TN 412: Digital Signal Processing

2: Discrete-Time Signals & Systems

Outline

- Discrete-Time Signal Representation
- Some Elementary Discrete-Time Signals
- Classification of Discrete-Time Signal
- Simple Manipulation of Discrete-Time Signal
- Classification of Discrete-Time System

INTRODUCTION

- A discrete-time signal is a sequence of numbers (real or complex) .
- Such a sequence represents the variation of some physical quantity as a function of a discrete-time index "n"
- Discrete-time signals are often derived by sampling a continuous-time signal, such as speech, with an analog-to-digital **(A/D) converter.**

Digital Signal Processing

Discrete-Time Signals & Systems

2.1. Discrete-Time Signals

2.1.1: Discrete-Time Signal Representation:

- Functional Representation:

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \qquad x(n) = \left(\frac{1}{3}\right)^n$$

- Tabular Representation:

n	$\cdots$	-2	-1	0	1	2	3	4	5	$\ldots$
$x(n)$	$\cdots$	0	0	0	1	4	1	0	0	$\ldots$

- Sequence Representation:
  - Infinite duration sequence

$$x(n) = \{\ldots 0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\}$$

  - Finite duration sequence

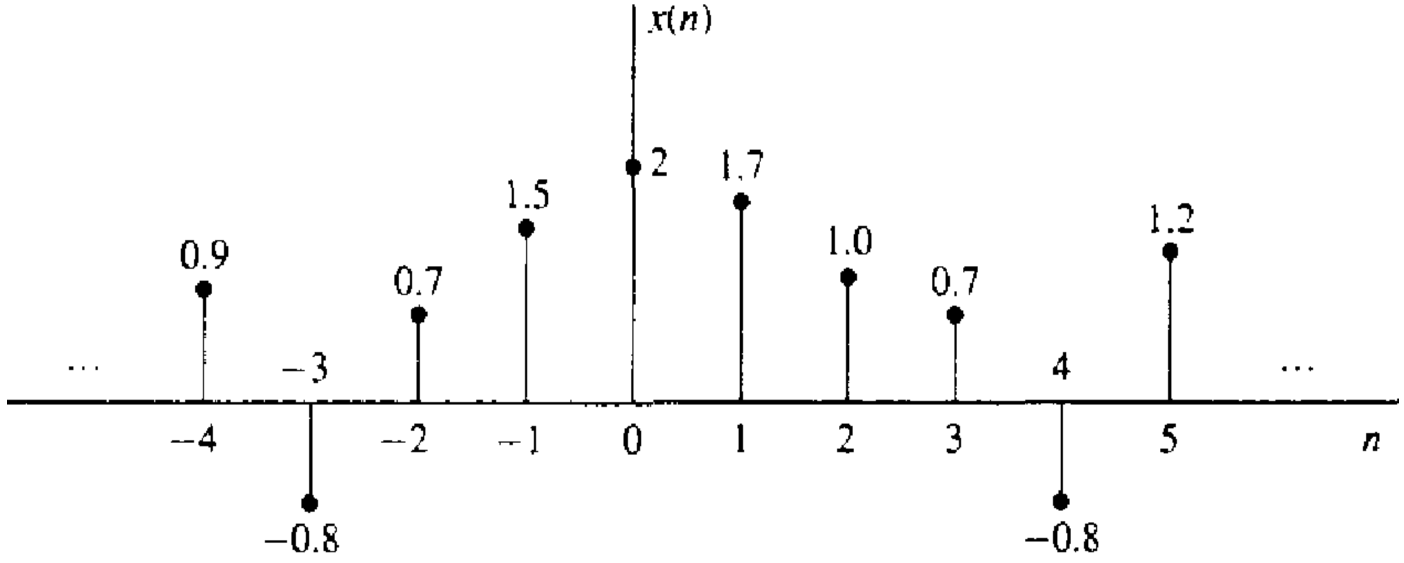
$$x(n) = \{3, -1, \underset{\uparrow}{-2}, 5, 0, 4, -1\}$$

Digital Signal Processing

Discrete-Time Signals & Systems

2.1. Discrete-Time Signals

- Graphical Representation



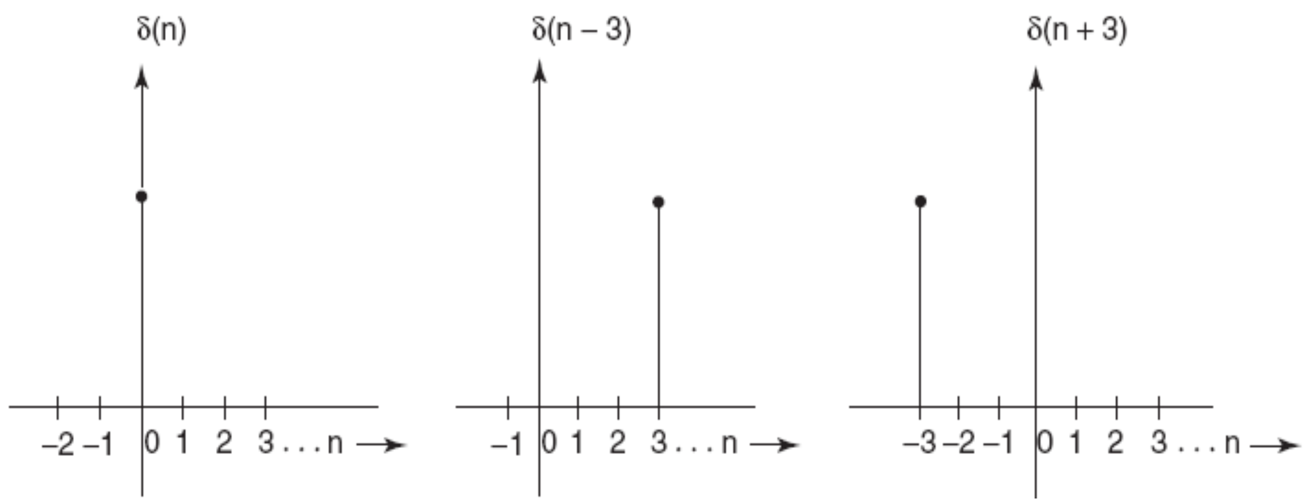
2.1. Discrete-Time Signals

2.1.2. Some Elementary Discrete-Time Signals:

- Unit Sample Sequence (Unit Impulse)

