TN 412: Digital Signal Processing

3: Discrete-Time Signals & Systems

Discrete-Time Signals & Systems

3.1. Analysis of DT Linear Time Invariant Systems

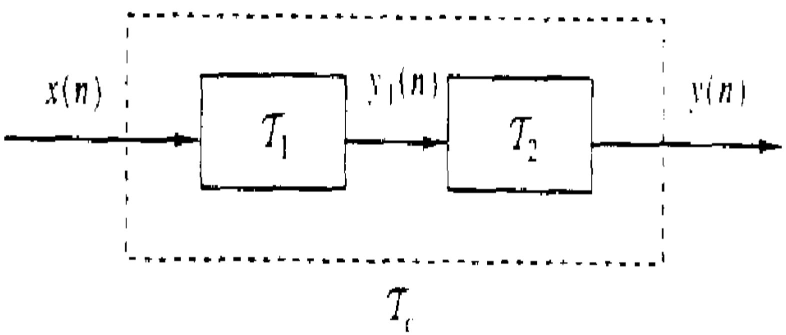
Motivation for the emphasis on the study of LTI

- There is a large collection of mathematical techniques that can be applied to the analysis of LTI systems.
- Many practical systems are either LTI systems or can be approximated by LTI systems.
- We demonstrate that such systems are characterized in the time domain simply by their response to a unit impulse sequence
- We also demonstrate that any arbitrary input signal can be decomposed and represented as a weighted sum of unit sample sequences.

3.3. Analysis of DT Linear Invariant Systems

System Interconnections:

Cascade



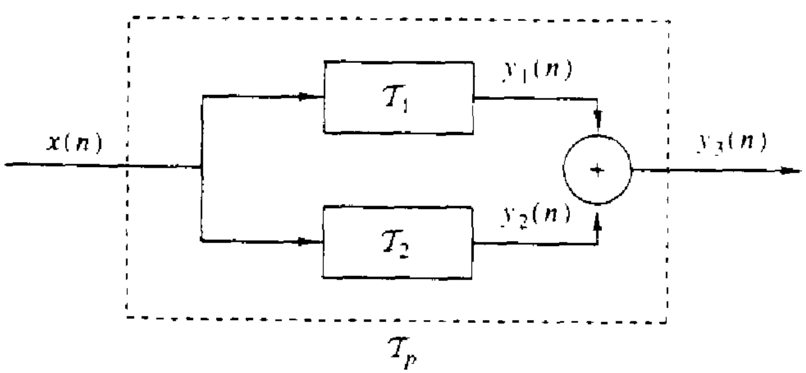
$$y_1(n) = T_1\{x(n)\}$$

$$y(n) = T_2\{y_1(n)\} = T_2\{T_1\{x(n)\}\}$$

$$T_c = T_2T_1 \neq \mathrm{T}_1\mathrm{T}_2$$

Specifically, for LTI Systems: $T_2T_1 = \mathrm{T}_1\mathrm{T}_2$

Parallel



$$y_1(n) = T_1\{x(n)\} \qquad y_2(n) = T_2\{x(n)\}$$

$$y_3(n) = y_1(n)+y_2(n) = T_1\{x(n)\} + T_2\{x(n)\}$$

$$y(n) = (T_2 + T_1)x(n)$$

$$T_p = (T_2 + T_1)$$

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3.1. Analysis of DT Linear Invariant Systems

In this section we will analyze the Linear Time-Invariant (LTI) systems.

3.3.1. System Analysis Techniques:

Two methods are presented in for analyzing the behavior/response of a system to a given input:

- **Direct Solution of the Input-Output Equation (or difference equation).**
- **Signal Decomposition (Convolution).**
  - **Decompose the input signal into sum of elementary signals.**

3.3. Analysis of DT Linear Invariant Systems

3.3.1. System Analysis Techniques:

- **System Input output equation:**

The general input output equation for any system:

$$y(n) = F[y(n-1), y(n-2), \ldots, y(n-N), x(n), x(n-1), \ldots, x(n-M)]$$

where F[ ] denotes some functions of the quantities.

We can rewrite the general input output equation as

$$y(n) = -\sum_{k=1}^{N} a_k(n)\cdot y(n-k) + \sum_{k=-L}^{M} b_k(n)\cdot x(n-k)$$

where a_k and b_k are constant parameters

If the system is casual L = 0, so the equation:

$$y(n) = -\sum_{k=1}^{N} a_k(n)\cdot y(n-k) + \sum_{k=0}^{M} b_k(n)\cdot x(n-k)$$

Note that both a and b vary with time

This system is linear , time-variant system. This system called Adaptive Linear System.

3.3. Analysis of DT Linear Invariant Systems

3.3.1. System Analysis Techniques:

Linear Time-Invariant (LTI) Systems:

If a and b are constant over time, then the previous equation further simplifies into the general equation for causal, Linear, Time-Invariant (LTI) Systems:

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k \cdot x(n-k)$$

Note: a_k and b_k are independent of time (n) or simply constant for all time n.

Input, Output Equation is called Difference Equation.

The order for the LTI system is N.

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3.3. Analysis of DT Linear Invariant Systems

3.3.1. System Analysis Techniques:

Non-linear Difference Equation:

Difference Equations can also describe non linear systems:

$$y(n) = |x(n)|$$

$$y(n) = 2 \cdot x(n) - 3 \cdot x^2(n)$$

$$y(n) = \sqrt{x(n)} + 3 \cdot \log(x(n))$$

$$y(n) = A \cdot x(n) + B$$

3.3. Analysis of DT Linear Invariant Systems

3.3.1. System Analysis Techniques:

➢ **Signal Decomposition:**

Decompose input into weighted basis functions:

$$x(n)=\sum_k c_k \cdot x_k(n)$$

Generally we choose the basis functions and compute C_k.

