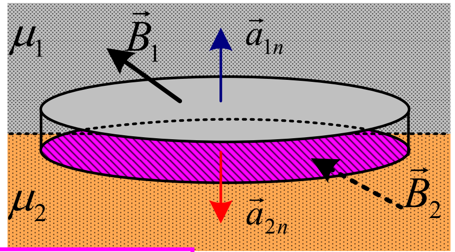
§ 5.5 Boundary Conditions

Boundary Condition 1. (in normal direction)

Make an auxiliary closed surface of a very very flat box.

from $\oint_S \vec{B} \bullet d\vec{S} = 0$



$\boxed{B_{1n} = B_{2n}}$ $\boxed{\vec{B}_1 \bullet \vec{a}_n = \vec{B}_2 \bullet \vec{a}_n}$

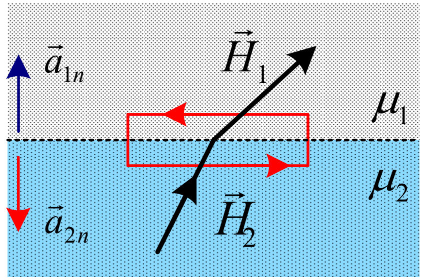
Normal components of M-flux density are equal at boundary.

Boundary Conditions 2. (in tangential direction)

Construct a closed rectangular path

$$\oint_C \vec{H} \bullet d\vec{l} = I$$

$$\boxed{H_{1t} - H_{2t} = J_{sFree}}$$



In case of no free surface current, tangential M-intensity is continuous.

$$\boxed{H_{1t} = H_{2t}}$$

Boundary Conditions 2. (in tangential direction)

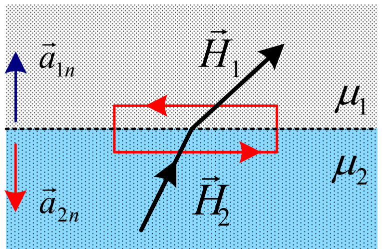
Construct a closed rectangular path

$$\oint_C \vec{H} \bullet d\vec{l} = I \quad H_{1t} - H_{2t} = J_{sFree}$$

$$(\vec{H}_1 - \vec{H}_2) \bullet \vec{a}_t = J_{sFree} \qquad \vec{a}_t = \vec{a}_s \times \vec{a}_n$$

$$\vec{a}_s \bullet [\vec{a}_n \times (\vec{H}_1 - \vec{H}_2)] = J_{sFree}$$

$$\boxed{\vec{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_{sFree}}$$



$$\boxed{\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{c})}$$

In case of no free surface current, tangential M-intensity is continuous.

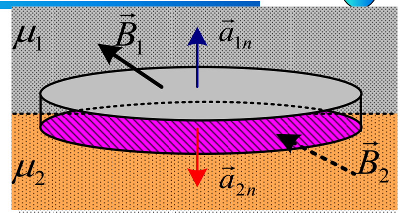
$$\vec{a}_n \times \vec{H}_1 = \vec{a}_n \times \vec{H}_2 \qquad \boxed{H_{1t} = H_{2t}}$$

Boundary Conditions

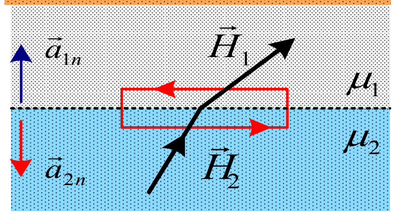
By Coulomb's Gauge

$$\boxed{\nabla \bullet \vec{A} = 0}$$

$$\oint_S \vec{A} \bullet d\vec{S} = 0 \qquad \boxed{A_{1n} = A_{2n}}$$



$$\boxed{\int_S \nabla \times \boldsymbol{A} \cdot d\boldsymbol{S} = \oint_l \boldsymbol{A} \cdot d\boldsymbol{l} = \int_S \boldsymbol{B} \cdot d\boldsymbol{S} = \Phi \approx 0}$$



