Queen Mary
University of London

EBU6018
Advanced Transform Methods

Week 3.3 – Wavelet Transform From Filterbanks

Dr Yixuan Zou

Lecture Outline

- Filterbanks
  - MRA
  - Downsampling
  - Signal Recovery

Note

The full derivation and proof of the following recursive equations for calculating the coefficients of the approximation and fine detail of the resulting filter bank are given in the book “Introduction to Time-frequency and Wavelet Transforms” by Shie Qian, Chapter 5.

The detailed proof is NOT required for this module, what is important is the fact that wavelet transforms can be applied using low pass and high pass filters.

These filters are based on the scaling (and wavelet) functions of the wavelet being used.

Wavelet Transform from Filter Banks

From Multiresolution Analysis (MRA),

if signal $s(t)$ is in V_m for fininte m, then $s(t)$ determined by

$$s(t) = \sum_{n=-\infty}^{\infty} c_{m,n}\phi_{m,n}(t) \quad \left[\sum_k c_k^j \phi(2^j t - k) \quad \text{in MRA lecture} \right]$$

Since $V_m = V_{m-1} \oplus W_{m-1}$ this can be written

$$s(t) = \sum_n c_{m_0,n}\phi_{m_0,n}(t) + \sum_{k=m_0}^{m-1} \sum_n d_{k,n}\psi_{k,n}(t) \qquad m > m_0$$

where coefficients $d_{m,n}$ and $c_{m,n}$ are

inner products between $\psi_{m,n}(t)$ and $\phi_{m,n}(t)$ respectively.

Queen Mary University of London

Approximating the Signal

Using Parseval, we have

$$c_{m,n} = 2^{m/2}\int_{-\infty}^{\infty} s(t)\phi^*(2^m t - n)dt$$

$$= \frac{1}{2\pi} 2^{-m/2}\int_{-\infty}^{\infty} S(\omega)\Phi^*(2^{-m}\omega)e^{-j2^{-m}\omega n}d\omega$$

For large scale m and $\Phi(0)=1$ (i.e. $\phi(t)$ normalised), have

$$c_{m,n} \approx \frac{1}{2\pi} 2^{-m/2}\int_{-\infty}^{\infty} S(\omega)e^{-j2^{-m}\omega n}d\omega = 2^{-m/2}s(2^{-m}n)$$

(since $\Phi(\omega/2^m) \approx \Phi(0) = 1$ over the range where $S(\omega)$ exists).

Thus $c_{m,n}$ approximates

$s(t)$ at $t = 2^{-m}n$ with a scaling factor of $2^{-m/2}$.

Recursive Computation of Coeffs

Dilation equation $\phi(t) = \sum_k p_k \phi(at - k)$ for integer k

From MRA, if φ(t/2) ∈ V_{-1} then φ(t/2) ∈ V_0

Let us define

$$c_{m,n} \equiv s[n] \equiv s(t)\big|_{t=2^{-m}n} \quad \text{for large } m.$$

And define a filter h_0[n]
Such that we get the
refinement equation:

Using the dilation equation $\phi(t/2) = 2\sum_n h_0[n]\phi(t-n)$ we get

$$c_{m-1,n} = \int_{-\infty}^{\infty} s(t)\phi^*_{m-1,n}(t)dt = 2^{(m-1)/2}\int_{-\infty}^{\infty} s(t)\phi^*\left(\frac{2^m t - 2n}{2}\right)dt$$

$$= 2^{(m-1)/2}\int_{-\infty}^{\infty} s(t)2\sum_i h_0[i]\phi^*(2^m t - 2n - i)dt$$

$$= \sqrt{2}\sum_i h_0[i]\int_{-\infty}^{\infty} s(t)\phi^*_{m,2n+i}(t)dt = \sqrt{2}\sum_i h_0[i]c_{m,2n+i}$$

i.e. $$c_{m-1,n} = \sqrt{2}\sum h_0[i-2n]c_{m,i} \quad ***$$

Filtering and Downsampling

Given this eqn $c_{m-1,n} = \sqrt{2}\sum_i h_0[i-2n]c_{m,i}$

once $c_{m,n}$ is known, we can compute $c_{k,n}$ for $k < m$,

using a low pass filter $H_0*(\omega)$ and downsampling $2n = i$.