Decomposition to Impulses:

Consider $x(n)=\{3,5,2,1\}$

Decompose input sequence to sum of unit samples.

$$x_1(n)=\{3,0,0,0\}=3\cdot\delta(n)$$

$$x_2(n)=\{0,5,0,0\}=5\cdot\delta(n-1)$$

$$x_3(n)=\{0,0,2,0\}=2\cdot\delta(n-2)$$

$$x_4(n)=\{0,0,0,1\}=\delta(n-3)$$

$$x(n)=x_1(n)+x_2(n)+x_3(n)+x_4(n)$$

$$x(n)=3\cdot\delta(n)+5\cdot\delta(n-1)+2\cdot\delta(n-2)+\delta(n-3)$$

3.3. Analysis of DT Linear Invariant Systems

3.3.1. System Analysis Techniques:

Decomposition to Impulses:

We can write the following equation for the previous expression:

$$x(n)=\sum_{k=0}^{3} x(k)\cdot\delta(n-k)$$

In general form

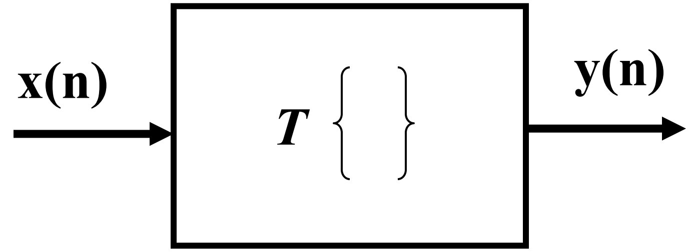
$$\boxed{x(n)=\sum_{k=-\infty}^{\infty} x(k)\cdot\delta(n-k)}$$

where $\quad c_k = x(k)\ ,\ x_k(n)=\delta(n-k)$

3.3. Analysis of DT Linear Invariant Systems

Convolution Summation:

Consider the following system:



Recall input can be decomposed as follows:

$$x(n)=\sum_{k=-\infty}^{\infty} x(k)\cdot\delta(n-k)$$

Therefore: $$y(n) = \mathcal{T}[x(n)] = \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}[\delta(n-k)]$$

If *T* system is Linear:

$$= \sum_{k=-\infty}^{\infty} x(k)h(n,k)$$

Impulse Response: $$y(n,k) \equiv h(n,k) = \mathcal{T}[\delta(n-k)]$$

Useless, infinite set of responses:
(dependent on both n and k)

$$y(n)=\sum_{k=-\infty}^{\infty} x(k)\cdot h_k(n)$$

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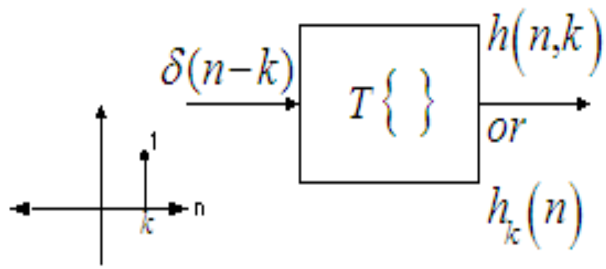
Convolution Summation:

Impulse Response



Note – if system is time invariant

$$h(n,k)=T\{\delta(n-k)\}$$ and $$h(n-k)=T\{\delta(n-k)\}$$

Therefore $$h(n,k)=h(n-k)$$

For LTI systems, the convolution sum is as follows:

$$y(n)=\sum_{k=-\infty}^{\infty} h(k)x(n-k)=\sum_{k=-\infty}^{\infty} x(k)h(n-k)=h(n)*x(n)=x(n)*h(n)$$

3.3. Analysis of DT Linear Invariant Systems

Convolution Summation Calculation:

Graphical method:

To calculate $y(n_0)$ using the convolution summation, do the following steps:

- ✓ **Folding**: Fold h(k) about k = 0 to obtain h(-k).
- ✓ **Shifting**: Shift h(-k) by n_0 to the right (left) if n_0 is positive (negative), to obtain $h(n_0 - k)$
- ✓ **Multiplication**: Multiply x(k) by $h(n_0 - k)$ to obtain the product sequence $x(k)h(n_0 - k)$
- ✓ **Summation**: Sum all the values of the product sequence $x(k)h(n_0 - k)$ to obtain the value of the output at time $n = n_0$

3.3. Analysis of DT Linear Invariant Systems

Graphical Convolution Example

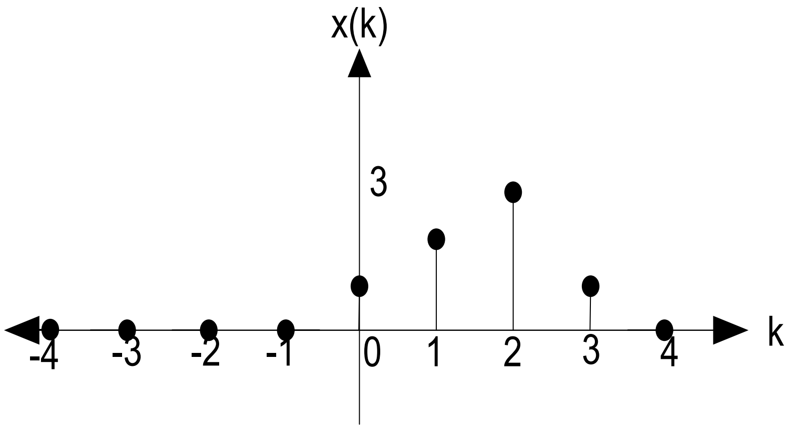
Consider the following input and impulse response

