SOLUTIONS

Module:	Advanced Transform Methods		
Module Code	EBU6018	Paper	A
Time allowed	2hrs	Filename	**Solutions_2122_EBU6018_A**
Rubric	ANSWER ALL FOUR QUESTIONS		
Examiners	Mr Andy Watson	Dr Cunhua PAN	

Question 1

a) A set of 3 mutually orthogonal vectors can be used as a BASIS for R^3.

$$\text{Matrix } A = \begin{bmatrix} 2 & -2 & x \\ -3 & -1 & y \\ 1 & 1 & z \end{bmatrix}$$

i) Find the values of x, y and z so that the columns of A form an orthogonal basis for R^3.

[9 marks]

ii) If vector $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find c_1, c_2 and c_3 so that $u = c_1v_1 + c_2v_2 + c_3v_3$. [7 marks]

Answer:

i) Put $v_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ [1 mark]

Then $<v_1, v_3> = [2 \quad -3 \quad 1]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2x - 3y + z$ [2 marks]

And $<v_2, v_3> = [-2 \quad -1 \quad 1]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2x - y + z$ [2 marks]

Combining these: z=2y [1 mark]

Substitute for z then gives y=2x [1 mark]

Put y = 1, then z = 2 and x = 0.5 [1 mark]

[This is an arbitrary choice, any other values would give scalar multiples of this one]

So, $v_3 = \begin{bmatrix} 0.5 \\ 1 \\ 2 \end{bmatrix}$ [1 mark]

Confirming orthogonality gives $<v_1, v_3> = 0$, $<v_2, v_3> = 0$, $<v_1, v_2> = 0$.

ii)

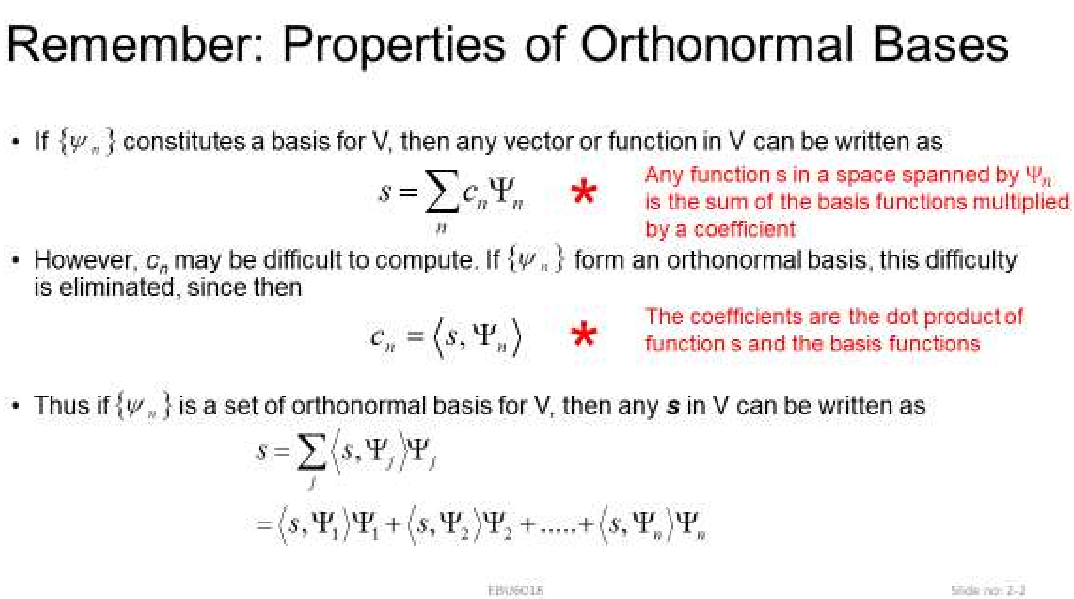
$c_1 = uv_1 = [1 \quad 2 \quad 3]\begin{vmatrix} 2 \\ -3 \\ 1 \end{vmatrix} = -1$ [2 marks]

