EBU6018 Past Paper Revision Tut Qs Dec 2018

1213 A

Question 2.

a) The Fast Fourier Transform (FFT) is a method for reducing the time taken to perform a Discrete Fourier Transform (DFT). Describe, using a 16-point sequence as an example, the process of implementing an FFT. Refer to radix-2 decimation in time.

[10 marks]

Answer:

a) A radix-2 sequence is one whose number of elements is a power of 2 [1 mark]. A sequence is split into two [1 mark], one of which is the even numbered elements and one the odd numbered elements [1 mark]. This process is continued till we have individual elements [1 mark].

For example, suppose we have a 16-point signal as follows (the numbers are the positions of the elements, not the values of the elements):

[0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15]

This is split into:

[0 2 4 6 8 10 12 14] and [1 3 5 7 9 11 13 15]

And so on:

[0 4 8 12] [2 6 10 14] [1 5 9 13] [3 7 11 15]

[0 8] [4 12] [2 10] [6 14] [1 9] [5 13] [3 11] [7 15]

[0] [8] [4] [12] [2] [10] [6] [14] [1] [9] [5] [13] [3] [11] [7] [15]   [2 marks]

Re-ordering the sequence in this way can be performed using bit-reversal, as each position is the reverse of the binary value of the original position. Element values are swapped accordingly, eg the value in position 3 is swapped with the value in position 12 [1 mark].

Each 1 point signal is then transformed to the frequency domain, nothing is required to do this step [1 mark].

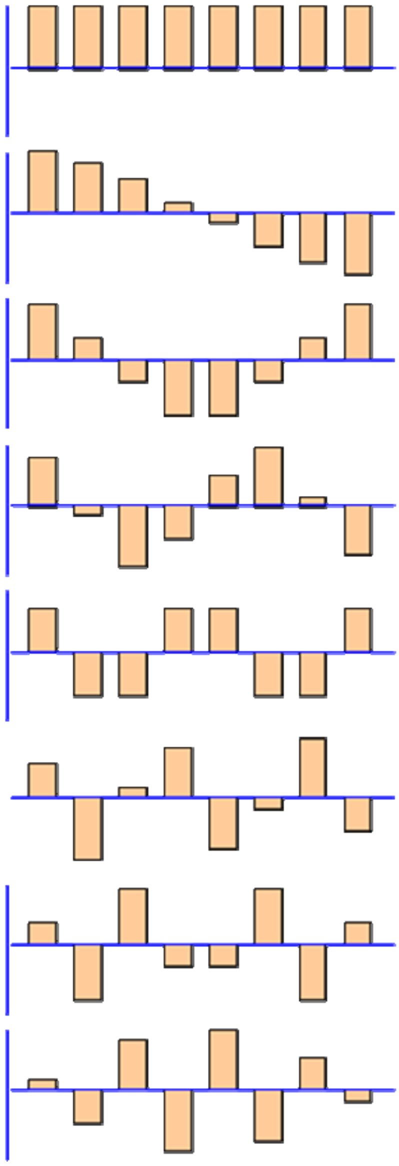
The individual frequency values are then combined by reversing, in steps, the decomposition process: [1 mark]

In the above case, the 16 individual frequency points are grouped into 8 groups of 2 points each. This is carried out by shifting, adding zeroes in the gaps and then adding [1 mark].

b) The Discrete Cosine Transform (DCT) transforms functions of time (discrete signals) and functions of position (images) into functions of frequency. Explain how it does this.

[6 marks]

Answer: The DCT correlates the input data [1 mark]with a series of cosine functions of increasing frequency, starting with dc [1 mark]. These discrete cosine functions are the basis functions. The basis functions for an 8-point DCT are [2 marks]:



Correlation gives a large value of output when there is a similarity between the input data distribution and a particular basis function [1 mark]. Therefore the output sequence from the DCT is related to the frequency of the distribution of data in the input sequence [1 mark].

c) Karhunen-Loeve Transform (KLT) compression is an example of Linear Transform Coding (LTC). Briefly explain the process of LTC and the role of the KLT.

[9 marks]

Answer: The image or signal to be compressed is divided into N blocks of pixels or samples. [1 mark]. Each block is then an N-dimensional vector, so we have a sequence of vectors [1 mark]. Each vector, x, is then transformed by multiplying by a linear matrix A [1 mark]. We then discard the less significant values from the transformed data [1 mark] and multiply by another matrix B to reconstruct the image or signal, $\hat{x}$ [1 mark]. We then find the mean squared error, J, between x and $\hat{x}$ [1 mark]. A and B are chosen to minimise J [1 mark]. The KLT minimises J by decorrelating the input data [1 mark] and discarding the set with the least variance [1 mark]

1213 A

Question 4

a) Data Compression is used to reduce the storage capacity needed and to allow higher rates of data transmission.Briefly explain what is meant by

i) Lossless compression

ii) Lossy compression.

[4 marks]

b) Multiresolution Analysis (MRA) is used to separate data into course and fine detail.

Apply the transform defined by

$$x_{n-1,i} = (x_{n,2i}+x_{n,2i+1})/2$$

$$d_{n-1,i} = (x_{n,2i}-x_{n,2i+1})/2$$

to the sequence

$$[x_{n,i}] = [10, 14, 26, 24, 30, 20, 8, 16]$$

Where i = 0…..7, is the index position in the sequence, and

n is the level. The next level is n-1.

At each level, concatenate the sequencies for $x_{n-1,i}$ and $d_{n-1,i}$

Continue till no further levels are possible.

i) Including the given sequence, how many levels are there?

ii) State the significance of the first element in the final level.

iii) Excluding the first element in the final level, what is the significance of the other elements?

iv) Has any information been lost in the process?

v) Comment on how this process could be used to compress the data.

vi) State the equations that can be used to reconstruct the original data from the final level.

vii) Relate the process of averaging and differencing to the Discrete Wavelet Transform.

[21 marks]

Answer:

a) i) Lossless compression is where fewer bits are required to store the data but no information is discarded. [1 mark]. This is achieved by eliminating redundancy [1 mark]

ii) Lossy compression results in a loss of information [1 mark]. This is achived by discarding marginally important information [1 mark].

b) Applying the transform:

n=3 [10, 14, 26, 24, 30, 20, 8, 16]

n=2 [12, 25, 25, 12, -2, 1, 5, -4]

n=1 [18.5, 18.5, -6.5, 6.5, -2, 1, 5, -4]

n=0 [18.5, 0, -6.5, 6.5, -2, 1, 5, -4] [6 marks: 2 for each row]

i) 4 levels [1 mark]

ii) The first element is the average of all the elements in the original sequence [1 mark].

iii) The remaining 7 elements are the fine detail in the original data. [1 mark]

iv) No information has been lost [1 mark]

v) Because most of the values in the final level are small, potentially fewer bits would be required to store it [1 mark]. Where there are zeroes, they do not need to be stored, although the positions of the other values would need to be stored [1 mark]. Small values could be replaced by zeroes without significant loss of detail [1 mark], this can be done by applying a threshold value, the bigger the threshold the greater the loss of detail [1 mark].

vi) The equations are

$$x_{n,2i} = x_{n-1,i} + d_{n-1,i}$$

$$x_{n,2i+1} = x_{n-1,i} - d_{n-1,i}$$ [4 marks: 2 each equation]

vii) The Haar DWT is used to perform averaging and differencing [1 mark], the coarse values by the Haar scaling function [1 mark] and the fine detail by the Haar Wavelet function [1 mark].