$$c_{m,n} \longrightarrow \boxed{H_0*(\omega)} \longrightarrow \boxed{\downarrow 2} \longrightarrow c_{m-1,n}$$

Low-pass filter for approximations

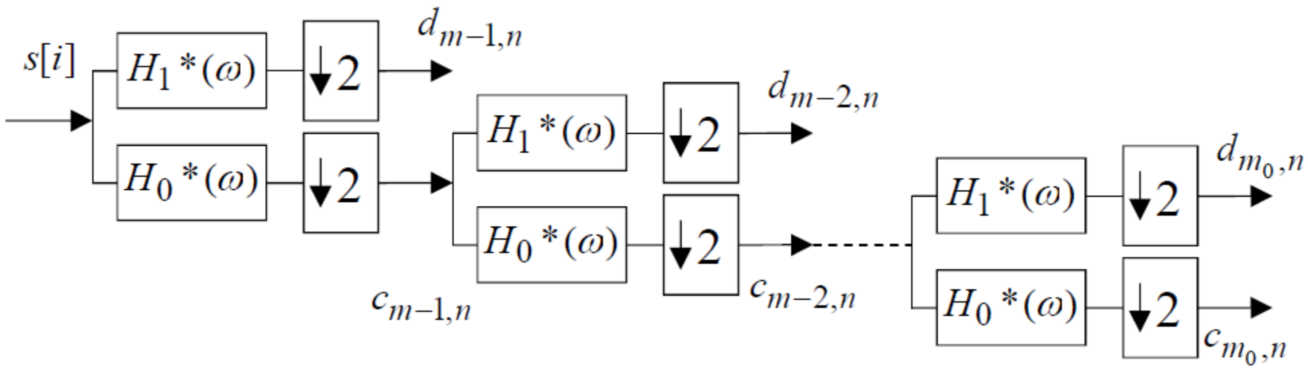
Similarly, we can show $d_{m-1,n} = \sqrt{2}\sum_i h_1[i-2n]c_{m,i}$ ***

i.e. a high pass filter $H_1*(\omega)$ and downsampling.

$$c_{m,n} \longrightarrow \boxed{H_1*(\omega)} \longrightarrow \boxed{\downarrow 2} \longrightarrow d_{m-1,n}$$

High-pass filter for fine detail

Filter Bank for Wavelet Series Coeffs

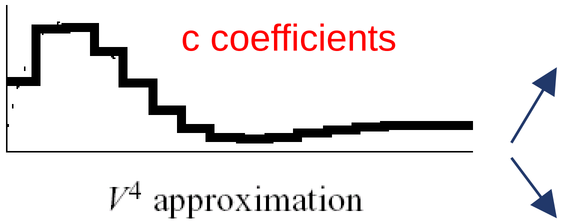


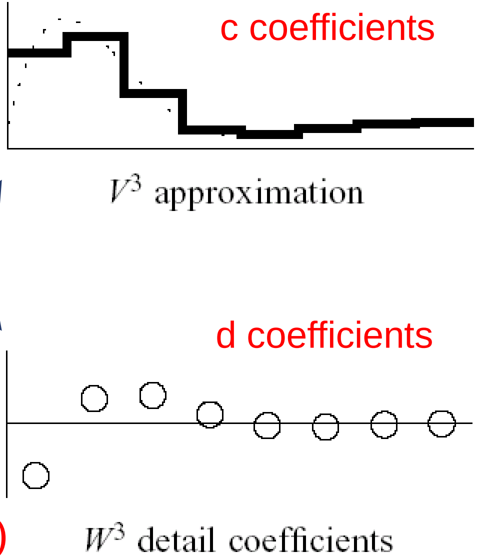
So for discrete-time samples $s[i]$, can compute wavelet transform directly by applying filter banks. No need to compute the mother wavelet $\psi(t)$.

