Digital Signal Processing

8: DIGITAL FILTERS STRUCTURE AND DESIGN

Digital Signal Processing

Digital Filter Design

Introduction

In signal processing, the functions of a ***filter*** are:

- ➢ Removing unwanted parts of the signal, such as random noise;
- ➢ Extracting useful parts of the signal, such as the components lying within a certain frequency range.

There are two main kinds of filter:
analog **and** ***digital***

Digital Signal Processing

Digital Filter Design

Introduction

Analog Filters:

An *analog filter* uses analog electronic circuits made up from components such as resistors and capacitors to produce the required filtering effect. Such filter circuits are widely used in such applications as noise reduction, signal enhancement, and many other areas.

***Advantages*:**
- simple and consolidated methodologies of plan;
- fast and simple realization;

Disadvantages:
- little stable and sensitive to temperature variations;
- expensive to realize in large amounts.

Digital Signal Processing

Digital Filter Design

Introduction

Digital Filters:

A *digital filter* uses a digital processor to perform numerical calculations on sampled values of the signal. The processor may be a general-purpose computer such as a PC, or a specialized DSP (Digital Signal Processor) chip.

Advantages:

- A digital filter is *programmable*,

- Digital filters are easily *designed*, *tested* and *implemented* on computer or workstation.

- Digital filters are extremely *stable* with respect both to time and temperature.

- Digital filters can handle *low frequency* signals accurately.

Digital Filter Design

Introduction

- **In digital signal processing, there are *two* important types of systems:**
  - **Digital filters: perform signal filtering in the time domain.**
  - **Spectrum analyzers: provide signal representation in the frequency domain.**
- **In this topic we will study several basic design algorithms for both FIR and IIR filters.**
- **These designs are mostly of the *frequency selective* type**
  - **lowpass, highpass, bandpass and bandstop filters.**

Digital Filter Design

Introduction

- In the filter design process, we determine the coefficients of causal FIR or IIR filter that closely approximates the desired frequency response specifications.
- The issue of whether to design FIR or IIR, depends on the nature of the problem and on the specifications of the desired frequency response.

Digital Signal Processing

Introduction

- **The design of a digital filter is carried out in three steps:**
  - **Specifications: they are determined by the applications**
  - ***Approximations*: once the specification are defined, we use various concepts and mathematics that we studied so far to come up with a filter description that approximates the given set of specifications.**
  - **Implementation: The product of the above step is a filter description in the form of either a difference equation, or a system function H(z), or an impulse response h(n). From this description we implement the filter in hardware or through software on a computer.**

Introduction

Filter Specifications:

- Specifications are required in the frequency-domain in terms of the desired *magnitude* and *phase* response of the filter. Generally a linear phase response in the passband is desirable.
  - In the case of FIR filters, It is possible to have exact linear phase.
  - In the case of IIR filters, a linear phase in the passband is not achievable.

Digital Signal Processing

Introduction

Magnitude Specifications:

- ***Absolute* specifications**
  - **Provide a set of requirements on the magnitude response function $|H(e^{jw})|$.**
  - **Generally used for FIR filters.**
- ***Relative* specifications**
  - **Provide requirements in *decibels* (dB), given by**

$$dB\ scale = -20\log_{10}\frac{|H(e^{jw})|}{|H(e^{jw})|_{\max}} \geq 0$$

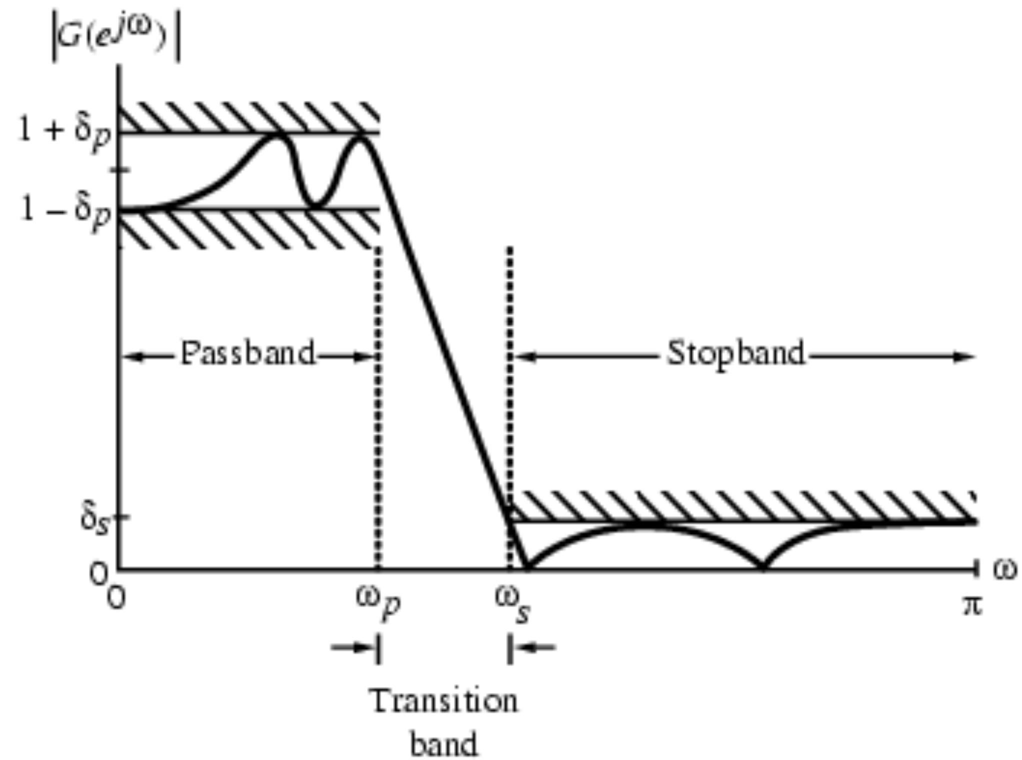
  - **Used for both FIR and IIR filters.**

Digital Signal Processing

Introduction

Absolute Specifications:

- Band [0,w_p] is called the ***passband***, and ***delta* p** is the tolerance (or ripple) that we are willing to accept in the ideal passband response.
- Band [w_s,*pi*] is called the ***stopband***, and ***delta s*** is the corresponding tolerance (or ripple)
- Band [w_p, w_s] is called the ***transition band***, and there are no restriction on the magnitude response in this band.



Magnitude response for physical realizable filter

Filter Specifications

- In the **passband** $0 \le \omega \le \omega_p$ we require that $\left|G(e^{j\omega})\right| \cong 1$ with a deviation $\pm\delta_p$

$$1-\delta_p \le \left|G(e^{j\omega})\right| \le 1+\delta_p, \quad |\omega| \le \omega_p$$

- In the **stopband** $\omega_s \le \omega \le \pi$ we require that $\left|G(e^{j\omega})\right| \cong 0$ with a deviation δ_s

$$\left|G(e^{j\omega})\right| \le \delta_s, \quad \omega_s \le |\omega| \le \pi$$

Filter Specifications

- In any Filter design problem we specify
- ω_p - **passband edge frequency**
- ω_s - **stopband edge frequency**
- δ_p - **peak ripple value** in the passband
- δ_s - **peak ripple value** in the stopband
- Based on these specifications , we can select parameters a_k and b_k in frequency response characteristics.

Filter Specifications

- Practical specifications are often given in terms of **loss function (in dB)**
- $\mathcal{G}(\omega) = -20\log_{10}\left|G(e^{j\omega})\right|$
- **Peak passband ripple**

  $\alpha_p = -20\log_{10}(1-\delta_p)$ dB
- **Minimum stopband attenuation**

  $\alpha_s = -20\log_{10}(\delta_s)$ dB

Filter Specifications

- In practice, passband edge frequency F_p and stopband edge frequency F_s are specified in Hz
- For digital filter design, normalized band edge frequencies need to be computed from specifications in Hz using

$$\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = 2\pi F_p T$$

$$\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} = 2\pi F_s T$$

Filter Specifications

- Example - Let $F_p = 7$ kHz, $F_s = 3$ kHz, and $F_T = 25$ kHz
- Then

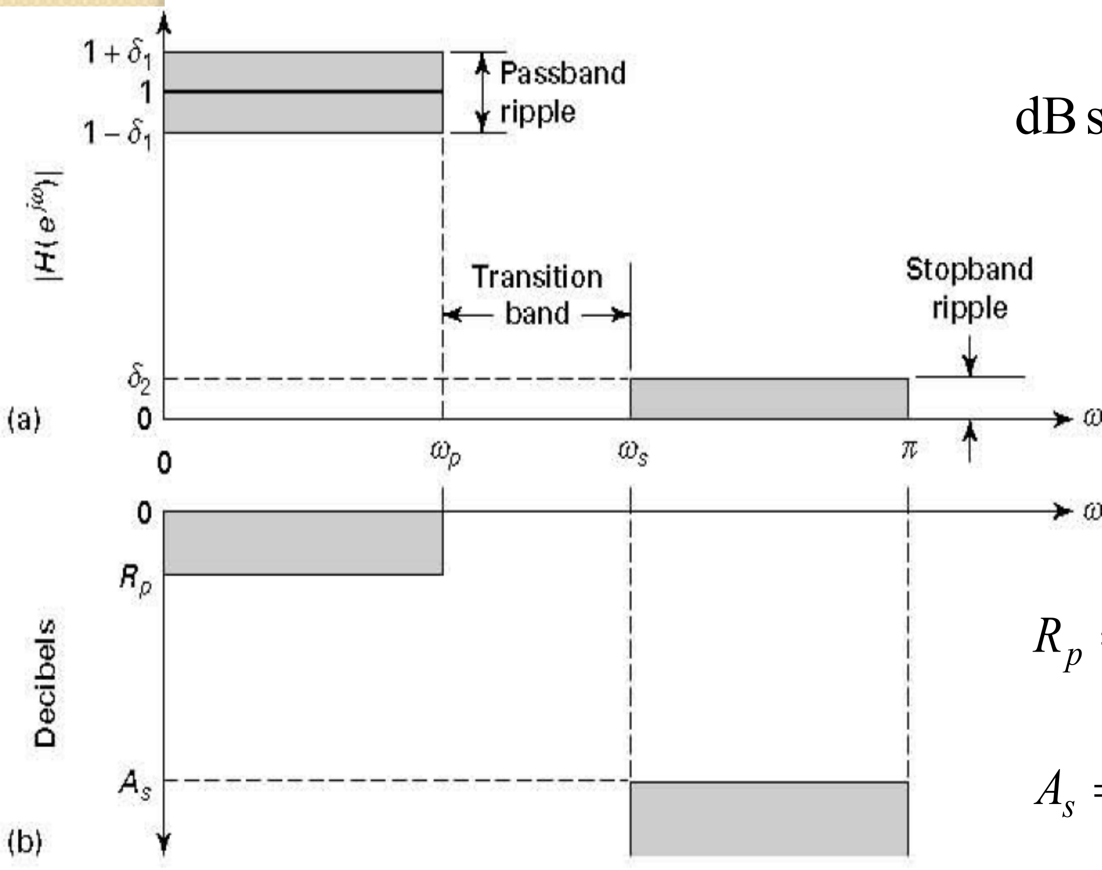
$$\omega_p = \frac{2\pi(7\times 10^3)}{25\times 10^3} = 0.56\pi$$

$$\omega_s = \frac{2\pi(3\times 10^3)}{25\times 10^3} = 0.24\pi$$

Digital Filter Design

Introduction

Relative (DB) Specifications:



$$\text{dB scale} = -20\log_{10}\frac{|H(\omega)|}{|H(\omega)|_{\max}} \geq 0$$

$$\alpha_p = -20\log_{10}(1-\delta_p)$$

$$\alpha_s = -20\log_{10}(\delta_s)$$

$$R_p = -20\log_{10}\frac{1-\delta_1}{1+\delta_1} > 0 \qquad \textit{for passband}$$

$$A_s = -20\log_{10}\frac{\delta_2}{1+\delta_1} > 0 \qquad \textit{for stopband}$$

Digital Filter Design

FIR Filter Design

A Finite Impulse Response (FIR) digital filter is one whose impulse response is of finite duration.

