Advanced Transform Methods

The Wavelet Transform

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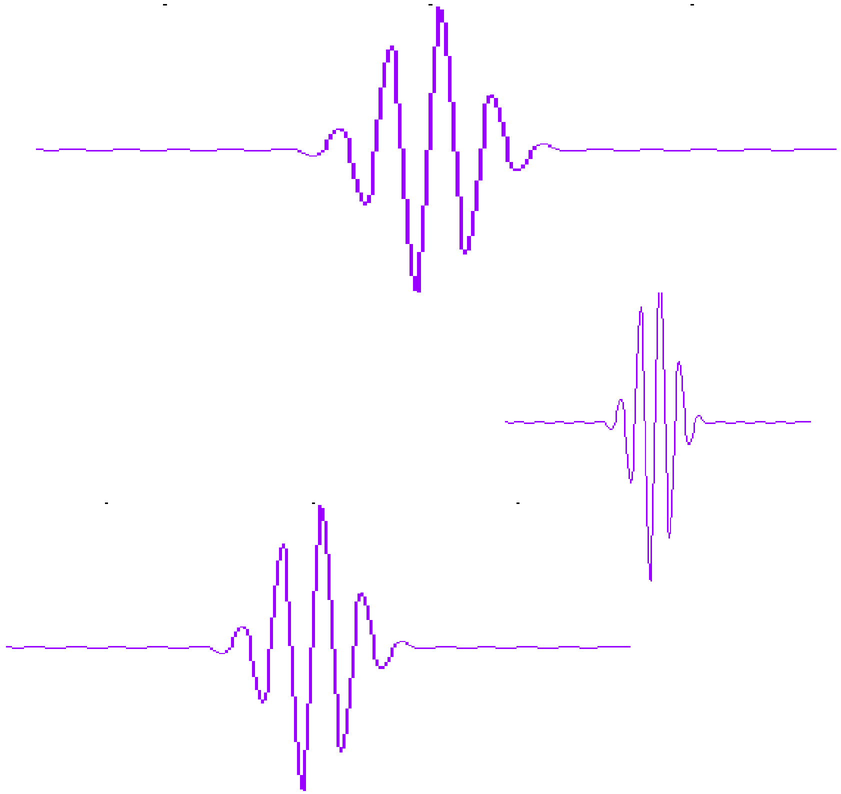
The Wavelet Transform

What is a "**Wavelet**"?

- a "small wave" or a "wave packet".

We can make a *family of wavelets* by:

- scaling and shifting a base or *mother* wavelet
- to create *daughter* wavelets (sometimes called *baby* wavelets)



Same wavelet,
just scaled and time-shifted

Wavelets in Application

- New way of evaluating and processing signals
- Works on nonstationary data
- Useful in many types of applications
  - Pattern recognition
    - Biotech: distinguish normal from pathological membranes
    - Biometrics: facial/corneal/fingerprint recognition
  - Feature extraction
    - Metallurgy: characterization of rough surfaces
  - Trend detection:
    - Finance: exploring variation of stock prices
  - Perfect reconstruction
    - Communications: wireless channel signals
  - Video compression – JPEG 2000

The Wavelet

- Consider scaling and translating the function ψ

$$\psi(t) \to \psi\left(\frac{t-b}{a}\right)$$

- a determines the centre frequency.
- b determines the translation.
- Time frequency centre of $\psi\left((t-b)/a\right)$
  are b (time centre)
  and $\langle\omega\rangle / a$ (frequency centre)

$\langle\omega\rangle$ is mean freq of ψ

- Daughter wavelets:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$$

Mother Wavelet

Continuous Wavelet Transform

$$CWT(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} s(t)\psi^*\left(\frac{t-b}{a}\right)dt$$

Scale

Translation

$$= \int_{-\infty}^{\infty} s(t)\psi_{a,b}{}^*(t)dt = \langle s, \psi_{a,b} \rangle$$

- The continuous wavelet transform, $CWT(a,b)$ is a function of two real variables.
- Compare short-time Fourier Transform:

$$STFT(t,\omega) = \int_{-\infty}^{\infty} s(\tau)\gamma^*(\tau - t)e^{-j\omega t}d\tau$$

- Have $\psi_{a,b}{}^*(t)$ instead of $\gamma^*(\tau - t)e^{-j\omega t}$

CWT: Time-Frequency Analysis

- CWT provides a time-frequency as well as time-scale representation.

