Modèles des langages de programmation (MPRI 2.2)

Stable and linear functions between clique domains

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In this problem, we call graph $A = (V, E)$ a set V of vertices equipped with a reflexive and symmetric relation $E \subseteq V \times V$ describing the edges. Recall that by a symmetric and reflexive relation, we mean that

$$\forall a \in V, \qquad (a, a) \in E$$

$$\forall a \in V, \forall a' \in V, \qquad (a, a') \in E \quad \Rightarrow \quad (a', a) \in E.$$

A clique of a graph A is defined as a subset $u \subseteq V$ such that

$$\forall a \in u, \forall a' \in u, \qquad (a, a') \subseteq E.$$

We recall that a continuous function is monotone by definition.

Question 1. Show that the set of cliques of A ordered by inclusion

$$u \leq_A v \qquad \overset{def}{\Longleftrightarrow} \qquad u \subseteq v$$

defines a domain $(\mathrm{D}_A, \leq_A)$.

Question 2. Show that a continuous function

$$f \quad : \quad \mathrm{D}_A \quad \longrightarrow \quad \mathrm{D}_B$$

is entirely determined by its restriction

$$!A \quad \longrightarrow \quad \mathrm{D}_A \quad \xrightarrow{f} \quad \mathrm{D}_B$$

to the set (noted $!A$) of the finite cliques of the graph A.

Question 3. From this, deduce the existence of a bijection between the set of continuous functions from D_A to D_B and the set of monotone functions from $!A$ to D_B — and describe how the bijection works.

Question 4. For every continuous function $f : \mathrm{D}_A \to \mathrm{D}_B$, one defines the set

$$\mathrm{Tr}(f) \quad \subseteq \quad !A \times B$$

of pairs (u, b) which satisfy the two properties below:

- $b \in f(u)$,
- $b \notin f(v)$ for every clique $v \in \mathrm{D}_A$ strictly included in u.

Show that the equality

$$f(u) \quad = \quad \Big\{ \quad b \in B \quad | \quad \exists\, v \ \in !A, \quad v \leq_A u \quad \text{et} \quad (v, b) \in \mathrm{Tr}(f) \quad \Big\}.$$

holds for every clique u of the graph A.

Question 5. Two cliques u and v of the graph A are compatible (notation: $u \uparrow v$) when there exists a clique w which contains both of them:

$$u \uparrow v \quad \stackrel{def}{\Longleftrightarrow} \quad \exists w. \quad u \leq w \quad \text{and} \quad v \leq w.$$

A continuous function $f : \mathrm{D}_A \to \mathrm{D}_B$ is called *stable* when

$$\forall u, v \in \mathrm{D}_A, \qquad u \uparrow v \quad \Rightarrow \quad f(u \cap v) = f(u) \cap f(v).$$

Suppose that f is stable. Show that (u, b) is an element of $\mathrm{Tr}(f)$ if and only if, for every clique v compatible with u, the equivalence below holds:

$$u \leq v \qquad \Longleftrightarrow \qquad b \in f(v).$$

Question 6. A continuous function $f : \mathrm{D}_A \to \mathrm{D}_B$ is called *linear* when it is stable and satisfies the two properties below:

$$\begin{array}{lrcl} (1) & f(\emptyset) & = & \emptyset \\ (2) \quad \forall u, v \in \mathrm{D}_A, \quad u \uparrow v \quad \Rightarrow & f(u \cup v) & = & f(u) \cup f(v). \end{array}$$

Show that a stable function $f : \mathrm{D}_A \to \mathrm{D}_B$ is linear if and only if every element (u, b) of the trace of f is of the form $(\{a\}, b)$.

Question 7. Show that the set of linear functions from D_A to D_B, ordered as follows:

$$f \leq g \quad \stackrel{def}{\Longleftrightarrow} \quad \forall u \in \mathrm{D}_A, \quad f(u) \leq_B g(u).$$

defines a domain. We write $\mathrm{D}_A \multimap \mathrm{D}_B$ for the domain of linear functions just defined.

Question 8. Define a graph $A \multimap B$ such that the equality holds:

$$\mathrm{D}_{A \multimap B} \quad = \quad \mathrm{D}_A \multimap \mathrm{D}_B$$

Question 9. Let 1 denote the graph with a unique vertex $*$. Show that the trace of a stable function

$$f \quad : \quad \mathrm{D}_A \quad \longrightarrow \quad \mathrm{D}_1 = \{\bot, \top\}$$

is of the form

$$\mathrm{Tr}(f) \quad = \quad \{ \quad (u, *) \quad | \quad u \in U \quad \}$$

where U is a set of finite and pairwise incompatible cliques of A.

Question 10. The ordered set $!A$ of finite cliques of the graph A defines a graph (also noted $!A$) where two finite cliques u and v of the graph A are connected by an edge precisely when $u \cup v$ is a clique of A. Construct a bijection between the set of stable functions from D_A to D_B and the set of linear functions from $\mathrm{D}_{!A}$ to D_B.

Question 11. Show that

$$\mathrm{D}_{(!A) \multimap B} \quad = \quad \mathrm{D}_{!A} \multimap \mathrm{D}_B \quad = \quad \mathrm{D}_A \Rightarrow \mathrm{D}_B$$

where $\mathrm{D}_A \Rightarrow \mathrm{D}_B$ denotes the set of stable functions from D_A to D_B, equipped with the ordering relation

$$f \leq_s g \quad \stackrel{def}{\Longleftrightarrow} \quad \forall u, v \in \mathrm{D}_A, \quad u \leq_A v \Rightarrow f(v) \cap g(u) = f(u).$$

Show in particular that

$$f \leq_s g \qquad \Longleftrightarrow \qquad \mathrm{Tr}(f) \subseteq \mathrm{Tr}(g).$$