Rafael Bombelli
Murray R. Spiegel.

* Complex number:

A number of the form $a+ib$ where a and b are real numbers and $i=\sqrt{-1}$ (imaginary quantity) is called a complex number.

b is called imaginary part
a is called real part.

Complex variable:

A variable z is said to be a complex variable if it can take away value from a set of complex numbers.

Addition, substraction, Multiplication and division:

Let $z_1 = x_1+iy_1$

$z_2 = x_2+iy_2$

$$\therefore z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

$$z_1 z_2 = (x_1+iy_1)(x_2+iy_2)$$

$$= x_1x_2 + iy_1x_2 + ix_1y_2 + i^2y_1y_2$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

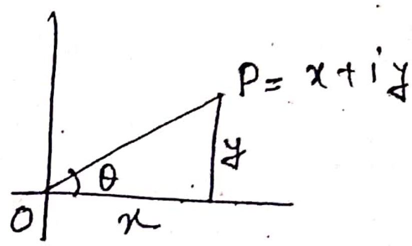
$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}$$

Scanned with CamScanner

$$= \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

* **Modulus and Argument:**

Let $z = x + iy$ be a complex number located at P. Then modulus of z or mod z or $|z|$ is the length



$$OP = \sqrt{x^2 + y^2}$$

$$\therefore r = |z| = \sqrt{x^2 + y^2}$$

The angle θ made by OP with +ve direction of x-axis is called argument of z or

$$\arg(z) = \theta = \tan^{-1}\frac{y}{x}.$$

In polar form
$x = r\cos\theta$
$y = r\sin\theta$
$z = r\cos\theta + ir\sin\theta$
$= r(\cos\theta + i\sin\theta)$
$= re^{i\theta}$.

Conjute of complex number:

Conjugate of $z = x + iy$

is $\bar{z} = x - iy$.

* Jean-Robert Argand
1806 → graphical representation of C.N.
Switzerland.

Properties:

$\bar{\bar{z}} = z$

(2) $|z|^2 = z\bar{z}$

let $z = x+iy$

$|z| = \sqrt{x^2+y^2}$

$|z|^2 = x^2+y^2$

$= (x+iy)(x-iy)$

$= z\bar{z}$

(3) $|z| = |\bar{z}|$

let $z = x+iy$, $\bar{z} = x-iy$

$|\bar{z}| = \sqrt{x^2+(-y)^2}$

$= \sqrt{x^2+y^2}$

$= |z|$

(4) $|z_1 z_2| = |z_1||z_2|$

**

let $z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2}$

$= r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\therefore |z_1 z_2| = r_1 r_2$

$= |z_1||z_2|$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Let, $z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

(6)* $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\therefore \arg(z_1 z_2) = \theta_1 + \theta_2$

$= \arg z_1 + \arg z_2$

(7) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$\therefore \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$

$= \arg z_1 - \arg z_2$

(8) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2$

$= (x_1+x_2) + i(y_1+y_2)$

Scanned with CamScanner

$$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2)$$
$$= (x_1 - iy_1) + (x_2 - iy_2)$$
$$= \bar{z}_1 + \bar{z}_2$$

(9) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Let, $z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

$\bar{z}_1 = r_1 e^{-i\theta_1}$

$\bar{z}_2 = r_2 e^{-i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$
$$\overline{z_1 z_2} = r_1 r_2 e^{-i(\theta_1 + \theta_2)}$$
$$= (r_1 e^{-i\theta_1})(r_2 e^{-i\theta_2})$$
$$= \bar{z}_1 \bar{z}_2$$

(10) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$
$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{r_1}{r_2} e^{-i(\theta_1 - \theta_2)}$$
$$= \frac{r_1 e^{-i\theta_1}}{r_2 e^{-i\theta_2}}$$
$$= \frac{\bar{z}_1}{\bar{z}_1}$$

$$z + \bar{z} = 2\,Re(z)$$

$z = x + iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x$

$= 2\,Re(z)$

$z = x + iy$

$Re(z) = x$

$|z| = \sqrt{x^2 + y^2}$

2) $z - \bar{z} = 2i\,Im(z)$

$z - \bar{z} = 2iy$

$= 2i\,Im(z)$.

Triangular Inequality :

For two complex numbers z_1 and z_2

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$

(ii) $|z_1 - z_2| \geq |z_1| - |z_2|$

Proof :

(i) $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2}$

$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$.

$\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$

$= |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2|$

$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$= (|z_1| + |z_2|)^2$

$\Rightarrow |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$

$\therefore |z_1 + z_2| \leq |z_1| + |z_2|$

(Proved)

(ii) $|z_1| = |(z_1 - z_2) + z_2|$

$\leq |z_1 - z_2| + |z_2|$ [using (i)]

$\therefore |z_1 - z_2| \geq |z_1| - |z_2|$

(Proved)

or any two complex numbers ...

that,

(i) $|z_1+z_2|^2+|z_1-z_2|^2 = 2|z_1|^2+2|z_2|^2$

(ii) $|z_1+\sqrt{z_1^2-z_2^2}|+|z_1-\sqrt{z_1^2-z_2^2}| = |z_1+z_2|+|z_1-z_2|$.

