Theory and operational rules for the discrete Hankel transform

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Previous definitions of a discrete Hankel transform (DHT) have focused on methods to approximate the continuous Hankel integral transform. In this paper, we propose and evaluate the theory of a DHT that is shown to arise from a discretization scheme based on the theory of Fourier–Bessel expansions. The proposed transform also possesses requisite orthogonality properties which lead to invertibility of the transform. The standard set of shift, modulation, multiplication, and convolution rules are derived. In addition to the theory of the actual manipulated quantities which stand in their own right, this DHT can be used to approximate the continuous forward and inverse Hankel transform in the same manner that the discrete Fourier transform is known to be able to approximate the continuous Fourier transform. © 2015 Optical Society of America

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1. INTRODUCTION

The Hankel transform has seen applications in many areas of science and engineering. In optics, it has seen a variety of applications in the propagation of optical beams and wavefields [1,2], generation of light diffusion profiles of layered turbid media [3], measurement of particles via a Fraunhofer diffraction pattern [4], propagation of cylindrical electromagnetic fields [5], reconstruction of optical fields [6], imaging through layered lenses [7], deflection tomographic reconstructions [8], design of beam shapers [9], and properties of microlenses [10].

Given the need for numerical computation with the Hankel transform, there have been correspondingly many attempts to define a discrete Hankel transform (DHT) in the literature. However, prior work has focused on proposing methods to approximate the calculation of the continuous Hankel integral. This stands in stark contrast to the approach taken with the Fourier transform, where the discrete Fourier transform (DFT) is a transform in its own right, with its own mathematical theory of the manipulated quantities. To quote Bracewell [11], “We often think of this as though an underlying function of a continuous variable really exists and we are approximating it. From an operational viewpoint, however, it is irrelevant to talk about the existence of values other than those given and those computed (the input and output). Therefore, it is desirable to have a mathematical theory of the actual quantities manipulated.” An additional feature of a carefully derived DFT is that it can be used to approximate the continuous Fourier transform, with relevant sampling and interpolation theories that can be used.

A similar set of theories does not exist for the Hankel transform, that is, a DHT as a complete and orthogonal transform that possesses its own mathematical theory, which in turn can be shown to be useful in approximating its continuous counterpart. Thus, the goal of this paper is to propose and evaluate the mathematical theory for the DHT, using the same ideas that have shown the DFT to be so useful in various disciplines. The standard set of shift, modulation, multiplication, and convolution rules are derived and presented. In addition, we show that this proposed DHT can be used to approximate the continuous Hankel transform in the same manner that the DFT is known to be able to approximate the continuous Fourier transform.

2. LITERATURE REVIEW

DHTs have received less attention than DFTs, although there is still a good body of literature on the subject. However, it should be noted that in all cases there has been no proposal of a true DHT in the same manner as the DFT, in the sense of a discrete transform that stands on its own, with its own set of rules. Rather, what has been considered in the literature to date are various ways to numerically implement the continuous Hankel transform. This is potentially somewhat confusing since what is often referred to as a DHT is, in fact, a discrete approximation to the continuous Hankel transform. This stands in contrast to the Fourier transforms, where the DFT is a transform that stands alone, and its use to approximate the continuous Fourier transform is well understood.

The seminal work on numerical computation of Hankel Transforms was by Siegman [12], where a nonlinear change of variables was used to convert the one-sided Hankel transform integral into a two-sided cross-correlation integral, which was then evaluated with an FFT. Agrawal and Lax suggested an end correction to Siegman’s approach [13]. Agnesi *et al.* found analytical formulations to the approach taken by Siegman to help improve accuracy without lower end corrections [14]. Cree and Bones reviewed several approaches to the numerical evaluation of Hankel transforms [15] and concluded that the performance of all algorithms

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depends on the type of function to be transformed. They also reported that projection based methods provided acceptable accuracy with better efficiency than numerical quadrature.

There have been many further attempts at a discretized version of the continuous Hankel transform integral. Suter and Hedges [16] showed that the algorithm proposed by Ferrari [17,18] could be interpreted as an application of the projection slice theorem, reducing the computation of the Hankel transform to a combination of Tchebychev and Fourier transforms [19]. In a similar vein, Oppenheim *et al.* exploited symmetry and the projection slice theorem to once again use an FFT to numerically compute the Hankel transform [20,21]. Hansen computed the Hankel transform by using a combination of a fast Abel transform and a fast Fourier transform [22,23]. Mook followed a similar approach of evaluating the Hankel transform as an Abel transform followed by a Fourier transform [24]. Barakat *et al.* proposed a method for the zeroth order Hankel transform based on Filon quadrature [25]. They compared their work to that of Magni *et al.* [1], who used a sampling scheme in combination with an FFT for their calculation of the zeroth order Hankel transform. Murphy and Gallagher also used a sampling scheme in combination with an FFT for their computation of the Hankel transform [26]. Markham and Conchello evaluated seven different numerical approaches to the numerical computation of a Hankel transform [27]. These were based on Filon quadrature, five different projection slice methods, and another approach based on the Abel transform. For the oscillating functions they considered in their paper, they found that one of the projection slice methods offered the best compromise between accuracy and computational efficiency. Candel took a completely different approach by using a combination of a Fourier-selection-summation (FSS) method in conjunction with a Bessel function large argument asymptotic expansion [28]. Higgins and Munson extended Candel's original approach of taking a one-dimensional Fourier transform followed by repeated summations of preselected Fourier components [29] to higher integer order Hankel transforms [30].

Garg *et al.* proposed a continuous but finite Hankel transform [31], for application to continuous problems on finite or semi-infinite domains, extending concepts first introduced by Sneddon [32]. Other generalized finite Hankel transforms, for example, have also been introduced [33]. Various other kernels have been proposed to evaluate the continuous Hankel transform [34–37]. Gupta *et al.* used an orthonormal exponential approximation [34]; Singh *et al.* used wavelets [35], Haar wavelets [38], linear Legendre multi-wavelets [39], and a hybrid of Block-pulse and Legendre polynomials [40]; Bisseling and Kosloff used an FFT [36]; and Knockaert used fast sine and cosine transforms [37]. Cavanagh and Cook used Gaussian–Laguerre polynomial expansions for their numerical approach to the Hankel transform [41]. In all cases, the starting point for the work is a focus on developing a discrete approximation of a continuous Hankel transform and does not start with a definition of a discrete Hankel transform entity that has its own rules and properties.

Interestingly, Guizar-Sicairos and Gutiérrez-Vega [2] use a discretization approach that involves the zeros of the Bessel functions. Their work is an extension to n-ordered Hankel transforms of an approach developed for zero-order Hankel transforms by Yu *et al.* [42], and termed the "Quasi-discrete Hankel Transform." This is highly interesting in its similarity to the approach taken by Fisk-Johnson [43], although neither Guizar-Sicairos and Gutiérrez-Vega nor Yu *et al.* seem to have been aware of this prior work by Fisk-Johnson. The works of Yu *et al.* [42] and Guizar-Sicairos and Gutiérrez-Vega [44] were the first to demonstrate a discrete version of the Parseval theorem for the Hankel transforms. Jerri [45] came close to defining a DHT that follows in the path of the DFT. However, Jerri missed a crucial orthogonality relationship, and thus his proposed transform approach is neither complete nor orthogonal.

In different research areas, there has been work done on discrete matrix transforms known as Bessel transforms that bear a resemblance to a Hankel transform. These have been proposed and demonstrated by Lemoine in [46,47], and Layton and Stade in [48], and have generally been applied to quantum mechanical eigenvalue problems. These latter two approaches differ in their choices of boundary conditions on the chosen Bessel functions. Both papers impose the condition that the functions must remain bounded at the origin. However, these two sets of authors differ in the treatment of the basis Bessel functions at the outer radial boundary in both physical and spatial frequency space. Layton and Stade assume a vanishing derivative of the Bessel functions at the outer boundary, whereas Lemoine assumes a vanishing of the basis Bessel function itself. Other choices of boundary conditions at the outer radii are possible in both physical space and spatial frequency space, leading to other possibilities for the actual structure of the matrix transform.

In this paper, a discrete Hankel transform is proposed and, following in the steps of the DFT, the accompanying set of orthogonality, shift, modulation, multiplication, convolution and Parseval rules are also derived. In addition, we demonstrate that this DHT can be used to approximate the continuous Hankel transform, in parallel to the way in which the DFT is known to be able to approximate the continuous Fourier transform.

3. HANKEL TRANSFORMS AND BESSEL SERIES

For the sake of completeness, we define the Hankel transform and Fourier–Bessel series as used in this work.