The general difference equation for a FIR digital filter is:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

Where:

- y(*n*) is the filter output at discrete time instance *n.*
- b_k is the *k*-th feed-forward tap, or filter coefficient.
- x(*n-k*) is the filter input delayed by *k* samples.
- *M* is the number of feed-forward taps in the FIR filter.

Note: the FIR filter output depends only on the previous *M* inputs. This feature is why the impulse response for a FIR filter is finite.

Digital Filter Design

FIR Filter Design

Advantages in using an FIR filter:

- **Can be designed with exact linear phase.**
- **Filter structure always stable with quantized coefficients.**

Disadvantages in using an FIR filter

• Order of an FIR filter is considerably higher than that of an equivalent IIR filter meeting the same specifications; this leads to higher computational complexity for FIR

Design objectives

•To obtain filter coefficients, b_k such that the magnitude of H(z) will approximate the desired magnitude frequency response such as that of ideal filters.

Design of FIR filters:

Symmetric and Anti-symmetric FIR filters:

Symmetry in filter impulse response will ensure Linear phase

- **The number of filter coefficients that specify the frequency response in (M+1)/2 when is M odd and M/2 when M is even in case of symmetric conditions.**

- **In case of impulse response anti-symmetric h(M-1/2)=0 so that there are (M-1/2) filter coefficients when M is odd and M/2 coefficients when M is even**

Digital Filter Design

FIR Filter Design

Methods of designing FIR filters:

- Fourier series based method
- Window based method
- Frequency sampling method

Digital Filter Design

FIR Filter Design

1. Design of Linear Phase FIR filter based on Fourier series method:

Motivation: Since the desired freq response $H_d(e^{j\omega})$ is a periodic function in ω with period 2π, it can be expressed as Fourier series expansion

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

where $h_d(n)$ *are fourier series coefficients*

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \quad \text{for } -\infty \prec n \prec \infty$$

This expansion results in impulse response coefficients which are infinite in duration and non causal.

- It can be made finite duration by truncating the infinite length
- The linear phase can be obtained by introducing symmetric property in the filter impulse response, i.e., h(n) = h(-n)
- It can be made causal by introducing sufficient delay (depends on filter length)

FIR Filter Design

Stepwise procedure:

- **From the desired frequency response using inverse FT relation obtain $h_d(n)$**
- **Truncate the infinite length of the impulse response to finite length with (assuming M odd)**

$$
\begin{aligned}
h(n) &= h_d(n) \quad for\ -(M-1)/2 \le n \le (M-1)/2 \\
&= 0 \quad otherwise
\end{aligned}
$$

- **Introduce h(n) = h(-n) for linear phase (Symmetrical) characteristics**
- **Write the expression for H(z)**

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$.........+h(-2)z^{2} + h(-1)z^{1} + h(0) + h(1)z^{-1} + h(2)z^{-2}$$

Cont…..

- **To obtain causal realization delay the truncated impulse response h(n) by (M-1)/2.**

$$= z^{-(M-1)/2} H(z)$$

  - The causal FIR filter will be

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}$$

Fourier Series Coefficients For the Ideal Filters

- The desired impulse response approximation of ideal low passfilter can be solved as

$$For\ \text{n=0 h(n)=}\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw})e^{jw(0)}dw$$

$$=\frac{1}{2\pi}\int_{-w_c}^{w_c} 1dw=\frac{w_c}{\pi}$$

FIR Filter Design

Fourier Series Coefficients For the Ideal Filters:

LPF

$$c_{\mathrm{LP}}(n) = \begin{cases} \dfrac{\omega_c}{\pi}; & n = 0 \\ \dfrac{\sin(\omega_c n)}{\pi n}; & |n| > 0 \end{cases}$$

BPF

$$c_{\mathrm{BP}}(n) = \begin{cases} \dfrac{\omega_{c2} - \omega_{c1}}{\pi}; & n = 0 \\ \dfrac{1}{\pi n}[\sin(\omega_{c2} n) - \sin(\omega_{c1} n)]; & |n| > 0 \end{cases}$$

HPF

$$c_{\mathrm{HP}}(n) = \begin{cases} 1 - \dfrac{\omega_c}{\pi}; & n = 0 \\ -\dfrac{\sin(\omega_c n)}{\pi n}; & |n| > 0 \end{cases}$$

BSF

$$c_{\mathrm{BS}}(n) = \begin{cases} 1 - \dfrac{(\omega_{c2} - \omega_{c1})}{\pi}; & n = 0 \\ \dfrac{1}{\pi n}[\sin(\omega_{c1} n) - \sin(\omega_{c2} n)]; & |n| > 0 \end{cases}$$

Digital Filter Design

FIR Filter Design

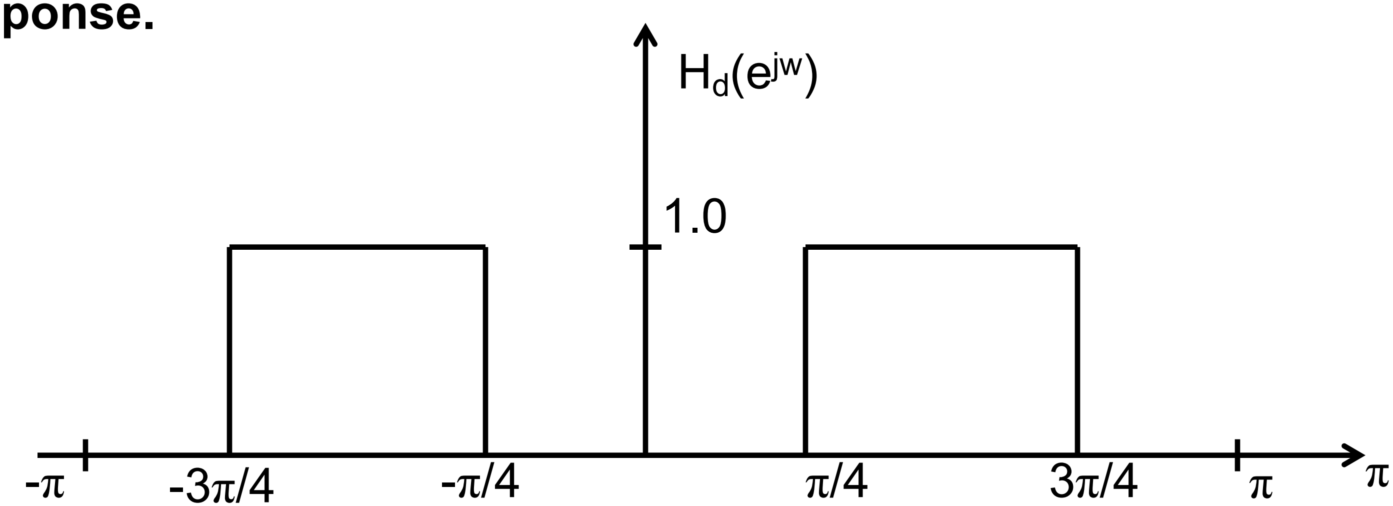
Example:

Design an ideal bandpass filter with a frequency response:

$$H_d(e^{j\omega}) = 1 \qquad for\ \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4}$$

$$= 0 \qquad otherwise$$

Find the values of h(n) for M = 11 and plot the magnitude frequency response.



Digital Filter Design

FIR Filter Design

Example: cont.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-3\pi/4}^{-\pi/4} e^{j\omega n}d\omega + \int_{\pi/4}^{3\pi/4} e^{j\omega n}d\omega\right]$$

$$= \frac{1}{\pi n}\left[\sin\frac{3\pi}{4}n - \sin\frac{\pi}{4}n\right] \quad -\infty \le n \le \infty$$

$$truncating\ to\ 11\ samples\ we\ have\ h(n) = h_d(n)\ \ for\ |n| \le 5$$

$$= 0 \quad otherwise$$

Or by using the ready equation

$$c_{\mathrm{BP}}(n) = \begin{cases} \dfrac{\omega_{c2} - \omega_{c1}}{\pi}; & n = 0 \\ \dfrac{1}{\pi n}[\sin(\omega_{c2}n) - \sin(\omega_{c1}n)]; & |n| > 0 \end{cases}$$

Digital Filter Design

FIR Filter Design

Example: cont.

For n = 0 the value of h(n) is separately evaluated from the basic integration

$h(0) = 0.5$

Other values of h(n) are evaluated from h(n) expression

$h(1) = h(-1) = 0$

$h(2) = h(-2) = -0.3183$

$h(3) = h(-3) = 0$

$h(4) = h(-4) = 0$

$h(5) = h(-5) = 0$

Digital Filter Design

FIR Filter Design

Example: cont.

The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{(N-1)/2} [h(n)\{z^{n} + z^{-n}\}]$$
$$= 0.5 - 0.3183(z^{2} + z^{-2})$$

the transfer function of the realizable filter is

$$H'(z) = z^{-5}[0.5 - 0.3183(z^{2} + z^{-2})]$$
$$= -0.3183z^{-3} + 0.5z^{-5} - 0.3183z^{-7}$$

the filter coeff are

$$h(0) = h(10) = h(1) = h(9) = h(2) = h(8) = h(4) = h(6) = 0$$
$$h(3) = h(7) = -0.3183$$
$$h(5) = 0.5$$

Digital Filter Design

FIR Filter Design

Example: cont.