$$\boxed{A_{1t} = A_{2t}}$$

Expression in form of M-vector Potential: $\boxed{\vec{A}_1 = \vec{A}_2}$

Summary: Boundary Conditions

electrostatics

1. normal $\boxed{D_{1n} - D_{2n} = \sigma_{fc}}$ $\quad \varepsilon_1 \cdot \frac{\partial \psi_1}{\partial n} = \varepsilon_2 \cdot \frac{\partial \psi_2}{\partial n} \quad (\text{if } \sigma_S = 0)$

2. tangential $\boxed{E_{1t} = E_{2t}}$ $\quad \psi_1 = \psi_2$

magnetostatics

1. normal $\boxed{B_{1n} = B_{2n}}$

2. tangential $\boxed{H_{1t} - H_{2t} = J_{sFree}}$ $\quad \vec{A}_1 = \vec{A}_2$

A Summary of Boundary Conditions

	normal	tangential
Static E-field	$\boxed{D_{1n} - D_{2n} = \sigma_{fc}}$ $\varepsilon_1 \cdot \frac{\partial \psi_1}{\partial n} = \varepsilon_2 \cdot \frac{\partial \psi_2}{\partial n} \quad (\text{if } \sigma_S = 0)$	$\boxed{E_{1t} = E_{2t}}$ $\psi_1 = \psi_2$
SC E-field	$\boxed{J_{1n} = J_{2n}}$ $\sigma_1 E_{1n} = \sigma_2 E_{2n}$ $\sigma_1 \frac{\partial \psi_1}{\partial n} = \sigma_2 \frac{\partial \psi_2}{\partial n}$	$\boxed{E_{1t} = E_{2t}}$ $J_{1t}/\sigma_1 = J_{2t}/\sigma_2$ $\psi_1 = \psi_2$
Static M-field	$\boxed{B_{1n} = B_{2n}}$ $\vec{A}_1 = \vec{A}_2$	$\boxed{H_{1t} - H_{2t} = J_{sFree}}$

A Summary of Boundary Conditions

Scalar form	normal	tangential
Static E-field	$\boxed{D_{1n} - D_{2n} = \sigma_{fc}}$	$\boxed{E_{1t} = E_{2t}}$
SC E-field	$\boxed{J_{1n} = J_{2n}}$	$\boxed{E_{1t} = E_{2t}}$
Static M-field	$\boxed{B_{1n} = B_{2n}}$	$\boxed{H_{1t} - H_{2t} = J_{sFree}}$

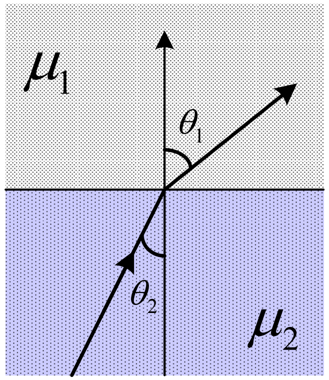
Applications of Boundary Conditions

In isotropic media

$$\vec{B}_1 \bullet \vec{a}_n = \vec{B}_2 \bullet \vec{a}_n$$

$$\vec{a}_n \times \vec{H}_1 = \vec{a}_n \times \vec{H}_2$$

$$\frac{tg\theta_1}{tg\theta_2} = \frac{\mu_1}{\mu_2}$$



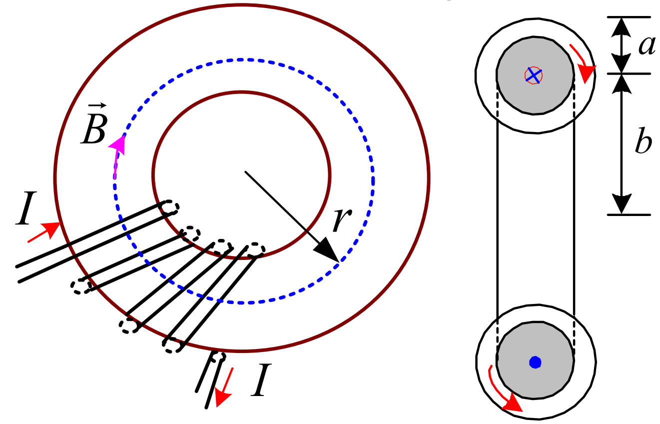
Similar to electrostatics $\boxed{\frac{tg\theta_1}{tg\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}}$

