

§ 2.9 Boundary Conditions for Electrostatics



➤ The Question

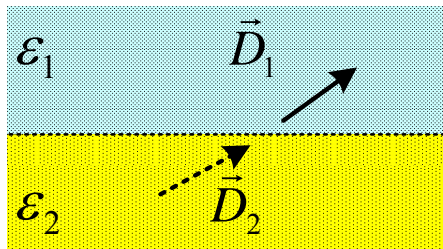
- In general, E-intensity or potential may be obtained through two equations:
 - ✦ In source free region --- Laplace's Equation
 - ✦ Otherwise --- Poisson's Equation
- **Boundary conditions are useful to**
 - ✦ determine the *undecided constants* in the solution;
 - ✦ turn the general solution into *specific solution*.
- **Boundary Value Problem:** solving partial differential equations under given boundary conditions.
 - ✦ Poisson's Equation subjected to boundary conditions
 - ✦ Laplace's Equation subjected to boundary conditions

What is Boundary Conditions?

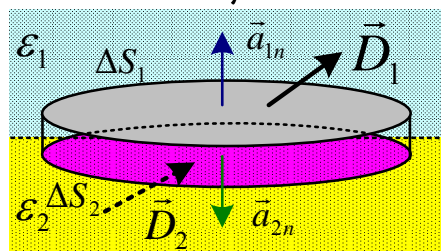


- **Equations governing the behavior of fields at the boundary (interface) between 2 mediums.**
- **Categories of Boundary Conditions:**
 - First: the field potential at boundary is given
 - ✦ *Dirichlet Problems* 狄利赫利问题
 - Second: the normal component of the derivative of the field potential at boundary is known
 - ✦ *Neumann Problems* 纽曼问题
 - Third: *Hybrid Problems* 混合问题

1. Boundary Conditions in Normal Direction



Construct an auxiliary closed surface of a very flat box.



Applying Gauss's Law

$$\oint_S \vec{D} \cdot d\vec{S} = \rho_s \cdot \Delta S$$

$$\vec{D}_1 \cdot \Delta \vec{S}_1 + \vec{D}_2 \cdot \Delta \vec{S}_2 = D_{1n} \Delta S - D_{2n} \Delta S = \rho_s \Delta S$$

$$\therefore D_{1n} - D_{2n} = \rho_s$$

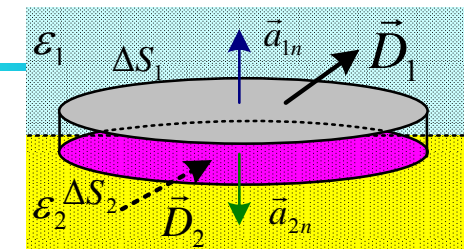
---surface density of free charges

$$D_{1n} - D_{2n} = \rho_s$$

$$\therefore \vec{E} = -\nabla \psi$$

$$\therefore \vec{D} \cdot \vec{a}_n = (\epsilon \vec{E}) \cdot \vec{a}_n = \epsilon (-\nabla \psi) \cdot \vec{a}_n = -\epsilon \cdot \frac{\partial \psi}{\partial n}$$

$$\therefore \epsilon_2 \cdot \frac{\partial \psi_2}{\partial n} - \epsilon_1 \cdot \frac{\partial \psi_1}{\partial n} = \rho_s$$



Discussions

$$D_{1n} - D_{2n} = \rho_s$$

$$\epsilon_2 \cdot \frac{\partial \psi_2}{\partial n} - \epsilon_1 \cdot \frac{\partial \psi_1}{\partial n} = \rho_s$$

➔ If there is no free charge at the boundary,

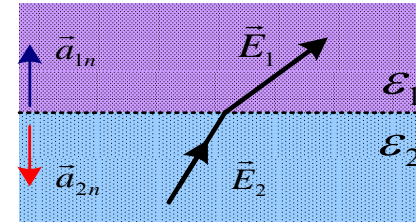
$$D_{1n} = D_{2n}$$

$$\epsilon_1 \cdot \frac{\partial \psi_1}{\partial n} = \epsilon_2 \cdot \frac{\partial \psi_2}{\partial n}$$