The magnitude response is given

$$H(\omega) = H_r(\omega)e^{-j\alpha\omega}$$

$$H_r(\omega) = h(\frac{M-1}{2}) + \sum_{n=0}^{\frac{M-3}{2}} 2h(n)\cos[\omega(\frac{M-1}{2} - n)]$$

The magnitude response function is

$|H(e^{j\omega})| = 0.5 - 0.6366 \cos 2\omega$

which can plotted for various values of ω

ω in degrees =

[0 20 30 45 60 75 90 105 120 135 150 160 180];

$|H(e^{j\omega})|$ in dBs =

[-17.3 -38.17 -14.8 -6.02 -1.74 0.4346 1.11 0.4346 -1.74 -6.02 -14.8 -38.17 -17.3];

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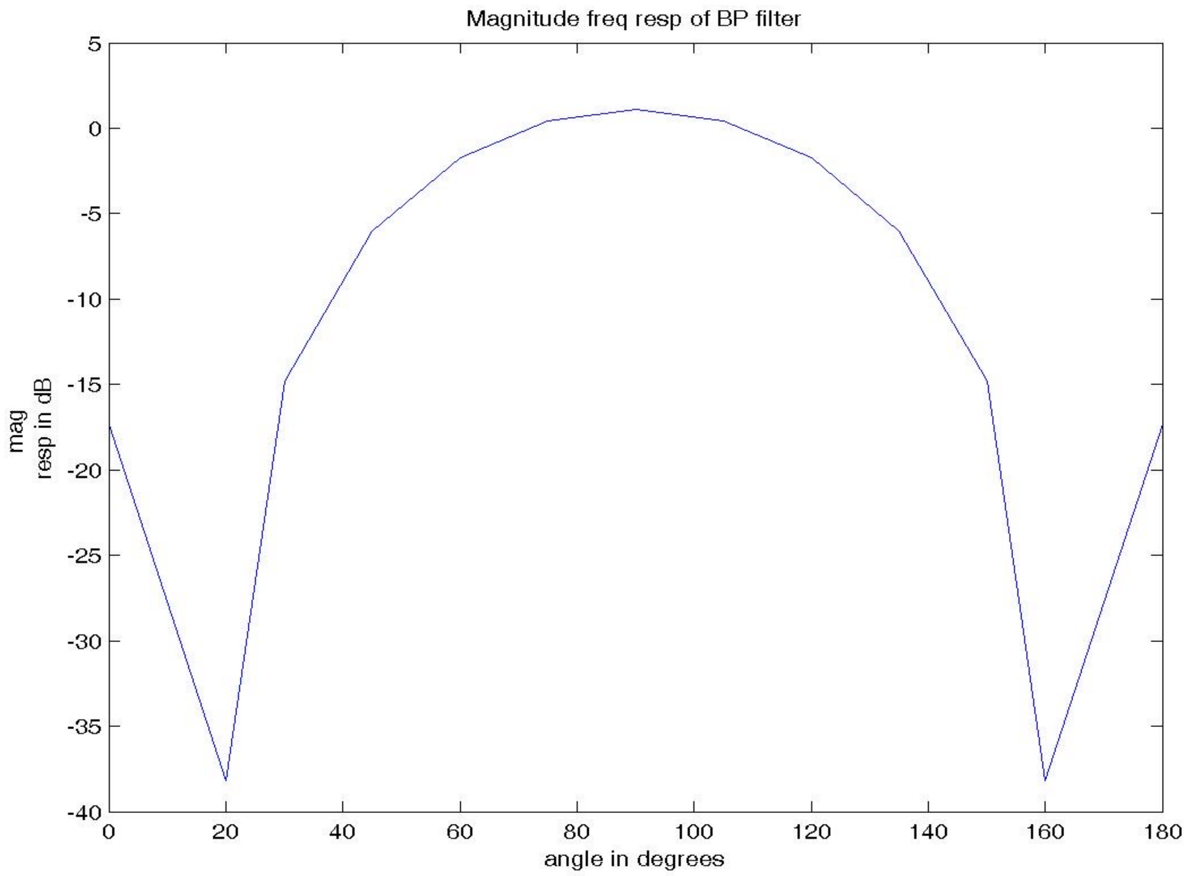
Digital Filter Design

FIR Filter Design

Example: cont.



WINDOW METHOD

- Window method
  - This was developed to remedy the undesirable oscillations in the passband and stopband originate from abrupt truncation of the infinite-length coefficient sequence.
  - The window function is symmetrical and can gradually weight the designed FIR coefficients down to zeroes at both ends,

Cont…

- Applying the window sequence to the filter coefficients will give

$$h_w(n) = h(n) * w(n)$$

- Where w(n) designates the window function.

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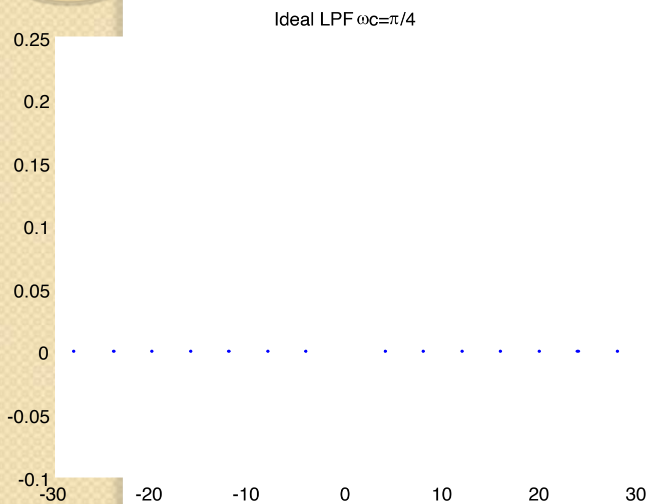
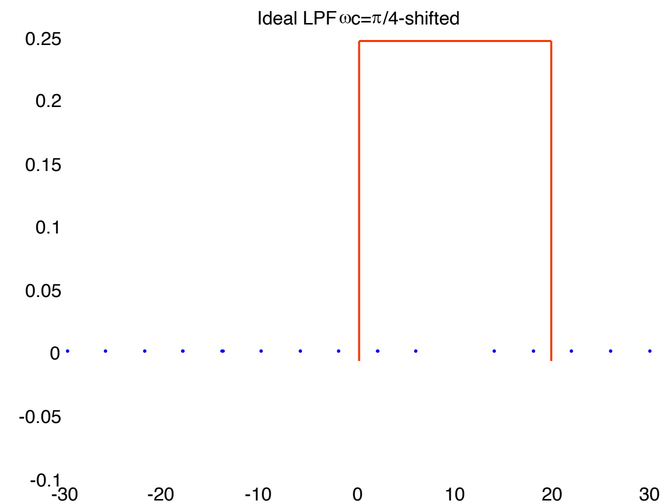
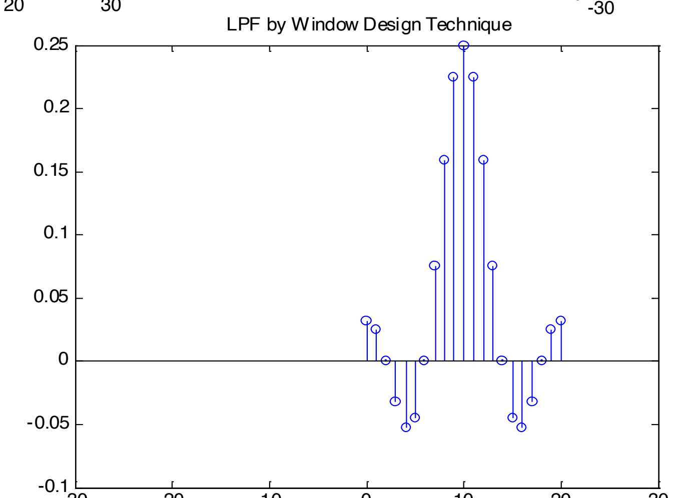
Window based method:



Shifting





Digital Filter Design

FIR Filter Design

General Design Procedures:

$H_{IDEAL}(\omega)$ **Ideal frequency response (given)**

Step 1

$$H_d(\omega) = H_{IDEAL}(\omega)e^{-j\alpha\omega}$$

$$\alpha = \frac{M-1}{2}$$

Step 2

$$h_d(n) = IDFT[H_d(\omega)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega$$

Step 3

$$h(n) = h_d(n)w(n), \quad 0 \le n \le M\text{-}1$$

Window function

- **symmetric about α over** $0 \le n \le M\text{-}1$
- **0 otherwise**

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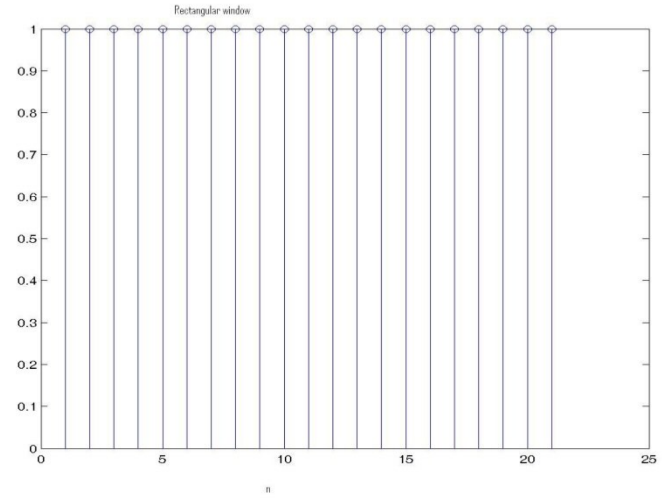
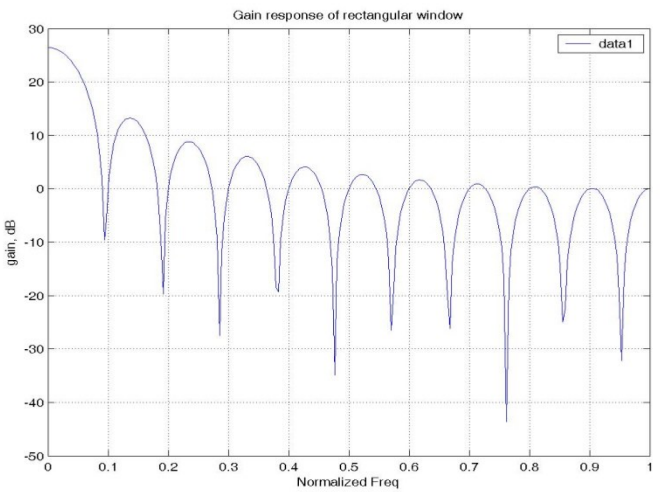
Commonly used Windows