$$CWT(a,b) = TF(t = b, \omega = \langle \omega \rangle / a)$$

- We can define the *Scalogram*

$$SCAL(a,b) = \left| CWT(a,b) \right|^2$$

- Compare Spectrogram: $\left| STFT(t,\omega) \right|^2$

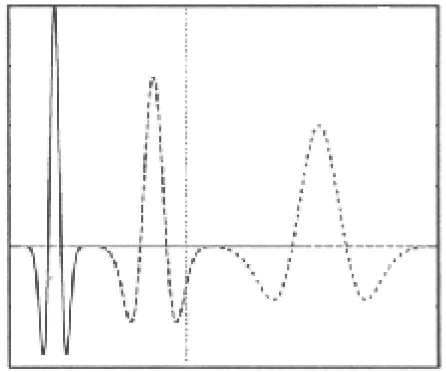
CWT versus STFT

CWT: Variable time-frequency resolution

$$CWT(a,b) = \langle s, \psi_{a,b} \rangle$$

Scale | Translation

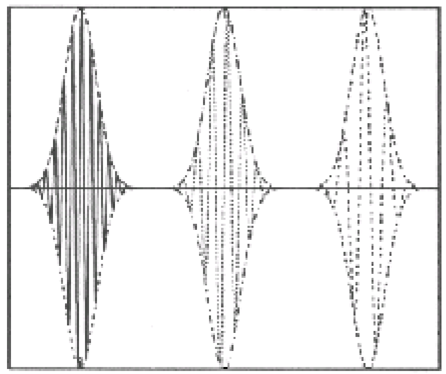
Different width;
Same no of cycles



STFT: Constant time-frequency resolution

$$STFT(u,b) = \langle s, g_{u,b} \rangle$$

Frequency | Translation



Same width;
Different no of cycles

Scaling of a signal

Consider time-scaling a signal: $r(t) = s(t/\alpha)$

This changes Fourier Transform: $R(\omega) = \alpha S(\alpha\omega)$

So changes energy: $E_r = \int_{-\infty}^{\infty} \left|s(t/\alpha)\right|^2 dt = \int_{-\infty}^{\infty} \left|s(\tau)\right|^2 d(\tau\alpha) = \alpha E$

New centre freq:

$$\left\langle \omega \right\rangle_R = \frac{1}{2\pi E_R} \int_{-\infty}^{\infty} \omega \left|R(\omega)\right|^2 d\omega$$

$$= \frac{1}{2\pi\alpha E} \int_{-\infty}^{\infty} \omega \left|\alpha S(\alpha\omega)\right|^2 d\omega \qquad R(\omega)=\alpha S(\alpha\omega)$$

$$= \frac{1}{2\pi\alpha E} \int_{-\infty}^{\infty} \frac{\Omega}{\alpha} \left|\alpha S(\Omega)\right|^2 d\,\frac{\Omega}{\alpha} \qquad \Omega=\alpha\omega$$

$$= \frac{1}{2\pi\alpha E} \int_{-\infty}^{\infty} \Omega \left|S(\Omega)\right|^2 d\Omega \quad = \frac{\left\langle \omega \right\rangle}{\alpha}$$

Scaled centre freq

Scaling (cont)

New frequency width:

$$\Delta_\omega^2(R) = \frac{1}{2\pi E_R}\int_{-\infty}^{\infty}\omega^2\left|R(\omega)\right|^2 d\omega - \left\langle\omega\right\rangle_R{}^2$$

$$= \frac{1}{2\pi\alpha E}\int_{-\infty}^{\infty}\omega^2\left|\alpha S(\alpha\omega)\right|^2 d\omega - (\left\langle\omega\right\rangle/\alpha)^2$$

$$= \frac{1}{2\pi\alpha E}\int_{-\infty}^{\infty}(\Omega/\alpha)^2\left|\alpha S(\Omega)\right|^2 d\frac{\Omega}{\alpha} - \left\langle\omega\right\rangle^2/\alpha^2$$