A. Hankel Transform

The nth order Hankel transform $F(\rho)$ of the function $f(r)$ of a real variable, $r \geq 0$, is defined by the integral [49]

$$F(\rho) = \mathbb{H}_n(f(r)) = \int_0^\infty f(r) J_n(\rho r) r \mathrm{d}r, \tag{1}$$

where $J_n(z)$ is the nth order Bessel function as shown in Eq. (1). If n is real and $n > -1/2$, the transform is self-reciprocating and the inversion formula is given by

$$f(r) = \int_0^\infty F(\rho) J_n(\rho r) \rho \mathrm{d}\rho. \tag{2}$$

Thus, Hankel transforms take functions in the spatial r domain and transform them to functions in the frequency ρ domain, $f(r) \Leftrightarrow F(\rho)$. The notation $\Leftrightarrow$ is used to indicate a Hankel transform pair.

B. Fourier–Bessel Series (Transform)

Functions defined on a finite portion of the real line $[0,R]$ can be expanded in terms of a Fourier–Bessel series [50] given by

$$f(r)=\sum_{k=1}^{\infty}f_kJ_n\left(\frac{j_{nk}r}{R}\right), \tag{3}$$

where the order of the Bessel function is arbitrary and j_{nk} denotes the kth root of the nth Bessel function. The Fourier–Bessel coefficients f_k of the function $f(r)$ can be found from

$$f_k=\frac{2}{R^2J_{n+1}^2(j_{nk})}\int_0^R f(r)J_n\left(\frac{j_{nk}r}{R}\right)r\mathrm{d}r. \tag{4}$$

Equations (3) and (4) can be considered to be a transform pair where the continuous function $f(r)$ is forward-transformed to the discrete vector f_k, given by the finite integral in (4). The inverse transformation, which returns $f(r)$ when starting with f_k, is then given by the summation in Eq. (3). The Fourier–Bessel series is to the Hankel transform as the Fourier series is to the Fourier transform. Just as the Fourier series is defined for a finite interval and has an infinite integral counterpart, the continuous Fourier transform, the Fourier–Bessel series similarly has a counterpart over an infinite interval, namely the Hankel transform.

4. INTUITIVE DISCRETIZATION SCHEME VIA SAMPLING THEOREMS

A. Discretization Scheme for a Band-Limited Function

Suppose that the function $f(r)$ is band-limited in the frequency Hankel domain so that its spectrum $F(\rho)$ is zero outside an interval $[0,2\pi W]$. The interval is written in this form since W would typically be quoted in units of Hz (cycles per second) if using temporal units, or cycles per meter if using spatial units. Therefore, the multiplication by 2π ensures that the final units are in s^{-1} or m^{-1}. Because the spectrum $F(\rho)$ is defined on a finite portion of the real line $[0,2\pi W]$, it can be expanded in terms of a Fourier–Bessel series so that

$$F(\rho)=\sum_{k=1}^{\infty}F_kJ_n\left(\frac{j_{nk}\rho}{2\pi W}\right). \tag{5}$$

The Fourier–Bessel coefficients can be found from (4) to give

$$F_k=\frac{2\int_0^{2\pi W}F(\rho)J_n\left(\frac{j_{nk}\rho}{2\pi W}\right)\rho\mathrm{d}\rho}{4\pi^2W^2J_{n+1}^2(j_{nk})}=\frac{1}{2\pi^2W^2J_{n+1}^2(j_{nk})}f\left(\frac{j_{nk}}{2\pi W}\right). \tag{6}$$

In (6), we have used the fact that $f(r)$ can be written in terms of its inverse Hankel transform, Eq. (2), in combination with the fact that the function is assumed band-limited. Equation (6) states that the values $f(\frac{j_{nk}}{2\pi W})$ determine the Fourier coefficients F_k in the series expansion of $F(\rho)$. Therefore, the samples $f(\frac{j_{nk}}{2\pi W})$ determine the function $f(r)$ completely since (i) $F(\rho)$ is determined on $[0,2\pi W]$ through Eqs. (5) and (6), and $F(\rho)$ is zero otherwise, and (ii) $f(r)$ is known if $F(\rho)$ is known. Another way of looking at this is that band-limiting, a function to $[0,2\pi W]$, results in information about the original function in space at samples $r_{nk}=j_{nk}/2\pi W$.

B. Discretization Scheme for a Space-Limited Function

By the same token, suppose that we assume that the function $f(r)$ is space-limited so that $f(r)$ is zero outside an interval $[0,R]$. Since the function $f(r)$ is defined on a finite portion of the real line $[0,R]$, then it can be expanded in terms of a Fourier–Bessel series so that

$$f(r)=\sum_{k=1}^{\infty}f_kJ_n\left(\frac{j_{nk}r}{R}\right). \tag{7}$$

As before, the Fourier–Bessel coefficients can be found from

$$f_k=\frac{2}{R^2J_{n+1}^2(j_{nk})}\int_0^R f(r)J_n\left(\frac{j_{nk}r}{R}\right)r\mathrm{d}r=\frac{2}{R^2J_{n+1}^2(j_{nk})}F\left(\frac{j_{nk}}{R}\right). \tag{8}$$

Here, we have used the definition of the Hankel transform $F(\rho)$, Eq. (1), in the right-hand side of Eq. (8). Equation (8) states that the values of $f(r)$ are determined by $F(j_{nk}/R)$. This follows, since for positions greater than R, $f(r)$ is zero and for smaller positions smaller than R, $f(r)$ is determined if its Fourier–Bessel coefficients are determined. Therefore, the samples $F(j_{nk}/R)$ determine the function $f(r)$ and its transform $F(\rho)$ completely. Another way of looking at this is that space limiting a function to $[0,R]$ implies discretization in spatial frequency space, at frequencies $\rho_{nk}=j_{nk}/R$.

C. Intuitive Discretization Scheme for the Hankel Transform

1. Forward Transform

We demonstrated above that a band-limited function, with $\rho\le W_\rho=2\pi W$, can be written as

$$F(\rho)=\sum_{k=1}^{\infty}\frac{2}{W_\rho^2J_{n+1}^2(j_{nk})}f\left(\frac{j_{nk}}{W_\rho}\right)J_n\left(\frac{j_{nk}\rho}{W_\rho}\right). \tag{9}$$

Evaluating the previous Eq. (9) at the sampling points $\rho_{nm}=\frac{j_{nm}W_\rho}{j_{nN}}$ gives

$$F\left(\frac{j_{nm}W_\rho}{j_{nN}}\right)=\sum_{k=1}^{\infty}\frac{2f\left(\frac{j_{nk}}{W_\rho}\right)}{W_\rho^2J_{n+1}^2(j_{nk})}J_n\left(\frac{j_{nk}}{W_\rho}\frac{j_{nm}W_\rho}{j_{nN}}\right). \tag{10}$$

For $m<N$, $\rho_{nm}=\frac{j_{nm}W_\rho}{j_{nN}}<W_\rho$, Eq. (10), with the summation over infinite k, is exact.

Now suppose we choose to terminate our series at $k=N$. Noting that at $k=N$, the last term in (10) is $J_n(\frac{j_{nN}j_{nm}}{j_{nN}})=J_n(j_{nm})=0$ and substituting $\rho_{nm}=\frac{j_{nm}W_\rho}{j_{nN}}$, $r_{nk}=\frac{j_{nk}}{W_\rho}$ for the sampling points, then Eq. (10) becomes

$$F(\rho_{nm})=\sum_{k=1}^{N-1}\frac{2}{W_\rho^2J_{n+1}^2(j_{nk})}f(r_{nk})J_n\left(\frac{j_{nk}j_{nm}}{j_{nN}}\right). \tag{11}$$

In this case, the truncated sum in Eq. (11) does not represent $F(\rho_{nm})$ exactly because of the truncation at N terms, but should provide an approximation since the Fourier–Bessel series is known to converge. This also provides a good motivation for a discrete Hankel transform formulation.