Proof :

(i) L.H.S. $= |z_1+z_2|^2+|z_1-z_2|^2$

$= (z_1+z_2)\overline{(z_1+z_2)} + (z_1-z_2)\overline{(z_1-z_2)}$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2) + (z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$= z_1\bar{z}_1 + \cancel{z_1\bar{z}_2} + \cancel{z_2\bar{z}_1} + z_2\bar{z}_2 + z_1\bar{z}_1 - \cancel{z_1\bar{z}_2}$

$- \cancel{z_2\bar{z}_1} + z_2\bar{z}_2$

$= 2z_1\bar{z}_1 + 2z_2\bar{z}_2$

$= 2|z_1|^2+2|z_2|^2$

$=$ R.H.S.

$\therefore$ L.H.S. $=$ R.H.S.

(Proved)

$|-3|^2 = |3^2| = |-3|^2$

(ii) $\left(|z_1+\sqrt{z_1^2-z_2^2}|+|z_1-\sqrt{z_1^2-z_2^2}|\right)^2$

$= |z_1+\sqrt{z_1^2-z_2^2}|^2+|z_1-\sqrt{z_1^2-z_2^2}|^2$

$+ 2|z_1+\sqrt{z_1^2-z_2^2}||z_1-\sqrt{z_1^2-z_2^2}|$

$= 2|z_1|^2 + 2|\sqrt{z_1^2-z_2^2}|^2 + 2|z_1^2-(z_1^2-z_2^2)|$

[using (1)]

$|-2||3| = |(-2)(3)| = |-6| = 6$

Scanned with CamScanner

$2|z_1|^2 + 2|z_1+z_2||z_1-z_2| + 2|z_2|^2$

$= 2|z_1|^2 + 2|z_2|^2 + 2|z_1+z_2||z_1-z_2|$

$= |z_1+z_2|^2 + |z_1-z_2|^2 + 2|z_1+z_2||z_1-z_2|$ [using (1)]

$= \left(|z_1+z_2| + |z_1-z_2|\right)^2$

Hence,

$$\left|z_1+\sqrt{z_1^2-z_2^2}\right| + \left|z_1-\sqrt{z_1^2-z_2^2}\right| = |z_1+z_2| + |z_1-z_2|$$

(Proved)

Given Problems

For any two complex number z_1, z_2, show that

(i) $|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

ii) $|z_1 + \sqrt{z_1^2-z_2^2}| + |z_1-\sqrt{z_1^2-z_2^2}| = |z_1+z_2| + |z_1-z_2|$

Proof:

(i) $L.H.S. = |z_1+z_2|^2 + |z_1-z_2|^2$

$= (z_1+z_2)\overline{(z_1+z_2)} + (z_1-z_2)\overline{(z_1-z_2)}$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2) + (z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$= z_1\bar{z}_1 + \cancel{z_1\bar{z}_2} + \cancel{z_2\bar{z}_1} + z_2\bar{z}_2 + z_1\bar{z}_1 - \cancel{z_1\bar{z}_2} - \cancel{z_2\bar{z}_1} + z_2\bar{z}_2$

$= 2z_1\bar{z}_1 + 2\cdot z_2\bar{z}_2$

$= 2|z_1|^2 + 2|z_2|^2$

$= R.H.S.$

$\therefore L.H.S. = R.H.S.$

(Proved)

(ii) $\left(|z_1+\sqrt{z_1^2-z_2^2}| + |z_1-\sqrt{z_1^2-z_2^2}|\right)^2$

$= |z_1+\sqrt{z_1^2-z_2^2}|^2 + |z_1-\sqrt{z_1^2-z_2^2}|^2 + 2|z_1+\sqrt{z_1^2-z_2^2}|\,|z_1-\sqrt{z_1^2-z_2^2}|$

Here, we know,

$|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

So using this we get,

$2|z_1|^2 + 2|\sqrt{z_1^2-z_2^2}|^2 + 2|z_1^2-(z_1^2-z_2^2)|$

$= 2|z_1|^2 + 2|z_1+z_2||z_1-z_2| + 2|z_2|^2$

$= 2|z_1|^2 + 2|z_2|^2 + 2|z_1+z_2||z_1-z_2|$

$= |z_1+z_2|^2 + |z_1-z_2|^2 + 2|z_1+z_2||z_1-z_2|$

$= (|z_1+z_2| + |z_1-z_2|)^2$

$\therefore |z_1 + \sqrt{z_1^2-z_2^2}| + |z_1 - \sqrt{z_1^2-z_2^2}| = |z_1+z_2| + |z_1-z_2|$

$\therefore$ L.H.S. = R.H.S.