1213 B

Question 4

a) For the Continuous Wavelet Transform (CWT), wavelet functions are of many different forms depending on the application. State and briefly explain the condition required for the wavelet function if the transform is to be reversed.

[4 marks]

b) i) When we perform a Discrete Wavelet Transform(DWT), we consider each wavelet to have a wavelet function $\psi(t)$ and a scaling function $\varphi(t)$. The DWT can be carried out using multiresolution analysis (MRA).

State the roles of $\psi(t)$ and $\varphi(t)$ in MRA.

[2 marks]

ii) The simplest wavelet function is the Haar wavelet. Sketch the wavelet and scaling functions for the Haar wavelet.

[2 marks]

iii) A signal can be filtered into its coarse and fine detail (analysed) by using these two functions.

Sketch and explain the block diagram for one stage of a dyadic wavelet transform. How could this process be used for a non-dyadic sequence?

[8 marks]

iv) What condition is required to ensure that the original signal can be synthesised by reversing the process?

[2 marks]

c) Suppose we have a Haar wavelet transform analysis filterbank which uses a low-pass filter $h_0[0] = h_0[1] = \frac{1}{2}$ and a high-pass filter $h_1[0] = \frac{1}{2}$, $h_1[1] = -\frac{1}{2}$. Use the recursive equations:

$$c_{m-1,n} = \sqrt{2} \cdot \frac{1}{2}(c_{m,2n} + c_{m,2n+1})$$
$$= \frac{1}{\sqrt{2}}(c_{m,2n} + c_{m,2n+1})$$

$$d_{m-1,n} = \frac{1}{\sqrt{2}}(c_{m,2n} - c_{m,2n+1})$$

to calculate the Haar wavelet transform for a sampled signal s[n] = [2 1 3 -1] after 1 and 2 stages of the transform filterbank.

[7 marks]

Answer:

a) The condition is called the Admissibility Condition [1 mark]

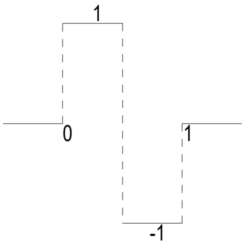
$$C_\Psi = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$$ [1 mark]

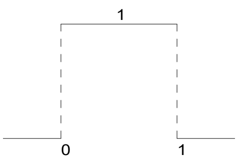
The admissibility condition gives that:

- Square of the Fourier transform must decay faster than 1/*w* (the wavelet must be compact in frequency domain) [1 mark]
- Admissibility implies zero average (ie, the wavelet must be oscillatory) [1 mark]

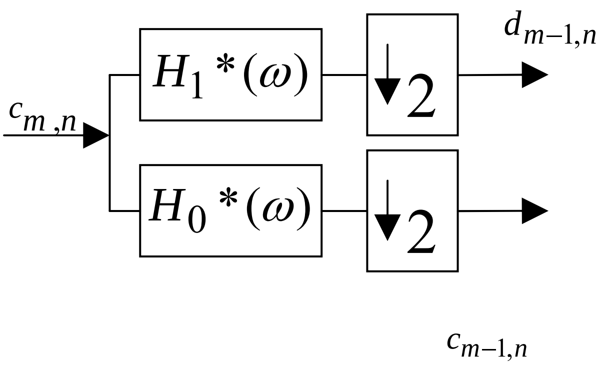
b) i) The wavelet function gives the fine detail in the signal [1 mark] and the scaling function gives the coarse detail [1 mark].

ii)





The first is the wavelet function ψ(t) and the second is the scaling function φ(t). [2 marks: 1 each] iii)



[2 marks]

Dyadic means that the number of values in the sequence being transformed is a power of 2 [1 mark]. The inputs to this stage are the coarse details from the previous stage [1 mark]. H_0 is a low pass filter to obtain the next level of coarse detail [1 mark]. H_1 is a high pass filter to obtain the next level of fine detail [1 mark]. Downsampling is carried out so that at each level there is half the number of values of each as at the previous stage [1 mark]. If the sequence length was not a power of 2, it can be zero padded to make it so [1 mark].

iv) If the wavelet functions are orthogonal, then the same filters can be used as in the analysis [1 mark]. If the wavelet functions are not orthogonal then filters that are biorthogonal with the analysis filters must be used for synthesis [1 mark].

c) Start with the signal is the finest resolution coefficient,

S[n] = [2 1 3 -1]

First level:

$$c_{1,0} = \tfrac{1}{\sqrt{2}}(2+1) = 3/\sqrt{2}$$
$$c_{1,1} = \tfrac{1}{\sqrt{2}}(3+(-1)) = 2/\sqrt{2}$$
$$d_{1,0} = \tfrac{1}{\sqrt{2}}(2-1) = 1/\sqrt{2}$$
$$d_{1,1} = \tfrac{1}{\sqrt{2}}(3-(-1)) = 4/\sqrt{2}$$

Hence the first level of the wavelet transform is $\dfrac{1}{\sqrt{2}}[3\ 2\ 1\ 4]$

Second level:

$$c_{0,0} = \tfrac{1}{\sqrt{2}}.\tfrac{1}{\sqrt{2}}(3+2) = 5/2$$
$$d_{0,0} = \tfrac{1}{\sqrt{2}}.\tfrac{1}{\sqrt{2}}(3-2) = 1/2$$

Hence the second level of the wavelet transform is $\frac{1}{2}[5\ \ 1\ \ \sqrt{2}\ \ 4\sqrt{2}]$

[7 marks:1 for each calculation and 1 for the final answer]

1213 C

Question 1.

c) i) Explain what is meant by the term “Biorthogonal Bases”.

ii) Dual bases are biorthogonal. Show that the following two bases are dual bases:

$\{\psi_n\} = \{(2,0), (1,2)\}$

$\{\hat{\Psi}_n\} = \{(0.5,-0.25), (0,0.5)\}$

iii) In the context of basis functions. Explain what is meant by the term “Frame”.

[13 marks]

Answer: i) Biorthogonal bases are a pair of bases [1 mark], as follows:

If $\{\psi_n\}$ and $\{\hat{\Psi}_n\}$ are both basis vectors themselves for V [1 mark], and satisfy

$$\langle \Psi_i, \hat{\Psi}_j \rangle = \delta_{ij} \quad [1 \text{ mark}]$$

then any **s** in V can be written as $s = \sum_{j=1}^{n} \langle s, \Psi_j \rangle \hat{\Psi}_j$ [1 mark]

$\{\psi_n\} = \{(2, 0), (1, 2)\}$

$\{\hat{\psi}_n\} = \{(0.5, -0.25), (0, 0.5)\}$

THEN $<\psi_1, \hat{\psi}_1> = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.25 \end{bmatrix} = \underline{\underline{1}}$ [1 MARK]

$<\psi_2, \hat{\psi}_2> = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} = \underline{\underline{1}}$ [1 MARK]

$<\psi_1, \hat{\psi}_2> = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} = \underline{\underline{0}}$ [1 MARK]

$<\psi_2, \hat{\psi}_1> = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.25 \end{bmatrix} = \underline{\underline{0}}$ [1 MARK]

FOR BIORTHOGONALITY, $<\psi_i, \hat{\psi}_j> = \sum_k \psi_i(k) \hat{\psi}_j(k) = \delta_{ij}$

[1 MARK]

δ_{jk} = KRONECKER DELTA

$\therefore \{\psi_n\}$ AND $\{\hat{\psi}_n\}$ ARE DUAL BASES [1 MARK]

iii) a frame is a set of vectors in vector space V that contains more vectors than the order of the space [1 mark], and that are not orthogonal or linearly independent [1 mark]. The frame vectors can be used to represent any other vector in the space [1 mark].

