Complex number: A number of the form $a+ib$ where a and b are real numbers and $i=\sqrt{-1}$ (imaginary quantity) is called a complex number.

Complex Variable:

A variable z is said to be a complex variable if it can take away value from a set of complex numbers.

Conjugate of complex number:

Conjugate of the complex number $z=x+iy$ is $\bar{z}=x-iy$.

<u>properties:</u>

① $\bar{\bar{z}}=z$

② $|z|^2=z\bar{z}$

③ $|z|=|\bar{z}|$

④ $|z_1 z_2|=|z_1||z_2|$

⑤ $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$

⑥ $\arg(z_1 z_2)=\arg z_1+\arg z_2$

⑦ $\arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2$

(8) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(9) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(10) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(11) $z + \bar{z} = 2Re(z)$

(12) $z - \bar{z} = 2i\,Im(z)$

(13) $Re(z) \leq |z|$

Theorem: For two complex numbers z_1 and z_2 prove that

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$

(ii) $|z_1 - z_2| \geq |z_1| - |z_2|$

Proof:
$$|z_1 + z_2|^2 = (z_1 + z_2)\,(\overline{z_1 + z_2})$$
$$= (z_1 + z_2)\,(\bar{z}_1 + \bar{z}_2)$$
$$= z_1\bar{z}_1 + z_2\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_2$$
$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2}$$
$$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$$
$$\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$$
$$= |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2|$$
$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$
$$= (|z_1| + |z_2|)^2$$
$$\Rightarrow |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$
$$\therefore |z_1 + z_2| \leq |z_1| + |z_2| \quad \text{(proved)}$$

(ii) $|z_1| = |(z_1 - z_2) + z_2|$

$\leq |z_1 - z_2| + |z_2|$ [using (i)]

$\Rightarrow |z_1 - z_2| \geq |z_1| - |z_2|$

(proved)

Theorem: For any two complex numbers z_1 and z_2

(i) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

(ii) $|z_1 + \sqrt{z_1^2 - z_2^2}| + |z_1 - \sqrt{z_1^2 - z_2^2}| = |z_1 + z_2| + |z_1 - z_2|$

proof:

(i) L.H.S $= |z_1 + z_2|^2 + |z_1 - z_2|^2$

$= (z_1 + z_2)(\overline{z_1 + z_2}) + (z_1 - z_2)(\overline{z_1 - z_2})$

$= (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) + (z_1 - z_2)(\overline{z}_1 - \overline{z}_2)$

$= z_1\overline{z}_1 + \overline{z}_1 \cancel{z_2} + \cancel{z_1\overline{z}_2} + z_2\overline{z}_2 + z_1\overline{z}_1$

$- \cancel{z_2\overline{z}_1} - \cancel{z_1\overline{z}_2} + z_2\overline{z}_2$

$= 2z_1\overline{z}_1 + 2z_2\overline{z}_2$

$= 2|z_1|^2 + 2|z_2|^2$

$=$ R.H.S

$\therefore$ L.H.S $=$ R.H.S

(proved)

(11)

$$(|z_1+\sqrt{z_1^2-z_2^2}| + |z_1-\sqrt{z_1^2-z_2^2}|)^2$$

$$= |z_1+\sqrt{z_1^2-z_2^2}|^2 + |z_1-\sqrt{z_1^2-z_2^2}|^2$$
$$+ 2|z_1+\sqrt{z_1^2-z_2^2}||z_1-\sqrt{z_1^2-z_2^2}|$$

Here we know,

$$|z_1+z_2|^2+|z_1-z_2|^2 = 2|z_1|^2+2|z_2|^2$$

So using this we get

$$= 2|z_1|^2 + 2|\sqrt{z_1^2-z_2^2}|^2 + 2|z_1^2-(\sqrt{z_1^2-z_2^2})^2|$$

$$= 2|z_1|^2 + 2|z_1+z_2||z_1-z_2| + 2|\cancel{z_1^2}-\cancel{z_1^2}+z_2^2|$$

$$= 2|z_1|^2 + 2|z_1+z_2||z_1-z_2| + 2|z_2|^2$$

$$= 2|z_1|^2+2|z_2|^2+2|z_1+z_2||z_1-z_2|$$

$$= |z_1+z_2|^2+|z_1-z_2|^2+2|z_1+z_2||z_1-z_2|$$

[using (1)]

$$= (|z_1+z_2| + (z_1-z_2)|^2$$

Hence. $|z_1+\sqrt{z_1^2-z_2^2}|+|z_1-\sqrt{z_1^2-z_2^2}| = |z_1+z_2|+|z_1-z_2|$

(Proved)

Modulus and argument of a complex number:

Let r and θ be polar co-ordinates at a point $z=(x,y)$. For $z \neq 0$,

Let $x = r\cos\theta$ and $y = r\sin\theta$ then

$$x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta)$$

$$\Rightarrow x^2+y^2 = r^2$$

$$\therefore r = \sqrt{x^2+y^2}$$

and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{r\sin\theta}{r\cos\theta} = \dfrac{y}{x}$

$$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

r is called the modulus or th absolute value and θ is called the amplitude or argument of the complex number z.

Problem: Find the modulus and argument:

$z = 2+i$.

