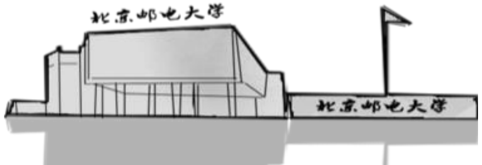
Chapter 7

Source and Source Coding

School of Information and Communication Engineering

Beijing University of Posts and Telecommunications



Digitization of analog signals

- **Sampling**
  - Sampling is the process of converting a continuous signal into a discrete sequence in the time domain.
  - This time-discrete sequence has a continuous amplitude.
- **Quantization**
  - Quantization, on the other hand, converts the amplitude-continuous sequence into an amplitude-discrete sequence.
- **Coding**
  - Finally, coding is the process of converting the amplitude-discrete sequence into a code sequence.
  - Compression coding might also be used.

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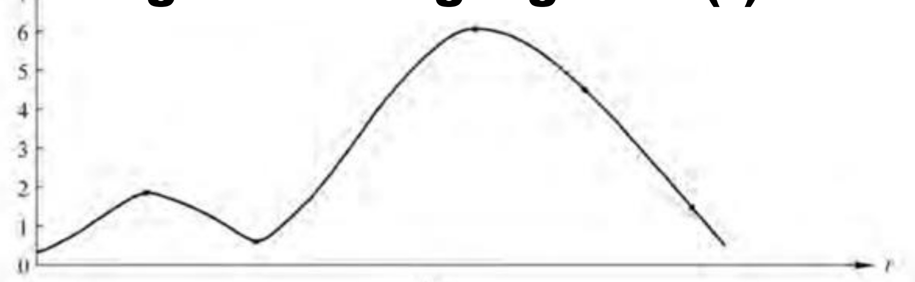
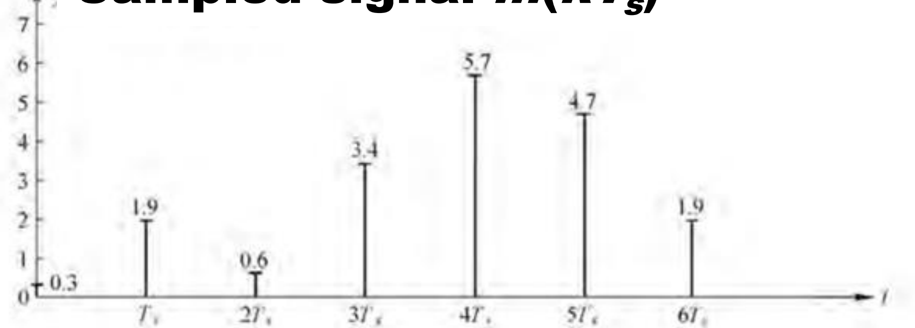
Digitization of analog signals

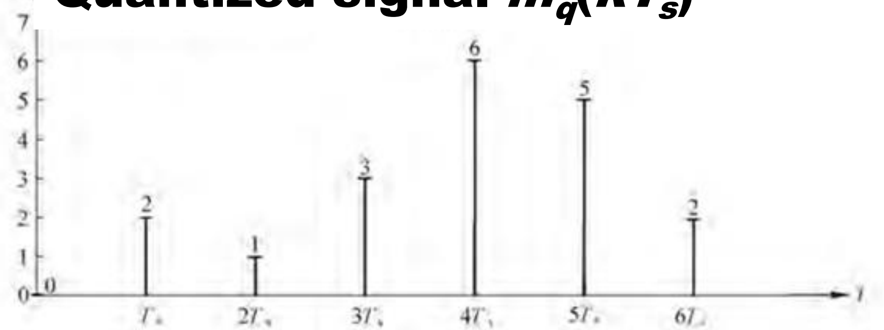
Original analog signal *m*(*t*)



Sampled signal *m*(*kT*$_s$)

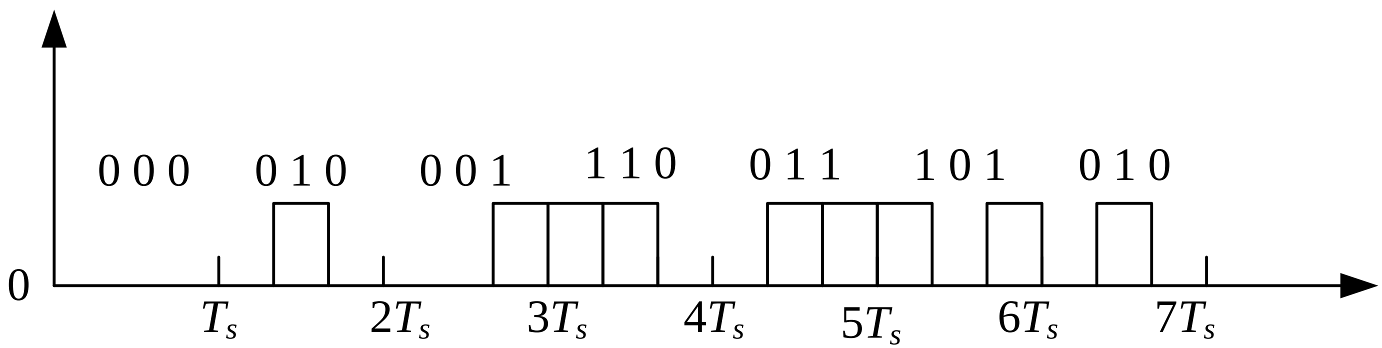


Quantized signal *m*$_q$(*kT*$_s$)

