静电场，恒定电流，静磁场，时变电磁场

- **What we have done?**

  The generation of EM field;

- **What we have gotten?**

  The Maxwell equations;

- **What we will do?**

  The propagation of EM field in source-free medium;

  Far away the source

- **How to do?**

  Using Maxwell equations in special medium conditions;

# Chpt. 8 Plane Wave Propagation

## Contents

- 1. Wave Eq. & Plane Wave ※
- 2. Plane Wave in Perfect Dielectrics ※
  - Polarization (极化,偏振)
- 3. Plane Wave in Conducting Medium
- 4. Plane Wave in Good Dielectrics
- 5. Plane Wave in Good Conductors ※
  - Skin effect
  - Surface resistance of good conductors
  - Loss of plane wave

学习本章时，请大家注意在头脑中**勾画波型**，把**抽象思维**与**形象思维**结合起来，加深理解。

# Foreword

- **Plane wave** is the simplest and most fundamental EM-wave.
- Fields and waves in this chapter refer to those in steady state and time harmonic.
- We apply vectors of $\mathcal{E}$ and $\mathcal{H}$ for the reason that
  - $\mathcal{E}\times\mathcal{H}$ is the power flow density
  - $\mathcal{E}/\mathcal{H}$ is of the unit of impedance

# Foreword

- Time-varying E- & M- fields may be mutually induced and thus form EM-wave.

E → M $\oint_C \vec{H} \bullet d\vec{l} = \int_S (\vec{J} + \partial\vec{D}/\partial t) \bullet d\vec{S}$ $\nabla \times \vec{H} = \vec{J} + \partial\vec{D}/\partial t$

M → E $\oint_C \vec{E} \bullet d\vec{l} = -\int_S \partial\vec{B}/\partial t \bullet d\vec{S}$ $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$

If there was no loss, or if the loss is rather little, the EM-wave might/may propagate far and far away.

EM-wave in this chapter are assumed to be far from the sources.

# In Perfect dielectric

1. From Maxwell Eq. getting Wave Eq.
2. The solutions of wave Eq.

# § 8.1.1 *Wave Equations*

**In the Medium we study:**

source-free $\rho = 0 \quad \vec{J} = 0$

region of isotropic homogeneous: $\varepsilon$ and $\mu$ are constants.

lossless medium $\sigma = 0$

$$\begin{cases} \nabla \times \vec{H} = \dfrac{\partial \vec{D}}{\partial t} \\ \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \bullet \vec{D} = 0 \\ \nabla \bullet \vec{B} = 0 \end{cases} \qquad \begin{cases} \vec{D} = \varepsilon \vec{E} \\ \vec{B} = \mu \vec{H} \end{cases} \Longrightarrow \begin{cases} \nabla \times \vec{H} = \varepsilon \, \partial \vec{E} / \partial t \\ \nabla \times \vec{E} = -\mu \, \partial \vec{H} / \partial t \\ \nabla \bullet \vec{H} = 0 \\ \nabla \bullet \vec{E} = 0 \end{cases}$$

**How to solve the equations?**

$\nabla \times$

# § 8.1.1 *Wave Equations*

**In the Medium we study:**

source-free $\rho = 0 \quad \vec{J} = 0$

region of isotropic homogeneous: $\varepsilon$ and $\mu$ are constants.

lossless medium $\sigma = 0$

$$\nabla \times \quad \begin{cases} \boxed{\begin{array}{l} \nabla \times \vec{H} = \varepsilon \, \partial \vec{E} / \partial t \\ \nabla \times \vec{E} = -\mu \, \partial \vec{H} / \partial t \end{array}} \\ \nabla \bullet \vec{H} = 0 \\ \nabla \bullet \vec{E} = 0 \end{cases}$$

$\nabla\times$

$$\begin{cases}\nabla\times\vec{H}=\varepsilon\,\partial\vec{E}/\partial t\\ \boxed{\nabla\times\vec{E}=-\mu\,\partial\vec{H}/\partial t}\\ \nabla\bullet\vec{H}=0\\ \nabla\bullet\vec{E}=0\end{cases}$$

$$\nabla\times\left(\nabla\times\vec{E}\right)=\nabla\times[-\mu\,\partial\vec{H}/\partial t]=-\mu\,\partial(\nabla\times\vec{H})/\partial t$$

$$\nabla\times\left(\nabla\times\vec{E}\right)=\nabla\left(\nabla\bullet\vec{E}\right)-\nabla^2\vec{E}$$

$$=0-\nabla^2\vec{E}$$

$$-\mu\,\partial(\nabla\times\vec{H})/\partial t=-\mu\varepsilon\,\partial^2\vec{E}/\partial t^2$$

$$\nabla\bullet\vec{E}=0$$

$$\boxed{\nabla^2\vec{E}=\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}}$$

## So, we get:

$\nabla \times$

$$\begin{cases} \nabla \times \vec{H} = \varepsilon\, \partial \vec{E} / \partial t \\ \nabla \times \vec{E} = -\mu\, \partial \vec{H} / \partial t \\ \nabla \bullet \vec{H} = 0 \\ \nabla \bullet \vec{E} = 0 \end{cases}$$

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

By similar way, we can also get:

$\nabla \times$

$$\begin{cases} \nabla \times \vec{H} = \varepsilon\, \partial \vec{E} / \partial t \\ \nabla \times \vec{E} = -\mu\, \partial \vec{H} / \partial t \\ \nabla \bullet \vec{H} = 0 \\ \nabla \bullet \vec{E} = 0 \end{cases}$$

$$\nabla^2 \vec{H} = \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

# § 8.1.1 *Wave Equations*

**In the Medium :**

source-free $\rho = 0 \quad \vec{J} = 0$

region of isotropic homogeneous: $\varepsilon$ and $\mu$ are constants.

lossless medium $\sigma = 0$

Wave Eq. with aspect to $\vec{E}$ and $\vec{H}$

$$\begin{cases} \nabla^2 \vec{E} - \mu\varepsilon \dfrac{\partial^2 \vec{E}}{\partial t^2} = 0 \\ \nabla^2 \vec{H} - \mu\varepsilon \dfrac{\partial^2 \vec{H}}{\partial t^2} = 0 \end{cases}$$

Both are 2nd-order partial diff. equations in vector forms.

They are wave equs. in simplest media.

Wave equs in other media are rather complex.

# Solution to Wave Eq.

- **How to solve the Wave Eqs?**

$$\begin{cases} \nabla^2 \vec{E} - \mu\varepsilon \dfrac{\partial^2 \vec{E}}{\partial t^2} = 0 \\ \nabla^2 \vec{H} - \mu\varepsilon \dfrac{\partial^2 \vec{H}}{\partial t^2} = 0 \end{cases}$$

**The two equations have same form, so the solution of $\vec{E}$ and $\vec{H}$ should same in form**

# From Maxwell Eq. getting Wave Eq.

## Maxwell Equations

$$\begin{cases} \nabla \times \vec{H} = \varepsilon\, \partial \vec{E} / \partial t \\ \nabla \times \vec{E} = -\mu\, \partial \vec{H} / \partial t \\ \nabla \bullet \vec{H} = 0 \\ \nabla \bullet \vec{E} = 0 \end{cases} \Longrightarrow$$

## Helmholtz equations

$$\begin{cases} \nabla^2 \vec{E} - \mu\varepsilon \dfrac{\partial^2 \vec{E}}{\partial t^2} = 0 \\ \nabla^2 \vec{H} - \mu\varepsilon \dfrac{\partial^2 \vec{H}}{\partial t^2} = 0 \end{cases}$$

**Time-harmonic EM-Fields** $\dfrac{\partial}{\partial t} = j\omega, \quad \dfrac{\partial^2}{\partial t^2} = -\omega^2$

$$\begin{cases} \nabla \times \vec{H} = j\omega\varepsilon \vec{E} \\ \nabla \times \vec{E} = -j\omega\mu \vec{H} \\ \nabla \bullet \vec{H} = 0 \\ \nabla \bullet \vec{E} = 0 \end{cases} \qquad \begin{cases} \nabla^2 \vec{E} + \omega^2 \mu\varepsilon \vec{E} = 0 \\ \nabla^2 \vec{H} + \omega^2 \mu\varepsilon \vec{H} = 0 \end{cases} \Longrightarrow \begin{cases} \nabla^2 \vec{E} + k^2 \vec{E} = 0 \\ \nabla^2 \vec{H} + k^2 \vec{H} = 0 \\ k^2 = \omega^2 \mu\varepsilon \end{cases}$$