Question 2

b) The Karhunen-Loeve Transform (KLT) is used to decorrelate the input data and allow compression to be carried out with minimum error. It uses Principal Component Analysis (PCA) to do this. The basis vectors for the KLT depend on the input data.

The following is a 2 dimensional data set:

x	y
2	2
4	3
5	4
5	5
3	4
2	3

i) For this data, use PCA to find the normalised basis vectors.

ii) List the steps to perform the KLT.

[17 marks]

b) i) [13 marks….see below]

x	y
2	2
4	3
5	4
5	5
3	4
2	3

FIND MEANS: $\bar{x} = 3.5 \quad \bar{y} = 3.5$ [1 MARK]

FIND VARIANCES AND CO-VARIANCE:

$$VAR_x = \frac{1}{5}\sum_{i=0}^{5}(x_i - \bar{x})^2$$

$$= \frac{1}{5}\left((-1.5)^2 + (0.5)^2 + (1.5)^2 + (1.5)^2 + (0.5)^2 + (-1.5)^2\right)$$

$$= \underline{1.9}$$ [1 MARK]

$$VAR_y = \frac{1}{5}\sum_{i=0}^{5}(y_i - \bar{y})^2$$

$$= \frac{1}{5}\left((-1.5)^2 + (-0.5)^2 + (0.5)^2 + (1.5)^2 + (0.5)^2 + (-0.5)^2\right)$$

$$= \underline{1.1}$$ [1 MARK]

$$COV_{xy} = \frac{1}{5}\sum_{i=0}^{5}(x_i - \bar{x})(y_i - \bar{y}) = \underline{1.1}$$ [1 MARK]

$$\text{COV. MATRIX} = R_{xx} = \begin{bmatrix} 1.9 & 1.1 \\ 1.1 & 1.1 \end{bmatrix}$$ [1 MARK]

EIGENVALUES ARE ROOTS OF: $|R_{xx} - \lambda I| = 0$

$$\begin{vmatrix} 1.9-\lambda & 1.1 \\ 1.1 & 1.1-\lambda \end{vmatrix} = 0$$ [1 MARK]

$$\lambda^2 - 3\lambda + 0.88 = 0$$

$$\underline{\lambda_1 = 2.67} \qquad \underline{\lambda_2 = 0.33}$$ [1 MARK]

FIND NORMALISED EIGENVECTORS :

$$\begin{bmatrix} -0.77 & 1.1 \\ 1.1 & -1.57 \end{bmatrix}\begin{bmatrix}\phi_{11}\\ \phi_{21}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$ [1 MARK]

$$\rightarrow \phi_{11} = 1.43\,\phi_{21}$$ [1 MARK]

$$\begin{bmatrix} 1.57 & 1.1 \\ 1.1 & 0.77 \end{bmatrix}\begin{bmatrix}\phi_{12}\\ \phi_{22}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$ [1 MARK]

$$\rightarrow \phi_{12} = -0.7\,\phi_{22}$$ [1 MARK]

TO NORMALISE :

$$\langle \phi_1, \phi_1 \rangle = 1 \rightarrow \phi_{11}^2 + \phi_{21}^2 = 1$$ [1 MARK]

$$\langle \phi_2, \phi_2 \rangle = 1 \rightarrow \phi_{12}^2 + \phi_{22}^2 = 1$$

$$\rightarrow \phi = \begin{bmatrix} 0.82 & -0.57 \\ 0.57 & 0.82 \end{bmatrix}$$ [1 MARK]

ii) Once the basis vectors have been calculated, the steps to perform the KLT are:

Transform the input data to the basis vectors

Remove data with least energy (smallest eigenvalue), and replace by zeroes

Invert the transform with the reduced data

Compare with the original data (mean square error) [4 marks: 1 each statement]

1415 A

Question 4

a) State the mathematical definition of the Discrete Cosine Transform (DCT) and hence the expression for the basis vector elements $\psi_k[n]$ of an N-point DCT.

[6 marks]

	Do not write in this column
[1 mark] $DCT[k] = c(k)\sum_{n=0}^{N-1} s[n]\cos\frac{\pi(2n+1)k}{2N}$	
Comparing the DCT formula to that of the general form for a linear transform :	
$S[k] = \langle s, \psi_k \rangle = \sum_{n=0}^{N-1} s[n]\psi_k^*[n]$ where $\psi_k[n]$ are the basis vector elements [2 marks]	
Gives $\psi_k[n] = c(k)\left[\cos\frac{\pi(2n+1)k}{2N}\right]$ where k=0,1,2...	
and $c(k) = \begin{cases} \sqrt{1/N} & k=0 \\ \sqrt{2/N} & k \neq 0 \end{cases}$ [3 marks]	
	6 marks

b) For N=3, calculate the DCT basis vectors $\psi_k = (\psi_k[0], \psi_k[1], \psi_k[2])$ for $k = 0,1,2$.

[9 marks]

	Do not write in this column
For k=0, $\psi_0[n]$=c(k)cos0=$\sqrt{1/N} = \sqrt{1/3}$	
So, $\psi_0 = \frac{1}{\sqrt{3}}(1,1,1)$ [3 marks]	

For k=1, $\psi_1[n]$=c(1)$\cos\frac{\pi(2n+1)}{6} = \sqrt{\frac{2}{3}}\cos\frac{\pi(2n+1)}{6}$

So, $\psi_1 = \sqrt{\frac{2}{3}}\left(\cos\frac{\pi}{6}, \cos\frac{3\pi}{6}, \cos\frac{5\pi}{6}\right)$

$\psi_1 = \sqrt{\frac{2}{3}}\left(\cos\frac{\pi}{6}, 0, -\cos\frac{\pi}{6}\right)$ [3 marks]

For k=2, $\psi_2[n]$=c(2)$\cos\frac{2\pi(2n+1)}{6} = \sqrt{\frac{2}{3}}\cos\frac{\pi(2n+1)}{3}$

So, $\psi_2 = \sqrt{\frac{2}{3}}\left(\cos\frac{\pi}{3}, \cos\frac{3\pi}{3}, \cos\frac{5\pi}{6}\right)$

$\psi_2 = \sqrt{\frac{2}{3}}\left(\frac{1}{2}, -1, 1/2\right)$ [3 marks]

9 marks

c) Calculate the DCT of the 3-point signal [2 0 3] **[6 marks]**

Do not write in this column

DCT [0]= $\langle s\ \psi_0\rangle$=[2 0 3]$\begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}$=5/√3 [2 marks]

DCT [1]= $\langle s\ \psi_1\rangle$=[2 0 3]$\sqrt{\frac{2}{3}}\begin{bmatrix}\cos\frac{\pi}{6}\\ 0\\ -\cos\frac{\pi}{6}\end{bmatrix} = \sqrt{\frac{2}{3}}$ (-cosπ/6) [2 marks]

DCT [2]= $\langle s\ \psi_2 \rangle$==$[1 \quad 0 \quad 1]\sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}$=5/(3√2) [2 marks]

6 marks

d) By using an appropriate diagram, show how a 1-D DCT can be applied to a 2-D image. **[4 marks]**

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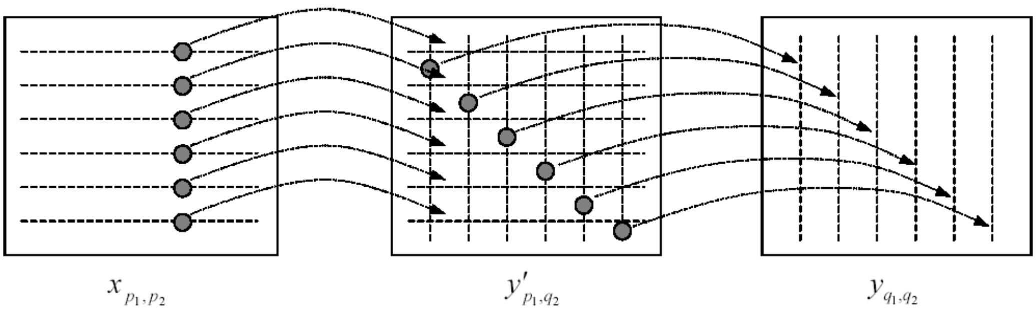
Separable Transforms

May be implemented by applying the one dimensional transform first to the rows of the image and then to its columns (note that changing the application order does not change the result).