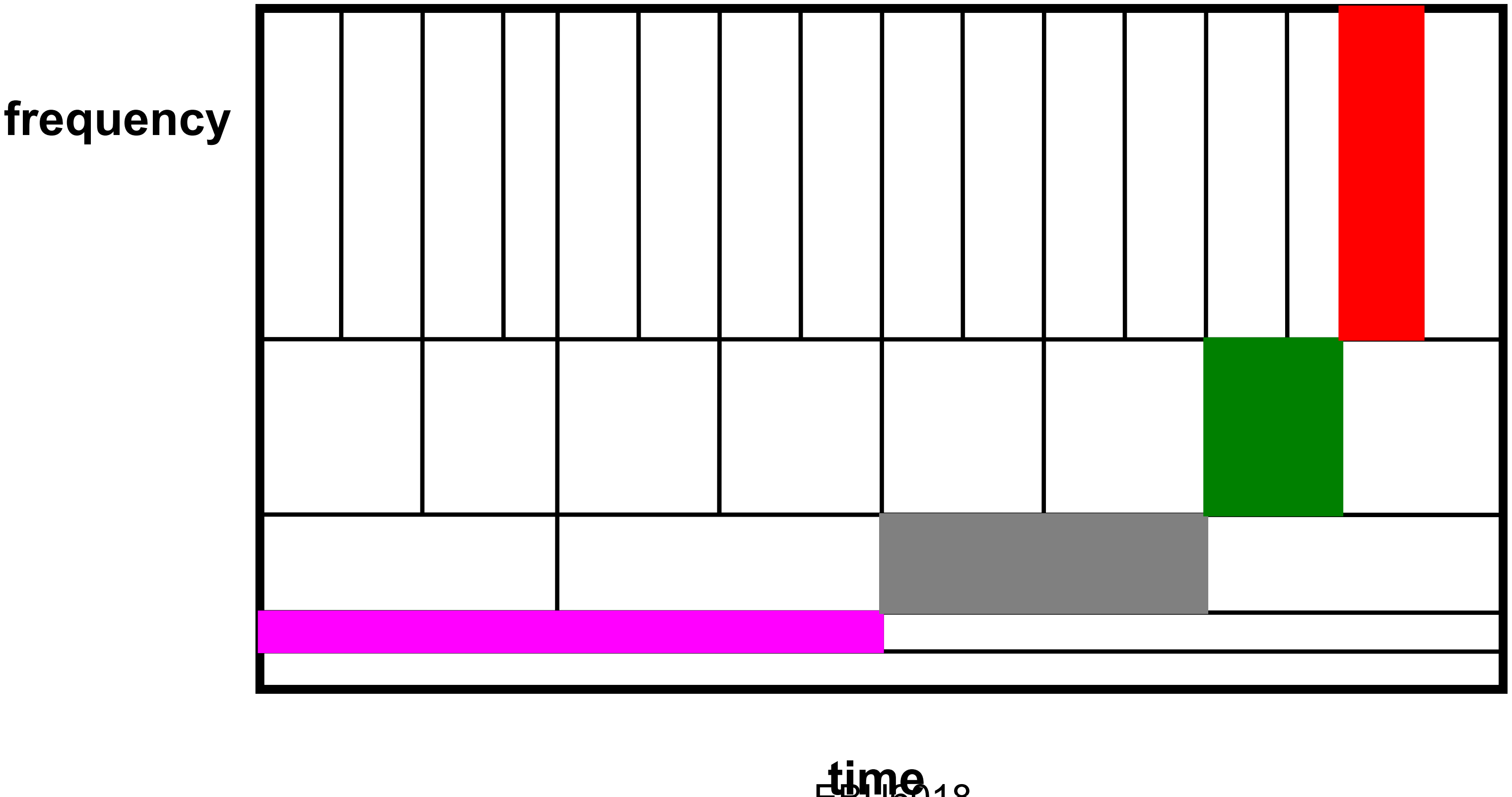
$$= \frac{1}{2\pi\alpha^2 E}\int_{-\infty}^{\infty}\Omega^2\left|S(\Omega)\right|^2 d\Omega - \left\langle\omega\right\rangle^2/\alpha^2$$