亥姆赫兹方程实际为场简谐变化下、复数形式的Maxwell方程

# Solution to Wave Eq.

- **Decompose vector Eq. to scalar Eq.**

$$\vec{E} = E_x\vec{e}_x + E_y\vec{e}_y + E_z\vec{e}_z$$

$$\vec{k} = k_x\vec{e}_x + k_y\vec{e}_y + k_z\vec{e}_z$$

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

$$\boxed{\nabla^2\vec{E} + k^2\vec{E} = 0}$$

$$(\nabla^2 E_x + k^2 E_x)\vec{e}_x + (\nabla^2 E_y + k^2 E_y)\vec{e}_y + (\nabla^2 E_z + k^2 E_z)\vec{e}_z = 0$$

$$\Rightarrow \left\{ \boxed{\begin{aligned} (\nabla^2 E_x + k^2 E_x) &= 0 \\ (\nabla^2 E_y + k^2 E_y) &= 0 \\ (\nabla^2 E_z + k^2 E_z) &= 0 \end{aligned}} \right.$$

**Assume E in x direction and Ex only changes with z**

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0$$

# Scalar solution to HPW

$$(\nabla^2 + k^2)\begin{Bmatrix}\vec{E}\\ \vec{H}\end{Bmatrix} = 0$$

A sub-equation is
$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \qquad k^2 = \omega^2 \mu\varepsilon$$

Solution takes form of
$$E_x = E_0 \mathrm{e}^{-\mathrm{j}kz} + E_0' \mathrm{e}^{\mathrm{j}kz}$$

incident wave
in +z direction

reflected wave
in –z direction

$$E_x = E_0 e^{-jkz} + E_0' e^{jkz}$$

incident wave
in +z direction

reflected wave
in –z direction

In order to give a complete expression, we now add time factor $e^{j\omega t}$ to solution above, and thus

$$E_x = E_0 e^{j(\omega t - kz)} + E_0' e^{j(\omega t + kz)}$$

$$\frac{\partial E_x}{\partial z} = -jkE_0 e^{j(\omega t - kz)} + jkE_0' e^{j(\omega t + kz)} = -\mu \frac{\partial H_y}{\partial t} = -j\omega\mu H_y$$

$$H_y = \frac{E_0}{\sqrt{\mu / \varepsilon}} e^{j(\omega t - kz)} - \frac{E_0'}{\sqrt{\mu / \varepsilon}} e^{j(\omega t + kz)}$$

# Solution to Wave Eq.

**Maxwell Eq.**

$$\begin{cases} \nabla\times\vec{H} = j\omega\varepsilon\vec{E} \\ \nabla\times\vec{E} = -j\omega\mu\vec{H} \\ \nabla\bullet\vec{H} = 0 \\ \nabla\bullet\vec{E} = 0 \end{cases}$$

**Helmholtz eq.**

$$\begin{cases} \nabla^2\vec{E} + k^2\vec{E} = 0 \\ \nabla^2\vec{H} + k^2\vec{H} = 0 \\ k^2 = \omega^2\mu\varepsilon \end{cases}$$

**Solution of eq.**

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 e^{j(\omega t - \vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon}}\vec{e}_k \times \vec{E}(\vec{r},t)$$

**In a special coordinate case, the solution of eq.**

$$\vec{E} = E_x\vec{e}_x$$

$$\vec{H} = H_y\vec{e}_y$$

$$\vec{k} = k_z\vec{e}_k$$

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - kz)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - kz)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon}} E_x(\vec{r},t)$$

# § 8.1.2 Plane Wave

- Directions of mutually induced E- & M-fields are normal to each other.
- Vectors $\vec{E}$ & $\vec{H}$ constitute a surface, named as Constant Phase Surface:
  - Phase of E-field at any point on this surface is identical. The same is true to M-field.
  - direction of wave at a point is normal to constant phase surface.
- *If constant phase surface is a plane, we call it a Plane Wave, or TEM wave (Transverse ElectroMagnetic wave 横电磁波).*

# Chpt. 8 Plane Wave Propagation

## Plane wave and Homogenous Plane Wave

**Plane wave:** **the components of $\vec{E}$ and $\vec{H}$ are lie in transverse plane, a plane perpendicular to the direction of propagation of wave.-TEM**

$$\vec{E} \perp \vec{H} \perp \vec{K}$$

where $\vec{K}$ is the wave vector

**Homogenous Plane wave:** **at any given time, a field has the same magnitude and direction in a plane containing it.**

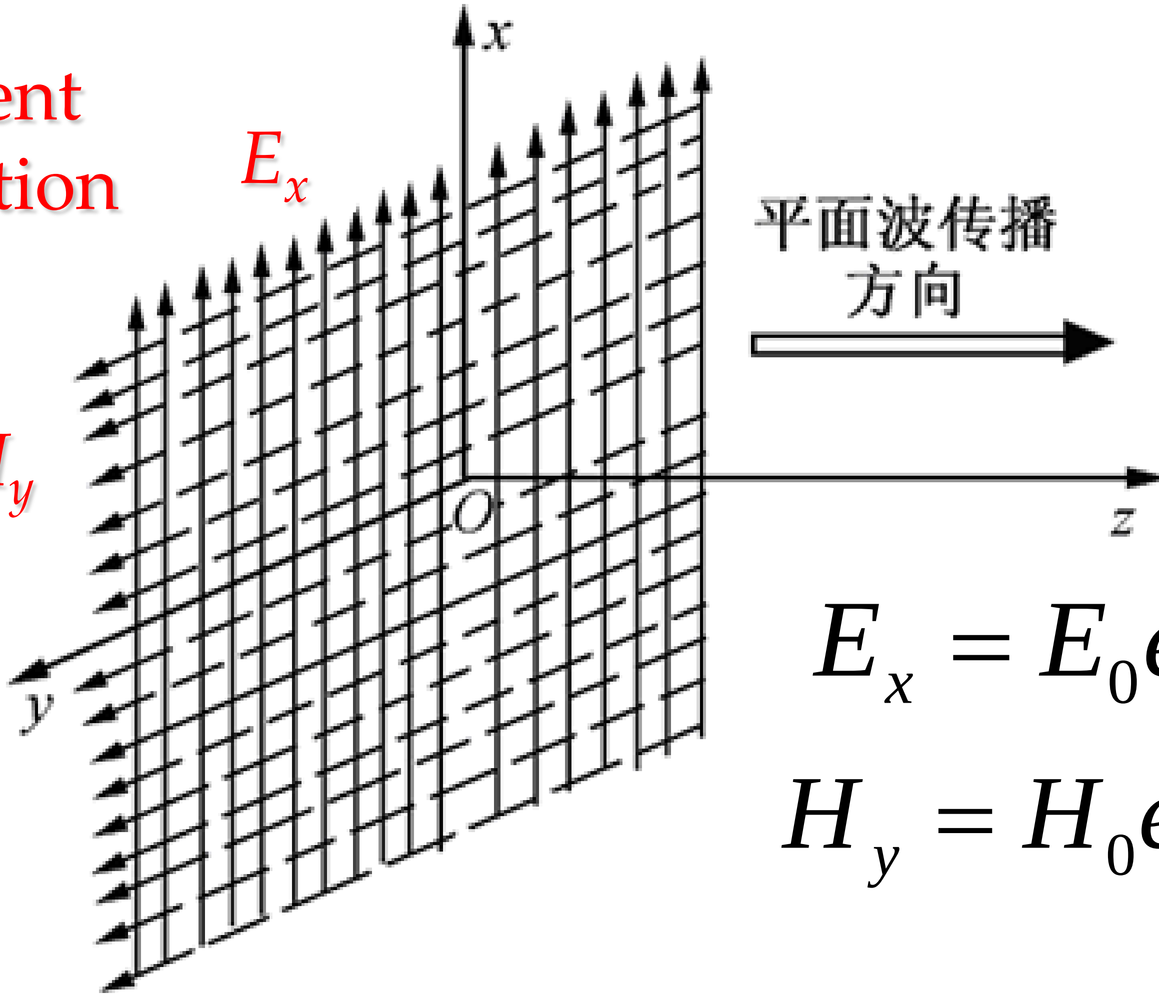
$\vec{E}$ and $\vec{H}$ are not function of x and y

