Cauchy - Riemann equations:

The partial differential equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are called Cauchy - Riemann equations of the complex function

$f(z) = u(x,y) + iv(x,y)$.

① Verify that Cauchy - Riemann equations are satisfied for the function $f(z) = (y^3 - 3x^2y) + i(x^3 - 3xy^2 + 2)$

Soln: Given that,

$$f(z) = (y^3 - 3x^2y) + i(x^3 - 3xy^2 + 2)$$

$$\Rightarrow u + iv = (y^3 - 3x^2y) + i(x^3 - 3xy^2 + 2)$$

where $f(z) = u + iv$

Equating the real and imaginary part we get,

$u = y^3 - 3x^2y \cdots (i)$

$v = x^3 - 3xy^2 + 2 \cdots (ii)$

Partially differentiating equations (i) and (ii) with respect to x and y respectively we get,

$\frac{\partial u}{\partial x} = -6xy \cdots (iii)$

$\frac{\partial u}{\partial y} = 3y^2 - 3x^2 \cdots (iv)$

Again,

$$\frac{\partial v}{\partial x} = 3x^2 - 3y^2$$

$$= -(3y^2 - 3x^2) \quad \cdots (v)$$

and $\frac{\partial v}{\partial y} = -6xy \quad \cdots (vi)$

From (iii) and (vi) we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \cdots (vii)$$

From (iv) and (v) we get,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \cdots (viii)$$

The above equations (vii) and (viii) shows that Cauchy Riemann equations are satisfied for the given function.

(ii) $f(z) = \frac{1}{x+iy}$

$$= \frac{(x-iy)}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2 - i^2y^2}$$

$$= \frac{x-iy}{x^2+y^2}, \text{ since } i^2 = -1$$

$$\Rightarrow u+iv = \frac{x-iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Equating the real and imaginary parts we get,

$$u = \frac{x}{x^2+y^2} \quad \cdots (i)$$

$$v = -\frac{y}{x^2+y^2} \quad \cdots (ii)$$

Differentiating (i) and (ii) w.r.t x and y partially we get,

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)\cdot\frac{\partial}{\partial x}(x) - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2 - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \cdots (iii)$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)\cdot\frac{\partial}{\partial y}(x) - x\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)\cdot 0 - x\cdot 2y}{(x^2+y^2)^2}$$

$$= -\frac{2xy}{(x^2+y^2)^2} \quad \cdots (iv)$$

$$\frac{\partial v}{\partial x} = -\frac{(x^2+y^2)\cdot\frac{\partial}{\partial x}(y) - y\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= -\frac{0 - y\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2} \quad \cdots (v)$$

$$\frac{\partial v}{\partial y} = -\frac{(x^2+y^2)\frac{\partial}{\partial y}(y) - y\cdot\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= -\frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}$$

$$= -\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= -\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} \quad \cdots (vi)$$

From (III) and (VI) we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

From (IV) and (V) we get,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So cauchy Riemann equations are Satisfied for the given function.

(III) $f(z) = e^x(\cos y + i\sin y)$

Soln: Given that,

$$f(z) = e^x(\cos y + i\sin y)$$

$\Rightarrow u + iv = e^x(\cos y + i\sin y)$, where $f(z) = u + iv$

Equating real and imaginary parts

we get,

$u = e^x \cos y \quad \cdots (I)$

$v = e^x \sin y \cdots (II)$

partially differentiating equation (I) and (II)

with respect to x and y respectively

we get,

$$\frac{\partial u}{\partial x} = e^x \cos y \cdots (III)$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \cdot (IV)$$

Again, $\dfrac{\partial v}{\partial x} = e^x \sin y \cdot\cdot (V)$

$$\frac{\partial v}{\partial y} = e^x \cos y \cdot\cdot (VI)$$

From (III) and (VI) we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \cdot\cdot (VII)$$

From (IV) and (V) we get,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \cdot (VIII)$$

The above equations (VII) and (VIII) show that Cauchy Riemann equations are satisfied for the given function.

H.W

(i) $f(z) = z^2$

(ii) $f(z) = e^{y}(\cos x + i\sin x)$

Harmonic function:

Any real function of two variables (x and y) is said to be harmonic in a domain D, if throughout D it has continuous partial derivatives and satisfies the Laplace equation

i.e $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Problem:

Determine which of the following functions u are harmonic

(i) $u = xe^{x}\cos y - ye^{x}\sin y$.

(ii) $u = \frac{1}{2}\ln(x^2+y^2)$

(iii) $u = e^{-x}(x\sin y - y\cos y)$

(1) $u = xe^x \cos y - ye^x \sin y$

Soln: Given that,

$$u = xe^x \cos y - ye^x \sin y \quad \cdots (1)$$

$$\frac{\partial u}{\partial x} = xe^x \cos y + e^x \cos y - ye^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = xe^x \cos y + e^x \cos y + e^x \cos y - ye^x \sin y$$
$$= xe^x \cos y + 2e^x \cos y - ye^x \sin y \quad (11)$$

And, $$\frac{\partial u}{\partial y} = -xe^x \sin y - e^x \sin y - ye^x \cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^x \cos y - e^x \cos y - e^x \cos y + ye^x \sin y$$
$$= -xe^x \cos y - 2e^x \cos y + ye^x \sin y$$
$$\cdots (111)$$

Adding (11) and (111) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xe^x \cos y + 2e^x \cos y - ye^x \sin y$$
$$- xe^x \cos y - 2e^x \cos y - ye^x \sin y$$

$$= 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which shows that the given function u is harmonic.

(11) Given that,

$$u = \frac{1}{2}\ln(x^2+y^2) \quad \cdots (1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x$$

$$= \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} \quad \cdots (11)$$

Again, $\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}}\frac{1}{x^2+y^2}\cdot\cancel{2}y$

$$= \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2} \quad \cdots (111)$$

Adding (11) and (111) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}$$

$$= 0$$

(iii) $u = \frac{y}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)\cdot 0 - y\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)^2\cdot(-2y) - (-2xy)\cdot 2(x^2+y^2)\cdot 2x}{\{(x^2+y^2)^2\}^2}$$

$$= \frac{(x^2+y^2)\{(x^2+y^2)(-2y) + (2xy)\cdot 2\cdot 2x\}}{(x^2+y^2)^4}$$

$$= \frac{-2x^2y - 2y^3 + 8x^2y}{(x^2+y^2)^3}$$

$$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

Again, $\frac{\partial u}{\partial y} = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)^2\cdot(-2y) - (x^2-y^2)\cdot 2(x^2+y^2)\cdot 2y}{\{(x^2+y^2)^2\}^2}$$

$$= \frac{(x^2+y^2)\{(x^2+y^2)(-2y) - (x^2-y^2)\cdot 2\cdot 2y\}}{(x^2+y^2)^4}$$

$$= \frac{-2x^2y - 2y^3 - 4x^2y + 4y^3}{(x^2+y^2)^3}$$

$$= \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3} + \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$$

$$= \frac{6x^2y - 2y^3 - 6x^2y + 2y^3}{(x^2+y^2)^3}$$

$$= \frac{0}{(x^2+y^2)^3} = 0$$

Since, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, So u is a harmonic function.