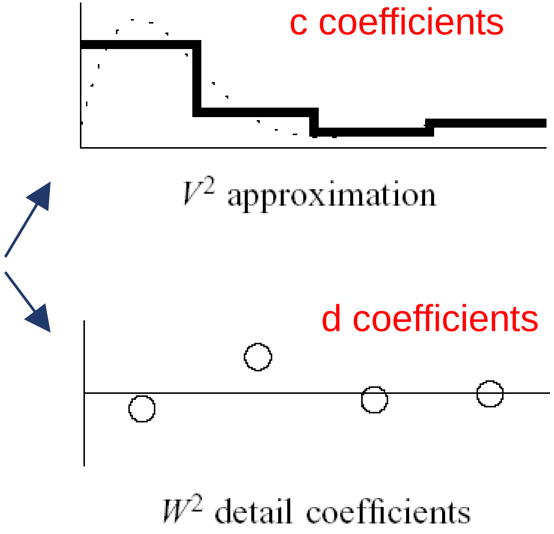
Piecewise Approximation

Basic Concept: Decompose a fine-resolution signal into

1. A coarse-resolution version of the signal, and
2. The differences left over.







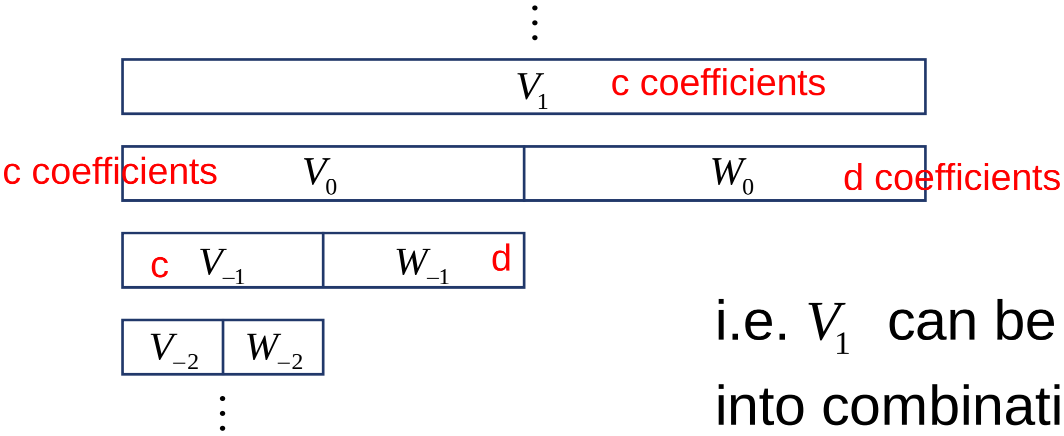
A sampled signal in V space could be constructed from a series of scaled and shifted scaling functions $\varphi(t)$

Multiresolution Analysis (MRA)

Subspaces $\{V_j\}$ are nested

while subspaces $\{W_j\}$ are mutually orthogonal. Consequently,

$$\begin{cases} V_j \cap V_l = V_j & l > j; \\ W_j \cap W_l = \{0\} & j \neq l; \\ V_j \cap W_l = \{0\} & j \leq l. \end{cases}$$



