函数的极限推导证明

目录

1　函数的极限

1.1　一些重要的极限

1.1.1　证明当 $a > 1$ 时，$\lim\limits_{x\to 0} a^x = 1$

解：$\forall \varepsilon > 0$，令 $|a^x - 1| < \varepsilon$, 即 $1-\varepsilon < a^x < 1+\varepsilon$ ，因此只需要 $\log_a 1 - \varepsilon$

1.1.2　证明 $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$

解法一：$\forall \varepsilon > 0$，令 $\left|\dfrac{\ln(1+x)}{x} - 1\right| < \varepsilon$，则有 $1-\varepsilon < \dfrac{\ln_{(1+x)}}{x} < 1+\varepsilon$

解法二：

$$
\begin{aligned}
&\lim_{x\to 0} \frac{\ln(1+x)}{x} \\
=&\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} \\
=&\ln \lim_{x\to 0} (1+x)^{\frac{1}{x}} \\
=&\ln e \\
=&1
\end{aligned}
$$

1.1.3　证明 $\lim\limits_{n\to+\infty} \left(1+\dfrac{x}{n}\right)^n = e^x$

$$
\begin{aligned}
\because \left(1+\frac{x}{n}\right)^n &= e^{n\ln\left(1+\frac{x}{n}\right)} \\
\lim_{n\to+\infty} n\ln\left(1+\frac{x}{n}\right) &= x \\
\therefore \lim_{n\to+\infty} \left(1+\frac{x}{n}\right)^n &= e^x
\end{aligned}
$$

1.1.4 证明 $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

1.1.5 证明 $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$ **有界**

解法一：夹逼法

$$
\begin{aligned}
\left(1+\frac{1}{x}\right)^x &= \sum_{n=0}^{\infty} \binom{n}{x} \frac{1}{x^n} \\
&= 1 + \frac{x}{1!} \cdot \frac{1}{x} + \frac{x(x-1)}{2!} \cdot \frac{1}{x^2} + \cdots + \frac{x(x-1) \cdot \cdots \cdot (x-x+1)}{x!} \cdot \frac{1}{x^x} \\
&= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{x}\right) + \frac{1}{3!}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right) + \cdots \\
&+ \frac{1}{x!}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\cdots\left(1-\frac{x-1}{x}\right)
\end{aligned}
$$

设 $X_n = \left(1+\frac{1}{n}\right)^n$ 时，有

$$
\begin{aligned}
X_{n+1} = &\left(1+\frac{1}{n+1}\right)^{n+1} \\
=&1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots \\
&+ \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)\left(1-\frac{n}{n+1}\right)
\end{aligned}
$$

$X_n < X_{n+1}$，即 X_n 单调递增，并且有

$$
X_n \leqslant 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n - 1}
$$