[2 marks for separable statement and 2 for diagram: 4 marks]

4 marks

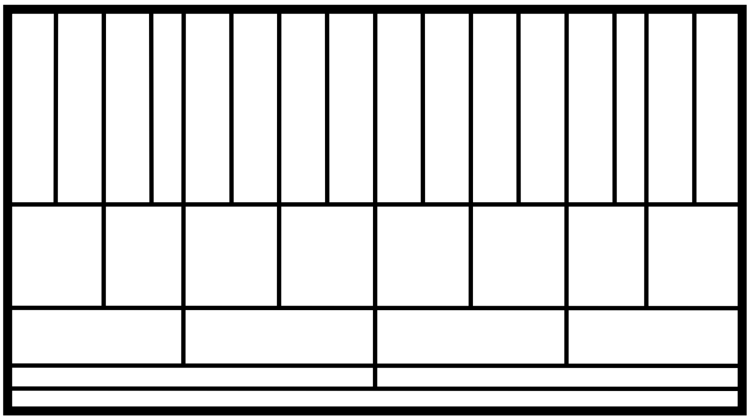
Question 3

a) With the aid of diagrams, compare the time-frequency tiling of the Wavelet Transform (WT) with that of the short-time Fourier transform (STFT).

[6 marks]

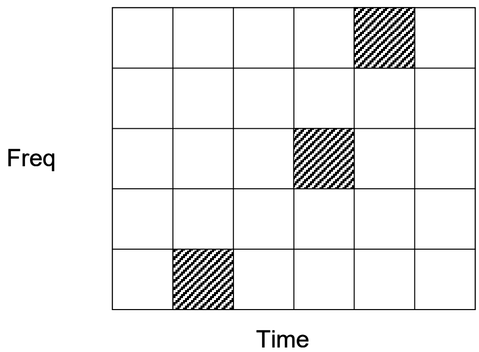
	Do not write in this column

Time-frequency tiling of wavelet transform:



[1 mark]

Compare t-f tiling of STFT:



[1 mark]

For STFT, have constant time and frequency resolution at all frequencies. [1 mark]

For WT, have variable time and frequency resolution, depending on centre frequency. [1 mark] At low freq, have fine freq resolution, but coarse time resolution. [1 mark]

At high freq, have coarse freq resolution, but fine time resolution. [1 mark]

6 marks

b) Sketch a diagram to illustrate a wavelet analysis filterbank. **[2 marks]**

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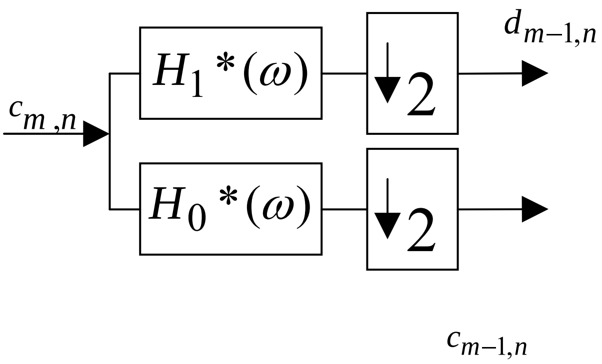


[2 marks]

2 marks

c) In the wavelet transform, the scaling function coefficients $c_{m,n}$ and wavelet series coefficients $d_{m,n}$ can be calculated recursively according to the following equations:

$$c_{m-1,n} = \sqrt{2}\sum_i h_0[i-2n]c_{m,i}$$
$$d_{m-1,n} = \sqrt{2}\sum_i h_1[i-2n]c_{m,i}$$

This process can be reversed using the recursive equation

$$c_{m,n} = \sqrt{2}\left(\sum_i h_0[n-2i]c_{m-1,i} + \sum_i h_1[n-2i]d_{m-1,i}\right).$$

Suppose we have a Haar wavelet transform analysis filterbank which uses a low-pass filter $h_0[0] = h_0[1] = \frac{1}{2}$ and a high-pass filter $h_1[0] = \frac{1}{2}$, $h_1[1] = -\frac{1}{2}$.

Using these recursive equations, calculate the Haar wavelet transform for a sampled signal $s[n] = [1, 2, 0, -2]$ after 1 and 2 stages of the transform filterbank, with the initial setting $c_{2,n} = s[n]$ for $n = 0 \ldots 3$.

Calculate the inverse wavelet transform of your result using the resynthesis filterbank, to confirm that the result of the inverse transform is the original $s[n]$.

[17 marks]

	Do not write in this column
Start with the signal is the finest resolution coefficient, $s[n] = [1, 2, 0, -2] = c_{2,i}$ or $h_0[i-2n] = \begin{cases} \frac{1}{2} & \text{if } i = 2n \text{ or } 2n+1 \\ 0 & \text{otherwise} \end{cases}$ [1 mark] Therefore $c_{m-1,n} = \sqrt{2} \cdot \frac{1}{2}(c_{m,2n} + c_{m,2n+1}) = \frac{1}{\sqrt{2}}(c_{m,2n} + c_{m,2n+1})$ [1 mark] similarly $d_{m-1,n} = \frac{1}{\sqrt{2}}(c_{m,2n} - c_{m,2n+1})$ First level: $c_{1,0} = \frac{1}{\sqrt{2}}(1+2) = 3/\sqrt{2}$ $c_{1,1} = \frac{1}{\sqrt{2}}(0+(-2)) = -2/\sqrt{2}$ $d_{1,0} = \frac{1}{\sqrt{2}}(1-2) = -1/\sqrt{2}$ $d_{1,1} = \frac{1}{\sqrt{2}}(0-(-2)) = 2/\sqrt{2}$ [4 marks: 1 each] Hence the first level of the wavelet transform is $\frac{1}{\sqrt{2}}(3, -2, -1, 2)$. Second level:	

$$c_{0,0} = \tfrac{1}{\sqrt{2}}.\tfrac{1}{\sqrt{2}}(3+(-2)) = 1/2$$
$$d_{0,0} = \tfrac{1}{\sqrt{2}}.\tfrac{1}{\sqrt{2}}(3-(-2)) = 5/2$$

[2 marks]

Hence the second level of the wavelet transform is $\frac{1}{2}(1, 5, -\sqrt{2}, 2\sqrt{2})$

(the *d* wavelet coefficients in the right hand half are unchanged).

