Coherence spaces

Coherence spaces

This model is at the origin of linear logic (1986)

Linear decomposition of the category with

- ▷ dI-domains
- ▷ stable functions

Coherence spaces

Since then, several linearizations have been achieved:

Concrete data structures Berry-Curien 1985	$\Rightarrow$	Sequential games Lamarche 1992
dI-domains with coherence and strongly stable functions Bucciarelli-Ehrhard 1991	$\Rightarrow$	Hypercoherence spaces Ehrhard 1993
Bidomains Berry 1979	$\Rightarrow$	Bistructures Curien-Plotkin-Winskel 1996

Coherence spaces

A coherence space is a pair $A = (|A|, \frown\!\!\!\!\smile_A)$ consisting of

- ▷ a set $|A|$ called the web of A
- ▷ a reflexive and symmetric relation

$$\boxed{\frown\!\!\!\!\smile_A \quad \subseteq \quad |A| \times |A|}$$

called its coherence.

So, coherence space is a pedantic to say graph.

Notation: one writes

- ▷ $\boxed{a \frown_A a'}$ if $a \frown\!\!\!\!\smile_A a'$ and $a \neq a'$.
- ▷ $\boxed{a \asymp_A a'}$ if $\underbrace{\neg\,(a \frown\!\!\!\!\smile_A a')}$ or $\underbrace{\boxed{a = a'.}}$

Coherence spaces

Example 1. the coherence spaces $0 = \top$ of empty web and $1 = \bot$ of singleton web.

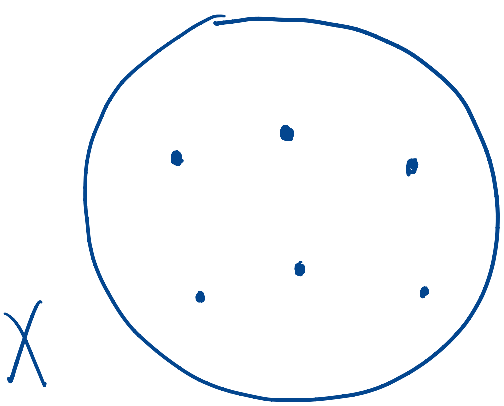
Example 2. for every set X, the **discrete** coherence space

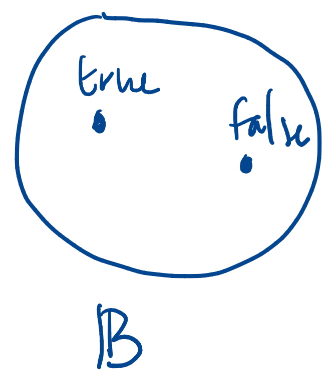
$$(X, =)$$

In particular,

$$B = (\{V, F\}, =)$$

$$N = (\mathbb{N}, =)$$





Interaction

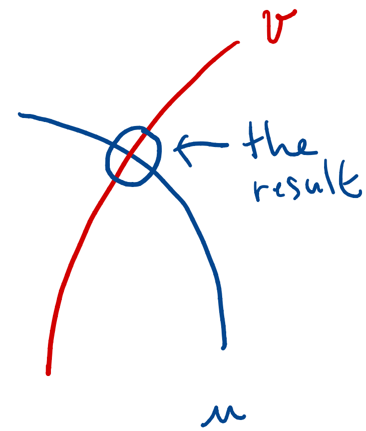
A clique u in a graph A is a subset of $|A|$ such that

$$\forall (a, a') \in u, \qquad a \overset{\frown}{\smile}_A a'$$

An anticlique v in a graph A is a subset of $|A|$ such that

$$\forall (a, a') \in v, \qquad a \asymp_A a'$$

We are going to interpret



- the simple types of the λ-calculus as graphs,
- the programs u of type A as cliques of A,
- the counter-programs v of type A as anti-cliques of A
- the interaction between u and v as the intersection $u \cap v$.

Remark: $u \cap v$ contains at most one element (= the result)

Negation

Let A be a coherence space.

The negation $A^\perp$ is defined as its dual graph:

- $|A^\perp| = |A|$

incoherence

- $\boxed{a \frown_{A^\perp} a'}$ if and only if $\boxed{a \asymp_A a'.}$

Remark: an anti-clique of A is a clique of $A^\perp$.

So, one makes a clique of A interact with a clique of $A^\perp$.

This reveals a fundamental duality between Player and Opponent:

$$\boxed{A \quad = \quad (A^\perp)^\perp}$$

reverses
the point of view

The sum (plus)

The sum of two coherence spaces A and B

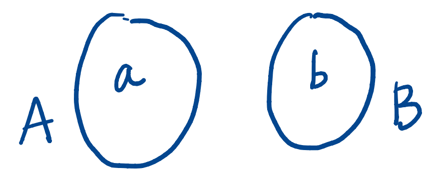
$$\boxed{A \oplus B}$$

is defined as their sum as graphs:

$\mathrm{in}_l\, a \smile_{A\oplus B} \mathrm{in}_l\, a'$

— $|A \oplus B| = |A| + |B|$

— $\boxed{a \smile_{A\oplus B} a'}$ if and only if $a \smile_A a'$,



— $\boxed{b \smile_{A\oplus B} b'}$ if and only if $b \smile_B b'$,

$\mathrm{in}_r\, b \smile_{A\oplus B} \mathrm{in}_r\, b'$

— $\boxed{a \smile_{A\oplus B} b}$ never.

Exercise. Show that the graphs $A \oplus 0$ and A are isomorphic.

The product (with)

The product of two coherence spaces A and B

$$A \mathbin{\&} B$$

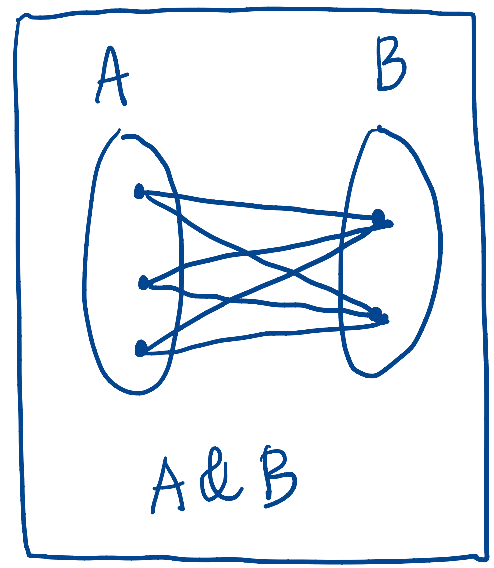
is defined as an « alternative » sum of the two graphs:

— $|A\&B| = |A| + |B|$

— $a \smile_{A\&B} a'$ if and only if $a \smile_A a'$,

— $b \smile_{A\&B} b'$ if and only if $b \smile_B b'$,

— $a \smile_{A\&B} b$ always.



Exercise. Show that

$$\boxed{A\&B \;=\; (A^{\perp} \oplus B^{\perp})^{\perp}}$$

de Morgan duality

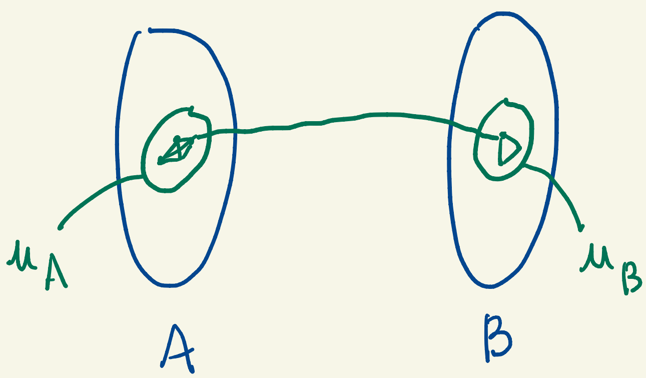
D_A domain of cliques of A.

$$D_{A\&B} = D_A \times D_B$$

what is a clique of $A\&B$?

$\boxed{\begin{array}{l} A\&B \\ \uparrow \\ \text{negative} \\ \text{connective} \end{array}}$

<u>Prf</u>



<u>Remark</u>:
every subset $u \subseteq |A\&B|$ is of the form $u = u_A \uplus u_B$ where $u_A \subseteq |A|$, $u_B \subseteq |B|$.