More Examples

- Examples about the windings（线圈）
- Calculation of M-field parameters
- For each example, please try A-C law at first, and then check if there is boundary conditions to be used.

Example 1. toroidal winding（环形绕组）

A closely spaced toroidal winding with N turns. The radii of each turn and the winding are a and b. Current I is input. Please determine the M-flux density within the winding.



Analysis

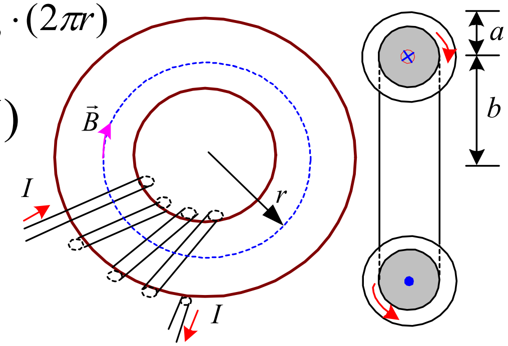
Applying A-C Law

Construct an auxiliary circle with radius r, $(b\text{-}a) < r < (a\text{+}b)$

$$\oint_C \vec{B} \bullet d\vec{l} = \oint_C B_\varphi \cdot dl = B_\varphi \cdot (2\pi r)$$

$$\mu_0 \cdot "I" = \mu_0 \cdot (N \cdot I)$$

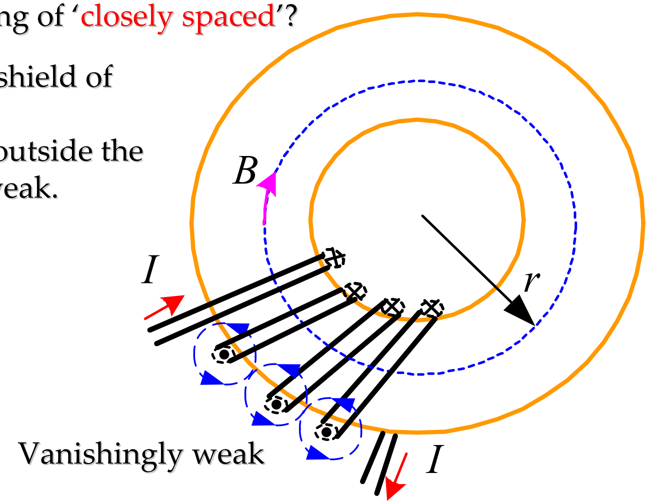
$$\therefore \vec{B} = \begin{cases} \dfrac{\mu_0 NI}{2\pi r}, & \cdots \\ 0, & \cdots \end{cases}$$



About the M-field Outside

What's the meaning of 'closely spaced'?

The wire forms a shield of high quality.
Thus the M-field outside the winding is very weak.
$r>(a+b)$ or $r<(b-a)$

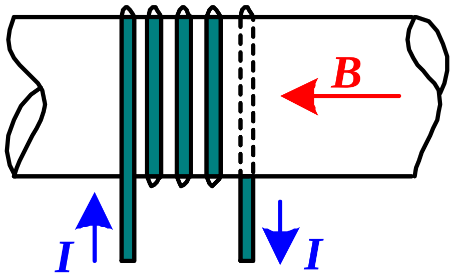


Example 2. Infinite Straight Solenoid（螺线管）

Closely spaced, n turns per meter, radius R for each turn, current I

Please determine M-field distribution around the solnoid.