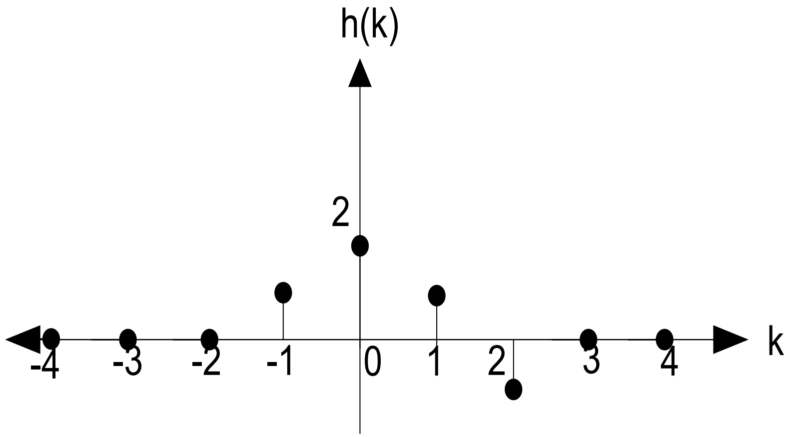
$$x(n)=\{1,2,3,1\} \quad \uparrow$$

Input Sequence

$$h(n)=\{1,2,1,-1\} \quad \uparrow$$

Impulse Response





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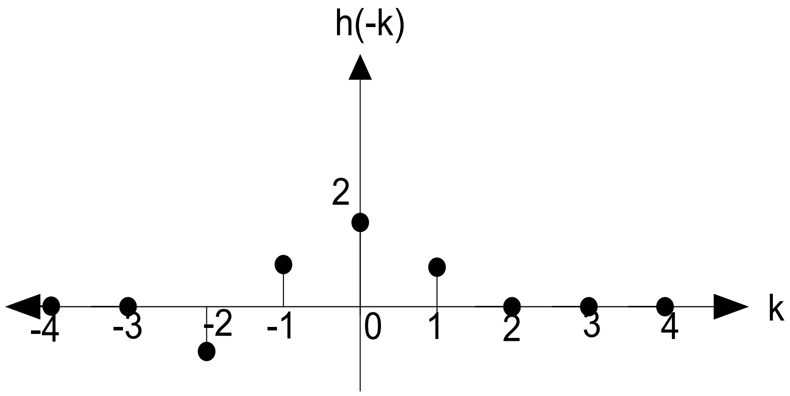
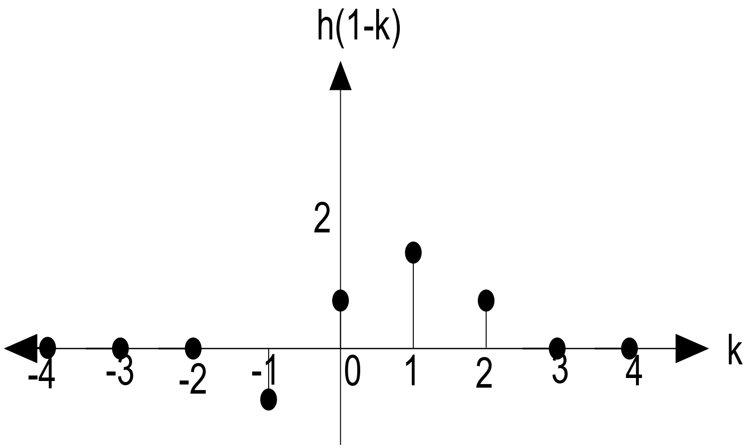
Graphical Convolution Example Cont.



Folded Impulse



Shifted Impulse

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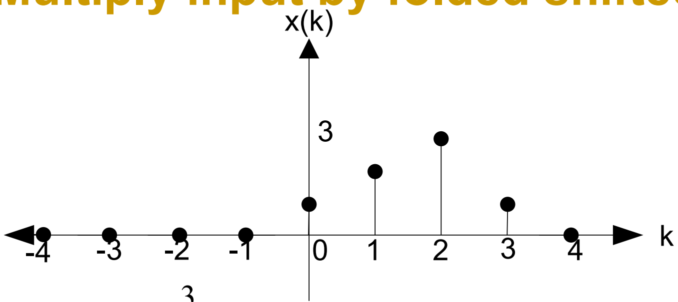
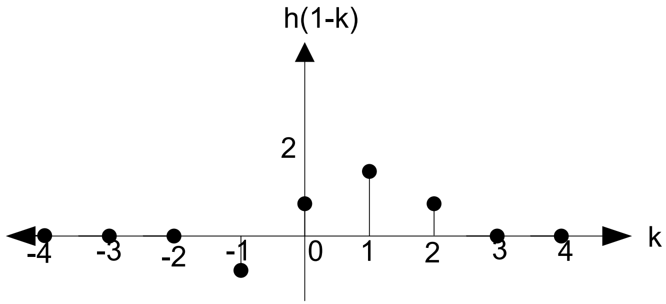
Graphical Convolution Example Cont.

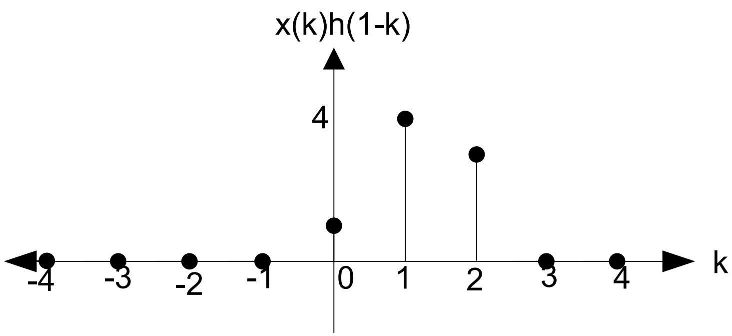
Multiply input by folded shifted h(n):



$$y(1)=\sum_{k=0}^{3} x(k)\cdot h(1-k)=1+4+3=8$$



Product Sequence – x(k)h(n-k):



Sum of Product Sequence:

3.3. Analysis of DT Linear Invariant Systems

Mathematical Convolution Example

Consider the following input and impulse response

x(n) = [5 ,2 , 4 , -1] **h(n) = [5 , 2 , 4 , -1]**

y[0]	y[1]	y[2]	y[3]	y[4]	y[5]	y[6]
25	20	44	6	12	-8	1

h(k) x(k)	5	2	4	-1
5	25	10	20	-5
2	10	4	8	-2
4	20	8	16	-4
-1	-5	-2	-4	1

y(n) = [25, 20 , 44 , 6 , 12 , -8 , 1]

Note: The convolution of two FINITE-LENGTH sequences is that if x(n) is of length L_1 and h(n) is of length L_2 , y(n) = x(n)*h(n) will be of length : $L = L_1 + L_2 - 1$

3.3. Analysis of DT Linear Invariant Systems

Convolution Properties

Convolution Symbol:

$$y(n)=\sum_{k=-\infty}^{\infty} x(k)\cdot h(n-k)=x(k)*h(n)$$

Convolution is Commutative:

$$x(k)*h(n)=h(n)*x(k)$$

$$\sum_{k=-\infty}^{\infty} x(k)\cdot h(n-k)=\sum_{k=-\infty}^{\infty} h(k)\cdot x(n-k)$$

Convolution is Distributive:

$$h(n)*\left[x_1(n)+x_2(n)\right]=h(n)*x_1(n)+h(n)*x_2(n)$$

Convolution is Associative:

$$\left[x_1(n)*x_2(n)\right]*x_3(n)=x_1(n)*\left[x_2(n)*x_3(n)\right]$$

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Useful Geometric Summation Formulas

$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$	$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \|a\| < 1$
$\sum_{n=0}^{N-1} na^n = \frac{(N-1)a^{N+1} - Na^N + a}{(1-a)^2}$	$\sum_{n=0}^{\infty} na^n = \frac{a}{(1-a)^2} \quad \|a\| < 1$
$\sum_{n=0}^{N-1} n = \frac{1}{2}N(N-1)$	$\sum_{n=0}^{N-1} n^2 = \frac{1}{6}N(N-1)(2N-1)$

3.3. Analysis of DT Linear Invariant Systems

Causal Linear Time-Invariant Systems

❖ **An LTI system is Causal IFF its impulse response is 0 for negative values of n.**

$$h(n)=0,\ for\ n<0$$

❖ **Convolution sum for Causal, LTI system reduces to the following:**

$$y(n)=\sum_{k=0}^{\infty} h(k)\cdot x(n-k)$$

❖ **Also, for Causal LTI:** $$y(n)=\sum_{k=-\infty}^{n} x(k)\cdot h(n-k)$$

❖ **If the input is Causal, LTI system is Causal then the Convolution further simplifies to:**

$$y(n)=\sum_{k=0}^{n} x(k)\cdot h(n-k)$$

3.3. Analysis of DT Linear Invariant Systems

Causal Linear Time-Invariant Systems

If Future Inputs are not needed then Causal LTI system reduces as follows:

$$y(n)=\sum_{k=-\infty}^{\infty} x(n-k)\cdot h(k)$$

$$y(n)=\underbrace{\sum_{k=-\infty}^{-1} x(n-k)\cdot h(k)}_{}^{\substack{future \\ inputs}}+\sum_{k=0}^{\infty} x(n-k)\cdot h(k)$$

For causality, the reference to future inputs must be equal to 0, or simply there must not be future inputs existent in the system's difference equation.