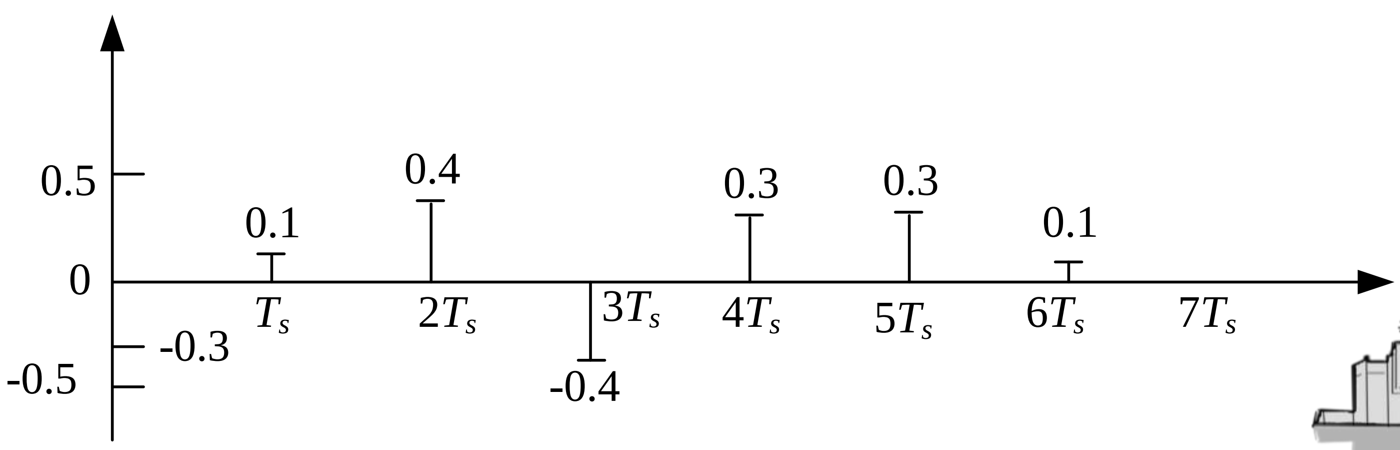


Digitization of analog signals

Binary sequence



Quantization error sequence

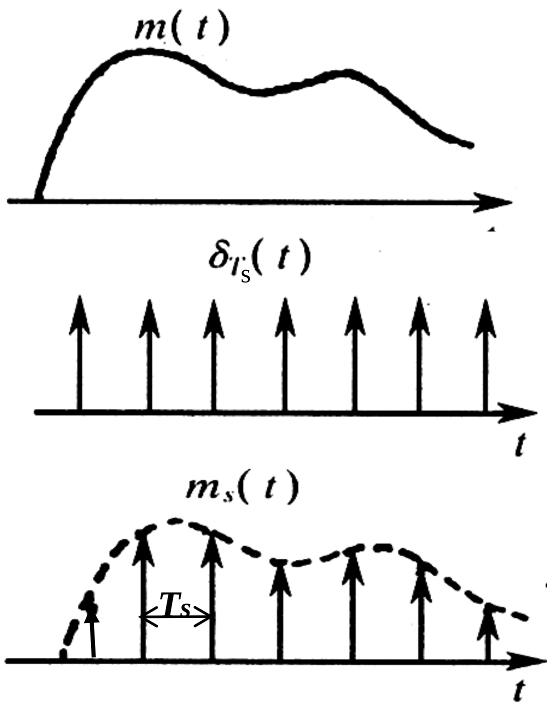


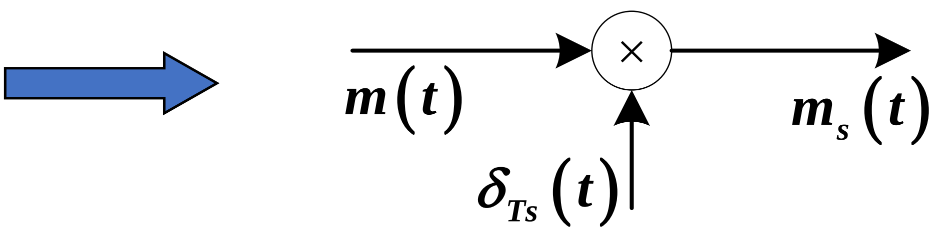
Distortion-Limited Coding of Continuous Sources

- Sampling theorem
- Quantization
- Pulse-code modulation(PCM)
- Time Division Multiplexing(TDM)

Sampling Theorem

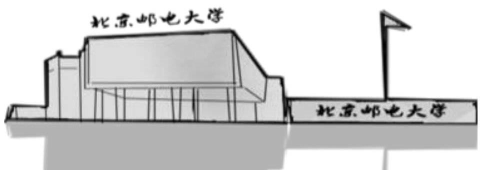
- **Formulation of sampling**



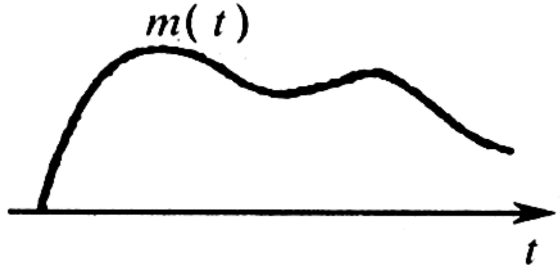


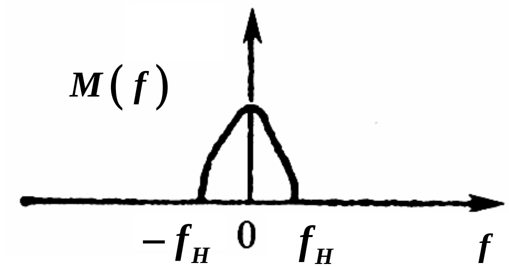
$$m_s(t) = m(t)\delta_{T_s}(t)$$

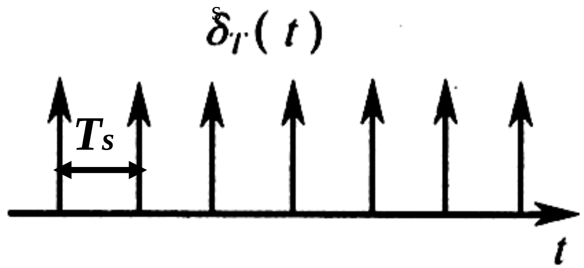
$$= \sum_{n=-\infty}^{\infty} m_n \delta(t - nT_s)$$



Sampling Theorem

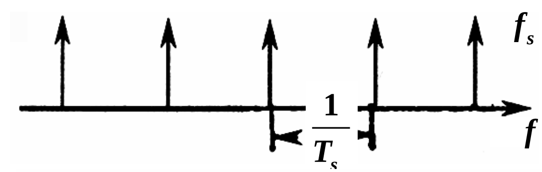


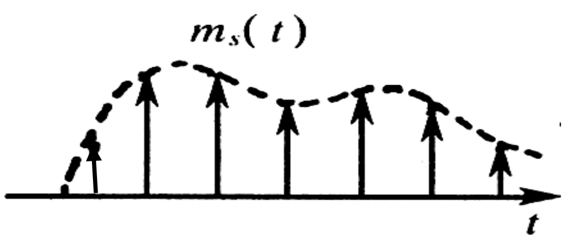


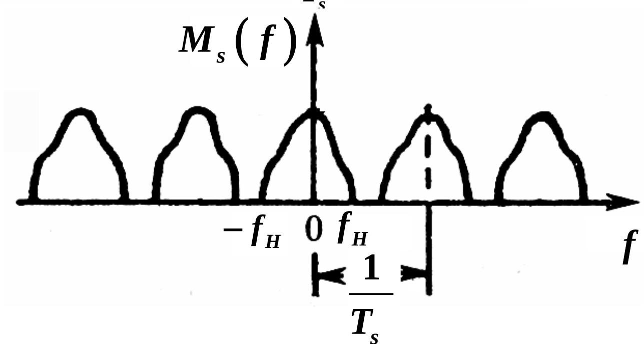


$$\delta_{f_s}(f) = f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s),$$

$$f_s = \frac{1}{T_s}$$







$$M_s(f) = M(f) * \delta_{f_s}(f)$$

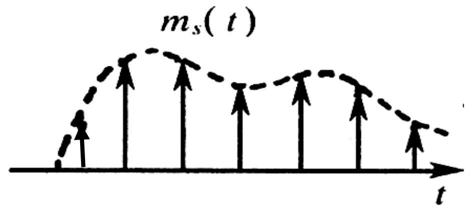
$$= \frac{1}{T_s}\left[M(f) * \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right]$$

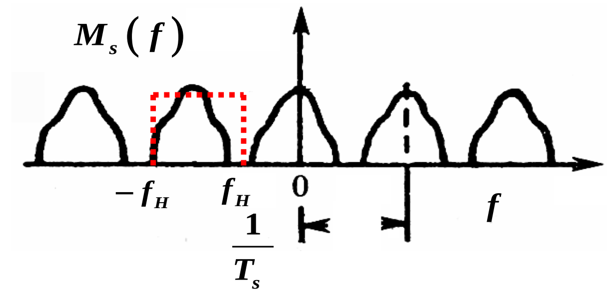
$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} M(f - nf_s)$$

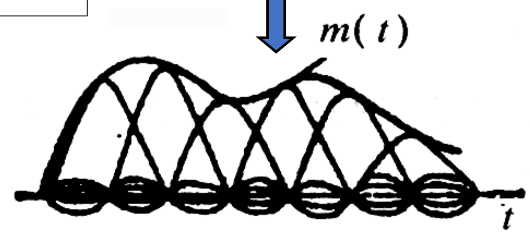
when $\frac{1}{T_s} \geq 2f_H$, $M(f)$ **periodically repeats, but dose not superimpose.**

Sampling Theorem

- **Reconstruction of $m(t)$ from $M_s(f)$**







$$M_s(f)\cdot rect\left(\frac{f}{2f_H}\right)=\frac{1}{T_s}\sum_{n=-\infty}^{\infty}M(f-nf_s)\cdot rect\left(\frac{f}{2f_H}\right)=\frac{1}{T_s}M(f)$$

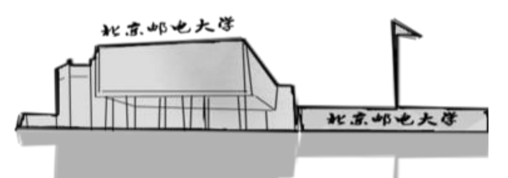
$$\therefore \quad m(t)=T_s\left[m_s(t)*2f_H\mathrm{sinc}(2f_Ht)\right] \xrightarrow{T_s=1/2f_H}$$

$$=\sum_{n=-\infty}^{\infty}m_n\delta(t-nT_s)*\mathrm{sinc}(2f_Ht)=\sum_{n=-\infty}^{\infty}m_n\mathrm{sinc}\left[2f_H(t-nT_s)\right]$$

