

# Interpretation of the simply-typed $\lambda$ -calculus in a CCC

# The simply-typed $\lambda$ -calculus

The simple types  $A, B$  are constructed by the grammar:

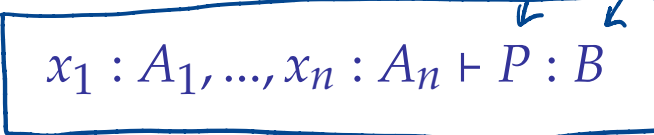
$$A, B ::= \alpha \mid A \Rightarrow B. \mid A \times B$$

A **typing context**  $\Gamma$  is a finite sequence

$$\Gamma = (x_1 : A_1, \dots, x_n : A_n)$$

where each  $x_i$  is a variable and each  $A_i$  is a simple type.

A **sequent** is a triple

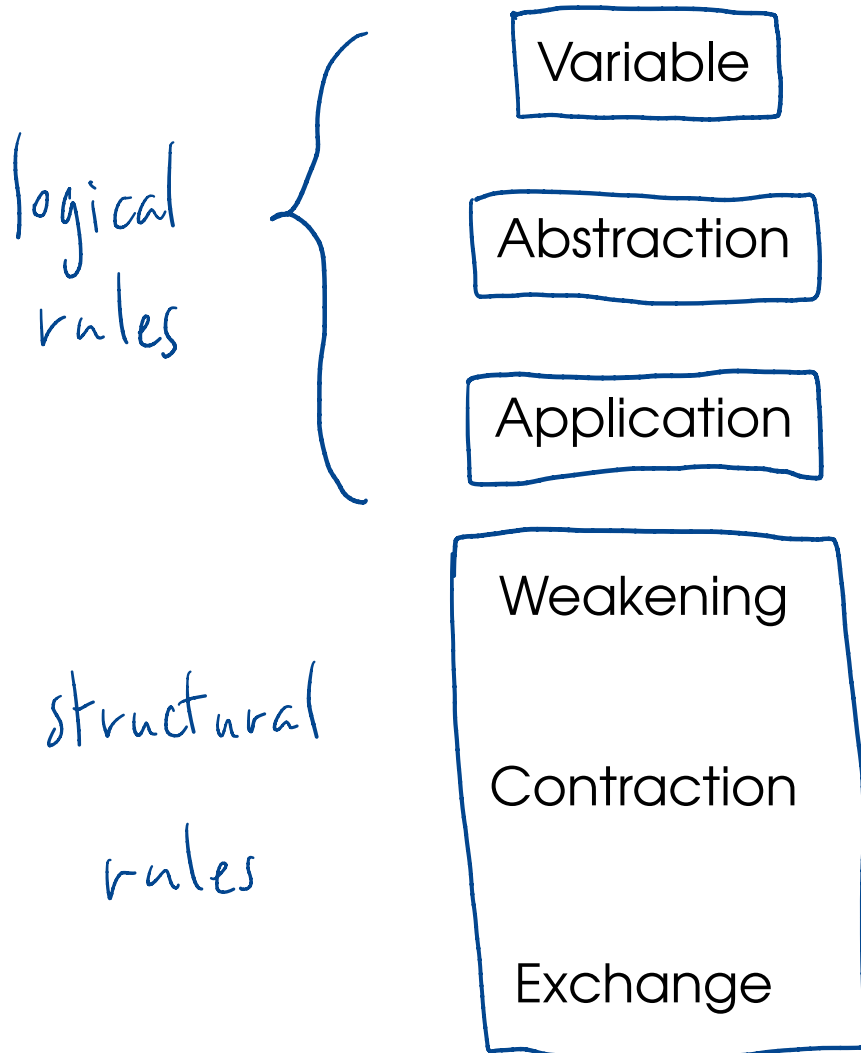

$$x_1 : A_1, \dots, x_n : A_n \vdash P : B$$

where

$$x_1 : A_1, \dots, x_n : A_n$$

is a typing context,  $P$  is a  $\lambda$ -term and  $B$  is a simple type.

# The simply-typed $\lambda$ -calculus



$$\frac{}{x : A \vdash x : A}$$

$$\frac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x. P : A \Rightarrow B}$$

$$\frac{\Gamma \vdash P : A \Rightarrow B \quad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$$

$$\frac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$$

$$\frac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$$

$$\frac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$$

# Interpretation of the $\lambda$ -calculus

**Step 1.** We suppose given a function

$$\xi : \alpha \mapsto \boxed{\xi(\alpha)} \text{ is an object of } \mathcal{C}.$$

which associates an object  $\xi(\alpha)$  to every type variable  $\alpha$ .

**Step 2.** Every type  $\boxed{A}$  is then interpreted as an object

$$\boxed{\llbracket A \rrbracket} \text{ of the category } \mathcal{C}$$

of the cartesian closed category by structural induction:

$$\begin{cases} \boxed{\llbracket \alpha \rrbracket} = \boxed{\xi(\alpha)} \\ \llbracket A \times B \rrbracket = \llbracket A \rrbracket \times \llbracket B \rrbracket \\ \llbracket A \Rightarrow B \rrbracket = \llbracket A \rrbracket \Rightarrow \llbracket B \rrbracket \end{cases}$$

↑  
syntax  
of types

↑  
semantics  
structure of ccc.

## Interpretation of the $\lambda$ -calculus

**Step 3.**

Every sequent

$$x_1 : A_1, \dots, x_n : A_n \vdash (t) : B$$

is interpreted as a morphism

$$\llbracket t \rrbracket : \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket \longrightarrow \llbracket B \rrbracket$$

by structural induction on the derivation tree which produced it.