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\delta(n-k) = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$



Digital Signal Processing

Discrete-Time Signals & Systems

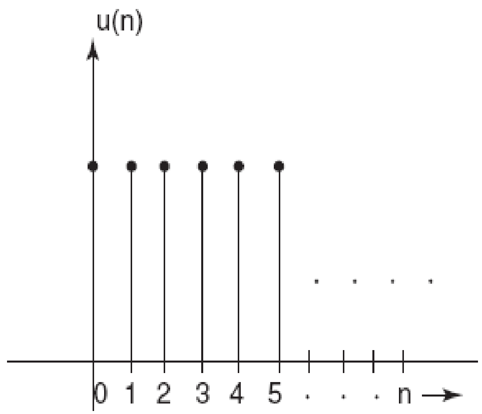
2.1. Discrete-Time Signals

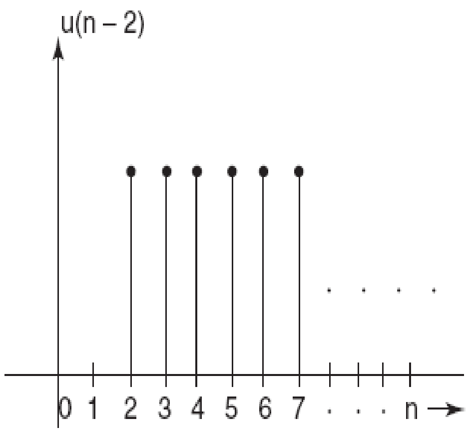
2.1.2. Some Elementary Discrete-Time Signals:

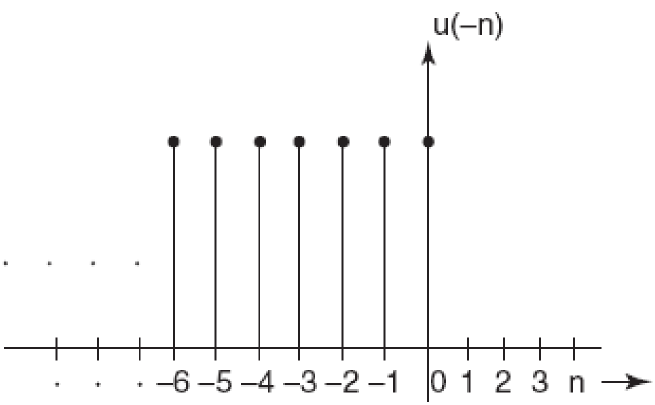
- Unit Step Signal

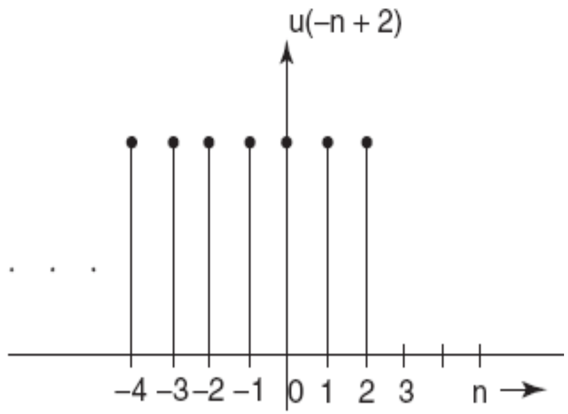
$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

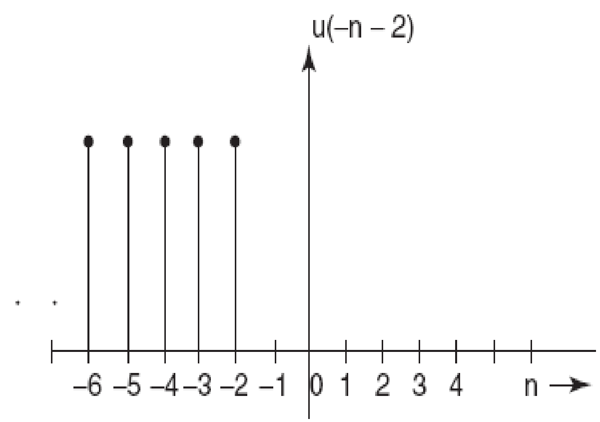
$$u(n-k) = \begin{cases} 1 & n \geq k \\ 0 & n < k \end{cases}$$









