TN 412: DIGITAL SIGNAL PROCESSING

The z-Transform and its Application to Analysis of LTI systems

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- Introduction
- *z*-Transform
- Region of Convergence
- Zeros and Poles
- Important *z*-Transform Pairs
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- Inverse *z*-Transform
- Application of Z Transform in LTI systems

The z-Transform

Introduction

Why z-Transform?

- Transform techniques are an important tool in analysis of signals and LTI systems.
- Z-transform plays the same role in the analysis of discrete time signals and LTI systems as Laplace transform does in the analysis of continuous-time signals and LTI systems.
- Z-transform provides us with a means of characterizing an LTI system, and its response to various signals, by its pole-zero locations.
- From mathematical point of view, is an alternative representation of a signal.

Z-Transform: Definition

- The z-transform of sequence $x(n)$ is defined as the power series

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- Let $z = e^{-j\omega}$.

Fourier Transform

$$\Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Z-Transform: Definition

- The *z*-transform of sequence *x*(*n*) is defined as the power series

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- ❖ The X(z) is sometimes called direct z-transform because it transforms the time-domain signal x(n) into its complex-plane representation .
- ❖ Since the z-transform is an infinite power series, it exists only for those values of z for which this series converges.
- ❖ Region of Convergence (ROC) of X(z) is a set of all values of z for which X(z) attains a finite value.

Z-Transform

Example

Determine the z-transforms of the following *finite-duration* signals.

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

(b) $x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

(c) $x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\}$

(d) $x_4(n) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\}$

(e) $x_5(n) = \delta(n)$

(f) $x_6(n) = \delta(n - k), k > 0$

(g) $x_7(n) = \delta(n + k), k > 0$

Z-Transform

Example

(a) $X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$, ROC: entire z-plane except $z = 0$

(b) $X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$, ROC: entire z-plane except $z = 0$ and $z = \infty$

(c) $X_3(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$, ROC: entire z-plane except $z = 0$

(d) $X_4(z) = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}$, ROC: entire z-plane except $z = 0$ and $z = \infty$

(e) $X_5(z) = 1$[i.e., $\delta(n) \overset{z}{\longleftrightarrow} 1$], ROC: entire z-plane

(f) $X_6(z) = z^{-k}$[i.e., $\delta(n-k) \overset{z}{\longleftrightarrow} z^{-k}$], $k > 0$, ROC: entire z-plane except $z = 0$

(g) $X_7(z) = z^k$[*i.e.*, $\delta(n+k) \overset{z}{\longleftrightarrow} z^k$], $k > 0$, ROC: entire z-plane except $z = \infty$

It is easily seen that the RC of a finite duration signal is the entire z-plane, except possibly the points z=0 and/or z=∞. These points are excluded, because z^k(k > 0) becomes unbounded for z=∞ and z^{-k}(k > 0)
The coefficient of z^{-n}, in a given transform, is the value of a signal at time n.

Z-Transform

Example

Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Solution From the definition (3.1.1) we have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

If $|\alpha z^{-1}| < 1$ or equivalently, $|z| > |\alpha|$, this power series converges to $1/(1 - \alpha z^{-1})$.

Z-Transform

Example

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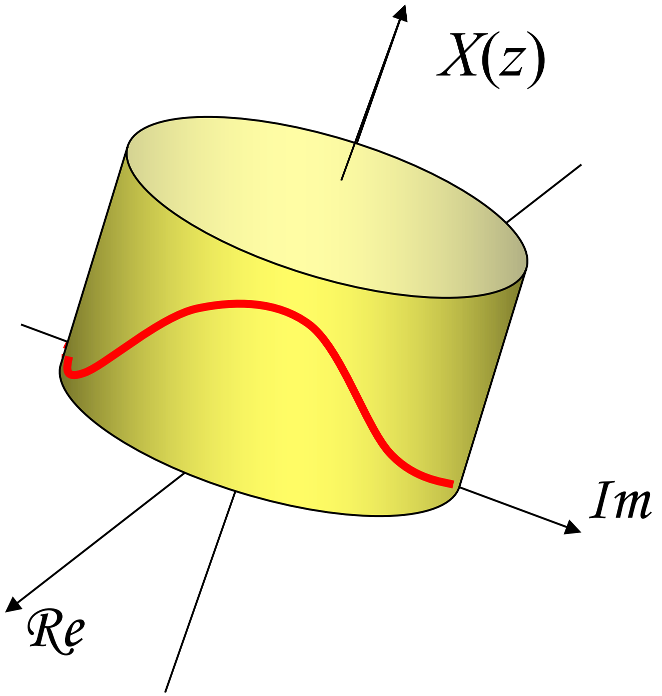
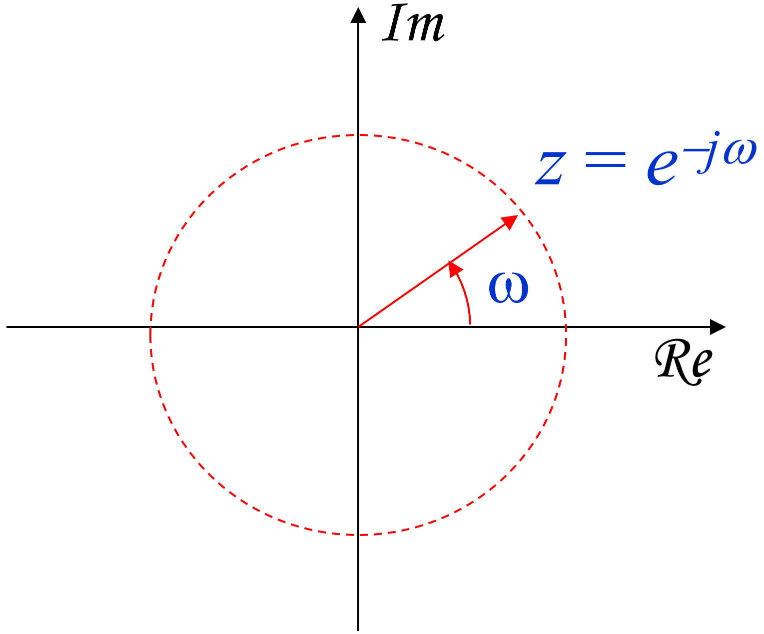
z-Plane

- **Once the poles and zeros have been found for a given Z-Transform, they can be plotted onto the Z-Plane.**
- **The z-plane is a complex plane with an imaginary and real axis referring to the complex-valued variable z.**
- **The position on the complex plane is given by $re^{i\omega}$ and the angle from the positive, real axis around the plane is denoted by ω**
- **When mapping poles and zeros onto the plane, poles are denoted by an "x" and zeros by an "o".**