2. Inverse Transform

Similarly, for a space-limited function we stated that for $r \le R$, then

$$f(r) = \sum_{m=1}^{\infty} \frac{2}{R^2 J_{n+1}^2(j_{nm})} F\left(\frac{j_{nm}}{R}\right) J_n\left(\frac{j_{nm} r}{R}\right). \quad (12)$$

Evaluating (12) at $r_{nk} = \frac{j_{nk} R}{j_{nN}}$ gives

$$f\left(\frac{j_{nk} R}{j_{nN}}\right) = \sum_{m=1}^{\infty} \frac{2}{R^2 J_{n+1}^2(j_{nm})} F\left(\frac{j_{nm}}{R}\right) J_n\left(\frac{j_{nm}}{R}\frac{j_{nk} R}{j_{nN}}\right). \quad (13)$$

Again, Eq. (13) with its infinite summation is exact. Terminating the series at N, further recalling that $J_n(\frac{j_{nN} j_{nm}}{j_{nN}}) = J_n(j_{nm}) = 0$, and using $\rho_{nm} = \frac{j_{nm}}{R}$, then Eq. (13) simplifies to

$$f(r_{nk}) = \sum_{m=1}^{N-1} \frac{2}{R^2 J_{n+1}^2(j_{nm})} F(\rho_{nm}) J_n\left(\frac{j_{nm} j_{nk}}{j_{nN}}\right). \quad (14)$$

As before, the truncated sum in Eq. (14) does not represent $f(r_{nk})$ exactly, but does provide a good motivation for the inverse discrete Hankel transform formulation.

3. Intuitive Discretization Scheme and Kernel

The preceding development shows that a natural, N-dimensional discretization scheme in finite space $[0, R]$ and finite frequency space $[0, W_\rho]$ is given by

$$\left.\begin{aligned} r_{nk} &= \frac{j_{nk}}{W_\rho} = \frac{j_{nk} R}{j_{nN}} \\ \rho_{nk} &= \frac{j_{nk}}{R} = \frac{j_{nk} W_\rho}{j_{nN}} \end{aligned}\right\} \quad k = 1 \ldots N-1. \quad (15)$$

The relationship $W_\rho = \frac{j_{nN}}{R}$ can be used to change from a finite frequency domain to a finite space domain. Furthermore, the preceding Eqs. (11) and (14) show that both forward and inverse discrete versions of the transforms contain an expression of the form

$$\frac{2}{J_{n+1}^2(j_{nk})} J_n\left(\frac{j_{nk} j_{nm}}{j_{nN}}\right). \quad (16)$$

This leads to a natural choice of kernel for the discrete transforms, as shall be outlined below.

5. DISCRETE ORTHOGONALITY OF THE BESSEL FUNCTIONS

It is shown in [43] that the following discrete orthogonality relationship is true:

$$\sum_{k=1}^{N-1} \frac{4 J_n(j_{nm} j_{nk}/j_{nN}) J_n(j_{nk} j_{ni}/j_{nN})}{j_{nN}^2 J_{n+1}^2(j_{nm}) J_{n+1}^2(j_{nk})} = \delta_{mi}$$
$$1 \le m, \quad i \le N-1, \quad (17)$$

where j_{nm} represents the mth zero of the nth order Bessel function $J_n(x)$, and δ_{mi} is the Kronecker-delta function, defined as

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}. \quad (18)$$

If written in matrix notation, then the Kronecker-delta of Eq. (18) is the identity matrix.

Fisk-Johnson discussed the analytical derivation of Eq. (17) in the appendix of [43]. Equation (17) is exactly true in the limit as $N \to \infty$ and is true for $N > 30$ within the limits of computational error ($\pm 10^{-7}$). For smaller values of N, Eq. (17) holds with the worst case for the smallest value of N giving $\pm 10^{-3}$.

6. TRANSFORMATION MATRICES

A. Transformation Matrix

With inspiration from the notation in [43], and an additional scaling factor of $1/j_{nN}$, we define an $(N-1) \times (N-1)$ transformation matrix with the (m, k)th entry given by

$$Y_{m,k}^{nN} = \frac{2}{j_{nN} J_{n+1}^2(j_{nk})} J_n\left(\frac{j_{nm} j_{nk}}{j_{nN}}\right) \quad 1 \le m, \quad k \le N-1. \quad (19)$$

In Eq. (19), the superscripts n and N refer to the order of the Bessel function and the dimension of the space that are being considered, respectively. The subscripts m and k refer to the (m, k)th entry of the transformation matrix.

Furthermore, the orthogonality relationship, Eq. (17), states that

$$\sum_{k=1}^{N-1} Y_{i,k}^{nN} Y_{k,m}^{nN} = \sum_{k=1}^{N-1} \frac{4 J_n(j_{ni} j_{nk}/j_{nN}) J_n(j_{nk} j_{nm}/j_{nN})}{j_{nN}^2 J_{n+1}^2(j_{nk}) J_{n+1}^2(j_{nm})}$$
$$= \delta_{im}. \quad (20)$$

Equation (20) states that the rows and columns of the matrix $Y_{m,k}^{nN}$ are orthonormal and can be written in matrix form as

$$Y^{nN} Y^{nN} = \mathbf{I}, \quad (21)$$

where $\mathbf{I}$ is the $N-1$ dimensional identity matrix, and we have written the $N-1$ square matrix $Y_{m,k}^{nN}$ as Y^{nN}. Clearly, this implies that the inverse of Y^{nN} is given by itself:

$$(Y^{nN})^{-1} = Y^{nN}. \quad (22)$$

The forward and inverse truncated and discretized transforms given in Eqs. (11) and (14) can be expressed in terms of Y^{nN}. The forward transform, Eq. (11), can be written as

$$F(\rho_{nm}) = \frac{R^2}{j_{nN}} \sum_{k=1}^{N-1} Y_{m,k}^{nN} f(r_{nk}). \quad (23)$$

Similarly, the inverse transform, Eq. (14), can be written as

$$f(r_{nk}) = \frac{j_{nN}}{R^2} \sum_{m=1}^{N-1} Y_{k,m}^{nN} F(\rho_{nm}). \quad (24)$$

B. Another Choice of Transformation Matrix

Following the notation in [2], we can also define a different $(N-1) \times (N-1)$ transformation matrix with the (m, k)th entry given by

$$T_{m,k}^{nN} = 2 \frac{J_n(j_{nm} j_{nk}/j_{nN})}{J_{n+1}(j_{nm}) J_{n+1}(j_{nk}) j_{nN}} \quad 1 \le m, \quad k \le N-1. \quad (25)$$

In Eq. (25), the superscripts n and N refer to the order of the Bessel function and the dimension of the space that are being considered, respectively. The subscripts m and k refer to the (m, k)th entry of the matrix. From (19), it can be seen that $T_{m,k}^{nN} = T_{k,m}^{nN}$, therefore T^{nN}, is a real symmetric matrix. The relationship between the $T_{m,k}^{nN}$ and $Y_{m,k}^{nN}$ matrices is given by

$$T_{m,k}^{nN} \frac{J_{n+1}(j_{nm})}{J_{n+1}(j_{nk})} = Y_{m,k}^{nN}. \quad (26)$$

The orthogonality relationship, Eq. (17), can be written as

$$\sum_{k=1}^{N-1} \frac{4J_n(j_{nm}j_{nk}/j_{nN})J_n(j_{nk}j_{ni}/j_{nN})}{J_{n+1}^2(j_{nm})J_{n+1}^2(j_{nk})j_{nN}^2} = \sum_{k=1}^{N-1} T_{m,k}^{nN} T_{k,i}^{nN} = \delta_{mi}. \quad (27)$$

Equation (27) states that the rows and columns of the matrix T^{nN} are orthonormal so that T^{nN} is an orthogonal matrix. Furthermore, T^{nN} is also symmetric. Equation (27) can be written in matrix form as

$$T^{nN} T^{nN} = T^{nN}(T^{nN})^T = \mathbf{I}. \quad (28)$$

Therefore, the T^{nN} matrix is unitary and furthermore orthogonal since the entries are real.

Using the symmetric, orthogonal transformation matrix T^{nN}, the forward transform from Eq. (11) can be written in as

$$F(\rho_{nm}) = \frac{R^2}{j_{nN}} \sum_{k=1}^{N-1} T_{m,k}^{nN} \frac{J_{n+1}(j_{nm})}{J_{n+1}(j_{nk})} f(r_{nk}), \quad (29)$$

or, in a more symmetrical form, as

$$\frac{F(\rho_{nm})}{J_{n+1}(j_{nm})} = \frac{R^2}{j_{nN}} \sum_{k=1}^{N-1} T_{m,k}^{nN} \frac{f(r_{nk})}{J_{n+1}(j_{nk})}. \quad (30)$$

The symmetrical form of (30) highlights the fact that $T_{m,k}^{nN}$ is the kernel of the discrete transform.

