Cauchy integral formula:

If $f(z)$ is analytic within and on a closed contour C and if 'a' is any point within C then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

$$\Rightarrow f(a)\, 2\pi i = \oint_C \frac{f(z)}{z-a} dz$$

Cauchy Integral formula for derivatives:

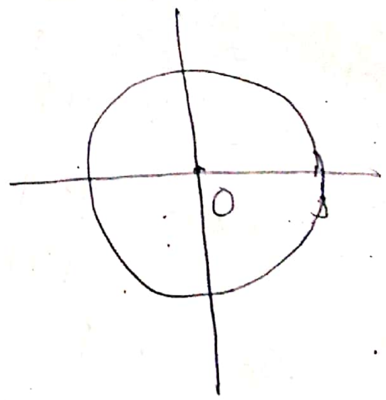
$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz$$

$$f^n(a) = \frac{\lfloor n}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}},$$

where $n = 0, 1, 2, ..$

(1) Using Cauchy's integral formula evaluate

$$\int_C \frac{1}{z(z^2+9)} dz$$ where C is the circle $|z| = 3$



Soln: Given that,

$$\int_C \frac{1}{z(z^2+9)} dz$$

Let, $f(z) = \dfrac{1}{z^2+9}$ ... (1)

$z = 0$ lies inside the circle $|z| = 3$

Now the given integral can be written as

$$\int_c \frac{f(z)}{z}\,dz$$

$$= \int_c \frac{f(z)}{z-0}\,dz$$

$= 2\pi i f(0)$ [by Cauchy's integral formula]

$= 2\pi i \times \frac{1}{9}$, Since, $f(z) = \frac{1}{z^2+9}$

$= \frac{2\pi i}{9}$ (Ans) $\qquad f(0) = \frac{1}{9}$

(ii) $\displaystyle\int_c \frac{dz}{(z^2+1)(z^2+9)}$ where $|z+3| = 3$

let, $f(z) = \frac{1}{z^2+9}$

Here $|z+3| = 3$ is a circle with center $(-3,0)$ and radius 3

$|x+iy+3| = 3$

$\sqrt{(x+3)^2+y^2} = 3$

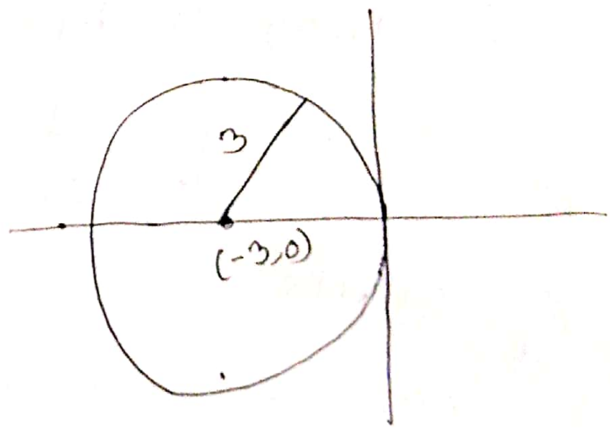
$\Rightarrow (x+3)^2 + y^2 = 3^2$

Now,

$z^2+1 = 0$

$\Rightarrow z^2 = -1$

$\therefore z = \pm i$



Here $|z| = |i| = 1 < 3$

$|z| = |-i| = 1 < 3$

So $z = i$ and $z = -i$ lies inside the circle $|z+3| = 3$

Then the given integral can be written as

$$\int_c \frac{dz}{(z^2+1)(z^2+9)} = \int_c \frac{f(z)\,dz}{z^2+1}$$

$$= \int_c \frac{f(z)\,dz}{z^2-i^2} \quad [i^2=-1]$$

$$= \int_c \frac{f(z)\,dz}{(z+i)(z-i)}$$

$$= \int_c \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right] f(z)\,dz$$

$$= \frac{1}{2i}\int_c \frac{f(z)}{z-i}\,dz - \frac{1}{2i}\int_c \frac{f(z)\,dz}{z-(-i)}$$

$$= \frac{1}{\cancel{2i}} \times \cancel{2\pi i} f(i) - \frac{1}{2i} \times 2\pi i f(-i)$$

[By cauchy's integral formula]

$$= \pi\frac{1}{8} - \pi\frac{1}{8}$$

[Since,

$f(z) = \frac{1}{z^2+9}$

$f(i) = \frac{1}{i^2+9} = \frac{1}{8}$

$f(-i) = \frac{1}{i^2+9} = \frac{1}{8}$

$$\therefore \int_c \frac{dz}{(z^2+1)(z^2+9)} = 0.$$

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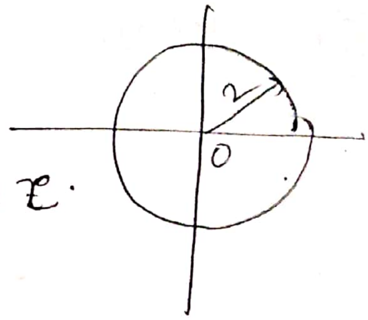
(iii) $\int_c \frac{z}{(9-z^2)(z+i)}$, $|z|=2$

Soln: Let,

$f(z) = \frac{z}{9-z^2}$, which is analytic inside and on the circle C.