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp[j(\omega t - \vec{k}\cdot\vec{r})]$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 \exp[j(\omega t - \vec{k}\cdot\vec{r})]$$

# HPW

At a moment
and a position




$$E_x = E_0 e^{j(\omega t - kz)}$$

$$H_y = H_0 e^{j(\omega t - kz)}$$

$$E_x = E_0 \cos(\omega t - kz), \quad H_y = H_0 \cos(\omega t - kz)$$

# Characters of HPW in perfect dielectric

$$\vec{E}(\vec{r},t)=\vec{E}_0 e^{j(\omega t-\vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t)=\vec{H}_0 e^{j(\omega t-\vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t)=\frac{1}{\sqrt{\mu/\varepsilon}}\vec{e}_k\times\vec{E}(\vec{r},t)$$

$$E_x(\vec{r},t)=E_0 e^{j(\omega t-kz)}$$

$$H_y(\vec{r},t)=H_0 e^{j(\omega t-kz)}$$

$$H_y(\vec{r},t)=\frac{1}{\sqrt{\mu/\varepsilon}}E_x(\vec{r},t)$$

1. E, H and k are normal each other, and in right hand rule
2. E and H are in phase

# Characters of HPW in perfect dielectric

$$\vec{E}(\vec{r},t)=\vec{E}_0 e^{j(\omega t-\vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t)=\vec{H}_0 e^{j(\omega t-\vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t)=\frac{1}{\sqrt{\mu/\varepsilon}}\vec{e}_k\times\vec{E}(\vec{r},t)$$

$$E_x(\vec{r},t)=E_0 e^{j(\omega t-kz)}$$

$$H_y(\vec{r},t)=H_0 e^{j(\omega t-kz)}$$

$$H_y(\vec{r},t)=\frac{1}{\sqrt{\mu/\varepsilon}}E_x(\vec{r},t)$$

3. k is angle frequency in space domain, just like the $\omega$ is angle frequency in time domain; where z is to t

$$\vec{k}=k_x\vec{e}_x+k_y\vec{e}_y+k_z\vec{e}_z$$

$$k^2=\omega^2\mu\varepsilon$$

$$k^2=k_x^2+k_y^2+k_z^2$$

# Characters of HPW in perfect dielectric

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 e^{j(\omega t - \vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon}} \vec{e}_k \times \vec{E}(\vec{r},t)$$

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - kz)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - kz)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon}} E_x(\vec{r},t)$$

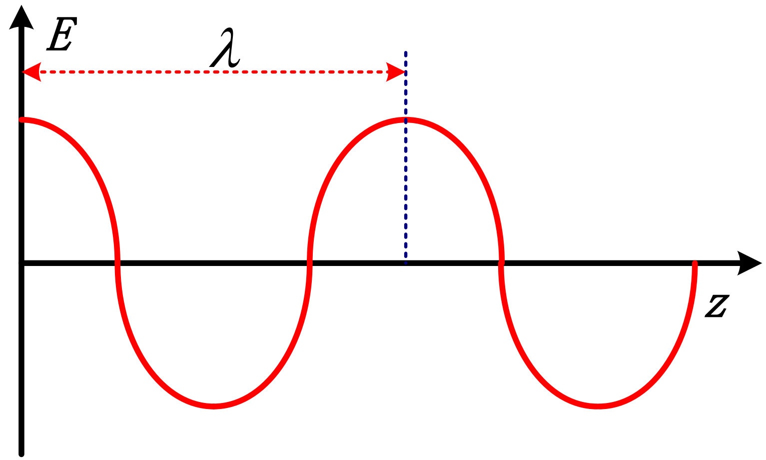
**4.** $\lambda$ Wavelength: the distance between two planes when the phase difference between them at any given time is $2\pi$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$$

$$T = \frac{2\pi}{\omega}$$

$k$

$\omega$



# Characters of HPW in perfect dielectric

$$\vec{E}(\vec{r},t)=\vec{E}_0 e^{j(\omega t-\vec{k}\bullet\vec{r})}$$
$$\vec{H}(\vec{r},t)=\vec{H}_0 e^{j(\omega t-\vec{k}\bullet\vec{r})}$$
$$\vec{H}(\vec{r},t)=\frac{1}{\sqrt{\mu/\varepsilon}}\vec{e}_k\times\vec{E}(\vec{r},t)$$

$$E_x(\vec{r},t)=E_0 e^{j(\omega t-kz)}$$
$$H_y(\vec{r},t)=H_0 e^{j(\omega t-kz)}$$
$$H_y(\vec{r},t)=\frac{1}{\sqrt{\mu/\varepsilon}}E_x(\vec{r},t)$$

**5.** $v_p$ Phase speed: speed of plane of constant phase

$\omega t-kz=\text{const}$ Take diff. as to *t*: $\omega-k\cdot\frac{dz}{dt}=0$

$$\therefore v_p=\frac{dz}{dt}=\frac{\omega}{k}=\frac{1}{\sqrt{\mu\varepsilon}}=c\frac{1}{\sqrt{\mu_r\varepsilon_r}}=\frac{c}{n}$$

$$v_p=\frac{1}{\sqrt{\mu\varepsilon}}=\left|\frac{\vec{E}}{\vec{B}}\right|=c/n$$

$$n=\sqrt{\mu_r\varepsilon_r}$$

Nondispersive medium

# Characters of HPW in perfect dielectric

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 e^{j(\omega t - \vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon}}\vec{e}_k \times \vec{E}(\vec{r},t)$$

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - kz)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - kz)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon}} E_x(\vec{r},t)$$

**6.** Intrinsic Impedance : $\eta$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \left|\frac{\vec{E}}{\vec{H}}\right|$$

This parameter is of unit of an impedance, and is called intrinsic impedance.

In free space, or air, we have

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \approx 377(\Omega)$$

# Characters of HPW in perfect dielectric

$$\vec{E}(\vec{r},t)=\vec{E}_0 e^{j(\omega t-\vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t)=\vec{H}_0 e^{j(\omega t-\vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t)=\frac{1}{\sqrt{\mu/\varepsilon}}\vec{e}_k\times\vec{E}(\vec{r},t)$$

$$E_x(\vec{r},t)=E_0 e^{j(\omega t-kz)}$$

$$H_y(\vec{r},t)=H_0 e^{j(\omega t-kz)}$$

$$H_y(\vec{r},t)=\frac{1}{\sqrt{\mu/\varepsilon}}E_x(\vec{r},t)$$

## 6. Average power density

$$\langle\vec{S}\rangle=\frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right]=\langle\vec{S}\rangle_z=\frac{1}{2\sqrt{\mu/\varepsilon}}E_0^2\vec{e}_z=\frac{1}{2\eta}E_0^2\vec{e}_z$$

$$\langle\vec{S}\rangle=\frac{1}{2\sqrt{\mu/\varepsilon}}E_0^2\vec{e}_z=\frac{\sqrt{\varepsilon}}{2\sqrt{\mu}}E_0^2\vec{e}_z=\frac{\varepsilon}{2\sqrt{\varepsilon\mu}}E_0^2\vec{e}_z=\frac{1}{2}\varepsilon E_0^2 v_p\vec{e}_z$$

The velocity of EM energy is equal to the phase velocity in perfect dielectric

# Problem of HPW in perfect dielectric

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{k}\bullet\vec{r} + \varphi_0)}$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 e^{j(\omega t - \vec{k}\bullet\vec{r} + \varphi_0)}$$

$$\vec{H}(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon}}\vec{e}_k \times \vec{E}(\vec{r},t)$$

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - kz + \varphi_0)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - kz + \varphi_0)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon}} E_x(\vec{r},t)$$

- Typical problems:
  - Given either $E$ or $H$, please find the other, and also $k$ & $S$.