$\boxed{u \text{ is a clique of } A\&B \iff u_A \text{ is a clique of } A \text{ and } u_B \text{ is a clique of } B.}$

<u>hence</u> a "program" (clique) of type $A\&B$ is a pair of a clique of type A and a clique of type B.

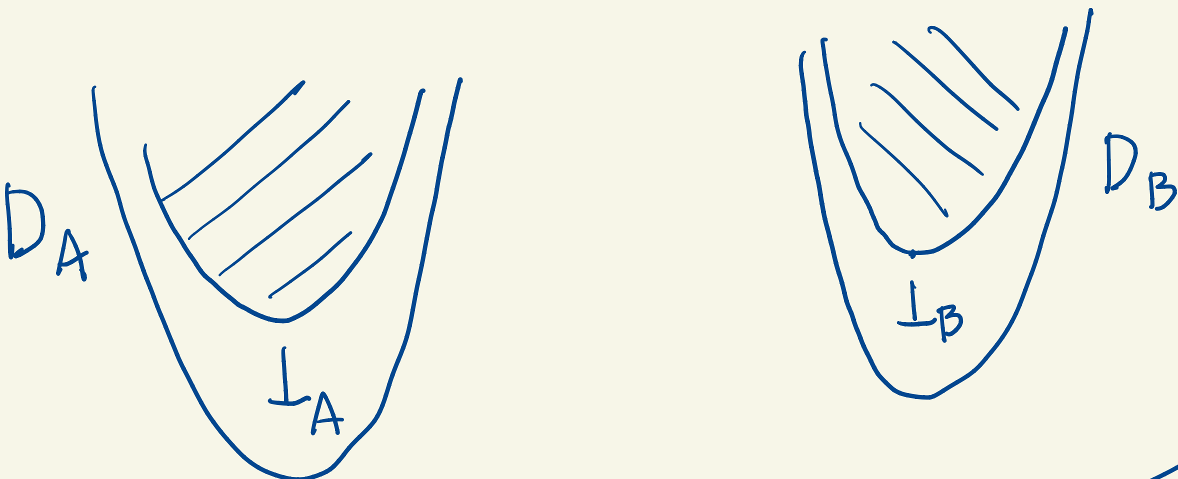
Similarly,

$$D_{A \oplus B} = D_A \oplus D_B$$

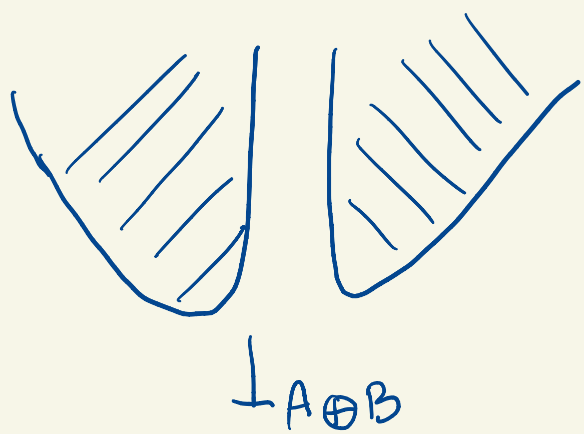
$A \oplus B$

positive

where $\oplus$ is the lifted sum of D_A, D_B



$D_{A \oplus B}$



non empty
a clique of $A \oplus B$ <u>makes</u>/<u>reveals</u> <u>a "choice"</u> between A and B

$$D_{A \oplus B} = D_A + D_B \big/ \perp_A = \perp_B$$

Tensor product

The tensor product of two coherence spaces A and B

$$\boxed{A \otimes B}$$

is defined as their product as graphs:

— $|A \otimes B| = |A| \times |B|$

— $\underbrace{(a,b)}_{a \otimes b} \coh_{A\otimes B} \underbrace{(a',b')}_{a' \otimes b'}$ if and only if $\boxed{a \coh_A a' \text{ and } b \coh_B b'}$.

and

Exercise. Show that the graphs $A \otimes 1$ and A are isomorphic.

$a \otimes b \frown_{A\otimes B} a' \otimes b'$

Parallel product, or par

The parallel product of two coherence spaces A and B

$$\boxed{A \parr B}$$

is defined as an « alternative » product of the two graphs:

— $|A \parr B| = |A| \times |B|$

— $(a, b) \frown_{A \parr B} (a', b')$ if and only if $\boxed{a \frown_A a'}$ ou $\boxed{b \frown_B b'}$.

or

$a \parr b \frown_{A \parr B} a' \parr b'$

Exercise. Show that

$$\boxed{A \parr B \;=\; (A^{\perp} \otimes B^{\perp})^{\perp}}$$

de Morgan dual

Distributivity laws

can be deduced by duality

$$\boxed{A \otimes (B \oplus C) \;\cong\; (A \otimes B) \oplus (A \otimes C)}$$

$$\boxed{A \parr (B \& C) \;\cong\; (A \parr B) \& (A \parr C)}$$

Reminiscent of

$$\boxed{A \times (B + C) \cong (A \times B) + (A \times C)}$$

in the category **Set**. Thus, one calls

0 $\oplus$

$\top$ $\&$

— $\boxed{\text{additives}}$ the connectives $\boxed{\oplus}$ and $\boxed{\&}$, and their units 0 and $\top$,

— $\boxed{\text{multiplicatives}}$ the connectives $\boxed{\otimes}$ and $\boxed{\parr}$, and units 1 and $\bot$.

1 $\otimes$ $\bot$ $\parr$

Remark: the sign $\cong$ means graph-isomorphism, or isomorphism in the category $\mathbf{Coh}$ constructed later.

$$\boxed{\begin{aligned} \top &= 0^\perp \\ \bot &= 1^\perp \end{aligned}}$$

Linear map

The linear map of two coherence spaces A and B

$$\boxed{A \multimap B}$$

$$\boxed{a \multimap b \ \overline{\smile}_{A \multimap B} \ a' \multimap b'}$$

is defined as

— $\quad |A \multimap B| = |A| \times |B|$

$$— \ (a, b) \ \overline{\smile}_{A \multimap B} \ (a', b') \text{ iff } \begin{cases} \boxed{a \ \overline{\smile}_{A} \ a'} \text{ implies } \boxed{b \ \overline{\smile}_{B} \ b'} \\ \text{and} \\ \boxed{b \ \overline{\smile}_{B^\perp} \ b'} \text{ implies } \boxed{a \ \overline{\smile}_{A^\perp} \ a'} \end{cases}$$

bidirectional nature of computations

Exercise. Show that

$$\boxed{A \multimap B = A^\perp \parr B = (A \otimes B^\perp)^\perp}$$

The category $\mathbf{Coh}$

The category $\mathbf{Coh}$ is defined as the category

— whose objects are $\boxed{\text{the coherence spaces}}$

— whose morphisms $\boxed{f : A \longrightarrow B}$ are $\boxed{\text{the cliques of } \boxed{A \multimap B}.}$

The identity

$$\boxed{id_A = \{(a,a) \in |A \multimap A|\}}$$

$id_A = \{ a \multimap a \mid a \in |A| \}$

The composition of $f : A \longrightarrow B$ and $g : B \longrightarrow C$.

$$\boxed{g \circ f = \{(a,c) \in |A \multimap C| \mid \exists b \in |B| \quad (a,b) \in f \text{ et } (b,c) \in g\}}$$

Exercise. Check that this defines a category.

given two sets A and B

a binary relation $R : A \nrightarrow B$

between A and B is defined as

a <u>subset</u> R of $A \times B$.

