Queen Mary
University of London

EBU6018
Advanced Transform Methods

Tutorial – Transform Matrices

Dr Yixuan Zou

Lecture Outline

- Tutorial
  - Discrete Fourier Transform
  - Discrete Cosine Transform
  - Discrete Wavelet Transform
    - Filterbank vs. Transform Matrix
  - Comparing DFT, DCT, DWT

Lecture Outline

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk} = X(\omega)\Big|_{\omega=\frac{2\pi}{N}k}$$

- **Tutorial**
  - Discrete Fourier Transform
  - Discrete Cosine Transform
  - Discrete Wavelet Transform
    - Filterbank vs. Transform Matrix
  - Comparing DFT, DCT, DWT

Discrete Fourier Transform - Tutorial

1. Let F_N denotes the N-point DFT matrix, state the (k,n)-th entry of F_N.

Discrete Fourier Transform - Tutorial

1. Let F_N denotes the N-point DFT matrix, state the (k,n)-th entry of F_N.

$$F_N[k,n] = {W_N}^{nk} = e^{-\frac{j2\pi nk}{N}}, \quad W_N = e^{-\frac{j2\pi}{N}}$$

Or, $$F_N[k,n] = {W_N}^{-nk} = e^{-\frac{j2\pi nk}{N}}, \quad W_N = e^{\frac{j2\pi}{N}}$$

Discrete Fourier Transform - Tutorial

2. Based on the given equation, state the range of n and k

$$F_N[k,n] = {W_N}^{nk} = e^{-\frac{j2\pi nk}{N}}, \quad W_N = e^{-\frac{j2\pi}{N}}$$

Discrete Fourier Transform - Tutorial

2. Based on the given equation, state the range of n and k

$$F_N[k,n] = {W_N}^{nk} = e^{-\frac{j2\pi nk}{N}}, \quad W_N = e^{-\frac{j2\pi}{N}}$$

$$n = 0,1,\ldots,N-1$$
$$k = 0,1,\ldots,N-1$$

Discrete Fourier Transform - Tutorial

3. Derive the
    - 2x2 DFT matrix
    - normalized 2x2 DFT matrix

Discrete Fourier Transform - Tutorial

3. Derive the
   - 2x2 DFT matrix
   - normalized 2x2 DFT matrix

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

unnormalized

$$F_2[k,n] = W_2{}^{nk} = e^{-\frac{j2\pi nk}{2}}, \quad \text{where } n,k = \{0,1\}$$

$$F_2[0{,}0] = F_2[0{,}1] = F_2[1{,}0] = W_2{}^{0} = e^{-\frac{j2\pi}{2}\times 0} = 1$$

$$F_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

normalized

$$F_2[1{,}1] = W_2{}^{1} = e^{-j\pi} = \cos(\pi) - i\sin(\pi) = -1$$

*All rows have the same norm=$\sqrt{1^2 + 1^2} = \sqrt{2}$

Discrete Fourier Transform - Tutorial

4. Derive the
   - 4x4 DFT matrix
   - normalized 4x4 DFT matrix

Discrete Fourier Transform - Tutorial

4. Derive the
   - 4x4 DFT matrix
   - normalized 4x4 DFT matrix

$F_4[k,n] = W_4{}^{nk} = e^{-\frac{j2\pi nk}{4}}$, where $n, k = \{0,1,2,3\}$

$F_4[k,n] = 1, \forall n = 0\ or\ k = 0$, i.e. 1st row and 1st column

$F_4[1,1] = W_4{}^1 = e^{-\frac{j\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right) = -i$

$F_4[1,2] = W_4{}^2 = e^{-j\pi} = \cos(\pi) - i\sin(\pi) = -1$

$F_4[1,3] = W_4{}^3 = e^{-\frac{j3\pi}{2}} = \cos\left(\frac{3\pi}{2}\right) - i\sin\left(\frac{3\pi}{2}\right) = i$

$$\Rightarrow \quad F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

unnormalized

Discrete Fourier Transform - Tutorial

4. Derive the
   - 4x4 DFT matrix
   - normalized 4x4 DFT matrix

unnormalized

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$$F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

normalized

Norm of complex vector $[x_1 + iy_1, x_2 + iy_2, x_3 + iy_3]$ is

$$\sqrt{(x_1^2+y_1^2) + (x_2^2+y_2^2) + (x_3^2+y_3^2)}$$

Norm of

- Row 1: $\sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$
- Row 2: $\sqrt{1^2 + (-1)^2 + (-1)^2 + 1^2} = \sqrt{4} = 2$
- Same for row 3 and 4

