The Basics of Matrix Algebra

A Matrix is a Rectangular Array of Numbers (called *Entries*)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

- The general form of the matrix is $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$
- The dimension of the array is
  # or rows x # of columns
- The dimension of the array above is 2x3.
- The dimension of $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ is 3x2.

Matrix / Row Vector / Column Vector

- [m x n] matrix $$\mathbf{A} = \begin{bmatrix} a_{11} & ,\ldots, & a_{1n} \\ a_{21} & ,\ldots, & a_{2n} \\ \ldots & \ldots & \ldots\ \ldots \\ a_{m1} & ,\ldots, & a_{mn} \end{bmatrix} = \{A_{ij}\}$$
- Row Vector = [1 x n] matrix $A\left[a_1 a_2, \ldots, a_n\right] = \{a_j\}$
- Column Vector = [m x 1] matrix $$A = \begin{bmatrix} a_1 \\ a_2 \\ \ldots \\ a_m \end{bmatrix} = \{a_i\}$$

Special Matrices

- A *square matrix* is has the same number of rows as columns.
- The nxn *identity matrix, I,* is a square matrix with n rows and n columns with 1's along the main diagonal and 0's everywhere else.
- The 3x3 identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- The identity matrix is a special type of diagonal matrix. A *diagonal matrix* is a square matrix whose entries off the main diagonal are all 0.

Square Matrix

Same number of rows and columns

$$B = \begin{bmatrix} 5 & 4 & 7 \\ 3 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Identity Matrix

Square matrix with ones on the diagonal and zeros elsewhere.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Transpose of a Matrix

- To find the *transpose* of a matrix, A, denoted A^T, switch the rows and the columns in the matrix.
- If A = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then A^T = $\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

Transpose Matrix

Rows become columns and columns become rows

$$A' = \begin{bmatrix} a_{11} & a_{21} & ,\ldots, & a_{m1} \\ a_{12} & a_{22} & ,\ldots, & a_{m2} \\ \ldots & \ldots & \ldots & \ldots \quad \ldots \\ a_{1n} & a_{2n} & ,\ldots, & a_{mn} \end{bmatrix}$$

Reverse rotations

- To undo a rotation of θ, $R(\theta)$
  - apply the inverse of the rotation $R^{-1}(\theta) = R(-\theta)$

- To construct $R^{-1}(\theta) = R(-\theta)$
  - Inside the rotation matrix: $\cos(-\theta) = \cos(\theta)$
    - The cosine elements of the inverse rotation matrix are unchanged
  - The sign of the sine elements will flip $\sin(-\theta) = -\sin(\theta)$

$$\rightarrow R^{-1}(\theta) = R(-\theta) = R^{T}(\theta)$$

Basic 3D transformations

Rotate around Z axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & 0 \\ \sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around Y axis

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ \boldsymbol{z}' \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} \cos\Theta & 0 & \sin\Theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\Theta & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \end{bmatrix}$$

Rotate around X axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta & 0 \\ 0 & \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Adding (or Subtracting) Matrices

- You can only add two matrices if they are of the same dimension.
- In order to add matrices (of the same dimension), just add corresponding entries.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 6 \\ 9 & 12 & 15 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 9 \\ 13 & 17 & 21 \end{pmatrix}$$

- Addition of matrices is commutative, that is A+B=B+A

Addition

If $\quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

and $\quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

then $\quad C = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$

Matrix Addition Example

$$A + B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix} = C$$

Scalar Multiplication

- In order to multiply any matrix by a scalar, a real number, multiply each entries by that number.

$$4A = 4\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 12 \\ 16 & 20 & 24 \end{pmatrix}$$

Multiplying 2 Matrices

- If you want to multiply two matrices : A (with dimension mxn) and B (with dimension pxq), the number of rows in A must equal the number of columns in B.
- The answer that you get will be a matrix with dimension mxq

$$A_{m \times n} \; B_{p \times q}$$

Matrix Multiplication

Matrices A and B can be multiplied if:

[r x c] and [s x d]

c = s

Matrix Multiplication

The resulting matrix will have the dimensions:

[r x c] and [s x d]

r x d

Computation: A x B = C (2x3 example)

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad [2 \text{ x } 2]$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \quad [2 \text{ x } 3]$$

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

[2 x 3]

Computation: A x B = C (3x3 example)

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$[3 \times 2]$ $[2 \times 3]$

A and B can be multiplied

$$C = \begin{bmatrix} 2*1+3*1=5 & 2*1+3*0=2 & 2*1+3*2=8 \\ 1*1+1*1=2 & 1*1+1*0=1 & 1*1+1*2=3 \\ 1*1+0*1=1 & 1*1+0*0=1 & 1*1+0*2=1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 8 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$[3 \times 3]$

Computation: A x B = C (3x3 example)

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$[3 \times 2]$ $[2 \times 3]$

Result is 3 x 3

$$C = \begin{bmatrix} 2*1+3*1=5 & 2*1+3*0=2 & 2*1+3*2=8 \\ 1*1+1*1=2 & 1*1+1*0=1 & 1*1+1*2=3 \\ 1*1+0*1=1 & 1*1+0*0=1 & 1*1+0*2=1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 8 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$[3 \times 3]$

Multiplying 2 Matrices (2x2 example)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 28 \\ 49 & 641 \end{pmatrix}$$

$$m_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}$$
$$m_{12} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}$$
$$m_{21} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31}$$
$$m_{22} = a_{21} \cdot b_{21} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32}$$

Multiplying 2 Matrices

- Multiplication of 2 matrices is not commutative $AB \neq BA$

Matrix Inversion

$$B^{-1}B = BB^{-1} = I$$

Like a reciprocal
in scalar math

Like the number one
in scalar math

Finding the Inverse of a Matrix

- If **A** is a square matrix, and **B** is a matrix such that **AB**=**I** and **BA**=**I**, then **A** is *nonsingular* (invertible) and **B** is the inverse of matrix **A**. If no such matrix **B** exists we say that matrix **A** is *singular*.

Finding the Inverse of Matrix

- Construct the augmented matrix $\left[A \mid I\right]$
- Perform row operations until you have an augmented matrix of the form

$$\left[I \mid B\right]$$

Finding the Inverse of a Matrix (example)

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Exercise

1) **Translation : $t_x = -3$; $t_y = 2$**
2) **Scaling : $s_x = 1/3$; $s_y = 2$**
3) **Rotation : $\Theta = -30$**

How can these transformations be combined?