If k is negative then x(n-k) corresponds to future input references

Therefore:

$$\sum_{k=-\infty}^{-1} x(n-k)\cdot h(k)=0 \quad \substack{future \\ inputs}$$

Then for the causal system the original equation can be rewritten:

$$\boxed{y(n)=\sum_{k=0}^{\infty} x(n-k)\cdot h(k)}$$

3.3. Analysis of DT Linear Invariant Systems

Non-Causal Systems

Non Causal systems will produce an output BEFORE the input is applied.

Consider the following non-causal equation:

$$y(n) = 2 \cdot x(n+1)$$

If the input is applied at n=0, then at n= -1, we already have an output (before the input is applied):

$$y(-1) = 2 \cdot x(0)$$

The output of course is dependant on a future input (which does not make much sense in real time implementations).

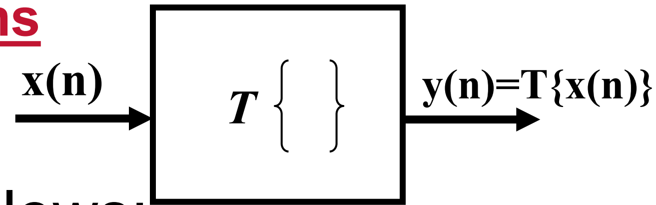
Stability of Linear Time-Invariant Systems

- BIBO Stable Defined as:
  - For all bounded inputs x(n), the outputs y(n) are bounded if the system is BIBO stable:
  - $|x(n)| \le M_x < \infty$ and $|y(n)| \le M_y < \infty$
    - Where M_x and M_y are finite numbers
- Reminder – Triangular or Schwartz Inequality: $|x_1 + x_2| \ne |x_1| + |x_2|$ $\quad |x_1 + x_2| \le |x_1| + |x_2|$

3.3. Analysis of DT Linear Invariant Systems

Stability of Linear Time-Invariant Systems

- Consider the following system:



- Note the output is defined as follows:

$$\left|y(n)\right| = \left|\sum_{k=-\infty}^{\infty} h(k)\cdot x(n-k)\right| \le \sum_{k=-\infty}^{\infty} \left|h(k)\cdot x(n-k)\right| = \sum_{k=-\infty}^{\infty} \left|h(k)\right|\cdot\left|x(n-k)\right|$$

- Assume the input is stable: $\left|x(n-k)\right| \le M_x$

- The above can then be rewritten as follows:

  - $$\sum_{k=-\infty}^{\infty} \left|h(k)\right|\cdot\left|x(n-k)\right| \le \sum_{k=-\infty}^{\infty} \left|h(k)\right|\cdot M_x$$

- Therefore: $\left|y(n)\right| \le M_x \cdot \sum_{k=-\infty}^{\infty} \left|h(k)\right|$

3.3. Analysis of DT Linear Invariant Systems

Stability of Linear Time-Invariant Systems

- **Now, if the system T is stable then the following holds true:**
  - $$|y(n)| \le M_x \cdot \sum_{k=-\infty}^{\infty} |h(k)| \le M_y$$
- **Multiplying both sides and further simplifying:**

$$\frac{1}{M_x} \cdot M_x \cdot \sum_{k=-\infty}^{\infty} |h(k)| \le M_y \cdot \frac{1}{M_x}$$

- **Defining new finite integer:** $$M_h \equiv M_y \cdot \frac{1}{M_x}$$
- **Therefore if the system T is stable and the input is bounded, then the following relation must be true:**

$$\sum_{k=-\infty}^{\infty} |h(k)| \le M_h < \infty$$

3.3. Analysis of DT Linear Invariant Systems

Stability of Linear Time-Invariant Systems

- Therefore a stable system's impulse response is absolutely summable:

$$\sum_{k=-\infty}^{\infty} |h(k)| \le M_h < \infty$$

- That summation may go to infinity (the system will be unstable) if any of the following occur:
  - The Summation is Infinite and does not converge
  - The Summation is finite but contains at least one term which is infinite
- If a system is FIR of length N then h(n) is absolutely summable (stable):
  - $\sum_{k=0}^{N} |h(k)| < \infty$ assuming $h(n) \neq \infty$ for all n

3.3. Analysis of DT Linear Invariant Systems

System with Finite Duration & Infinite Duration Impulse Response:

If impulse response, h(n), is of finite length, then the system is categorized as finite impulse response – FIR

- **Causal FIR System**

$$h(n) = 0 \qquad n < 0 \text{ and } n \geq M \qquad y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

If impulse response, h(n), is of infinite length, then the system is categorized as infinite impulse response – IIR

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

3.3. Analysis of DT Linear Invariant Systems

- **Recursive systems use previous outputs to compute present output:**

$$y(n) = H\{y(n-1),\ldots,x(n),x(n-1),\ldots,x(n+1),\ldots\}$$

- **Non-recursive systems do NOT use previous outputs to compute present output:**

$$y(n) = H\{x(n),x(n-1),\ldots,x(n+1),\ldots\}$$

3.3. Analysis of DT Linear Invariant Systems

Difference Equations:

- Convolution summation suggests a means for the realization of the system.
- In case of FIR systems, such a realization involves additions, multiplications, and a finite number of memory locations and hence FIR can be realized by Convolution summation.
- If the system IIR, its practical implementations as implied by convolution is clearly impossible, since it requires an infinite number of memory locations, multiplications and additions.
- IIR systems can be realized by difference equations.
- IIR systems are useful in a variety of practical applications, including implementation of digital filters and modeling of physical phenomenon and physical systems.