Discrete Fourier Transform - Tutorial

5. Perform DFT on the given input sequence

$$S[n] = [2, -3]$$

Discrete Fourier Transform - Tutorial

5. Perform DFT on the given input sequence

$$S[n] = [2, -3]$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{bmatrix}$$

Discrete Fourier Transform - Tutorial

6. Perform DFT on the given input sequence

$$S[n] = [1, 2, -3, -5]$$

Discrete Fourier Transform - Tutorial

6. Perform DFT on the given input sequence

$$S[n] = [1, 2, -3, -5]$$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -3 \\ -5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+2-3-5 \\ 1-2i+3-5i \\ 1-2-3+5 \\ 1+2i+3+5i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -5 \\ 4-7i \\ 1 \\ 4+7i \end{bmatrix} = \begin{bmatrix} -2.5 \\ 2-3.5i \\ 0.5 \\ 2+3.5i \end{bmatrix}$$

Discrete Fourier Transform - Tutorial

7. Explain the what DFT does to the input sequence, with reference to the DFT matrix.

Discrete Fourier Transform - Tutorial

7. Explain the what DFT does to the input sequence, with reference to the DFT matrix.

Each row of the DFT Matrix corresponds to a cosine wave and a sine wave of the same frequency. Higher rows of the DFT matrix corresponds to a higher frequency.

By multiplying the DFT matrix with the input sequence, it computes the correlation between the input data and a series of cosine and sine waves of increasing frequency.

The output of DFT approximates the frequency spectrum of the signal that we sampled from.

Lecture Outline

$$DCT[k] = c(k) \sum_{n=0}^{N-1} s[n] \cos \frac{\pi(2n+1)k}{2N}$$

- Tutorial
  - Discrete Fourier Transform
  - **Discrete Cosine Transform**
  - Discrete Wavelet Transform
    - Filterbank vs. Transform Matrix
  - Comparing DFT, DCT, DWT

$$c(k) = \begin{cases} \sqrt{1/N} & k = 0 \\ \sqrt{2/N} & k \neq 0 \end{cases}$$

$$k = 0,1,2 \ldots N-1$$

Discrete Cosine Transform – Tutorial

2. State the assumption of DCT on the input signal

Discrete Cosine Transform – Tutorial

2. State the assumption of DCT on the input signal

DCT assumes the input signal to be even and periodic

*an even signal is one that is symmetric to the y-axis, i.e. $x(-t) = x(t)$

Discrete Cosine Transform – Tutorial

4. Derive the

   - normalized 2x2 DCT matrix , in terms of cosine functions

N-point DCT: $DCT[k] = c(k)\sum_{n=0}^{N-1} s[n]\cos\frac{\pi(2n+1)k}{2N}$ $\quad c(k) = \begin{cases} \sqrt{1/N} & k = 0 \\ \sqrt{2/N} & k \neq 0 \end{cases}$

$$k = 0,1,2\ldots N-1$$

Discrete Cosine Transform – Tutorial

4. Derive the
   - normalized 2x2 DCT matrix , in terms of cosine functions

N-point DCT: $DCT[k] = c(k)\sum_{n=0}^{N-1} s[n]\cos\frac{\pi(2n+1)k}{2N}$ $c(k) = \begin{cases} \sqrt{1/N} & k = 0 \\ \sqrt{2/N} & k \neq 0 \end{cases}$

$$k = 0,1,2\ldots N-1$$

In matrix form: $\Psi_N[k,n] = c(k)\cos\frac{\pi(2n+1)k}{2N}$ $n,k = 0,1$

2-point DCT matrix: $\Psi_2[k,n] = c(k)\cos\frac{\pi(2n+1)k}{4}$ $c(k) = \begin{cases} 1/\sqrt{2} & k = 0 \\ 1 & k \neq 0 \end{cases}$