Solution 1. via A-C Law



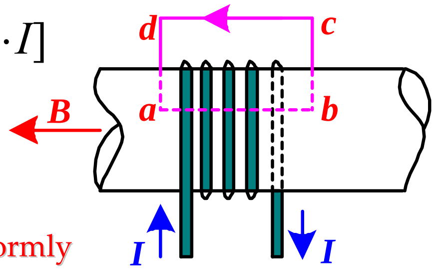
Solution 1. via A-C Law

1. Identify the axial symmetry
2. Construct a closed path *a-b-c-d* with a direction.
3. Neglect the field outside due to 'closely spaced'.

$$\oint_C \vec{B} \bullet d\vec{l} = -\oint_C B \cdot dl = -B \cdot L_{ab}$$

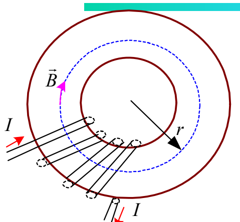
$$\mu_0 \cdot "I" = \mu_0 \cdot \sum I = \mu_0 \cdot [(n \cdot L_{ab}) \cdot I]$$

$$\therefore \vec{B} = \begin{cases} -\mu_0 \cdot (n \cdot I) \\ \text{the direction} \end{cases}$$



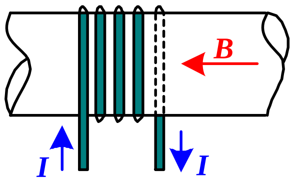
Field within the solenoid is uniformly distributed, independent of radius.

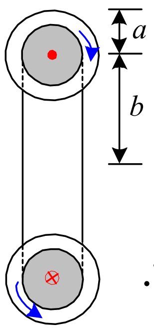
Solution 2. from the results of Example 1.



break it up

$b \to \infty$





If b → ∞, we feel the toroidal winding as if a straight solenoid, similar to the equator（赤道） and the ground.

$$b \to \infty \text{ and } r \to \infty, \quad \therefore r \to b$$

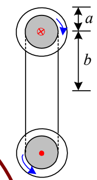
$$\text{since} \quad B = \frac{\mu_0 \cdot NI}{2\pi \cdot r}, \quad (b-a) < r < (b+a)$$

$$\therefore B = \frac{\mu_0 \cdot NI}{2\pi b} = \mu_0 \cdot I \cdot \left(\frac{N}{2\pi b}\right) = \mu_0 \cdot I \cdot n$$

Example 3. Toroidal Winding with Air Gap

A toroidal winding, closely spaced N turns, ferromagnetic core, radii $b \gg a$, an air gap in length of l

Please determine M-field within the winding & the gap.

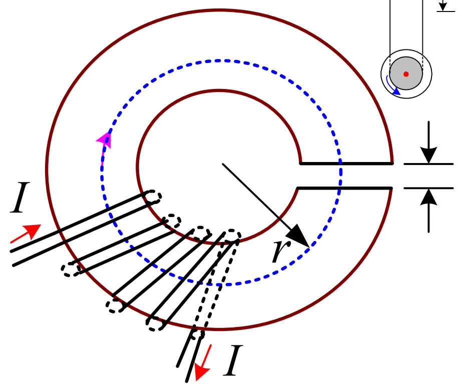


Analysis: why $b \gg a$?