Example:

FIR $$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

IIR $$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

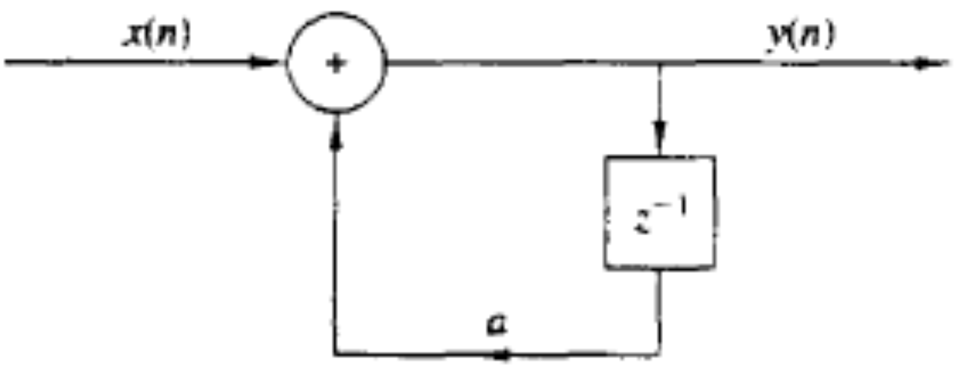
3.3. Analysis of DT Linear Invariant Systems

Difference Equations:

- System described by constant-coefficient linear difference equations are a subclass of the recursive and non-recursive system.
- Suppose we have recursive system with an input-output relation :

$$y(n) = ay(n-1) + x(n)$$

- The system has constant coefficient (independent of time) and can be realized by block diagram as follows:



3.3. Analysis of DT Linear Invariant Systems

Difference Equations:

LTI System is characterized by a linear Constant-Coefficient difference equation (LCCDE):

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=-L}^{M} b_k \cdot x(n-k)$$

Order of difference equation is N

Example:

$$y(n) = \frac{5}{6} \cdot y(n-1) - \frac{1}{6} \cdot y(n-2) + x(n)$$

n=0

$$y(0) = \frac{5}{6} \cdot y(-1) - \frac{1}{6} \cdot y(-2) + x(0)$$

Order: N=2

Mathematical Initial Conditions: $y(-1)$ **and** $y(-2)$

3.3. Analysis of DT Linear Invariant Systems

Difference Equations:

For a given difference equation, there exists as many initial conditions as the order of the difference equation (N initial conditions for a difference equation of order N).

- **Output sequence computed recursively:**

n = 1 $$y(1)=\frac{5}{6}\cdot y(0)-\frac{1}{6}\cdot y(-1)+x(1)$$

n = 2 $$y(2)=\frac{5}{6}\cdot y(1)-\frac{1}{6}\cdot y(0)+x(2)$$

n = 3 $$y(3)=\frac{5}{6}\cdot y(2)-\frac{1}{6}\cdot y(1)+x(3)$$

etc.

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Difference Equations:

Linear Constant-Coefficient difference equation (LCCDE):

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=-L}^{M} b_k \cdot x(n-k)$$

Total Solution: $y(n) = y_h(n) + y_p(n)$

The procedure for computing the solution of LCCDE is:

$y_h(n)$ is the **homogenous** solution

$y_p(n)$ is the **particular** solution

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Difference Equations:

The Homogenous Solution:

Now set input **x(n) = 0**

$$\sum_{k=0}^{N} a_k \cdot y(n-k) = 0$$

Assume exponential solution: $y(n) = \lambda^n$ where λ is constant to be determined.

Make substitutions into D.E.:

$$\sum_{k=0}^{N} a_k \cdot \lambda^{n-k} = 0$$

or

$$\lambda^{n-N} \cdot \left(\lambda^N + a_1 \cdot \lambda^{N-1} + a_2 \cdot \lambda^{N-2} + \ldots + a_{N-1} \cdot \lambda + a_N\right) = 0$$

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The Homogenous Solution:

❖ **From the previous:**

$$\lambda^{n-N}\cdot\left(\lambda^{N}+a_1\cdot\lambda^{N-1}+a_2\cdot\lambda^{N-2}+\ldots+a_{N-1}\cdot\lambda+a_N\right)=0$$

❖ **The polynomial in parenthesis is called characteristic polynomial**

Has N roots:

$$\lambda_1,\lambda_2,\ldots,\lambda_N$$

❖ **In practice, typically coefficients (a_1,a_2,…….a_N) are real.**

The general form of the homogenous solution is:

$$y_h(n)=C_1\cdot\lambda_1^n+C_2\cdot\lambda_2^n+C_3\cdot\lambda_3^n+\ldots C_N\cdot\lambda_N^n$$

Recursive Difference Equations Stability:

Assuming N-distinct roots, the homogeneous solution is written as:

$$y_h(n) = C_1 \cdot \lambda_1^n + C_2 \cdot \lambda_2^n + C_3 \cdot \lambda_3^n + \dots C_N \cdot \lambda_N^n$$

If $$\left.\begin{array}{c} y_h(n) \to 0 \\ n \to \infty \end{array}\right\} \begin{array}{c} system\ \ is \\ stable \end{array}$$

For $y_h(n) \to 0$ **as** $n \to \infty$ **then all the roots must satisfy the following relation:**

$$|\lambda_i| < 1, for\ \ all\ \ i = 1, 2, \dots N$$

Recursive Difference Equations Stability Example:

- Consider: $$y(n)=\frac{5}{6}\cdot y(n-1)-\frac{1}{6}\cdot y(n-2)+x(n)$$
- Assume an exponential solution: $$y(n)=\lambda^n$$
  - Where λ is a constant to be determined
- Plug into difference equation: $$\frac{1}{6}\cdot\lambda^{n-2}-\frac{5}{6}\cdot\lambda^{n-1}+\lambda^n=0$$
  -
- Note above must = 0 (input = 0)
- Factor:

$$\lambda^{n-2}\cdot\left[\lambda^2-\frac{5}{6}\cdot\lambda^1+\frac{1}{6}\right]=0$$

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Recursive Difference Equations Stability Example:

- Characteristic Equation: $$\lambda^2 - \frac{5}{6}\cdot\lambda + \frac{1}{6} = 0$$
- Eliminate Fractions: $$6\cdot\lambda^2 - 5\cdot\lambda + 1 = 0$$
- Quadratic Equation: $$a\cdot\lambda^2 + b\cdot\lambda + c = 0$$
  - Solution: $$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4\cdot a\cdot c}}{2\cdot a}$$
- Our Example: $$\lambda_{1,2} = \frac{5 \pm \sqrt{25-24}}{12} = \frac{5 \pm 1}{12} = \frac{1}{2}, \frac{1}{3}$$

3.3. Analysis of DT Linear Invariant Systems

Recursive Difference Equations Stability Example:

- Two Solutions: $y_1(n)=\left(\frac{1}{2}\right)^n, y_2(n)=\left(\frac{1}{3}\right)^n$
- The Solution is Linear Combination of two solutions:
  - $y(n)=C_1\cdot y_1(n)+C_2\cdot y_2(n)$
  -

$$y(n)=C_1\cdot\left(\frac{1}{2}\right)^n+C_2\cdot\left(\frac{1}{3}\right)^n$$

- Note C1 and C2 are arbitrary constants computed from initial conditions

Recursive Difference Equations Stability Example:

- $y_h(n) =$ homogeneous solution
- If $\left.\begin{array}{c} y_h(n) \to \infty \\ n \to \infty \end{array}\right\}$ $\begin{array}{l} \textit{system is} \\ \textit{unstable} \end{array}$
- If $\left.\begin{array}{c} y_h(n) \to 0 \\ n \to \infty \end{array}\right\}$ $\begin{array}{l} \textit{system is} \\ \textit{stable} \end{array}$
- For → $y(n) = C_1 \cdot \left(\frac{1}{2}\right)^n + C_2 \cdot \left(\frac{1}{3}\right)^n$
  - System is stable

3.3. Analysis of DT Linear Invariant Systems

The Particular Solution:

The particular solution $y_p(n)$ is required to satisfy the DE for the specific input signal x(n), n>0:

$$\sum_{k=0}^{N} a_k y_p(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad a_0 = 1$$

For example if $x(n) = a^n u(n)$ the particular solution will be in the form:

$$y_p(n) = Ca^n u(n)$$

3.3. Analysis of DT Linear Invariant Systems

The Particular Solution:

The particular solution to a DE for different several inputs:

Term in $x(n)$	Particular Solution
C	C_1
Cn	$C_1 n + C_2$
Ca^n	$C_1 a^n$
$C\cos(n\omega_0)$	$C_1\cos(n\omega_0) + C_2\sin(n\omega_0)$
$C\sin(n\omega_0)$	$C_1\cos(n\omega_0) + C_2\sin(n\omega_0)$
$Ca^n\cos(n\omega_0)$	$C_1 a^n\cos(n\omega_0) + C_2 a^n\sin(n\omega_0)$
$C\delta(n)$	None

3.3. Analysis of DT Linear Invariant Systems

Example:

Determine the total solution for n ≥ 0 of a DT system characterized by the following difference equation:

$$y(n) - 0.25y(n-2) = x(n)$$

For x(n) = u(n) assuming the initial conditions of y(-1) = 1 and y(-2) = 0.

Solution:

Particular solution:

For x(n) = u(n)

$$y_p(n) = C_1$$

Substitute this solution into the DE

$$C_1 - 0.25C_1 = 1$$

$$C_1 = \frac{1}{1-0.25} = \frac{4}{3}$$

Example: (cont.)

Homogenous solution:

set $y(n)=\lambda^n$ substitute $\lambda^n - 0.25\lambda^{(n-2)} = 0$

$$\lambda^{(n-2)}(\lambda^2 - 0.25) = 0$$

$$(\lambda - 0.5)(\lambda + 0.5) = 0$$

then

$$\lambda_1 = 0.5$$

$$\lambda_2 = -0.5$$

$$y_1(n) = (0.5)^n$$

$$y_2(n) = (-0.5)^n$$

$$y_h(n) = A_1 y_1(n) + A_2 y_2(n)$$

The homogenous solution $y_h(n) = A_1(0.5)^n + A_2(-0.5)^n$

3.3. Analysis of DT Linear Invariant Systems

Example: (cont.)

Total Solution:

$$y(0) - 0.25y(-2) = x(0) = 1$$
$$y(1) - 0.25y(-1) = x(1) = 1$$

The total solution is:

$$y(n) = \tfrac{4}{3} + A_1(0.5)^n + A_2(-0.5)^n \qquad n \geq 0$$

at n = 0 and n = 1

$$y(0) = \tfrac{4}{3} + A_1 + A_2 = 1$$
$$y(1) = \tfrac{4}{3} + \tfrac{1}{2}A_1 - \tfrac{1}{2}A_2 = 1$$

$$A_1 = -\tfrac{1}{2} \qquad A_2 = \tfrac{1}{6}$$

The solution is:

$$y(n) = \tfrac{4}{3} - (0.5)^{n+1} + \tfrac{1}{6}(-0.5)^n \qquad n \geq 0$$

Digital Signal Processing

Discrete-Time Signals & Systems

3.3. Analysis of DT Linear Invariant Systems

Difference Equations:

Zero-Input & Zero-State Response:

An alternate approach to determining the total solution of DE.

$$y(n) = y_{zi}(n) + y_{zs}(n)$$

$y_{zi}(n)$: zero-input response

$y_{zs}(n)$: zero-state response

- **$y_{zi}(n)$ is obtained by solving DE by setting the input $x(n) = 0$.**

- **$y_{zs}(n)$ is obtained by solving DE by applying the specified input with all initial conditions set to zero (0).**

Discrete-Time Signals & Systems

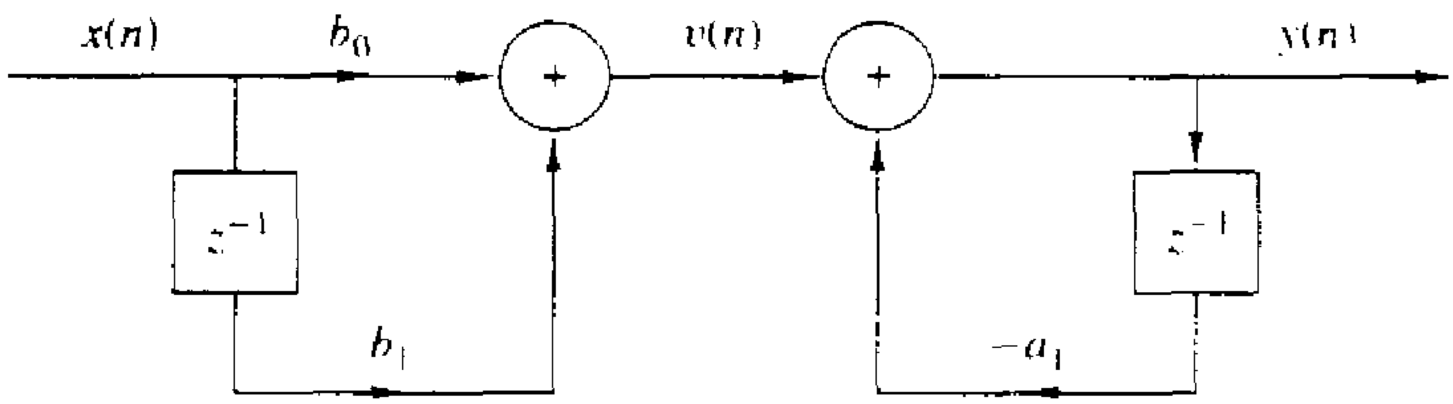
3.4. implementation of DT Systems

Structure for the Realization of LTI Systems:

Here, the LCCDE structure for the realization of systems is described, additional structures for these system will introduce in later chapters.