Queen Mary University of London

Discrete Cosine Transform – Tutorial

4. Derive the
   - normalized 2x2 DCT matrix, in terms of cosine functions

2-point DCT matrix: $\Psi_2[k,n] = c(k)\cos\frac{\pi(2n+1)k}{4}$

$$n,k = 0,1$$

$$c(k) = \begin{cases} 1/\sqrt{2} & k = 0 \\ 1 & k \neq 0 \end{cases}$$

- $\Psi_2[0,0] = \Psi_2[0,1] = c(0)\cos(0) = c(0) = \frac{1}{\sqrt{2}}$
- $\Psi_2[1,0] = c(1)\cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$
- $\Psi_2[1,1] = c(1)\cos\left(\frac{3\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$

$$\Rightarrow \quad \Psi_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\left(\frac{\pi}{4}\right) & \cos\left(\frac{3\pi}{4}\right) \end{bmatrix}$$

Queen Mary University of London

Discrete Cosine Transform – Tutorial

5. Derive the
   - normalized 4x4 DCT matrix , in terms of cosine functions

N-point DCT: $DCT[k] = c(k)\sum_{n=0}^{N-1} s[n]\cos\frac{\pi(2n+1)k}{2N}$ $c(k) = \begin{cases} \sqrt{1/N} & k = 0 \\ \sqrt{2/N} & k \neq 0 \end{cases}$

$$k = 0,1,2\ldots N-1$$

Discrete Cosine Transform – Tutorial

5. Derive the
   - normalized 4x4 DCT matrix , in terms of cosine functions

N-point DCT matrix: $\Psi_N[k,n] = c(k)\cos\frac{\pi(2n+1)k}{2N}$ $\quad k = 0,1,2\ldots N-1 \quad c(k) = \begin{cases} \sqrt{1/N} & k = 0 \\ \sqrt{2/N} & k \neq 0 \end{cases}$

4-point DCT matrix: $\Psi_4[k,n] = c(k)\cos\frac{\pi(2n+1)k}{8}$ $\quad k = 0,1,2,3 \quad c(k) = \begin{cases} 1/2 & k = 0 \\ 1/\sqrt{2} & k \neq 0 \end{cases}$

- $\Psi_4[0,0] = \Psi_4[0,1] = \Psi_4[0,2] = \Psi_4[0,3] = c(0)\cos(0) = c(0) = \frac{1}{2}$
- $\Psi_4[1,0] = c(1)\cos\left(\frac{\pi}{8}\right) = \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{8}\right)$; $\Psi_4[1,1] = \frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{8}\right)$; $\Psi_4[1,2] = \frac{1}{\sqrt{2}}\cos\left(\frac{5\pi}{8}\right)$; $\Psi_4[1,3] = \frac{1}{\sqrt{2}}\cos\left(\frac{7\pi}{8}\right)$
- $\Psi_4[2,0] = c(2)\cos\left(\frac{2\pi}{8}\right) = \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4}\right)$; …
- …

Discrete Cosine Transform – Tutorial

5. Derive the

   ➢ normalized 4x4 DCT matrix , in terms of cosine functions

4-point DCT matrix: $\Psi_4[k,n] = c(k)\cos\frac{\pi(2n+1)k}{8}$ $\quad k = 0,1,2,3 \quad$ $c(k) = \begin{cases} 1/2 & k = 0 \\ 1/\sqrt{2} & k \neq 0 \end{cases}$

$$
\begin{array}{c|c}
 & \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \end{array} \\
\begin{array}{c} k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} &
\begin{bmatrix}
1/2 & 1/2 & 1/2 & 1/2 \\
\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{5\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{7\pi}{8}\right) \\
\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{4}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{6\pi}{4}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{7\pi}{4}\right) \\
\frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{5\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{7\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{9\pi}{8}\right)
\end{bmatrix}
\end{array}
$$