(Proved)

2. Q: $|z| = 3$, prove that $\left|\dfrac{z^3+3z-5}{z^4+27}\right| \leq \dfrac{41}{54}$

Solution:

$|z^3+3z-5| = |z^3+3z+(-5)|$

$\leq |z|^3 + 3|z| + |-5|$

$= 3^3 + 3\cdot 3 + 5$

$= 41$

Again $|z^4+27| = |z^4 - (-27)|$

$\geq |z|^4 - |-27|$

$= 3^4 - 27$

$= 54$

So, $\left|\dfrac{1}{z^4+27}\right| \leq \dfrac{1}{54}$

Scanned with CamScanner

Hence, $\left|\dfrac{z^3+3z-5}{z^4+27}\right|$

$$= |z^3+3z-5|\left|\frac{1}{z^4+27}\right|$$

$$\leq 41 \cdot \frac{1}{54}$$

$$= \frac{41}{54}$$

$$\left|\frac{z^3+3z-5}{z^4+27}\right| \leq \frac{41}{54}$$

(Proved)

3. On $|z|=2$, Prove that $\left|\dfrac{z^2-2z+4}{z^3-6}\right| \leq 6$

Solution:

$$|z^2-2z+4| = |z^2+2(-z)+4|$$

$$\leq |z|^2+2|z|+|4|$$

$$= 2^2+2\cdot 2+4$$

$$= 12$$

Again, $|z^3-6| = |z^3-6|$

$$\geq |z|^3-|6|$$

$$= 8-6$$

$$= 2$$

$$\therefore \left|\frac{1}{z^3-6}\right| \leq \frac{1}{2}$$

Hence, $\left|\dfrac{z^2-2z+4}{z^3-6}\right| \leq \dfrac{|z^3-2z+4|}{|z^3-6|}$

$$= |z^3 - 2z + 4| \left| \frac{1}{z^3 - 6} \right|$$

$$= 12 \cdot \frac{1}{2}$$

$$= 6$$

$$\therefore \left| \frac{z^2 - 2z + 4}{z^3 - 6} \right| \le 6.$$

(Proved)

4. On $|z| = 3$ Prove that $\left| \dfrac{z^3 + 3z - 5}{z^3 - 2z + 2} \right| \le \dfrac{41}{19}$

Solution:

$$|z^3 + 3z - 5| = |z^3 + (+3z) - 5|$$

$$\le |z|^3 + 3|z| + |-5|$$

$$= 3^3 + 3 \cdot 3 + 5$$

$$= \cancel{23}\ 41$$

Again $|z^3 - 2z + 2| = |z^3 + (+2z) - (-2)|$

$$\ge |z|^3 - 2|z| - |-2|$$

$$= 3^3 - 2 \cdot 3 - 2$$

$$= 27 - 8$$

$$= 19$$

$$\therefore \left| \frac{1}{z^3 - 2z + 2} \right| \le \frac{1}{19}$$

$$\therefore \left| \frac{z^3 + 3z - 5}{z^3 - 2z + 2} \right| \le |z^3 + 3z - 5| \left| \frac{1}{z^3 - 2z + 2} \right|$$

$$\le 23 \cdot \frac{1}{19} = \frac{23}{19}$$

: (Proved)

$|z+i| \le 3$

Solution:

$$|z+i| \le 3$$

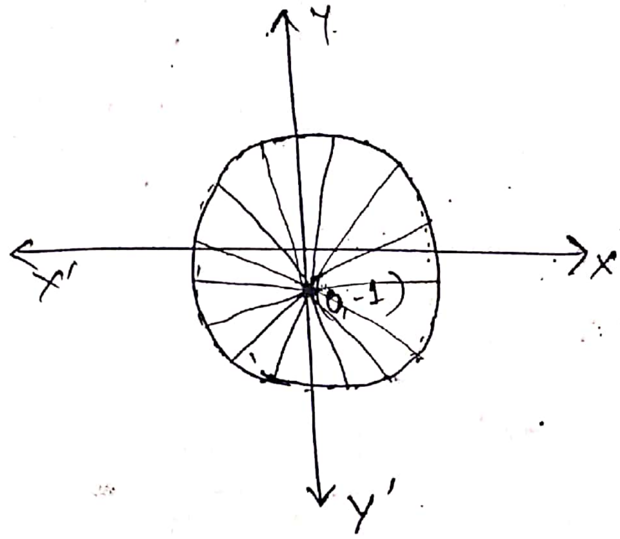
$$\Rightarrow |x+iy+i| \le 3$$

$$\Rightarrow |x+i(y+1)| \le 3$$

$$\Rightarrow \sqrt{x^2+(y+1)^2} \le 3$$

$$\Rightarrow x^2+(y+1)^2 \le 3^2$$

which is a circle with centre $(0,-1)$ and radius 3. | This represents the set of all internal points of the circle whose centre is $(0,-1)$ and radius is 3.