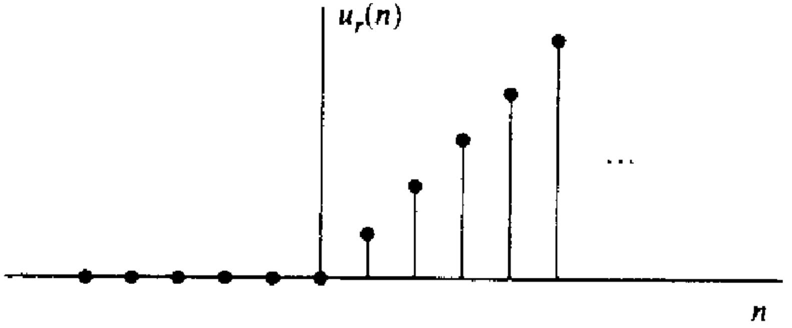


2.1. Discrete-Time Signals

2.1.2. Some Elementary Discrete-Time Signals:

- Unit Ramp Signal

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$



Digital Signal Processing

Discrete-Time Signals & Systems

2.1. Discrete-Time Signals

2.1.2. Some Elementary Discrete-Time Signals:

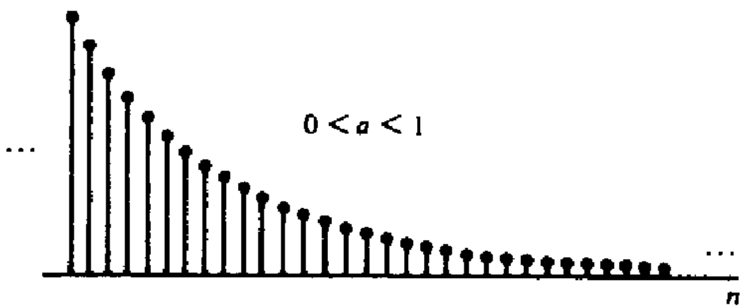
➢ Exponential Signal

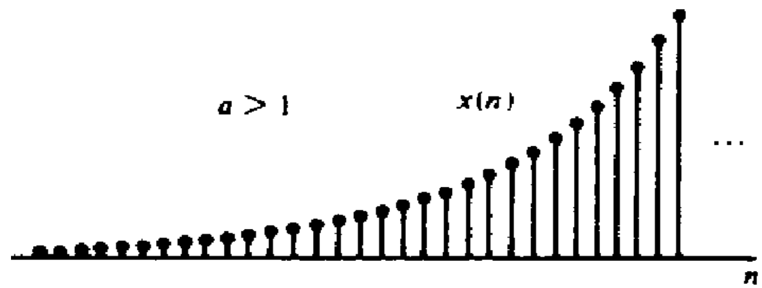
$$x(n) = a^n \qquad \text{for all } n$$

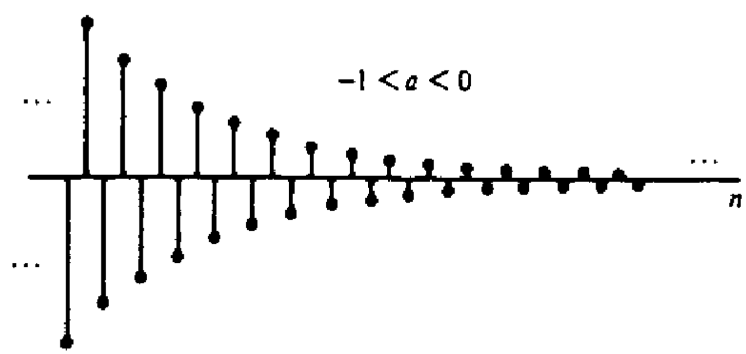
If a is real, x(n) real.

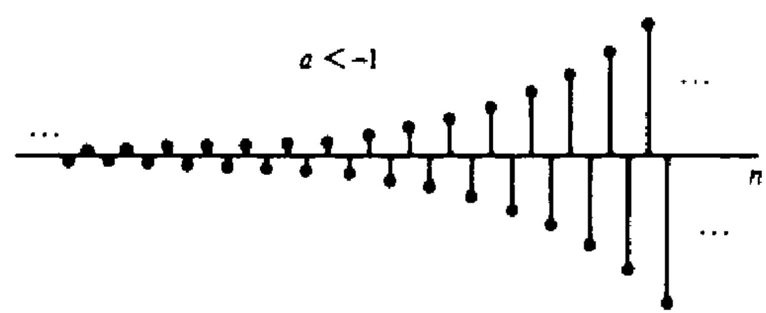
When a is complex valued,

$$a \equiv re^{j\theta} \qquad x(n) = r^n e^{j\theta n}$$
$$= r^n(\cos\theta n + j\sin\theta n)$$









Digital Signal Processing

Discrete-Time Signals & Systems

2.1. Discrete-Time Signals

2.1.2. Some Elementary Discrete-Time Signals:

Imaginary Properties

$$\left(a+j\cdot b\right)+\left(c+j\cdot d\right)=\left(a+c\right)+j\cdot\left(b+d\right)$$

$$\left(a+j\cdot b\right)\cdot\left(c+j\cdot d\right)=a\cdot c-b\cdot d+j\cdot\left(b\cdot c+a\cdot d\right)$$

$$r_1\cdot e^{j\cdot\theta_1}\cdot r_2\cdot e^{j\cdot\theta_2}=r_1\cdot r_2\cdot e^{j\cdot\left(\theta_1+\theta_2\right)}$$

$$\left|z\right|^n=\left|z^n\right| \qquad \left|z_1\cdot z_2\right|=\left|z_1\right|\cdot\left|z_2\right|$$

$$\left|\frac{z_1}{z_2}\right|=\frac{\left|z_1\right|}{\left|z_2\right|} \qquad \left|z_1+z_2\right|\neq\left|z_1\right|+\left|z_2\right| \qquad \left|z_1+z_2\right|\leq\left|z_1\right|+\left|z_2\right|$$

Rectangular / Cartesian Form — Polar Form

$$a+j\cdot b=r\cdot e^{j\cdot\theta}$$

$a=$ real part $\qquad a=r\cdot\cos\left(\theta\right)$

$b=$ imaginary part $\qquad b=r\cdot\sin\left(\theta\right)$

$r=$ magnitude $\qquad r=\sqrt{a^2+b^2}$

$\theta=$ phase $\qquad \theta=\tan^{-1}\left(\frac{b}{a}\right)$

Digital Signal Processing

Discrete-Time Signals & Systems

2.1. Discrete-Time Signals

2.1.2. Some Elementary Discrete-Time Signals:

Complex Conjugate

$$z=\frac{a+j\cdot b}{c+j\cdot d} \qquad z^{*}=\frac{a-j\cdot b}{c-j\cdot d}$$

$$z=a+j\cdot b \qquad z^{*}=a-j\cdot b$$

$$z=\frac{3-2\cdot e^{j\cdot 2}+j\cdot 4}{3\cdot j+2\cdot e^{-j}} \qquad z^{*}=\frac{3-2\cdot e^{-j\cdot 2}-j\cdot 4}{-3\cdot j+2\cdot e^{j}}$$

$$z\cdot z^{*}=\left|z\right|^{2}$$

2.1. Discrete-Time Signals

2.1.3. Classification of Discrete-Time Signals:

- Energy Signals & Power Signals

- The energy E of a signal x(n) is defined as,

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2$$

If the E is finite ($0<E<\infty$), then x(n) is called an **energy signal**.