Discrete Cosine Transform – Tutorial

6. Calculate the
   - normalized 4x4 DCT matrix, to 2 decimal points

Discrete Cosine Transform – Tutorial

6. Calculate the
   - normalized 4x4 DCT matrix, to 2 decimal points

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{5\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{7\pi}{8}\right) \\ \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{4}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{5\pi}{4}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{7\pi}{4}\right) \\ \frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{9\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{15\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{21\pi}{8}\right) \end{bmatrix} \Rightarrow \begin{bmatrix} 0.50 & 0.50 & 0.50 & 0.50 \\ 0.65 & 0.27 & -0.27 & -0.65 \\ 0.50 & -0.50 & -0.50 & 0.50 \\ 0.27 & -0.65 & 0.65 & -0.27 \end{bmatrix}$$

Discrete Cosine Transform – Tutorial

7. Describe what DCT does to the input signal, with reference to the DCT matrix

Discrete Cosine Transform – Tutorial

7. Describe what DCT does to the input signal, with reference to the DCT matrix

   Each row of the DCT matrix consists of samples from a cosine wave. Higher rows in the DCT matrix represents cosine waves of increasing frequencies.

   When applied to the input sequence, DCT calculates the correlation between the input and the cosine waves at different frequencies.

   The output of DCT describes the rate of change in the input sequence

Discrete Cosine Transform – Tutorial

8. Based on the given DCT output, explain what it implies on the input sequence.

$$\begin{bmatrix} 2.1 \\ 0.6 \\ -1.2 \\ 1.9 \\ -0.1 \\ 2.6 \\ -1.7 \\ 10.3 \end{bmatrix}$$

Discrete Cosine Transform – Tutorial

8. Based on the given DCT output, explain what it implies on the input sequence.

$$\begin{bmatrix} 2.1 \\ 0.6 \\ -1.2 \\ 1.9 \\ -0.1 \\ 2.6 \\ -1.7 \\ 10.3 \end{bmatrix}$$

The first output implies that the input sequence has a mean value of $2.1/\sqrt{8}$

The last output has a significantly larger magnitude than all other outputs. It indicates that the input has a strong correlation with this cosine waveform, which implies that the input sequence has a fast rate of change in values.

Discrete Cosine Transform – Tutorial

9. What is main application of DCT? Describe how DCT can be employed for that application.

Discrete Cosine Transform – Tutorial

9. What is main application of DCT? Describe how DCT can be employed for that application.

- The DCT is used to perform image compression to produce jpeg format.
- For this format, an image is sub-divided into 8x8 blocks of data.
- The transform is then the dot-product of the 8x8 DCT matrix with an 8-point input sequence to produce an 8-point output sequence. Repeat this step for all blocks of data.
- This is effectively correlation of the input data with a range of cosine waves of different frequency.
- Compression is performed by setting a threshold on the DCT output, where outputs below the threshold are set to 0.
- Then, inverse DCT is performed on the compressed DCT output to obtain an accurate approximation of the original image

Lecture Outline

- **Tutorial**
  - Discrete Fourier Transform
  - Discrete Cosine Transform
  - **Discrete Wavelet Transform**
    - **Filterbank vs. Transform Matrix**
  - Comparing DFT, DCT, DWT

Discrete Wavelet Transform - Tutorial

1. What are the definitions of wavelets and a wavelet family?

Discrete Wavelet Transform - Tutorial

1. What are the definitions of wavelets and a wavelet family?

Wavelets are a class of functions which are of short duration and are oscillatory. They are used in the wavelet transform to perform time-frequency analysis.

A wavelet family is defined through the following equation:

$$\psi(t) \rightarrow \psi\left(\frac{t-b}{a}\right)$$

where $\Psi(t)$ is the basic wavelet function, b is the translation parameter, and a is the scaling parameter

Discrete Wavelet Transform - Tutorial

2. State the unnormalized 8x8 Haar matrix.

Discrete Wavelet Transform - Tutorial

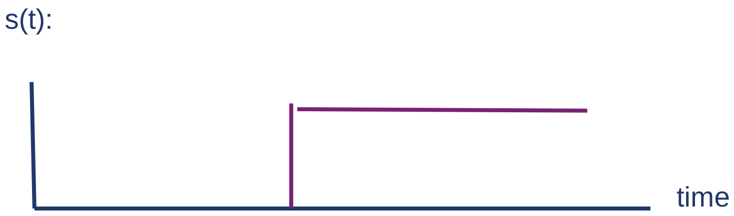
2. State the unnormalized 8x8 Haar matrix.