z-Plane

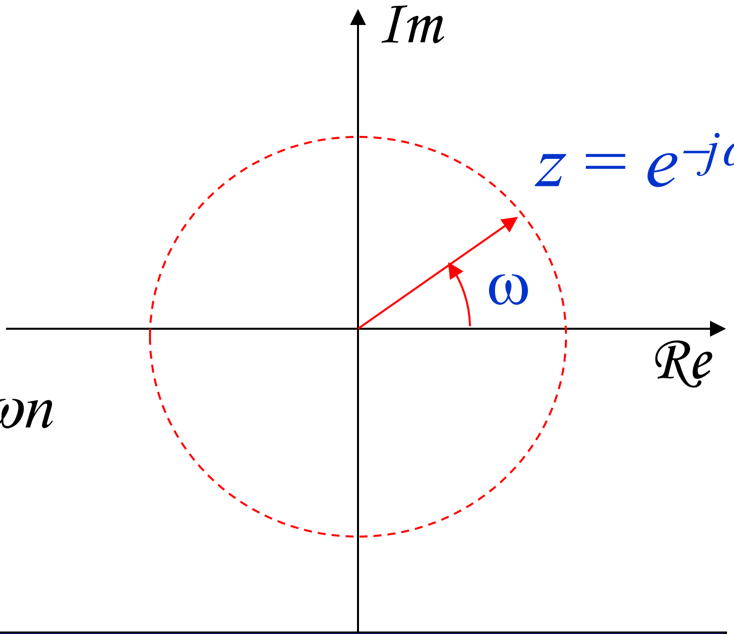




z-Plane

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$



Fourier Transform is to *evaluate z-transform on a unit circle.*

The z-Transform

Region of Convergence

Definition

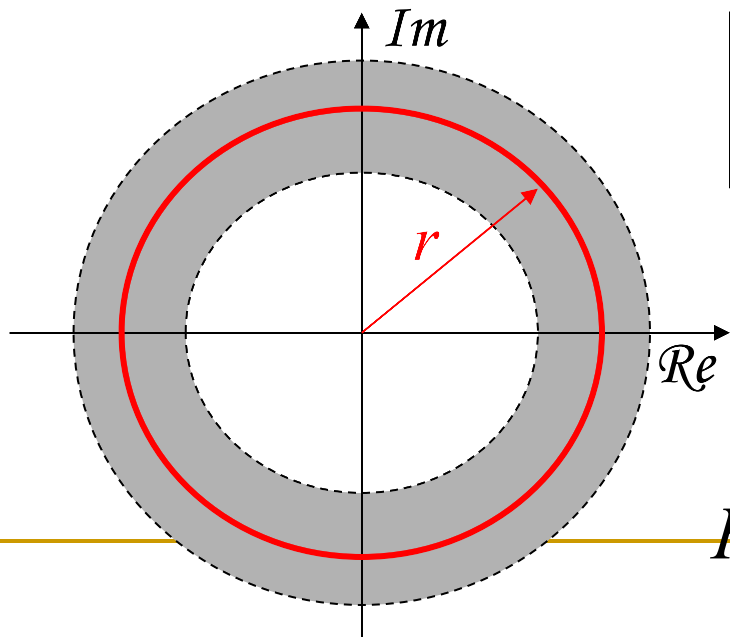
- Give a sequence, the set of values of z for which the z-transform converges, i.e., $|X(z)|<\infty$, is called the region of convergence.

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| = \sum_{n=-\infty}^{\infty} |x(n)| |z|^{-n} < \infty$$

ROC is centered on origin and consists of a set of rings.

Example: Region of Convergence

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| = \sum_{n=-\infty}^{\infty} |x(n)| |z|^{-n} < \infty$$



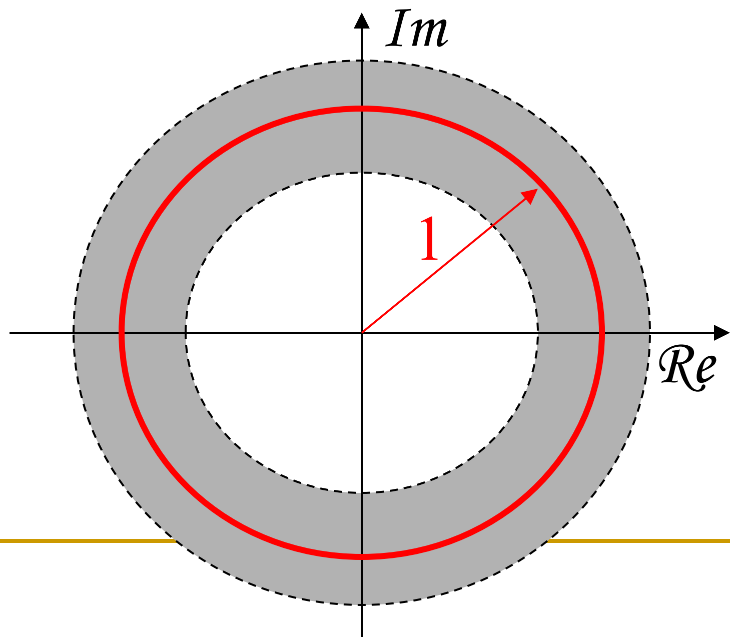
ROC is an annual ring centered on the origin.

$$R_{x-} < |z| < R_{x+}$$

$$ROC = \{ z = re^{j\omega} \mid R_{x-} < r < R_{x+} \}$$

Stable Systems

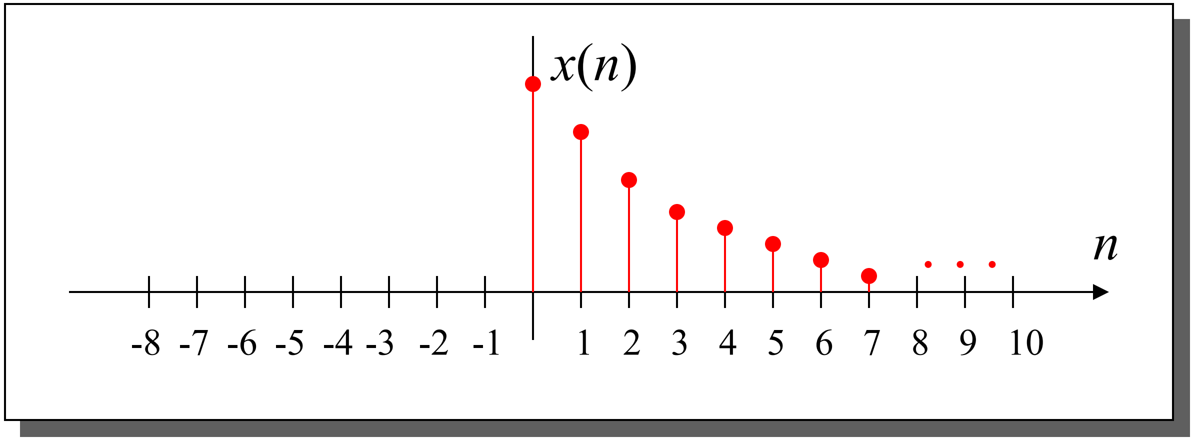
- A stable system requires that its Fourier transform is uniformly convergent.



- Fact: Fourier transform is to evaluate z-transform on a unit circle.
- A stable system requires the ROC of z-transform to include the unit circle.

Example: A right sided Sequence

$$x(n) = a^n u(n)$$



Example: A right sided Sequence

$$\boxed{x(n) = a^n u(n)}$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}| < \infty \Rightarrow |az^{-1}| < 1$$

$$\Rightarrow |z| > |a|$$

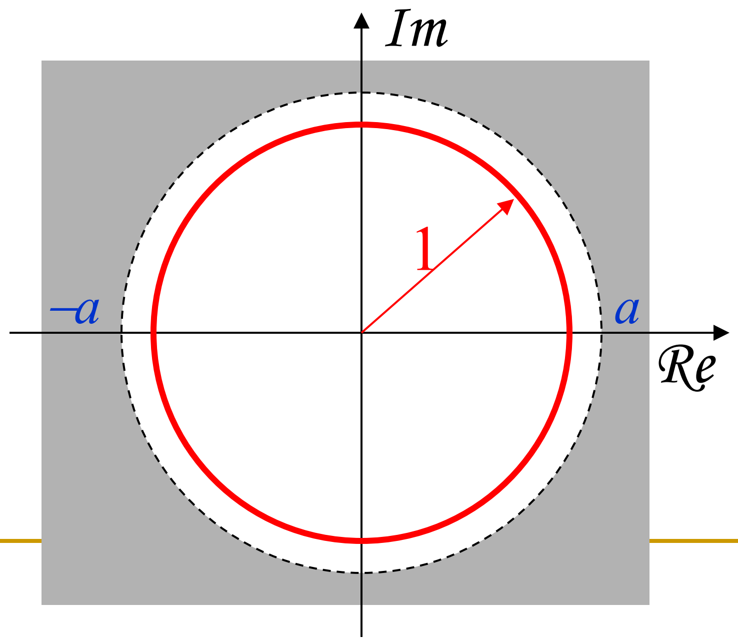
$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