- The average power of x(n) is defined as,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

- If E is finite, P = 0.
- If E is infinite, P may be either finite or infinite.
- If P is finite (and nonzero), the signal is called a **power signal**.

2.1.3. Classification of Discrete-Time Signals:

- **Energy Signals & Power Signals**

Example:

Determine the power & energy of the unit step sequence.

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} u^2(n) = \lim_{N \to \infty} \frac{N+1}{2N+1} = \lim_{N \to \infty} \frac{1+1/N}{2+1/N} = \frac{1}{2}$$

Consequently, the unit step signal is a power signal. Its energy is infinite.

2.1. Discrete-Time Signals

2.1.3. Classification of Discrete-Time Signals:

- **Periodic Signals & Non-periodic Signals**

Signal is periodic if and only if

$$x(n+N)=x(n)$$ For all n

The smallest value of N which satisfy the above prosperity is called Fundamental Period.

The sinusoidal signal $x(n) = A\sin 2\pi f_0 n$ **is periodic when f_0 is a rational number** $f_0 = \frac{k}{N}$

If x(n) is periodic signal with fundamental period N the average power is:

$$P = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2$$

Digital Signal Processing

Discrete-Time Signals & Systems

2.1. Discrete-Time Signals

2.1.3. Classification of Discrete-Time Signals:

- Symmetric (Even) Signals & Anti-symmetric (Odd) Signals

Even Signal (Symmetric):

Signal is even IIF

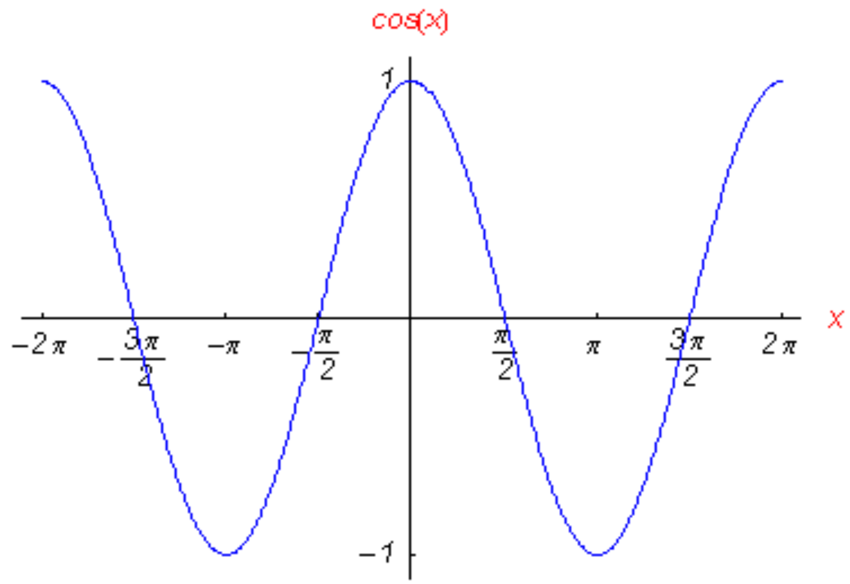
$$x(-n)=x(n)$$

Any arbitrary signal can be expressed as sum of two signal components even and odd, the even component:

$$x_e(n)=\frac{1}{2}\cdot\left[x(n)+x(-n)\right]$$

Example:

cos(x) = cos(-x)



Digital Signal Processing

Discrete-Time Signals & Systems

2.1. Discrete-Time Signals

2.1.3. Classification of Discrete-Time Signals:

- Symmetric (Even) Signals & Anti-symmetric (Odd) Signals

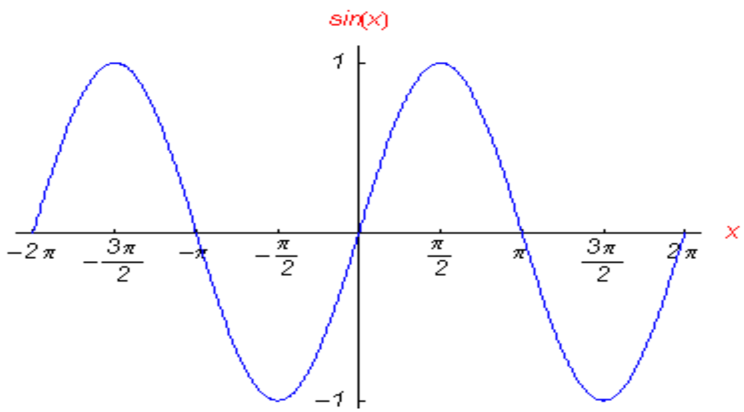
Odd Signal (Anti-symmetric):

Signal is odd IIF

$$x(-n) = -x(n) \quad \& \quad x(0) = 0$$

Example:

sin(-x) = -sin(x)



Any arbitrary signal can be expressed as sum of two signal components even and odd, the odd component:

$$x_o(n) = \frac{1}{2} \cdot [x(n) - x(-n)]$$

$$x(n) = x_e(n) + x_o(n)$$

The sum of 2 component form the signal x(n)

Digital Signal Processing

Discrete-Time Signals & Systems

2.1. Discrete-Time Signals

2.1.4. Simple Manipulation of Discrete-Time Signals:

- Transformation of independent Variable

Time Shifting:

Delay: Delay always possible (refers to past samples) $n \rightarrow n - n_0$

Advance: Time advance only possible if signal is stored. $n \rightarrow n + n_0$

- Refers to future samples (in reference to present sample)
- Time advance impossible in real time.