Key to solution:

$$\begin{cases} \nabla\times\vec{H} = j\omega\varepsilon\vec{E} \\ \nabla\times\vec{E} = -j\omega\mu\vec{H} \\ \nabla\bullet\vec{H} = 0 \\ \nabla\bullet\vec{E} = 0 \end{cases}$$

$$\begin{cases} \nabla^2\vec{E} - \mu\varepsilon\dfrac{\partial^2\vec{E}}{\partial t^2} = 0 \\ \nabla^2\vec{H} - \mu\varepsilon\dfrac{\partial^2\vec{H}}{\partial t^2} = 0 \end{cases}$$

# An Example

A sinusoidally HPW propagate in +z direction in perfect uniform dielectric medium with frequency 100MHz, the parameters of the medium is $\varepsilon_r = 4, \mu_r = 1, \sigma = 0$. If E-field in x direction, $\vec{E} = E_x \vec{e}_x$; when t=0, and z=1/8m, the E-field gets it maximum amplitude $E_m = 10^{-4} V/m$. Determine a) E(z,t) and H(z,t); b) phase speed; c) average power density.

Solution: a) The E-field can be written in cosusoidally form

$$\vec{E}(z,t) = E_x(z,t)\vec{e}_x$$

$$= E_m \cos(\omega t - kz + \phi_0)\vec{e}_x \qquad E_m = 10^{-4} V/m$$

$$k = \omega\sqrt{\mu\varepsilon} = 2\pi f\sqrt{4\mu_0\varepsilon_0}$$

$$= 2\pi \times 100 \times 10^6 \times 2\sqrt{\mu_0\varepsilon_0} = \frac{4}{3}\pi rad/m$$

# An Example

For: $E_x(\frac{1}{8},0) = E_m = 10^{-4}$ so $\omega t - kz + \phi_0 = 0$

We get $\phi_0 = kz = \frac{4\pi}{3} \times \frac{1}{8} = \frac{\pi}{6} rad$

so $\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r \mu_0}{\varepsilon_r \varepsilon_0}} = \frac{1}{2}\eta_0 = \frac{1}{2} \times 120\pi = 60\pi\Omega$

$$\vec{E}(z,t) = 10^{-4}\cos(2\pi \times 10^8 t - \frac{4\pi}{3}z + \frac{\pi}{6})\vec{e}_x$$

$$\vec{H}(z,t) = H_y \vec{e}_y = \frac{E_x}{\eta}\vec{e}_y = \frac{E_x}{60\pi}\vec{e}_y$$
$$= \frac{1}{60\pi}10^{-4}\cos(2\pi \times 10^8 t - \frac{4\pi}{3}z + \frac{\pi}{6})\vec{e}_y$$

# An Example

## b) The velocity of phase

$$v_p = \sqrt{\frac{1}{\mu\varepsilon}} = \sqrt{\frac{1}{4\mu_0\varepsilon_0}} = \frac{1}{2}c = 1.5\times10^8 m/s$$

## c) The power density

$$\vec{S}_{avg} = \frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right]$$

$$\vec{E} = 10^{-4} e^{j\left(2\pi\times10^8 t - \frac{4\pi}{3}z + \frac{\pi}{6}\right)}\vec{e}_x$$

$$\vec{H} = \frac{10^{-4}}{60\pi} e^{j\left(2\pi\times10^8 t - \frac{4\pi}{3}z + \frac{\pi}{6}\right)}\vec{e}_y$$

$$\vec{S}_{avg} = \frac{1}{2}\mathrm{Re}\left[10^{-4} e^{j\left(2\pi\times10^8 t - \frac{4\pi}{3}z + \frac{\pi}{6}\right)}\vec{e}_x \times \frac{10^{-4}}{60\pi} e^{-j\left(2\pi\times10^8 t - \frac{4\pi}{3}z + \frac{\pi}{6}\right)}\vec{e}_y\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[10^{-4}\times\frac{10^{-4}}{60\pi}\vec{e}_z\right] = \frac{10^{-8}}{120\pi}\vec{e}_z W/m^2$$

# An Example

A HPW propagate in perfect uniform boundless dielectric medium .

$$\vec{E} = 300(\vec{e}_x + E_{y0}\vec{e}_y + \sqrt{5}\vec{e}_z)\cos(30\pi \times 10^8 t + 4\pi(\sqrt{5}x + 2y - 4z))$$

Determine

1) Direction of EM-wave propagation

2) Frequency, wavelength, and phase velocity

3) Assume $\mu_r = 1$ find $\varepsilon_r$

4) $E_{y0}$

5) $\vec{H}$

6) $\vec{S}_{avg} = \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right]$

# An Example

Solution 1) Direction of EM-wave propagation

$$\vec{k} \bullet \vec{r} = -4\pi(\sqrt{5}x + 2y - 4z) = k_x x + k_y y + k_z z$$

$$\vec{k} = 4\pi(-\sqrt{5}\vec{e}_x - 2\vec{e}_y + 4\vec{e}_z)$$

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} = 4\pi\sqrt{5+4+16} = 20\pi$$

$$\vec{e}_k = \frac{\vec{k}}{k} = \frac{1}{5}(-\sqrt{5}\vec{e}_x - 2\vec{e}_y + 4\vec{e}_z)$$

2) Frequency, wavelength, and phase velocity

$$f = \frac{\omega}{2\pi} = 15 \times 10^8\, Hz \qquad \lambda = \frac{2\pi}{k} = 0.1m$$

$$v_p = \frac{\omega}{k} = 1.5 \times 10^8\, m/s$$

# An Example

Solution 3) $$v_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = \frac{c}{\sqrt{\varepsilon_r}} = 1.5\times10^8 m/s$$

$$\varepsilon_r = 4$$

4) $$\vec{E}\bullet\vec{k} = 0$$

$$= (\vec{e}_x + E_{y0}\vec{e}_y + \sqrt{5}\vec{e}_z)\bullet(\sqrt{5}\vec{e}_x + 2\vec{e}_y - 4\vec{e}_z)) = 0$$

$$E_{y0} = \frac{3}{2}\sqrt{5}$$

5) $$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r}} = \frac{\eta_0}{\sqrt{\varepsilon_r}} = 60\pi$$

$$\vec{H} = \frac{1}{\eta}\vec{e}_k \times \vec{E}$$

# An Example

Solution 5) $\vec{H}=\frac{1}{\eta}\vec{e}_k\times\vec{E}$

$$=\frac{1}{60\pi}*\frac{1}{5}(-\sqrt{5}\vec{e}_x-2\vec{e}_y+4\vec{e}_z)\times(\vec{e}_x+\frac{3}{2}\sqrt{5}\vec{e}_y+\sqrt{5}\vec{e}_z)e^{-j\vec{k}\bullet\vec{r}}\times 300$$

$$=\frac{1}{\pi}(-8\sqrt{5}\vec{e}_x+9\vec{e}_y-\frac{11}{2}\vec{e}_z)e^{-j\vec{k}\bullet\vec{r}}$$

6) $\vec{S}_{avg}=\frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right]$

$$=\frac{150}{\pi}(\vec{e}_x+\frac{3}{2}\sqrt{5}\vec{e}_y+\sqrt{5}\vec{e}_z)\times(-8\sqrt{5}\vec{e}_x+9\vec{e}_y-\frac{11}{2}\vec{e}_z)$$

$$=\frac{5175}{2\pi}(-\sqrt{5}\vec{e}_x-2\vec{e}_y+4\vec{e}_z)$$

# Polarization of EM wave

## What is polarization?

The locus of the tip of the E-field vector at a given point as a function of time.

本质上是：在波传输的某一点上看，由初振幅和初相位决定的电场强度矢量的顶点在等相位面上的运动轨迹

## How to describe polarization?

Fundamental Polarization Type:

1. *Linear Polarization*
2. *Circular Polarization*
3. *Elliptical Polarization*

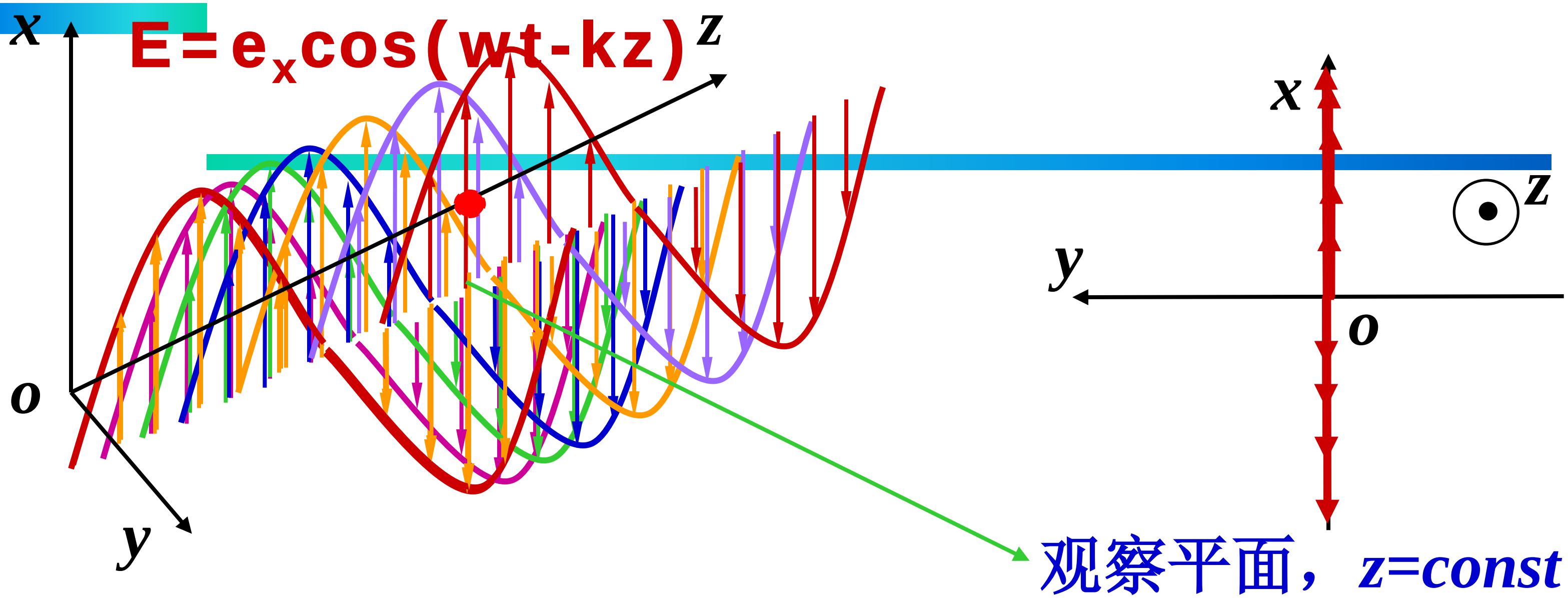
$$E_x(\vec{r},t) = E_0 e^{j(\omega t - kz + \varphi_0)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - kz + \varphi_0)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon}} E_x(\vec{r},t)$$

The other polarization types may be the combination of the above. And vice versa.

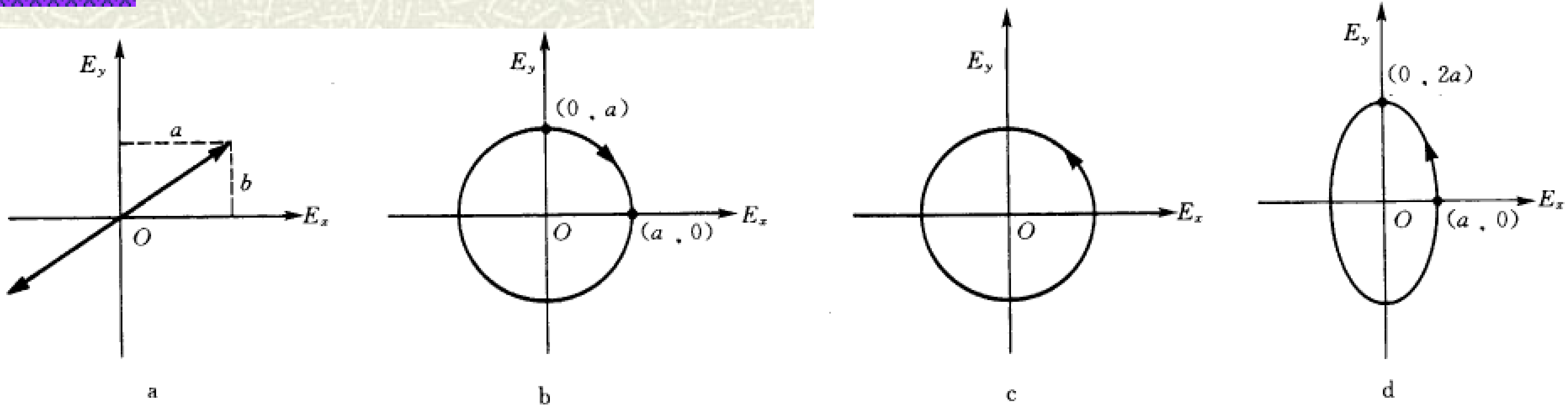
**电磁波的极化方式由辐射源（即天线）的性质决定。**




$E=e_x\sin(wt-kz)$电场的振动方向始终是沿$x$轴方向，所以这是一个沿$x$方向的线极化波。

## 极化的判断

- 两个相互正交的线极化波叠加，合成得到不同的极化方式。
- 由电磁波电场场量或者磁场场量，可以判断波的极化方式。




*Linear* *Circular* *Elliptical*

# How to analyze Polarization of EM wave

**Assume the EM wave propagates in Z direction, for a given Z position,** The locus of the tip of **the EM-field will changes with time in Constant phase plane**

**Check the combination of 2 single polarized TEM waves**

$$\vec{E}(\vec{r},t) = E_x\vec{e}_x + E_y\vec{e}_y$$

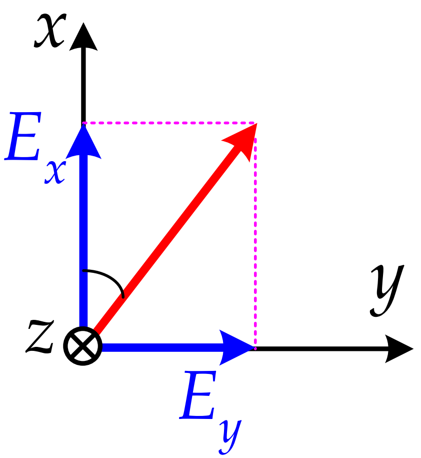
$$E_x = E_{xm}\cos(\omega t - kz + \phi_x)$$

$$E_y = E_{ym}\cos(\omega t - kz + \phi_y)$$

**Assume z=0**

$$E_x = E_{xm}\cdot\cos(\omega t + \phi_x)$$

$$E_y = E_{ym}\cdot\cos(\omega t + \phi_y)$$

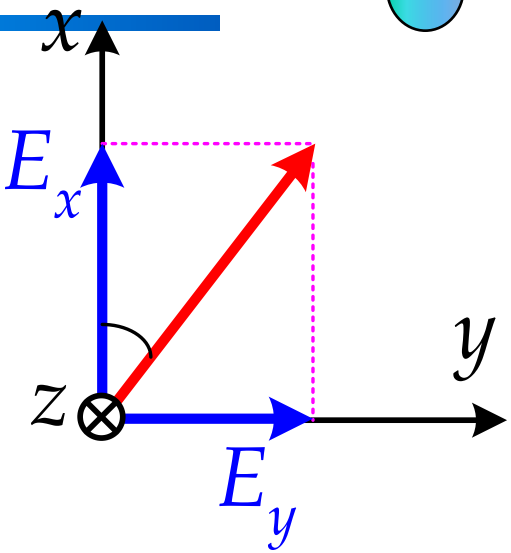


# How to analyze Polarization of EM wave

$$\vec{E}(\vec{r},t) = E_x\vec{e}_x + E_y\vec{e}_y$$

$$E_x = E_{xm}\cdot\cos(\omega t + \phi_x)$$

$$E_y = E_{ym}\cdot\cos(\omega t + \phi_y)$$




合成电场的模及其与x轴夹角为：

$$|\vec{E}| = \sqrt{E_x^2 + E_y^2}$$

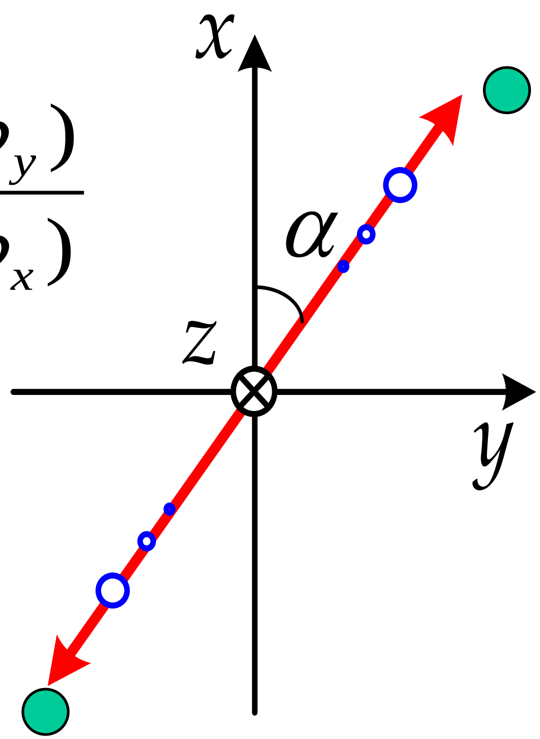
$$tg\alpha = \frac{E_y}{E_x} = \frac{E_{ym}\cos(\omega t + \phi_y)}{E_{xm}\cos(\omega t + \phi_x)}$$

$$\text{Amplitude}: \ |\mathrm{E}(\vec{\mathrm{r}},\mathrm{t})| = E_m = \sqrt{E_{xm}^2 + E_{ym}^2}$$