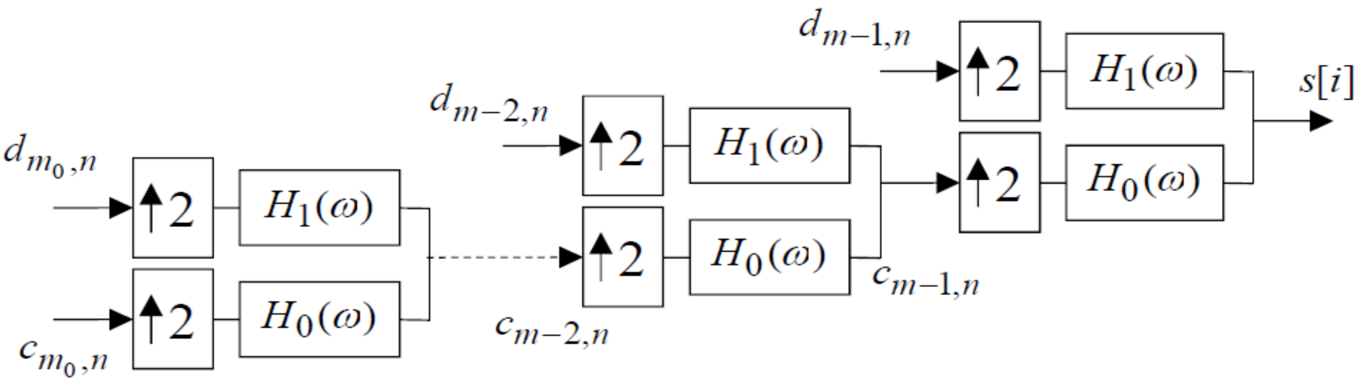
i.e. V_1 can be decomposed

into combination of V_0 and W_0, etc.

Signal Recovery Filterbank

Can also compute high-res coeffs from low-res coeffs:

$$c_{m,n} = \sqrt{2}\left(\sum_i h_0[n-2i]c_{m-1,i} + \sum_i h_1[n-2i]d_{m-1,i}\right)$$



(For proof, see e.g. Qian)

So – don't need scaling functions or wavelets, just filter banks!

Filterbank – Example 1

Suppose we have a wavelet transform analysis filterbank which uses a low-pass filter

$$h_0[0] = h_0[1] = \tfrac{1}{2}$$

And a high pass filter

$$h_1[0] = \tfrac{1}{2} \quad h_1[1] = -\tfrac{1}{2}$$

Use the recursive equations:

$\frac{1}{2}$ from filter

$\sqrt{2}$ from recursive equation

$$c_{m-1,n} = \sqrt{2} \cdot \frac{1}{2}(c_{m,2n} + c_{m,2n+1})$$

$$= \frac{1}{\sqrt{2}}(c_{m,2n} + c_{m,2n+1})$$

$\frac{1}{\sqrt{2}}$ normalises the Haar Transform

$$d_{m-1,n} = \frac{1}{\sqrt{2}}(c_{m,2n} - c_{m,2n+1})$$

to calculate the Haar wavelet transform for a sampled signal s[n] = [2, 5, -3, 7] after 1 and 2 stages of the transform filterbank.

Filterbank – Haar Wavelet

Remember for the normalised Haar Functions:

$$\varphi = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}$$

$$\psi = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}$$

Using Recursive Equation: $\sqrt{2}(\ldots\ldots\ldots)$

So call the Haar Filters: $h_0 = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}$, $h_1 = \begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\end{bmatrix}$

Filterbank – Example 1 Solutions

Start with the signal in the finest resolution coefficient,

$S[n] = [2, 5, -3, 7]$

First level:

$C_{1,0} = 1/\sqrt{2}(2+5) = 7/\sqrt{2}$

$C_{1,1} = 1/\sqrt{2}(-3+7) = 4/\sqrt{2}$

$D_{1,0} = 1/\sqrt{2}(2-5) = -3/\sqrt{2}$

$D_{1,1} = 1/\sqrt{2}(-3-7) = -10/\sqrt{2}$

Hence the first level of the wavelet transform is

$\frac{1}{\sqrt{2}}[7, 4, -3, -10]$

Second level:

$\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$

$C_{0,0} = ½(7+4) = 11/2$

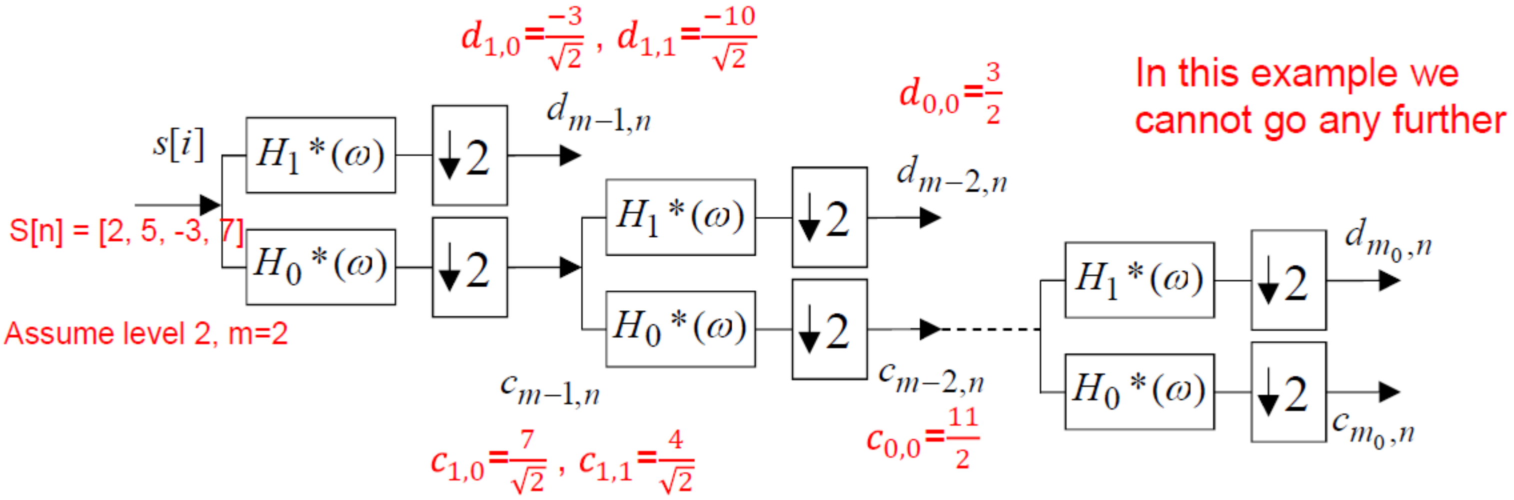
$D_{0,0} = ½(7-4) = 3/2$

Hence the second level of the wavelet transform is

$[11/2, 3/2, -3/\sqrt{2}, -10/\sqrt{2}]$

$[c_{0,0}, d_{0,0}, d_{1,0}, d_{1,1}]$

Filterbank – Wavelet Series Coeffs



So for discrete - time samples $s[i]$, can compute wavelet transform directly by applying filter banks. No need to compute the mother wavelet $\psi(t)$.

Filterbank – Example 2

A Haar wavelet transform is implemented using an analysis filterbank using normalised low-pass and high-pass filters:

$$h_0 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \quad h_1 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$

In this example we have the obtained the normalised Haar Functions by combining the $\frac{1}{2}$ of the previous question filter with the $\sqrt{2}$ of the recursive equation

Calculate the Haar Transform for 2 levels of decomposition, and then invert the transform, for the input sequence:

s[n] = [1, 2, 3, 4]

Filterbank – Example 2 Solutions

$$s[n] = [1\ \ 2\ \ 3\ \ 4]$$

Going down to 1st level:

Multiply sum by $\frac{1}{\sqrt{2}}$

$$\left[\frac{1+2}{\sqrt{2}}\quad \frac{3+4}{\sqrt{2}}\quad \frac{1-2}{\sqrt{2}}\quad \frac{3-4}{\sqrt{2}}\right]$$

Multiply difference by $\frac{1}{\sqrt{2}}$

$$\left[\frac{3}{\sqrt{2}}\quad \frac{7}{\sqrt{2}}\quad \frac{-1}{\sqrt{2}}\quad \frac{-1}{\sqrt{2}}\right]$$