Examples:

- Advance by 2 samples: $y(n) = x(n+2)$
- Delay by M samples: $y(n) = x(n-M)$

In the above, y(n) is output, x(n) is input

Digital Signal Processing

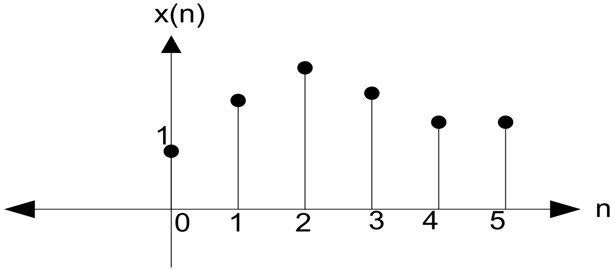
Discrete-Time Signals & Systems

2.1. Discrete-Time Signals

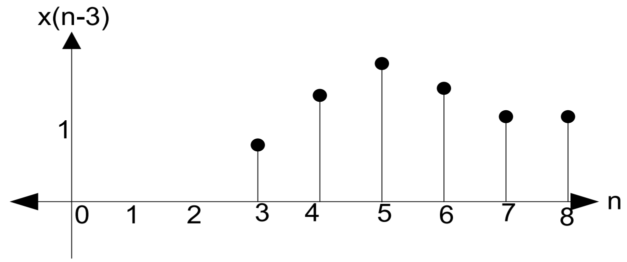
2.1.4. Simple Manipulation of Discrete-Time Signals:

Delay versus advance time shifting:

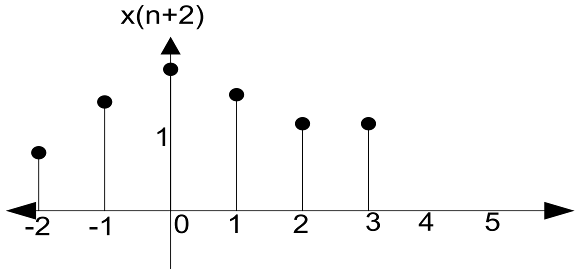
Original signal



Time delay (3 sample)



Time advance (2 sample)

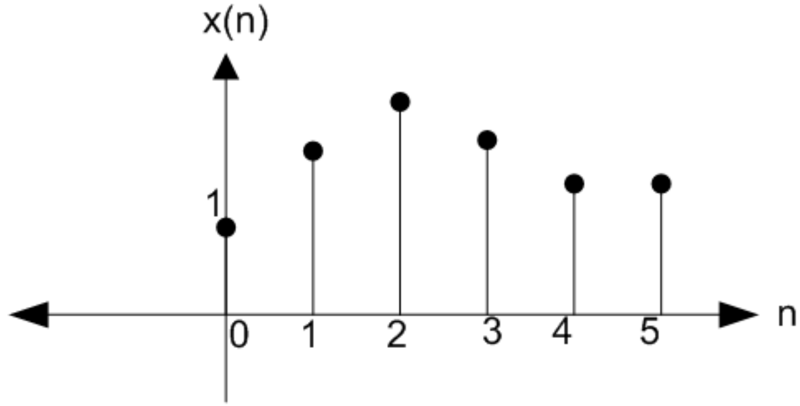


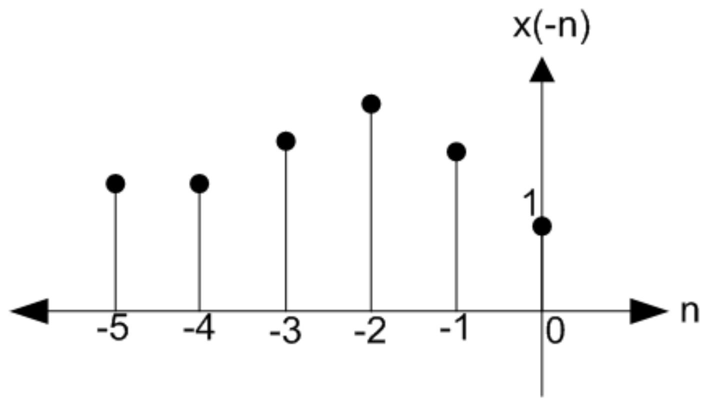
2.1. Discrete-Time Signals

2.1.4. Simple Manipulation of Discrete-Time Signals:

Time Reversal:

Original signal





Time reversal
or Reflected

Digital Signal Processing

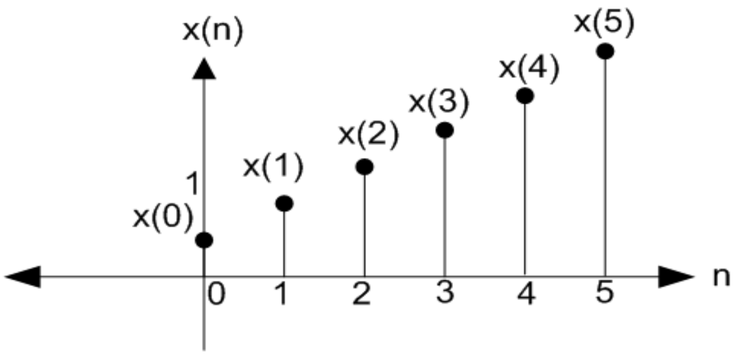
Discrete-Time Signals & Systems

2.1. Discrete-Time Signals

2.1.4. Simple Manipulation of Discrete-Time Signals:

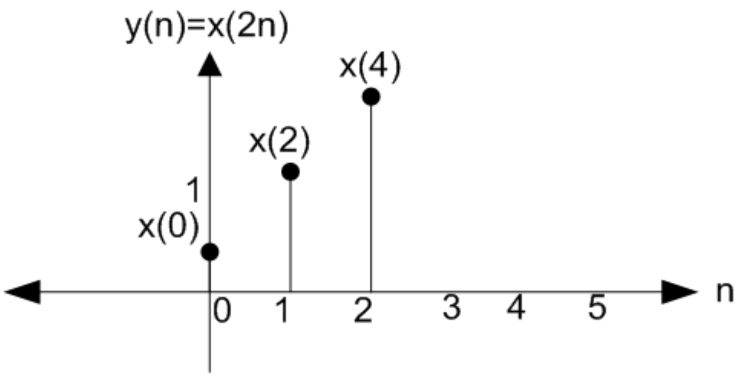
Down Sampling or Decimation:

Decreasing the sampling rate by a constant factor:



μ Is an integer greater than 0

n	0	1	2	3	4	5	...
x(n)	$\boxed{x(0)}$	x(1)	$\boxed{x(2)}$	x(3)	$\boxed{x(4)}$	x(5)	...
y(n)=x(2n)	x(0)	x(2)	x(4)	x(6)	x(8)	x(10)	...



Down sampled by factor 2

$$y(n) = x(2n)$$

Digital Signal Processing

Discrete-Time Signals & Systems

2.1. Discrete-Time Signals

2.1.4. Simple Manipulation of Discrete-Time Signals:

Signal Multiplication & Addition:

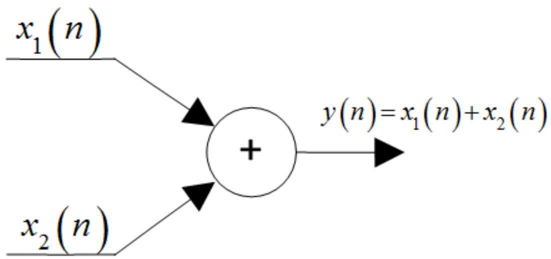
- **Signal Multiplication by constant (k):** $$y(n)=k\cdot x(n)$$
- **Signal Addition by constant (k):** $$y(n)=k+x(n)$$
- **Multiple Signal Addition:** $$y(n)=x_1(n)+x_2(n)$$
- **Multiple Signal Multiplication:** $$y(n)=x_1(n)\cdot x_2(n)$$

2.1. Discrete-Time Signals

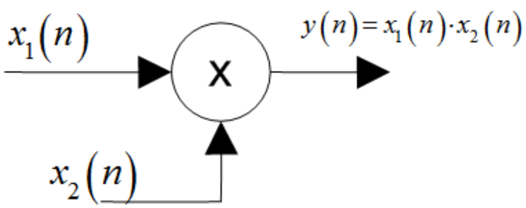
2.1.4. Simple Manipulation of Discrete-Time Signals:

Graphical Representation:

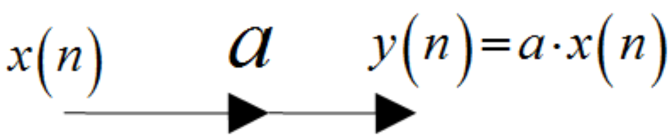
Signal Adder



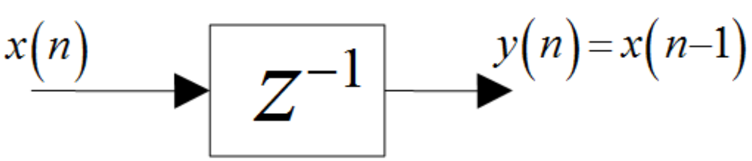
Signal Multiplier



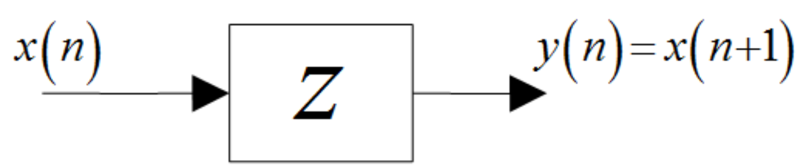
Constant Multiplier



Unit Delay



Unit Advance

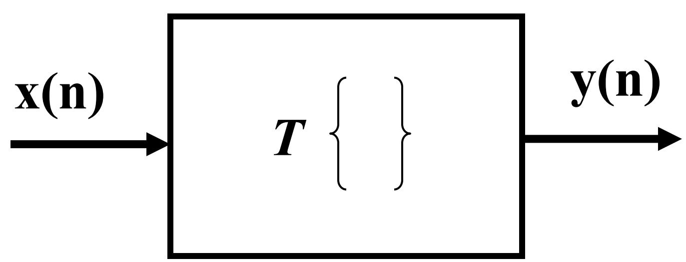


2.2. Discrete-Time Systems

A discrete-time system is a device or algorithm that operates on discrete–time signal called the input (Excitation), according to well defined rule, to produce another discrete-time signal called the output (Response) of the system.

