Interpretation of the simply-typed λ-calculus in a CCC

The simply-typed λ-calculus

The simple types A, B are constructed by the grammar:

$$A, B \ ::= \ \alpha \mid A \Rightarrow B. \quad \mid \quad A \times B$$

A typing context Γ is a finite sequence

$$\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$$

where each x_i is a variable and each A_i is a simple type.

A sequent is a triple

$$\boxed{x_1 : A_1, \ldots, x_n : A_n \vdash P : B}$$

where

$$x_1 : A_1, \ldots, x_n : A_n$$

is a typing context, P is a λ-term and B is a simple type.

The simply-typed λ-calculus

logical rules

Variable	$$\frac{}{x : A \vdash x : A}$$
Abstraction	$$\frac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x.P : A \Rightarrow B}$$
Application	$$\frac{\Gamma \vdash P : A \Rightarrow B \qquad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$$

structural rules

Weakening	$$\frac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$$
Contraction	$$\frac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$$
Exchange	$$\frac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$$

Interpretation of the λ-calculus

Step 1. We suppose given a function

$$\xi \quad : \quad \alpha \quad \mapsto \quad \boxed{\xi(\alpha)}$$

is an object of $\mathscr{C}$.

which associates an object $\xi(\alpha)$ to every type variable α.

Step 2. Every type $\boxed{A}$ is then interpreted as an object

$$\boxed{[\![A]\!]}$$

of the category $\mathscr{C}$

of the cartesian closed category by structural induction:

$$\begin{array}{rcl}
\boxed{[\![\alpha]\!]} & = & \boxed{\xi(\alpha)} \\
[\![A \times B]\!] & = & [\![A]\!] \times [\![B]\!] \\
[\![A \Rightarrow B]\!] & = & [\![A]\!] \Rightarrow [\![B]\!]
\end{array}$$

syntax of types

semantics structure of ccc.

Interpretation of the λ-calculus

Step 3. Every sequent

$$x_1 : A_1 , \ldots , x_n : A_n \quad \vdash \quad t : B$$

is interpreted as a morphism

$$[\![t]\!] \quad : \quad [\![A_1]\!] \times \cdots \times [\![A_n]\!] \quad \longrightarrow \quad [\![B]\!]$$

by structural induction on the derivation tree which produced it.

The logical rules

- Variable: $[\![A]\!] \xrightarrow{id_{[\![A]\!]}} [\![A]\!]$
- Lambda:

$$A \times \Gamma \xrightarrow{f} B$$

becomes

$$\Gamma \xrightarrow{\phi_{A,\Gamma,B}(f)} A \Rightarrow B$$

$A \times -$ is left adjoint to $A \Rightarrow -$

- Application:

$$\Gamma \xrightarrow{\boxed{f}} A \quad \text{and} \quad \Delta \xrightarrow{\boxed{g}} A \Rightarrow B$$

become

$$\underbrace{\Gamma \times \Delta \xrightarrow{f \times g} A \times (A \Rightarrow B)}_{\text{here we use the fact that } - \times - : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C} \text{ defines a functor}} \xrightarrow{eval_{A,B}} B$$

the counit ε of the adjunction

use the ccc structure of $\mathscr{C}$.

Every adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

induces a pair of <u>natural</u> transformations

left as exercise

<u>unit</u> $\left(\eta_A : A \longrightarrow RLA \right)_{A \in \mathrm{Ob}\,\mathcal{A}} : \mathrm{Id}_{\mathcal{A}} \Longrightarrow R \circ L$

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\mathrm{Id}"] \arrow[dr, "L"'] & \Downarrow \eta & \mathcal{A} \\ & \mathcal{B} \arrow[ur, "R"'] & \end{tikzcd}$$

<u>counit</u> $\left(\varepsilon_B : LRB \longrightarrow B \right)_{B \in \mathrm{Ob}\,\mathcal{B}} : L \circ R \Longrightarrow \mathrm{Id}_{\mathcal{B}}$

$$\begin{tikzcd} & \mathcal{A} \arrow[dr, "L"] & \\ \mathcal{B} \arrow[ur, "R"] \arrow[rr, "\mathrm{Id}_{\mathcal{B}}"'] & \Downarrow \varepsilon & \mathcal{B} \end{tikzcd}$$

$$\frac{LA \xrightarrow{\mathrm{id}_{LA}} LA}{A \xrightarrow{\boxed{\eta_A}} RLA} \;\; \phi_{A,LA} \qquad \phi^{-1}_{RB,B} \;\; \frac{RB \xrightarrow{\mathrm{id}_{RB}} RB}{LRB \xrightarrow{\boxed{\varepsilon_B}} B}$$

in the case of a cartesian closed category:

we get two natural transformations

for each object A:

$$\eta_{AB} : \quad B \xrightarrow{\text{coeval}} A \Rightarrow (A \times B)$$

$$b : B \longmapsto \lambda a.(a,b) : A \Rightarrow (A \times B)$$

$$\varepsilon_{AB} : \quad A \times (A \Rightarrow B) \xrightarrow{\text{eval}} B$$

$$a : A,\ f : A \Rightarrow B \longmapsto \underline{fa} : B$$

Exercise: show that the families are also natural in the object A.

The structural rules

$\delta_A \times \Gamma = \delta_A \times id_\Gamma$

- ▷ Contraction:

$$A \times A \times \Gamma \xrightarrow{f} B$$

becomes

$$A \times \Gamma \xrightarrow{\boxed{\delta_A \times \Gamma}} A \times A \times \Gamma \xrightarrow{f} B$$

$A \xrightarrow{\delta_A} A \times A$
diagonal

- ▷ Weakening:

$$\Gamma \xrightarrow{f} B$$

becomes

$$A \times \Gamma \xrightarrow{\boxed{\varepsilon_A \times \Gamma}} 1 \times \Gamma \xrightarrow{\sim} \Gamma \xrightarrow{f} B$$

$A \longrightarrow 1$
projection

- ▷ Permutation:

$$\Gamma \times A \times B \times \Delta \xrightarrow{f} B$$

becomes

$$\Gamma \times B \times A \times \Delta \xrightarrow{\boxed{\Gamma \times \gamma_{A,B} \times \Delta}} \Gamma \times A \times B \times \Delta \xrightarrow{f} B$$

$A \times B \xrightarrow{\gamma_{AB}} B \times A$
symmetry

$F : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$ a pair of families of functors

$F(C, -) : \mathcal{D} \to \mathcal{E}$ $\quad F(-, D) : \mathcal{C} \to \mathcal{E}$ $\quad$ + ...

Soundness theorem

Theorem.

In every cartesian closed category $\mathcal{C}$, the interpretation $[\![-]\!]$ is an invariant modulo β, η.

▷ If $\Gamma \vdash (\lambda x.M) : A \Rightarrow B$ and $\Delta \vdash N : A$, then

$$[\![\Gamma, \Delta \vdash (\lambda x.M)N : B]\!] \quad = \quad [\![\Gamma, \Delta \vdash M[x := N] : B]\!]$$

▷ If $\Gamma \vdash M : A \Rightarrow B$ then

$$[\![\Gamma \vdash (\lambda x.Mx) : A \Rightarrow B]\!] \quad = \quad [\![\Gamma \vdash M : A \Rightarrow B]\!]$$

PCF

Exercise. Establish the soundness theorem.

$[\![B]\!] = \{\mathrm{true}, \mathrm{false}\}$ in Set

$$[\![B]\!] = \begin{tikzcd} \mathrm{true} & & \mathrm{false} \\ & \bot \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd} \quad \text{in Dom}$$

flat domain