$$H_8 = \begin{array}{cl} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} & \begin{array}{c} \varphi_0(\mathrm{t}) \\ \psi_0(\mathrm{t}) \\ \psi_{1,0}(\mathrm{t}) \\ \psi_{1,1}(\mathrm{t}) \\ \psi_{2,0}(\mathrm{t}) \\ \psi_{2,1}(\mathrm{t}) \\ \psi_{2,2}(\mathrm{t}) \\ \psi_{2,3}(\mathrm{t}) \end{array} \end{array}$$

Discrete Wavelet Transform - Tutorial

3. Determine the output of the unnormalised 8x8 Haar Transform for the input sequence:

S[n] = [1, 1, 1, 1, 10, 10, 10, 10]



NOTE: I am using the UNNORMALISED matrix because I am looking for any significant feature in the data. Its exact transform is not required for this. The unnormalised arithmetic is simpler.

Discrete Wavelet Transform - Tutorial

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 44 \\ -36 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \varphi_0(t) \\ \psi_0(t) \\ \psi_{1,0}(t) \\ \psi_{1,1}(t) \\ \psi_{2,0}(t) \\ \psi_{2,1}(t) \\ \psi_{2,2}(t) \\ \psi_{2,3}(t) \end{matrix}$$

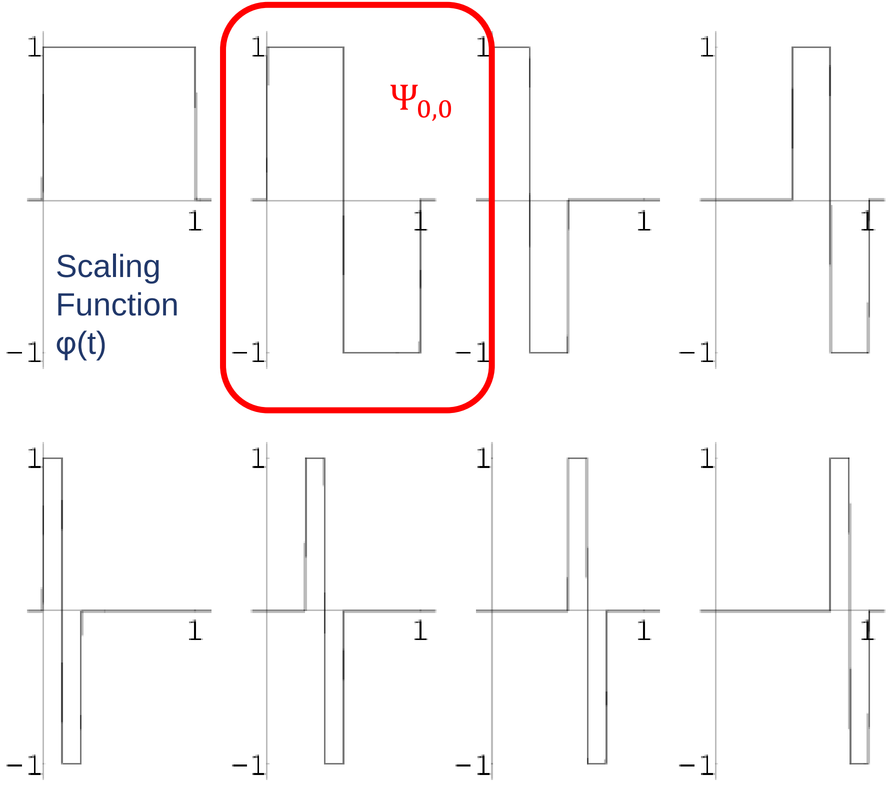
What does the output implies?