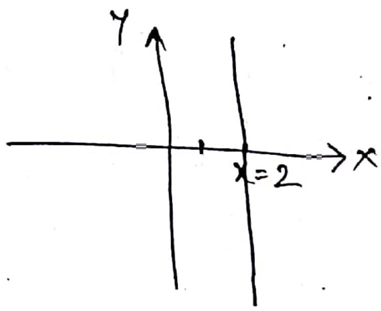
* (ii) $Re(\bar{z}-i)=2$

Solution: * $Re(\bar{z}-i)=2$

$$\Rightarrow Re(x-iy-i)=2$$

$$\Rightarrow Re\{x+i(-y-1)\}=2$$

$$\Rightarrow x=2$$



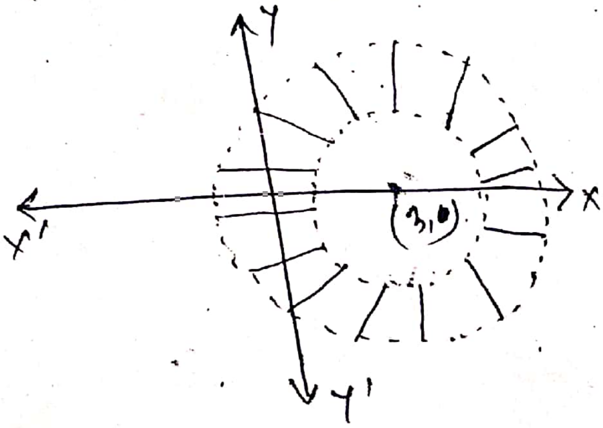
$\sqrt{7} \le |z+4| \le 7$

* (iii) * $2 < |z-3| < 4$

Solution: $2 < |z-3| < 4$

$$\Rightarrow 2 < |x+iy-3| < 4$$

$$\Rightarrow 2 < |(x-3)+iy| < 4$$



$\Rightarrow 2 < \sqrt{(x-3)^2+y^2} < 4$

$\Rightarrow 2^2 < (x-3)^2+y^2 < 4^2$

which is the region between two circle with centree $(3,0)$ and radius 2 & 4

(iv) $Im\,|\bar{z}-i| = 3$

Solution :

$Im\,|\bar{z}-i| = 3$

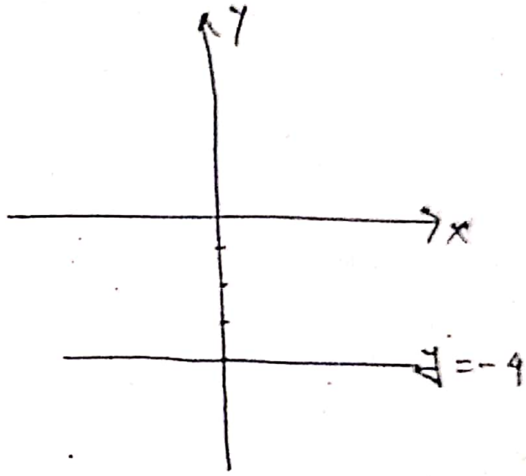
$\Rightarrow Im\,|x-iy-i| = 3$

$\Rightarrow Im\,|x+i(-y-1)| = 3$

$\Rightarrow -y-1 = 3$

$\Rightarrow -y = 4$

$\therefore y = -4$



(v) $|z-2| + |z+2| < 6$
≥ 6

Solution :

$|z-2| + |z+2| < 6$

$\Rightarrow |z-2| < 6 - |z+2|$

$\Rightarrow |(x-2)+iy| < 6 - |(x+2)+iy|$

$\Rightarrow \sqrt{(x-2)^2+y^2} < 6 - \sqrt{(x+2)^2+y^2}$

...aring both sides we get,

$$x^2-4x+4+y^2 < 36 - 12\sqrt{x^2+4x+4+y^2} + x^2+4x+4+y^2$$

$$\Rightarrow 12\sqrt{x^2+4x+4+y^2} < 36+8x$$

$$\Rightarrow 3\sqrt{x^2+4x+4+y^2} < 9+2x$$

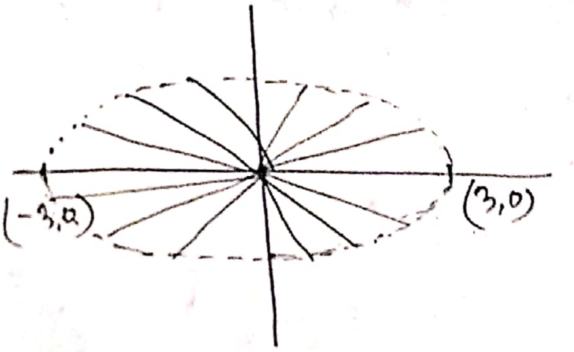
squaring both sides we have,

$$9x^2+3\cancel{6}x+36+9y^2 < 81+3\cancel{6}x+4x^2$$

$$\Rightarrow 5x^2+9y^2 < 45$$

$$\Rightarrow \frac{x^2}{9}+\frac{y^2}{5} < 1$$

$$\Rightarrow \frac{x^2}{3^2}+\frac{y^2}{(\sqrt{5})^2} < 1$$



which is the region inside an elipse whose length of majority axis $= 2.3 = 6$.

(VI) $|z-i| = |z+i|$

Solution:

$$|z-i| = |z+i|$$

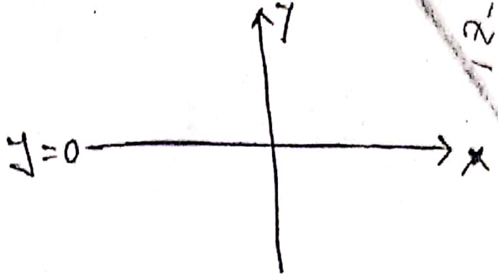
$$\Rightarrow |x+iy-i| = |x+iy+i|$$

$$\Rightarrow |x+i(y-1)| = |x+i(y+1)|$$

$$\Rightarrow \sqrt{x^2+(y-1)^2} = \sqrt{x^2+(y+1)^2}$$

$$\Rightarrow \cancel{x^2}+(y-1)^2 = \cancel{x^2}+(y+1)^2$$

$\Rightarrow \cancel{y^2} - 2y + \cancel{1} = \cancel{y^2} + 2y + \cancel{1}$



$\Rightarrow 4y = 0$

$\Rightarrow y = 0$

(vi). $|2z+1| = |\bar{z}+2|$

Solution: $|2z+1| = |\bar{z}+2|$

$\Rightarrow |2(x+iy)+1| = |x-iy+2|$

$\Rightarrow |2x+2iy+1| = |x+2-iy|$

$= \sqrt{(2x+1)^2+(2y)^2} = \sqrt{(x+2)^2+(-y)^2}$

$\Rightarrow (2x+1)^2 + 4y^2 = (x+2)^2 + y^2$

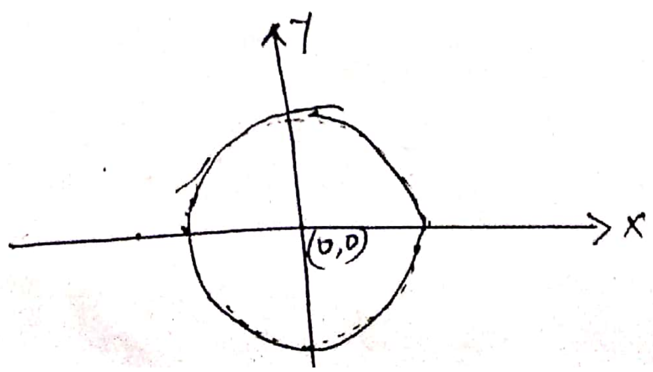
$\Rightarrow 4x^2 + \cancel{4x} + 1 + 4y^2 = x^2 + \cancel{4x} + 4 + y^2$

$\Rightarrow 4x^2 + 1 + 4y^2 - x^2 - 4 - y^2 = 0$

$\Rightarrow 3x^2 + 3y^2 - 3 = 0$

$\Rightarrow x^2 + y^2 = 1$

which is a circle with centere $(0,0)$ and radius 1.



$|z-2| + |z+2| = 6$

ution: $|z-2| + |z+2| = 6$

$\Rightarrow |z+2| = 6 - |z-2|$

$\Rightarrow |x+iy+2| = 6 - |x+iy-2|$

$\Rightarrow |(x+2)+iy| = 6 - |(x-2)+iy|$

$\Rightarrow \sqrt{(x+2)^2+y^2} = 6 - \sqrt{(x-2)^2+y^2}$

$\Rightarrow (x+2)^2+y^2 = \{6-\sqrt{(x-2)^2+y^2}\}^2$

$\Rightarrow (x+2)^2+\cancel{y^2} = 36 - 12\sqrt{(x-2)^2+y^2} + (x-2)^2+\cancel{y^2}$

$\Rightarrow \cancel{x^2}+4x+\cancel{4} = 36 - 12\sqrt{(x-2)^2+y^2} + \cancel{x^2}-4x+\cancel{4}$

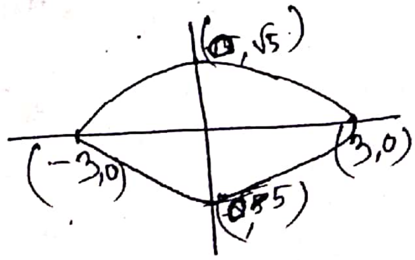
$\Rightarrow 8x-36 = -12\sqrt{x^2-4x+4+y^2}$

$\Rightarrow 2x-9 = -3\sqrt{x^2-4x+4+y^2}$

$\Rightarrow 4x^2-\cancel{36x}+81 = 9x^2-\cancel{36x}+36+9y^2$

$\Rightarrow 5x^2+9y^2 = 45$

$\Rightarrow \frac{x^2}{9}+\frac{y^2}{5} = 1$



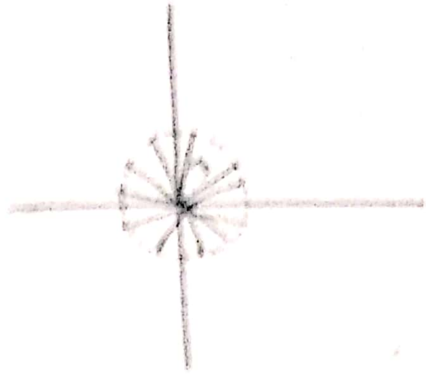
$\Rightarrow \frac{x^2}{3^2}+\frac{y^2}{(\sqrt{5})^2} = 1$

which is an ellipse.

length of major axis $= 2a = 2\cdot 3 = 6$.

Ex: $|z| < 3$

Solution:



$|z| < 3$

$\Rightarrow |x+iy| < 3$

$\Rightarrow \sqrt{x^2+y^2} < 3$

$\Rightarrow x^2+y^2 < 3^2$

which is inside a circle with centre $(0,0)$ and radius 3.

* $|z-3| > 2$

Solution: $|z-3| > 2$

$\Rightarrow |x+iy-3| > 2$

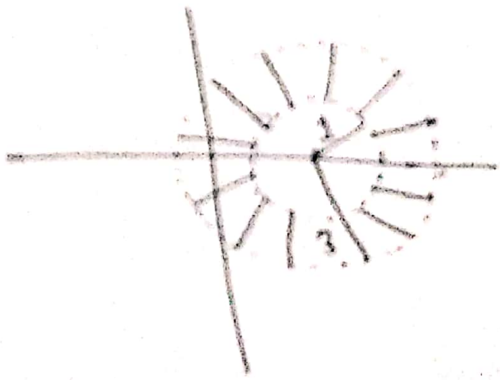
$\Rightarrow |(x-3)+iy| > 2$

$\Rightarrow \sqrt{(x-3)^2+y^2} > 2$

$\Rightarrow (x-3)^2+y^2 > 2^2$

Which is outside circle with centre $(3,0)$ and radius 2.