Sampling Theorem

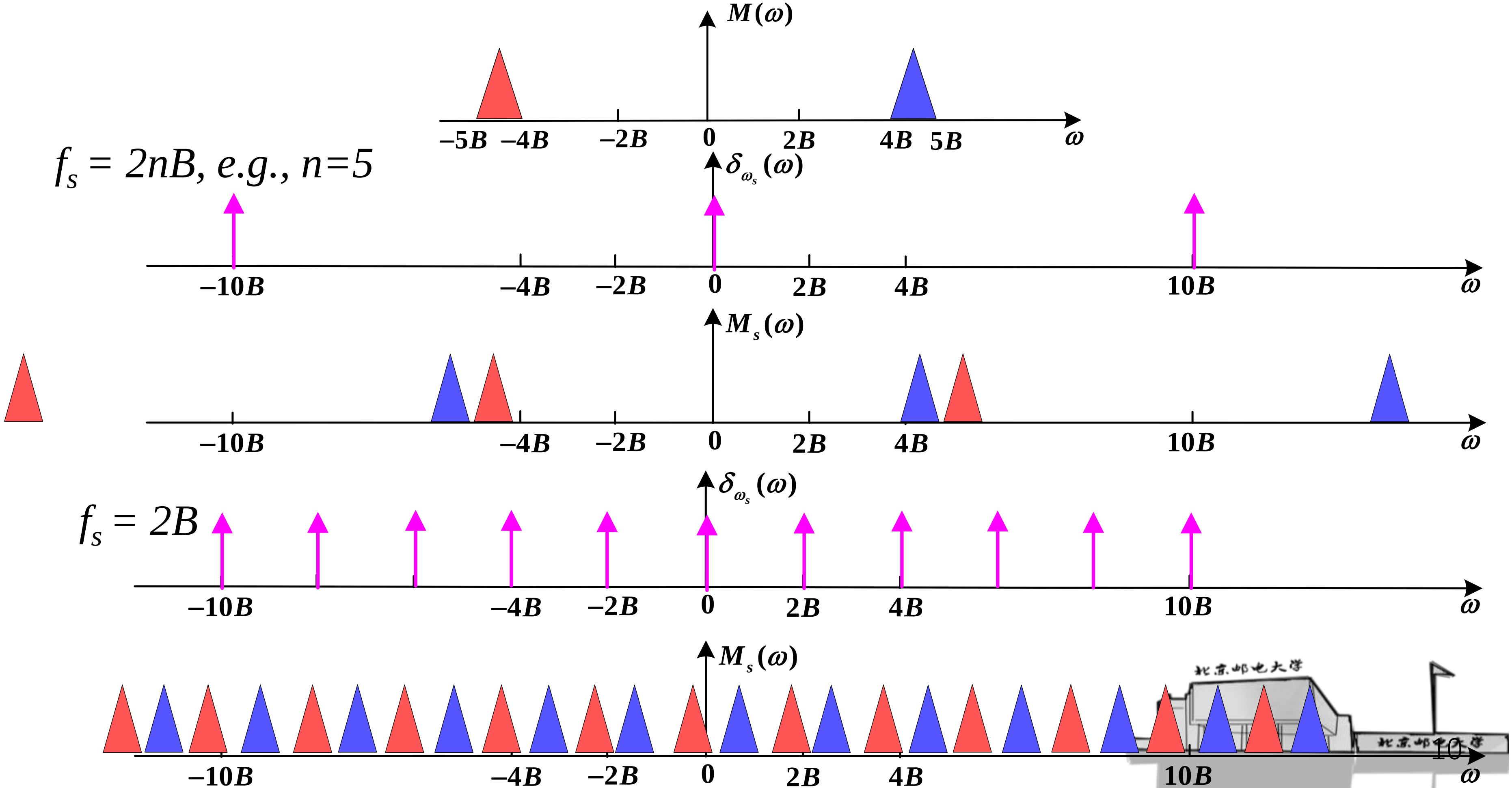
- **Sampling theorem of baseband signal with bandwidth f_{H}.**

> **For accurate reconstruction of the original signal, the sampling frequency $fs \geq 2f_{H}$. Correspondingly, the sampling interval $Ts \leq 2/f_{H}$.**



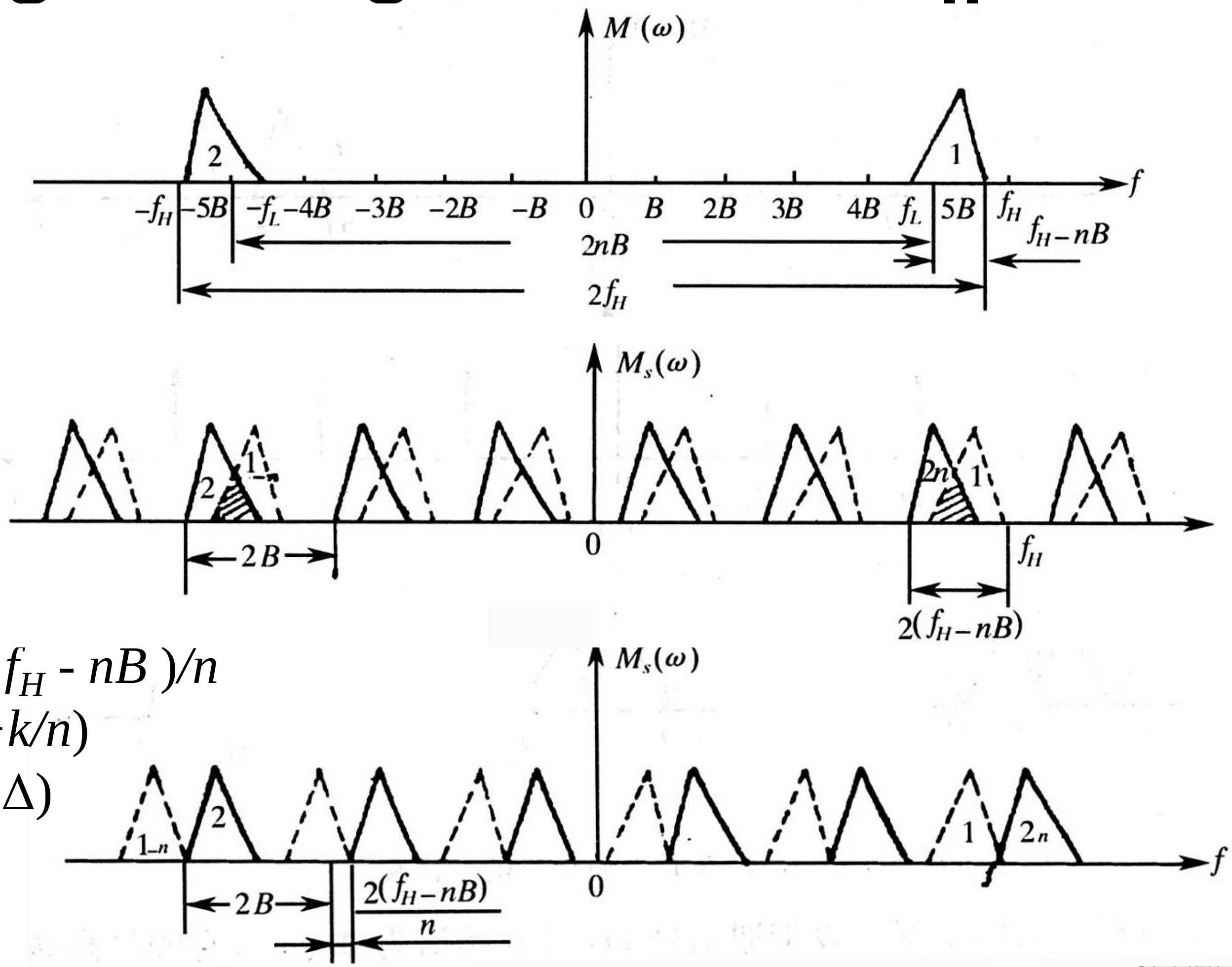
Sampling Theorem

- **For a bandpass signal with a spectrum ranging in [f_L,f_H], its bandwidth $B = f_H - f_L$.**
- **If $f_H = nB$, n is an integer.**