Discrete Wavelet Transform - Tutorial

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 44 \\ -36 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} \varphi_0(\mathrm{t}) \\ \psi_0(\mathrm{t}) \\ \psi_{1,0}(\mathrm{t}) \\ \psi_{1,1}(\mathrm{t}) \\ \psi_{2,0}(\mathrm{t}) \\ \psi_{2,1}(\mathrm{t}) \\ \psi_{2,2}(\mathrm{t}) \\ \psi_{2,3}(\mathrm{t}) \end{array}$$

Output indicates a discontinuity corresponding to $\Psi_{0,0}$.
That is, in the centre of the input sequence.

Discrete Wavelet Transform - Tutorial



Wavelet Function:

$$\psi(x) \equiv \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ -1 & \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_{jk}(x) \equiv \psi(2^j x - k),$$

$\phi_{00} = \phi\,(x)$

$\psi_{00} = \psi\,(x)$

$\psi_{10} = \psi\,(2x)$

$\psi_{11} = \psi\,(2x\text{-}1)$

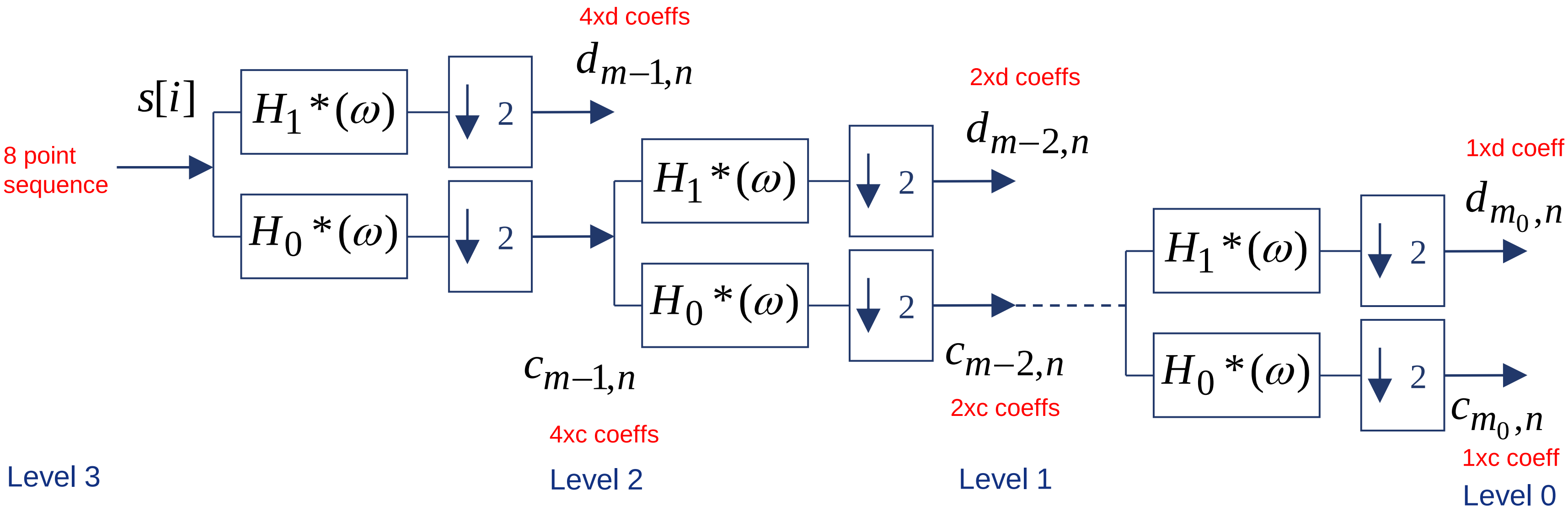
$\psi_{20} = \psi\,(4x)$

$\psi_{21} = \psi\,(4x\text{-}1)$

$\psi_{22} = \psi(4x\text{-}2)$

$\psi_{23} = \psi(4x\text{-}3)$

Discrete Wavelet Transform - Tutorial



- The 8x8 Haar Transform Matrix is performing 3 levels of decomposition.
- Input is at level 3, then decomposing to level 2, then to level 1 then to level 0.
- d coefficients are detail coefficients