Similarly, the inverse discrete transform of Eq. (14) can be written as

$$f(r_{nk}) = \frac{j_{nN}}{R^2} \sum_{m=1}^{N-1} T_{k,m}^{nN} \frac{J_{n+1}(j_{nk})}{J_{n+1}(j_{nm})} F(\rho_{nm}), \quad (31)$$

or, symmetrically, as

$$\frac{f(r_{nk})}{J_{n+1}(j_{nk})} = \frac{j_{nN}}{R^2} \sum_{m=1}^{N-1} T_{k,m}^{nN} \frac{F(\rho_{nm})}{J_{n+1}(j_{nm})}. \quad (32)$$

7. DISCRETE FORWARD AND INVERSE HANKEL TRANSFORM

From the previous section, it is clear that the two natural choices of kernel for a DFT are either $Y_{m,k}^{nN}$ or $T_{m,k}^{nN}$. $Y_{m,k}^{nN}$ is closer to the discretized version of the continuous Hankel transform that we hope the discrete version emulates. However, $T_{m,k}^{nN}$ is an orthogonal and symmetric matrix; therefore it is energy preserving and will be shown to lead to a Parseval-type relationship if chosen as the kernel for the DHT. Thus, to define a discrete Hankel transform (DHT), we can use either formulation:

$$F_m = \sum_{k=1}^{N-1} Y_{m,k}^{nN} f_k \quad \text{or} \quad F_m = \sum_{k=1}^{N-1} T_{m,k}^{nN} f_k. \quad (33)$$

Here, the transform is of *any* $N - 1$ dimensional vector f_k to *any* $N - 1$ dimensional vector F_m for the integers m, k, where $1 \le m, k < N - 1$. This can be written in matrix form as

$$\mathbf{F} = Y^{nN}\mathbf{f} \quad \text{or} \quad \mathbf{F} = T^{nN}\mathbf{f}, \quad (34)$$

where $\mathbf{F}$ is *any* $N - 1$ dimensional column vector and $\mathbf{f}$ is also *any* column vector, defined in the same manner.

The inverse discrete Hankel transform (IDHT) is then given by

$$f_k = \sum_{m=1}^{N-1} Y_{k,m}^{nN} F_m \quad \text{or} \quad f_k = \sum_{m=1}^{N-1} T_{k,m}^{nN} F_m. \quad (35)$$

This can also be written in matrix form as

$$\mathbf{f} = Y^{nN}\mathbf{F} \quad \text{or} \quad \mathbf{f} = T^{nN}\mathbf{F}. \quad (36)$$

We note that the forward and inverse transforms are the same.

Proof

We show the proof for the Y^{nN} formulation, but it proceeds similarly if Y^{nN} is replaced with T^{nN}. Substituting Eq. (35) into the right-hand side of (33) gives

$$\sum_{k=1}^{N-1} Y_{m,k}^{nN} f_k = \sum_{k=1}^{N-1} Y_{m,k}^{nN} \left[\sum_{p=1}^{N-1} Y_{k,p}^{nN} F_p \right]. \quad (37)$$

Switching the order of the summation in Eq. (37) gives

$$\sum_{p=1}^{N-1} \underbrace{\sum_{k=1}^{N-1} Y_{m,k}^{nN} Y_{k,p}^{nN}}_{\delta_{mp}} F_p = \sum_{p=1}^{N-1} \delta_{mp} F_p = F_m. \quad (38)$$

The inside summations, as indicated in Eq. (38), are recognized as yielding the Dirac-delta function, the orthogonality property of Eq. (20) [or Eq. (27) if using T^{nN}], which in turn yields the desired result. This proves that the DHT given by (33) can be inverted by (35).

8. GENERALIZED PARSEVAL THEOREM

Inner products are preserved and thus energies are preserved under the T^{nN} matrix formulation. To see this, consider any two vectors given by the transform $\mathbf{g} = T^{nN}\mathbf{G}$, $\mathbf{h} = T^{nN}\mathbf{H}$, then

$$\mathbf{g}^T\mathbf{h} = (T^{nN}\mathbf{G})^T T^{nN}\mathbf{H} = \mathbf{G}^T \underbrace{(T^{nN})^T T^{nN}}_{=I} \mathbf{H} = \mathbf{G}^T\mathbf{H}. \quad (39)$$

On the other hand, the Y^{nN} matrix formulation does not directly preserve inner products:

$$\mathbf{g}^T\mathbf{h} = (Y^{nN}\mathbf{G})^T Y^{nN}\mathbf{H} = \mathbf{G}^T (Y^{nN})^T Y^{nN}\mathbf{H}. \quad (40)$$

However, under the Y^{nN} formulation, the inner product between $\frac{g_k}{J_{n+1}(j_{nk})}$ and $\frac{h_k}{J_{n+1}(j_{nk})}$ is preserved. To see this, we calculate the inner product between them as

$$\begin{aligned}&\sum_{k=1}^{N-1}\frac{g_k}{J_{n+1}(j_{nk})}\frac{h_k}{J_{n+1}(j_{nk})}\\&=\sum_{k=1}^{N-1}\frac{1}{J_{n+1}^2(j_{nk})}\sum_{p=1}^{N-1}Y_{k,p}^{nN}G_p\sum_{q=1}^{N-1}Y_{k,q}^{nN}H_q\\&=\sum_{p=1}^{N-1}\sum_{q=1}^{N-1}\sum_{k=1}^{N-1}\frac{1}{J_{n+1}^2(j_{nk})}Y_{k,p}^{nN}Y_{k,q}^{nN}H_qG_p\\&=\sum_{p=1}^{N-1}\sum_{q=1}^{N-1}\frac{1}{J_{n+1}^2(j_{np})}\underbrace{\sum_{k=1}^{N-1}\frac{4(j_{nk}j_{np}/j_{nN})J_n(j_{nk}j_{nq}/j_{nN})}{j_{nN}^2J_{n+1}^2(j_{nk})J_{n+1}^2(j_{nq})}}_{\delta_{pq}}H_qG_p.\end{aligned}\tag{41}$$

Making use of the now-present Dirac-delta function, Eq. (41) simplifies to give a modified Parseval relationship:

$$\sum_{k=1}^{N-1}\left(\frac{g_k}{J_{n+1}(j_{nk})}\right)\left(\frac{h_k}{J_{n+1}(j_{nk})}\right)=\sum_{p=1}^{N-1}\left(\frac{H_p}{J_{n+1}(j_{np})}\right)\left(\frac{G_p}{J_{n+1}(j_{np})}\right).\tag{42}$$

In other words, under a DHT that uses the Y^{nN} matrix, inner products of the scaled functions are preserved but not the inner products of the functions themselves.

A. Parseval Theorem

As a consequence of the orthogonality property of T^{nN}, the T^{nN} based DHT is energy preserving, meaning that Parseval's theorem is satisfied:

$$\sum_{m=1}^{N-1}|F_m|^2=\sum_{k=1}^{N-1}|f_k|^2.\tag{43}$$

In matrix notation, this can be written as

$$\bar{\mathbf{F}}^T\mathbf{F}=\bar{\mathbf{f}}^T\mathbf{f},\tag{44}$$

where the overbar indicates a conjugate transpose and the superscript T indicates a transpose.

This follows from (40) by substituting $\mathbf{g}=\bar{\mathbf{f}}, \mathbf{h}=\mathbf{f}$ and $\mathbf{G}=\bar{\mathbf{F}}, \mathbf{H}=\mathbf{F}$. For the formulation with Y^{nN} as the transformation kernel, the equivalent expression is

$$\bar{\mathbf{F}}^T\mathbf{F}=(Y^{nN}\bar{\mathbf{f}})^TY^{nN}\mathbf{f}=\bar{\mathbf{f}}^T(Y^{nN})^TY^{nN}\mathbf{f}.\tag{45}$$

Although it is obvious from Eq. (42) that if we define the "scaled" vector

$$f_k^{\text{Scaled}}=\frac{f_k}{J_{n+1}(j_{nk})},\quad F_p^{\text{Scaled}}=\frac{F_p}{J_{n+1}(j_{np})},\tag{46}$$

then by straightforward substitution of scaled vectors and their conjugates for g and h functions and their transforms, it follows that

$$\overline{(\mathbf{F}^{\text{Scaled}})}^T\mathbf{F}^{\text{Scaled}}=(\bar{\mathbf{f}}^{\text{Scaled}})^T\mathbf{f}^{\text{Scaled}}.\tag{47}$$

9. TRANSFORM RULES

In keeping with the development of the well-known DFT, we develop the standard toolkit of rules for the discrete Hankel transform. In the following, the Y^{nN} is used, but all expressions apply equally if Y^{nN} is replaced with T^{nN}.