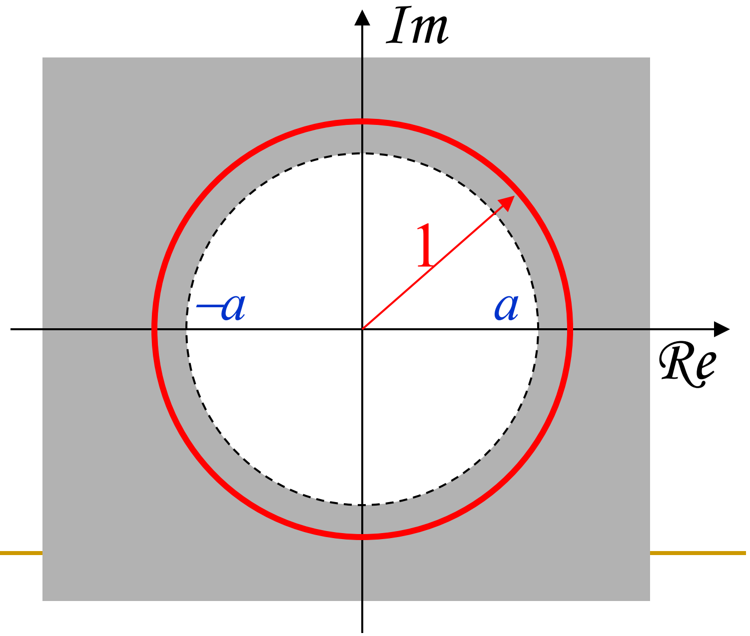
$$|z| > |a|$$

Example: A right sided Sequence ROC for $x(n)=a^n u(n)$

$$X(z) = \frac{z}{z-a}, \qquad |z| > |a|$$

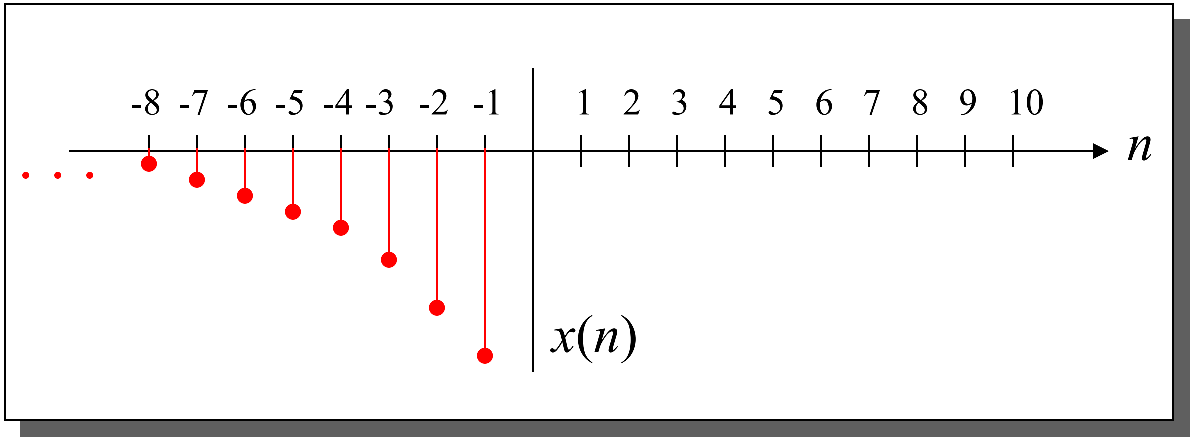
Which one is stable?





Example: A left sided Sequence

$$x(n) = -a^n u(-n-1)$$



Example: A left sided Sequence

$$\boxed{x(n) = -a^n u(-n-1)}$$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} a^{-n} z^n$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |a^{-1}z| < \infty \Rightarrow |a^{-1}z| < 1$$

$$\Rightarrow |z| < |a|$$

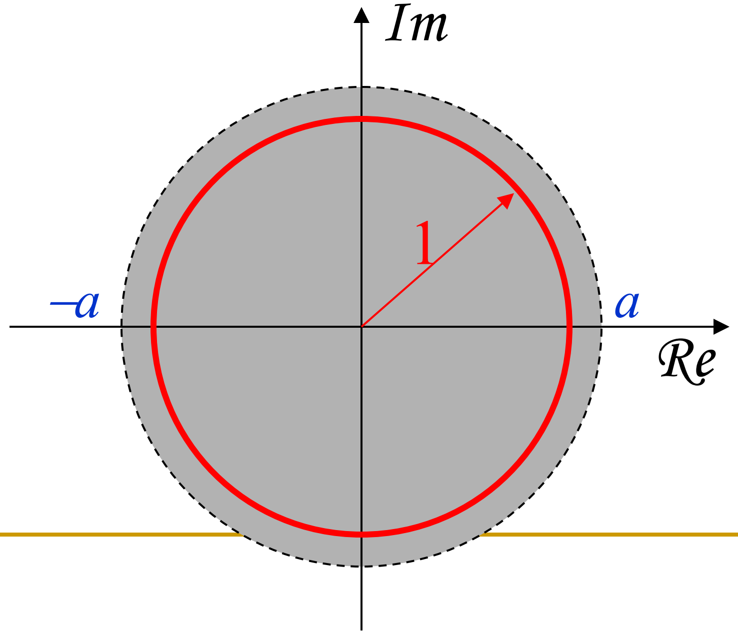
$$X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{z}{z-a}$$

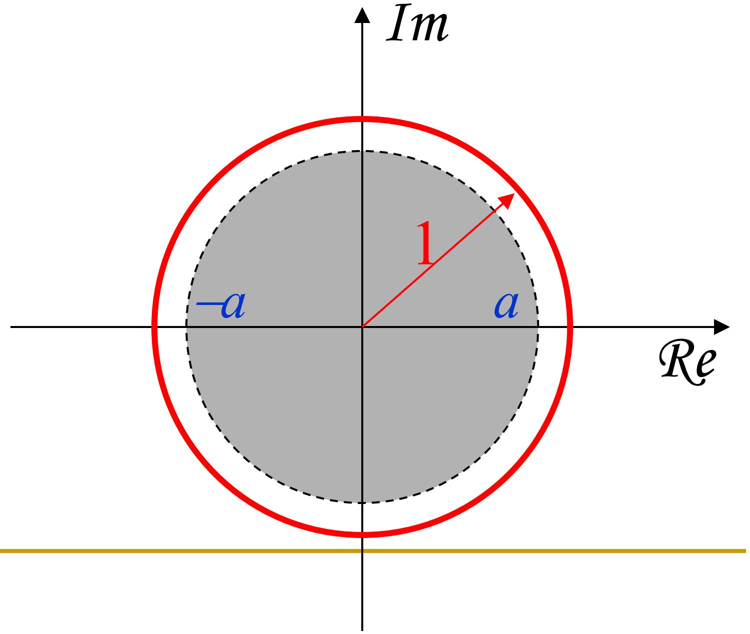
$$|z| < |a|$$

Example: A left sided Sequence ROC for $x(n) = -a^n u(-n-1)$

$$X(z) = \frac{z}{z-a}, \qquad |z| < |a|$$

Which one is stable?





The z-Transform

Zeros and Poles

Represent z-transform as a Rational Function

$$X(z) = \frac{P(z)}{Q(z)}$$

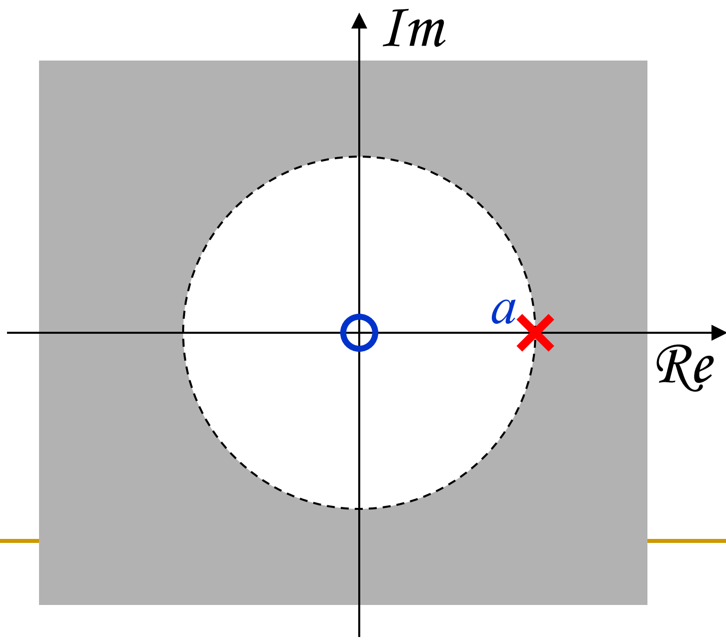
where $P(z)$ and $Q(z)$ are polynomials in z.

