathrm{Set}(A, UM)$$

$$\phi_{A,M} : \mathcal{B}(LA, M) \cong \mathcal{A}(A, RM)$$

$\mathcal{A} \xrightarrow{L} \mathcal{B} \qquad\qquad \mathcal{B} \xrightarrow{R} \mathcal{A}$

What does natural bijection ϕ exactly mean?

By ϕ natural, one means that the two families of sets

$$\boxed{\mathbf{Hom}_{\mathcal{B}}(LA, B)} \qquad\qquad \mathbf{Hom}_{\mathcal{A}}(A, RB)$$

define functors

$$\mathbf{Hom}_{\mathcal{B}}(L-,-) \quad : \quad \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathbf{Set}$$

$$\mathbf{Hom}_{\mathcal{A}}(-,R-) \quad : \quad \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathbf{Set}$$

and that the family of bijections ϕ defines a **natural transformation**

$$\phi : \boxed{\mathbf{Hom}_{\mathcal{B}}(L-,-)} \cong \boxed{\mathbf{Hom}_{\mathcal{A}}(-,R-)} : \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathbf{Set}$$

between them.

$$\begin{tikzcd} \mathcal{A}^{op} \times \mathcal{B} \arrow[r, bend left=50, "\mathrm{Hom}_{\mathcal{B}}(L-,-)"] \arrow[r, bend right=50, "\mathrm{Hom}_{\mathcal{A}}(-,R-)"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathrm{Set} \end{tikzcd}$$

$$\begin{array}{c} \mathrm{Hom}_{\mathcal{B}}(LA, B) \\ \downarrow \phi_{A,B} \\ \mathrm{Hom}_{\mathcal{A}}(A, RB) \end{array}$$

Starting point.

Every category $\mathcal{C}$ comes equipped with a functor

$$\boxed{\begin{array}{rcl} \mathrm{Hom} : \mathcal{C}^{op} \times \mathcal{C} & \longrightarrow & \mathbf{Set}. \\ (A, B) & \longmapsto & \mathrm{Hom}(A,B) \end{array}}$$

$$\mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathbf{Set}$$

$$\begin{tikzcd}
(A,B) \arrow[d, "{(h_A, h_B)}"'] & A \arrow[d, "\boxed{h_A}"'] & A & B \arrow[d, "h_B"] \arrow[r, phantom, "\overset{?}{\longmapsto}"] & \mathrm{Hom}(A,B) \arrow[d, "{\mathrm{Hom}(h_A, h_B)}"] \\
(A', B') & A' \arrow[d, dash] & A' \arrow[u, "h_A"'] \arrow[d, dash] & B' \arrow[d, dash] & \mathrm{Hom}(A', B') \arrow[d, dash] \\
 & \text{in } \mathcal{C}^{op} & \text{in } \mathcal{C} & \text{in } \mathcal{C} & \text{a function between the two hom-sets}
\end{tikzcd}$$

$$\boxed{\begin{array}{ccc} \mathrm{Hom}(A,B) & \longrightarrow & \mathrm{Hom}(A',B') \\ f & \longmapsto & h_B \circ f \circ h_A \end{array}}$$

$$\boxed{\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "h_B"] \\ A' \arrow[u, "h_A"] \arrow[r, dashed, "h_B \circ f \circ h_A"'] & B' \end{tikzcd}}$$

<u>Same recipe</u>: given a functor $\boxed{L : \mathcal{A} \longrightarrow \mathcal{B}}$

(left as exercise)

we get a functor

$$(A,B) \mapsto \mathrm{Hom}_{\mathcal{B}}(LA,B) : \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathrm{Set}$$

What does natural bijection ϕ exactly mean?

Natural in A and B thus means that every commutative diagram

$$\begin{tikzcd} LA \arrow[r, "g"] \arrow[d, leftarrow, "Lh_A"'] & B \arrow[d, "h_B"] \\ LA' \arrow[r, "f"'] & B' \end{tikzcd}$$

is transformed into a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi_{A,B}(g)"] \arrow[d, leftarrow, "h_A"'] & RB \arrow[d, "Rh_B"] \\ A' \arrow[r, "\phi_{A',B'}(f)"'] & RB' \end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{A}^{op} \times \mathcal{B} \arrow[r, bend left=50, "\mathrm{Hom}_{\mathcal{B}}(L-,-)"] \arrow[r, bend right=50, "\mathrm{Hom}_{\mathcal{A}}(-,R-)"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathbf{Set}
\end{tikzcd}
$$

$\boxed{\phi \text{ natural transformation}}$ what does that mean?

