Random variable and probability function

Formulas: In case of one discrete random variable:

(i) Condition of the probability function

$\Sigma P(x) = 1$.

(ii) Distribution function $F(x) = \sum^{x} P(x)$

Problem: If $P(x) = \frac{x+1}{14}$, $x = 1, 2, 3, 4$ then show that $P(x)$ is a probability function and find the value of $P(2 \leq x < 4)$.

Soln: Given,

$$P(x) = \frac{x+1}{14}, \quad x = 1, 2, 3, 4$$

We know, $P(x)$ is a probability function if $\Sigma P(x) = 1$.

then
$$\Sigma P(x) = \sum_{x=1}^{4} \frac{x+1}{14}$$

$$= \frac{1+1}{14} + \frac{2+1}{14} + \frac{3+1}{14} + \frac{4+1}{14}$$

$$= \frac{14}{14} = 1$$

So $P(x)$ is a probability function.

Again, $P(2 \le x < 4) = P(x=2) + P(x=3)$

$$= \frac{2+1}{14} + \frac{3+1}{14}$$

$$= 0.5$$

Problem: The probability distribution of discrete random variable x is given below:

$$P(x) = \frac{2x+k}{56}, \quad x = -3, -2, -1, 0, 1, 2, 3$$

(i) Find the value of k

(ii) Show that $P(x)$ is a probability function

(iii) Find the distribution function and show that $P(0 < x \le 2) = F(2) - F(0)$

(iv) Find the value of $P(-2 \le x \le 2)$

Soln: Given that,

$$P(x) = \frac{2x+k}{56}, \quad x = -3, -2, -1, 0, 1, 2, 3$$

(i) We know,

$$\sum P(x) = 1$$

$$\Rightarrow \sum_{x=-3}^{3} \frac{2x+k}{56} = 1$$

$$\Rightarrow \frac{2(-3)+k}{56} + \frac{2(-2)+k}{56} + \frac{2(-1)+k}{56} + \frac{2(0)+k}{56}$$

$$+ \frac{2 \cdot 2+k}{56} + \frac{2 \cdot 3+k}{56} = 1$$

$$\Rightarrow \frac{-6+k}{56} + \frac{-4+k}{56} + \frac{-2+k}{56} + \frac{0+k}{56} + \frac{2+k}{56}$$

$$+ \frac{4+k}{56} + \frac{6+k}{56} = 1$$

$$\Rightarrow \frac{-6+k-4+k-2+k+0+k+2+k+4+k+6+k}{56} = 1$$

$$\Rightarrow \frac{0+7k}{56} = 1$$

$$\Rightarrow 7k = 56$$

$$\therefore k = 8$$

(ii) We have,

$$p(x) = \frac{2x+8}{56} \ ; \ x = -3, -2, -1, 0, 1, 2, 3$$

$$= \frac{x+4}{28}$$

We know, $p(x)$ is a probability function if $\Sigma P(x) = 1$.

Now $\Sigma P(x) = \sum_{x=-3}^{3} \frac{x+4}{28}$

$$= \frac{-3+4}{28} + \frac{-2+4}{28} + \frac{-1+4}{28} + \frac{0+4}{28}$$

$$+ \frac{1+4}{28} + \frac{2+4}{28} + \frac{3+4}{28}$$

$$= \frac{28}{28} = 1$$

So $p(x)$ is a probability function.

(III) We have,

$$p(x) = \frac{x+4}{28}, \quad x = -3, -2, -1, 0, 1, 2, 3$$

The probability distribution and distribution function of x are given below:

x	-3	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{28}$	$\frac{2}{28}$	$\frac{3}{28}$	$\frac{4}{28}$	$\frac{5}{28}$	$\frac{6}{28}$	$\frac{7}{28}$
$F(x)$	$\frac{1}{28}$	$\frac{3}{28}$	$\frac{6}{28}$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{21}{28}$	$\frac{28}{28}=1$

$$L.H.S = P(0 < x \leq 2)$$

$$= P(x=1) + P(x=2)$$

$$= \frac{5}{28} + \frac{6}{28} = \frac{11}{28}$$

$$R.H.S = F(2) - F(0)$$

$$= \frac{21}{28} - \frac{10}{28}$$

$$= \frac{11}{28}$$

$\therefore$ L.H.S = R.H.S

(iv) $P(-2 \le x \le 2) = \sum_{x=-2}^{2} P(x) = \sum_{x=-2}^{2} \frac{x+4}{28}$

$= \frac{-2+4}{28} + \frac{-1+4}{28} + \frac{0+4}{28} + \frac{1+4}{28} + \frac{2+4}{28}$