$c_2 = uv_2 = [1 \quad 2 \quad 3]\begin{vmatrix} -2 \\ -1 \\ 1 \end{vmatrix} = -1$ [2 marks]

$c_3 = uv_3 = [1 \quad 2 \quad 3]\begin{vmatrix} 0.5 \\ 1 \\ 2 \end{vmatrix} = 8.5$ [2 marks]

[NOTE: the following row is incomplete because the basis vectors should be normalised]

so, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -1\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} - 1\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + 8.5\begin{bmatrix} 0.5 \\ 1 \\ 2 \end{bmatrix}$ [1 mark]



b) The Fast Fourier Transform (FFT) is a method for reducing the time taken to perform a Discrete Fourier Transform (DFT). Using the given 8-point sequence, describe the process of implementing an FFT. Refer to radix-2 decimation in time.

[3, -4, 8, -2, -7, 9, 13, 6]

[9 marks]

Answer:

b) A radix-2 sequence is one whose number of elements is a power of 2 [1 mark]. A sequence is split into two [1 mark], one of which is the even numbered elements and one the odd numbered elements [1 mark]. This process is continued till we have individual elements [1 mark].
For example, for the 8-point signal as follows:
[3, -4, 8, -2, -7, 9, 13, 6]
This is split into:
[3 8 -7 13] and [-4 -2 9 6]
And so on:
[3 -7] [8 13] [-4 9] [-2 6]
[3] [-7] [8] [13] [-4] [9] [-2] [6] [3 marks, 1 for each row]
Re-ordering the sequence in this way can be performed using bit-reversal, as each position is the reverse of the binary value of the original position. Element values are swapped accordingly, eg the value in position 3 (binary 011) [-2] is swapped with the value in position 6 (binary 110) [13] [1 mark].
Each 1 point signal is then transformed to the frequency domain, nothing is required to do this step [1 mark].

Question 2

a) The Discrete Fourier Transform can be defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{-nk}$$

where $W_N = e^{\frac{j2\pi}{N}}$

An N-point DFT can be written as X = Wx

where x is the N-point input sequence of samples of a continuous signal, W is the N-by-N DFT matrix and X is the DFT of the signal

Show that the normalised 4x4 Fourier Matrix is:

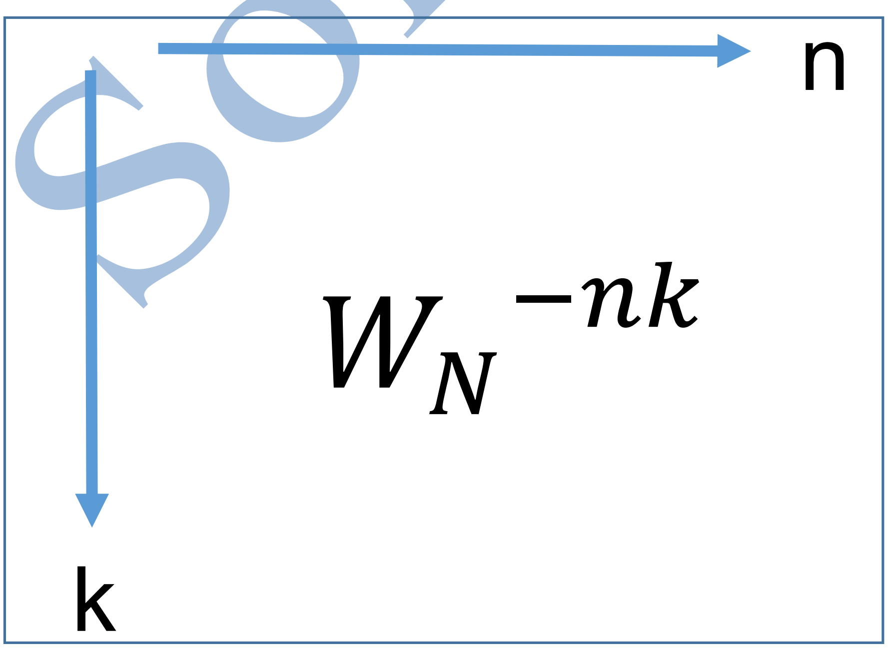
$$F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