Going down to 2nd level:

$$\left[\frac{3+7}{\sqrt{2}\sqrt{2}}\quad \frac{3-7}{\sqrt{2}\sqrt{2}}\quad \frac{-1}{\sqrt{2}}\quad \frac{-1}{\sqrt{2}}\right]$$

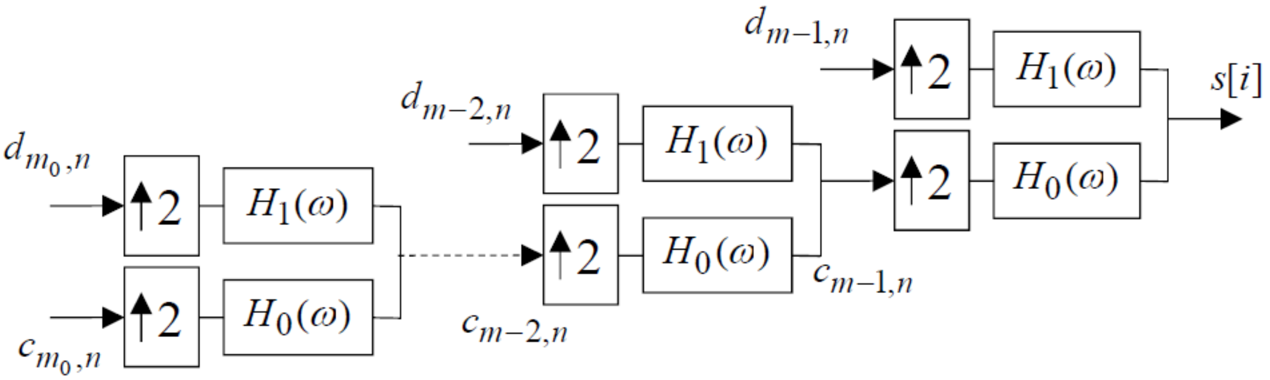
$$\left[\frac{10}{\sqrt{2}\sqrt{2}}\quad \frac{-4}{\sqrt{2}\sqrt{2}}\quad \frac{-1}{\sqrt{2}}\quad \frac{-1}{\sqrt{2}}\right]$$

Now invert:

Filterbank – Signal Recovery

Can also compute high - res coeffs from low - res coeffs :

$$c_{m,n} = \sqrt{2}\left(\sum_i h_0[n-2i]c_{m-1,i} + \sum_i h_1[n-2i]d_{m-1,i}\right)$$



(For proof, see e.g. Qian)

So – don't need scaling functions or wavelets, just filter banks!

Filterbank – Signal Recovery

Going up one level: $\left[\frac{10-4}{\sqrt{2}\sqrt{2}\sqrt{2}} \quad \frac{10+4}{\sqrt{2}\sqrt{2}\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}}\right]$

$$\left[\frac{3}{\sqrt{2}} \quad \frac{7}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}}\right]$$

Going up another level:

$$\left[\left(\frac{3}{\sqrt{2}\sqrt{2}}-\frac{1}{\sqrt{2}\sqrt{2}}\right) \quad \left(\frac{3}{\sqrt{2}\sqrt{2}}+\frac{1}{\sqrt{2}\sqrt{2}}\right) \quad \left(\frac{7}{\sqrt{2}\sqrt{2}}-\frac{1}{\sqrt{2}\sqrt{2}}\right) \quad \left(\frac{7}{\sqrt{2}\sqrt{2}}+\frac{1}{\sqrt{2}\sqrt{2}}\right)\right]$$

$$=\left[\frac{2}{2} \quad \frac{4}{2} \quad \frac{6}{2} \quad \frac{8}{2}\right]$$

$$=[1 \quad 2 \quad 3 \quad 4]$$

If we want to compress the data, we could make $d_{0,0}$= 0, for example, then invert.

Queen Mary
University of London