# *Linearly Polarized*

Amplitude: $\mathrm{E_m} = \sqrt{E_{xm}^2 + E_{ym}^2}$

Phase: $\mathrm{tg}\alpha = \dfrac{E_y}{E_x} = \dfrac{E_{ym}\cos(\omega t + \varphi_y)}{E_{xm}\cos(\omega t + \varphi_x)}$




**If** the trace of endpoint of E is a line,

$$tg\alpha = \frac{E_y}{E_x} = \text{Constant}$$

Condition: $\varphi_x = \varphi_y$ or $\varphi_x = \varphi_y + \pi$

Vector: $\mathrm{E} = \sqrt{E_{xm}^2 + E_{ym}^2}\cos\omega t$

# Circularly Polarized

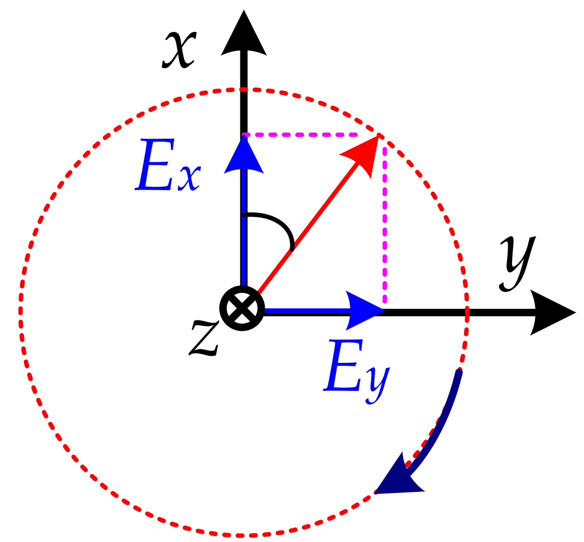
☞ 当$\varphi_x - \varphi_y = \pm\dfrac{\pi}{2}$ 且$E_{xm} = E_{ym}$时

$$E_x = E_{xm}\cos(\omega t + \varphi_x)$$

$$E_y = E_{ym}\cos(\omega t + \varphi_x \mp \frac{\pi}{2}) = \pm E_{ym}\sin(\omega t + \varphi_x)$$

合成电场的模及其与x轴夹角为:

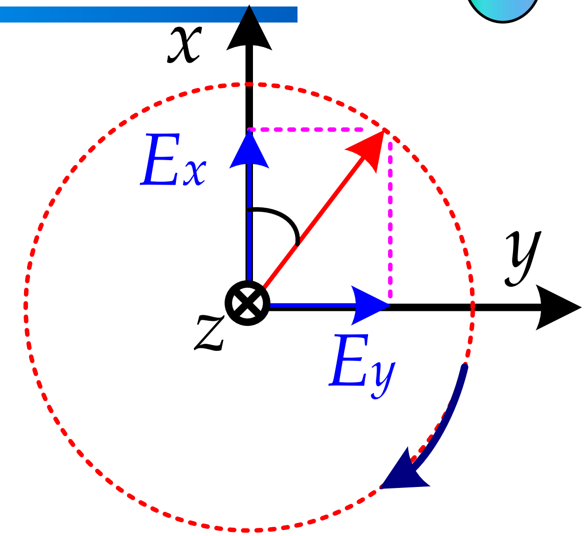
$$\left|\vec{E}\right| = \sqrt{E_x^2 + E_y^2} = E_{xm} = const$$



# Circularly Polarized

Amplitude: $\left|\vec{E}\right| = \sqrt{E_x^2 + E_y^2} = E_{xm} = const$

Phase: $\operatorname{tng}\alpha = \dfrac{E_y}{E_x} = \dfrac{E_{ym}\cos(\omega t + \phi_y)}{E_{xm}\cos(\omega t + \phi_x)}$




If Condition: $E_{xm} = E_{ym}$ and $\left|\varphi_x - \varphi_y\right| = \dfrac{\pi}{2}$

$E = \text{Constant}$ $\qquad tg\alpha = \dfrac{E_y}{E_x} = tg(\omega t)$ $\qquad \alpha(t) = \omega t$

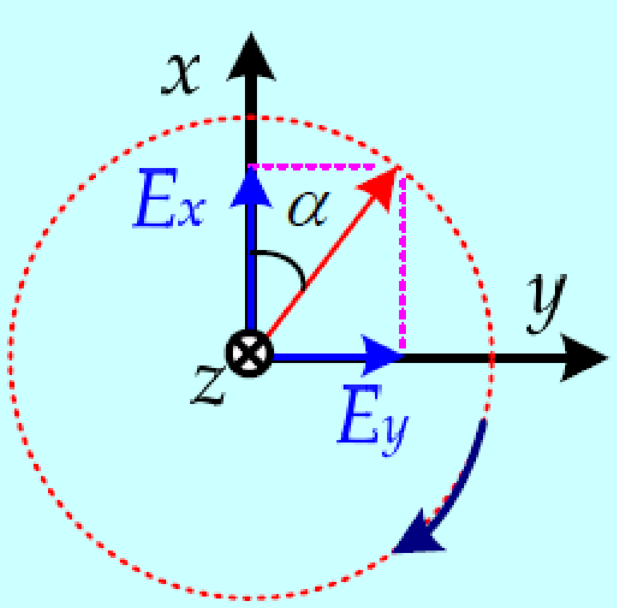
Sub-type: left-handed and right-handed circularly polarized

Judgment: put thumb to direction of wave, and check if the trace of endpoint of E-field is coincident with left hand or right hand.

Vector: $\mathrm{E} = \sqrt{E_x^2 + E_y^2} = E_m$

# Right-handed circularly polarized

**Observation along wave propagation direction（Z axis direction）**




$$\frac{E_y}{E_x} = tg\alpha = \frac{E_{ym}\cos(\omega t - \beta z + \varphi_y)}{E_{xm}\cos(\omega t - \beta z + \varphi_x)}$$

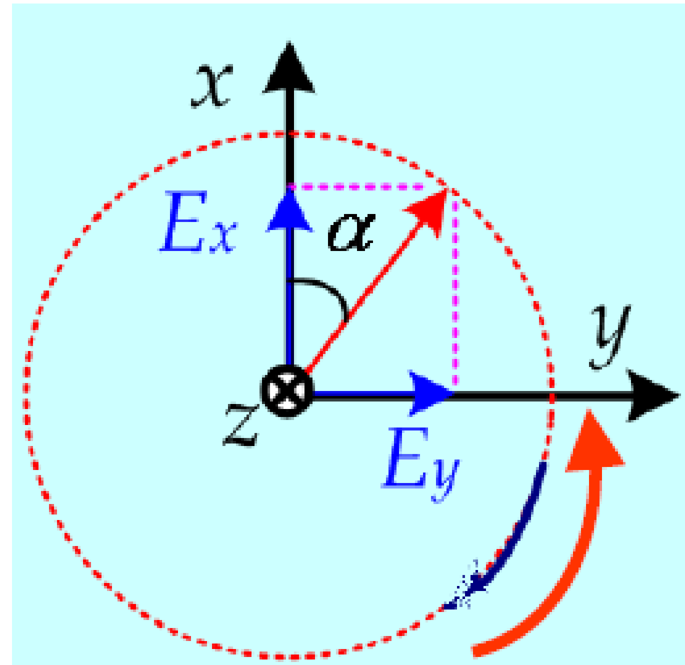
$$\varphi_y = \varphi_x - \frac{\pi}{2} \qquad E_{xm} = E_{ym}$$

$$\frac{E_y}{E_x} = tg(\omega t - \beta z + \varphi_x) = tg\alpha \quad \rightarrow \quad \alpha = \omega t - \beta z + \varphi_x$$