Sampling Theorem

- **If $f_H = nB+kB$, $0 \le k < 1$, n is the largest integer less than f_H / B**



$f_s = 2B$

$f_s = 2B + 2(f_H - nB)/n$

$= 2B\,(1 + k/n)$

$= 2B\,(1 + \Delta)$

$= 2B'$

$B' = f_H - f_L'$

The modified $f_L' = f_L - \Delta$

Distortion-Limited Coding of Continuous Sources

- Sampling theorem
- **Quantization**
- Pulse-code modulation(PCM)
- Time Division Multiplexing(TDM)

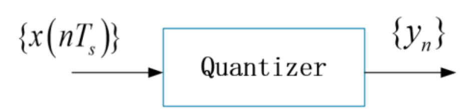
Quantization

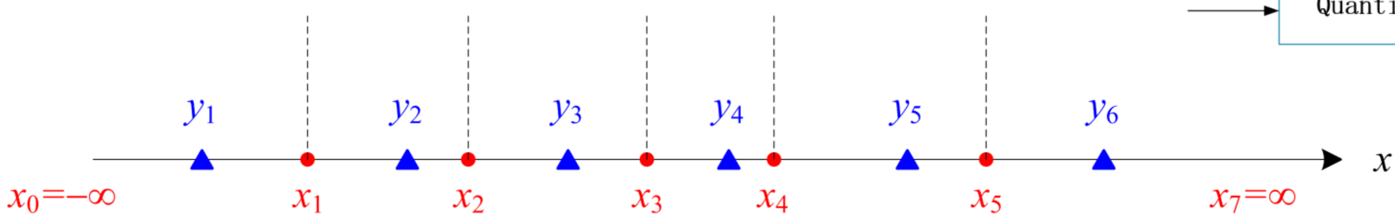
- **Quantization converts continuous amplitudes into discrete M-ary levels.**

$$y = Q(x) \qquad x \in [-A, +A] \qquad y \in \{y_1, y_2, \cdots, y_M\}$$

$$for \quad x = [x_{k-1}, x_k), y = y_k, \qquad k = 1, 2, \ldots, M$$

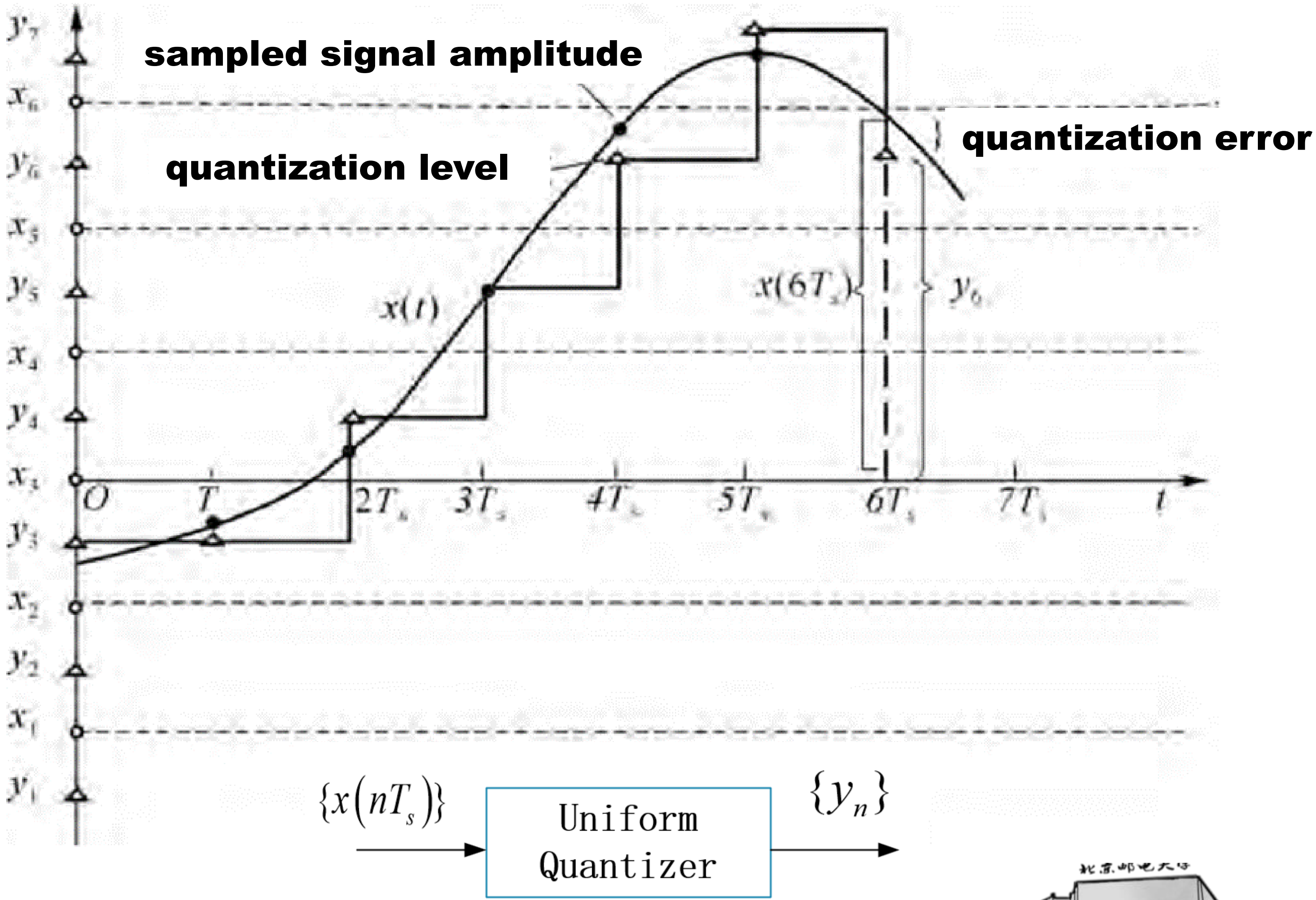
- **(x_{k-1}, x_k) : the kth Quantization interval,**
- **y_k : quantization level of the kth quantization interval,**
- **x_{k-1}, x_k : bounds of the kth quantization interval.**





An example of scalar quantization

Quantization



Quantization

- **Average power of input signal**

$$S = E\left[x^2\right] = \int_{-\infty}^{\infty} p(x) x^2 \mathrm{d}x$$

- **Probability of $x \in (x_{k-1}, x_k)$**

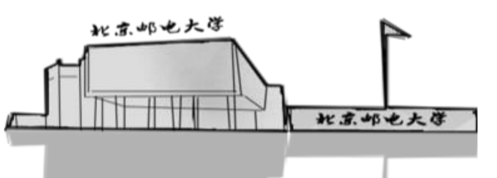
$$q_k = \int_{x_{k-1}}^{x_k} p(x) \mathrm{d}x$$

- **Inner-quantization-interval probability (conditional probability)**

$$p_k(x) = \begin{cases} \dfrac{p(x)}{q_k}, x \in [x_{k-1}, x_k) \\ 0, x \notin [x_{k-1}, x_k) \end{cases}$$

- **Average power of output signal**

$$S_q = \sum_{k=1}^{M} q_k y_k^2$$



Quantization

- **Quantization noise**

$$e_{\mathrm{q}} = y - x = Q(x) - x$$

$$y = x + e_q$$

- **Average power of quantization noise**

$$N_{\mathrm{q}} = \mathrm{E}\left[e_{\mathrm{q}}^2\right] = \int_{-\infty}^{\infty} (y-x)^2 \, p(x) \, \mathrm{d}x$$

$$= \sum_{k=1}^{M} q_k \int_{x_{k-1}}^{x_k} (y_k - x)^2 \, p_k(x) \, \mathrm{d}x$$

$$= \sum_{k=1}^{M} q_k N_{q,k}$$

$$N_{q,k} = E\left[(y-x)^2 \middle| y = y_k\right] = \int_{x_{k-1}}^{x_k} (y_k - x)^2 \, p_k(x) \, dx$$