$= \frac{5}{7}$.

Format 2: In case of two discrete random variable

(i) In the case of finding the value of the constant (k/a/b etc)

we know $\sum_x \sum_y P(x,y) = 1$ or $\sum_y \sum_x P(x,y) = 1$

(ii) (a) The marginal probability function of x is $P(x) = \sum_y P(x,y)$

(b) The marginal probability function of y is $P(y) = \sum_x P(x,y)$

(iii) (a) Conditional probability of x given y is $P(x|y) = \frac{P(x,y)}{P(y)}$

(b) The conditional probability of y given x is $P(y|x) = \frac{P(x,y)}{P(x)}$

Problem: The joint probability function of two random variables x and y is given below:

$$P(x,y) = \frac{x+2y}{16}, \quad x=0,1, \quad y=0,1,2,3$$

(i) Find the marginal probability function of x and y.

(ii) Find the conditional probability of x given y.

Soln: Given,

$$P(x,y) = \frac{x+2y}{16}, \quad x=0,1$$
$$y=0,1,2,3$$

The marginal probability of x is

$$P(x) = \sum_{y} P(x,y) = \sum_{y=0}^{3} \frac{x+2y}{16}$$

$$= \frac{x+2\cdot 0}{16} + \frac{x+2\cdot 1}{16} + \frac{x+2\cdot 2}{16} + \frac{x+2\cdot 3}{16}$$

$$= \frac{x+x+2+x+4+x+4}{16}$$

$$= \frac{4x+12}{16} = \frac{4(x+3)}{16} = \frac{x+3}{4}$$

The marginal probability of y is

$$P(y) = \sum_{x} P(x,y) = \sum_{x=0}^{1} \frac{x+2y}{16}$$

$$= \frac{0+2y}{16} + \frac{1+2y}{16}$$

$$= \frac{4y+1}{16}$$

Scanned with CamScanner

(iii) The conditional probability of x given y is

$$P(x|y) = \frac{P(x,y)}{P(y)}$$

$$= \frac{\frac{x+2y}{16}}{\frac{4y+1}{16}} = \frac{x+2y}{4y+1}$$

Format 3:

In case of one continuous random Variable:

(i) In order to find the value of the constant (k/a/b etc) or condition of the probability density function

We know, $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

i.e $$\int_{\text{lowest value of } x}^{\text{highest value of } x} f(x)\,dx = 1$$

(ii) Distribution function

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

(iii) $$P(a<x<b) = \int_a^b f(x)\,dx$$

$$P(x \geq a) = \int_a^{\infty} f(x)\,dx$$

$$P(x<b) = \int_{-\infty}^{b} f(x)\,dx$$

(III) The conditional probability of x given y is

$$P(x|y) = \frac{P(x,y)}{P(y)}$$

$$= \frac{\frac{x+2y}{16}}{\frac{4y+1}{16}} = \frac{x+2y}{4y+1}$$

Format 3:

In case of one continuous random variable:

(I) In order to find the value of the constant (k/a/b etc) or condition of the probability density function

we know, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e $\int_{\text{lowest value of } x}^{\text{highest value of } x} f(x)\,dx = 1$