# The logical rules

▷ Variable:  $\llbracket A \rrbracket \xrightarrow{id} \llbracket A \rrbracket$

▷ Lambda:

$$A \times \Gamma \xrightarrow{f} B$$

becomes

$$\Gamma \xrightarrow{\phi_{A,\Gamma,B}(f)} A \Rightarrow B$$

$A \times -$  is left adjoint  
to  $A \Rightarrow -$

▷ Application:

$$\Gamma \xrightarrow{f} A$$

and

$$\Delta \xrightarrow{g} A \Rightarrow B$$

become

$$\Gamma \times \Delta \xrightarrow{f \times g} A \times (A \Rightarrow B) \xrightarrow{eval_{A,B}} B$$

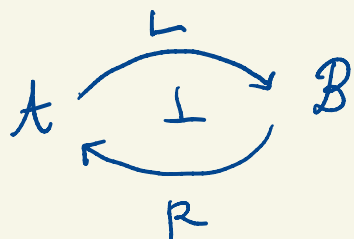
the counit  $\epsilon$   
of the adjunction

use the ccc  
structure of  $\mathcal{C}$ .

here we use the fact

that  $- \times - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  defines a functor

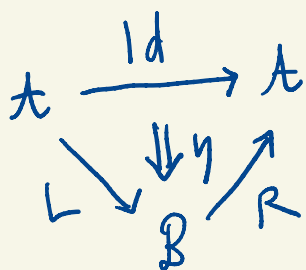
Every adjunction



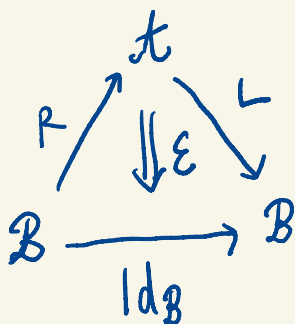
left  
as exercise

induces a pair of natural transformations

unit  $\left( \eta_A : A \longrightarrow RLA \right)_{A \in \text{Ob } \mathcal{A}} : \text{Id}_{\mathcal{A}} \Rightarrow R \circ L$



counit  $\left( \epsilon_B : LRB \longrightarrow B \right)_{B \in \text{Ob } \mathcal{B}} : L \circ R \Rightarrow \text{Id}_{\mathcal{B}}$



$$\begin{array}{c}
 LA \xrightarrow{\text{id}_{LA}} LA \\
 \hline
 A \boxed{\eta_A} \rightarrow RLA
 \end{array}
 \quad
 \begin{array}{c}
 \phi_{A, LA} \\
 \downarrow
 \end{array}
 \quad
 \begin{array}{c}
 RB \xrightarrow{\text{id}_{RB}} RB \\
 \hline
 LRB \boxed{\epsilon_B} \rightarrow B
 \end{array}
 \quad
 \begin{array}{c}
 \phi_{RB, B}^{-1}
 \end{array}$$

in the case of a cartesian closed category:

we get two natural transformations  
for each object  $A$ :

$$\eta_{AB}: B \xrightarrow{\text{coeval}} A \Rightarrow (A \times B)$$

$$b:B \vdash \lambda a. (a, b): A \Rightarrow (A \times B)$$

$$\epsilon_{AB}: A \times (A \Rightarrow B) \xrightarrow{\text{eval}} B$$

$$a:A, f:A \Rightarrow B \vdash \underbrace{fa}: B$$

Exercise: show that the families  
are also natural in the object  $A$ .



# The structural rules

$$\delta_A \times \Gamma = \delta_A \times \text{id}_\Gamma$$

▷ Contraction:

$$A \times A \times \Gamma \xrightarrow{f} B$$

becomes

$$A \times \Gamma \xrightarrow{\delta_A \times \Gamma} A \times A \times \Gamma \xrightarrow{f} B$$

$$A \xrightarrow{\delta_A} A \times A$$

diagonal

▷ Weakening:

$$\Gamma \xrightarrow{f} B$$

becomes

$$A \times \Gamma \xrightarrow{\varepsilon_A \times \Gamma} 1 \times \Gamma \xrightarrow{\sim} \Gamma \xrightarrow{f} B$$

$$A \longrightarrow \mathbb{1}$$

projection

▷ Permutation:

$$\Gamma \times A \times B \times \Delta \xrightarrow{f} B$$

becomes

$$\Gamma \times B \times A \times \Delta \xrightarrow{\Gamma \times \gamma_{A,B} \times \Delta} \Gamma \times A \times B \times \Delta \xrightarrow{f} B$$

$$A \times B \xrightarrow{\gamma_{A,B}} B \times A$$

symmetry

$F: \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$  a pair of families of functors

$$F(C, -): \mathcal{D} \rightarrow \mathcal{E}$$

$$F(-, D): \mathcal{C} \rightarrow \mathcal{E} \quad (+ \dots)$$

# Soundness theorem

## Theorem.

In every cartesian closed category  $\mathcal{C}$ , the interpretation  $\llbracket - \rrbracket$  is an invariant modulo  $\beta, \eta$ .

▷ If  $\Gamma \vdash (\lambda x.M) : A \Rightarrow B$  and  $\Delta \vdash N : A$ , then

$$\llbracket \Gamma, \Delta \vdash (\lambda x.M)N : B \rrbracket = \llbracket \Gamma, \Delta \vdash M[x := N] : B \rrbracket$$

▷ If  $\Gamma \vdash M : A \Rightarrow B$  then

$$\llbracket \Gamma \vdash (\lambda x.Mx) : A \Rightarrow B \rrbracket = \llbracket \Gamma \vdash M : A \Rightarrow B \rrbracket$$

PCF

**Exercise.** Establish the soundness theorem.

$\llbracket B \rrbracket = \{\text{true}, \text{false}\}$   
in Set

$\llbracket B \rrbracket =$   
true false  
└─┬─  
  1  
flat domain

in Dom