Baye's theorem;

Let $\{A_1, A_2, \ldots A_i, \ldots, A_k\}$ be a set of mutually exclusive and exhaustive events form a partition of the sample space S such that $A_1 \cup A_2 \cup \ldots \cup A_k = S$ and $P(A_i) > 0$. Again let the event B of S such that $P(B) > 0$ then

$$P(A_i|B) = \frac{P(A_i)\,P(B|A_i)}{\sum_{i=1}^{k} P(A_i)\,P(B|A_i)}$$

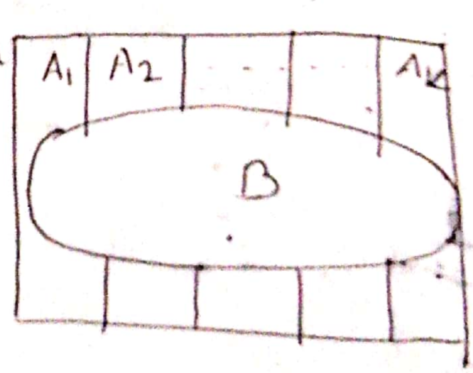
$i = 1, 2, \ldots, k$

which is Baye's theorem.

B

Proof: According to the given theorem, A_i and B are dependent.

then by using multiplication rule of probability for dependent events we have



$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} \quad \ldots (i)$$

$$= \frac{P(A_i)\,P(B|A_i)}{P(B)} \quad (1)$$

we have, $B = S \cap B$

$$= (A_1 \cup A_2 \cup \cdots \cup A_k) \cap B$$

$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots (A_k \cap B)$$

$$P(B) = P[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_k \cap B)]$$

$$= P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_k \cap B)$$

[since $(A_1 \cap B)$, $(A_2 \cap B)$, ... $(A_k \cap B)$ are mutually exclusive

$$= \sum_{i=1}^{k} P(A_i \cap B)$$

$$= \sum_{i=1}^{k} P(A_i) P(B|A_i)$$

Now putting the value of $P(B)$ in equation (1) we have

$$P(A_i|B) = \frac{P(A_i)\, P(B|A_i)}{\sum_{i=1}^{k} P(A_i)\, P(B|A_i)}$$

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Problem: Three machines M_1, M_2 and M_3 produce ~~M_1, M_2 and M_3~~ respectively 40%, 25% and 35% of the total number of items of a factory. The percentages of defective items of these machines are 2%, 4% and 5%.

(i) If an item is selected at random, find the probability that the item is defective.

(ii) If an item is selected at random, find the probability that the defective item was produced by machine M_1.

Soln:

Let,

A_1: Machine M_1 produce the item

A_2: Machine M_2 produce the item

A_3: Machine M_3 produce the item

And event B: The item is defective

According to the question we have

$P(A_1) = 40\% = 0.40$

$P(A_2) = 25\% = 0.25$

$P(A_3) = 35\% = 0.35$

$P(B|A_1) = 2\% = 0.02$

$P(B|A_2) = 4\% = 0.04$

$P(B|A_3) = 5\% = 0.05$

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(i) The probability that the item is defective $= P(B)$

$$= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$$

$$= 0.40 \times 0.02 + 0.25 \times 0.04 + 0.35 \times 0.05$$

$$= 0.0355$$

(ii) By using Bayes' theorem, the probability that the defective item was produced by machine M_1

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)}$$

$$= \frac{0.40 \times 0.02}{0.40 \times 0.02 + 0.25 \times 0.04 + 0.35 \times 0.05}$$

$$= \frac{0.008}{0.0355}$$

$$= 0.22504$$

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Problem: A manufacturing company produces plastic pipes in 3 plants with daily production volumes 2000, 1000 and 500. Among their daily thousand production 10, 8 and 5 items are defective respectively. If a pipe is chosen at random and it found defective, find out

(i) From which plant for this defective pipe, the probability is highest.

(ii) What is the probability that it came from the third plant?

Soln: Let us define the events as follows:

A_1: production volume of first plant

A_2: production volume of second plant

A_3: production volume of third plant.

and B: a defective pipe

From the given information, we have

$$P(A_1) = \frac{2000}{2000+1000+500} = \frac{2000}{3500} = 0.5714$$

$$P(A_2) = \frac{1000}{2000+1000+500} = \frac{1000}{3500} = 0.2857$$

$$P(A_3) = \frac{500}{2000+1000+500} = \frac{500}{3500} = 0.1429$$

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$P(B \mid A_1) = \frac{10}{1000}$

$= 0.01$

$P(B \mid A_2) = \frac{8}{1000}$

$= 0.008$

$P(B \mid A_3) = \frac{5}{1000} = 0.005$

(i) By using Bayes' theorem we have

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)}$$

$$= \frac{0.5714 \times 0.01}{0.5714 \times 0.01 + 0.2857 \times 0.008 + 0.1429 \times 0.005)}$$

$$= 0.6567$$

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)}$$

$$= \frac{0.002287}{0.00868} = 0.2627$$

$$P(A_3 \mid B) = \frac{P(A_3)P(B \mid A_3)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)}$$

$$= \frac{0.0007}{0.00868} = 0.0806$$

Since $P(A_1 \mid B)$ has the highest probability so the defective pipe has been from the first plant.

(ii) The required probability that the defective pipe came from the third plant is given by $P(A_3 \mid B)$ is 0.0806

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