(II) Distribution function

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

(III) $P(a<x<b) = \int_{a}^{b} f(x)\,dx$

$P(x \geq a) = \int_{a}^{\infty} f(x)\,dx$

$P(x<b) = \int_{-\infty}^{b} f(x)\,dx$

Problem: The probability density function of a continuous random variable is given below:

$$f(x) = k(x+1) \; ; \; 0<x<1$$
$$= 0 \; ; \; \text{Otherwise}$$

(i) Find the value of k

(ii) Find the distribution function and show that $F(1) - F(0) = P(x>0)$

Soln: Given that,

$$f(x) = k(x+1) \; ; \; 0<x<1$$
$$= 0 \; ; \; \text{Otherwise}$$

(1) We know,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} k(x+1)\,dx = 1$$

$$\Rightarrow k\left[\frac{x^2}{2} + x\right]_0^1 = 1$$

$$\Rightarrow k\left\{\left(\frac{1}{2} + 1 - 0 - 0\right)\right\} = 1$$

$$\Rightarrow k \times \frac{3}{2} = 1$$

$$\therefore k = \frac{2}{3}$$

(ii) Distribution function,

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{0}^{x} k(x+1)\,dx$$

$$= k\left[\frac{x^2}{2} + x\right]_0^x$$

$$= k\left(\frac{x^2}{2} + x\right) = \frac{2}{3}\left(\frac{x^2+2x}{2}\right)$$

$$= \frac{x^2+2x}{3}$$

Distribution function,

$$F(x) = \frac{x^2+2x}{3}, \quad 0 < x < 1$$

$$F(1) = \frac{1^2+2.1}{3} = \frac{3}{3} = 1$$

$$F(0) = \frac{0}{3} = 0$$

$$L.H.S = F(1) - F(0) = 1 - 0 = 1$$

$$R.H.S = P(x > 0)$$

$$= \int_0^{\infty} f(x)\,dx$$

$$= \int_0^1 k(x+1)\,dx$$

$$= k\left[\frac{x^2}{2} + x\right]_0^1 = k\left[\left(\frac{1^2}{2}+1\right) - \left(\frac{0^2}{2}+0\right)\right]$$

$$= k\left(\frac{1}{2}+1\right)$$

$$= \frac{2}{3} \times \frac{3}{2} = 1$$

$\therefore F(1) - F(0) = P(x > 0)$ (Proved)

Problem: Show that $f(x) = \frac{1}{30}(3+2x)$; $2<x<5$ is a probability density function and find the value of $P(x \geq 4)$.

Soln: We know,

$f(x)$ is a probability density function if

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Now,
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{2}^{5} \frac{1}{30}(3+2x)\,dx$$
$$= \frac{1}{30}\int_{2}^{5}(3+2x)\,dx$$
$$= \frac{1}{30}\left[3x + 2\frac{x^2}{2}\right]_2^5$$
$$= \frac{1}{30}\left\{(3\cdot5+5^2) - (3\cdot2+2^2)\right\}$$
$$= \frac{1}{30}\left\{(15+25) - (6+4)\right\}$$
$$= \frac{1}{30} \times 30 = 1$$

$\therefore$ $f(x)$ is a probability density function.

$$P(x \geq 4) = \int_{4}^{\infty} f(x)\,dx$$
$$= \frac{1}{30}\int_{4}^{5}(3+2x)\,dx$$
$$= \frac{1}{30}\left[3x + 2\frac{x^2}{2}\right]_4^5$$
$$= \frac{2}{5}$$

H.W

(1) The probability density function of a continuous random variable is given below;

$$f(x) = ax^2;\quad 0<x<4$$
$$= 0;\quad \text{otherwise}$$

Find the value of a and $P(1<x<3)$

Ans: $a = \frac{3}{64}$, $P(1<x<3) = \frac{13}{32}$

(2) The probability density function of a continuous random variable x is given below;

$$f(x) = kx^2 + kx + \frac{1}{8},\quad 0<x<2$$
$$= 0;\quad \text{otherwise}$$

(i) Find the value of k

(ii) Find the value of $P(1<x<2)$

Ans: $k = \frac{9}{56}$, $P(1<x<2) = \frac{83}{112}$

Format 4: (I) In case of the joint probability density function

$$\int_y \int_x f(x,y)\,dx\,dy = 1 \quad \text{or} \quad \int_x \int_y f(x,y)\,dy\,dx = 1$$

(II) (a) The marginal probability density function of x is $f(x)$ or $g(x) = \int_y f(x,y)\,dy$

(b) The marginal probability density function of y is $f(y)$ or $h(y) = \int_x f(x,y)\,dx$.