- **Rectangular Windows:**

$$w_r(n) = 1 \ for \ 0 \le n \le M-1$$



- **MATLAB function:** w=boxcar (M)



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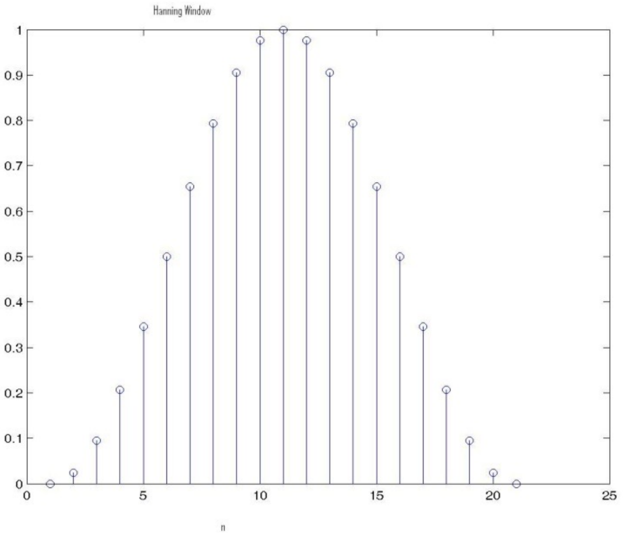
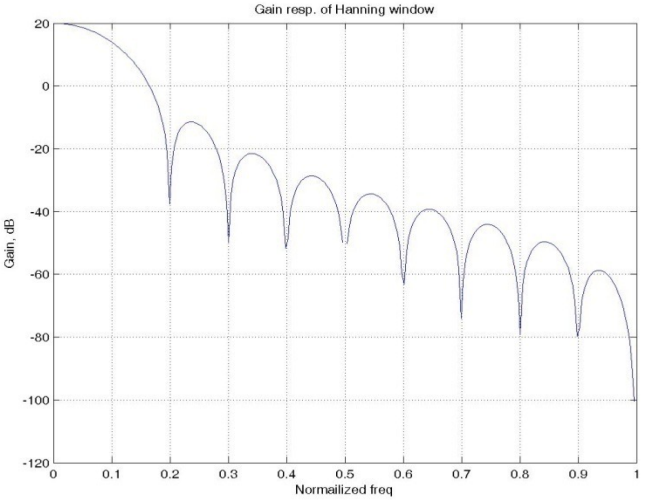
➢ **Hanning Windows:**

$$w_{han}(n) = 0.5\left(1 - \cos\frac{2\pi n}{M-1}\right) \quad for\ 0 \le n \le M-1$$



- **MATLAB function: w=hann (M)**



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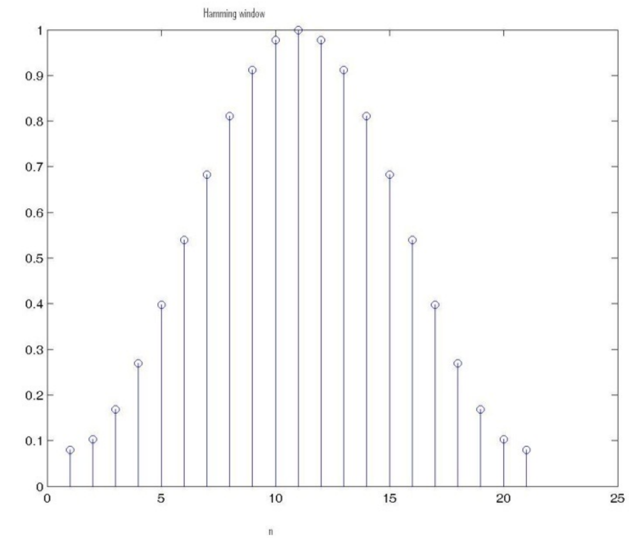
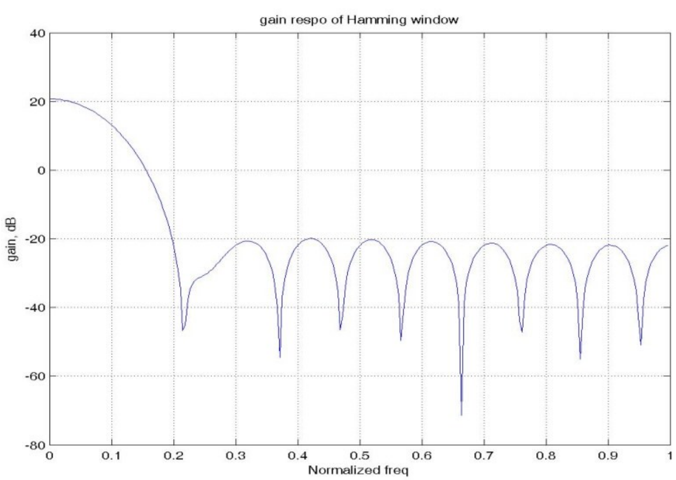
➢ **Hamming Windows:**

$$w_{ham}(n) = 0.54 - 0.46\cos\frac{2\pi n}{M-1} \quad for\ 0 \le n \le M-1$$



- **MATLAB function: w=hamming (M)**



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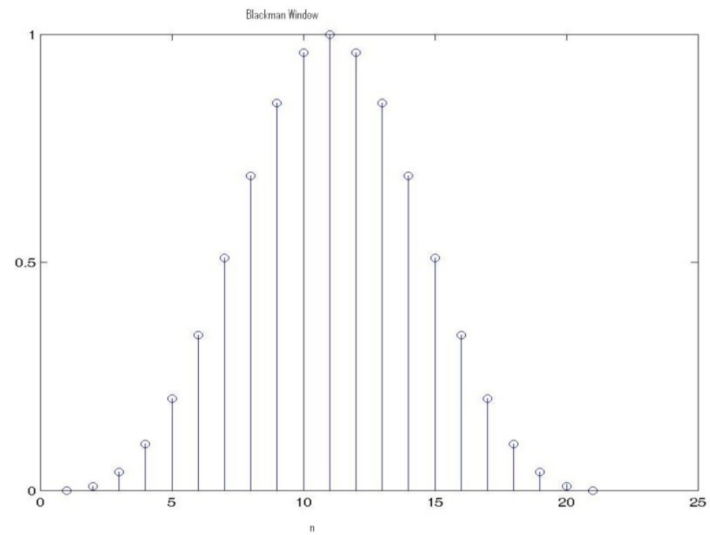
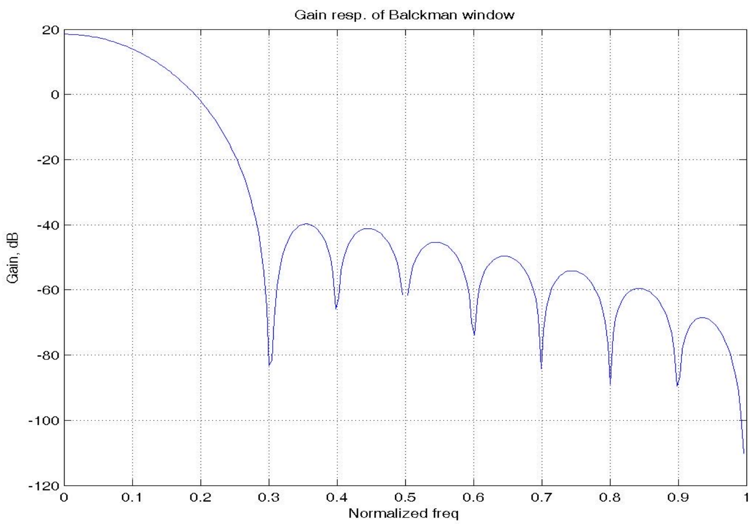
➤ **Blackman Windows:**

$$w_{blk}(n) = 0.42 - 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1} \quad for\ 0 \le n \le M-1$$



- **MATLAB function:** **w=blackman (M)**

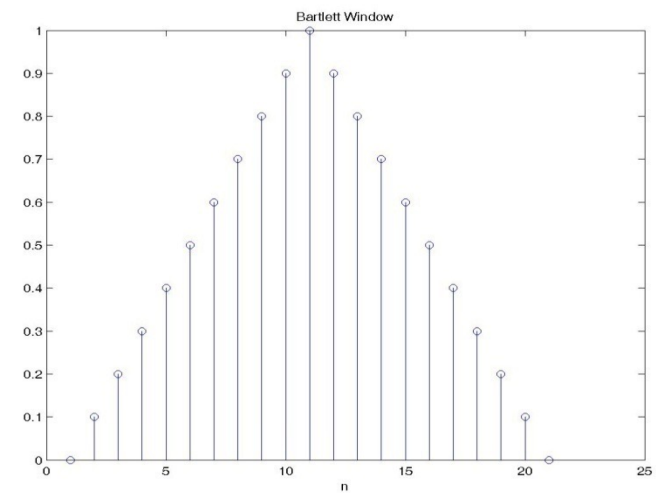


Digital Filter Design

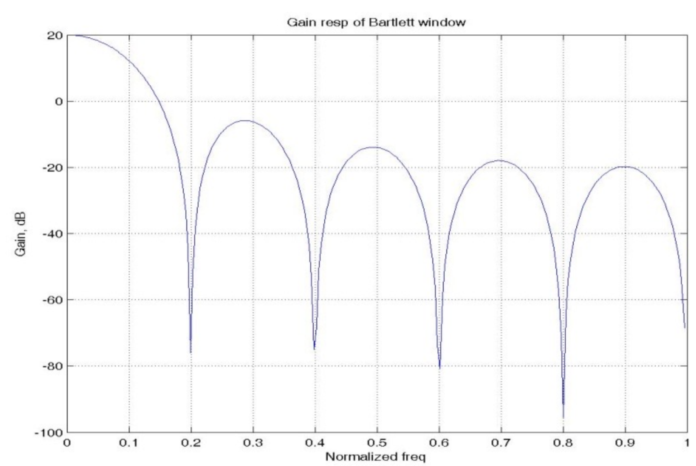
FIR Filter Design

➢ Bartlett Windows:

$$
w(n)=\begin{cases} \dfrac{2n}{M-1}, & 0\le n\le \dfrac{M-1}{2} \\ 2-\dfrac{2n}{M-1}, & \dfrac{M-1}{2}\le n\le M-1 \\ 0, & \text{otherwise} \end{cases}
$$