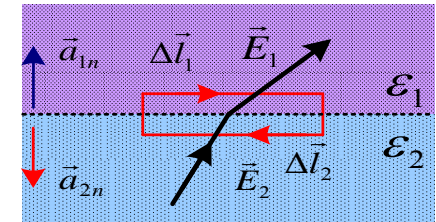
➔ If the boundary is the interface of a conductor $D_{2n} = 0$

$$D_{1n} = \rho_s$$

2. Boundary Conditions in Tangential Direction



Make an auxiliary closed path in rectangular shape, $\Delta h \rightarrow 0$



Applying the conservative law

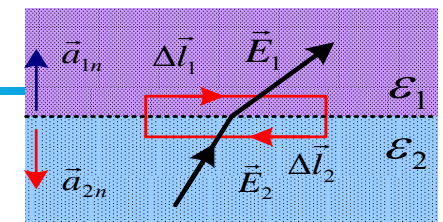
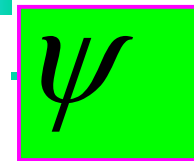
$$\oint_c \vec{E} \cdot d\vec{l} = 0 \quad \therefore \vec{E}_1 \cdot \Delta \vec{l}_1 + \vec{E}_2 \cdot \Delta \vec{l}_2 = E_{1t} \cdot \Delta l - E_{2t} \cdot \Delta l = 0$$

$$\therefore E_{1t} = E_{2t}$$

2. Boundary Conditions in Tangential Direction

➔ The tangential components of E-intensity are **always continuous** across the boundary.

$$E_{1t} = E_{2t}$$



$$\Delta \psi = \psi_1 - \psi_2 = \int_a^{a+\Delta h} \vec{E} \cdot d\vec{l} \rightarrow 0 \quad \Delta h \rightarrow 0$$

$$\therefore \psi_1 = \psi_2$$

➔ E-potential is **continuous** across the boundary;

➔ In other word, E-potential is an integral function and all integral functions are continuous within their domains of definition.

Summary --- Boundary Conditions



1. In normal direction

$$D_{1n} - D_{2n} = \sigma_{fc}$$

$$\epsilon_2 \cdot \frac{\partial \psi_2}{\partial n} - \epsilon_1 \cdot \frac{\partial \psi_1}{\partial n} = \sigma_{fc}$$

2. In tangential direction

$$E_{1t} = E_{2t}$$

$$\psi_1 = \psi_2$$

3. Applications of Boundary Conditions

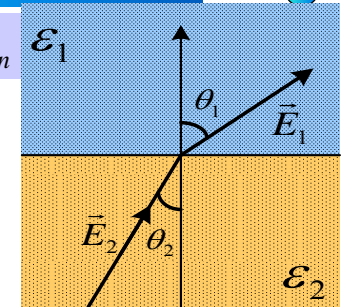


$$\because D_{1n} = D_{2n} \Rightarrow \epsilon_1 \cdot E_{1n} = \epsilon_2 \cdot E_{2n}$$

$$E_{1t} = E_{2t}$$

$$\because \begin{cases} E_1 \cdot \sin \theta_1 = E_2 \cdot \sin \theta_2 \\ \epsilon_1 \cdot E_1 \cdot \cos \theta_1 = \epsilon_2 \cdot E_2 \cdot \cos \theta_2 \end{cases}$$

$$\therefore \frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_1}{\epsilon_2}$$



Similar to the refraction of light

Exercise 3.27 in textbook p127

Example 2.



- ➔ A conductor ball. Radius a . E-potential U .
- ➔ Please determine the potential outside the ball.
- ➔ Analysis:
 - ✦ Any Symmetry? — Yes, point symmetry.
 - ✦ How many approaches to determine E-potential?
 - ✦ Direct solution — via integral or sum
 - ✦ Via E-intensity
 - ✦ Via differential equations

Solution 1.



Because ??? we obtain $\nabla^2 \psi = 0$

Because ??? we infer $\psi = \psi(r)$



Express Laplace's Equ. in spherical coordinates

$$\nabla^2 \psi = \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\psi}{dr} = 0$$

Through integral of above equ. $\psi = -\frac{C_1}{r} + C_2$

Via boundary conditions

$$\psi = \begin{cases} r > a & \frac{a}{r} \cdot U \\ r = a & U \\ r < a & U \end{cases}$$

$$\psi = -\frac{C_1}{r} + C_2$$

Determine the undecided constants via Boundary Conditions

If $r \rightarrow \infty$, we know $\psi = 0$, and then $C_2 = 0$

If $r = a$ and $\psi = U$, we infer $C_1 = -aU$

$$\psi = \begin{cases} r > a, & \frac{a}{r} \cdot U \\ r \leq a, & U \end{cases}$$

➡ Please check that E-potential is continuous across the sphere of the conductor ball, i.e. $\psi_1 = \psi_2$



Now, let's go on --->>>