* $1 < |z-2| < 3$



Solution: $1 < |z-2| < 3$

$\Rightarrow 1 < |x+iy-2| < 3$

$\Rightarrow 1 < |(x-2)+iy| < 3$

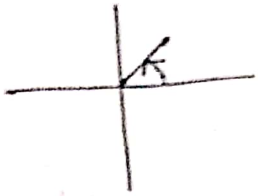
odulus & Argument : Sketch

Find the modulus and argument and express them in polar form.

① * $z = 1 + \sqrt{3}i$

solution:

$z = 1 + \sqrt{3}i$



$|z| = \sqrt{1^2 + (\sqrt{3})^2}$

$= \sqrt{4} = 2$

$\arg z = \tan^{-1} \frac{\sqrt{3}}{1} = \pi/3$

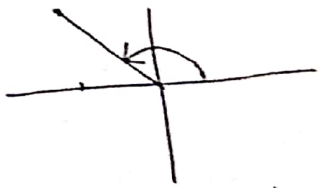
$\therefore$ Polar form of $z = 2e^{\frac{\pi}{3}i}$

(ii)* $z = -2 + 2i$

Solution: $z = -2 + 2i$

$|z| = \sqrt{(-2)^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\arg z = \tan^{-1}\left(\frac{2}{-2}\right)$



$= \pi - \pi/4$

$= \frac{3\pi}{4}$

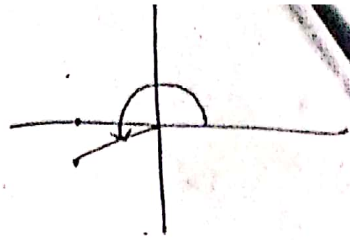
$\therefore$ Polar form of $z = 2\sqrt{2}\, e^{\frac{3\pi}{4}i}$

(iii) $z = -3 - \sqrt{3}i$

Solution: $z = -3 - \sqrt{3}i$

$|z| = \sqrt{(-3)^2 + (-\sqrt{3})^2} = \sqrt{9 + 3} = \sqrt{12} = 2\sqrt{3}$

$\arg z = \tan^{-1} \frac{-\sqrt{3}}{-3} = \tan^{-1} \frac{1}{\sqrt{3}}$



$= \pi + \pi/6$

$= \frac{7\pi}{6}$

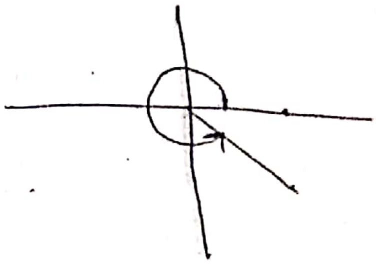
$\therefore$ Polar form of $z = 2\sqrt{3}\, e^{\frac{7\pi}{6} i}$

(iv) $z = 1 - i$

Solution: $z = 1 - i$

$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\arg z = \tan^{-1}\left(\frac{-1}{1}\right)$



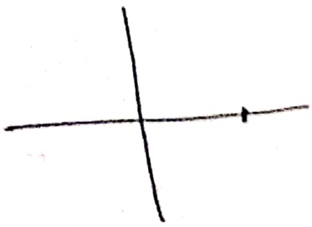
$= 2\pi - \frac{\pi}{4}$

$= \frac{7\pi}{4}$

$\therefore$ Polar form of $z = \sqrt{2}\, e^{\frac{7\pi}{4} i}$

(v) $z = 4$

Solution $|z| = \sqrt{4^2} = 4$



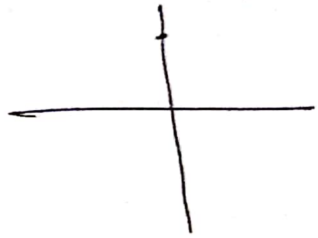
$\arg z = \tan^{-1}\left(\frac{0}{4}\right) = 0$

$\therefore$ Polar form of $z = 4$

$z = 2i$

Solution:

$|z| = \sqrt{2^2} = 2$

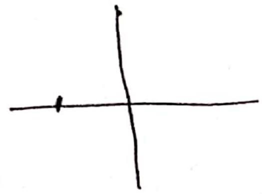


$\arg z = \tan^{-1}\left(\frac{2}{0}\right) = \pi/2$

Polar form of $z = 2e^{\pi/2 i}$

(VII) $z = -3$

Solution: $|z| = \sqrt{(-3)^2} = 3$



$\arg z = \tan^{-1}\left(\frac{0}{-3}\right)$

$= \pi$

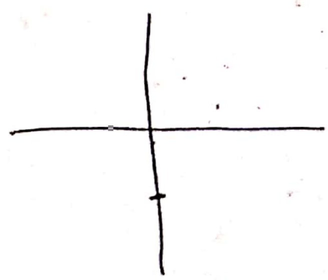
$\therefore$ polar form of $z = 3 e^{\pi i}$

(VIII) $z = -4i$

Solution!