A. Transform of Kronecker-Delta Function

The discrete counterpart of the Dirac-delta function is the Kronecker-delta function, δ_{kk_0}. We recall that the DHT as defined above is a matrix transform from an $N-1$ dimensional vector to another. The vector δ_{kk_o} is interpreted as the vector as having zero entries everywhere except at position $k=k_0$ (k_0 fixed so δ_{kk_0} is a vector), or in other words, the k_0th column of the $N-1$ sized identity matrix. The DHT of the Kronecker-delta can be found from the definition of the forward transform via

$$\mathbb{H}(\delta_{kk_o})=\sum_{k=1}^{N-1}Y_{m,k}\delta_{kk_o}=Y_{m,k_0}^{nN}.\tag{48}$$

The symbol $\mathbb{H}(\cdot)$ is used to denote the operation of taking the discrete Hankel transform. This gives us our first DHT transform pair of order n dimension $N-1$, and we denote this relationship as

$$\delta_{kk_o}\Leftrightarrow Y_{m,k_0}^{n,N}.\tag{49}$$

Here, $f_k\Leftrightarrow F_m$ is used to denote a transform pair and Y_{m,k_0}^{nN} is k_0th column of the transformation matrix Y^{nN}.

B. Inverse Transform of the Kronecker-Delta Function

From Eq. (49), we can deduce that the vector f_k that transforms *to* the Kronecker-delta vector δ_{mm_o} function. Namely, we take the forward transform of

$$f_k=Y_{k,m_0}^{n,N}.\tag{50}$$

As before, Y_{k,m_0}^{nN} represents the m_0th column of the transformation matrix Y^{nN}. From the forward definition of the transform, Eq. (33), the transform of $Y_{k,m_0}^{n,N}$ is given by

$$F_m=\sum_{k=1}^{N-1}Y_{m,k}^{nN}f_k=\sum_{k=1}^{N-1}Y_{m,k}^{nN}Y_{k,m_0}^{nN}=\delta_{mm_0},\tag{51}$$

where we have used the orthogonality relationship (20). This gives us another DHT pair:

$$Y_{k,m_0}^{n,N}\Leftrightarrow\delta_{mm_o}.\tag{52}$$

C. Generalized-Shift Operator

For a one-dimensional Fourier transform, one of the known transform rules is the shift rule, which states that

$$f(x-a)=\mathbb{F}^{-1}\{e^{-ia\omega}\hat{f}(\omega)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{e^{-ia\omega}\hat{f}(\omega)\}e^{i\omega t}\mathrm{d}\omega.\tag{53}$$

In Eq. (53), $\hat{f}(\omega)$ is the Fourier transform of $f(x)$, $\mathbb{F}^{-1}$ denotes an inverse Fourier transform, and $e^{-ia\omega}$ is the kernel of the Fourier transform operator. Motivated by this result, we define a generalized-shift operator by finding the inverse

DHT of the DHT of the function multiplied by the DHT kernel. This is a discretized version of the definition of a generalized shift operator as proposed by Levitan [51]. (He suggested the complex conjugate of the Fourier operator, which for Fourier transforms is the inverse transform operator.) We thus propose the definition of the generalized-shifted function to be given by

$$f_{k,k_o}^{\text{shift}} = \sum_{p=1}^{N-1} Y_{k,p}^{nN} \underbrace{\{Y_{p,k_o}^{nN} F_p\}}_{\text{shift in Hankel domain}}, \tag{54}$$

where $1 \le k, k_o \le N-1$. For a single, fixed value of k_o, then f_{k,k_o}^{shift} is another $N-1$ vector, with the notation f_{k,k_o}^{shift} implying a k_0 -shifted version of f_k. This generalizes the notion of the shift, usually denoted f_{k-k_o}, which inevitably encounters difficulty when the subscript $k-k_o$ falls outside the range $[1, N-1]$. We note that, if all possible shifts k_o are considered, then f_{k,k_o}^{shift} is an $N-1$ square matrix (in other words, a two-dimensional structure), whereas the original unshifted f_k is an $N-1$ vector. For the DFT, when the shifted subscript $k-k_o$ falls outside the range of the indices, it is usually interpreted modulo the size of the DFT. However, the kernel of the Fourier transform is periodic so this does not create difficulties for the DFT. The Bessel functions are not periodic so the same trick cannot be used with the Hankel transform. In fact, this lack of periodicity and lack of simple relationship between $J_n(x-y)$ and $J_n(x)$ is the reason that the continuous Hankel transform does not have a convolution-multiplication rule [52]. Therefore, the notation f_{k-k_o} would not make mathematical sense when used with the DHT. With the definition given by Eq. (54), no such confusion arises since the definition is unambiguous for all allowable values of k and k_o.

The shifted function f_{k,k_o}^{shift} can also be expressed in terms of the original unshifted function f_k. Using the definition of F_m from Eq. (33) and a dummy change of variable, then Eq. (54) becomes

$$f_{k,k_o}^{\text{shift}} = \sum_{p=1}^{N-1} Y_{k,p}^{nN} Y_{p,k_o}^{nN} F_p = \sum_{p=1}^{N-1} Y_{k,p}^{nN} Y_{p,k_o}^{nN} \sum_{m=1}^{N-1} Y_{p,m}^{nN} f_m. \tag{55}$$

Changing the order of summation gives

$$f_{k,k_o}^{\text{shift}} = \sum_{p=1}^{N-1} Y_{k,p}^{nN} Y_{p,k_o}^{nN} F_p = \sum_{m=1}^{N-1} \underbrace{\sum_{p=1}^{N-1} Y_{k,p}^{nN} Y_{p,k_o}^{nN} Y_{p,m}^{nN}}_{\text{shift operator}} f_m. \tag{56}$$

As indicated in Eq. (56), the quantity in brackets can be considered to be a type of shift operator acting on the original unshifted function. We can define this as

$$S_{k,k_o,m}^{nN} = \sum_{p=1}^{N-1} Y_{k,p}^{nN} Y_{p,k_o}^{nN} Y_{p,m}^{nN}. \tag{57}$$

It then follows that Eq. (56) can be written as

$$f_{k,k_o}^{\text{shift}} = \sum_{m=1}^{N-1} S_{k,k_o,m}^{nN} f_m. \tag{58}$$

This triple-product shift operator is similar to previous definitions of shift operators for multidimensional Fourier transforms that rely on Hankel transforms [53,54] and of generalized Hankel convolutions [55–57].

D. Transform of the Generalized Shift

We now consider the forward DHT transform of the shifted function f_{k,k_o}^{shift}. From the definition, the DHT of the shifted function can be found from

$$\sum_{k=1}^{N-1} Y_{m,k}^{nN} f_{k,k_o}^{\text{shift}} = \sum_{k=1}^{N-1} Y_{m,k}^{nN} \sum_{p=1}^{N-1} Y_{k,p}^{nN} Y_{p,k_o}^{nN} F_p. \tag{59}$$

Changing the order of summation gives

$$\sum_{p=1}^{N-1} \underbrace{\sum_{k=1}^{N-1} Y_{m,k}^{nN} Y_{k,p}^{nN}}_{=\delta_{mp}} Y_{p,k_o}^{nN} F_p = \sum_{p=1}^{N-1} \delta_{mp} Y_{p,k_o}^{nN} F_p = Y_{m,k_o}^{nN} F_m. \tag{60}$$

This yields another transform pair and is the shift-modulation rule. This rule is analogous to the shift-modulation rule for regular Fourier transforms, whereby a shift in the spatial domain is equivalent to modulation in the frequency domain:

$$f_{k,k_o}^{\text{shift}} \Leftrightarrow Y_{m,k_o}^{nN} F_m. \tag{61}$$

Note that Eq. (61) does *not* imply a summation over the m index. For a fixed value of k_o on the left-hand side, the corresponding transformed value of F_m is multiplied by the (m, k_o)th entry of the Y^{nN} matrix.