$$R \subseteq A \times B.$$

given two coherence spaces A and B,

$\boxed{\text{a clique } f \text{ of } A \multimap B}$

defines a binary relation

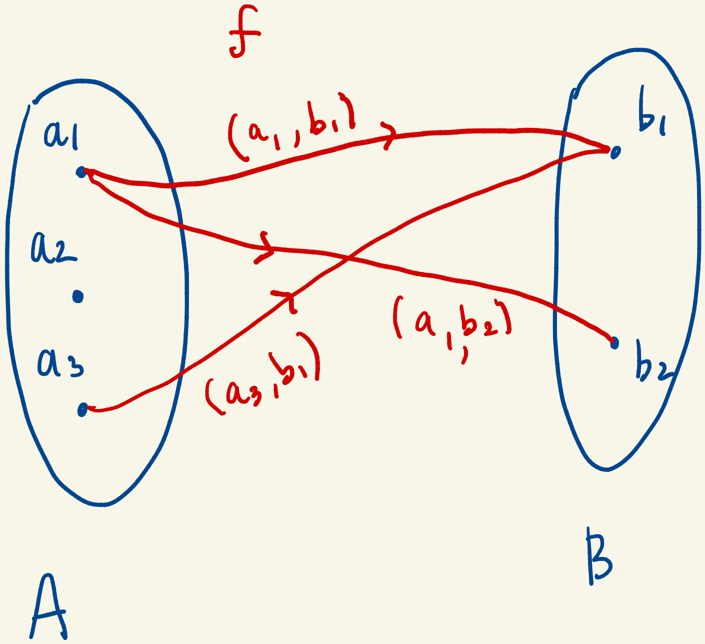
between $|A|$ and $|B|$

because $|A \multimap B| = |A| \times |B|$

(f a clique of $A \multimap B$

consists of elements of

the form (a, b) $a \in |A|$
$b \in |B|$)



f is a relation

or

a

"multi valued function"

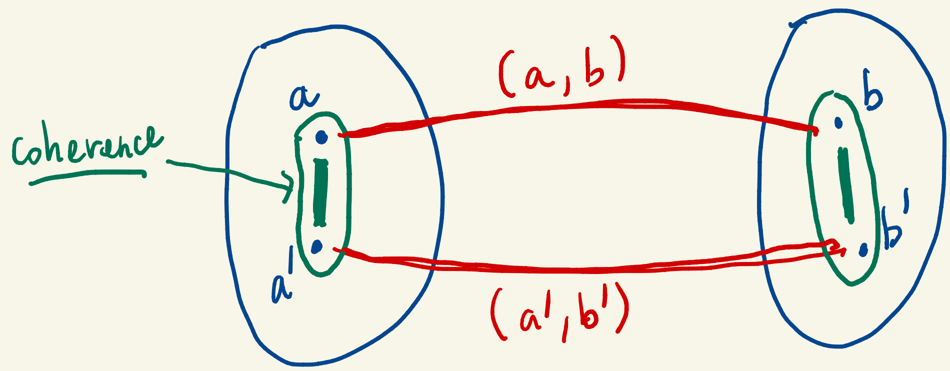
$$f = \{ (a_1, b_1), (a_1, b_2), (a_3, b_1) \}$$

When is a binary relation between $|A|$ and $|B|$

$$f \subseteq |A| \times |B|$$

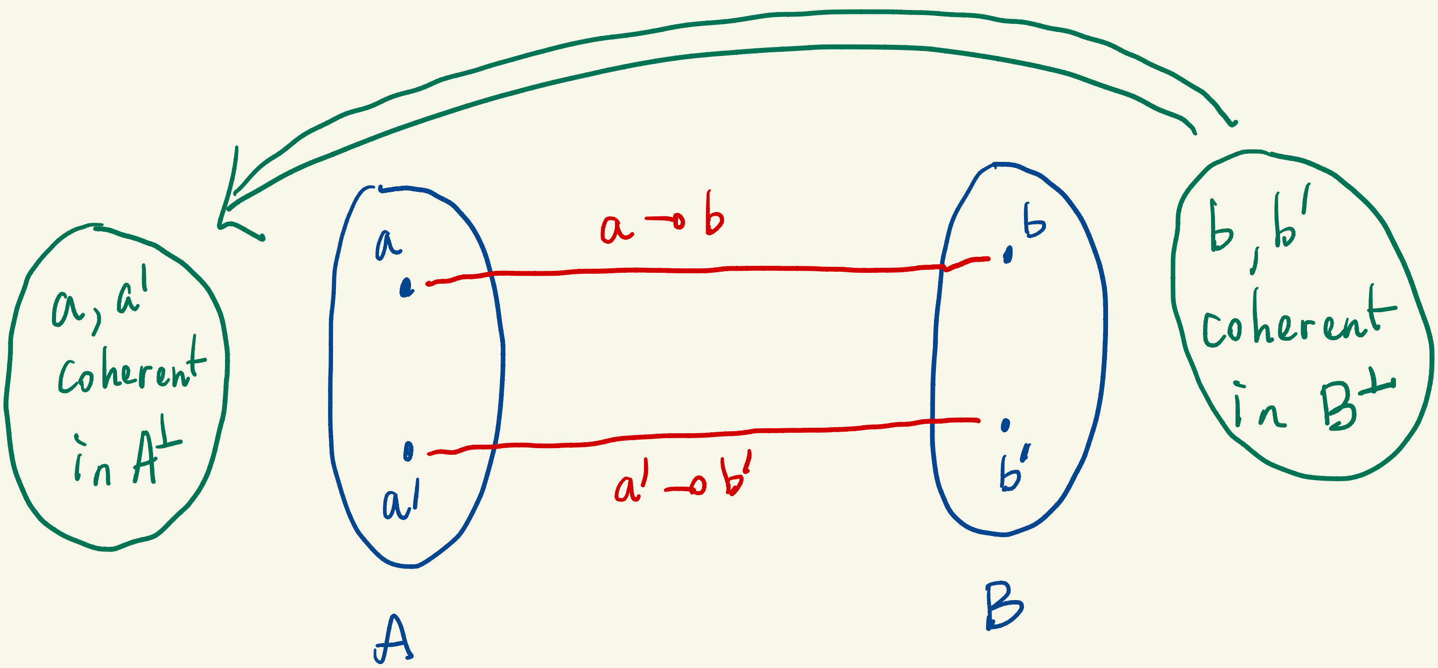
a clique of $\boxed{A \multimap B}$?

$$a \multimap b \asymp_{A \multimap B} a' \multimap b' \iff \left\{ \begin{array}{l} \boxed{a \asymp_A a' \implies b \asymp_B b'} \\ \boxed{b \asymp_{B^\perp} b' \implies a \asymp_{A^\perp} a'} \end{array} \right.$$



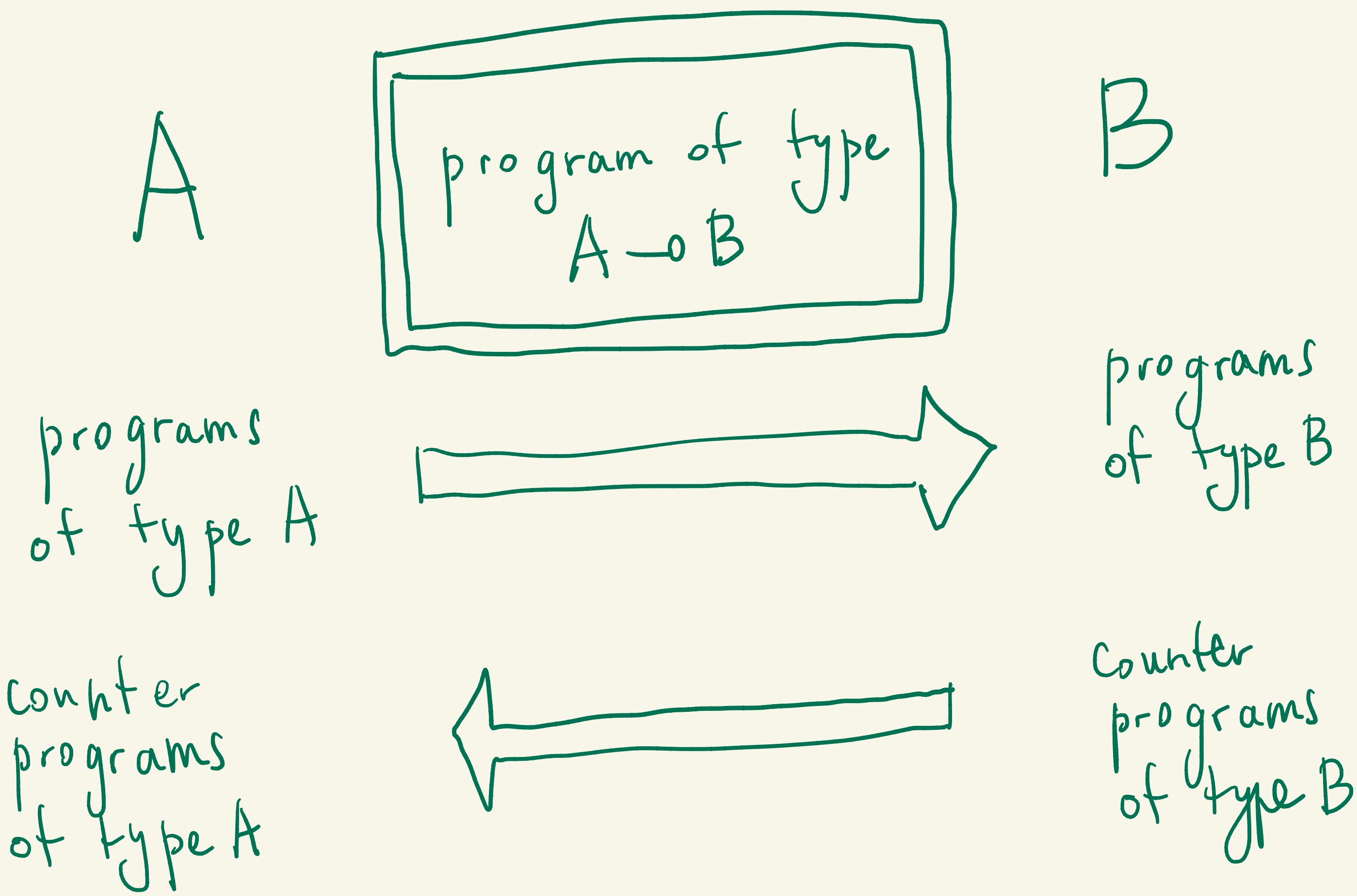
cliques are "propagated" in the "forward" direction forward propagation of cliques