Zeros: The values of z's such that $X(z) = 0$

Poles: The values of z's such that $X(z) = \infty$

Example: A right sided Sequence

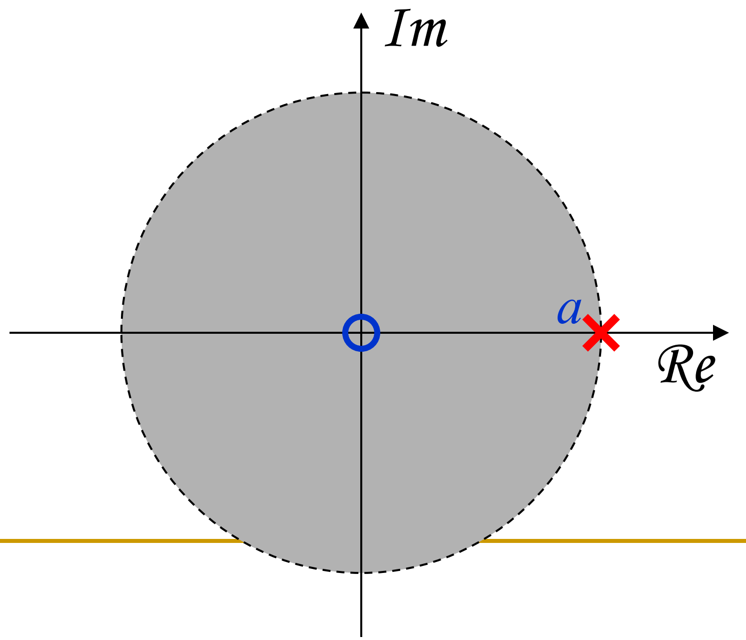
$$x(n) = a^n u(n) \Rightarrow X(z) = \frac{z}{z-a}, \qquad |z| > |a|$$



ROC is bounded by the pole and is the exterior of a circle.

Example: A left sided Sequence

$$x(n) = -a^n u(-n-1) \quad \Rightarrow \quad X(z) = \frac{z}{z-a}, \qquad |z| < |a|$$

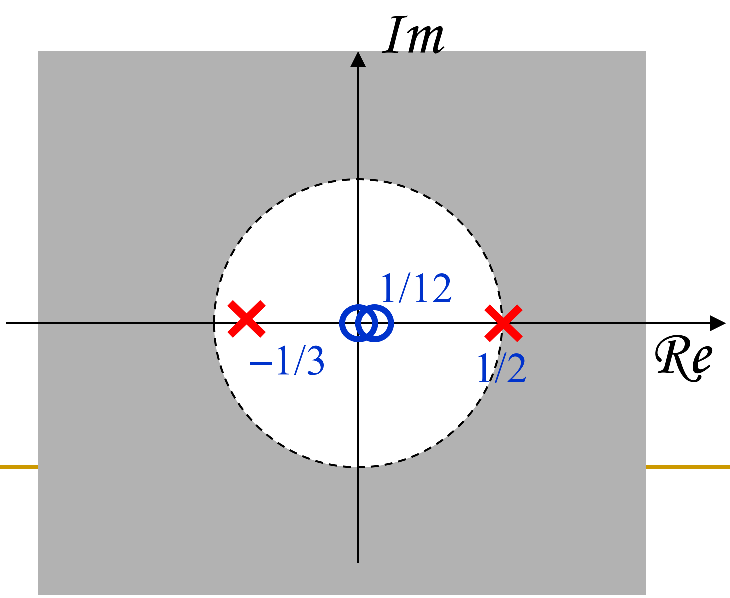


ROC is bounded by the pole and is the interior of a circle.

Example: Sum of Two Right Sided Sequences

$$x(n) = (\tfrac{1}{2})^n u(n) + (-\tfrac{1}{3})^n u(n)$$

$$\Rightarrow \quad X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

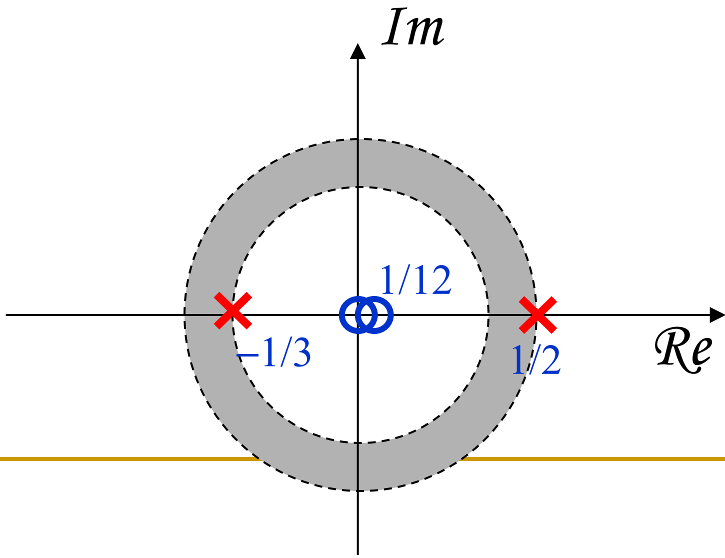


ROC is bounded by poles and is the exterior of a circle.

Example: A Two Sided Sequence

$$x(n) = (-\tfrac{1}{3})^n u(n) - (\tfrac{1}{2})^n u(-n-1)$$

$$X(z) = \frac{z}{z+\frac{1}{3}} + \frac{z}{z-\frac{1}{2}} \qquad = \frac{2z(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})}$$



ROC is bounded by poles and is a ring.

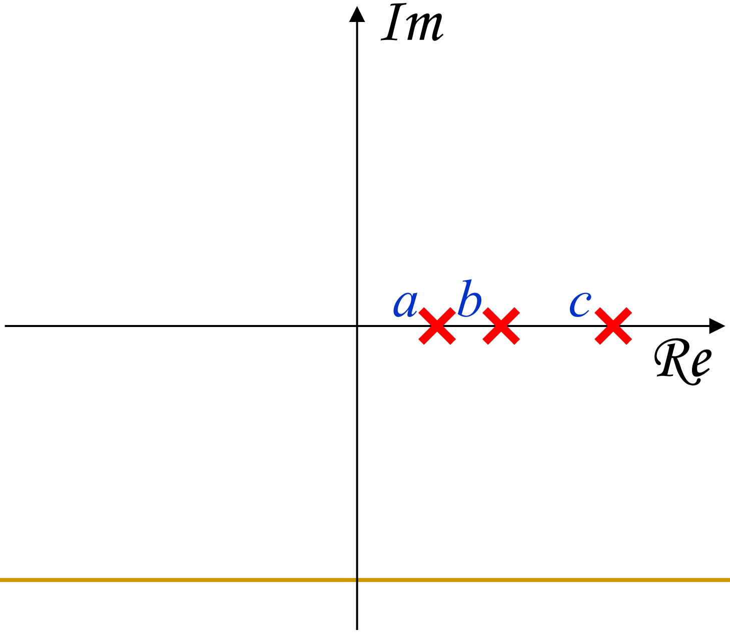
Properties of ROC

- A ring or disk in the z-plane centered at the origin.
- The Fourier Transform of $x(n)$ is converge absolutely iff the ROC includes the unit circle.
- The ROC cannot include any poles
- Finite Duration Sequences: The ROC is the entire z-plane except possibly $z=0$ or $z=\infty$.
- Right sided sequences: The ROC extends outward from the outermost finite pole in $X(z)$ to $z=\infty$.
- Left sided sequences: The ROC extends inward from the innermost nonzero pole in $X(z)$ to $z=0$.

More on Rational z-Transform

Consider the rational z-transform with the pole pattern:

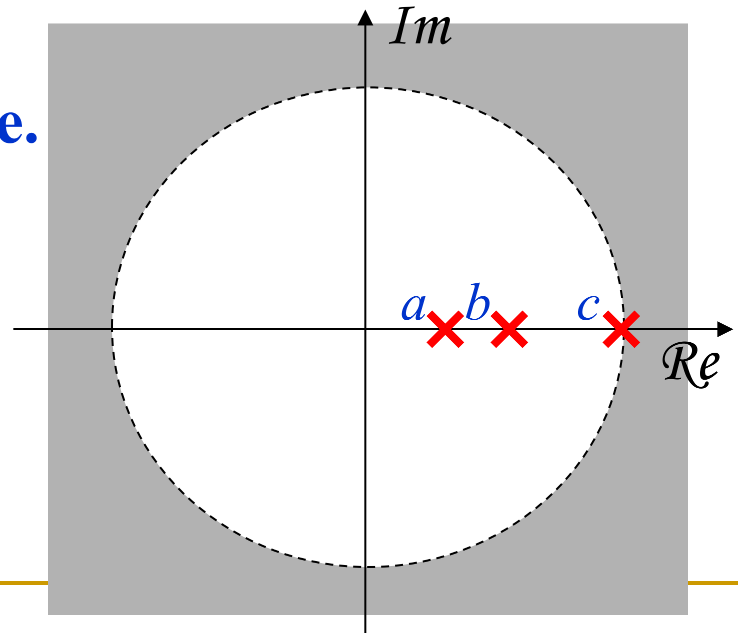
Find the possible ROC's



More on Rational *z*-Transform

Consider the rational z-transform with the pole pattern:

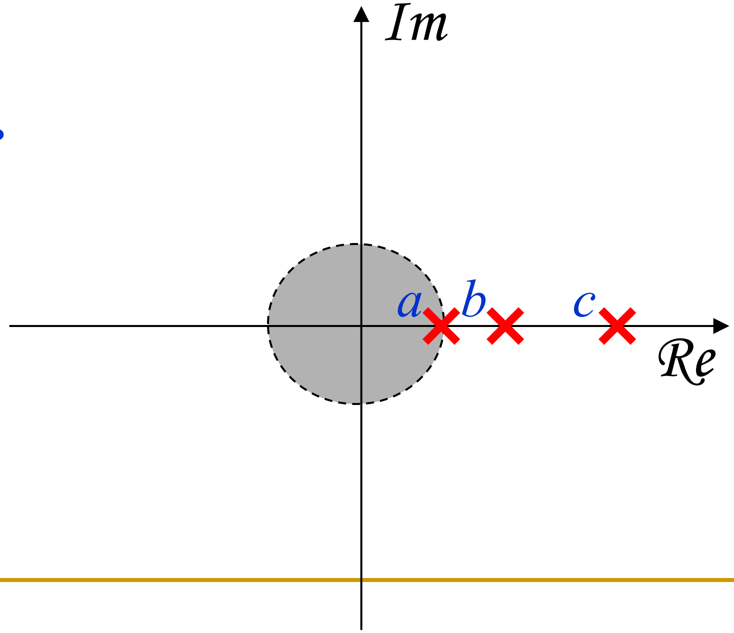
Case 1: A right sided Sequence.



More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

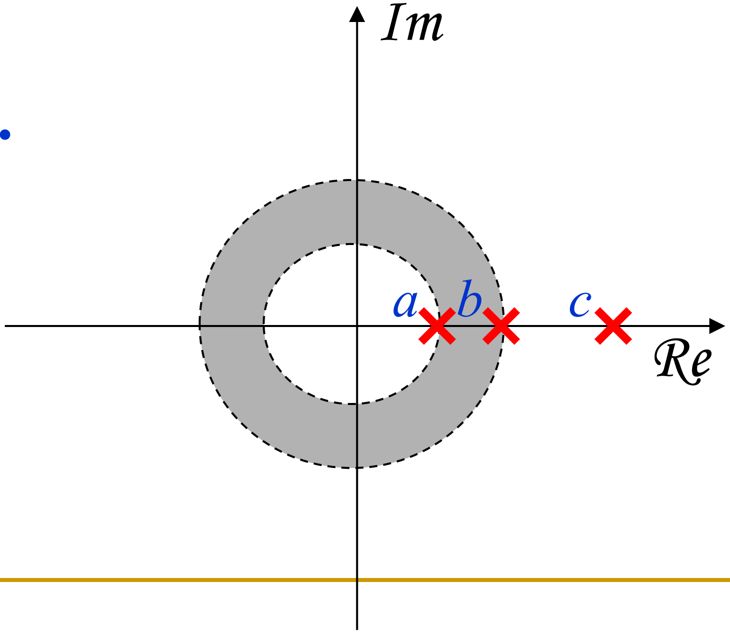
Case 2: A left sided Sequence.



More on Rational z-Transform

Consider the rational z-transform with the pole pattern:

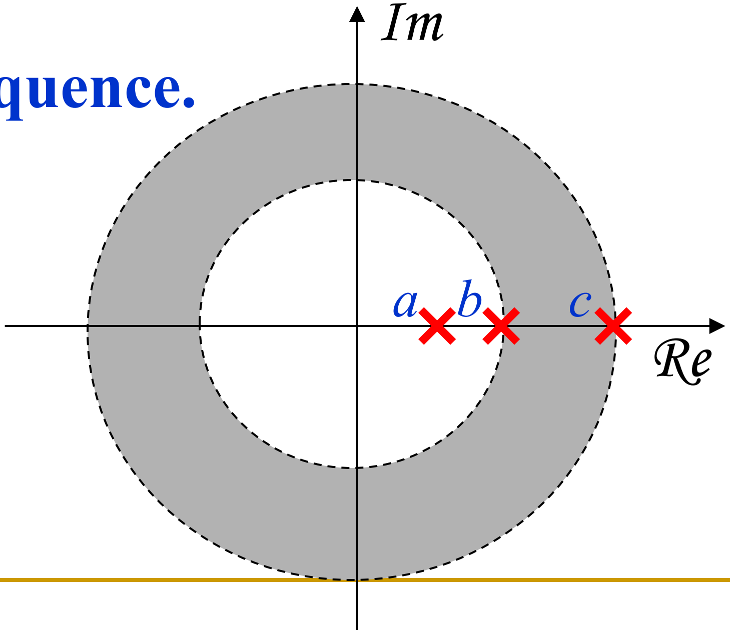
Case 3: A two sided Sequence.



More on Rational z-Transform

Consider the rational z-transform with the pole pattern:

Case 4: Another two sided Sequence.



The z-Transform

Important

z-Transform Pairs

Z-Transform Pairs

Sequence	z-Transform	ROC
$\delta(n)$	1	All z
$\delta(n-m)$	z^{-m}	All z except 0 (if m>0) or ∞ (if m<0)
$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z \rvert > 1$
$-u(-n-1)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z \rvert < 1$
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z \rvert > \lvert a \rvert$
$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z \rvert < \lvert a \rvert$

Z-Transform Pairs

Sequence	z-Transform	ROC
$[\cos\omega_0 n]u(n)$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
$[\sin\omega_0 n]u(n)$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
$[r^n\cos\omega_0 n]u(n)$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>r$
$[r^n\sin\omega_0 n]u(n)$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>r$
$\begin{cases} a^n & 0\le n\le N-1 \\ 0 & \text{otherwise}\end{cases}$	$\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert>0$

The z-Transform

z-Transform Theorems and Properties

Linearity

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathcal{Z}[y(n)] = Y(z), \qquad z \in R_y$$

$$\mathcal{Z}[ax(n) + by(n)] = aX(z) + bY(z), \qquad \underbrace{z \in R_x \cap R_y}_{\text{Overlay of the above two ROC's}}$$

Overlay of the above two ROC's

Shift

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathcal{Z}[x(n+n_0)] = z^{n_0} X(z) \qquad z \in R_x$$

Multiplication by an Exponential Sequence

$$\mathcal{Z}[x(n)] = X(z), \qquad R_{x-} < |z| < R_{x+}$$

$$\mathcal{Z}[a^n x(n)] = X(a^{-1}z) \qquad z \in |a| \cdot R_x$$

Differentiation of $X(z)$

$$Z[x(n)] = X(z), \qquad z \in R_x$$

$$Z[nx(n)] = -z\frac{dX(z)}{dz} \qquad z \in R_x$$

Conjugation

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathcal{Z}[x^*(n)] = X^*(z^*) \qquad z \in R_x$$

Reversal

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathcal{Z}[x(-n)] = X(z^{-1}) \qquad z \in 1/R_x$$

Real and Imaginary Parts

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathcal{Re}[x(n)] = \tfrac{1}{2}[X(z) + X^*(z^*)] \qquad z \in R_x$$

$$Im[x(n)] = \tfrac{1}{2j}[X(z) - X^*(z^*)] \qquad z \in R_x$$

Initial Value Theorem

$$x(n) = 0, \qquad \text{for } n < 0$$

$$x(0) = \lim_{z \to \infty} X(z)$$

Convolution of Sequences

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathcal{Z}[y(n)] = Y(z), \qquad z \in R_y$$

$$\mathcal{Z}[x(n) * y(n)] = X(z)Y(z) \qquad z \in R_x \cap R_y$$

Convolution of Sequences

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k) y(n-k)$$

$$Z[x(n) * y(n)] = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x(k) y(n-k) \right) z^{-n}$$