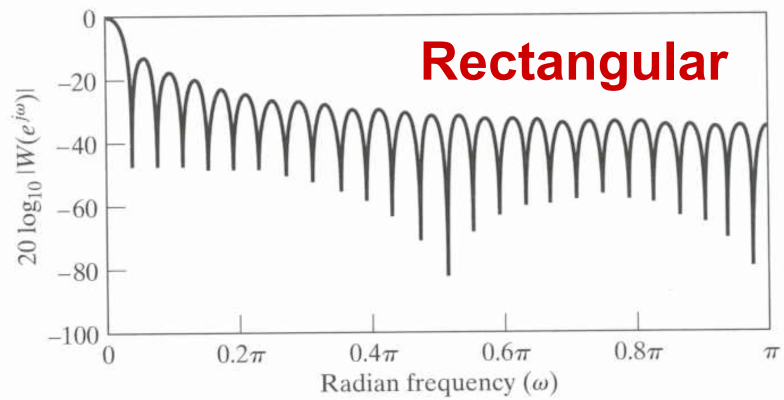


•**MATLAB function:** w=bartlett (M)

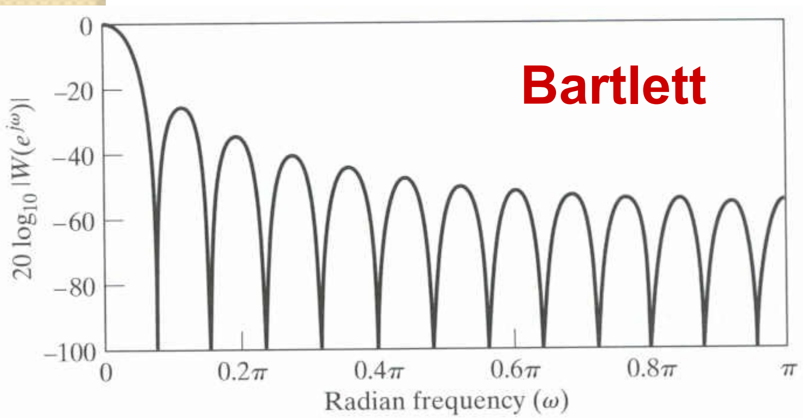


Frequency Spectrum of Windows

For M = 50



(a)



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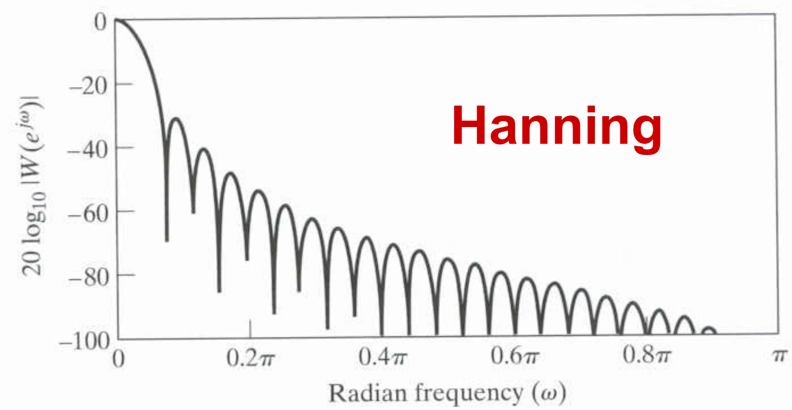

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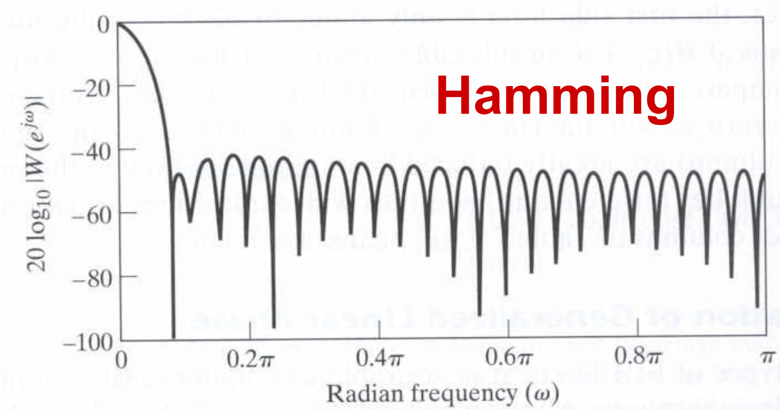

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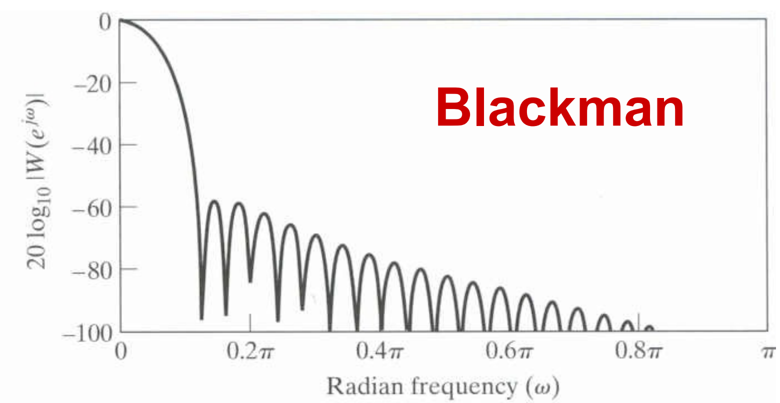

(e)

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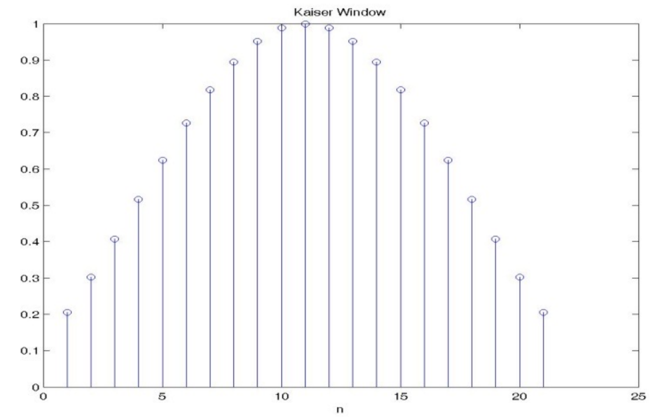
➢ **Kaiser Windows:**

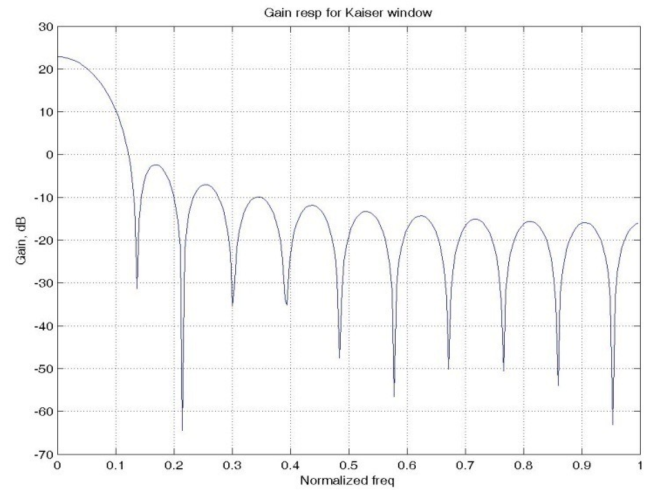
$$w_k[n] = \begin{cases} \dfrac{I_0[\beta(1-(\frac{n-\alpha}{\alpha})^2)^{1/2}]}{I_0(\beta)} & , \quad 0 \le n \le M,\ \alpha = \dfrac{M}{2} \\ 0 & , \quad \text{otherwise} \end{cases}$$



•**MATLAB function:** **w = kaiser(M,beta)**

$$I_0(x) = 1 + \sum_{k=1}^{\infty} \left[\frac{(x/2)^k}{k!} \right]^2$$

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7) & \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) & 21 \le \alpha_s \le 50 \\ 0.0 & \alpha_s < 21 \end{cases}$$



Digital Filter Design

FIR Filter Design

Procedure for designing linear-phase FIR filters using windows

1. From the desired freq response using inverse FT relation obtain $h_d(n)$
2. Truncate the infinite length of the impulse response to finite length with (assuming M odd) choosing proper window

$$h(n) = h_d(n)w(n) \quad where$$

$$w(n)\ is\ the\ window\ function\ defined\ for -(M-1)/2 \le n \le (M-1)/2$$

3. Introduce h(n) = h(-n) for linear phase characteristics
4. Write the expression for H(z); this is non-causal realization
5. To obtain causal realization $H'(z) = z^{-(M-1)/2} H(z)$

FIR Filter Design

Example:

Design a filter with a frequency response:

$$H_d(e^{j\omega}) = e^{-j3\omega} \qquad for -\frac{\pi}{4} \le \omega \le \frac{\pi}{4}$$

$$= 0 \qquad \frac{\pi}{4} < |\omega| \le \pi$$

using a Hanning window with M = 7

The frequency response is having a term $e^{-j\omega(M-1)/2}$ which gives h(n) symmetrical about n = M-1/2 = 3 i.e we get a causal sequence.

$$w_{han}(n) = 0.5\left(1 - \cos\frac{2\pi n}{M-1}\right) \; for \; 0 \le n \le M-1$$