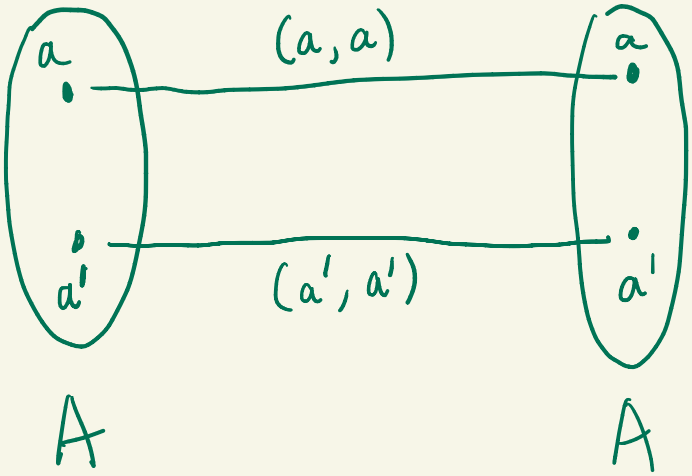
this captures the intuition that **a program of type $A \multimap B$** should transport programs of type A into programs of type B.



backward propagation of anti cliques

this captures the intuition that counter programs of type B are transported to counter programs of type A.





$$(a,a) \mathrel{\substack{\frown \\ \smile}}_{A \multimap A} (a',a')$$

hence the identity relation id_A

$$id_A = \{(a,a) \mid a \in |A|\} \subseteq |A| \times |A|$$

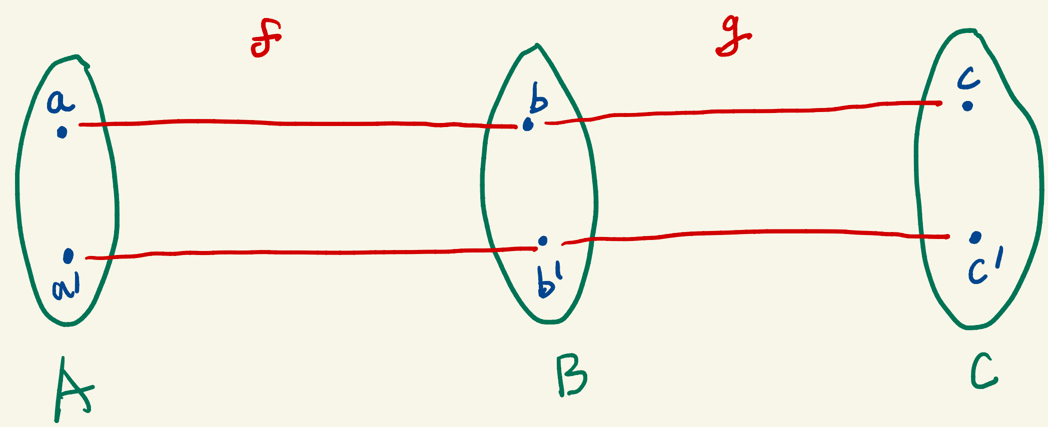
is a <u>clique</u> of $A \multimap A$.

This defines the identity morphism in $\boxed{Coh}$ on the coherence space A

Exercise: given a clique f of $A \multimap B$

(1) and a clique g of $B \multimap C$

show that $g \circ f$ is a clique of $A \multimap C$.



Exercise: what is a morphism

(*) $f : 1 \longrightarrow A$ in Coh?

what is a morphism

(**) $k : A \longrightarrow \perp$ in Coh?

what is composition of f and k doing?

(***) $1 \xrightarrow{f} A \xrightarrow{k} \perp$?

(*) what is a clique $1 \multimap A \cong A$

f is a clique of A

(**) what is a clique of $A \multimap \perp$

$$A \multimap \perp \cong A^{\perp} \parr \perp \cong A^{\perp}$$

(***)

$$\boxed{1 \xrightarrow{f} A \xrightarrow{k} \perp}$$

testing whether $f \cap k = \emptyset$ or singleton

$k \circ f =$ empty when $f \cap k = \emptyset$

$k \circ f = \{*\}$ when $f \cap k \neq \emptyset$.

singleton — the computation reached value in A

$$\begin{tikzcd} 1 \arrow[r, bend left, "\text{singleton}"] \arrow[r, bend right, "\text{empty}"'] & \perp \end{tikzcd}$$

empty — the computations produced a loop

Exercice

Exercise. Show that the category $\mathbf{Coh}$ contains the category of sets and partial functions as a full subcategory (see (MacLane) for the definition of full subcategory). To that purpose, consider the « discrete » coherence space

$$(X, =)$$

associated to the set X.

Show that the subcategory is closed by $\oplus$ and $\otimes$, but not closed under $\multimap$.

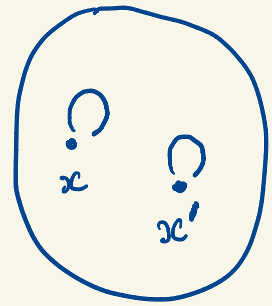
Show that the maximal anticliques of the coherence space

$$(X, =) \multimap (Y, =)$$

are the absciss lines.

Consider a set X

$$(X, =)$$



$D_{(X,=)}$

the flat domain associated to X

$$\mathbb{B} = \left(\{\text{true}, \text{false}\}, = \right)$$

$$D_{\mathbb{B}} = \begin{tikzcd} \{\text{true}\} \arrow[dr, dash] & & \{\text{false}\} \arrow[dl, dash] \\ & \perp & \end{tikzcd} \qquad \perp = \phi.$$

given X, Y two sets

$$(X, =) \xrightarrow{\;?\;} (Y, =) \quad \text{in Coh}$$

what are the cliques of

$$(X, =) \multimap (Y, =) \quad ?$$

let us think about it!

if we use the notation X and Y
for the coherence space
what are the <u>anti cliques</u> of $X \multimap Y$?

they are the <u>cliques</u> of $(X \multimap Y)^\perp$

$$(X \multimap Y)^\perp = X \otimes Y^\perp$$

<u>Reminder</u> : $\boxed{X \multimap Y} = (X \otimes Y^\perp)^\perp = \boxed{X^\perp \parr Y}$

conjunction

<u>disjunction</u>. of linear logic

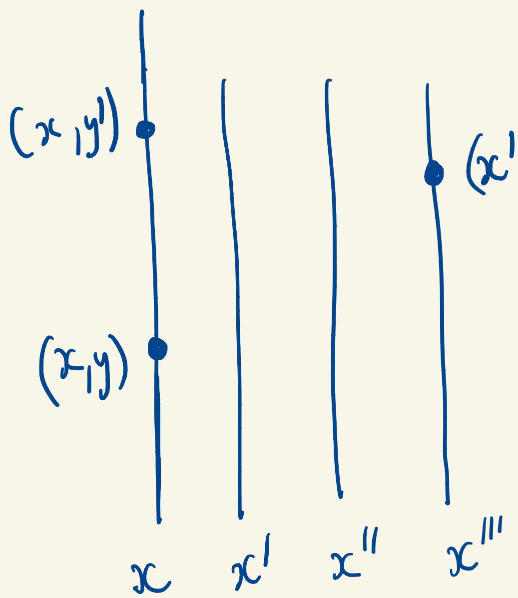
$Y^\perp$ = the coherence space
with the elements of the set Y
as elements of the web
all of them pairwise coherent.