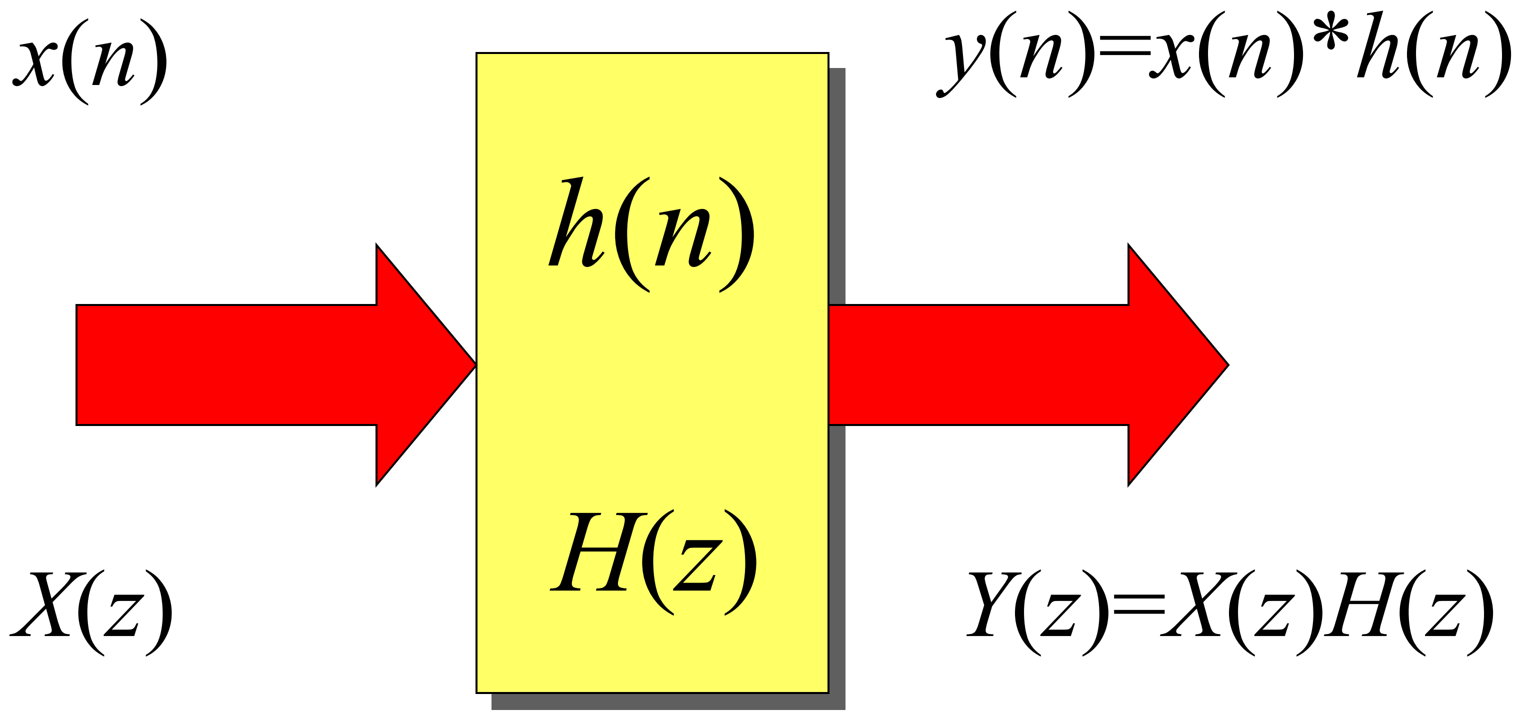
$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k) z^{-n} \quad = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= X(z) Y(z)$$

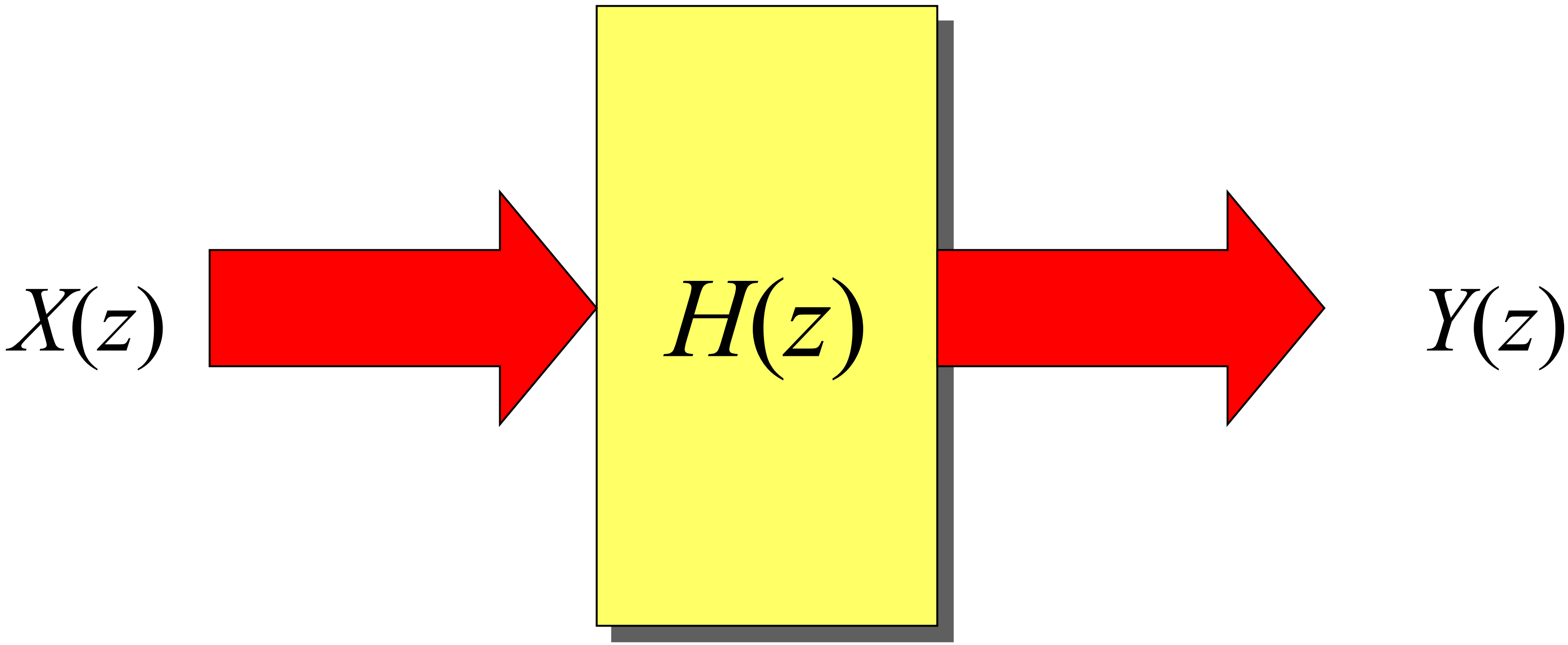
The z-Transform

System Function

Shift-Invariant System



Shift-Invariant System



$$H(z) = \frac{Y(z)}{X(z)}$$

N^{th}-Order Difference Equation

$$\boxed{\sum_{k=0}^{N} a_k y(n-k) = \sum_{r=0}^{M} b_r x(n-r)}$$

$$\Rightarrow \quad Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{r=0}^{M} b_r z^{-r}$$

$$\boxed{H(z) = \sum_{r=0}^{M} b_r z^{-r} \Big/ \sum_{k=0}^{N} a_k z^{-k}}$$

Representation in Factored Form

Contributes *poles* at 0 and *zeros* at c_r

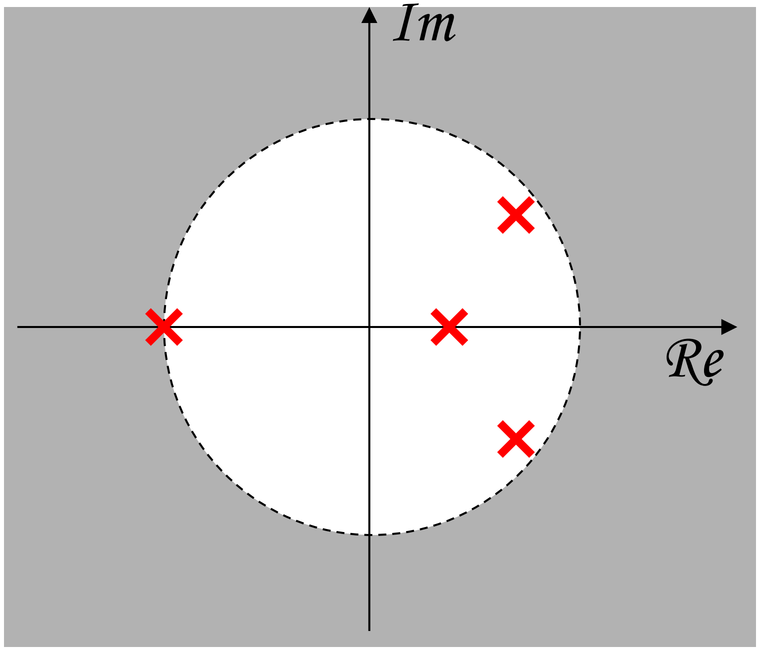
$$H(z) = \frac{A\prod_{r=1}^{M}(1 - c_r z^{-1})}{\prod_{k=1}^{N}(1 - d_r z^{-1})}$$

Contributes *zeros* at 0 and *poles* at d_r

Stable and Causal Systems

Causal Systems : ROC extends outward from the outermost pole.