Quantization

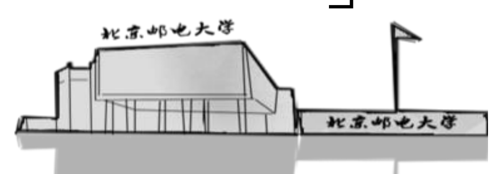
- Setting the quantization levels at the probability weighted centroids of the quantization intervals, we have

$$y_k = E\left[x \middle| y = y_k\right] = \int_{x_{k-1}}^{x_k} x p_k(x)\,dx$$

- then

$$\boxed{N_{\mathrm{q}} = \sum_{k=1}^{M} q_k N_{q,k} = S - S_q}$$

$$\begin{aligned}
N_{q,k} &= E\left[(y-x)^2 \middle| y = y_k\right] \\
&= E\left[(y_k - x)^2 \middle| x \in [x_{k-1}, x_k)\right] \\
&= E\left[y_k^2 - 2xy_k + x^2 \middle| x \in [x_{k-1}, x_k)\right] \\
&= y_k^2 - 2y_k E\left[x \middle| x \in [x_{k-1}, x_k)\right] + E\left[x^2 \middle| x \in [x_{k-1}, x_k)\right] \\
&= -y_k^2 + E\left[x^2 \middle| x \in [x_{k-1}, x_k)\right]
\end{aligned}$$



Quantization

- **Quantization SNR**

Average input signal power

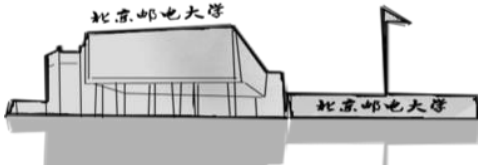
$$S = \mathrm{E}\left[x^2\right]$$

Average output signal power

$$S_q = \sum_{k=1}^{M} q_k y_k^2$$

Quantization SNR

$$\frac{S}{N_q} = \frac{E\left[x^2\right]}{E\left[e_q^2\right]}$$



Uniform Quantization

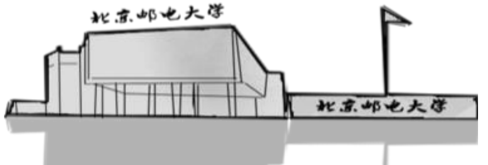
- **Uniform quantization adopts equal quantization interval, i.e., uniformly sets the quantization bounds, and with quantization levels at the centers of the quantization intervals.**

$$x \in [-A, A]$$

$$\Delta = \frac{2A}{M}$$

$$x_k = -A + k\Delta, k = 1, \ldots, M, x_0 = -A$$

$$y_k = \frac{x_{k-1} + x_k}{2} = x_{k-1} + \frac{\Delta}{2}$$



Uniform Quantization

- **Suppose x is a uniformly distributed random variable, then,**

the average quantization noise power

$$N_q = S - S_q = \sum_{k=1}^{M} q_k N_{q,k} = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta} x^2 \mathrm{d}x = \frac{\Delta^2}{12}$$

the average signal power

$$S = \mathrm{E}\left[x^2\right] = \int_{-A}^{A} \frac{1}{2A} x^2 \mathrm{d}x = \frac{A^2}{3} = \frac{M^2\Delta^2}{12}$$

the average output signal power

$$S_q = \sum_{k=1}^{M} q_k y_k^2 = \sum_{k=1}^{M} y_k^2 \int_{x_{k-1}}^{x_k} p(x)\mathrm{d}x = \frac{1}{M} \sum_{k=1}^{M} y_k^2$$

the quantization SNR $\dfrac{S}{N_q} = M^2$

$$\boxed{\begin{aligned} & 2A = M\Delta \\ \Leftrightarrow\ & \Delta = \frac{2A}{M} \\ \Leftrightarrow\ & \Delta^2 = \frac{4A^2}{M^2} \\ \Leftrightarrow\ & A^2 = \frac{M^2\Delta^2}{4} \end{aligned}}$$

Optimal Quantization

- The optimal quantization bounds are the centers of the quantization levels.

$$x_{k,\mathrm{opt}}=\frac{1}{2}(y_{k+1,\mathrm{opt}}+y_{k,\mathrm{opt}}),\quad k=1,2,\cdots,M-1$$

- The optimal quantization levels are the probability-weighted centroids.

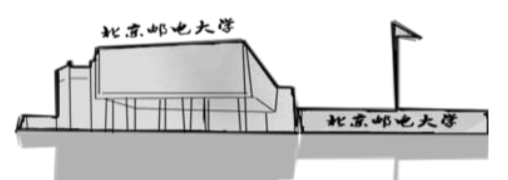
$$y_{k,opt}=\int_{x_{k-1,opt}}^{x_{k,opt}} xp(x)\mathrm{d}x \Big/ \int_{x_{k-1,opt}}^{x_{k,opt}} p(x)\mathrm{d}x$$

- It's evident that the design of optimal quantization is related to the distribution of the input signal x.

- It can be proved that, for uniformly distributed x, uniform quantization is the optimal quantization.

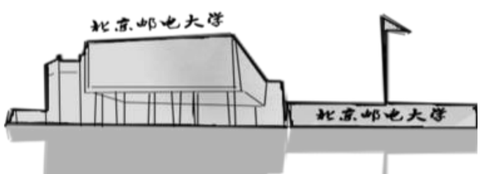
Logarithmic Quantization

- Logarithmic quantization divides the signal range into unequal intervals according to the distribution of the signal.
- For example, with the distribution of human voice, i.e., a low probability for large amplitude while a high probability for small amplitude, when we perform logarithmic quantization, we design small intervals for more precise quantization in small amplitude range, but large intervals in high amplitude range.
- With logarithmic quantization, we can achieve higher quantization SNR with less bit length.



Logarithmic Quantization

- **Two realizations of logarithmic quantization**
  - ✓ **Method 1**
    - **Using a compressor to perform nonlinear amplify, e.g., ln(1+Ax).**
    - **And then do uniform quantization.**
  - ✓ **Method 2**
    - **Perform non-uniform quantization directly.**
    - **Method 2 is usually adopted.**



Logarithmic Quantization

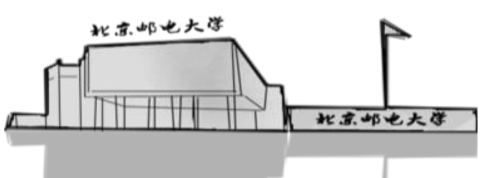
- **A law** and **μ law** logarithmic quantization are two PCM (Pulse Code Modulation) schemes in CCITT G.711.

✓ **A law is generally adopted in Europe and China.**

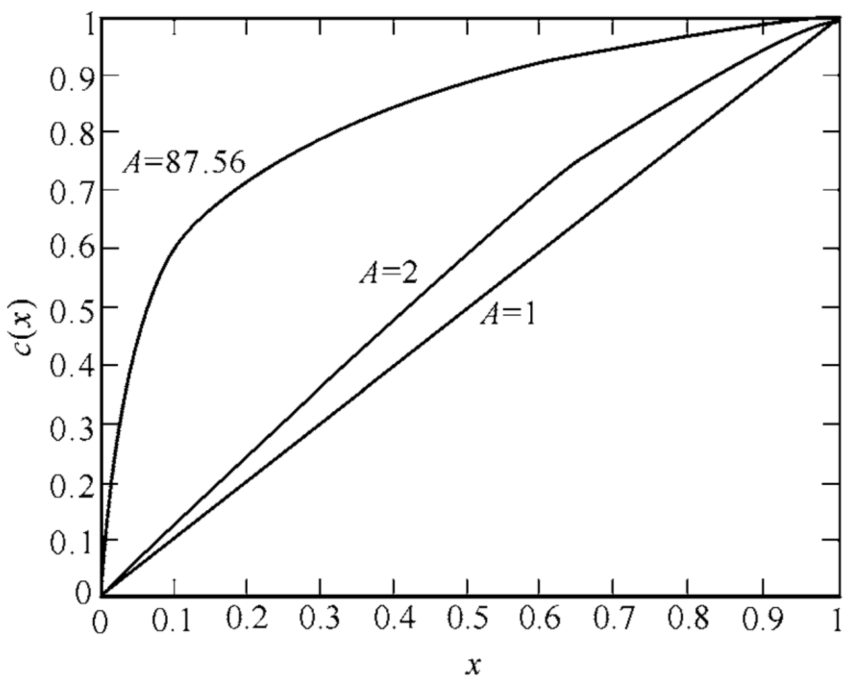
$$c(x)=\begin{cases}\dfrac{Ax}{1+\ln A}, & 0\leqslant x\leqslant\dfrac{1}{A}\\[2ex] \dfrac{1+\ln Ax}{1+\ln A}, & \dfrac{1}{A}\leqslant x\leqslant 1\end{cases}$$