$z = -4i$

$|z| = \sqrt{(-4)^2} = 4$



$\arg z = \text{———} \tan^{-1} \frac{-4}{0}$

$= \frac{3\pi}{2}$

$\therefore$ Polar form of $z = 4 e^{\frac{3\pi}{2} i}$

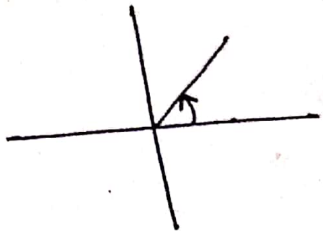
(IX) $z = 1 + i$

Solution: $z = 1 + i$

$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Scanned with CamScanner

$\arg z = \tan^{-1}\frac{1}{1} = \pi/4$



$\therefore$ Polar form is,

$$z = \sqrt{2}\, e^{\pi i/4}$$

(x) $z = -1+\sqrt{3}i$

Solution: $z = -1+\sqrt{3}i$

$$|z| = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$$

$$\arg z = \tan^{-1}\frac{\sqrt{3}}{-1}$$

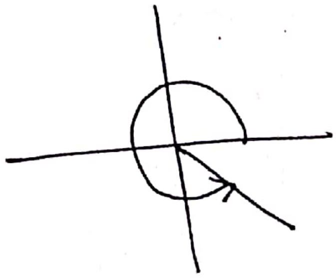
$$= \pi - \pi/3$$

$$= \frac{2\pi}{3}$$

Polar form is, $z = 2e^{\frac{2\pi}{3}i}$

(xi)** $z = 2-2\sqrt{3}i$

Solution: $z = 2-2\sqrt{3}i$



$$|z| = \sqrt{2^2+(2\sqrt{3})^2} = 4$$

$$\arg z = \tan^{-1}\frac{-2\sqrt{3}}{2}$$

$$= 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \text{ or } -\pi/3.$$

Find the modulus and argument of the following and then express them in Polar form.

(a). $\dfrac{2-2i}{-1+\sqrt{3}i}$

Solution:

Let $z_1 = 2-2i$

and $z_2 = -1+\sqrt{3}i$

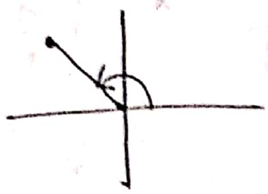
$\therefore |z_1| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$|z_2| = \sqrt{1+3} = \sqrt{4} = 2$

$\therefore |z| = \dfrac{|z_1|}{|z_2|} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

$\arg(z_1) = \tan^{-1}(-1) = 2\pi - \dfrac{\pi}{4}$

$= \dfrac{7\pi}{4}$



$\arg(z_2) = \tan^{-1}(-\sqrt{3})$

$= \pi - \pi/3$

$= \dfrac{2\pi}{3}$

$\therefore \arg z = \arg z_1 - \arg z_2$

$= \dfrac{7\pi}{4} - \dfrac{2\pi}{3} = \dfrac{21\pi - 8\pi}{12} = \dfrac{13\pi}{12}$

$\therefore$ In Polar form $z = \sqrt{2}\, e^{i\frac{13\pi}{12}}$.

Scanned with CamScanner

$|z_1| = 2\sqrt{3}$

$|z_2| = 4$

b. $(\sqrt{3}+3i)(2\sqrt{2}-2\sqrt{2}i)$ $\quad |z| = 8\sqrt{3}$

$\arg(z_1) = \pi/3$

$\arg(z_2) = 7\pi/4$

$\arg(z) = \frac{\pi}{3} + \frac{7\pi}{4}$

$= \frac{4\pi+21\pi}{12}$

$= \frac{25\pi}{12}$

e. $\dfrac{-2\sqrt{3}-2\sqrt{2}i}{-\sqrt{3}+3i}$

* Express $\dfrac{(1+2i)^2}{(2+i)^2}$ in the form of $A+iB$.

$$\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi}$$

$$\left(\frac{5-3i}{2+3i}\right)^2$$

* page - 34,

1.73, 1.74, 1.81, 1.92.

Problem :

1. If $x_n = \cos \pi/2^n + i \sin \pi/2^n$, then show that $x_1 x_2 x_3 \cdots \infty = -1$.

Sol: If $n = 1, 2, 3, \cdots\cdots$ then

$x_1 = \cos \pi/2 + i \sin \pi/2$, $x_2 = \cos \pi/2^2 + i \sin \pi/2^2$

$x_3 = \cos \pi/2^3 + i \sin \pi/2^3$, $\cdots\cdots$

$$\begin{aligned}
L.H.S &= x_1 x_2 x_3 \cdots\cdots \infty \\
&= \cos\left(\pi/2 + \pi/2^2 + \pi/2^3 + \cdots\right) + i \sin\left(\pi/2 + \pi/2^2 + \pi/2^3 + \cdots\right) \\
&= \cos \frac{\pi/2}{1 - 1/2} + i \sin \frac{\pi/2}{1 - 1/2} \\
&= \cos\pi + i \sin\pi \\
&= -1.
\end{aligned}$$

2. If $x_n = \cos \pi/3^n + i \sin \pi/3^n$, then show that $x_1 x_2 x_3 \cdots\cdots \infty = i$.

Sol:

$$\begin{aligned}
L.H.S &= x_1 x_2 x_3 \cdots\cdots \infty \\
&= \cos \frac{\pi/3}{1 - 1/3} + i \sin \frac{\pi/3}{1 - 1/3} \\
&= \cos \pi/2 + i \sin \pi/2 \\
&= i
\end{aligned}$$

3. $x_n = \cos \frac{a^n}{b^n}\pi + i \sin \frac{a^n}{b^n}\pi$

$\therefore x_1 x_2 x_3 \cdots\cdots \infty = \cos \frac{\pi a}{b-a} + i \sin \frac{\pi a}{b-a}$.

