Binomial Distribution

Definition: A discrete random variable x is said to have a binomial distribution if it has a probability function

$$P(x) = {}^nC_x\, p^x (1-p)^{n-x} = {}^nC_x\, p^x q^{n-x};\quad x = 0,1,2,\ldots,n$$

where $q = 1-p$ and ${}^nC_x = \binom{n}{x}$; n = Number of trials;

p = probability of sucess and $0 \le p \le 1$

Binomial random variable or binomial variate:

A discrete random variable x is said to have a binomial random variable or binomial variate if it has a probability function with $P(x) = {}^nC_x\, p^x q^{n-x}$; $x = 0,1,2,\ldots,n$

* **Special case:** If $n=1$, the distribution is known as unit (or point) binomial distribution or Bernoulli distribution.

Mean, variance and Standard deviation of Binomial distribution:

Let, x be a binomial variate with parameter n and p.

probability function,

$$P(x) = {}^nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

Now, mean, $E(x) = \sum_{x=0}^{n} x P(x)$

$$= 0 \cdot P(0) + 1 \cdot P(1) + 2 \cdot P(2) + \ldots + nP(n)$$

$$= 0 + {}^nC_1 p^1 q^{n-1} + 2 \cdot {}^nC_2 p^2 q^{n-2} + \ldots + n \cdot 1\, {}^nC_n p^n q^{n-n}$$

$$= npq^{n-1} + 2 \cdot \frac{n(n-1)}{2} p^2 q^{n-2} + \ldots + n \cdot 1 \cdot p^n \cdot 1$$

$$= npq^{n-1} + n(n-1) p^2 q^{n-2} + \ldots + np^n$$

$$= np[q^{n-1} + (n-1) pq^{n-2} + \ldots + p^{n-1}]$$

$$= np(q+p)^{n-1} \quad [\because q^n + {}^nC_1 q^{n-1} p + \ldots + p^n = (q+p)^n]$$

$$= np(1)^{n-1} \quad [\because q+p=1]$$

$$= np.$$

$\therefore$ Mean, $E(x) = np$.

(*) Variance, $V(x) = npq$.

(*) Standard deviation, $\sqrt{V(x)} = \sqrt{npq}$

Theorem:

Prove that the sum of all probabilities of a binomial distribution is one, that is $\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} = 1$

Proof: Let, x be a binomial variate with parameters n and p.

probability function,

$$p(x) = {}^nC_x p^x q^{n-x}; \quad x = 0, 1, 2, \ldots, n \text{ and } p+q=1$$

Now the sum of all probabilities =

$$\sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x}$$

$$= {}^nC_0 p^0 q^{n-0} + {}^nC_1 p^1 q^{n-1} + {}^nC_2 p^2 q^{n-2} + \ldots + {}^nC_n p^n q^{n-n}$$

$$= 1.1\, q^n + {}^nC_1 p^1 q^{n-1} + {}^nC_2 p^2 q^{n-2} + \ldots + 1.p^n.1$$

$$= q^n + {}^nC_1 q^{n-1} p + {}^nC_2 q^{n-2} p^2 + \ldots + p^n$$

$$= (q+p)^n$$

$$= (1)^n \quad [\because p+q=1]$$

$$= 1$$

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Problems

Format 1

In case of determine the parameter mean, variance, standard deviation and the probability function of binomial distribution:

Let x be a binomial variate with parameters n and p. Probability function $P(x) = {}^nC_x\, p^x q^{n-x}$; $x = 0, 1, 2, \ldots, n$

and $p+q=1$

* Mean $= np$

Variance $= npq$ [Mean > variance]

Standard deviation $= \sqrt{npq}$

N.B: The value of n can not be fraction and the negative values of p and q are not acceptable.

Problem 1: The mean and variance of a binomial distribution are 4 and $\frac{4}{3}$ respectively. Find (i) Probability function (ii) $P(x=0)$ (iii) $P(x \geq 1)$

Soln:

Let x be a binomial variate with parameters n and p.