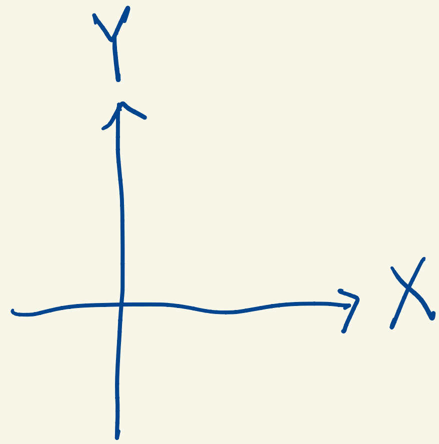
$X \otimes Y^\perp$ the "product" of X and $Y^\perp$
as graphs

From this follows that:

$$(x, y) \underset{X \otimes Y^{\perp}}{\overset{\frown}{\smile}} (x', y') \iff x = x'$$

$X \otimes Y^{\perp}$

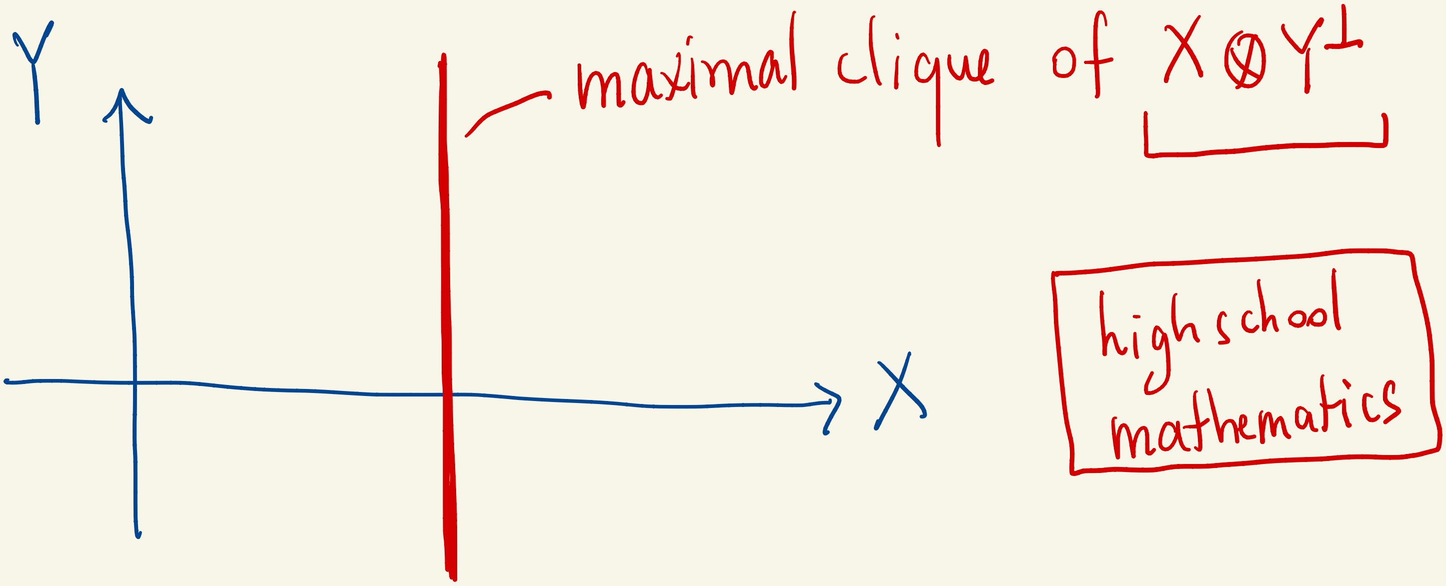




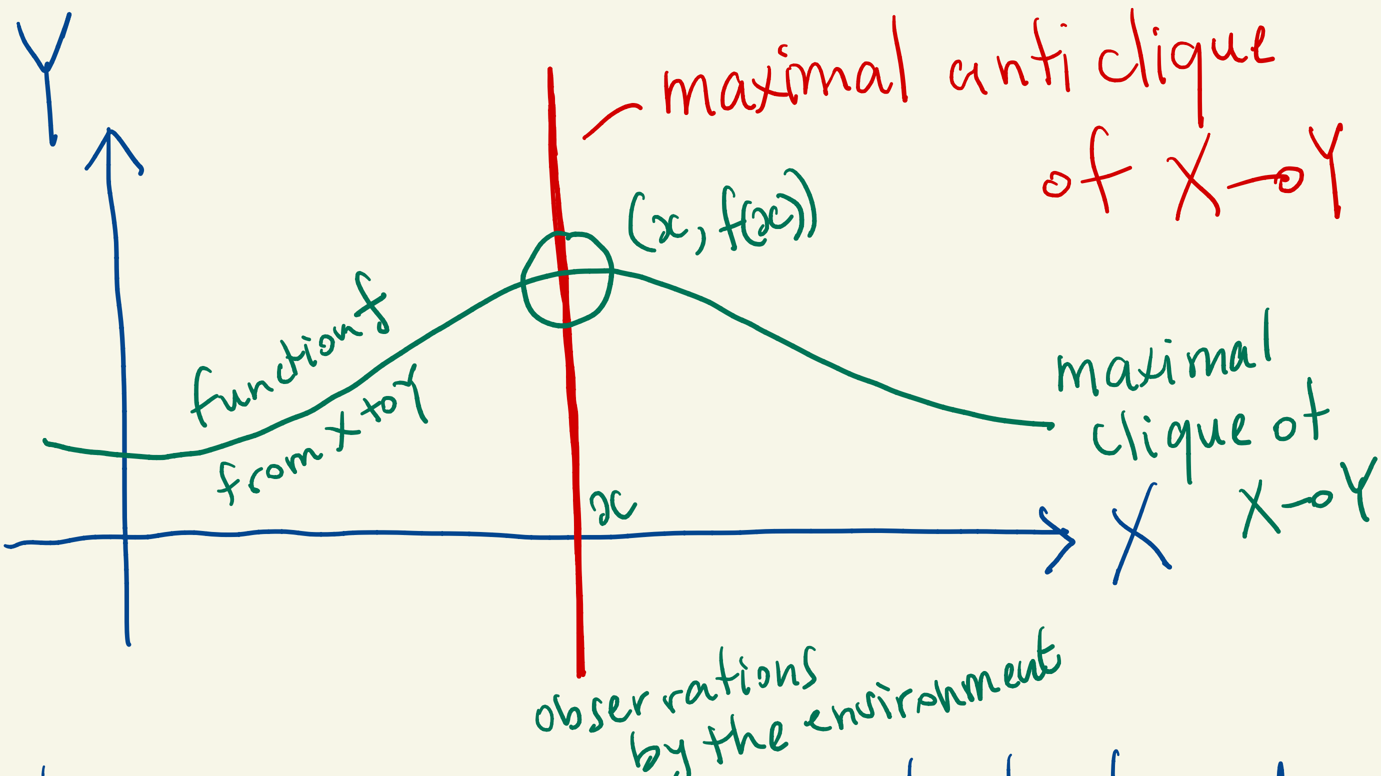
the cliques of $X \otimes Y^{\perp}$ are the sets of the form $x \otimes U$ for $U \subseteq Y$.

$$x \otimes U = \{ (x, u) \mid u \in U \}$$

in particular, the maximal cliques are of the form $x \otimes Y = \{ (x, y) \mid y \in Y \}$



the maximal cliques of $X \otimes Y^{\perp}$ are the "vertical lines" $x \otimes Y$



Rem: a clique in a graph is a subset of nodes which intersects every maximal anticlique at most once.

a clique f of $X \multimap Y$ is thus a subset of $X \times Y$ such that

$$\forall (x,y), (x',y') \in f$$

$$x = x' \implies y = y'$$

hence, a clique of $X \multimap Y$ is a partial function from X to Y.

a maximal clique of $X \multimap Y$ is a total function from X to Y.