Discrete Wavelet Transform - Tutorial

$$H_8 = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}}_{\text{8x8 Haar Transform Matrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} = \underbrace{\begin{bmatrix} c_{0,0} \\ d_{0,0} \\ d_{1,0} \\ d_{1,1} \\ d_{2,0} \\ d_{2,1} \\ d_{2,2} \\ d_{2,3} \end{bmatrix}}_{\text{Filterbank coef.}} \quad \begin{array}{c} \varphi_0(\mathrm{t}) \\ \psi_0(\mathrm{t}) \\ \psi_{1,0}(\mathrm{t}) \\ \psi_{1,1}(\mathrm{t}) \\ \psi_{2,0}(\mathrm{t}) \\ \psi_{2,1}(\mathrm{t}) \\ \psi_{2,2}(\mathrm{t}) \\ \psi_{2,3}(\mathrm{t}) \end{array}$$

Notes on Haar Transform

- To perform a Haar Transform (or in general any wavelet transform) we would use the normalised filters in a filterbank and the normalised functions in a matrix.

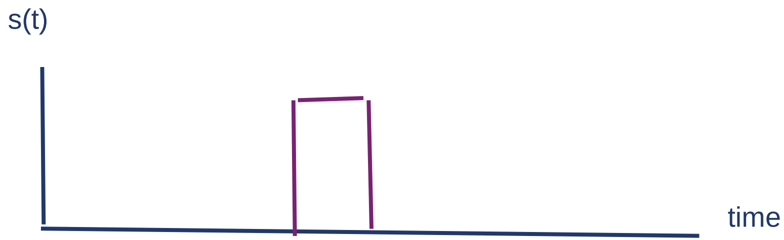
- For the examples given in the lecture on filterbanks using the Haar functions, the filter is given as $H = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$ because when multiplied by the $\sqrt{2}$ of the recursive equation then the filter is normalised.

- If the normalised filter is given, then there is no need to use the $\sqrt{2}$ of the recursive equation.

Discrete Wavelet Transform - Tutorial

4. Determine the output of the unnormalised 8x8 Haar Transform for the input sequence:

S[n] = [1, 1, 1, 10, 10, 1, 1, 1]



NOTE: I am using the UNNORMALISED matrix because I am looking for any significant feature in the data. Its exact transform is not required for this. The unnormalised arithmetic is simpler.

Discrete Wavelet Transform - Tutorial

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 10 \\ 10 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 26 \\ 0 \\ -9 \\ 9 \\ 0 \\ -9 \\ 9 \\ 0 \end{bmatrix} \begin{matrix} \varphi_0(\mathrm{t}) \\ \psi_0(\mathrm{t}) \\ \psi_{1,0}(\mathrm{t}) \\ \psi_{1,1}(\mathrm{t}) \\ \psi_{2,0}(\mathrm{t}) \\ \psi_{2,1}(\mathrm{t}) \\ \psi_{2,2}(\mathrm{t}) \\ \psi_{2,3}(\mathrm{t}) \end{matrix}$$

Can you explain the output?

Discrete Wavelet Transform - Tutorial

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 10 \\ 10 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 26 \\ 0 \\ -9 \\ 9 \\ 0 \\ -9 \\ 9 \\ 0 \end{bmatrix} \quad \begin{matrix} \varphi_0(t) \\ \psi_0(t) \\ \psi_{1,0}(t) \\ \psi_{1,1}(t) \\ \psi_{2,0}(t) \\ \psi_{2,1}(t) \\ \psi_{2,2}(t) \\ \psi_{2,3}(t) \end{matrix}$$

Output indicates discontinuities corresponding to shifted Ψ_1 and Ψ_2

That is, identifies a feature.

The resolution would be improved by using a larger Haar Matrix.

Queen Mary University of London

Discrete Wavelet Transform – Notes

- The resolution is not good in this example because for an 8-point sequence we have only three frequencies of the wavelet function.
- That is, only two changes of scale.
- So the scaled wavelet is not sufficiently narrow to identify narrow changes in the input signal.
- In practice, the input sequence would be much larger and we would use the relevant dimension of matrix giving more changes of scale and so much narrower functions.
- Also in practice, if we want to identify features of different shape we would use wavelet functions of a similar shape.