For the resynthesis, we have

$$c_{m,n} = \sqrt{2}\left(\sum_i h_0[n-2i]c_{m-1,i} + \sum_i h_1[n-2i]d_{m-1,i}\right)$$

i.e. with the Haar filters we have

$c_{m,n} = \sqrt{2}\left(\frac{1}{2}c_{m-1,n/2} + \frac{1}{2}d_{m-1,n/2}\right)$ for *n* even, or

$c_{m,n} = \sqrt{2}\left(\frac{1}{2}c_{m-1,(n-1)/2} - \frac{1}{2}d_{m-1,(n-1)/2}\right)$ for *n* odd. [2 marks]

From level 0 to level 1:

$$c_{1,0} = \sqrt{2}\left(\tfrac{1}{2}c_{0,0} + \tfrac{1}{2}d_{0,0}\right) = \sqrt{2}\left(\tfrac{1}{2}1/2 + \tfrac{1}{2}5/2\right) = \sqrt{2}(6/4) = 3/\sqrt{2}$$
$$c_{1,1} = \sqrt{2}\left(\tfrac{1}{2}c_{0,0} - \tfrac{1}{2}d_{0,0}\right) = \sqrt{2}\left(\tfrac{1}{2}1/2 - \tfrac{1}{2}5/2\right) = \sqrt{2}(-4/4) = -2/\sqrt{2}$$

[2 marks]

The *d* coefficients in the 2nd half of the WT are unchanged, so we have

$\frac{1}{\sqrt{2}}(3, -2, -1, 2)$ which is the same as the first stage transform in the analysis direction.

From level 1 to level 2 we have

$$c_{2,0} = \sqrt{2}\left(\tfrac{1}{2}c_{1,0} + \tfrac{1}{2}d_{1,0}\right) = \sqrt{2}\left(\tfrac{1}{2}3/\sqrt{2} + \tfrac{1}{2}(-1/\sqrt{2})\right) = \frac{1}{2}(3-1) = 1$$
$$c_{2,1} = \sqrt{2}\left(\tfrac{1}{2}c_{1,0} - \tfrac{1}{2}d_{1,0}\right) = \sqrt{2}\left(\tfrac{1}{2}3/\sqrt{2} - \tfrac{1}{2}(-1/\sqrt{2})\right) = \frac{1}{2}(3+1) = 2$$
$$c_{2,2} = \sqrt{2}\left(\tfrac{1}{2}c_{1,1} + \tfrac{1}{2}d_{1,1}\right) = \sqrt{2}\left(\tfrac{1}{2}.-2/\sqrt{2} + \tfrac{1}{2}2/\sqrt{2}\right) = \frac{1}{2}(-2+2) = 0$$
$$c_{2,3} = \sqrt{2}\left(\tfrac{1}{2}c_{1,1} - \tfrac{1}{2}d_{1,1}\right) = \sqrt{2}\left(\tfrac{1}{2}.-2/\sqrt{2} - \tfrac{1}{2}2/\sqrt{2}\right) = \frac{1}{2}(-2-2) = -2$$

[4 marks: 1 each]

so for the inverse transformed signal we have $[1, 2, 0, -2]$ which is our original signal successfully recovered. [1 mark]

17 marks

1516 B

Q 1 b) The Fast Fourier Transform (FFT) is a method for reducing the time taken to perform a Discrete Fourier Transform (DFT). Using the 16-point sequence below, describe the process of implementing a decimation-in-time FFT and hence obtain the DFT of this sequence. Describe the process of inverting the DFT.

16-point sequence: [10 8 5 7 11 6 4 -2 -5 1 0 3 9 12 16 14]

[13 marks]

	Do not write in this column
Process: a sequence is split into two [1 mark] one of which is the even numbered elements and the other the odd numbered elements [1 mark]. This is continued till we have individual elements [1 mark].	
For the inverse DFT, the individual frequency values are combined in steps by reversing the decomposition process [1 mark]. The 16 individual frequency points are grouped into 8 groups of 2 points each [1 mark]. This is carried out by shifting, adding zeroes in the gaps and then adding [1 mark].	

15-16 B
Q1(b)

16 POINT SEQUENCE

[10, 8, 5, 7, 11, 6, 4, -2, -5, 1, 0, 3, 9, 12, 16, 14]

SPLITS INTO:

[10 5 11 4 -5 0 9 16]

AND [8 7 6 -2 1 3 12 14] [1 MARK]

THEN [10 11 -5 9][5 4 0 16][8 6 1 12] [7 -2 3 14]

[1 MARK]

THEN [10 -5][11 9][5 0][4 16][8 1][6 12][7 3][-2 14]

[1 MARK]

THEN [10][-5][11][9][5][0][4][16][8][1][6][12][7][3][-2][14]

[1 MARK]

Reordering the sequence in this way can be performed using bit reversal as each position is the reverse of the binary value of the original position [1 mark]. Element values are swapped accordingly, eg, the value in position 3 is swapped with the value in position 12 [1 mark].

13 marks

Q 3 b) The Discrete Cosine Transform (DCT) is given by

$$DCT[k]=c(k)\sum_{n=0}^{N-1}s[n]\cos\frac{\pi(2n+1)k}{2N} \quad \text{where } c(k)=\begin{cases}\sqrt{1/N} & k=0\\ \sqrt{2/N} & k\neq 0\end{cases}$$

Determine the basis functions for a 4-point DCT and calculate the DCT of the 4-point sequence

S[n]=[0, 1, 0, 2].

[13 marks]

For the DCT transform $\psi_k[n]=\left[c(k)\cos\frac{\pi(2n+1)k}{2N}\right]^*=c(k)\cos\frac{\pi(2n+1)k}{2N}$ with $c(k)$ as above.

[1 mark]

$$\psi_k[n]=c(k)\cos\frac{\pi(2n+1)k}{2N}$$ i.e.

$\psi_0=\frac{1}{2}(1,1,1,1)$ and $\psi_k=\frac{1}{\sqrt{2}}\left(\cos\frac{\pi k}{8},\cos\frac{3\pi k}{8},\cos\frac{5\pi k}{8},\cos\frac{7\pi k}{8}\right)$ for $k\neq 0$ [1 mark]

Writing these out, we get

$$\psi_1=\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8},\cos\frac{3\pi}{8},\cos\frac{5\pi}{8},\cos\frac{7\pi}{8}\right)=\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8},\cos\frac{3\pi}{8},-\cos\frac{3\pi}{8},-\cos\frac{\pi}{8}\right)$$ [1 mark]

$$\psi_2=\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4},\cos\frac{3\pi}{4},\cos\frac{5\pi}{4},\cos\frac{7\pi}{4}\right)=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$
$$=\frac{1}{2}(1,-1,-1,1)$$

[1 mark]

$$\psi_3=\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{8},\cos\frac{9\pi}{8},\cos\frac{15\pi}{8},\cos\frac{21\pi}{8}\right)=\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{8},-\cos\frac{\pi}{8},\cos\frac{\pi}{8},-\cos\frac{3\pi}{8}\right)$$ [1 mark]

For an N-point transform based on a basis vector set $\{\psi_k\}$, we can write

$$S[k] = \langle s, \psi_k \rangle = \sum_{n=0}^{N-1} s[n]\psi_k^*[n].$$

So for the DCT of [0, 1, 0, 2] we have:

15-16 B

Q 3

4 POINT SEQUENCE [0, 1, 0, 2]

$$S[0] = \langle s, \psi_0 \rangle = \frac{1}{2}(0\times1 + 1\times1 + 0\times1 + 2\times1) = \underline{\underline{\frac{3}{2}}}$$

$$S[1] = \langle s, \psi_1 \rangle = \frac{1}{\sqrt{2}}\left(0\times\cos\frac{\pi}{8} + 1\times\cos\frac{3\pi}{8} - 0\times\cos\frac{3\pi}{8} - 2\cos\frac{\pi}{8}\right)$$

$$= \underline{\underline{\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{8} - 2\cos\frac{\pi}{8}\right)}}$$

$$S[2] = \langle s, \psi_2 \rangle = \frac{1}{2}(0\times1 - 1\times1 - 0\times1 + 2\times1) = \underline{\underline{\frac{1}{2}}}$$

$$S[3] = \langle s, \psi_3 \rangle = \frac{1}{\sqrt{2}}\left(0\times\cos\frac{3\pi}{8} - 1\times\cos\frac{\pi}{8} + 0\times\cos\frac{\pi}{8} - 2\cos\frac{3\pi}{8}\right)$$

$$= \underline{\underline{\frac{1}{\sqrt{2}}\left(-\cos\frac{\pi}{8} - 2\cos\frac{3\pi}{8}\right)}}$$

[8 MARKS: 2 MARKS EACH]

1516 C

Q 1 c) Perform Multiresolution Analysis (MRA) on the 8-point sequence

$[x_{n,i}]$ = [10, 13, 25, 26, 29, 21, 7, 15].