Field is uniformly distributed in the gap.

$$\oint_C \vec{H} \bullet d\vec{l} = NI$$

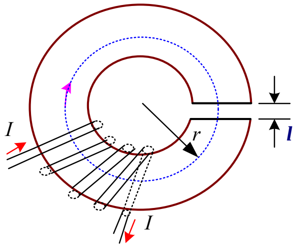
$$B_{iron} = B_{gas} = \left|\vec{a}_\varphi B_{iron}\right|$$



$$\oint_C \vec{H} \bullet d\vec{l} = NI \qquad \vec{B}_{iron} = \vec{B}_{gas} = \vec{a}_\varphi B_{iron}$$

In the core $\vec{H}_{iron} = \dfrac{\vec{B}_{iron}}{\mu} = \vec{a}_\varphi \dfrac{B_{iron}}{\mu}$

In the gap $\vec{H}_{gas} = \dfrac{\vec{B}_{gas}}{\mu_0} = \vec{a}_\varphi \dfrac{B_{iron}}{\mu_0}$



Note that $\oint_C \vec{H} \bullet d\vec{l} = \int_{C_{iron}} \ldots + \int_{C_{gas}} \ldots = NI$

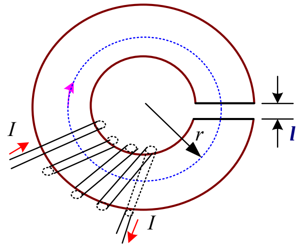
$$\therefore \frac{B_{iron}}{\mu} \cdot (2\pi \cdot b - l) + \frac{B_{gas}}{\mu_0} \cdot l = N \cdot I$$

$$\therefore B_{iron} = B_{gas} = \frac{\mu_0 \cdot \mu \cdot N \cdot I}{\mu_0 \cdot (2\pi \cdot b - l) + \mu \cdot l}$$

$$B_{iron} = B_{gas} = \frac{\mu_0 \cdot \mu \cdot N \cdot I}{\mu_0 \cdot (2\pi \cdot b - l_g) + \mu \cdot l_g}$$

In the core $\vec{H}_{iron} = \dfrac{\vec{B}_{iron}}{\mu} = \vec{a}_\varphi \dfrac{B_{iron}}{\mu} = \ldots$

In the gap $\vec{H}_{gas} = \dfrac{\vec{B}_{gas}}{\mu_0} = \vec{a}_\varphi \dfrac{B_{iron}}{\mu_0} = \ldots$



Where the M-field is stronger? In the iron core or the gap?

$$\frac{H_{gas}}{H_{iron}} = \frac{\mu}{\mu_0} >> 1$$

It's the principle of electromagnet crane.

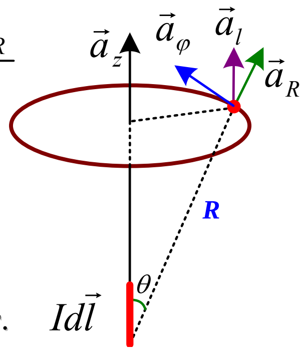
§ 5.6 Inductance（电感）

M-Flux Density $\vec{B} = \dfrac{\mu_0 I}{4\pi} \oint_C \dfrac{d\vec{l}_{\text{source}} \times \vec{a}_R}{R^2_{\text{source-spot}}}$

M-Flux $\Phi = \int_S \vec{B} \cdot d\vec{S}$

M-Flux Linkage (磁链) $\Psi = N \cdot \Phi$

Note that N is unnecessary to be an integer.



$$\vec{B} \propto I \qquad \Phi \propto I$$

Inductance depends on Ψ and I.

电感分类

- 自感Self-Inductance: 某电流产生的磁场与回路自身相交的磁链/该电流 $L = \Psi / I \ (H)$
  - 回路通单位电流时本身产生的磁链。
  - 外自感: Ψ是导体外部的磁链时…
  - 内自感: Ψ是导体内部的磁链时…
- 互感Mutual-Inductance: 某电流产生的磁场与其他回路相交的磁链/该电流
  - 一回路通单位电流时，另一回路所交链的磁链数

$$M_{12} = \Psi_{2-1} / I_1$$

Categories of Inductances

Inductance is rate of Ψ over *I*. Unit: 亨利 *Henry*

- Self-Inductance: Ψ is induced and ringed by *I* itself.
  - Outer Self-Inductance: Ψ is formed by M-field passing through the wire loop.
  - Inner Self-Inductance: Ψ is formed by M-field passing the conductor.

$$L = \Psi / I$$

- Mutual-Inductance: Ψ is induced by I_1 and area C_2, while the inductance is rate of Ψ over I_1.

$$M_{12} = \Psi_{2-1} / I_1$$

- Inductance is determined by the shape, size, number of turns, material of the loop. It is independent of whether the loop is input a current.