Digital Filter Design

FIR Filter Design

Example: cont.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j3\omega} e^{j\omega n} d\omega$$

$$= \frac{\sin \frac{\pi}{4}(n-3)}{\pi(n-3)}$$

$$\textit{this gives } h_d(0) = h_d(6) = 0.075$$

$$h_d(1) = h_d(5) = 0.159$$

$$h_d(2) = h_d(4) = 0.22$$

$$h_d(3) = 0.25$$

The Hanning window function values are given by

$w_{hn}(0) = w_{hn}(6) = 0$

$w_{hn}(1) = w_{hn}(5) = 0.25$

$w_{hn}(2) = w_{hn}(4) = 0.75$

$w_{hn}(3) = 1$

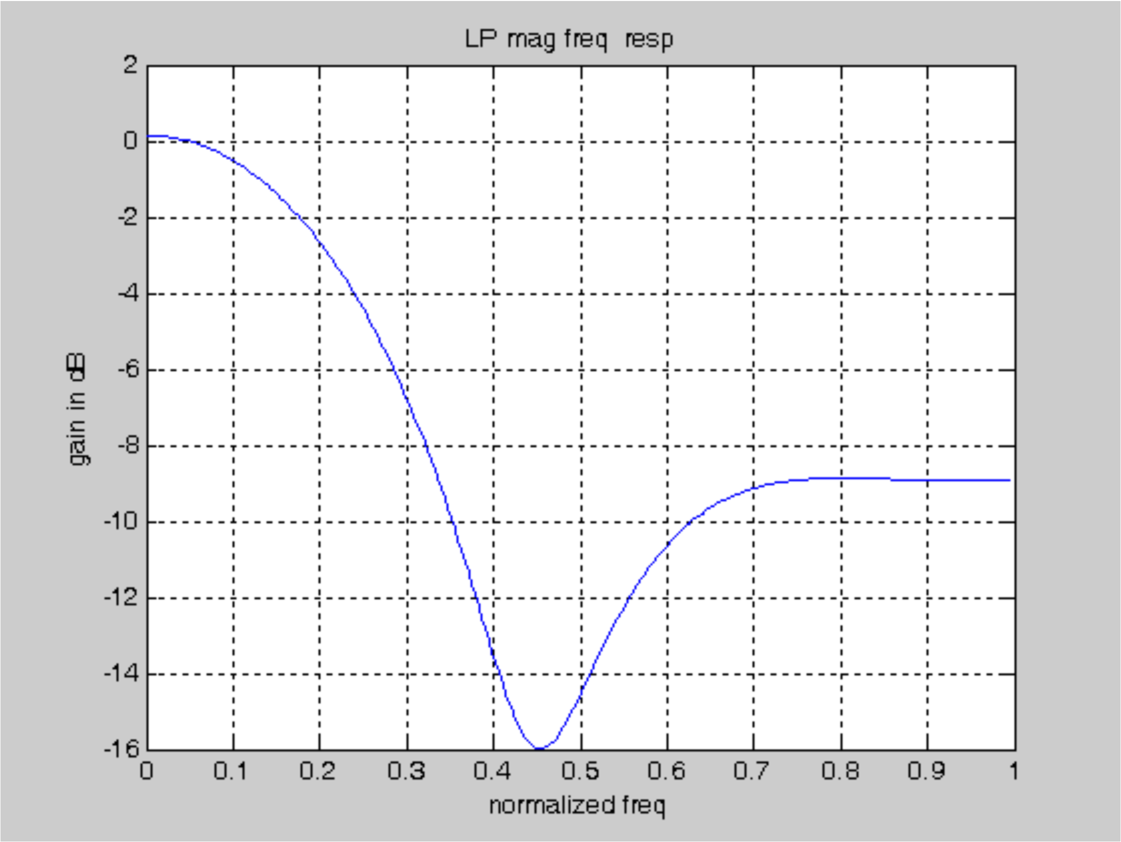
$h(n) = h_d(n)\, w_{hn}(n)$

h(n)=[0 0.03975 0.165 0.25 0.165 0.3975 0]

Digital Filter Design

FIR Filter Design

Example: cont.



Digital Signal Processing

Digital Filter Design

IIR Filter Design

The IIR filter is responsible for the infinite duration of the impulse response. The IIR filter is **recursive** system.

The general **difference equation** for an IIR digital:

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=1}^{M} b_k x[n-k]$$

- a_k is the k-th feedback tap depending on previous outputs. If $a_k = 0$ then the filter is a **FIR**.
- **N** is the number of feedback taps in the IIR filter.
- **M** is the number of feed-forward taps.

The transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} b_i z^{-i}}{1 + \sum_{i=1}^{M} a_i z^{-i}} = \frac{\sum_{i=0}^{N} b_i z^{-i}}{\sum_{i=0}^{M} a_i z^{-i}}$$

(with $a_0 = 1$)

yields *N-1* roots which are zeroes

yields *M-1* roots which are poles

Digital Signal Processing

Digital Filter Design

IIR Filter Design

We want to design digital filters that meet desired specifications in order to get as close to the ideal filter as possible, e.g., for a lowpass filter we need to specify the following:

There are several ways of designing $H(e^{j\omega})$ to approximate $H_{ideal}(e^{j\omega})$. We will be looking at obtaining $H(e^{j\omega})$ from an analog filter $H(j\Omega)$ using the techniques of:

1. **Impulse Equivalence Invariance**
2. **Bilinear Transform**

An analogue design is a mature and well developed field , so it is not surprising that we begin the design of a digital filter in the analog domain and then convert the design into the digital domain

Digital Filter Design

IIR Filter Design

Standard approach:

(1) Convert the digital filter specifications into an analogue prototype lowpass filter specifications.

(2) Determine the analogue lowpass filter transfer function $H_a(s)$

(3) Transform $H_a(s)$ by replacing the complex variable to the digital transfer function $G(z)$

Digital Signal Processing

Digital Filter Design

IIR Filter Design

Let an analogue transfer function be

$$H_a(s) = \frac{P_a(s)}{D_a(s)}$$

where the subscript "*a*" indicates the analogue domain

A digital transfer function derived from this is denoted as

$$G(z) = \frac{P(z)}{D(z)}$$

IIR Filter Design

- **Basic idea behind the conversion of $G(z)$ into $H_a(s)$ is to apply a mapping from the *s*-domain to the *z*-domain so that essential properties of the analogue frequency response are preserved.**
- **Thus mapping function should be such that**
  - **Imaginary $(j\Omega)$ axis in the *s*-plane be mapped onto the unit circle of the *z*-plane**
  - **Left-half plane (LHP) of the s-plane should map into the inside pf the unit circle in the z-plane. Leading to a stable analogue transfer function be mapped into a stable digital transfer function.**

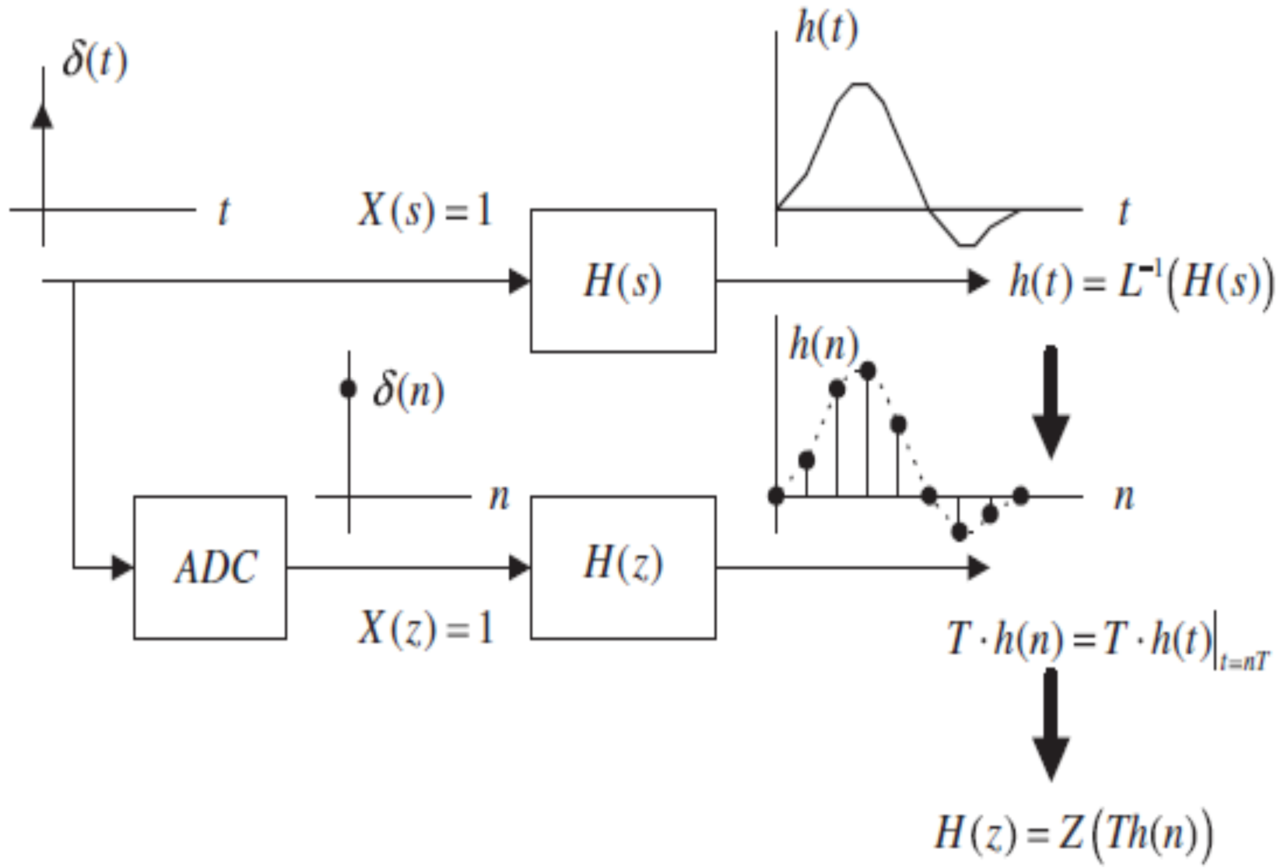
IIR Filter Design

- Transform difference equations into algebraic equations that are easier to solve
- Are complex-valued functions of a complex frequency variable
  - Laplace: $s = \sigma + j\,2\,\pi\,f$
  - *Z*: $z = r\,e^{j\,\omega}$

Impulse Invariance Method

- Given the transfer function of designed analog filter, an analog impulse response can be easily found by the inverse Laplace transform of the transfer function.
- To replace the analog filter by the equivalent digital filter, we apply an approximation in the time domain in which the digital filter impulse response must be equivalent to the analog filter impulse response.
- Therefore, we can sample the analog impulse response to get the digital impulse response, and take the z-transform of the sampled analog impulse response to obtain the transfer function of the digital filter.
- The Impulse Invariance method is appropriate for designing highpass filters due to spectrum aliasing that results from sampling

Cont….