**It is apparent that the angle $\alpha$ will increase with time and the tip of electric field rotates clockwise, i.e. right-handed circularly polarized wave!**

# Left-handed circularly polarized

**Observation along wave propagation direction（Z axis direction）**




$$\frac{E_y}{E_x} = tg\alpha = \frac{E_{ym}\cos(\omega t - \beta z + \varphi_y)}{E_{xm}\cos(\omega t - \beta z + \varphi_x)}$$

$$\varphi_y = \varphi_x + \frac{\pi}{2} \qquad E_{xm} = E_{ym}$$

$$\frac{E_y}{E_x} = tg(-\omega t + \beta z - \varphi_x) = tg\alpha \rightarrow \alpha = -(\omega t - \beta z + \varphi_x)$$

**It is apparent that the angle $\alpha$ will decrease with time and the tip of E field rotates anti-clockwise, i.e. left-handed circularly polarized wave!**

# Elliptically Polarized

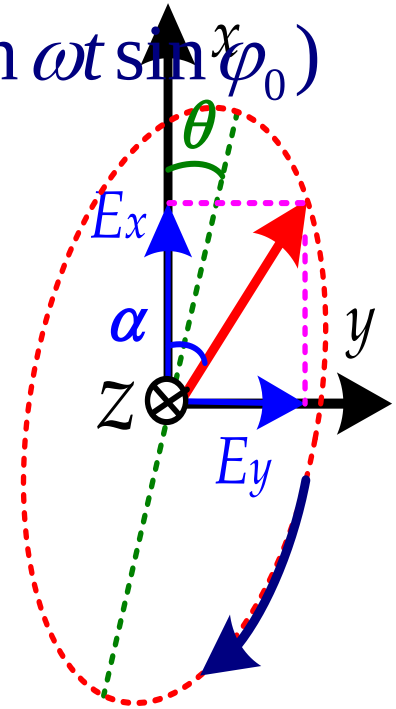
Condition: $E_{xm} \neq E_{ym}$ and $\varphi_x - \varphi_y = \varphi_0$

$$E_x = E_{xm} \cdot \cos(\omega t) \qquad E_y = E_{ym} \cdot \cos(\omega t - \varphi_0)$$

$$E_y = E_{ym} \cos(\omega t - \varphi_0) = E_{ym} (\cos \omega t \cos \varphi_0 + \sin \omega t \sin \varphi_0)$$

$$\frac{E_y}{E_{ym}} = \frac{E_x}{E_{xm}} \cos \varphi_0 + \sqrt{1 - (\frac{E_x}{E_{xm}})^2} \sin \varphi_0$$

$$\Rightarrow (\frac{E_y}{E_{ym}})^2 + (\frac{E_x}{E_{xm}})^2 - 2\frac{E_x}{E_{xm}} \cdot \frac{E_y}{E_{ym}} \cos \varphi_0 = \sin^2 \varphi_0$$




An equation for ellipse

Angle btwn long axis & $x$-axis

$$tg(2\theta) = \frac{2E_{xm}E_{ym}}{E_{xm}^2 - E_{ym}^2} \cos \varphi_0$$

# Example 1.

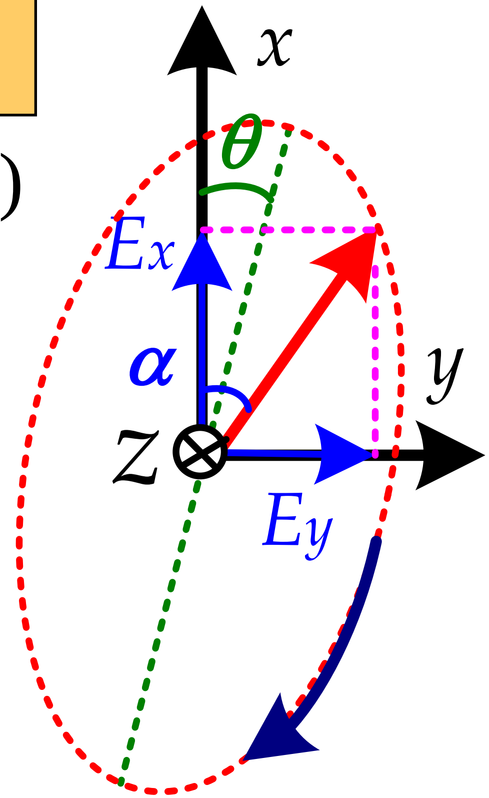
Condition: $E_{xm} \neq E_{ym}$ and $\varphi_x - \varphi_y = \varphi_0$

$$E_x = E_{xm} \cdot \cos(\omega t) \qquad E_y = E_{ym} \cdot \cos(\omega t - \varphi_0)$$

If $\varphi_0 > 0$, $E_x \downarrow$ and $E_y \uparrow$ with $t \uparrow$ .

Thus trace of endpoint of E is clockwise.

If TEM wave is in +z direction, we find it is right-handed elliptically polarized.




If $\varphi_0 < 0$, ……

If TEM wave is in –z direction, ……

# Example 2. Problem 8.24

Please prove: a linearly polarized wave is the combination of a left-handed and a right-handed circularly polarized waves with the same amplitude.

Condition: $E_{xm} = E_{ym}$ and $\left|\varphi_x - \varphi_y\right| = \dfrac{\pi}{2}$

Solution: assume TEM wave is in +z direction and the amplitude of E is $E_0$. Thus 2 circularly polarized waves are

$$(\vec{a}_x E_0 - j\vec{a}_y E_0)e^{j(\omega t - kz)} \qquad \varphi_x - \varphi_y = \pi/2 > 0 \qquad \text{Right-handed}$$

$$(\vec{a}_x E_0 + j\vec{a}_y E_0)e^{j(\omega t - kz)} \qquad \varphi_x - \varphi_y = -\pi/2 < 0 \qquad \text{Left-handed}$$

$$\vec{E} = E_0\left[\left(\vec{\boldsymbol{a}}_x - \mathrm{j}\vec{\boldsymbol{a}}_y\right) + \left(\vec{\boldsymbol{a}}_x + \mathrm{j}\vec{\boldsymbol{a}}_y\right)\right]\cdot \mathrm{e}^{\mathrm{j}(\omega t\text{-}kz)} = \vec{\boldsymbol{a}}_x(2E_0)\mathrm{e}^{\mathrm{j}(\omega t\text{-}kz)}$$

Their combination is linearly polarized.

Similar exercise --- Problem 8.25

# Polarization disassembly and synthesization

**Linearly polarized wave can be synthesized by two circularly polarized wave with same amplitudes, in which one is left-handed and the other is right-handed.**

$$E_x = E_m \cos(\omega t - \beta z + \varphi_x)$$

$$E_y = E_m \cos(\omega t - \beta z + \varphi_x + \frac{\pi}{2})$$

**Left-hand circularly**

$$E_x{}' = E_m \cos(\omega t - \beta z + \varphi_x)$$

$$E_y{}' = E_m \cos(\omega t - \beta z + \varphi_x - \frac{\pi}{2})$$

**Right-hand circularly**

$$\vec{E} = \vec{e}_x E_x = 2E_m \cos(\omega t - \beta z + \varphi_x)$$

**Linearly polarized wave**

# Polarization disassembly and synthesization

**Elliptically polarized wave can be synthesized by two circularly polarized wave with different amplitudes, in which one is left-handed and the other is right-handed.**

$$\boxed{\vec{E} = (\vec{e}_x E_1 + j\vec{e}_y E_2)e^{-j\beta z}}$$

$$(E_1' + E_2' = E_1;\ \ E_1' - E_2' = E_2)$$

$$= (\vec{e}_x + j\vec{e}_y)E_1' e^{-j\beta z} + (\vec{e}_x - j\vec{e}_y)E_2' e^{-j\beta z}$$

$$(E_1' = \frac{E_1 + E_2}{2};\ E_2' = \frac{E_1 - E_2}{2})$$

$$= \underline{(\vec{e}_x + j\vec{e}_y)}\,\underline{\frac{E_1 + E_2}{2}} e^{-j\beta z} + \underline{(\vec{e}_x - j\vec{e}_y)}\,\underline{\frac{E_1 - E_2}{2}} e^{-j\beta z}$$