[14 marks]

Hence, calculate the DFT of the 4-point input sequence x= [3, 0, 4, 2]. [4 marks]

Answer:

a) We can produce an N-by-N Fourier Matrix where n are input samples and k are output frequencies (with k=n):



[2 marks]

Call the N-by-N Fourier Matrix F_n (N = 0…..(n-1)):

$$F_n = \begin{bmatrix} 1 & 1 & 1 \dots\dots & 1 \\ 1 & \omega & \omega^2 \dots . & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 \dots . & \omega^{2(n-1)} \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{(n-1)^2} \end{bmatrix}$$ [4 marks, 1 mark each row]

Now, consider $\omega = e^{\frac{j2\pi}{N}} = [\cos\left(\frac{2\pi}{N}\right) + j\sin(\frac{2\pi}{N})]$ $so\ \omega^N = e^{j2\pi} = 1$ [2 marks]

For N = 4 (N = 0…..3), $\omega = e^{j2\pi/4}$ = i and ω^{-1} = -I [2 marks]

So the normalised 4x4 Fourier Matrix is:

$$F_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

[4 marks, 1 mark for each row]

Each row corresponds to an increasing frequency.

For x= [3, 0, 4, 2],

$$\text{DFT} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4.5 \\ -0.5 + i \\ 2.5 \\ -0.5 - i \end{bmatrix}$$ [4 marks, 1 mark for each row]

b) i) Explain the limitation of the Fourier Transform in the context of non-stationary signals.
[2 marks]

ii) Explain how a Short-Time Fourier Transform overcomes the limitation of the Fourier Transform and state two limitations of the Short-Time Fourier Transform.
[5 marks]

Answer:

b) i) The Fourier Transform gives the frequencies in the signal [1 mark], but in a non-stationary signal the frequency content changes with time so the Fourier Transform "loses" that time dependent information [1 mark]

ii) The Short-Time Fourier Transform uses a short moving window to "isolate" short segments of the signal [1 mark] by multiplying the window function and the FT complex exponential [1 mark] This then gives the Fourier Transform within the window and so gives the distribution of frequencies with time [1 mark]..

Two limitations of the STFT are: The basis functions are not orthogonal [1 mark] and the finite width of the window results in uncertainty in the location of the frequencies in time [1 mark].

Question 3

This question is about the Karhunen-Loeve Transform (KLT).

a) List the steps in performing a KLT. [13 marks]

b) Table Q3 b) shows a 2D data set which contains a sample of a larger set.

x	y
2.5	2.4
0.5	0.7
2.2	2.9
1.9	2.2
3.1	3.0
2.3	2.7
2.0	1.6
1.0	1.1
1.5	1.6
1.1	0.9

Table Q3 b)

Show that the covariance matrix is (to 4 decimal places):

$$cov_{x,y} = \begin{bmatrix} 0.6166 & 0.6154 \\ 0.6154 & 0.7166 \end{bmatrix}$$ [12 marks]

NOTE: you must show all your working.

Answer:

a)

1. Find the mean vector for the input data

$$E(\vec{x}) = \frac{1}{N}\sum_{i=0}^{N-1}\vec{x}_i$$

The mean vector is a Dx1 vector [3 marks: 1 for each item]

2. Find the covariance matrix

$$\mathbf{R}_{\mathbf{XX}} = \frac{1}{N-1}\sum_{i=0}^{N-1}(\vec{x}_i - E(\vec{x}))(\vec{x}_i - E(\vec{x}))^T$$

The covariance matrix is DxD [3 marks: 1 for each item]

3. Find eigenvalues of the covariance matrix

$$\left|\mathbf{R}_{\mathbf{XX}} - \lambda\mathbf{I}\right| = 0$$

There are D eigenvalues. Each one is a scalar [3 marks: 1 for each item]

4. Find eigenvectors of the covariance matrix

$$\left[\mathbf{R}_{\mathbf{XX}} - \lambda_i\mathbf{I}\right]\vec{\phi}_i = 0$$

[2 marks: 1 for each item]