2

Damnation: $\mathbf{Coh}$ is not cartesian closed!

Exercise. Show that

— $\boxed{A\&B}$ is the cartesian product of A and B in the category $\mathbf{Coh}$.

— the object $\top$ is terminal in $\mathcal{C}$.

Deduce that $(\mathbf{Coh}, \&, \top)$ defines a cartesian category.

Exercise. Show that the object $0 = \top$ admits a cartesien exponentiation in the cartesian category $(\mathbf{Coh}, \&, \top)$. (Use (1) the equality $0 = \top$, (2) that $\mathbf{Hom}(0, A)$ is singleton for all objects A, (3) that every object A exponentiable defines a bijection

$$\frac{A\&\top \longrightarrow B}{\top \longrightarrow A \Rightarrow B} \quad \phi_{\top,A,B}$$

to show that $\mathbf{Hom}(A, B)$ is singleton, for all object B.) Deduce that the category $(\mathbf{Coh}, \&, \top)$ is not cartesian closed.

$D_{A\&B}$ $\qquad 1 \xrightarrow[\text{of } A\&B]{\text{clique}} A\&B$ $\qquad 1 \xrightarrow[\text{of } A]{\text{clique}} A$, $\; 1 \xrightarrow[\text{of } B]{\text{clique}} B$ $\qquad D_A \times D_B$

$$D_{A\&B} \cong D_A \times D_B$$

$$\underbrace{\underline{\mathrm{Coh}}(1, A\&B)}_{\text{the set of maps from 1 to } A\&B \text{ (or domain of maps)}} \cong \underline{\mathrm{Coh}}(1,A) \times \underline{\mathrm{Coh}}(1,B)$$

the set of maps
from 1 to A & B
(or domain of maps)

a consequence of the fact that A&B is the cartesian product of A and B

given two coherence spaces A and B
the set of maps from A to B
is the set of cliques of $A \multimap B$
we can think of it as the domain $D_{A \multimap B}$

$\underbrace{A \times -}_{\text{left adjoint}} \dashv \underbrace{A \Rightarrow -}_{\text{right adjoint}}$

But nearly so...

Exercise. Use associativity and definition of ⅋ to show that

$$\boxed{(A \otimes B) \multimap C = B \multimap (A \multimap C)}$$

Deduce that there exists for every coherence space A a family of bijections $(\phi_{A,B,C})_{B,C}$ in **Coh**:

$$\frac{A \otimes B \longrightarrow C}{B \longrightarrow A \multimap C} \quad \phi_{A,B,C}$$

and show its naturality of B and C.

currification

$$\mathbf{Coh}(A \otimes B, C) \cong \mathbf{Coh}(B, A \multimap C)$$

$\boxed{\text{positive}}$ $\boxed{\text{left}}$ $A \otimes - \dashv A \multimap -$ $\boxed{\text{right}}$ $\boxed{\text{negative}}$

$\boxed{\text{adjunction}}$

$\mathscr{C}(A, B)$ hom-set set of maps

$A \Rightarrow B$ internal hom object of $\mathscr{C}$

$$\mathscr{C}(1, \underbrace{A \Rightarrow B}) \cong \mathscr{C}(A, B)$$

in this object
there is enough information
to reconstruct $\mathscr{C}(A, B)$

$D \Rightarrow E$

the domain
of
continuous
function.

$\boxed{\mathrm{Dom}(D, E)}$

set of continuous
functions
from D to E

A, B

$A \multimap B$

the
coherence space
itself

$\mathrm{Coh}(A, B)$

set of
cliques of $A \multimap B$

$$A \Rightarrow B \cong \underbrace{(!A)}_{\substack{\text{exponential} \\ \text{modality} \\ \text{of linear logic.}}} \multimap B$$

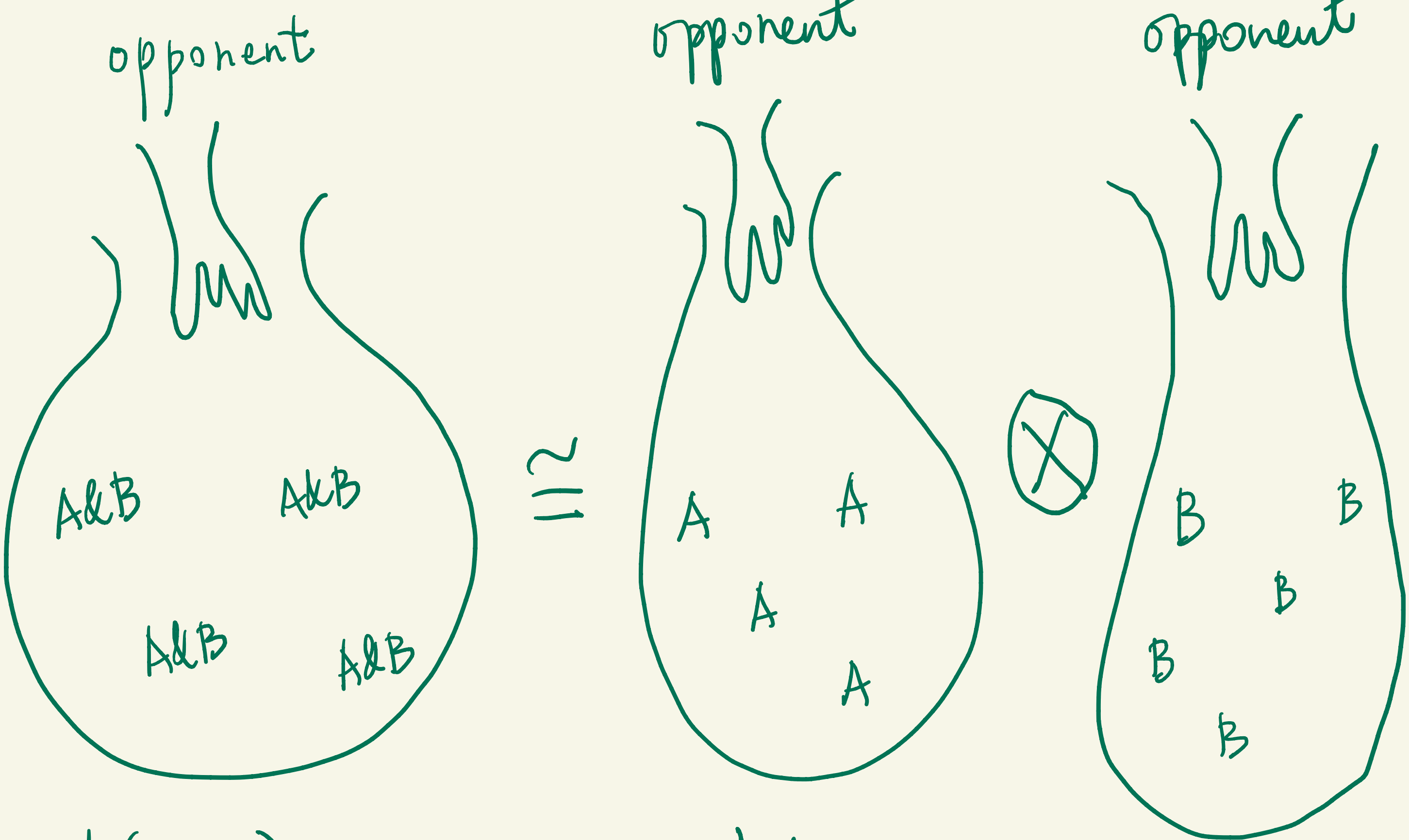
$A \& B$ cartesian product in Coh

$A \otimes B$ which has a closed structure

$$A \otimes - \dashv A \multimap -$$

"bang"

$$!(A \& B) \cong\ !A \otimes\ !B$$



$!(A \& B) \qquad !A \otimes !B$

A chocolate

B strawberry

opponent has control $\quad !(A \& B) \cong !A \otimes !B$

player has control $\quad ?(A \oplus B) \cong ?A \parr ?B$

But nearly so...