Take the mean of the numbers in pairs and the difference between the first number in a pair and their mean.

State the significance of the first value in the last row.

	Do not write in this column
First level of decomposition: [11.5, 25.5, 25, 11, -1.5, -0.5, 4, -4]	
Second level of decomposition: 18.5, 18, -7, 7, -1.5, -0.5, 4, -4]	
Third and final level of decomposition: [18.25, 0.25, -7, 7, -1.5, -0.5, 4, -4]	
[6 marks: 2 for each row]	
The significance of 18.25 is that it is the mean value of the numbers in the original sequence [1 mark].	
	7 marks

1617 A

Question 2

a) The Fast Fourier Transform (FFT) is a method for reducing the time taken to perform a Discrete Fourier Transform (DFT). Using the given 8-point sequence, describe the process of implementing an FFT and state how the reverse process is carried out. Refer to radix-2 decimation in time.

[1 3 2 7 6 10 9 5]

Answer:

a) A radix-2 sequence is one whose number of elements is a power of 2 [1 mark]. A sequence is split into two [1 mark], one of which is the even numbered elements and one the odd numbered elements [1 mark]. This process is continued till we have individual elements [1 mark].

For example, for the 8-point signal as follows:

[1 3 2 7 6 10 9 5]

This is split into:

[1 2 6 9] and [3 7 10 5]

And so on:

[1 6] [2 9] [3 10] [7 5]

[1] [6] [2] [9] [3] [10] [7] [5] [2 marks]

Re-ordering the sequence in this way can be performed using bit-reversal, as each position is the reverse of the binary value of the original position. Element values are swapped accordingly, eg the value in position 3 (binary 011) [7] is swapped with the value in position 6 (binary 110) [9]

[1 mark].

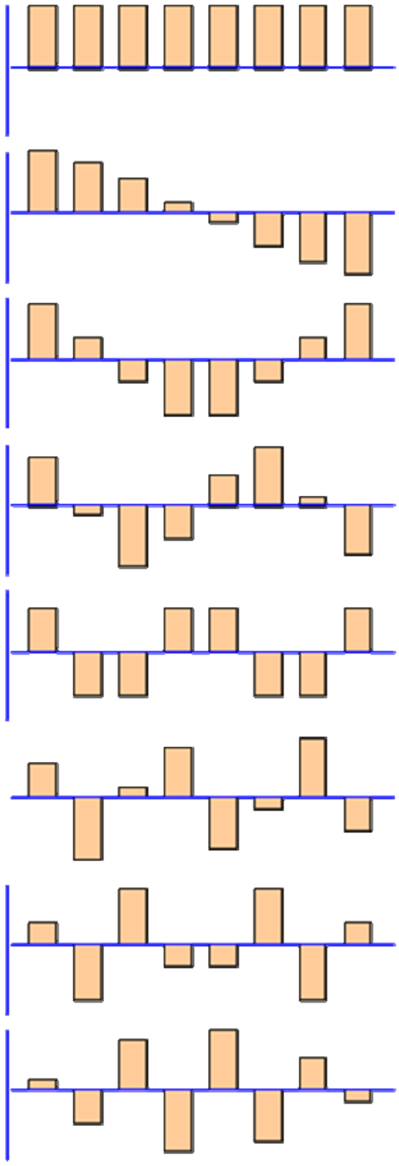
Each 1 point signal is then transformed to the frequency domain, nothing is required to do this step [1 mark].

The individual frequency values are then combined by reversing, in steps, the decomposition process: [1 mark]

In the above case, the 16 individual frequency points are grouped into 8 groups of 2 points each. This is carried out by shifting, adding zeroes in the gaps and then adding [1 mark].

b) The Discrete Cosine Transform (DCT) transforms functions of time (discrete signals) and functions of position (images) into functions of frequency. Explain how it does this.

Answer: The DCT correlates the input data [1 mark] with a series of cosine functions of increasing frequency, starting with dc [1 mark]. These discrete cosine functions are the basis functions. The basis functions for an 8-point DCT are [2 marks]:

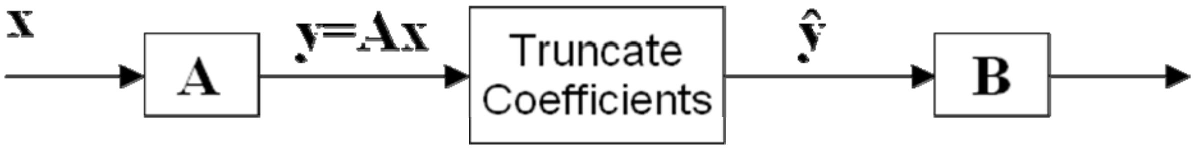


Correlation gives a large value of output when there is a similarity between the input data distribution and a particular basis function [1 mark]. Therefore the output sequence from the DCT is related to the frequency of the distribution of data in the input sequence [1 mark].

c) Karhunen-Loeve Transform (KLT) compression is an example of Linear Transform Coding (LTC). With reference to a suitable diagram, briefly explain the process of LTC and the role of the KLT.

[9 m

Answer:



[2 marks for diagram]

The image or signal to be compressed is divided into N blocks of pixels or samples. [1 mark]. Each block is then an N-dimensional vector, so we have a sequence of vectors [1 mark]. Each vector, x, is then transformed by multiplying by a linear matrix A [1 mark]. We then discard the less significant values from the transformed data [1 mark] and multiply by another matrix B to reconstruct the image or signal, $\hat{x}$ [1 mark]. The KLT minimises the mean squared error, J, between x and $\hat{x}$ by decorrelating the input data [1 mark] and discarding the set with the least variance [1 mark]

Question 4

a) Data Compression is used to reduce the storage capacity needed and to allow higher rates of data transmission.Briefly explain what is meant by

i) Lossless compression

ii) Lossy compression.

[4 marks]

b) Multiresolution Analysis (MRA) is used to separate data into course and fine detail.

Apply the transform defined by

$$x_{n-1},i = (x_{n,2i}+x_{n,2i+1})/2$$

$$d_{n-1},i = (x_{n,2i}-x_{n,2i+1})/2$$

to the sequence

$$[x_{n,i}] = [12, 8, 13, 10, 6, 7, 11, 14]$$

Where i = 0…..7, is the index position in the sequence, and

n is the level. The next level is n-1.

At each level, concatenate the sequencies for x_{n-1},i and d_{n-1},i

Continue till no further levels are possible.

i) Including the given sequence, how many levels are there?

ii) State the significance of the first element in the final level.

iii) Excluding the first element in the final level, what is the significance of the other elements?

iv) Has any information been lost in the process?

v) Comment on how this process could be used to compress the data.

vi) State the equations that can be used to reconstruct the original data from the final level.

vii) Explain the application of a specific Discrete Wavelet Transform to this process of averaging and differencing.