Here $|z|=2$ is a circle of centre $(0,0)$ and radius 2

Now,

$z+i=0$

$\Rightarrow z=-i$

$\therefore |z| = |-i| = 1$, lies inside the circle $|z|=2$.

The given integral can be written as

$$\int_c \frac{z\,dz}{(9-z^2)(z+i)} = \int_c \frac{f(z)\,dz}{z+i}$$

$$= \int_c \frac{f(z)\,dz}{z-(-i)}$$

$= 2\pi i f(-i)$ [By Cauchy's integral formula]

$= 2\pi i \times \frac{-i}{10}$, since, $f(z) = \frac{z}{9-z^2}$

$= \frac{-2\pi i^2}{10}$ $\quad f(-i) = \frac{-i}{9-(-i)^2}$

$= \frac{\pi}{5}$ $\quad = \frac{-i}{9+1} = -\frac{i}{10}$

$$\therefore \int_c \frac{z\,dz}{(9-z^2)(z+i)} = \frac{\pi}{5}.$$

$\int_C \frac{e^{3z}}{z+\pi i}\,dz$ where C is the circle $|z+1|=4$

<ins>Soln:</ins> Here the centre of the circle is $(-1,0)$ and radius 4.

$|x-iy+1|=4$
$\Rightarrow |(x+1)-iy|=4$
$\Rightarrow \sqrt{(x+1)^2-y^2}=4$
$(x+1)^2-y^2=4$

Here $z=-\pi i$

$\therefore |z|=|-\pi i|=3.1416<4$

So $|z|=|-\pi i|$ lies inside the circle $|z+1|=4$.

let, $f(z)=e^{3z}$ then $f(z)$ is analytic inside and on C.

Hence, by Cauchy's integral formula we get,

$$\int_C \frac{e^{3z}}{z+\pi i}\,dz = \int_C \frac{f(z)}{z-(-\pi i)}\,dz$$

$= 2\pi i f(-\pi i)$

$= 2\pi i\, e^{-i3\pi}$, since, $f(z)=e^{3z}$, $f(-\pi i)=e^{-3\pi i}$

$= 2\pi i(\cos 3\pi - i\sin 3\pi)$

$= 2\pi i(-1-0)$

$= -2\pi i$ (Ans)

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H.W $\int_c \frac{dz}{(z-4)^2}$, C is the circle $|z|=2$

Ans: $\frac{\pi i}{8}$.

Evaluate $\int_c \frac{e^z}{z^2+1} dz$ over the circular path $|z|=2$

Soln: Here $|z|=2$ is a circle with centre $(0,0)$ and radius 2.

Now, $z^2+1=0$

$\Rightarrow z^2=-1$

$\therefore z=\pm i$

$|z|=|i|=1$, lies inside the circle $|z|=2$

$|z|=|-i|=1$ " " $|z|=2$

let, $f(z)=e^z$ is analytic inside and on c.

$$\therefore \int_c \frac{e^z}{z^2+1}dz = \int_c \frac{f(z)}{(z+i)(z-i)} dz$$

$$= \frac{1}{2i}\int_c \left[\frac{1}{z-i} - \frac{1}{z+i}\right] f(z)\, dz$$

$$= \frac{1}{2i}\int_c \frac{f(z)}{z-i} dz - \frac{1}{2i}\int_c \frac{1}{z+i} f(z)\, dz$$

$$= \frac{1}{2i} \cdot 2\pi i f(i) - \frac{1}{2i} 2\pi i f(-i)$$

By Cauchy's integral formula

$= \pi e^{i} - \pi e^{-i}$

$= \pi (e^{i} - e^{-i})$

$= \pi (2i \sin 1)$ (Ans)

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$\sin 1 = \frac{e^{i} - e^{-i}}{2i}$

Since

$f(z) = e^{z}$

$f(i) = e^{i}$

$f(-i) = e^{-i}$

Problem: Show that $\oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi i e^{-2}}{3}$

where C is the circle $|z| = 3$

Soln: $|z| = 3$ is a circle with centre $(0,0)$ and radius 3.

$\therefore$ ~~$z+1=0, z=$~~

$\therefore z = -1$ lies inside the circle

$|z| = 1$

$|z| = 3$

let $f(z) = e^{2z}$ which is analytic inside and on C.

Then by Cauchy's integral formula for n-th derivative we get,

$$\oint_C \frac{e^{2z}}{(z+1)^4} dz = \oint_C \frac{f(z)}{\{z-(-1)\}^4} dz$$

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$$\Rightarrow \oint_C \frac{e^{2z}}{(z+1)^4}\,dz = 2\pi i\,\frac{f'''(-1)}{\lfloor 3} \quad \cdots (1)$$

Here, $f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$\therefore f'''(-1) = 8e^{-2}$

Then from (1) we get,

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz = 2\pi i\,\frac{8e^{-2}}{6}$$

$$= \frac{8\pi i}{3}\,e^{-2} \text{ (Ans)}$$