5. Normalise the eigenvectors [1 mark]

$$\left\langle \vec{\phi}_i, \vec{\phi}_i \right\rangle = 1$$

6. Transform the input [1 mark]

$$\mathbf{Y} = \phi^T\mathbf{X}$$

b) Mean value of x = 1.81
Mean value of y = 1.91 [2 marks, 1 mark each]

Subtract the mean of each column:

x	y
0.69	0.49
-1.31	-1.21
0.39	0.99
0.09	0.29
1.29	1.09
0.49	0.79
0.19	-0.31
-0.81	-0.81
-0.31	-0.31
-0.71	-1.01

[2 marks]

Calculate the variance in x: (working must be shown)

$$var_x = \frac{1}{N-1}\sum_0^{N-1}(x_i - x_{mean})^2 = 0.6166$$ [2 marks]

Calculate the variance in y: (working must be shown)

$$var_y = \frac{1}{N-1}\sum_0^{N-1}(y_i - y_{mean})^2 = 0.7166$$ [2 marks]

Calculate the covariance x,y: (working must be shown)

$$cov_{x,y} = \frac{1}{N-1}\sum_0^{N-1}(x_i - x_{mean})(y_i - y_{mean}) = 0.6154$$ [2 marks]

So,

$$cov_{x,y} = \begin{bmatrix} 0.6166 & 0.6154 \\ 0.6154 & 0.7166 \end{bmatrix}$$ [2 marks]

Question 4

a) In the wavelet transform, the scaling function coefficients $c_{m,n}$ and wavelet series coefficients $d_{m,n}$ can be calculated recursively according to the following equations:

$$c_{m-1,n} = \sqrt{2}\sum_i h_0[i-2n]c_{m,i}$$
$$d_{m-1,n} = \sqrt{2}\sum_i h_1[i-2n]c_{m,i}$$

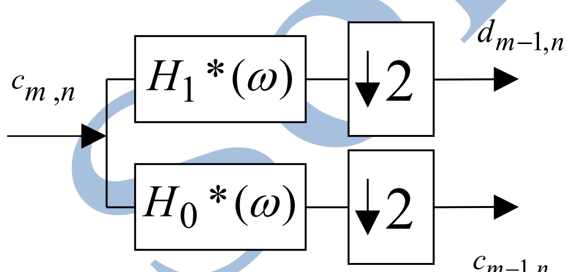
Explain how this can be interpreted in terms of filtering and downsampling, and hence leads to the concept of an *analysis filterbank*. Sketch a diagram to illustrate this filterbank. [6 marks]

Answer:

a) If $c_{m,i}$ are the scaling function coefficients at level m, we can calculate the scaling function coefficients and wavelet series coefficients $d_{m-1,n}$ at level m-1 recursively from these.
[2 marks]

The signals h_0 and h_1 are (time-reversed) low-pass and high-pass filters. [1 mark]
Steps of 1 in n correspond to steps of 2 in i, corresponding to downsampling by a factor of 2 from n to i. [1 mark]

Therefore these equations represent filtering followed by downsampling, as shown in the diagram:



[2 marks for diagram]

b) Suppose we have a Haar wavelet transform analysis filterbank which uses a low-pass filter

$h_0[0] = h_0[1] = \frac{1}{2}$ and a high-pass filter $h_1[0] = \frac{1}{2}$, $h_1[1] = -\frac{1}{2}$.

Use the recursive equations:

$$\begin{aligned} c_{m-1,n} &= \sqrt{2} \cdot \frac{1}{2}(c_{m,2n} + c_{m,2n+1}) \\ &= \frac{1}{\sqrt{2}}(c_{m,2n} + c_{m,2n+1}) \\ d_{m-1,n} &= \frac{1}{\sqrt{2}}(c_{m,2n} - c_{m,2n+1}) \end{aligned}$$

to calculate the Haar wavelet transform for a sampled signal s[n] = [4, 7, 6, -2] after 1 and 2 stages of the transform filterbank. [7 marks]