[21 marks]

Answer:

a) i) Lossless compression is where fewer bits are required to store the data but no information is discarded. [1 mark]. This is achieved by eliminating redundancy [1 mark]

ii) Lossy compression results in a loss of information [1 mark]. This is achieved by discarding marginally important information [1 mark].

b) Applying the transform:

n=3 [12, 8, 13, 10, 6, 7, 11, 14]

n=2 [10, 11.5, 6.5, 12.5, 2, 1.5, -0.5, -1.5]

n=1 [10.75, 9.5, -0.75, -3, 2, 1.5, -0.5, -1.5]

n=0 [10.125, 0.625, -0.75, -3, 2, 1.5, -0.5, -1.5] [6 marks: 2 for each row]

i) 4 levels [1 mark]

ii) The first element is the average of all the elements in the original sequence [1 mark].

iii) The remaining 7 elements are the fine detail in the original data. [1 mark]

iv) No information has been lost [1 mark]

v) Because most of the values in the final level are small, potentially fewer bits would be required to store it [1 mark]. Where there are zeroes, they do not need to be stored, although the positions of the other values would need to be stored [1 mark]. Small values could be replaced by zeroes without significant loss of detail [1 mark], this can be done by applying a threshold value, the bigger the threshold the greater the loss of detail [1 mark].

vi) The equations are

$$x_{n,2i} = x_{n-1,i} + d_{n-1,i}$$

$$x_{n,2i+1} = x_{n-1,i} - d_{n-1,i}$$ [4 marks: 2 each equation]

vii) The Haar DWT is used to perform averaging and differencing [1 mark], the coarse values by the Haar scaling function [1 mark] and the fine detail by the Haar Wavelet function [1 mark].

1617B

Question 2

a) Explain in what context the Discrete Cosine Transform (DCT) is suitable for lossy data compression.

[5 marks]

b) Why is the DCT more efficient than the Discrete Fourier Transform (DFT)?

[2 marks]

c) Most image and signal processing is carried out digitally. An image will already be in digital form, but we will need to sample the signal before it can be processed.

With the aid of relevant diagrams, explain the effect in the frequency domain of sampling a signal in the time domain.

[12 marks]

d) With the aid of a diagram, demonstrate the effect of the folding of an aliased signal. For a sampling frequency of 3000 samples/second, what is the apparent frequency resulting from sampling signals of respectively 1000 Hz, 5000 Hz, 7500 Hz and 11,500 Hz?

[6 marks]

Answer:

a) In the DCT, most of the signal information tends to be concentrated in a few low frequency components.[1 mark]. The amount of information that can be allowed to be lost during compression depends on the application [1 mark]. In visual images, much of the high frequency information (rapid spatial changes) can be discarded because the human eye/brain is far less sensitive to them than the lower frequency information [1 mark]. The DCT itself is not lossy [1 mark] but it does separate the frequency components to allow compression to take place [1 mark].

b) Because the DCT is based only on the cosine function, it is the real part of the DFT and complex arithmetic is not required in carrying out the DCT [1 mark]. The FT of of a real and even function is real and even, meaning the DCT is efficient for real data with even symmetry [1 mark].

c) Sampling in the time domain is multiplication of the signal with a comb of Dirac delta functions [1 mark]. This is equivalent to convolution with a comb of delta functions in the frequency domain [1 mark].

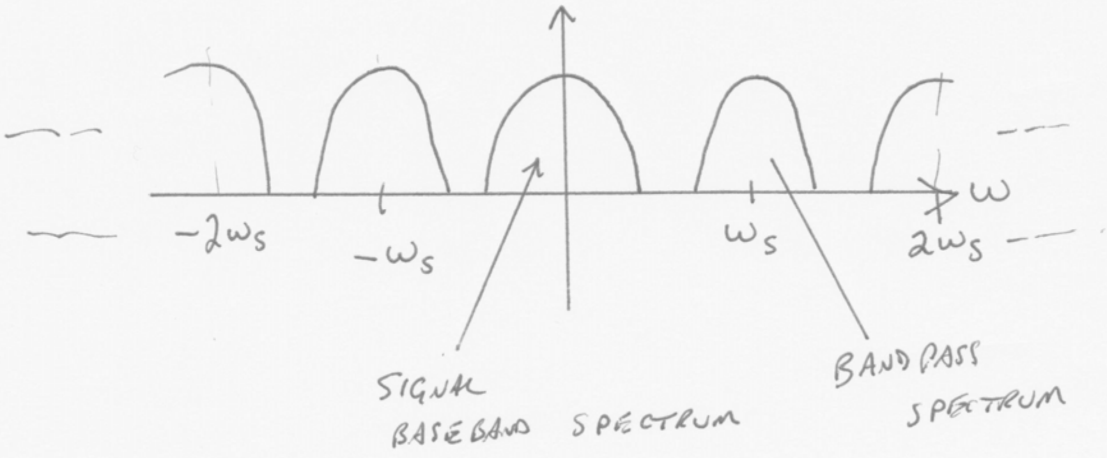
The result of this is that sampling replicates the frequency spectrum of the continuous time signal in the frequency domain at inter multiple of the sampling frequency [1 mark].

There is a gap between spectra if the signal is oversampled [1 mark], spectra are adjacent if the signal is critically sampled (nyquist sampling rate) [1 mark], spectra overlap if the signal is undersampled [1 mark].

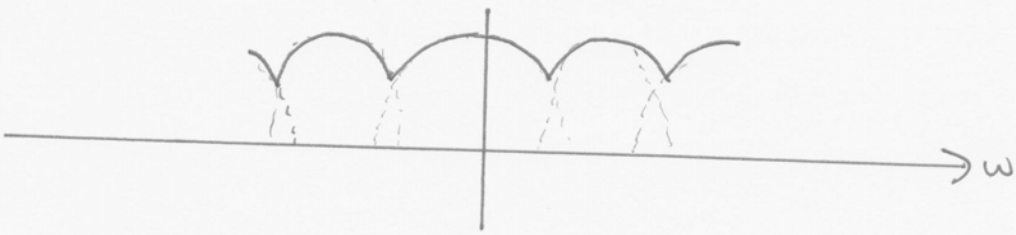
[2 marks: 1 mark for each sketch]

Aliasing occurs when the signal is sampled at less than the Nyquist limit [1 mark]. Aliasing is when an unwanted signal is mistaken for a wanted one [1 mark]. Once aliasing occurs it is not possible to remove it [1 mark].

The baseband spectrum can be obtained from the bandpass spectrum by using a low-pass filter [1 mark].