✓ **μ law is usually adopted in USA and Japan.**

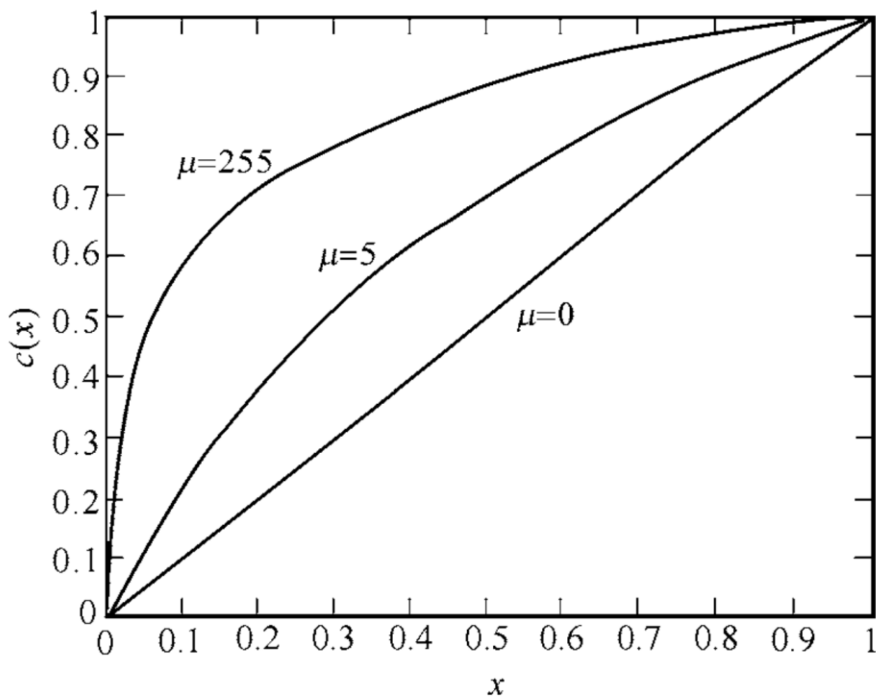
$$c(x)=\frac{\ln\left(1+\mu x\right)}{\ln\left(1+\mu\right)},\quad 0\leqslant x\leqslant 1$$



Logarithmic Quantization



A Law

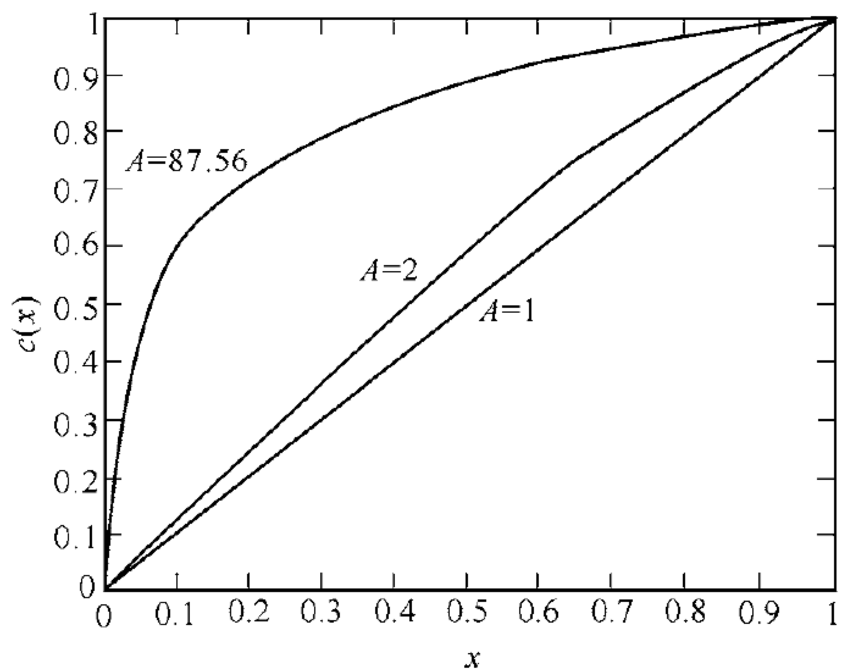


μ Law

Logarithmic Quantization

- **A law PCM**
  - **The A law curve $c(x)$ (A=87.56) is approximated by a polygonal line with 13 segments.**
  - **The procedure of A law PCM includes normalization, segmentation, and coding.**

$$c(x)=\begin{cases}\dfrac{Ax}{1+\ln A}, & 0\leqslant x\leqslant\dfrac{1}{A}\\[2ex] \dfrac{1+\ln Ax}{1+\ln A}, & \dfrac{1}{A}\leqslant x\leqslant 1\end{cases}$$