* Find two complex numbers whose sum is 4 and product is 8.

* Derive the Euler's formula for Complex number.

* Find ~~the~~ two C.N. whose sum is 2 and product is 5.

Sol: Let two complex numbers are

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$\therefore$ ~~x_1+iy~~

Given that

$z_1 + z_2 = 4$ and $z_1 z_2 = 8 \longrightarrow (I)$

$\therefore (z_1 - z_2)^2 = (z_1 + z_2)^2 - 4z_1z_2 = -16$

$\therefore z_1 - z_2 = \pm 4i \longrightarrow (II)$

$\therefore 2z_1 = 4 + 4i \Rightarrow z_1 = 2 + 2i$ and $z_2 = 2 - 2i$

again $2z_1 = 4 - 4i \Rightarrow z_1 = 2 - 2i$ and $z_2 = 2 + 2i$.

* Using Demoivres theorem solve the following equation $x^7+x^4+x^3+1 = 0$

Sol: $x^4(x^3+1)+1(x^3+1)=0$

$\Rightarrow (x^4+1)(x^3+1)=0$

$\therefore x^4+1=0$ | $x^3+1=0$

$\Rightarrow x^4=\cos\pi+i\sin\pi$ | $x^3=\cos\pi+i\sin\pi$

$\Rightarrow x^4=\cos(2k+1)\pi+i\sin(2k+1)\pi$

$\therefore x=\cos(2k+1)\pi/4+i\sin(2k+1)\pi/4$ | $x=\cos(2n+1)\pi/3+i\sin(2n+1)\pi/3$

$k=0,1,2,3.$ | $n=0,1,2.$

** $x^{12}-1=0$ — solve.

Sol. $x^{12}=\cos 2n\pi+i\sin 2n\pi$

$\therefore x=\cos n\pi/6+i\sin n\pi/6$, $n=0,1,2,\dots 11.$

* Solve $(x-1)^n=x^n$.

Sol. $\left(1-\frac{1}{x}\right)^n=1=\cos 2n\pi+i\sin 2n\pi$

$\Rightarrow 1-\frac{1}{x}=\cos\frac{2n\pi}{n}+i\sin\frac{2n\pi}{n}$

$\Rightarrow \frac{1}{x}=1-\cos\frac{2n\pi}{n}-i\sin\frac{2n\pi}{n}=2\sin^2\frac{n\pi}{n}-2i\sin\frac{n\pi}{n}\cos\frac{n\pi}{n}$

$=-2i\sin\frac{n\pi}{n}\left(\cos n\pi/n+i\sin n\pi/n\right)$

$\therefore x=\frac{i}{2}\left(\cot n\pi/n-1\right)$ for $n=0,1,2,\dots n-1.$

H.W. Solve the following:

i. $x^{49}=1$

ii. $x^7=1$

iii. $x^n=1$

iv. $x^4-x^2+1=0 \Rightarrow x^2=\frac{1\pm\sqrt{3}i}{2}=\cos\pi/3\pm i\sin\pi/3$
$=\cos(2n\pi+\pi/3)\pm i\sin(2n\pi+\pi/3) \Rightarrow x=\cos(6n+1)\pi/6\pm i\sin(6n+1)\pi/6$
$\therefore n=0,1.$

v. $x^6-9x^3+3=0$

vi.

Scanned with CamScanner

Euler's Formula

** we know, from maclaurin series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

when $x = i\theta$, we can arrive at the result

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots\cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots\cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta$, which is called Euler's formula.

** Demoivres Theorem:

positive, negative or rational.

For any complex number z and any integer n,

$$\left(r(\cos\theta + i\sin\theta)\right)^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

or for every real number θ and every positive integer n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

* Use De Moivre's theorem to compute $(1+i)^{12}$.

* sol: The polar form of $1+i$ is

$$1+i = \sqrt{2}\left(\cos\pi/4 + i\sin\pi/4\right)$$

$$\Rightarrow (1+i)^{12} = \left\{\sqrt{2}\left(\cos\pi/4 + i\sin\pi/4\right)\right\}^{12}$$

$$= (\sqrt{2})^{12}\left(\cos 3\pi + i\sin 3\pi\right)$$

$$= 2^6.\left(\cos\pi + i\sin\pi\right) = 64.(-1)$$

$$= -64 \text{ Ans.}$$

* $(\sqrt{3}+i)^5$, $(1-i)^6$ etc, $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^{100}$, ** $(1+\sqrt{3}i)^{2013}$

$-16\sqrt{3}+16i$ $8i$ $=1$ $=-2^{2013}$