- 

$\delta(t)$
t
$X(s)=1$
$H(s)$
$h(t)$
t
$h(t)=L^{-1}(H(s))$
$h(n)$
$\delta(n)$
n
n
ADC
$H(z)$
$X(z)=1$
$T \cdot h(n) = T \cdot h(t)\big|_{t=nT}$
$H(z)=Z(Th(n))$

Digital Signal Processing

Digital Filter Design

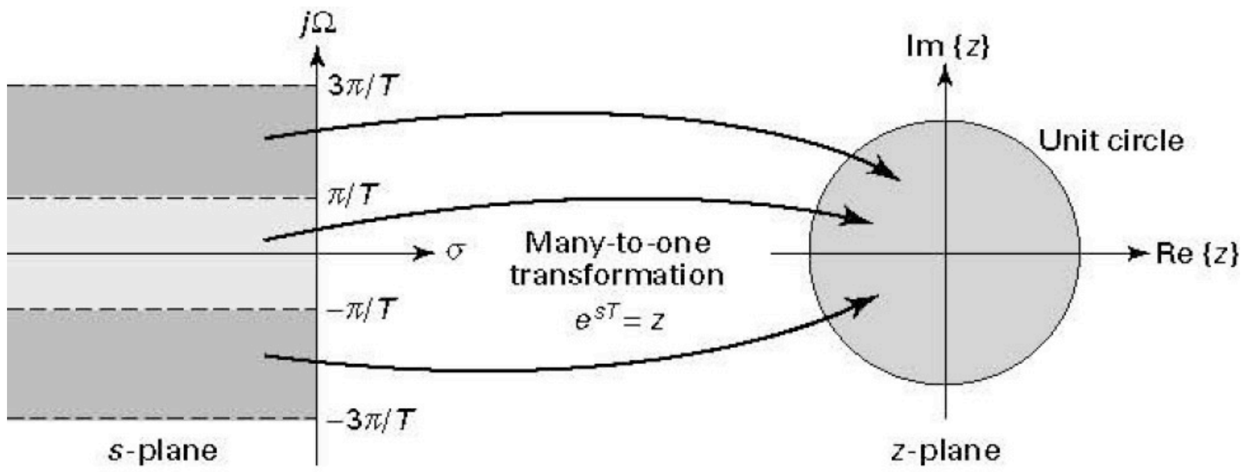
IIR Filter Design

Filter Design by Impulse Invariance: (MATLAB function: **impinvar**)

Impulse Invariance transformation:

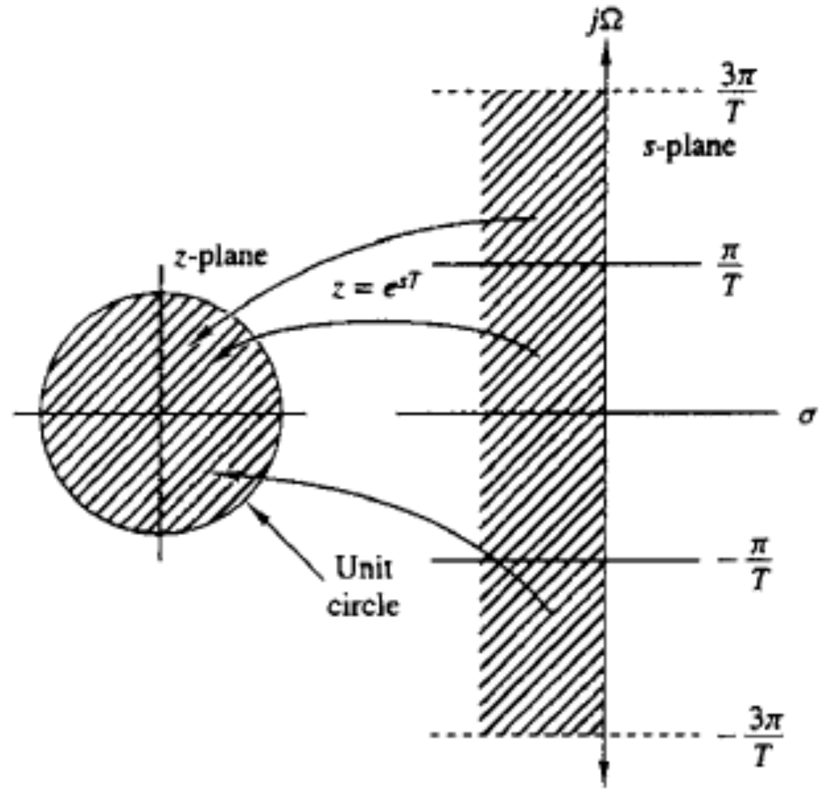
- **Preserve the shape of impulse response**

$$z = e^{sT}$$



Cont….

- The mapping from analogue frequency to Ω to frequency variable ω in digital domain is many to one which reflects the effects of aliasing due to sampling.



Digital Signal Processing

Digital Filter Design

IIR Filter Design

Filter Design by Impulse Invariance: (MATLAB function: **impinvar**)

1. **Choose T_d and determine the analog frequencies**

2. **Design an analog filter** $H_c(s)$ **using specifications** $\Omega_p, \Omega_s, R_p, \text{and } A_s$
   - **Corresponding impulse response**

3. **Partial fraction expansion** $H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$

   - **Corresponding impulse response**

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

4. **Transform analog poles** $\{s_k\}$ **into digital poles** $e^{s_k T_d}$

to obtain transfer function $H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$

Laplace transform pair

$$e^{-\alpha t} u(t) \longleftrightarrow \frac{1}{s + \alpha}$$

Example

- Consider the following transfer function:

$$H(s) = \frac{2}{s+2}$$

Determine H(z) using the impulse-invariant method if the sampling rate fs=10Hz.

Digital Signal Processing

Digital Filter Design

IIR Filter Design

Example:

Find the discrete equivalent system of the following transfer function:

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \Rightarrow H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{T_d s_k} z^{-1}}$$

$$\left.\begin{aligned} s_{p1} &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} j \\ s_{p2} &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} j \\ A_1 &= -\frac{\sqrt{2}}{2} j, A_2 = \frac{\sqrt{2}}{2} j \end{aligned}\right\} \Rightarrow H(z) = \frac{0.3078 z^{-1}}{1 - 1.0308 z^{-1} + 0.3530 z^{-2}}$$

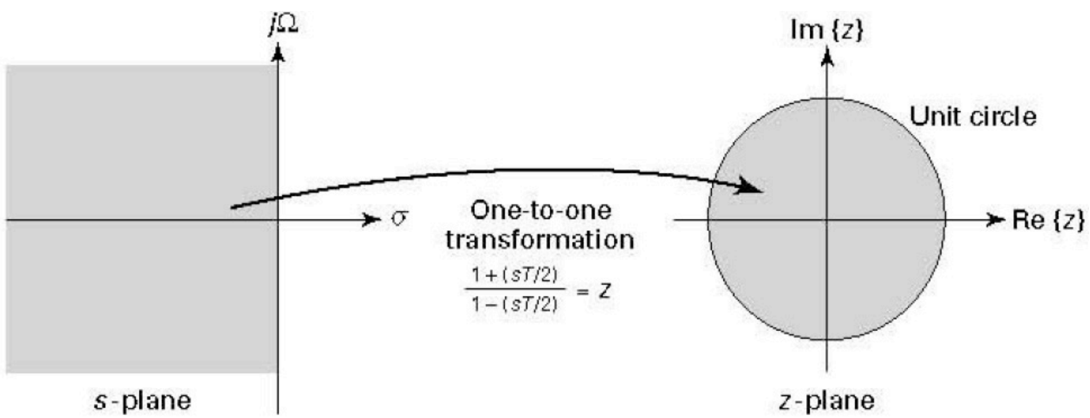
IIR Filter Design

Filter Design by the bilinear transformation:

(MATLAB function: bilinear)

- Bilinear transformation, overcomes the limitation of Impulse invariance method.
- The mapping between s plane and z-plane is conducted only once.
- Preserve the system function representation

$$z = \frac{1 + (sT)/2}{1 - (ST)/2} \qquad s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$



Digital Signal Processing

Digital Filter Design

IIR Filter Design

Filter Design by the bilinear transformation: (MATLAB function: bilinear)

1. Choose T and determine the analog frequencies

$$\Omega_p = \frac{2}{T}\tan(\frac{\omega_p}{2}) \qquad \Omega_s = \frac{2}{T}\tan(\frac{\omega_s}{2})$$

2. Design an analog filter $H_c(s)$ using specifications $\Omega_p, \Omega_s, R_p, \text{and } A_s$

3. Bilinear transformation

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$

- T_d cancels out so we can ignore it
- We can solve the transformation for z as

$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s} = \frac{1+\sigma T_d/2 + j\Omega T_d/2}{1-\sigma T_d/2 - j\Omega T_d/2} \qquad s = \sigma + j\Omega$$

IIR Filter Design

- Maps the left-half s-plane into the inside of the unit-circle in z
  - Stable in one domain would stay in the other

- On the unit circle the transform becomes

$$z = \frac{1 + j\Omega T_d / 2}{1 - j\Omega T_d / 2} = e^{j\omega}$$

- To derive the relation between ω and Ω

$$s = \frac{2}{T_d}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right) = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2} j\sin(\omega/2)}{2e^{-j\omega/2}\cos(\omega/2)}\right] = \frac{2j}{T_d}\tan\left(\frac{\omega}{2}\right)$$

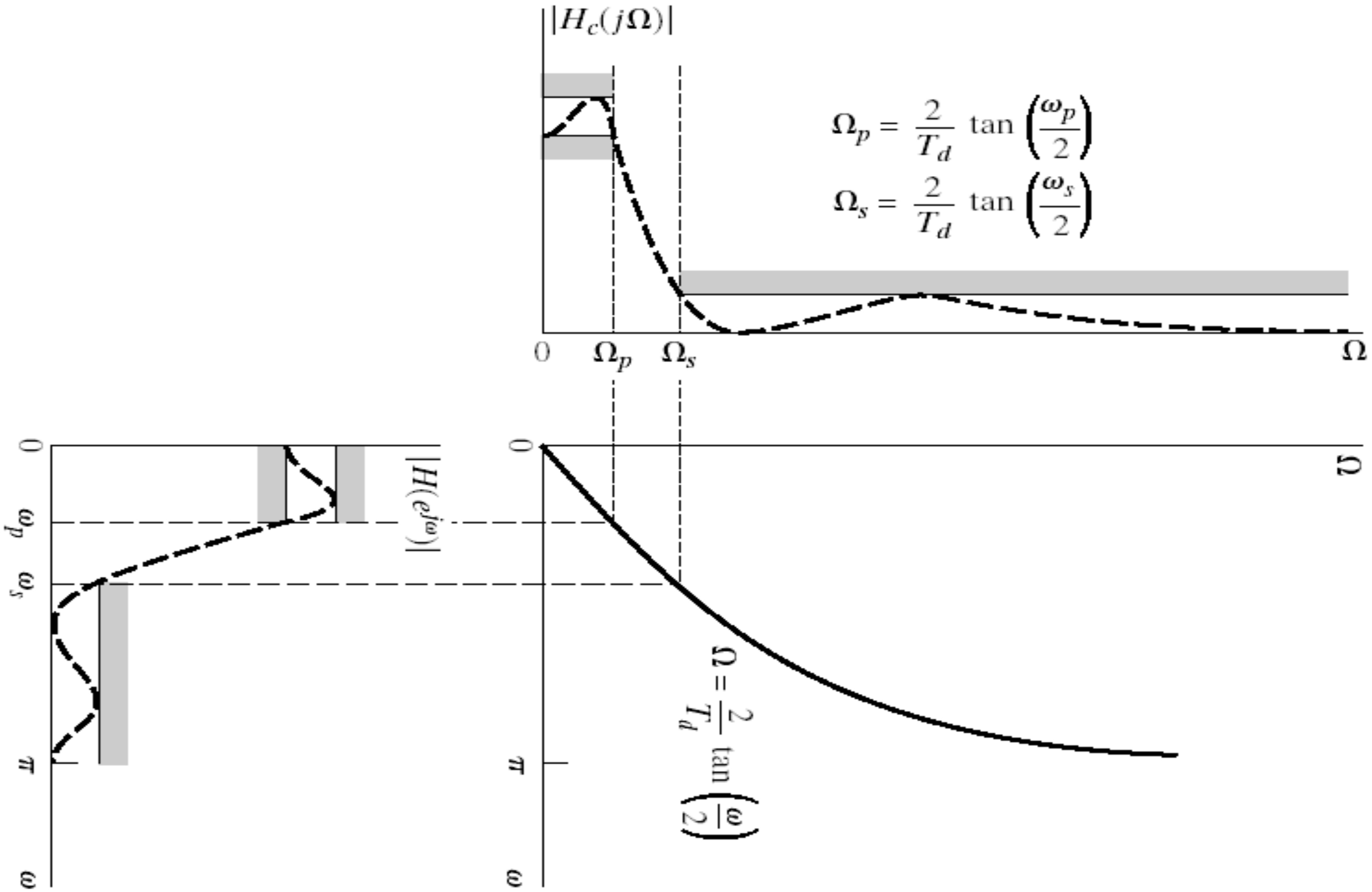
Which yields

$$\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right) \quad \text{or} \quad \omega = 2\arctan\left(\frac{\Omega T_d}{2}\right)$$