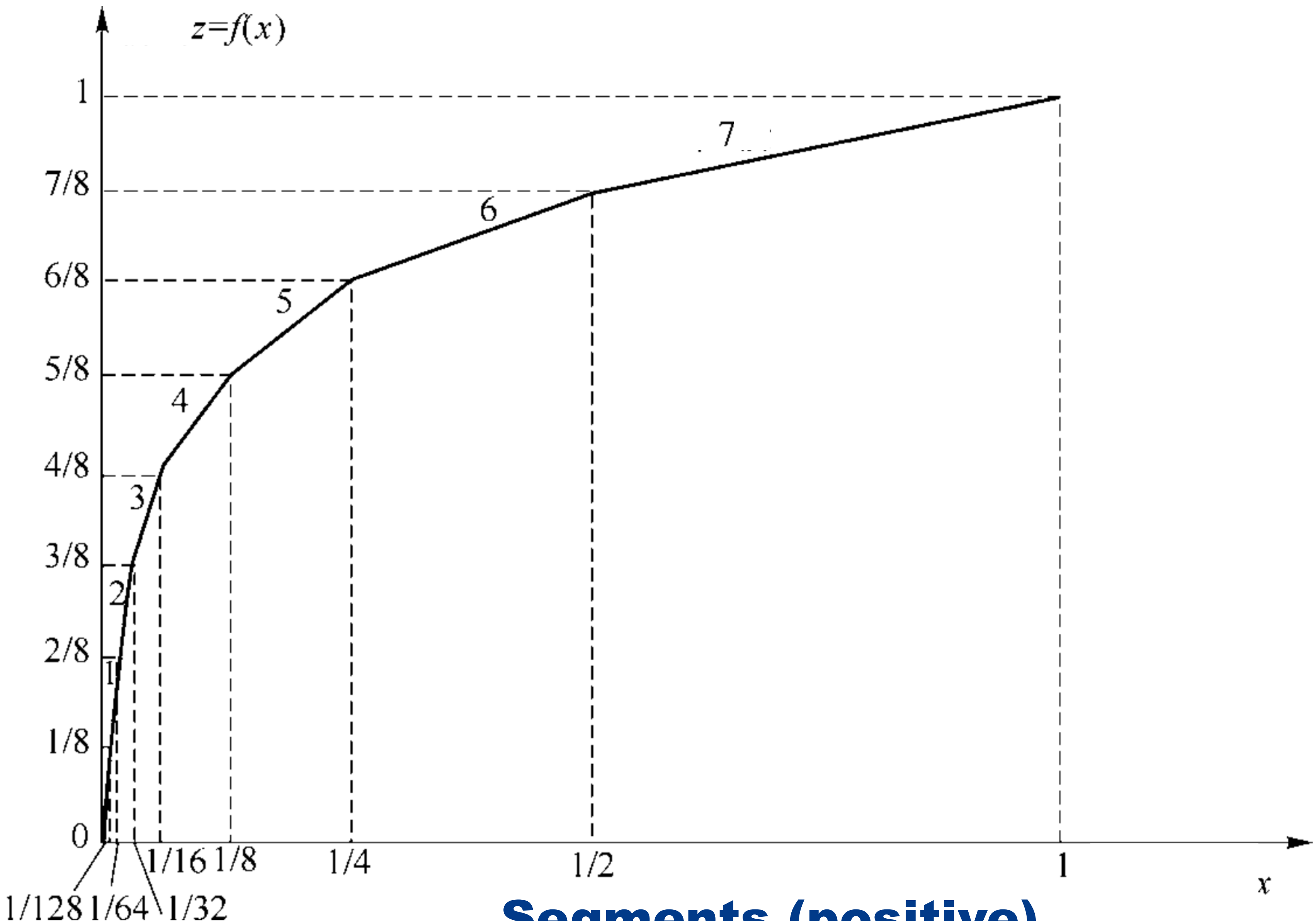


Logarithmic Quantization

The equivalent nonuniform coding

- The nonuniform coding method for digital signals involves using 8 bits for coding.
- The first bit represents polarity, where '1' is used for positive and '0' for negative.
- The second to fourth bits constitute the segment code. Sixteen segments are numbered using a 4-bit segment code. With the middle four segments having the same slope, the sixteen segments are reduced to thirteen. Apart from the first bit (the polarity bit), the remaining 3 bits represent the corresponding segment of the signal amplitude norm.
- The 6th to 8th bits constitute the inner-segment code, which indicates the number of quantization intervals in each segment. Three bits represent eight quantization intervals.
- For human voice signals whose frequency range is between 0 to 4000 Hz, with a minimal sample rate of 8kHz, the final data rate is 64kbps.

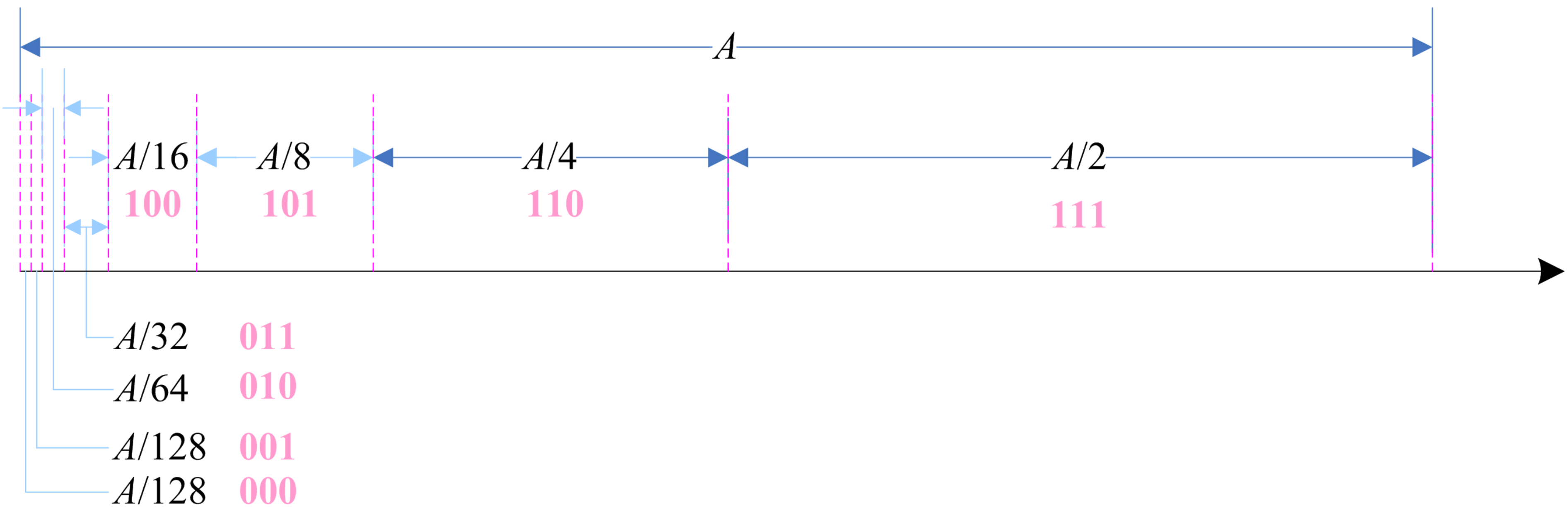
Logarithmic Quantization



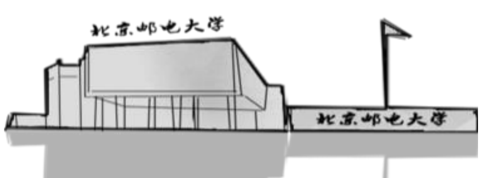
Segments (positive)

Logarithmic Quantization

- **A law PCM: 8-bit nonuniform coding**



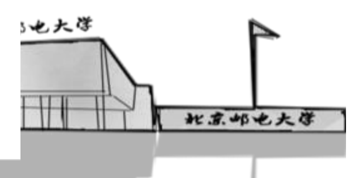
segment coding



A law PCM: 8-bit nonuniform coding

Inner-segment coding (Quantization interval coding)

Quantization interval No.		Natural binary codes	Folded binary codes	Gray binary codes
15	positive polarity	1 1 1 1	1 1 1 1	1 0 0 0
14		1 1 1 0	1 1 1 0	1 0 0 1
13		1 1 0 1	1 1 0 1	1 0 1 1
12		1 1 0 0	1 1 0 0	1 0 1 0
11		1 0 1 1	1 0 1 1	1 1 1 0
10		1 0 1 0	1 0 1 0	1 1 1 1
9		1 0 0 1	1 0 0 1	1 1 0 1
8		1 0 0 0	1 0 0 0	1 1 0 0
7	negative polarity	0 1 1 1	0 0 0 0	0 1 0 0
6		0 1 1 0	0 0 0 1	0 1 0 1
5		0 1 0 1	0 0 1 0	0 1 1 1
4		0 1 0 0	0 0 1 1	0 1 1 0
3		0 0 1 1	0 1 0 0	0 0 1 0
2		0 0 1 0	0 1 0 1	0 0 1 1
1		0 0 0 1	0 1 1 0	0 0 0 1
0		0 0 0 0	0 1 1 1	0 0 0 0

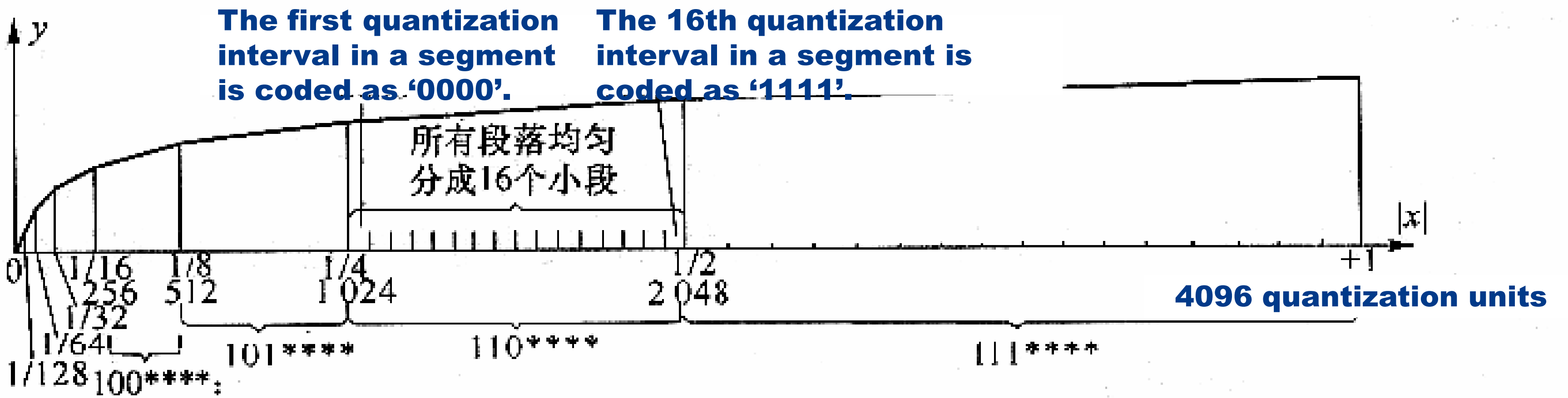


A law PCM: 8-bit nonuniform coding

- **The minimum quantization interval**

$$\Delta = \frac{1}{128} \times \frac{1}{16} = \frac{1}{2048} = 2 \text{ quantization units}$$