$$
\left(
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(LA, B) \arrow[d, "\phi_{AB}\ \text{function}"] \\
\mathrm{Hom}_{\mathcal{A}}(A, RB)
\end{tikzcd}
\right)_{\substack{A \in \mathrm{ob}\,\mathcal{A} \\ B \in \mathrm{ob}\,\mathcal{B}}}
$$

a family of functions (maps in Set)

the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(LA, B) \arrow[r, "\mathrm{Hom}_{\mathcal{B}}(Lh_A, h_B)"] \arrow[d, "\phi_{AB}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_{\mathcal{B}}(LA', B') \arrow[d, "\phi_{A'B'}"] \\
\mathrm{Hom}_{\mathcal{A}}(A, RB) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(h_A, Rh_B)"'] & \mathrm{Hom}_{\mathcal{B}}(A', RB')
\end{tikzcd}$$

should commute for all maps:

$$h_A : A' \longrightarrow A \quad \text{in } \mathcal{A}$$

$$h_B : B \longrightarrow B' \quad \text{in } \mathcal{B}$$

Cartesian exponentiation

Consider an object A in a cartesian category $(\mathcal{C}, \times, \mathbf{1})$.

A cartesian exponentiation of A is a pair consisting of a functor

$$(A \Rightarrow -) : \mathcal{C} \longrightarrow \mathcal{C}$$

"curryfication"

and of a family of bijections

$$\phi_{A,B,C} \quad : \quad \mathbf{Hom}(A \times B, C) \longrightarrow \mathbf{Hom}(B, A \Rightarrow C)$$

natural in B and C.

In other words, it is an **adjunction** between the functors

$$A \times - \quad \dashv \quad A \Rightarrow -$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "A\Rightarrow -"] \end{tikzcd}$$

$$A \times - : B \longmapsto A \times B \qquad\qquad A \Rightarrow - : B \longrightarrow A \Rightarrow B$$

$$\mathcal{C} \longmapsto \mathcal{C} \qquad\qquad \mathcal{C} \longrightarrow \mathcal{C}$$

What natural bijection means in that case

Naturality in B and C means that the family of bijections

$$\phi_{A,B,C} \quad : \quad \mathbf{Hom}(A \times B, C) \longrightarrow \mathbf{Hom}(B, A \Rightarrow C)$$

transforms every commutative diagram

$$\begin{tikzcd}
A \times B \arrow[rr, "g"] & & C \arrow[d, "h_C"] \\
A \times B' \arrow[u, "A\times h_B"] \arrow[rr, "f"'] & & C'
\end{tikzcd}$$

$f = h_C \, g(-, h_B -)$

into a commutative diagram

$$\begin{tikzcd}
B \arrow[rr, "\phi_{A,B,C}(g)"] & & A \Rightarrow C \arrow[d, "A\Rightarrow h_C"] \\
B' \arrow[u, "h_B"] \arrow[rr, "\phi_{A,B',C'}(f)"'] & & A \Rightarrow C'
\end{tikzcd}$$

$b' \longmapsto \lambda a'.\, f(a', b')$

Cartesian closed category

Definition.

A cartesian closed category (ccc) is a cartesian category

$$(\mathcal{C}, \times, \mathbf{1})$$

equipped with a cartesian exponentation

$$\frac{A \times B \longrightarrow C}{B \longrightarrow A \Rightarrow C} \quad \phi_{A,B,C}$$

for every object A of the category.

Parameter theorem

We have seen that the cartesian product defines a bifoncteur

$$A, B \quad \mapsto \quad A \times B \quad : \quad \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

in every cartesian category. In the same way,

Parameter theorem (MacLane)

The family of cartesian exponentiations

$$(A \Rightarrow -)_A \quad : \quad \mathcal{C} \quad \longrightarrow \mathcal{C}$$

defines a unique bifunctor

$$A, B \quad \mapsto \quad A \Rightarrow B \quad : \quad \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$$

such that the bijections $\phi_{A,B,C}$ are natural in A, B, C.

Parameter theorem

Here, natural in A, B, C means that the family of bijections

$$(\phi_{A,B,C})_{A,B,C} \quad : \quad \mathbf{Hom}(A \times B, C) \longrightarrow \mathbf{Hom}(B, A \Rightarrow C)$$

transforms every commutative diagram

$$\begin{tikzcd} A \times B \arrow[r, "g"] & C \arrow[d, "h_C"] \\ A' \times B' \arrow[u, "h_A \times h_B"] \arrow[r, "f"'] & C' \end{tikzcd}$$

in a commutative diagram:

$$\begin{tikzcd} B \arrow[r, "\phi_{A,B,C}(g)"] & A \Rightarrow C \arrow[d, "h_A \Rightarrow h_C"] \\ B' \arrow[u, "h_B"] \arrow[r, "\phi_{A',B',C'}(f)"'] & A' \Rightarrow C' \end{tikzcd}$$