E. Modulation

We consider the forward DHT of a function "modulated" in the space domain $f_k = Y_{k,k_o}^{nN} g_k$. Here, the interpretation of $f_k = Y_{k,k_o}^{nN} g_k$ is that the kth entry of the vector g is multiplied by the (k, k_o)th entry of Y^{nN} for a fixed value of k_o. No summation is implied, so this is not a dot product; both f_k and $Y_{k,k_o}^{nN} g_k$ are $N-1$ vectors. Again, we implement the definition of the forward transform,

$$\sum_{k=1}^{N-1} Y_{m,k}^{nN} f_k = \sum_{k=1}^{N-1} Y_{m,k}^{nN} Y_{k,k_o}^{nN} g_k, \tag{62}$$

and write g_k in terms of its inverse transform:

$$g_k = \sum_{p=1}^{N-1} Y_{k,p}^{nN} G_p. \tag{63}$$

Then Eq. (62) becomes

$$\sum_{k=1}^{N-1} Y_{m,k}^{nN} f_k = \sum_{k=1}^{N-1} Y_{m,k}^{nN} Y_{k,k_o}^{nN} g_k = \sum_{k=1}^{N-1} Y_{m,k}^{nN} Y_{k,k_o}^{nN} \sum_{p=1}^{N-1} Y_{k,p}^{nN} G_p. \tag{64}$$

Interchanging the order of summation gives

$$\sum_{p=1}^{N-1} \underbrace{\sum_{k=1}^{N-1} Y_{m,k}^{nN} Y_{k,k_o}^{nN} Y_{k,p}^{nN}}_{\text{shift operator}} G_p = G_{m,k_o}^{\text{shift}}. \tag{65}$$

By comparing Eq. (65) with Eqs. (56)–(57), we recognize the shift operator as indicated in (65). This produces a modulation-shift rule as would be expected, so that the forward DHT of a modulated function is equivalent to a generalized shift in the frequency domain. This yields another transform pair:

$$Y_{k,k_o}^{nN} g_k \Leftrightarrow G_{m,k_o}^{\text{shift}}. \tag{66}$$

In other words, Eq. (66) says that modulation in the space domain is equivalent to a shift in the frequency domain, as would be expected for a (generalized) Fourier transform.

F. Convolution

We consider the convolution using the generalized shifted function previously defined. The convolution of two functions is defined as

$$f_k = (g * h)_k = \sum_{k_0=1}^{N-1} g_{k_o} h_{k,k_o}^{\text{shift}}. \tag{67}$$

The meaning of Eq. (67) follows from the traditional definition of a convolution: multiply one of the functions by a shifted version of a second function and then sum over all possible shifts.

Subsequently, from the definition of the inverse transforms, we obtain

$$\begin{aligned}\sum_{k_0=1}^{N-1} g_{k_o} h_{k,k_o}^{\text{shift}} &= \sum_{k_0=1}^{N-1} \underbrace{\sum_{q=1}^{N-1} Y_{k_o,q}^{nN} G_q}_{g_{k_o}} \underbrace{\sum_{p=1}^{N-1} Y_{k,p}^{nN} Y_{p,k_o}^{nN} H_p}_{h_{k,k_o}^{\text{shift}}} \\ &= \sum_{q=1}^{N-1} \sum_{p=1}^{N-1} \underbrace{\sum_{k_0=1}^{N-1} Y_{p,k_o}^{nN} Y_{k_o,q}^{nN}}_{=\delta_{pq}} Y_{k,p}^{nN} H_p G_q. \end{aligned} \tag{68}$$

However, from the orthogonality relationship (20), the summation over k_0 gives the Kronecker-delta function, so that Eq. (68) becomes

$$\begin{aligned}(g * h)_k &= \sum_{k_0=1}^{N-1} g_{k_o} h_{k,k_o}^{\text{shift}} = \sum_{q=1}^{N-1} \sum_{p=1}^{N-1} \delta_{pq} Y_{k,p}^{nN} H_p G_q \\ &= \sum_{p=1}^{N-1} Y_{k,p}^{nN} (H_p G_p). \end{aligned} \tag{69}$$

The right-hand side of Eq. (69) is clearly the inverse transform of the product of the transforms $H_p F_p$. This gives us another transform pair:

$$(g * h)_k = \sum_{k_0=1}^{N-1} g_{k_o} h_{k,k_o}^{\text{shift}} \Leftrightarrow H_m G_m. \tag{70}$$

It follows from Eq. (69) that interchanging the roles of g and h will yield the same result, meaning

$$\sum_{k_0=1}^{N-1} g_{k,k_o}^{\text{shift}} h_{k_o} = \sum_{p=1}^{N-1} Y_{k,p}^{nN} G_p H_p. \tag{71}$$

Therefore, it follows that

$$(h * g)_k = \sum_{k_0=1}^{N-1} g_{k,k_o}^{\text{shift}} h_{k_o} = \sum_{k_0=1}^{N-1} g_{k_o} h_{k,k_o}^{\text{shift}} = (g * h)_k. \tag{72}$$

G. Multiplication

We now consider the forward transform of a product in the space domain $f_k = g_k h_k$ so that

$$\sum_{k=1}^{N-1} Y_{m,k}^{nN} g_k h_k = \sum_{k=1}^{N-1} Y_{m,k}^{nN} \underbrace{\sum_{q=1}^{N-1} Y_{k,q}^{nN} G_q}_{g_k} \underbrace{\sum_{p=1}^{N-1} Y_{k,p}^{nN} H_p}_{h_k}. \tag{73}$$

Rearranging gives

$$\begin{aligned}\sum_{k=1}^{N-1} Y_{m,k}^{nN} g_k h_k &= \sum_{q=1}^{N-1} G_q \sum_{p=1}^{N-1} \underbrace{\sum_{k=1}^{N-1} Y_{m,k}^{nN} Y_{k,q}^{nN} Y_{k,p}^{nN}}_{\text{shift operator}} H_p \\ &= \sum_{q=1}^{N-1} G_q H_{m,q}^{\text{shift}} = (G * H)_m. \end{aligned} \tag{74}$$

This gives us yet another transform pair that says that multiplication in the spatial domain is equivalent to convolution in the transform domain:

$$g_k h_k \Leftrightarrow \sum_{q=1}^{N-1} G_q H_{m,q}^{\text{shift}} = (G * H)_m. \tag{75}$$

Interchanging the roles of G and H in Eq. (75) demonstrates that convolution in the transform domain also commutes:

$$(G * H)_m = \sum_{q=1}^{N-1} G_q H_{m,q}^{\text{shift}} = \sum_{q=1}^{N-1} G_{m,q}^{\text{shift}} H_q = (H * G)_m. \tag{76}$$

10. USING THE DHT TO APPROXIMATE THE CONTINUOUS HANKEL TRANSFORM

Equations (10) and (13), with their infinite summations, are exact for band-limited and space-limited functions, respectively. Here, we propose using the discrete Hankel transform with finite domains for both space and frequency to approximate the continuous transform (where it is understood that a function could be band-limited or space-limited, but not both). In other words, we propose the following N-dimensional discretization scheme in finite space $[0, R]$ and finite frequency space $[0, W_\rho]$ given by

$$\left.\begin{aligned} r_{nk} &= \frac{j_{nk}}{W_\rho} = \frac{j_{nk} R}{j_{nN}} \\ \rho_{nk} &= \frac{j_{nk}}{R} = \frac{j_{nk} W_\rho}{j_{nN}} \end{aligned}\right\} \quad k = 1 \ldots N-1. \tag{77}$$

In Eq. (77), the order of the Bessel zeros must match the order of the discrete Hankel transform that is sought, in keeping with the derivation of the intuitive discretization scheme in Section 4. The relationship $W_\rho = \frac{j_{nN}}{R}$ can be used to relate the finite frequency domain to the finite space domain.

If the function $f(r)$ is defined between $[0, R]$ and we want to calculate a forward transform, then it is proposed to use the space and frequency space discretization scheme of

$$r_{nk} = \frac{j_{nk}R}{j_{nN}}, \quad \rho_{nm} = \frac{j_{nm}}{R} \quad 1 \leq m, \quad k \leq N-1. \tag{78}$$

To compute the forward transform, sample the function $f(r)$ on $r_{nk} = \frac{j_{nk}R}{j_{nN}}$ and assign this to f_k so that $f_k = f(r_{nk}) = f\left(\frac{j_{nk}R}{j_{nN}}\right)$. Then calculate F_m from the DHT:

$$F_m = \frac{R^2}{j_{nN}} \sum_{k=1}^{N-1} Y_{m,k}^{nN} f_k \quad \text{or} \quad \mathbf{F} = \frac{R^2}{j_{nN}} Y^{nN} \mathbf{f}. \tag{79}$$

The resulting values F_m are then an approximation to $F(\rho_{nm}) = F\left(\frac{j_{nm}}{R}\right)$.