IF SIGNAL IS UNDERSAMPLED:

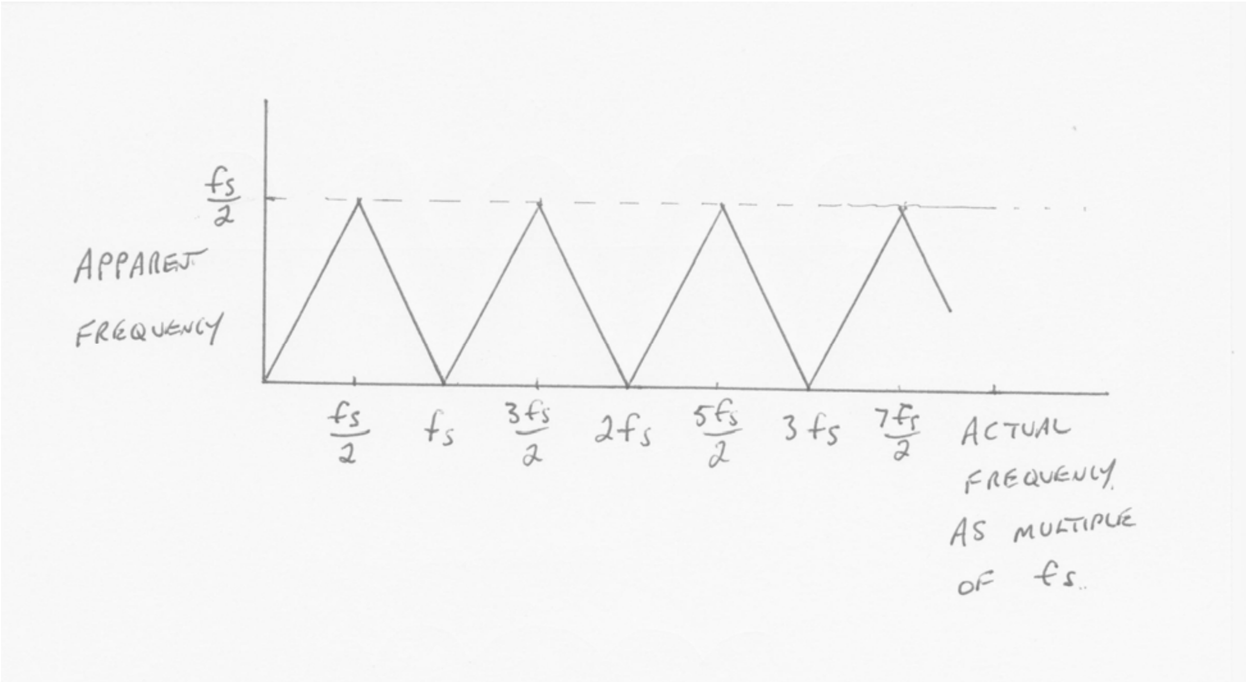


SPECTRA OVERLAP.

RESULTS IN <u>ALIASING</u>

d) The term folding arises because of the mirror effect about multiples of $f_s/2$ (f_s is the sampling frequency) [1 mark].

[1 mark for diagram]



If we sample at 3000 samples/second:

1000Hz appears to be 1000Hz (as it should). [1 mark].

5000Hz appears to be 1000Hz [1 mark]

7500Hz appears to be 1500Hz [1 mark]

11500Hz appears to be 500Hz [1 mark].

Question 4

a) For the Continuous Wavelet Transform (CWT), wavelet functions are of many different forms depending on the application. State and briefly explain the condition required for the wavelet function if the transform is to be reversed.

b) i) When we perform a Discrete Wavelet Transform(DWT), we consider each wavelet to have a wavelet function $\psi(t)$ and a scaling function $\varphi(t)$. The DWT can be carried out using multiresolution analysis (MRA).

State the roles of $\psi(t)$ and $\varphi(t)$ in MRA.

[2 marks]

ii) The simplest wavelet function is the Haar wavelet. Sketch the wavelet and scaling functions for the Haar wavelet.

[2 marks]

iii) A signal can be filtered into its coarse and fine detail (analysed) by using these two functions.

Sketch and explain the block diagram for one stage of a dyadic wavelet transform. How could this process be used for a non-dyadic sequence?

[8 marks]

iv) What condition is required to ensure that the original signal can be synthesised by reversing the process?

[2 marks]

c) Suppose we have a Haar wavelet transform analysis filterbank which uses a low-pass filter $h_0[0] = h_0[1] = \frac{1}{2}$ and a high-pass filter $h_1[0] = \frac{1}{2}$, $h_1[1] = -\frac{1}{2}$. Use the recursive equations:

$$c_{m-1,n} = \sqrt{2} \cdot \frac{1}{2}(c_{m,2n} + c_{m,2n+1})$$
$$= \frac{1}{\sqrt{2}}(c_{m,2n} + c_{m,2n+1})$$

$$d_{m-1,n} = \frac{1}{\sqrt{2}}(c_{m,2n} - c_{m,2n+1})$$

to calculate the Haar wavelet transform for a sampled signal s[n] = [3 2 5 -2] after 1 and 2 stages of the transform filterbank.

[7 marks]

Answer:

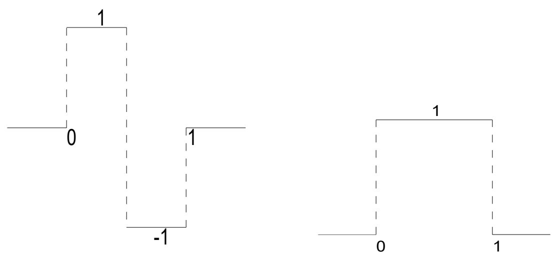
a) The condition is called the Admissibility Condition [1 mark]

The admissibility condition gives that: $$C_\Psi = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$$ [1 mark]

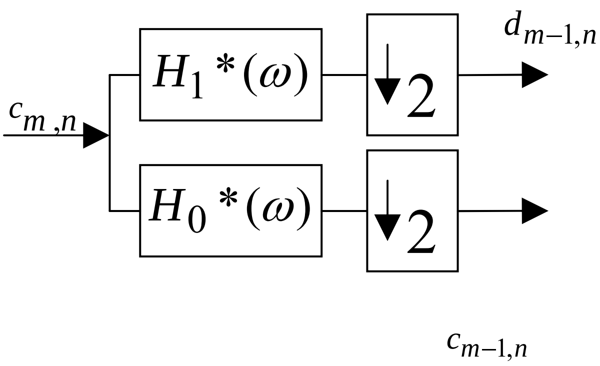
- Square of the Fourier transform must decay faster than 1/*w* (the wavelet must be compact in frequency domain) [1 mark]
- Admissibility implies zero average (ie, the wavelet must be oscillatory) [1 mark]

b) i) The wavelet function gives the fine detail in the signal [1 mark] and the scaling function gives the coarse detail [1 mark].

ii)



The first is the wavelet function ψ(t) and the second is the scaling function φ(t). [2 marks: 1 each] iii)



[2 marks]

Dyadic means that the number of values in the sequence being transformed is a power of 2 [1 mark]. The inputs to this stage are the coarse details from the previous stage [1 mark]. H_0 is a low pass filter to obtain the next level of coarse detail [1 mark]. H_1 is a high pass filter to obtain the next level of fine detail [1 mark]. Downsampling is carried out so that at each level there is half the number of values of each as at the previous stage [1 mark]. If the sequence length was not a power of 2, it can be zero padded to make it so [1 mark].

iv) If the wavelet functions are orthogonal, then the same filters can be used as in the analysis [1 mark]. If the wavelet functions are not orthogonal then filters that are biorthogonal with the analysis filters must be used for synthesis [1 mark].

c) Start with the signal is the finest resolution coefficient,

S[n] = [3 2 5 -2]

First level:

$C_{1,0} = 1/\sqrt{2}(3+2)= 5/\sqrt{2}$

$C_{1,0} = 1/\sqrt{2}(5+(-2)) = 3/\sqrt{2}$

$D_{1,0} = 1/\sqrt{2}(3-2) = 1/\sqrt{2}$

$D_{1,1} = 1/\sqrt{2}(5-(-2)) = 7/\sqrt{2}$

Hence the first level of the wavelet transform is $\frac{1}{\sqrt{2}}[5\ 3\ 1\ 7]$

Second level:

$C_{0,0} = ½(5+3) =4$

$D_{0,0} = ½(5-3) = 1$

Hence the second level of the wavelet transform is [4 1 $1/\sqrt{2}$ $7/\sqrt{2}$]

[7 marks:1 for each calculation and 1 for the final answer]