Lecture Outline

- **Tutorial**
  - Discrete Fourier Transform
  - Discrete Cosine Transform
  - Discrete Wavelet Transform
    - Filterbank vs. Transform Matrix
  - **Comparing DFT, DCT, DWT**

Transform Methods – Exercise 1

For a 4-point input sequence

$$S[n] = [6, 3, -2.5, 7]$$

Determine the output from each of the following discrete transforms:

a. DFT

b. DCT

c. DWT, using a Haar wavelet function.

Transform Methods – Exercise 1 Solution

$$\text{DFT} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ -2.5 \\ 7 \end{bmatrix} = \begin{bmatrix} 6.75 \\ 4.25 + 2i \\ -3.25 \\ 4.25 - 2i \end{bmatrix}$$

Transform Methods – Exercise 1 Solution

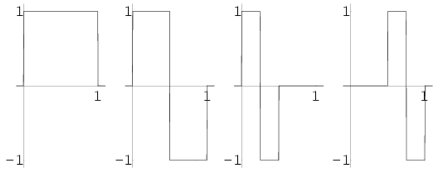
$$\text{DCT} = \begin{bmatrix} 0.50 & 0.50 & 0.50 & 0.50 \\ 0.65 & 0.27 & -0.27 & -0.65 \\ 0.50 & -0.50 & -0.50 & 0.50 \\ 0.27 & -0.65 & 0.65 & -0.27 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ -2.5 \\ 7 \end{bmatrix} = \begin{bmatrix} 6.75 \\ 0.835 \\ 6.25 \\ -3.845 \end{bmatrix}$$

Queen Mary University of London

Transform Methods – Exercise 1 Solution

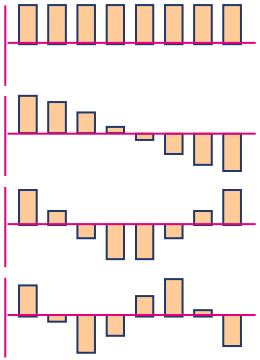
$$\text{DHT} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ -2.5 \\ 7 \end{bmatrix} = \begin{bmatrix} 6.75 \\ 2.25 \\ 2.12 \\ -6.72 \end{bmatrix}$$

Transform Methods – Exercise 1 Solution



Input: S[n] = [6, 3, -2.5, 7]

$$\text{DFT} = \begin{bmatrix} 6.75 \\ 4.25 + 2i \\ -3.25 \\ 4.25 - 2i \end{bmatrix} \qquad \text{DCT} = \begin{bmatrix} 6.75 \\ 0.835 \\ 6.25 \\ -3.845 \end{bmatrix}$$



$$\text{DHT} = \begin{bmatrix} 6.75 \\ 2.25 \\ 2.12 \\ -6.72 \end{bmatrix}$$

- ➢ The first output of all three methods are the same. i.e. the smoothed value, divide by $\sqrt{N}$ to obtain mean
- ➢ DFT is the only one that produces complex outputs

- ➢ Three transform methods are designed for different purposes
  - ➢ DFT – frequency spectrum as a function of time
  - ➢ DCT – rate of change in the input
  - ➢ DHT – identify short duration features

DFT/DCT/DHT – Tutorial

- State the applications of DFT, DCT, and DWT

DFT/DCT/DHT – Tutorial

- State the applications of DFT, DCT, and DWT

- **Fourier Transforms** are used to obtain the frequency spectrum of a function of time.
- **Cosine Transforms** are used to identify the rate-of-change of the input data.
- **Wavelet Transforms** are used to identify trends in the input data ("approximations") and to identify short-duration features or artifacts in the input data.

Notes on Comparing DFT/DCT/DHT

- DFT is the only transform method (among DFT,DCT,DHT) that outputs complex numbers
- DFT and DCT both assume the input to be periodic
- DCT further assumes the input to be even
- If the above assumption is not met, the output does not accurately represent the frequency information of the input signal that we sampled from
- The outputs of DFT and DCT on even functions are the same
- DHT (or any wavelet transform) is the only transform method (among DFT,DCT,DHT) that can perform time-frequency analysis

Thank you

Queen Mary
University of London

Queen Mary
University of London