Conversely, if the Hankel transform $F(\rho)$ is defined on $[0, W_\rho]$ and it is desired to calculate the inverse transform, then use the discretization scheme

$$\left.\begin{aligned} r_{nk} &= \frac{j_{nk}}{W_\rho} \\ \rho_{nm} &= \frac{j_{nm}W_\rho}{j_{nN}} \end{aligned}\right\} \quad 1 \leq m, \quad k \leq N-1. \tag{80}$$

Sample the function $F(\rho_{nm}) = F\left(\frac{j_{nm}W_\rho}{j_{nN}}\right)$ and assign this to F_m so $F_m = F(\rho_{nm}) = F\left(\frac{j_{nm}W_\rho}{j_{nN}}\right)$. Calculate f_k from the IDHT:

$$f_k = \frac{W_\rho^2}{j_{nN}} \sum_{m=1}^{N-1} Y_{k,m}^{nN} F_m \quad \text{or} \quad \mathbf{f} = \frac{W_\rho^2}{j_{nN}} Y^{nN} \mathbf{F}. \tag{81}$$

Then the resulting values f_k calculated via the IDHT are an approximation to $f(r_{nk}) = f\left(\frac{j_{nk}}{W_\rho}\right)$.

A. Numerical Tests

The calculations proposed to use the DHT to approximate the continuous Hankel transform were tested on two functions with known forward and inverse Hankel transforms. All simulations were performed using Matlab (Mathworks). The first function (modified Gaussian) is given by

$$f(r) = e^{-a^2 r^2} r^n. \tag{82}$$

Here, a is a real number and n is an integer. The continuous nth order Hankel transform of (82) [so that the order n of the Hankel transform is the same as the power of r in (82)] is known to be given by [49]

$$F(\rho) = \frac{\rho^n}{(2a^2)^{n+1}} e^{-\frac{\rho^2}{4a^2}}. \tag{83}$$

For the modified Gaussian, the closed form analytical expression for the nth order Hankel transform only exists when the order of the modification is the same as the order of the Hankel transform.

The second function to be tested is the sinc function, which is given by

$$f(r) = \frac{\sin(ar)}{ar}. \tag{84}$$

Its nth order Hankel transform is also known to be given by [49]

$$F(\rho) = \begin{cases} \dfrac{\cos\left(\frac{\pi n}{2}\right)}{a^2\sqrt{1-\frac{\rho^2}{a^2}}} \cdot \dfrac{\left(\frac{\rho}{a}\right)^n}{\left(1+\sqrt{1-\frac{\rho^2}{a^2}}\right)^n} & \text{for } \rho < a \\ \dfrac{\sin\left[n \arcsin\left(\frac{a}{\rho}\right)\right]}{a^2\sqrt{\frac{\rho^2}{a^2}-1}} & \text{for } \rho > a \end{cases}. \tag{85}$$

For the purpose of testing the accuracy of the DHT and IDHT, the dynamic error is used, defined as [2]

$$e(v) = 20\log_{10}\left[\frac{|f(v) - f^*(\nu)|}{\max|f^*(v)|}\right]. \tag{86}$$

This error function compares the difference between the exact function values $f(v)$ (evaluated from the continuous function) and the function values estimated via the discrete transform, $f^*(\nu)$, scaled with the maximum value of the discretely estimated samples. Equation (86) can be used to evaluate the computation of either forward or inverse Hankel transform via the DHT/IDHT, and compared with known continuous Hankel relationships. The dynamic error uses the ratio of the absolute error to the maximum amplitude of the function on a log scale. Therefore, negative decibel errors imply an accurate discrete estimation of the true transform value. It is noted that −320 dB corresponds to floating point numerical accuracy. The transform is also tested for accuracy on itself. This is performed by consecutive forward and then inverse transformation to verify that the transforms themselves do not add errors. For this evaluation, the average absolute error $\frac{1}{N}\sum_{i=1}^{N}|f_i - f_i^*|$ is used.

To maintain good sampling of the function for the discrete transform, it is important to maintain the $W_\rho = \frac{j_{nN}}{R}$ relationship. Although the functions chosen for testing the transforms are not exactly space or band-limited, they can be considered to be approximately limited since the function or its transform approaches zero at some point (it can be made as close to zero as we like). Moreover, the two functions were chosen so that one is effectively space limited and the other is effectively band-limited. Thus, for the purpose of the tests, the true function has been computed, and space and band-limits have been imposed. It is to be noted that the functions are evaluated at the (scaled) Bessel zeros in accordance with Eqs. (78) or (80), which implies that their evaluation does not start at zero on the plots.

1. Evaluation of the DHT of the Modified Gaussian

The first function, the modified Gaussian of Eq. (82), is tested with $a = 5$, and with two different orders of n, $n = 1$ and $n = 11$. The true functions are shown in Fig. 1.

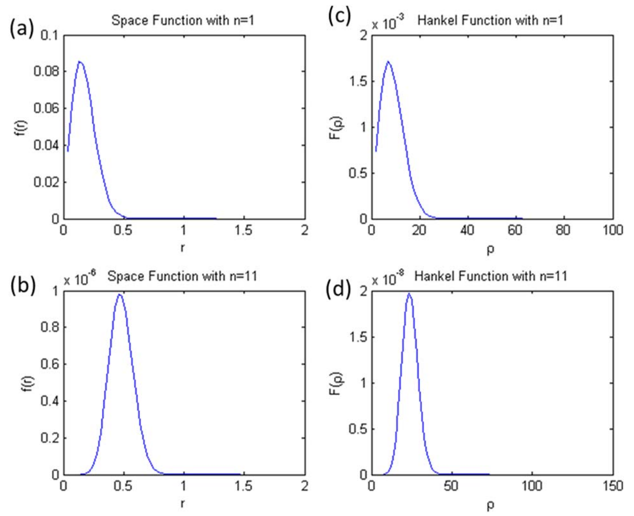

Fig. 1. (a) and (b) Scaled Gaussian and its (c) first- and (d) eleventh-order Hankel Transform.

From the graph of the function and its transform for $n = 1$, it can be assumed that $f(r)$ is space-limited at $r = 1$ and $F(\rho)$ is band-limited at $\rho = 50$. Taking a slightly larger domain for numerical tests, we choose $R = 2$ and $W_\rho = 100$. From $W_\rho = \frac{j_{nN}}{R}$, the closest Bessel function zero is $j_{nN} = 201.8455$, with $N = 64$. The same can be done with the $n = 11$ function. Consequently, values are taken as $R = 2$ and $W_\rho = 110$. This results in $N = 64$ and $j_{nN} = 217.2774$. Choosing the function as space limited results in using the scaling factors $\frac{R^2}{j_{nN}}$ for the DHT and $\frac{j_{nN}}{R^2}$ for the IDHT.

The results of performing the finite DHT and IDHT calculations and comparing these with evaluating the continuous transforms at the same sampling points are shown in Figs. 2 and 3, respectively. In the figures, the solid lines are the continuous transforms and the dotted lines are the discrete transforms.

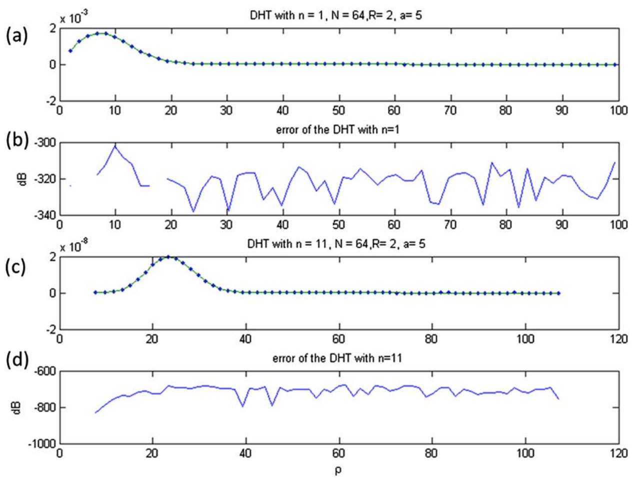

Fig. 2. (a) Continuous (solid line) and discrete (dotted line) $n = 1$ Hankel transform of the $n = 1$ modified Gaussian, and (b) the error between the continuous and discrete $n = 1$ HT. (c) Continuous (solid line) and discrete (dotted line) $n = 11$ Hankel transform of the $n = 11$ modified Gaussian, and (d) the error between the continuous and discrete $n = 11$ HT.