The general relation:

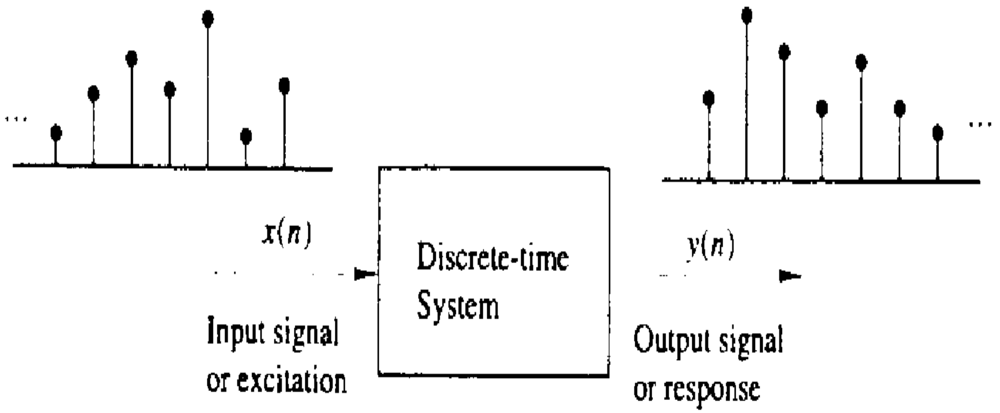
$$y(n) \equiv T[x(n)] \qquad x(n) \xrightarrow{T} y(n)$$



2.2. Discrete-Time Systems

2.2.1. Input-Output Description of Systems:

The input-output description of a DT system consists of a mathematical expression which defines the relation between the I/O signals.



Digital Signal Processing

Discrete-Time Signals & Systems

2.2. Discrete-Time Systems

2.2.1. Input-Output Description of Systems:

Examples of different Input/Output Relations

- **Pass Through:** $$y(n) = x(n)$$
- **Unit delay:** $$y(n) = x(n-1)$$
- **Amplified by factor of 2:** $$y(n) = 2 \cdot x(n)$$
- **Gain – Delay:** $$y(n) = 3 \cdot x(n-2)$$
- **Moving Average:** $$y(n) = \frac{1}{3} \cdot \left[x(n-1) + \cdot x(n) + x(n+1)\right]$$

2.2. Discrete-Time Systems

2.2.1. Input-Output Description of Systems:

Examples of different Input/Output Relations

- **Weighted Summation:** $$y(n)=\frac{1}{2}\cdot x(n-1)+\frac{1}{4}\cdot x(n)-\frac{1}{3}\cdot x(n+1)$$
- **Absolute Value:** $$y(n)=\left|x(n)\right|$$
- **Squarer:** $$y(n)=\left[x(n)\right]^2$$
- **Max Filter:** $$y(n)=\max\left\{x(n-1),x(n),x(n+1)\right\}$$
- **Accumulator:** $$y(n)=\sum_{k=-\infty}^{n} x(k)$$

Digital Signal Processing

Discrete-Time Signals & Systems

2.2. Discrete-Time Systems

2.2.1. Input-Output Description of Systems:

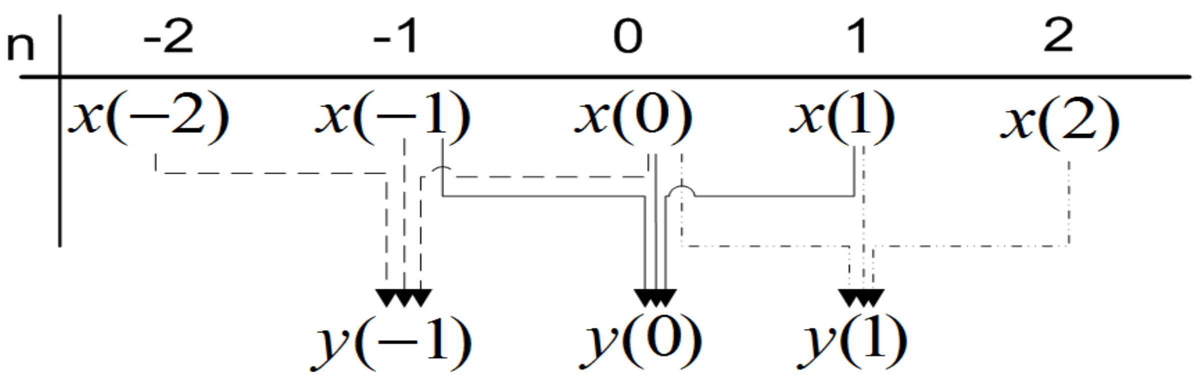
Moving Average

$$y(n)=\frac{1}{3}\cdot\left[x(n-1)+\cdot x(n)+x(n+1)\right]$$

$$y(-1)=\frac{1}{3}\cdot\left[x(-2)+x(-1)+x(0)\right]$$

$$y(0)=\frac{1}{3}\cdot\left[x(-1)+x(0)+x(1)\right]$$

$$y(1)=\frac{1}{3}\cdot\left[x(0)+x(1)+x(2)\right]$$



2.2. Discrete-Time Systems

2.2.1. Input-Output Description of Systems:

System may use present inputs: $$y(n) = x(n)$$

and/or past inputs: $$y(n) = x(n-1)$$

and/or future inputs: $$y(n) = \frac{1}{3} \cdot \left[x(n-1) + \cdot x(n) + x(n+1) \right]$$

to produce present output: $$y(n) = \frac{1}{2} \cdot x(n-1) + \frac{1}{4} \cdot x(n) - \frac{1}{3} \cdot x(n+1)$$

2.2. Discrete-Time Systems

2.2.2. Classification of Discrete-Time Systems:

Static vs. Dynamic Systems

Static:

Discrete system whose output at any time instant depends on the input sample at that same time instant. The system is memory less or static. The system's output is NOT dependent on past input or past output samples.

Dynamic:

The system is dependant on future and/or past input or previous output samples. The system has memory

2.2. Discrete-Time Systems

2.2.2. Classification of Discrete-Time Systems:

Static vs. Dynamic Systems

Static Discrete Time Systems

$$y(n)=x(n)$$ (dependent only on present input

$$y(n)=3\cdot x(n)$$

$$y(n)=x_1(n)+x_2(n)$$ (only on present inputs)

Dynamic Discrete Time Systems

$$y(n)=3\cdot x(n-2)$$ (dependent on previous inputs)

$$y(n)=\frac{1}{2}\cdot x(n-1)+\frac{1}{4}\cdot x(n)-\frac{1}{3}\cdot x(n+1)$$ (previous, present & future inputs)

$$y(n)=15\cdot x(n-1)+y(n-1)$$ (previous input & output)

2.2. Discrete-Time Systems

2.2.2. Classification of Discrete-Time Systems:

Time-Invariant vs. Time-Variant Systems

A system is called time invariant if its input-output characteristics do not change with time. A time invariant system, when presented with an input at present time, produces an output. That time invariant system will produce a delayed version of the output if the input were presented at a later (delayed) time.