$\therefore$ probability function, $p(x) = {}^nC_x\, p^x q^{n-x}$; $x = 0, 1, 2, \ldots, n$

We know,

mean $= np$ and variance $= npq$

According to the question,

$np = 4 \quad \cdots (i)$

and $npq = \frac{4}{3}$

$\Rightarrow 4q = \frac{4}{3}$

$\Rightarrow q = \frac{1}{3}$ and $p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$

Putting the value of p in eqn (i), we have $n \times \frac{2}{3} = 4$

$\Rightarrow n = \frac{12}{2}$

$\therefore n = 6.$

(i) The probability function,

$$P(x) = {}^6C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x}; \quad x = 0, 1, 2, \ldots, 6$$

(ii) $P(x=0) = {}^6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{6-0} = 1 \times 1 \times \left(\frac{1}{3}\right)^6$

$= 1 \times 1 \times 0.0014$

$= 0.0014$

(iii) $P(x \geq 1) = 1 - P(x=0)$

$= 1 - 0.0014$

$= 0.9986$.

Problem 2: The mean of a binomial distribution is 40 and standard deviation is 6. Calculate n, p and q.

Soln: Let, x be a binomial ~~variate~~ variate with parameters n and p.

We know, mean $= np$

Standard deviation $= \sqrt{npq}$

According to the question we have,

$np = 40 \cdots (1)$

and. $\sqrt{npq} = 6$

$\Rightarrow npq = 36$

$\Rightarrow 40q = 36$

$\Rightarrow q = 0.9$

$\therefore p = 1 - q = 1 - 0.9 = 0.1$

Putting the value of p in (1)

$n \times 0.1 = 40$

$\Rightarrow n = 400.$

Format 2

Required probability in question	To find out
(i) Exactly 2/only 2/ just 10	$P(x=2) = {}^{n}C_2 p^2 q^{n-2}$
(ii) at least 1/1 or more/ greater than or equal to 1/no less than 1	$P(x \geq 1) = 1 - P(x=0)$ $= 1 - {}^{n}C_0 p^0 q^{n-0}$
(iii) At best 2/at most 2/ 2 or fewer/no more than 2/not exceed 2/ less than or equal 2	$P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$ $= {}^{n}C_0 p^0 q^{n-0} + {}^{n}C_1 p^1 q^{n-1} + {}^{n}C_2 p^2 q^{n-2}$

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(iv) Above 1/ greater 1/ More than 1	$P(x>1) = P(x \geq 2)$ $= 1 - P(x=0) - P(x=1)$
(v) Less ~~to~~ than 2	$P(x<2) = P(x \leq 1)$ $= P(x=0) + P(x=1)$
(vi) Between 2 and 4	$P(2 \leq x \leq 4)$ $= P(x=2) + P(x=3) + P(x=4)$

problem: The probability that a bulb will fail before 420 hours is 0.2. Bulbs fail independently. If 15 bulbs are tested for life lengths, what is the probability that the number of failures before 420 hours does not exceed 3?

Soln: According to the given information, we have

$p = 0.2$

$\therefore q = 1 - p = 1 - 0.2 = 0.8$, $n = 15$

Let x be denoted the number of failures blubs.

The probability function,

$$p(x) = {}^{n}c_x \, p^x q^{n-x} \; ; \quad x = 0, 1, 2, \ldots, n$$

$$\Rightarrow p(x) = {}^{15}c_x \, (0.2)^x (0.8)^{15-x} \; ; \quad x = 0, 1, 2, \ldots, 15$$

$\therefore$ The required probability

$$P(x \leq 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$$

$$= {}^{15}C_0 (0.2)^0 (0.8)^{15-0} + {}^{15}C_1 (0.2)^1 (0.8)^{15-1} + {}^{15}C_2 (0.2)^2 (0.8)^{15-2} + {}^{15}C_3 (0.2)^3 (0.8)^{15-3}$$

$$= 1 \times 1 \times 0.0352 + 15 \times 0.2 \times 0.0439 + 105 \times 0.04 \times 0.0549 + 455 \times 0.008 \times 0.0687$$

$$= 0.6476$$

Problem: An unbiased coin is tossed 10 times. Find the probability of getting (i) 3 heads (ii) at least 1 head (iii) at most 1 head.

Soln: We have, the probability of getting head in a single toss, $p = \frac{1}{2}$

$\therefore q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

and the number of trial $n = 10$. So we may use binomial distribution.