Approaches & Steps to Get Inductance

Solution 1. Common Approach

1. Select the coordinates according to the loop shape.
2. Assume current *I* in the loop.
3. Get M-flux density directly, or via A-C law or B-S law
4. Get M-flux via its density
5. Get M-flux-linkage via M-flux
6. Get inductance by its definition

$$I \Rightarrow \vec{B} \Rightarrow \Phi = \int_S \vec{B} \cdot d\vec{S} \Rightarrow \Psi = N\Phi \Rightarrow L = \frac{\Psi}{I}$$

- Solution 2.

$$I \Rightarrow \vec{A} \Rightarrow \Phi = \oint_C \vec{A} \bullet d\vec{l} \Rightarrow \Psi = N\Phi \Rightarrow L = \frac{\Psi}{I}$$

- Solution 3.

$$I \Rightarrow \vec{H} \Rightarrow W_m = \begin{cases} \int_V \frac{1}{2} \mu_0 |\vec{H}|^2 dV \\ \frac{1}{2} L I^2 \end{cases} \Rightarrow L = ?$$

- Steps to get Mutual Inductance are similar to those for self-inductance.

Example 1. Inner Self-Inductance

- Please determine the inner self-inductance at per unit length of a straight wire, of which the radius is *a*.
- From example 4 in § 5.1, we know in the wire…

$$\vec{H} = \frac{\vec{a}_\varphi I r}{2\pi a^2}$$

- M-flux through the shadow area is

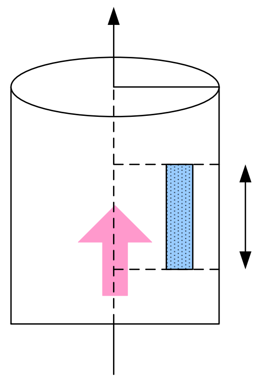
$$d\Phi = \vec{B} \bullet d\vec{S}_\varphi = \frac{\mu_0 I r}{2\pi a^2} \cdot dr \cdot 1$$

- Current ringed by this portion of M-flux is

$i = I \times r^2 / a^2$

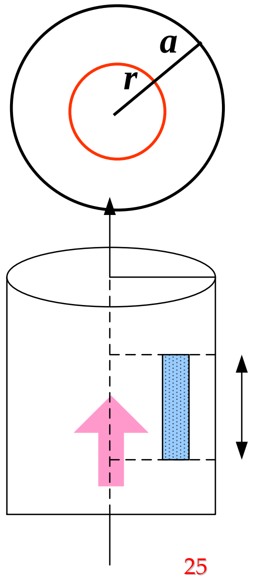
- The corresponding M-flux-linkage is ?

$$d\Psi = N \cdot d\Phi = \frac{\pi r^2}{\pi a^2} \cdot \left(\frac{\mu_0 \cdot I}{2\pi \cdot a^2} \right) \cdot r dr$$



The overall M-flux-linkage is

$$\Psi = \int d\psi = \int_0^a \frac{\mu_0 \cdot I}{2\pi \cdot a^4} \cdot r^3 dr = \frac{\mu_0 \cdot I}{8\pi}$$



The inner self-inductance is

$$L = \frac{\Psi}{I} = \frac{\mu_0}{8\pi}$$

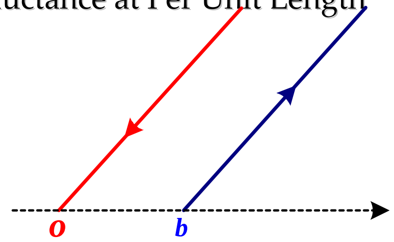
The inner self-inductance at per unit length of a straight wire.