Digital Signal Processing

Digital Filter Design

IIR Filter Design



IIR Filter Design

Example:

Design a single- pole LP IIR digital filter with a 3-dB bandwidth of 0.2π, using the bilinear transformation applied to the analog filter

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

Where Ω_c is the 3-dB bandwidth of the analog filter.

Solution:

The digital filter is specified to have its -3-dB gain at $\omega_c = 0.2\pi$. In the frequency domain of the analog filter $\omega_c = 0.2\pi$ corresponds to

$$\Omega_c = \frac{2}{T}\tan 0.1\pi = \frac{0.65}{T}$$

IIR Filter Design

Example: **cont.**

$$H(s) = \frac{0.65/T}{s + 0.65/T}$$

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right]$$

$$H(z) = \frac{0.245(1 + z^{-1})}{1 - 0.509z^{-1}}$$

Where the parameter T has been divided out.

Frequency response

$$H(\omega) = \frac{0.245(1 + e^{-j\omega})}{1 - 0.509e^{-j\omega}}$$

At $\omega = 0$, $H(0)=1$, and at $\omega = 0.2\pi$, we have $|H(0.2\pi)| = 0.707$, which is the desired response.

Digital Filter Design

IIR Filter Design

Commonly used analogue

- As we have seen, IIR digital filters can easily be designed from analog filter and then mapping to transform the s-pale to z-plane
- The desired characteristics of analog filters are preserved as much as possible.

Digital Filter Design

IIR Filter Design

Commonly used analogue

Butterworth ---- maximally flat amplitude.

Chebyshev type I ---- equiriple in the passband.

Chebyshev type II ---- equiriple in the stopband.

Elliptic ---- equiripple in both the passband and stopband.

Digital Filter Design

IIR Filter Design

Butterworth Lowpass Filters:

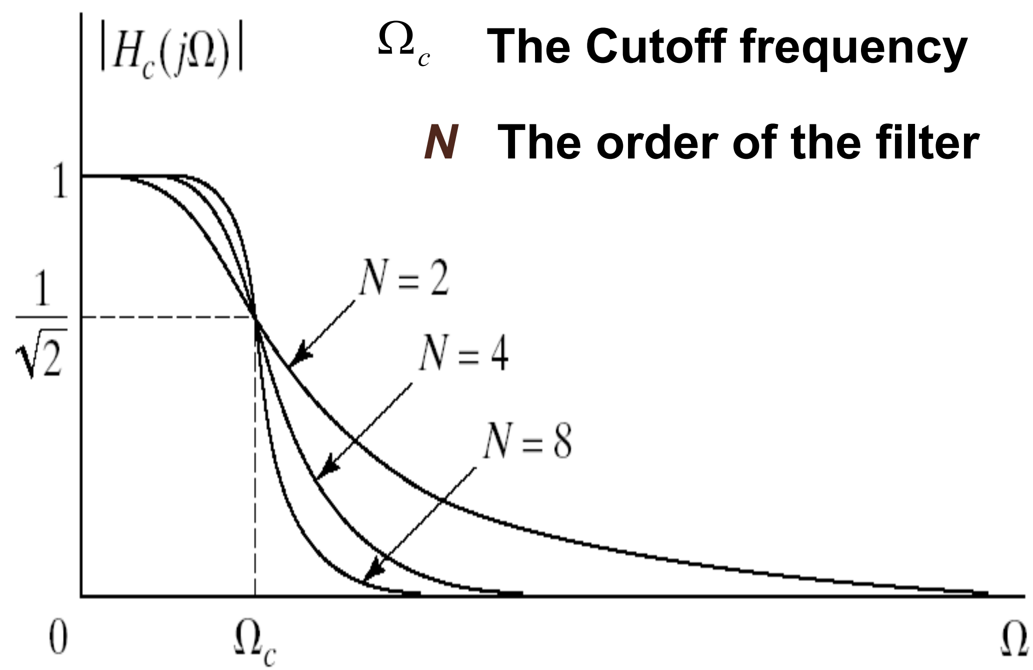
- Passband is designed to be maximally flat.
- The magnitude-squared function is of the form

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (j\Omega / j\Omega_c)^{2N}}$$

$$|H_c(s)|^2 = \frac{1}{1 + (s / j\Omega_c)^{2N}}$$

$$H_a(s) = \frac{\Omega_c^{\;N}}{\prod_{\text{LHP poles}} (s - s_k)}$$

$$s_k = \Omega_c e^{j\frac{\pi(2k+N+1)}{2N}}, \quad k = 0,1,\cdots 2N-1$$



Ω_c **The Cutoff frequency**

N **The order of the filter**

Digital Signal Processing

Digital Filter Design

IIR Filter Design

Chebyshev Filters:

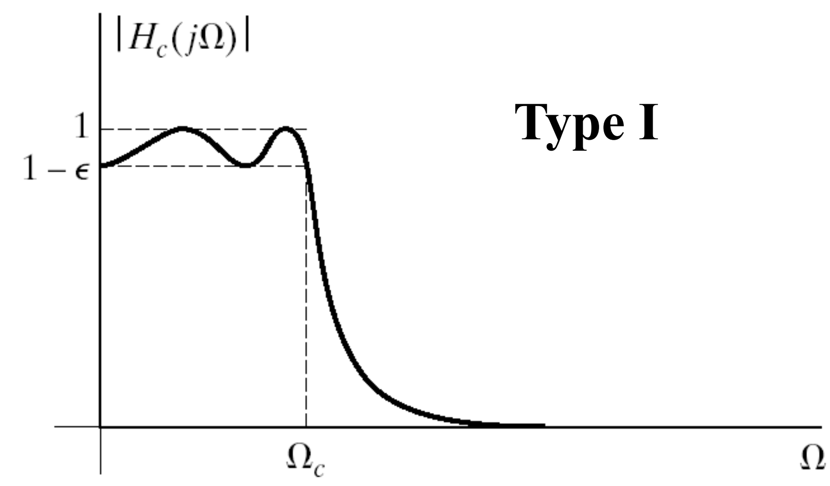
- **Equiripple in the passband and monotonic in the stopband.**

$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 V_N^2(\Omega/\Omega_c)}$$

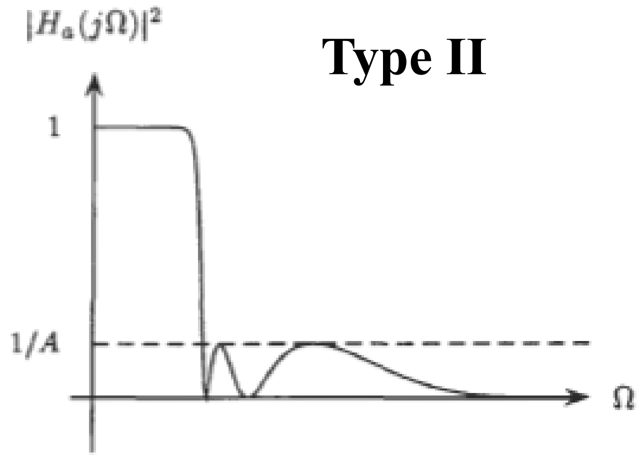
$$V_N(x) = \cos(N\cos^{-1}x)$$

- **Or equiripple in the stopband and monotonic in the passband.**

$$|H_c(j\Omega)|^2 = \frac{1}{1+[\varepsilon^2 V_N^{\,2}(\Omega/\Omega_c)]^{-1}}$$



Type I



Type II

Digital Filter Design

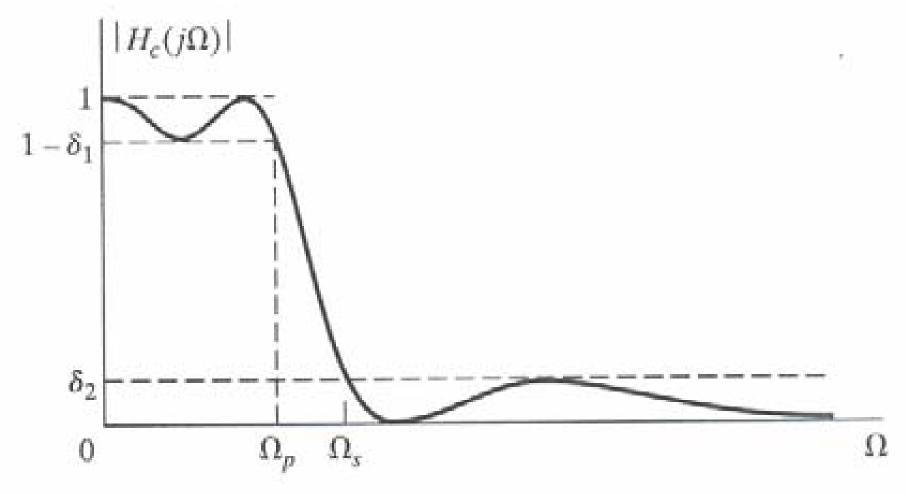
IIR Filter Design

Elliptic Filters:

Equiripple both in stopband and in the passband

$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N^2(\Omega)}$$

Jacobian Elliptic function



IIR Filter Design

Reading Assignment:

Describe how an FIR filter be designed using Frequency sampling method

End!!!!!!!