Conclusion:

— the cartesian structure is given by the additives $\&$ and $\top$

— the closed structure is given by the multiplicatives $\otimes$ and $\mathbf{1}$

Prescription:

— one needs an exponential to bridge the two worlds

Monoidal categories

The first step towards a categorical account of linear logic

Intuition

Redo everything as in a cartesian closed category...
but replace the cartesian product $\times$ by an arbitrary bifunctor $\otimes$.

1. replace the **universal properties** of $\times$ by **coherence diagrams** of $\otimes$.

One obtains in this way a **symmetric monoidal category.**

2. replace the adjunction

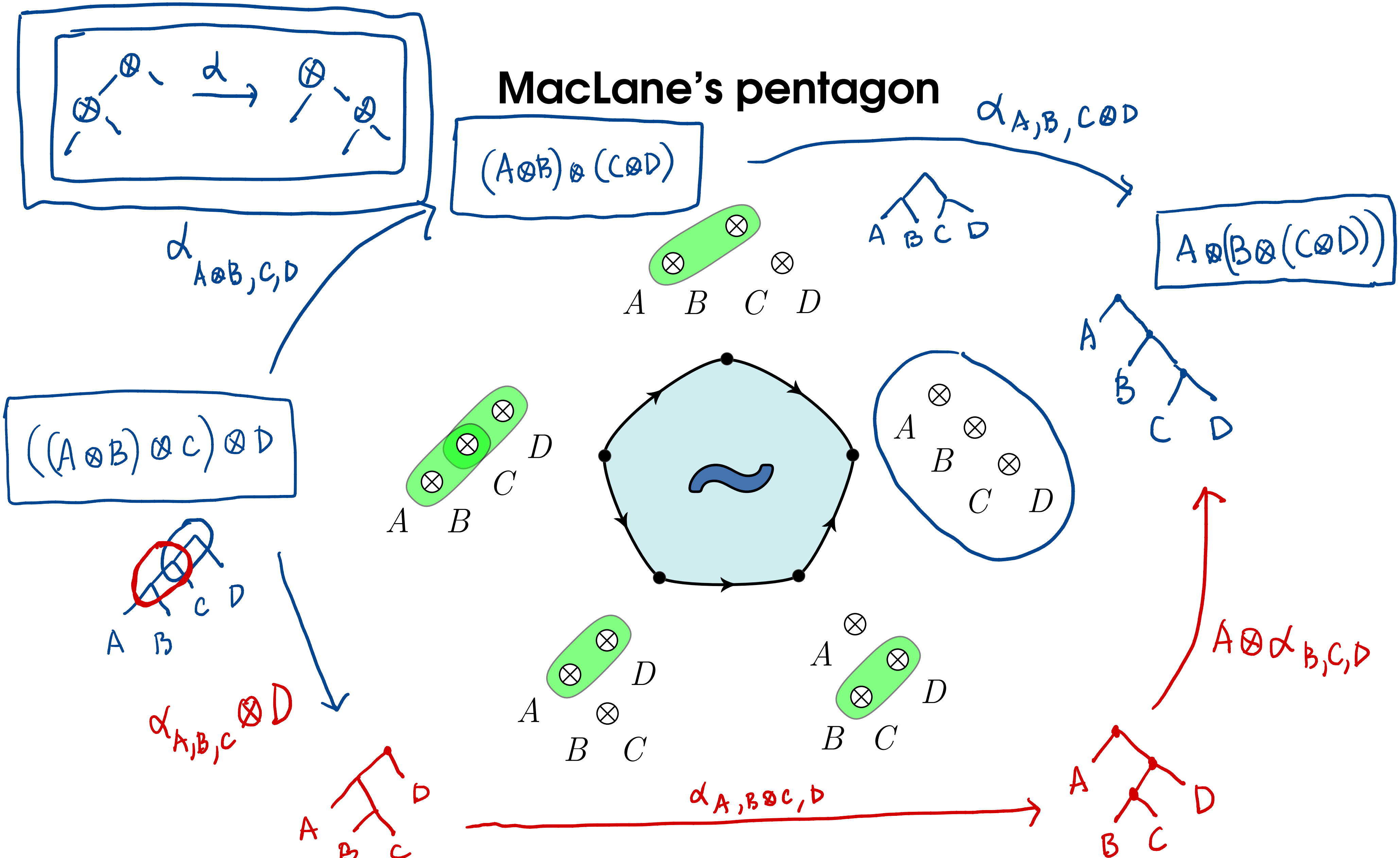
$$\frac{A \times B \longrightarrow C}{B \longrightarrow A \Rightarrow C}$$

by an adjunction

$$\frac{A \otimes B \longrightarrow C}{B \longrightarrow A \multimap C}$$

One obtains a **symmetric monoidal closed category (smcc)**
where **intuitionistic linear logic** (linear λ-calculus) may be interpreted.

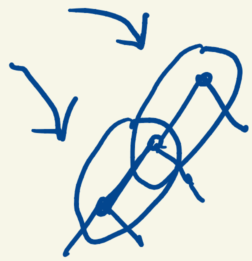
MacLane's pentagon

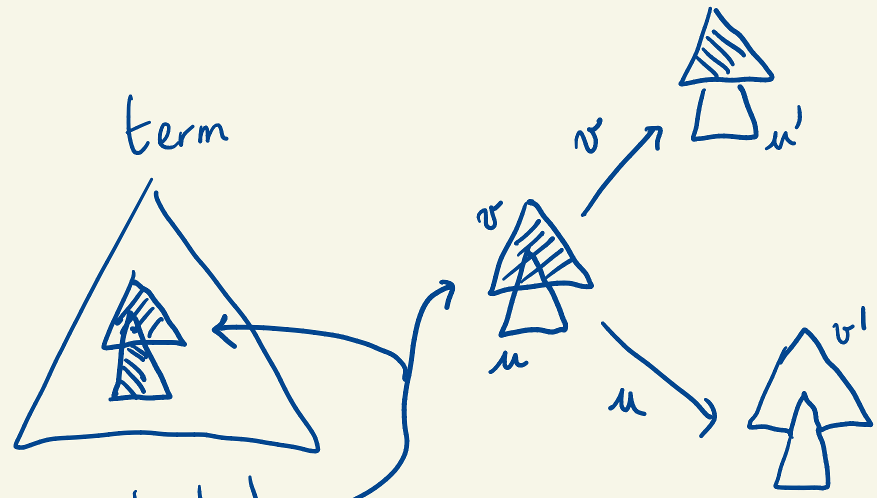


The associativity patterns are indicated in green

«local confluence» diagram for the associativity (seen as rewriting)