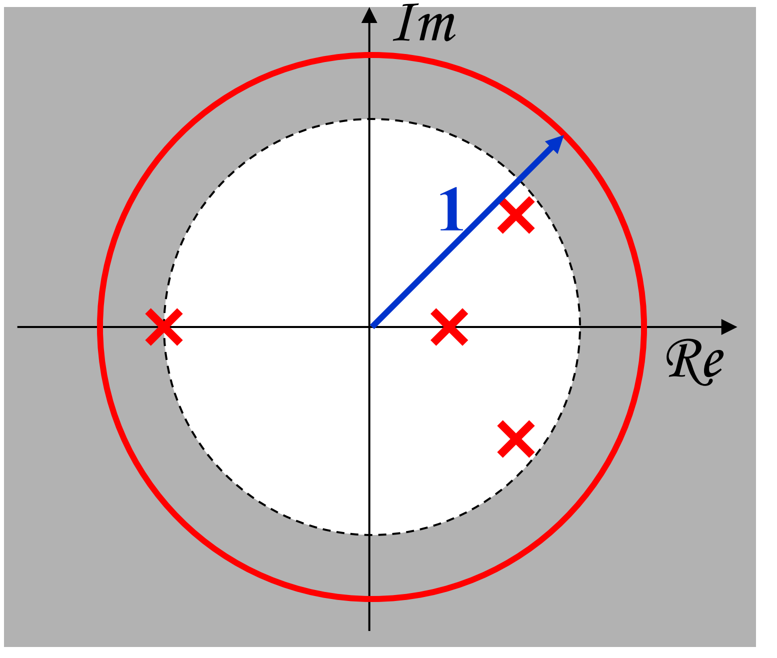
$$H(z) = \frac{A\prod_{r=1}^{M}(1-c_r z^{-1})}{\prod_{k=1}^{N}(1-d_r z^{-1})}$$



Stable and Causal Systems

Stable Systems : ROC includes the unit circle.

$$H(z) = \frac{A\prod_{r=1}^{M}(1 - c_r z^{-1})}{\prod_{k=1}^{N}(1 - d_r z^{-1})}$$



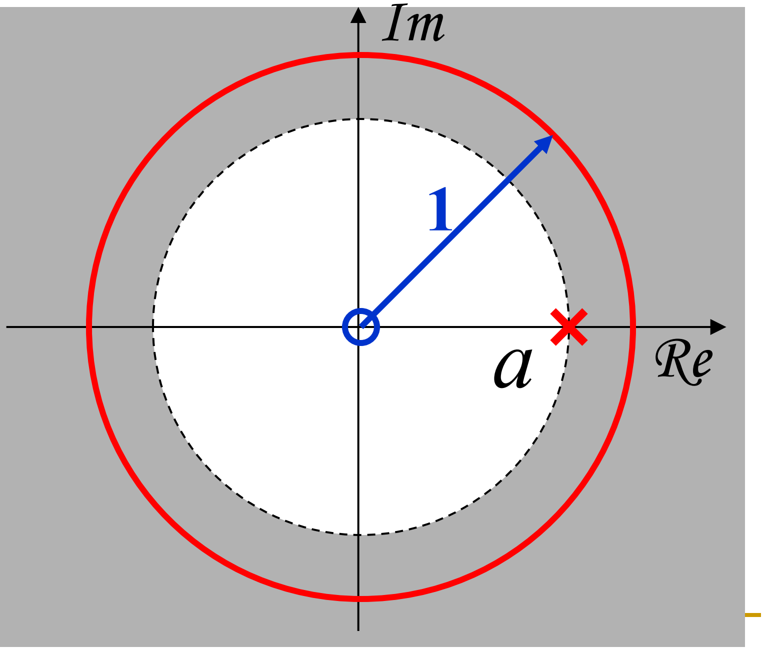
Example

Consider the causal system characterized by

$$y(n) = ay(n-1) + x(n)$$

$$\boxed{H(z) = \frac{1}{1 - az^{-1}}}$$

$$h(n) = a^n u(n)$$



Example

Transfer Function:

Systems described as difference equations have system functions of the form

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)}$$

Example:

$$H(z)=\frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)}=\frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}}=\frac{Y(z)}{X(z)}$$

$$\left(1+\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}\right)Y(z)=\left(1+2z^{-1}+z^{-2}\right)X(z)$$

$$y[n]+\frac{1}{4}y[n-1]+\frac{3}{8}y[n-2]=x[n]+2x[n-1]+x[n-2]$$

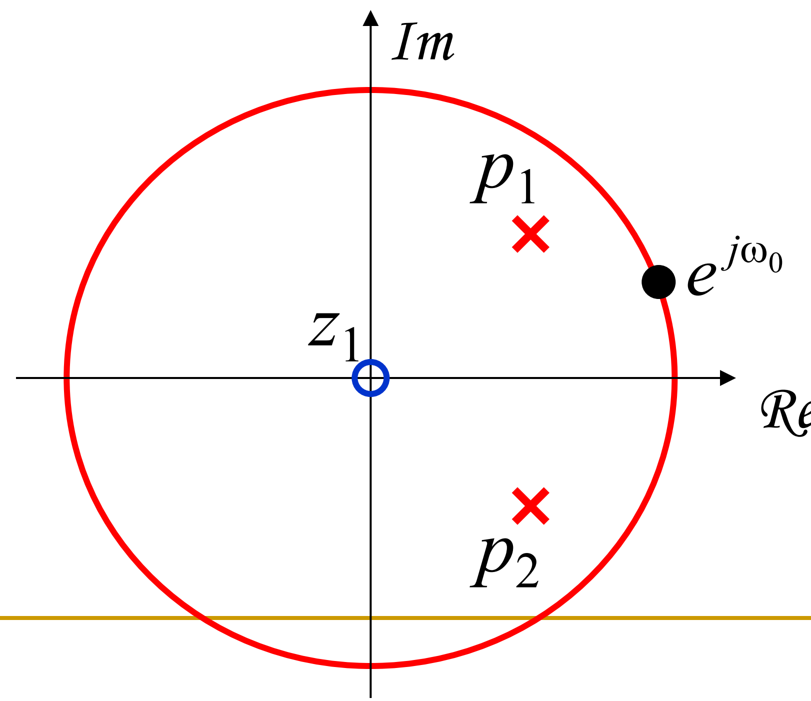
Determination of Frequency Response from pole-zero pattern

- A LTI system is completely characterized by its pole-zero pattern.