Consider the first-order system:

$$y(n) = -a_1 y(n-1) + b_0 x(n) + b_1 x(n-1)$$



Direct Form I Structure

3.4. implementation of DT Systems

Structure for the Realization of LTI Systems:

The previous system can be viewed as 2 LTI systems in cascade, the first is a non-recursive system described by the equation:

$$v(n) = b_0 x(n) + b_1 x(n-1)$$

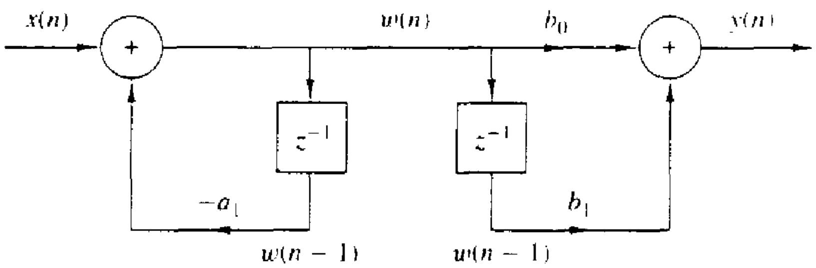
Whereas the second is a recursive system described by the equation:

$$y(n) = -a_1 y(n-1) + v(n)$$

3.4. implementation of DT Systems

Structure for the Realization of LTI Systems:

If we interchange the order of the recursive and non-recursive systems, we obtain an alternative structure for the realization of the system described previously:



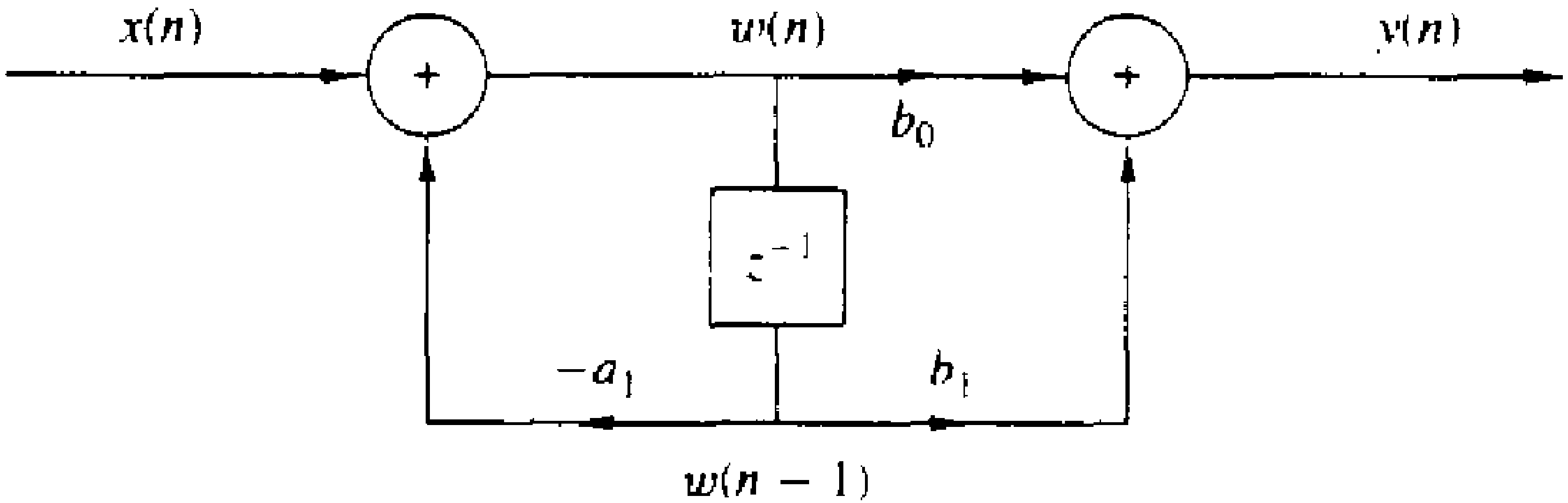
$$y(n) = b_0 w(n) + b_1 w(n-1)$$

$$w(n) = -a_1 w(n-1) + x(n)$$

3.4. implementation of DT Systems

Structure for the Realization of LTI Systems:

Minimizing the 2 delay in the structure form I to 1 delay in structure form II.

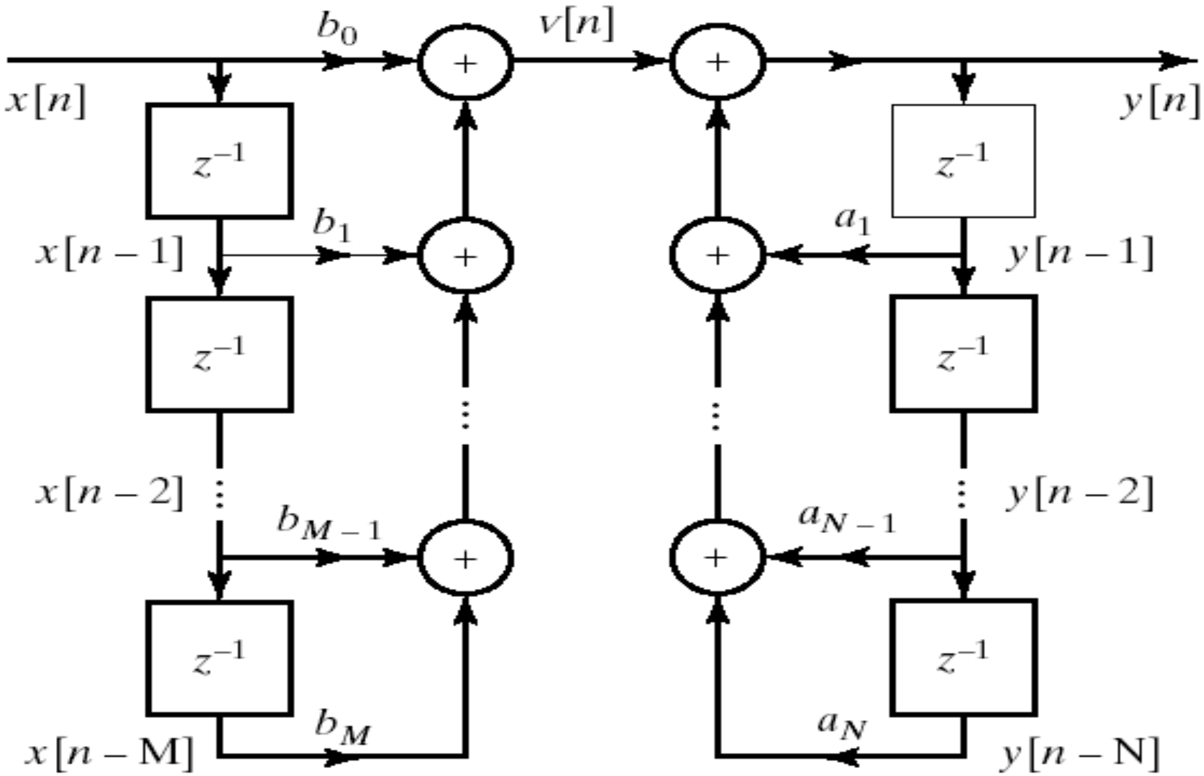


Direct Form II Structure

3.4. implementation of DT Systems

Structure for the Realization of LTI Systems:

In General $\sum_{k=0}^{N} \hat{a}_k y[n-k] = \sum_{k=0}^{M} \hat{b}_k x[n-k]$ **or** $y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$



Direct Form I Structure

Digital Signal Processing

Discrete-Time Signals & Systems

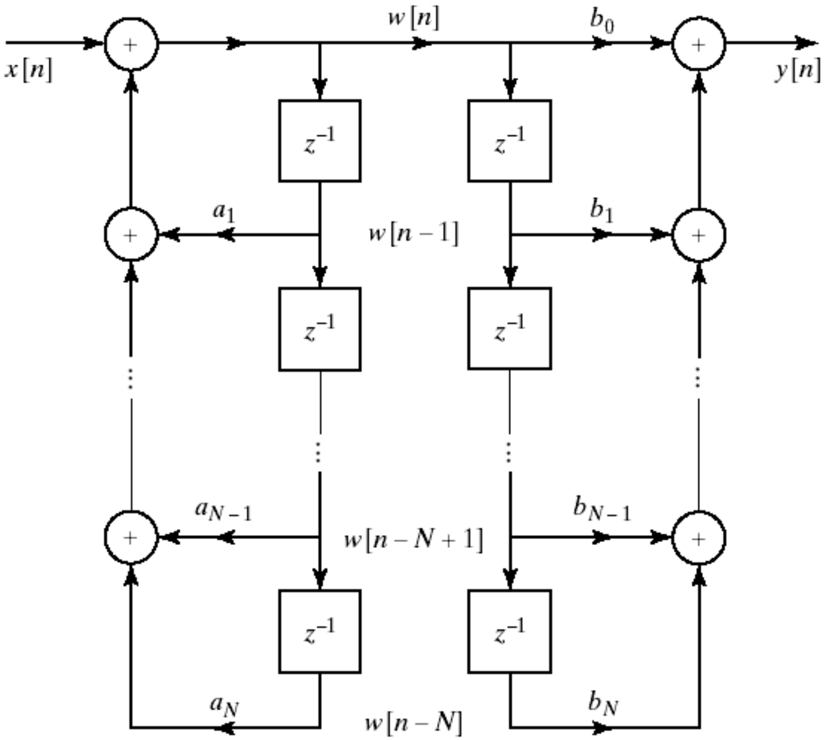
3.4. implementation of DT Systems

Structure for the Realization of LTI Systems:

We can change the order of the cascade systems.

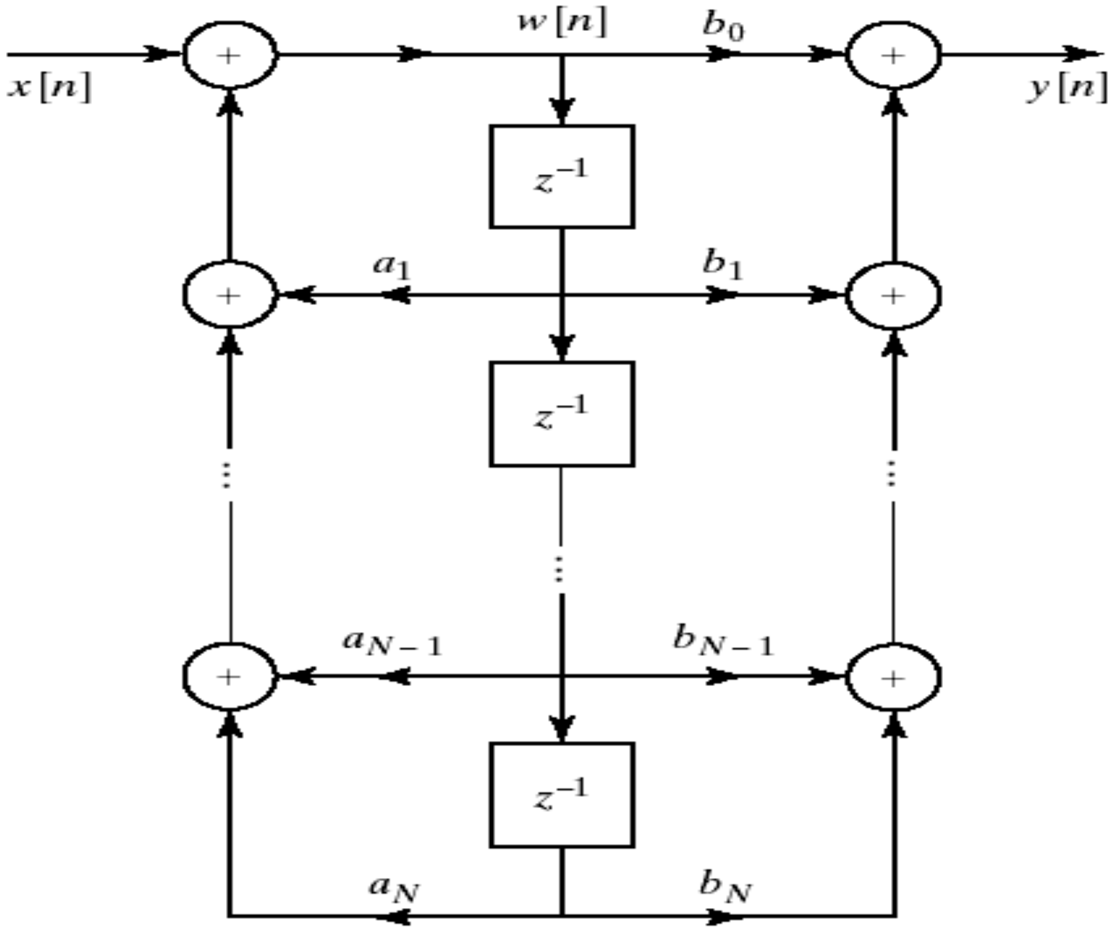
$$w[n] = \sum_{k=1}^{N} a_k w[n-k] + x[n]$$

$$y[n] = \sum_{k=0}^{M} b_k w[n-k]$$



3.4. implementation of DT Systems

Structure for the Realization of LTI Systems:



Direct Form II Structure

End!!!