Answer:

b) Start with the signal in the finest resolution coefficient,
S[n] = [4, 7, 6, -2]
First level:
$C_{1,0}$ = 1/√2(4+7)= 11/√2
$C_{1,1}$ = 1/√2(6-2) = 4/√2
$D_{1,0}$ = 1/√2(4-7) = -3/√2
$D_{1,1}$ = 1/√2(6+2) = 8/√2

Hence the first level of the wavelet transform is $\frac{1}{\sqrt{2}}$[11, 4, -3, 8]

Second level:
$C_{0,0}$ = ½(11+4) =15/2
$D_{0,0}$ = ½(11-4) = 7/2

Hence the second level of the wavelet transform is [15/2, 7/2, -3/√2, 8/√2]
[7 marks:1 for each calculation and 1 for the final answer]

c) With reference to the general filterbank block diagram shown in Figure xx, explain what is meant by the term "Perfect Reconstruction" and how this helps to design the filters.

[6 marks]

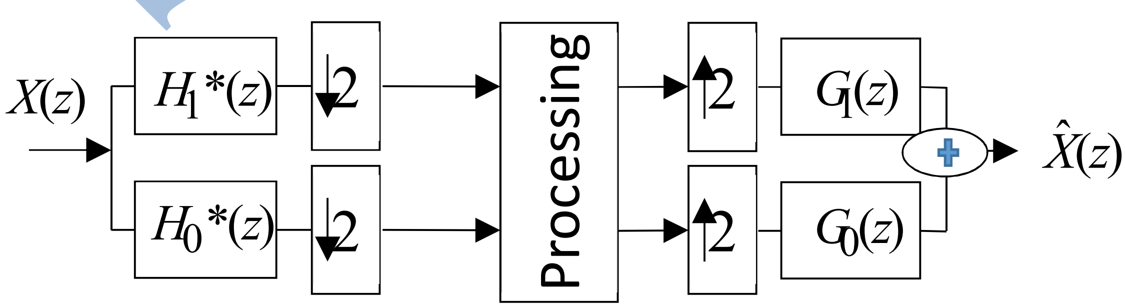


Figure Q4 c)

Answer:
c)

H and G are the analysis filters and the synthesis filters respectively. [1mark]

It is then possible to write $\hat{X}(z)$ in terms of the filters and $X(z)$ [1mark]

If we assume no processing, then perfect reconstruction means that $\hat{X}(z) = X(z)$ [1 mark]

However, there will be a pure delay between them [1 mark] and that allows us to find a relationship between the analysis filters and the synthesis filters [1 mark] and also between the analysis high pass filter and low pass filter. So if we know the coefficients of the analysis low pass filter then the other 3 filters can be determined. [1 mark]

d) Filter banks are used to implement wavelet transforms.

In the diagram of Q4c), the analysis low pass filter is referred to as H_0 and the high pass filter as H_1. **T**he synthesis low pass filter is referred to as G_0 and the high pass filter as G_1

For orthogonal analysis filters:

$$H_1(z) = (-z)^{-N} H_0(-z^{-1})$$

And for the synthesis filters

$$G_0(z) = H_1(-z)$$

$$G_1(z) = -H_0(-z)$$

Daubechies wavelets are orthogonal. For a Daubechies 2nd order wavelet,

$H_0[n]$ is defined by the sequence [0.483, 0.837, 0.224, -0.129].

Determine $H_1[n]$, $G_0[n]$ and $G_1[n]$. [6 marks]

Answer:

$H_o[n]=[H_0[0], H_0[1], H_0[2], H_0[3]]$

$H_1[n]=[H_0[3], - H_0[2], H_0[1], -H_0[0]] = [0.129, -0.224, 0.837, -0.483]$ [2 marks]

$G_0[n]=[H_0[3], H_0[2], H_0[1], H_0[0]] = [-0.129, 0.224, 0.837, 0.483]$ [2 marks]

$G_1[n]=[-H_0[0], H_0[1], -H_0[2], H_0[3]] = [-0.483, 0.837, -0.224, -0.129]$ [2 marks]