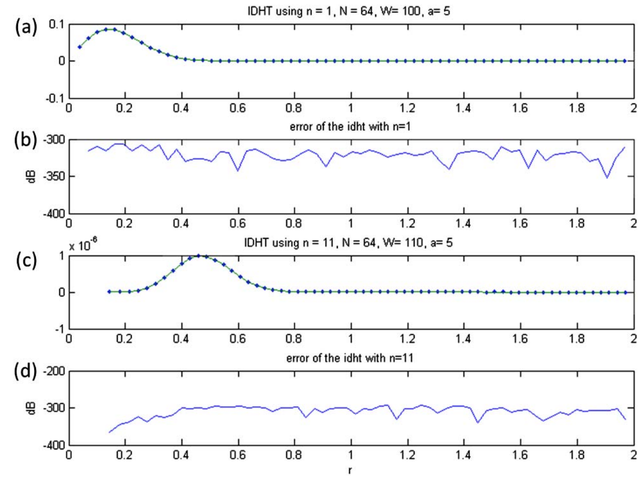

Fig. 3. (a) Continuous (solid line) and discrete (dotted line) $n = 1$ Inverse Hankel transform of the $n = 1$ modified Gaussian, and (b) the error between the continuous and discrete $n = 1$ IHT. (c) Continuous (solid line) and discrete (dotted line) $n = 11$ Inverse Hankel transform of the $n = 11$ modified Gaussian, and (d) the error between the continuous and discrete $n = 11$ IHT.

It can be noted that, for both the forward and inverse transforms, the computed error is very low, even for relatively small values of N. Performing the forward DHT and then the IDHT on the obtained result results in an average absolute error of 1.6926e-17 from the original function for $n = 1$ and 8.5249e-22 for $n = 11$, for $N = 64$ in both cases.

2. Evaluation of the DHT of the Sinc Function

For the sinc function of Eq. (84), the assumed limits are taken with respect to the true functions, as shown in Fig. 4.

In both cases, from Fig. 4 it can be seen that the Hankel function is effectively band-limited at $W_\rho = 30$. Thus, taking a sample size of $N = 256$ gives $j_{nN} = 805.0327$ for $n = 1$ and $j_{nN} = 820.6675$ for $n = 11$, we obtain $R = 26.75$ for $n = 1$ and $R = 27.5$ for $n = 11$.

Since the function is band-limited, the approximation to the continuous transform is done by using the frequency scaling factor $\frac{j_{nN}}{W_p^2}$ for the DHT and $\frac{W_p^2}{j_{nN}}$ for the IDHT. The results of performing the finite DHT and IDHT calculations and comparing

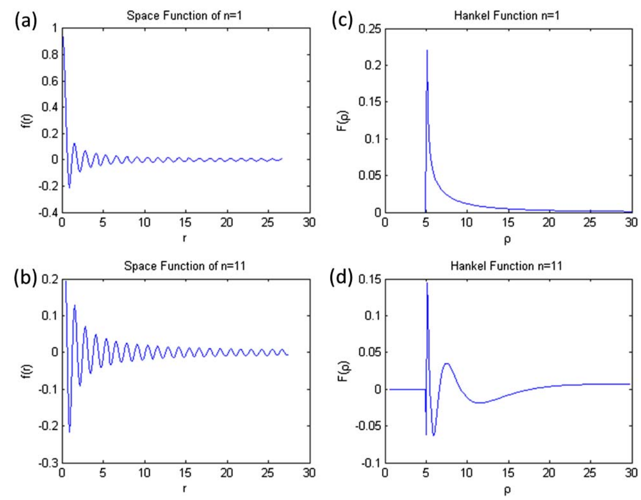

Fig. 4. (a) and (b) Sinc function and its (c) HT with $n = 1$ and (d) HT for $n = 11$.

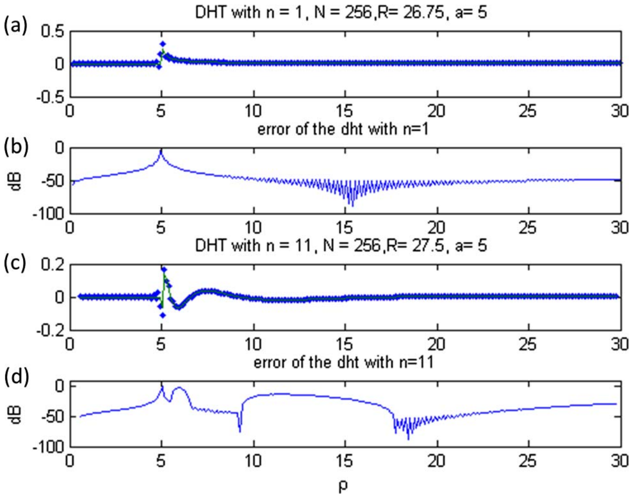

Fig. 5. (a) Comparison of continuous (solid line) and discrete (dotted line) $n = 1$ Hankel transform of a sinc, (b) the error in part (a), (c) continuous and discrete $n = 11$ HT of a sinc, and (d) the error in part (c).

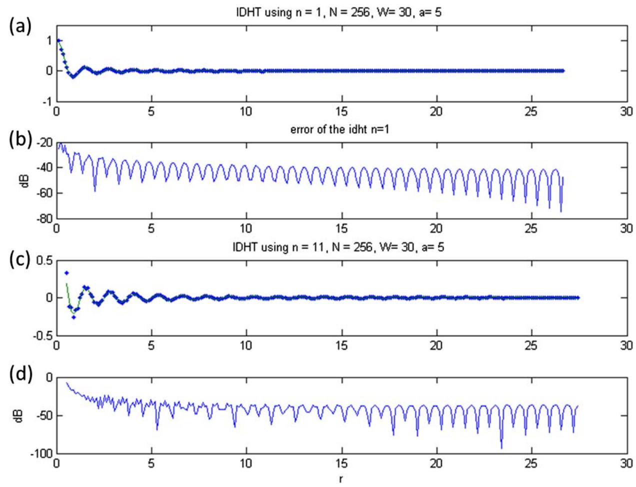

Fig. 6. (a) Comparison of continuous (solid line) and discrete (dotted line) $n = 1$ Inverse Hankel transform of a sinc, (b) the error in part (a), (c) continuous and discrete $n = 11$ IHT of a sinc, and (d) the error in part (c).

Table 1. Summary of DHT Relationships

Operation	f_k	F_m
Definition of forward transform	f_k	$\sum_{k=1}^{N-1} Y_{m,k}^{nN} f_k$
Definition of inverse transform	$\sum_{m=1}^{N-1} Y_{k,m}^{nN} F_m$	F_m
Dirac-delta in space	δ_{kk_o}	$Y_{m,k_0}^{n,N}$
Dirac-delta in frequency	$Y_{k,m_0}^{n,N}$	δ_{mm_o}
Generalized shift in space	$f_{k,k_o}^{\text{shift}} = \sum_{p=1}^{N-1} Y_{k,p}^{nN} Y_{p,k_o}^{nN}$	$Y_{m,k_o}^{nN} F_m$
	$F_p = \sum_{m=1}^{N-1} \underbrace{\sum_{p=1}^{N-1} Y_{k,p}^{nN} Y_{p,k_o}^{nN} Y_{p,m}^{nN}}_{\text{shift operator}} f_m$	
Generalized shift in frequency	$Y_{k,k_o}^{nN} g_k$	G_{m,k_o}^{shift}
Convolution in space	$(g * h)_k = \sum_{k_0=1}^{N-1} g_{k_o} h_{k,k_o}^{\text{shift}}$	$H_m G_m$
Multiplication in space	$g_k h_k$	$\sum_{m_0=1}^{N-1} G_{m_o} H_{m,m_0}^{\text{shift}} = (G * H)_m$

these with evaluating the continuous transforms at the same sampling points are shown in Figs. 5 and 6, respectively. In the graphs, the continuous transform is displayed as a solid line and the discrete transform as the dots.

The DHT suffers from Gibbs phenomenon at the discontinuity, as expected. However, the dynamic error for the rest of the function remains low. Furthermore, performing the forward DHT and then the IDHT on the obtained results in an average absolute error of 5.2274e-15 from the original function for $n = 1$ and 6.1430e-13 for $n = 11$, for $N = 256$ in both cases.

11. SUMMARY AND CONCLUSIONS

In summary, in this paper we have motivated, proposed, and evaluated the mathematical theory for the DHT, using the same ideas that have shown the DFT to be so useful in various disciplines. The standard set of shift, modulation, multiplication, and convolution rules were derived. The summary of these rules is shown in Table 1. In addition, we proposed the use of this DHT to approximate the continuous Hankel transform in the same manner that the DFT is known to be able to approximate the continuous Fourier transform, using specifically chosen sampling points in both spatial and frequency domains. The errors of using the DHT to approximate its continuous counterpart were shown to be low for the chosen effectively space-limited and effectively band-limited functions.

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