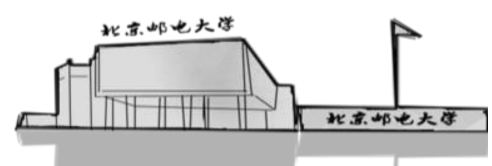
1 quantization unit=1/4096



A law PCM: 8-bit nonuniform coding

- **Segment coding**

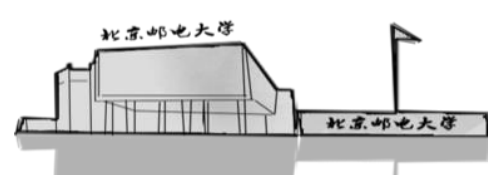
Segment No.	Segment coding	Length of quantization interval in each segment		Minimum signal level of each segment		The first quantization level in each segment	
7	1 1 1	64Δ	128	1024Δ	2048	1056Δ	2112
6	1 1 0	32Δ	64	512Δ	1024	528Δ	1056
5	1 0 1	16Δ	32	256Δ	512	264Δ	528
4	1 0 0	8Δ	16	128Δ	256	132Δ	264
3	0 1 1	4Δ	8	64Δ	128	66Δ	132
2	0 1 0	2Δ	4	32Δ	64	33Δ	66
1	0 0 1	Δ	2	16Δ	32	16.5Δ	33
0	0 0 0	Δ	2	0	0	0.5Δ	1



A law PCM: 8-bit nonuniform coding

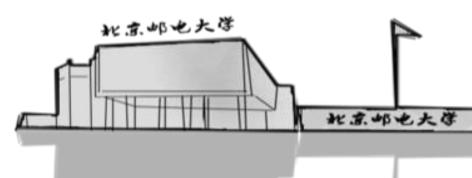
Inner segment coding

Quantization interval No.	Inner segment coding (with natural binary codes)
15	
14	1 1 1 0
13	1 1 0 1
12	1 1 0 0
11	1 0 1 1
10	1 0 1 0
9	1 0 0 1
8	1 0 0 0
7	0 1 1 1
6	0 1 1 0
5	0 1 0 1
4	0 1 0 0
3	0 0 1 1
2	0 0 1 0
1	0 0 0 1
0	0 0 0 0



Linear PCM: 13-bit folded coding

- **Linear PCM adopts 13 folded bits to denote the signal range, i.e., [-4096,4096] quantization units.**
  - **The 1^{st} bit is for polarity, i.e., '1' for positive signal, '0' for negative signal.**
  - **The 2^{nd} to 13^{th} bits are for absolute signal level, coded with natural binary code.**
- **Example**
  - **A signal level +2240 (in quantization unit) can be coded as '1 1000 1100 0000' with 13-bit folded binary code, and '1 111 001' with 8-bit Logarithmic code.**



A law PCM: 8-bit nonuniform coding

Example

- Suppose there is an A law PCM encoder whose input ranges in [-6,6](V). If input signal x=2.4V, find out the corresponding output code of the encoder, as well as the decoded signal level.
- **Solution:**

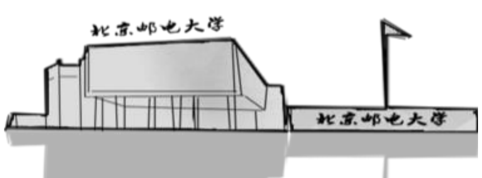
  x =-2.4V, thus, the polarity bit is '0'.

  -3V<-2.4V<-1.5V, x is in the 6th segment, thus the segment code is '110'.

  The length of quantization interval in the 6th segment is

$$\Delta = \frac{1.5}{16} = \frac{3}{32}$$

$$\frac{2.4\text{-}1.5}{\Delta} = 0.9 \div \frac{3}{32} = 0.9 \times \frac{32}{3} = 9.6$$



A law PCM: 8-bit nonuniform coding

- x is in the 9th quantization interval of the 6th segment, thus the inner-segment code is ‘1001’；

- The encoder output is ‘01101001’；

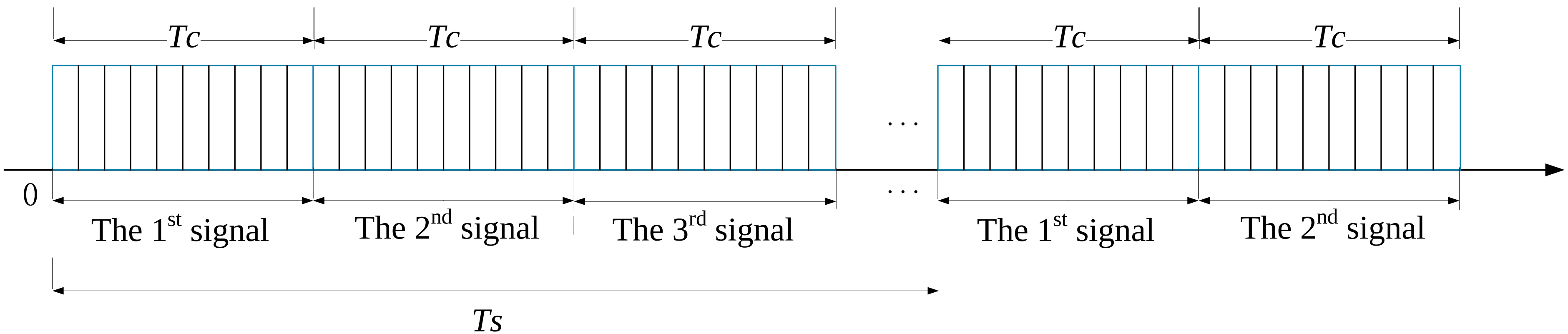
- The quantized signal level is

$$-\left(1.5+9*\frac{3}{32}+\frac{1}{2}*\frac{3}{32}\right)\approx -2.3906$$

- The quantization error is

$$-2.4-\left(-2.3906\right)=0.0094$$

Time division multiplexing (TDM)

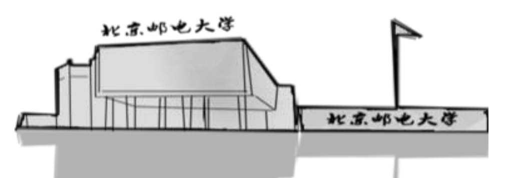


- **The TDM signal's data rate is the sum of the rates of involved signals.**

Time division multiplexing (TDM)

Example: E1 signal

- (1) Each human voice signal is sampled with an 8kHz sample rate, i.e., the sampling interval is 125ms, and 8-bit coding is adopted to denote the sampled signal, then the bit rate of each voice signal is 64kps.
- (2) The 125ms period is divided into 32 slots. The sampled and coded signal from one source can only be in one of the 32 slots, i.e., for each user, the data rate remains 64kbps.
- Since there are 32 slots in a 125ms period, the sum data rate is 32×64kbps=2.048Mbps.
- For the 32 slots in a 125ms period, 30 of them are for payload (standard PCM signals) transmission, the reserved 2 slots are for signaling and synchronization.



TDM

Example

TS0	TS1	TS2	…	TS15	TS16	TS17	…	TS30	TS31

125 μ s (32 slots)

$$R_b = f_s \cdot N \cdot n = 8\,000 \times 32 \times 8 = 2.048 \text{ Mbit/s}$$

unit	primary group	second - order multiplex	third-order multiplex	fourth-order multiplex
Kbit/s	2048	8448	34368	139264
No. of input signals	30	120	480	1920