Let, x denotes the number of heads

$\therefore$ The probability function of binomial variate x is

$$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x} \quad ; x = 0, 1, 2, \ldots, n$$

$$\Rightarrow P(x) = {}^{10}C_{x} \left(\tfrac{1}{2}\right)^{x} \left(\tfrac{1}{2}\right)^{10-x} ; x = 0, 1, 2, \ldots, 10$$

(i) The probability of getting 3 heads

$$P(x=3) = {}^{10}C_{3} \left(\tfrac{1}{2}\right)^{3} \left(\tfrac{1}{2}\right)^{10-3}$$

$$= {}^{10}C_{3} (0.5)^{3} (0.5)^{7}$$

$$= 120 \times 0.000977$$

$$= 0.11724$$

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(ii) The Probability of getting at least one head

$P(x \geq 1) = 1 - P(x=0)$

$= 1 - {}^{10}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10-0}$

$= 1 - (1 \times 1 \times 0.000977)$

$= 1 - 0.000977$

$= 0.999023$ (Ans)

(iii) Probability of getting at most (best) 1 head $P(x \leq 1) = P(x=0) + P(x=1)$

$= {}^{10}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10-0} + {}^{10}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{10-1}$

$= 1 \times 1 \times (0.5)^{10} + 10 \times (0.5)^{10}$

$= 0.000977 + 0.00977$

$= 0.010747$ (Ans)

Problem: The probability of a new-born baby will be boy or girl is equal; then among five new-born babies, what is the probability that

(i) at least one child is boy

(ii) at best two children are boy.

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Soln: We have,

The probability of boy,

$p = \frac{1}{2}$, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$, $n = 5$

Let, x denotes the number of new-born babies who are boy

$\therefore P(x) = {}^{n}C_x p^x q^{n-x}$; $x = 0, 1, 2, \ldots, n$

$\Rightarrow P(x) = {}^{5}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}$; $x = 0, 1, 2, \ldots, 5$

(i) The probability that at least one child is boy

$= P(x \geq 1) = 1 - P(x = 0)$

$= 1 - {}^{5}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{5-0}$

$= 1 - 1 \times 1 \times (0.5)^5$

$= 1 - 0.03125 = 0.96875$

(ii) The probability that at best two children are boy.

$$= P(x \le 2) = P(x=0) + P(x=1) + P(x=2)$$

$$= 5C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^{5-0} + 5C_1 \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^{5-1} + 5C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{5-2}$$

$$= [1 \times 1 \times (0.5)^5] + [5 \times (0.5) \times (0.5)^4] + [10 \times (0.5)^2 \times (0.5)^3]$$

$$= (0.5)^5 + [5 \times (0.5)^5] + [10 \times (0.5)^5]$$

$$= 0.5 \quad \text{(Ans)}$$

(H.W) In a community the probability that a newly child will be a girl is 0.4. Among 4 newly born children in that community, what is the probability that

.0256

(i) all the four girls Ans: 0.0265

(ii) at least 2 girls Ans: 0.5248

(iii) no girl Ans: 0.1296

Problem:

The probability that Bangladesh win a cricket test match against pakistan is given to be $\frac{1}{3}$. If Bangladesh and pakistan play three test matches, use binomical distribution to find the probability that,

(i) Bangladesh will loss all three test matches

(ii) Bangladesh will win at least one test match.

Soln: The probability that Bangladesh win test match, $p = \frac{1}{3}$, $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$, $n = 3$.

Let, x denote the number of test matches that Bangladesh wins.

$\therefore P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$; $x = 0, 1, 2, \ldots, n$

$\Rightarrow P(x) = {}^{3}C_{x} \left(\frac{1}{2}\right)^{x} \left(\frac{2}{3}\right)^{3-x}$; $x = 0, 1, 2, 3$

(i) The probability that Bangladesh will loss all three test matches

$$P(x=0) = {}^{3}C_{0} \left(\frac{1}{3}\right)^{0} \left(\frac{2}{3}\right)^{3-0}$$

$$= 1 \times 1 \times (0.6667)^{3}$$

$$= 0.2963$$

The probability that Bangladesh will win at least one test matches is

$$P(x \geq 1) = 1 - P(x=0)$$

$$= 1 - 0.2963$$

$$= 0.7037 \quad \text{(Ans)}$$