(III) (a) The conditional probability density function of x given y is

$$f(x|y) = \frac{f(x,y)}{h(y)} \quad \text{or} \quad f(x|y) = \frac{f(x,y)}{f(y)}$$

(b) The conditional probability density function of y given x is

$$f(y|x) = \frac{f(x,y)}{g(x)} \quad \cancel{\text{or}} \text{ or } \quad f(y|x) = \frac{f(x,y)}{f(x)}$$

(IV) x and y are independent if

$$f(x,y) = f(x) \cdot f(y).$$

problem: The joint probability density function of the two continuous random variable x, y is given below:

$$f(x,y) = k(8-x-y); \quad 0 \leq x \leq 2,\ 0 \leq y \leq 2$$
$$= 0 \quad ; \text{ otherwise}$$

(i) Find the value of k

(ii) Find the marginal probability density function of x and y.

(iii) Are x and y independent?

Soln: We know,

$$\int_y \int_x f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_0^2 \int_0^2 k(8-x-y)\,dx\,dy = 1$$

$$\Rightarrow k\int_0^2 \left\{ \int_0^2 (8-x-y)\,dx \right\} dy = 1$$

$$\Rightarrow k\int_0^2 \left[8x - \frac{x^2}{2} - yx\right]_0^2 dy = 1$$

$$\Rightarrow k\int_0^2 \left\{ \left(8\times 2 - \frac{2^2}{2} - 2y\right) - \left(8\times 0 - \frac{0^2}{2} - 0\times y\right) \right\} dy = 1$$

$$\Rightarrow k\int_0^2 (16-2-2y)\,dy = 1$$

$$\Rightarrow k\int_0^2 (14-2y)\,dy = 1$$

$$\Rightarrow k\left[14y - 2\frac{y^2}{2}\right]_0^2 = 1$$

$$\Rightarrow k\left[14y - y^2\right]_0^2 = 1$$

$$\Rightarrow k\left\{14\times 2 - 2^2\right\} = 1$$

$$\Rightarrow 24k = 1 \quad \therefore k = \frac{1}{24}$$

(ii) The marginal probability density function of x is

$$g(x) = \int_y f(x,y)\,dy$$

$$= \int_0^2 \frac{1}{24}(8-x-y)\,dy$$

$$= \frac{1}{24}\left[8y - xy - \frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{24}\left\{8\times2 - x\times2 - \frac{2^2}{2} - 0\right\}$$

$$= \frac{1}{24}(14-2x)$$

$$= \frac{1}{24}\cdot 2(7-x)$$

$$= \frac{1}{12}(7-x);\ 0\le x\le 2$$

The marginal probability density function of y is

$$h(y) = \int_x f(x,y)\,dx$$

$$= \int_0^2 \frac{1}{24}(8-x-y)\,dx$$

$$= \frac{1}{24}\left[8x - \frac{x^2}{2} - xy\right]_0^2$$

$$= \frac{1}{12}(7-y);\ 0\le y\le 2$$

(iii) We know x and y are independent if

$$f(x,y) = g(x)\,h(y)$$

Here, $f(x,y) = \frac{1}{24}(8-x-y)$; $0 \le x \le 2$, $0 \le y \le 2$

$g(x) = \frac{1}{12}(7-x)$; $0 \le x \le 2$

$h(y) = \frac{1}{12}(7-y)$; $0 \le y \le 2$

$$g(x)\,h(y) = \frac{1}{12}(7-x) \cdot \frac{1}{12}(7-y)$$

$$= \frac{1}{144}(49-7y-7y+xy)$$

$$\ne f(x,y)$$

i.e $f(x,y) \ne g(x)\,h(y)$

So x and y are not independent.

Scanned with CamScanner