Input at Present Produces Response: $x(n) \rightarrow y(n)$

Delayed Input Produces Delayed Response: $x(n-k) \rightarrow y(n-k)$

Digital Signal Processing

Discrete-Time Signals & Systems

2.2. Discrete-Time Systems

2.2.2. Classification of Discrete-Time Systems:

Time-Invariant vs. Time-Variant Systems

Time-Invariance Test:

- **Excite system (*T*) with an input – x(n)** $y(n) = T\{x(n)\}$
- **Excite the system (*T*) with same input but delayed by k – x(n-k)** $y(n,k) = T\{x(n-k)\}$
- **Delay the output of the system by k (replace all instances of n by n-k):** $y(n-k) = T\{x(n-k)\}$
- **If the system is time invariant:** $y(n,k) = y(n-k)$
- **Otherwise system is time variant:** $y(n,k) \neq y(n-k)$

2.2. Discrete-Time Systems

2.2.2. Classification of Discrete-Time Systems:

Time-Invariant vs. Time-Variant Systems

Time-Invariance Test Examples:

I. Is the following system time-invariant? $y(n)=x(n)-x(n-1)$

- Find system output for delayed input

$$y(n,k)=x(n-k)-x(n-1-k)$$

- Find delayed system output (replace all n by n-k):

$$y(n-k)=x(n-k)-x(n-1-k)$$

- Compare: $y(n,k)=y(n-k)$

System is TIME-INVARIANT

Digital Signal Processing

Discrete-Time Signals & Systems

2.2. Discrete-Time Systems

2.2.2. Classification of Discrete-Time Systems:

Time-Invariant vs. Time-Variant Systems

Time-Invariance Test Examples:

II. Is the following system time-invariant? $y(n) = n \cdot x(n)$

- Find system output for delayed input

$$y(n,k) = n \cdot x(n-k)$$

- Find delayed system output (replace all n by n-k):

$$y(n-k) = (n-k) \cdot x(n-k)$$

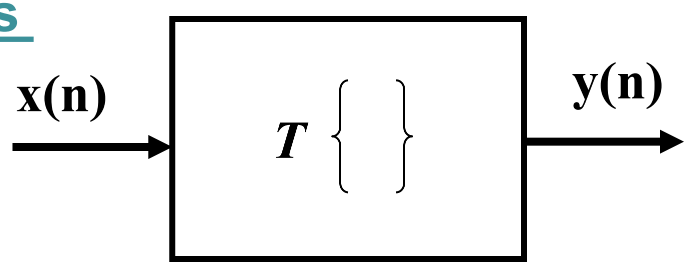
- Compare: $y(n,k) \neq y(n-k)$

System is TIME-VARIANT

2.2. Discrete-Time Systems

2.2.2. Classification of Discrete-Time Systems:

Linear vs. Nonlinear Systems



Given the system:

$$y(n) = T\{x(n)\}$$

System is Linear if and only if:

$$T\{k \cdot x(n)\} = k \cdot T\{x(n)\} = k \cdot y(n)$$

if $T\{x_1(n)\} = y_1(n)$

$$T\{x_2(n)\} = y_2(n)$$

Then

$$T\{x_1(n) + x_2(n)\} = y_1(n) + y_2(n)$$

Combined: Linear IIF $T\{k_1 \cdot x_1(n) + k_2 \cdot x_2(n)\} = k_1 \cdot y_1(n) + k_2 \cdot y_2(n)$

Digital Signal Processing

Discrete-Time Signals & Systems

2.2. Discrete-Time Systems

2.2.2. Classification of Discrete-Time Systems:

Causal vs. Non-causal Systems

Causal System is causal if the output of the system at any defined time instance depends only on present and/or past inputs and/or past outputs.

Causal systems do NOT use future inputs to calculate their output.

Examples:

$$y(n) = 15 \cdot x(n-1) + y(n-1)$$

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

$$y(n) = 3 \cdot x(n-1) + \left(n+12\right) \cdot x(n)$$

2.2. Discrete-Time Systems

2.2.2. Classification of Discrete-Time Systems:

Causal vs. Non-causal Systems

Non-Causal System is dependent on future input samples.

Non-Causal systems are impossible to implement in real-time due to the fact that the future information is unknown.

Examples:

$$y(n) = \frac{1}{3} \cdot \left[x(n-1) + \cdot x(n) + x(n+1) \right]$$

$$y(n) = \frac{1}{2} \cdot x(n-1) + \frac{1}{4} \cdot x(n) - \frac{1}{3} \cdot x(n+1)$$

$$y(n) = 3 \cdot x(n+1) + \left(n-12\right) \cdot x(n)$$

$$y(n) = x(n^2)$$

2.2. Discrete-Time Systems

2.2.2. Classification of Discrete-Time Systems:

Stable vs. Unstable Systems

BIBO: Bounded input bounded output

BIBO Stable: **A system is defined as BIBO stable if and only if every bounded input produces a bounded output.**

Bounded Input and Output Signal defined as follows:

$$|x(n)| \le M_x < \infty \qquad\qquad |y(n)| \le M_y < \infty$$

Where M_x **&** M_y **are finite number.**

If for some bounded inputs sequence x(n), the output is unbounded (infinite), the system is classified as unstable.

Tutorial

- Chapter 2: Problems by Proakis and Manolakis
  - 2.1, 2.2, 2.5, 2.6, 2.7 etc

End!!!