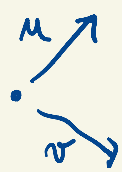
critical pair



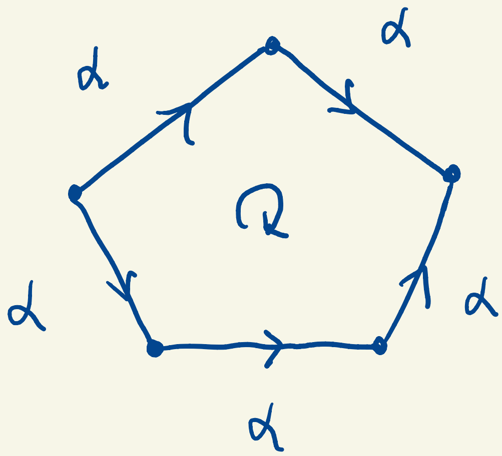


critical pair in rewriting theory

usually produce "non determinism" in rewriting

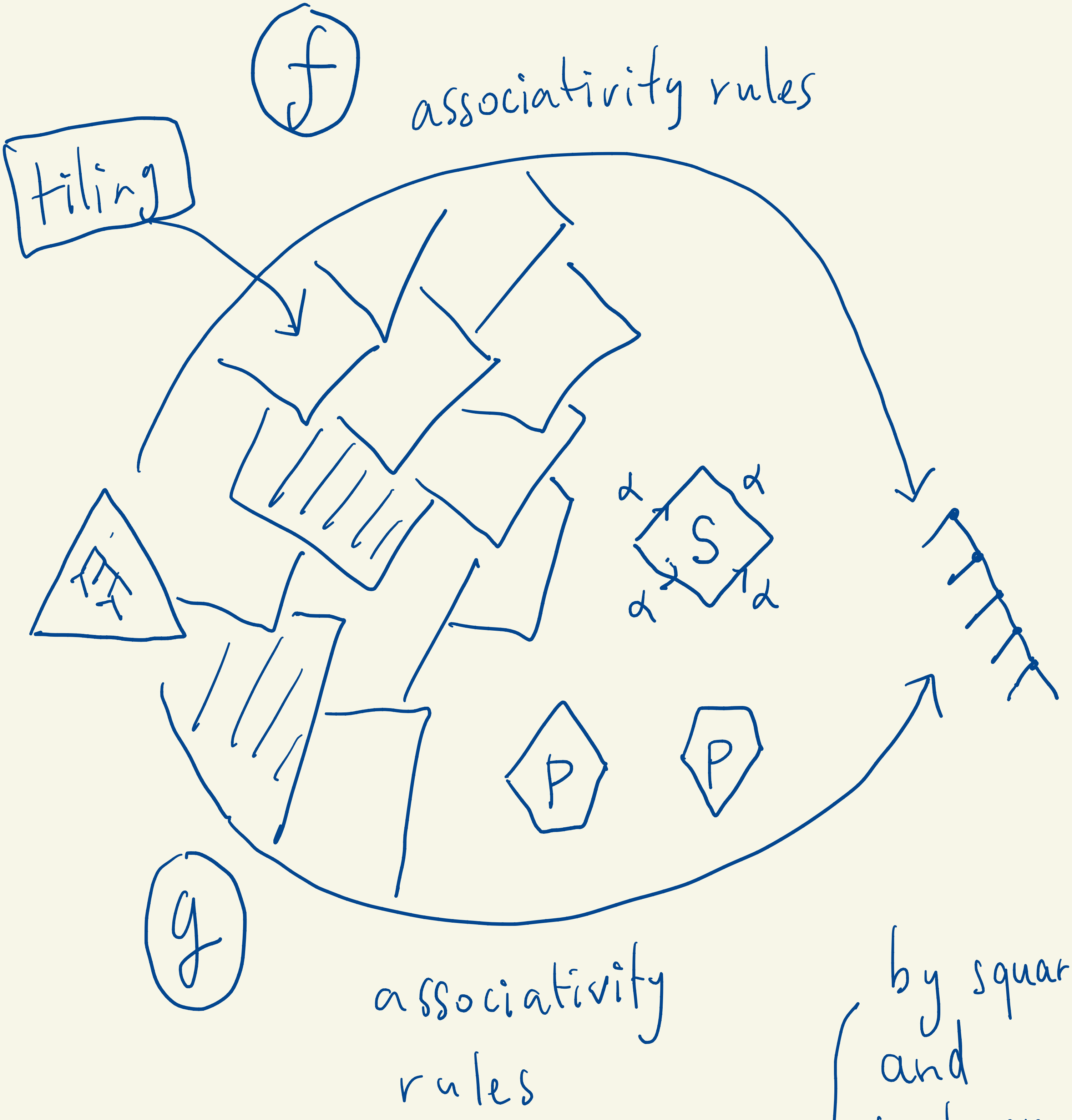


no confluence



confluence diagram

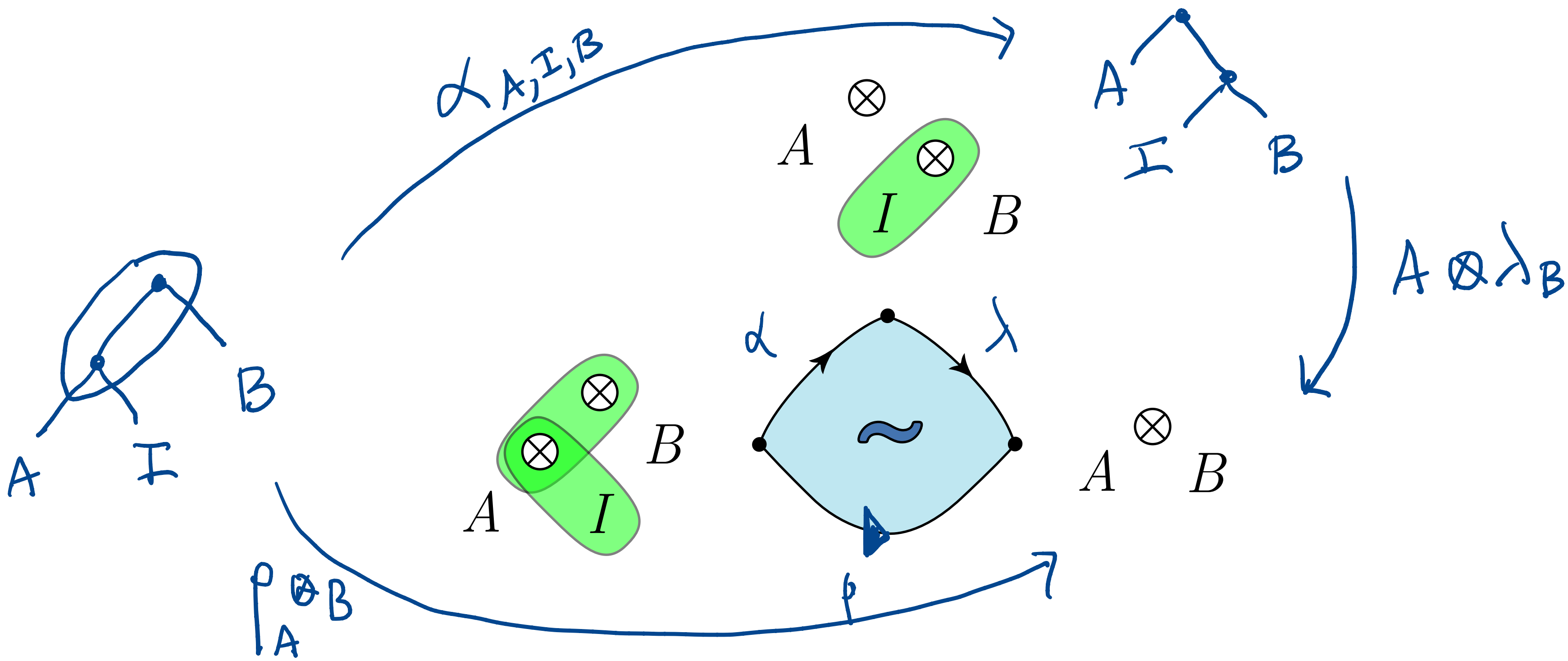
coherence property : the diagram induces the same composite morphism in $\mathcal{C}$



it is possible to reorganise f in order to obtain g

by square and pentagon permutations

MacLane's triangle



The associativity patterns are indicated in green

Coherence theorem

MacLane's book Categories for the Working mathematician

Idea: benefit from one consequence of the universality property... but without the universality property.

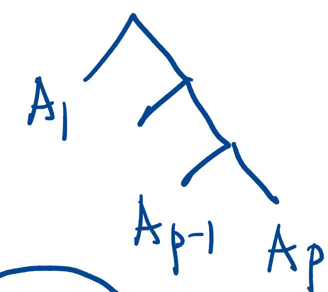
Given a sequence $A_1, \ldots, A_p$ of objects of a monoidal category $\mathcal{C}$, consider the words w over $A_1, \ldots, A_p$ defined as

- ▷ the object I when $p = 0$,
- ▷ the object $u \otimes v$ where $\begin{cases} u & \text{is a word on } (A_1, \ldots, A_m) \\ v & \text{is a word on } (A_{m+1}, \ldots, A_p) \end{cases}$

  for some $1 \leq m \leq p$.

Among these words, one finds the **canonical** word

$$\boxed{(\cdots(A_1 \otimes A_2) \otimes \cdots A_p)}$$



Coherence theorem.
There exists a $\boxed{\text{unique}}$ structural morphism built from "α, λ, ρ" from any word on $(A_1, \ldots, A_p)$ to the canonical word.