Example:

$$H(z) = \frac{z - z_1}{(z - p_1)(z - p_2)}$$

$$H(e^{j\omega_0}) = \frac{e^{j\omega_0} - z_1}{(e^{j\omega_0} - p_1)(e^{j\omega_0} - p_2)}$$



The z-Transform

Inverse *z*-Transform

Inverse z-Transform

- The inverse z-transform is formally given by

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- Where the integral is a contour integral over a closed path C that encloses the origin and lies within the ROC of $X(z)$ in the z-plane

Inverse *z*-Transform

- Method of evaluation for the inverse z-transformation.
  - Direct evaluation by contour integration
  - Expansion into series of terms in the variables z and z^{-1}
  - Partial-fraction and table lookup

Inverse *z*-Transform

- The inverse z-transform by Contour integration
  - Cauchy residue theorem is used to determine the inverse z-transform

Cauchy residue theorem. Let $f(z)$ be a function of the complex variable z and C be a closed path in the z-plane. If the derivative $df(z)/dz$ exists on and inside the contour C and if $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{z - z_0}\,dz = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \qquad \textbf{(1)}$$

Inverse z-Transform

- The inverse z-transform by Contour integration
  - More generally, if the (k+1)-order derivative of $f(z) exists\ and\ f(z)$ has no poles at $z = z_o$, then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-z_0)^k}dz = \begin{cases} \frac{1}{(k-1)!}\left.\frac{d^{k-1}f(z)}{dz^{k-1}}\right|_{z=z_0}, & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases} \quad \textbf{(2)}$$

Inverse z-Transform

- The inverse z-transform by Contour integration
  - The values of the right-hand side of (1) and (2) are called the residue of the poles at $z = z_o$.
  - The results in (1) and (2) are two forms of the Cauchy residue theorem.
  - Suppose that the integrand of the contour integral is $P(z) = f(z)/g(z), where\ \ f(z)$ has no poles inside the contour C and $g(z)$ is a polynomial with distinct (sample) roots $z_1, z_2,,,, z_n$ inside C.

Inverse *z*-Transform

- The inverse z-transform by Contour integration
  - Then

$$\begin{aligned}\frac{1}{2\pi j}\oint_C \frac{f(z)}{g(z)}dz &= \frac{1}{2\pi j}\oint_C \left[\sum_{i=1}^{n}\frac{A_i(z)}{z-z_i}\right]dz \\ &= \sum_{i=1}^{n}\frac{1}{2\pi j}\oint_C \frac{A_i(z)}{z-z_i}dz \\ &= \sum_{i=1}^{n} A_i(z_i)\end{aligned} \qquad \textbf{(3)}$$

  - Where $A_i(z) = (z - z_i)P(z) = (z - z_i)\dfrac{f(z)}{g(z)}$ **(4)**

Inverse z-Transform

- The inverse z-transform by Contour integration

The values $\{A_i(z_i)\}$ are residues of the corresponding poles at $z = z_i$, $i = 1, 2, \ldots, n$. Hence the value of the contour integral is equal to the sum of the residues of all the poles inside the contour C.

In the case of the inverse z-transform, we have

$$
\begin{aligned}
x(n) &= \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \\
&= \sum_{\text{all poles } \{z_i\} \text{ inside } C} [\text{residue of } X(z)z^{n-1} \text{ at } z = z_i] \\
&= \sum_i (z - z_i) X(z) z^{n-1} |_{z=z_i}
\end{aligned}
$$

Inverse *z*-Transform

Evaluate the inverse z-transform of

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

using the complex inversion integral.

Inverse z-Transform by Power Series Expansion

- Given a z-transform X(z) with corresponding ROC, we can expand X(z) into a power series of the form.

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n}$$

- Which converges in the given ROC. Then, by the uniqueness of the z-transform, $x(n) = c_n$ for all n. When $X(z)$ is a rational, the expansion can be performed by long division.

Inverse z-Transform by Power Series Expansion

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

when

(a) ROC: $|z| > 1$

(b) ROC: $|z| < 0.5$

Inverse *z*-Transform by Power Series Expansion

Determine the inverse z-transform of

$$X(z) = \log(1 + az^{-1}) \qquad |z| > |a|$$

Solution Using the power series expansion for $\log(1 + x)$, with $|x| < 1$, we have

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

Thus

$$x(n) = \begin{cases} (-1)^{n+1} \dfrac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

Expansion of irrational functions into power series can be obtained from tables.

Inverse z-Transform by Partial fraction Expansion

- The expression X(z) as a linear combination

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + \cdots + \alpha_K X_K(z)$$

where $X_1(z), \ldots, X_K(z)$ are expressions with inverse transforms $x_1(n), \ldots, x_K(n)$ available in a table of z-transform pairs. If such a decomposition is possible, then $x(n)$, the inverse z-transform of $X(z)$, can easily be found using the linearity property as

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) + \cdots + \alpha_K x_K(n)$$

- This approach is particularly useful if X(z) is a rational function. Without loss of generality, we assume that a_o =1, then

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

Inverse *z*-Transform by Partial fraction Expansion

- If $a_o \neq 1$ we can divide both numerator and denominator by a_o.
- A rational function is proper if $a_N \neq 0$ and M<N. It follows that this is equivalent to saying that the number of infinite zeros is less than the number of finite poles.
- An improper rational function (M$\geq N$) can always be written as the sum of a polynomial and a proper rational function.
- Example:

Express the improper rational transform

$$X(z) = \frac{1 + 3z^{-1} + \frac{11}{6}z^{-2} + \frac{1}{3}z^{-3}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

in terms of a polynomial and a proper function.

Inverse z-Transform by Partial fraction Expansion

In general, any improper rational function ($M \geq N$) can be expressed

$$X(z) = \frac{N(z)}{D(z)} = c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{N_1(z)}{D(z)} \qquad ($$

Eliminate negative powers of z by multiplying both numerator and denominator by z^N. Resulting to

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

Perform partial fraction expansion as a sum of simple fractions. Factor the denominator polynomial into factors that contains poles of X(z)

Distinct poles. Suppose that the poles $p_1, p_2, \ldots, p_N$ are all different (distinct). Then we seek an expansion of the form

Inverse *z*-Transform by Partial fraction Expansion

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

The problem is to determine the coefficients $A_1, A_2, \ldots, A_N$. There are two ways to solve this problem, as illustrated in the following example.

Inverse z-Transform by Partial fraction Expansion

Determine the partial-fraction expansion of the proper function

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \tag{3.4.16}$$

Solution First we eliminate the negative powers, by multiplying both numerator and denominator by z^2. Thus

$$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$$

The poles of $X(z)$ are $p_1 = 1$ and $p_2 = 0.5$. Consequently, the expansion of the form (3.4.15) is

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5} \tag{3.4.17}$$

A very simple method to determine A_1 and A_2 is to multiply the equation by the denominator term $(z-1)(z-0.5)$. Thus we obtain

$$z = (z-0.5)A_1 + (z-1)A_2 \tag{3.4.18}$$

Now if we set $z = p_1 = 1$ in (3.4.18), we eliminate the term involving A_2. Hence

$$1 = (1-0.5)A_1$$

Thus we obtain the result $A_1 = 2$. Next we return to (3.4.18) and set $z = p_2 = 0.5$, thus eliminating the term involving A_1, so we have

$$0.5 = (0.5-1)A_2$$

and hence $A_2 = -1$. Therefore, the result of the partial-fraction expansion is

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5} \tag{3.4.19}$$

Analysis of LTI Systems in Z-domain

- This section describe the use of system function in determination of the response of the system to some excitation signal.
- Our attention is on the important class of pole-zero system represented by linear constant-coefficient difference equations with arbitrary initial conditions.
- The ratio of system function can be presented as $H(z) = B(z)/A(z)$
- The input signal $x(n)$ has a rational z-transform $X(z)$ of the form $X(z) = N(z)/Q(z)$
- If the system is initially relaxed, that is, the initial conditions for the deference equation are zero, $y(-) = y(-2) = \cdots . y(-N) = 0, the\ z -$ transform of the output of the system has the form

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)}$$

Analysis of LTI Systems in Z-domain

- Then a partial-fraction expansion of

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1-p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1-q_k z^{-1}}$$

The inverse transform of Y(z) yields the output signal from the system in the form

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

The output sequence $y(n)$ can be divided into two parts.

The first part is a function of poles (p_k) of the system is called natural response of the system.

The influence of the input signal on this part of response is through the scale factor $\{A_k\}$.

The second part of the response is a function of the poles $\{q_k\}$ of the input signal and is called the forced response of the system.

Working Examples

A linear time-invariant system is characterized by the system function

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

$$= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

Specify the ROC of $H(z)$ and determine $h(n)$ for the following conditions:

(a) The system is stable.

(b) The system is causal.

(c) The system is anticausal.

Working Examples

Determine the response of the system

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$$

to the input signal $x(n) = \delta(n) - \frac{1}{3}\delta(n-1)$.

Working Examples

Compute the convolution of the following pair of signals in the time domain and using the one-sided z-transform.

(a) $x_1(n) = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$, $\quad x_2(n) = \{\underset{\uparrow}{1}, 1, 1\}$

End!!!!!