Soln: Given, $z = 2+i$

$|z| = \sqrt{2^2+1^2}$ Modulus of $(2+i)$ is $\sqrt{5}$

$= \sqrt{5}$

Argument of $z=2+i$ is $\tan^{-1}\frac{1}{2}$

(ii) $z = (2+3i)^2$

$= 4+12i+9i^2$

$= 4-9+12i$

$= -5+12i$

$\therefore |z| = \sqrt{(-5)^2+(12)^2}$

$= \sqrt{169}$

$= 13$

So modulus of $z = (2+3i)^2$ is 13

the argument of z is $\tan^{-1}\left(\frac{-12}{5}\right)$

(iii) $z = \dfrac{\sqrt{3}+i}{\sqrt{3}-i}$

$= \dfrac{(\sqrt{3}+i)(\sqrt{3}+i)}{(\sqrt{3}-i)(\sqrt{3}+i)}$

$= \dfrac{(\sqrt{3})^2+2\sqrt{3}i+i^2}{(\sqrt{3})^2-i^2}$

$= \dfrac{3+2\sqrt{3}i-1}{3+1}$

$= \dfrac{2+2\sqrt{3}i}{4}$

$= \dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$

$|z| = \left|\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right|$

$= \sqrt{\left(\dfrac{1}{2}\right)^2+\left(\dfrac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\dfrac{1}{4}+\dfrac{3}{4}} = \sqrt{1} = 1$

Modulus of z is 1

and argument of z is $\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)$

$= \tan^{-1}(\sqrt{3})$

$= \tan^{-1}\tan\left(\frac{\pi}{3}\right)$

$= \frac{\pi}{3}$

(iv) $\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)}$

$= \frac{(1-i)^2}{1-i^2}$

$= \frac{1-2i-1}{1+1}$

$= \frac{-2i}{2} = -i$

Modulus, $\left|\frac{1-i}{1+i}\right| = |-i| = \sqrt{0^2+(-1)^2}$

$= 1$

the argument of z is $= \tan^{-1}\left(\frac{-1}{0}\right)$

$= \tan^{-1}(\infty)$

$= \frac{\pi}{2}$

Problem: Express $\dfrac{(1+2i)^2}{(2+i)^2}$ in the form $A+iB$.

Also find its modulus and argument.

Soln: $\dfrac{(1+2i)^2}{(2+i)^2} = \dfrac{1+4i+4i^2}{4+4i+i^2}$

$= \dfrac{1+4i-4}{4+4i-1}$

$= \dfrac{-3+4i}{3+4i}$

$= \dfrac{(-3+4i)(3-4i)}{(3+4i)(3-4i)}$

$= \dfrac{-9+12i+12i-16i^2}{(3)^2-(4i)^2}$

$= \dfrac{-9+24i+16}{9+16}$

$= \dfrac{7+24i}{25}$

$= \dfrac{7}{25} + \dfrac{24}{25}i$

Modulus $= \left|\dfrac{(1+2i)^2}{(2+i)^2}\right| = \left|\dfrac{7}{25} + \dfrac{24}{25}i\right|$

$= \sqrt{\left(\dfrac{7}{15}\right)^2 + \left(\dfrac{24}{25}\right)^2}$

$= \sqrt{\dfrac{625}{625}} = 1$

The argument is $= \tan^{-1}\left(\dfrac{24/25}{7/25}\right)$

$= \tan^{-1}\left(\dfrac{24}{7}\right)$

Scanned with CamScanner

Problem: Find the modulus and argument

(i) $\left(\frac{2+i}{3-i}\right)^2$ (ii) $z=(-3+5i)^2$

Soln: $\left(\frac{2+i}{3-i}\right)^2 = \frac{4+4i+i^2}{9-6i+i^2}$

$= \frac{4+4i-1}{9-6i-1}$

$= \frac{3+4i}{8-6i}$

$= \frac{(3+4i)(8+6i)}{(8-6i)(8+6i)}$

$= \frac{24+18i+32i+24i^2}{(8)^2-(6i)^2}$

$= \frac{24+50i-24}{64+36}$

$= \frac{50i}{100} = \frac{i}{2}$

Modulus $= \left|\left(\frac{2+i}{3-i}\right)^2\right| = \left|\frac{i}{2}\right| = \sqrt{0^2+\left(\frac{1}{2}\right)^2}$

$= \sqrt{\frac{1}{4}}$

$= \frac{1}{2}$

The argument is $= \tan^{-1}\left(\frac{\frac{1}{2}}{0}\right)$

$= \tan^{-1}\left(\tan\frac{\pi}{2}\right)$

$= \frac{\pi}{2}$

(ii) $z = (-3+5i)^2$

$= 9 - 30i + 25i^2$

$= 9 - 25 - 30i$

$= -16 - 30i$

$|z| = |-16 - 30i|$

$= \sqrt{(-16)^2 + (-30)^2}$

$= \sqrt{1156}$

$= 34$

Modulus of z is 34

And argument of z is $\tan^{-1}\left(\frac{-30}{-16}\right)$

$= \tan^{-1}\left(\frac{15}{8}\right)$

(iii) $\frac{2-2i}{-1+\sqrt{3}i}$

Soln: Let,

$z_1 = 2 - 2i$

$z_2 = -1 + \sqrt{3}i$

$|z_1| = \sqrt{2^2 + (-2)^2}$

$= \sqrt{8}$

$= 2\sqrt{2}$

$|z_2| = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$= \sqrt{1+3}$

$= 2$

Modulus,
$$|z| = \frac{|z_1|}{|z_2|}$$
$$= \frac{2\sqrt{2}}{2}$$
$$= \sqrt{2}$$

$$\arg(z_1) = \tan^{-1}\left(\frac{-2}{2}\right)$$
$$= \tan^{-1}(-1)$$
$$= \tan^{-1}\tan\left(2\pi - \frac{\pi}{4}\right)$$
$$= \frac{7\pi}{4}$$

$$\arg(z_2) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$$
$$= \tan^{-1}\tan\left(\pi - \frac{\pi}{3}\right)$$
$$= \frac{2\pi}{3}$$

$$\arg z = \arg z_1 - \arg z_2$$
$$= \frac{7\pi}{4} - \frac{2\pi}{3}$$
$$= \frac{13\pi}{12} \text{ Ans}$$