# Method to estimate right or left handed wave

Observation along wave propagation direction（Z axis direction）:

X component leads Y component，right-handed；

Y component leads X component，left-handed；

Observation along wave propagation direction（-Z axis direction）:

X component leads Y component，left-handed；

Y component leads X component，right-handed；

How about wave along other propagation direction?

$$X-Y-Z-X-Y-Z$$

# Decide polarization state

(1) $\vec{E}(z)=\hat{e}_x jE_m e^{jkz}+e_y jE_m e^{jkz}$

$$\varphi_x=\varphi_y=\frac{\pi}{2},\ 故：\ \varphi_x-\varphi_y=0$$

Linear

(2) $\vec{E}(z,t)=\hat{e}_x E_m \sin(\omega t-kz)+e_y E_m \cos(\omega t-kz)$

$$\varphi_x=-\frac{\pi}{2},\varphi_y=0,\ 故：\ \varphi_x-\varphi_y=-\frac{\pi}{2}$$

$E_{xm}=E_{ym}=E_m$ Left circularly polarization

## (4) $\vec{E}(z) = \hat{e}_x E_m e^{-jkz} - e_y jE_m e^{jkz}$

$$\vec{E}(z,t) = \hat{e}_x E_m \cos(\omega t - kz) + e_y E_m \cos(\omega t - kz - \frac{\pi}{2})$$

$$\varphi_x = 0, \varphi_y = -\frac{\pi}{2} \quad \therefore \varphi_x - \varphi_y = \frac{\pi}{2}$$

$$E_{xm} = E_{ym} = E_m$$

Right circularly polarization

(5) $\vec{E}(z,t) = \hat{e}_x E_m \sin(\omega t - kz) + e_y E_m \cos(\omega t - kz + 40^\circ)$

***Elliptically Polarized***

# Homework

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- Exercises 8.6, 8.7, 8.15, 8.16, 8.17
- Problems 8.7, 8.8, 8.22, 8.23

## 2、平面波的传播速度

等相面在沿波矢 k 方向上距离与时间的关系为

$$\omega t - kr = C \Rightarrow v = \frac{dr}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}}$$

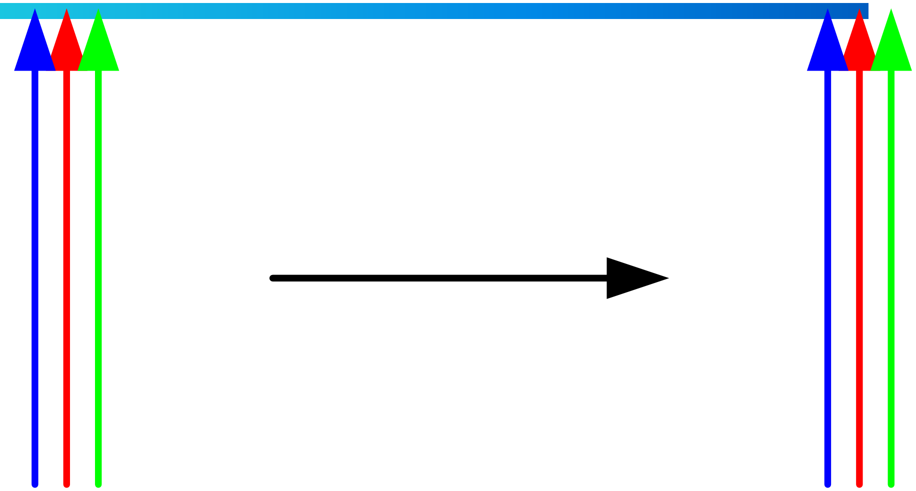
是为相速，即等相面沿其法线方向的传播速度，也即平面波的传播速度。如果媒质的特性与频率无关，即ε、μ与频率无关，则不同频率的平面电磁波的相速与频率无关，媒质称为无色散媒质。一般情况下，媒质总是存在色散。

相速：平面波等相面的传播速度

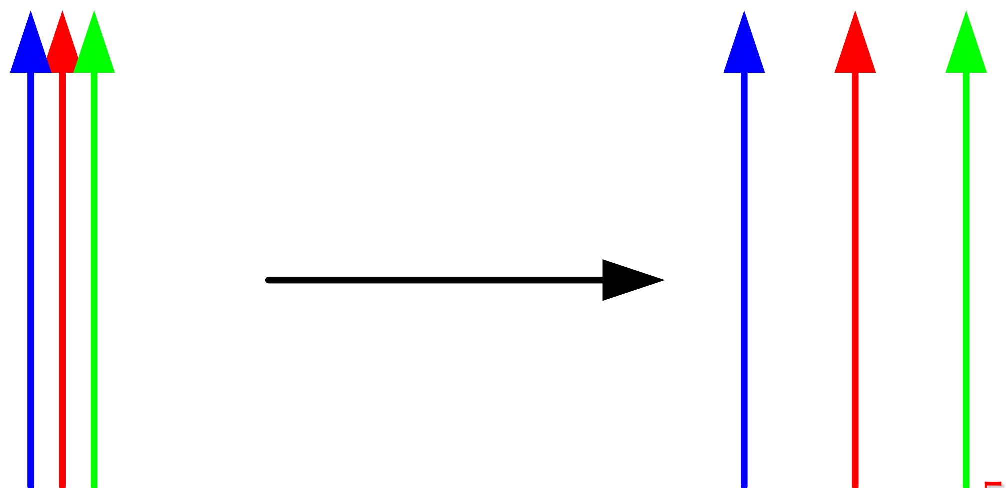
群速：信号能量传输的速度

信号总能被分解成具有不同频率的平面波的叠加，对每一个频率分量来说，其相速等于其能量传播的速度。若信号所在媒质为无色散媒质，即平面波的相速(或其能量传播速度)与频率无关，则信号的能量传播速度等于平面波的相速，即群速等于相速；一般情况下，媒质总是存在色散，则信号的群速不等于平面波的相速。

无色散媒质



色散媒质



考虑一个平面波信号，由频率为($\omega_0$-$\Delta\omega$，$\omega_0$+$\Delta\omega$)的平面波组成：

$$E(z,t)=\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega}E_0(\omega)e^{i[\omega t-k(\omega)z]}d\omega$$

将 $k(\omega)$在$\omega_0$ 处展开：

$$k(\omega)=k(\omega_0)+\left(\frac{dk}{d\omega}\right)_{\omega_0}\Delta\omega+\cdots$$

$$=k_0+\left(\frac{dk}{d\omega}\right)_0(\omega-\omega_0)+\cdots$$

保留到一阶项并带入信号表达式，则有：

$$E(z,t)=\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega}E_0(\omega)e^{i[\omega t-k(\omega)z]}d\omega$$

$$=\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega}E_0(\omega)e^{i[\omega_0 t-k_0 z+(\omega-\omega_0)t-k_0'(\omega-\omega_0)z]}d\omega$$

若 $\dfrac{dE_0(\omega)}{d\omega}$ 很小，则将各频率分量的振幅在$\omega_0$ 处展开：

$$E_0(\omega)=E_0(\omega_0)+\left(\frac{dE_0(\omega)}{d\omega}\right)_{\omega_0}\Delta\omega+\cdots\approx E_0(\omega_0)=E_0$$

则信号的表达式可演化为：

$$E(z,t)=\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega}E_0(\omega)e^{i[\omega_0 t-k_0 z+(\omega-\omega_0)t-k_0'(\omega-\omega_0)]}d\omega$$

$$=E_0e^{i(\omega_0 t-k_0 z)}\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega}e^{i[(\omega-\omega_0)t-k_{00}'(\omega-\omega_0)]}d\omega$$

$$=E_0e^{i(\omega_0 t-k_0 z)}\int_{\Delta\omega}^{\Delta\omega}e^{i[\Delta\omega\cdot t-\Delta\omega\cdot k_0' z]}d(\Delta\omega)$$

$$=2\Delta\omega E_0e^{i(\omega_0 t-k_0 z)}\cdot\frac{\sin[(t-k_0'z)\Delta\omega]}{(t-k_0'z)\Delta\omega}$$