$$= \frac{\Delta_\omega^2(S)}{\alpha^2}$$

Scaled frequency resolution

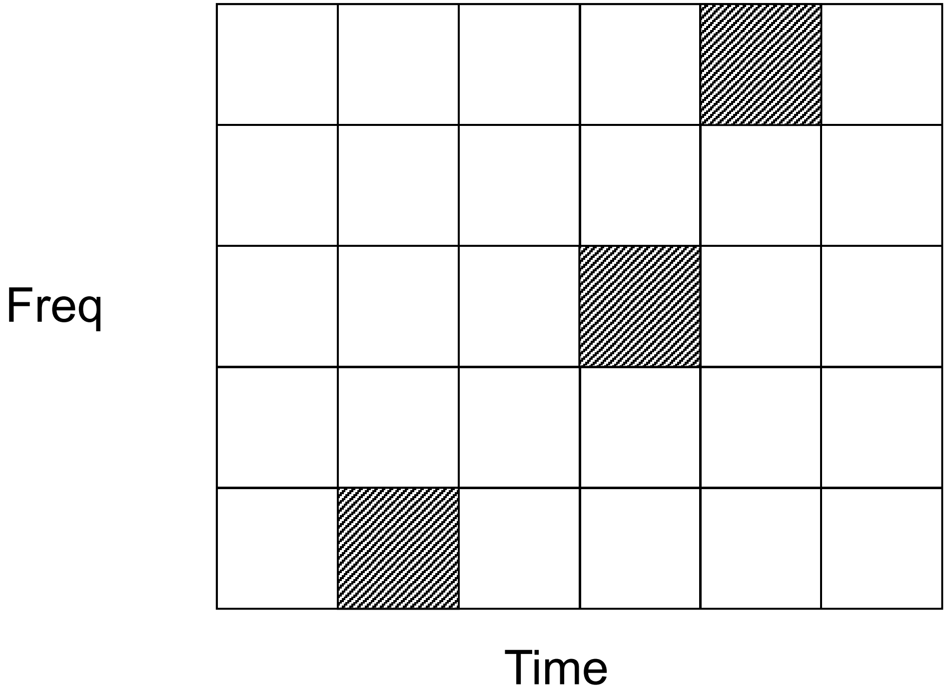
Partition of the time-frequency plane

- High scale (low frequency)
  - large window size, better frequency resolution
- Low scale (high frequency)
  - small window size, better time resolution.



Time-Freq Partition: STFT

FT: Equal time and frequency resolution



(WT: Logarithmic scale of frequency resolution)

Inverse CWT: The Admissability Criterion

- We can construct an Inverse FT to reconstruct *s*(*t*)
  Can we do the same for CWT?
- Yes: provided that the *Admissibility Condition* is satisfied:

$$C_\Psi = \int_{-\infty}^{\infty} \frac{\left|\Psi(\omega)\right|^2}{\left|\omega\right|} d\omega < \infty$$

where $\Psi(\omega) = \int_{-\infty}^{\infty} \psi(t)\, \mathrm{e}^{-j\omega t}\, dt$ is the Fourier Transform of $\psi(t)$

Reconstruction:

$$s(t) = \frac{1}{C_\Psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^2} CWT(a,b) \psi_{a,b}(t) da db$$

Admissibility Condition (cont)

- Square of the Fourier transform must decay faster than 1/*w*.
- Admissibility is measure of signal's band-limitedness.
- Admissibility implies zero average:

$$\Psi(0) = \int_{-\infty}^{\infty} \psi(t)\, \mathrm{e}^{-j0t}\, dt = \int_{-\infty}^{\infty} \psi(t) dt = 0$$

because otherwise $\frac{|\Psi(\omega)|^2}{|\omega|} \to \infty$ as $\omega \to 0$

Comparison of STFT and CWT

- Similarities:
  - signal is multiplied by a function, and the transform is computed separately for different segments of signals.
  - can be written in inner product form

$$STFT(b,\omega) = \left\langle s(t), \gamma(t-b)e^{j\omega t} \right\rangle \quad CWT(b,a) = \left\langle s(t), \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) \right\rangle$$

  - Time-frequency window area remains constant.

- Difference:
  - Fixed time duration and freq bandwidths of $\gamma(t)$
  - Variable time duration and bandwidth of $\psi(t)$

Comparison of Bases

- Fourier Transform
  - Basis is global (across all time)
  - Sinusoids with frequencies in arithmetic progression
- Gabor Transform (STFT)
  - Basis is local (in time)
  - Sinusoid times Gaussian
  - Fixed-width Gaussian “window”
- Wavelet Transform
  - Basis is local (in time)
  - Frequencies in geometric progression
  - Basis has constant shape independent of scale

Problems with CWT

Redundancy

- Basis functions for CWT are shifted and scaled versions of each other. Cannot form a very orthonormal base.

Infinite solution space

- The result holds an infinite number of wavelets:
  hard to solve and hard to find the desired results out of the transformed data.

Efficiency

- Most transforms cannot be solved analytically. Solutions have to be calculated numerically: time-consuming.
  Must find efficient algorithms.

Solution?

- *Multiresolution Analysis*