The inner self-inductance at 2 meters of such a straight wire ?

Example 2. Outer-Self-Inductance of Parallel Double Lines at Per Unit Length

Parallel double lines are shown as the figure, and $b \gg a$
Please determine the outer-self-inductance at Per Unit Length

Analysis:
(1) Proper coordinates?
(2) Assume current I in the wire.
(3) How to get M-flux density?
(via A-C law or B-S law)
(4) How to get the M-flux?
(5) How to obtain the M-flux-linkage?
(6) Determine the inductance according to its definition.

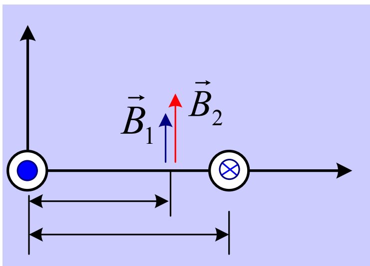


a

Solution:

M-flux density on the plane of the double lines is

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \vec{a}_y \frac{\mu_0 I}{2\pi}\left(\frac{1}{x} + \frac{1}{b-x}\right)$$



M-flux across area of per unit length

$$\Phi = \int_s \vec{B} \bullet d\vec{S} = \int_a^{b-a} B \cdot (1 \cdot dx) = \int_a^{b-a} \left[\frac{\mu_0 I}{2\pi} \cdot \left(\frac{1}{x} + \frac{1}{b-x}\right)\right] dx = ?$$

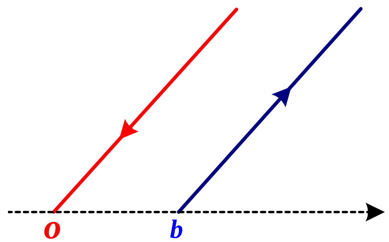
The linkage of 1 loop: $\Psi = 1 \cdot \Phi = \dfrac{\mu_0 I}{\pi} \ln \dfrac{b-a}{a}$

Outer-self-inductance $L = \dfrac{\Psi}{I} = \dfrac{\mu_0}{\pi} \ln \dfrac{b-a}{a} \approx \dfrac{\mu_0}{\pi} \ln \dfrac{b}{a}$

A Comparison for Parallel Double Lines

Self-inductance at per unit length $L \approx \dfrac{\mu_0}{\pi} \ln\left(\dfrac{b}{a}\right)$

Capacitance at per unit length $C \approx \dfrac{\pi \varepsilon_0}{\ln(b/a)}$



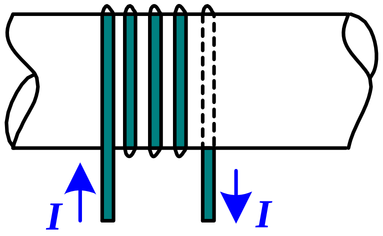
By comparison $\begin{cases} L \cdot C = ? \\ L/C = ? \end{cases}$ $\begin{cases} L \cdot C = \mu \cdot \varepsilon \\ L/C = Z \quad \text{(intrinsic impedance)} \end{cases}$

Parallel Double Lines are very useful for microwave transmission lines, unshielded twisted pair (UTP), shielded twisted pair (STP), and circuit at very high speed.

b

Exercise: determine self-inductance of infinite solenoid at per unit length.

Infinite solenoid, radius a for each turn, n turns per unit length



Steps:

(1) coordinates

(2) Assume I

(3) Get B via A-C law

(4) Get Φ

(5) Get Ψ

(6) Get L via definition

Application: make the inductance of your telecomm. system

How to determine the mutual inductance?

Steps: two loops C1 and C2

(1) Assume I_1 in C1 and get B_1

(2) Integrate B_1 over S_2

(3) Get Φ_{12} by C1 through C2

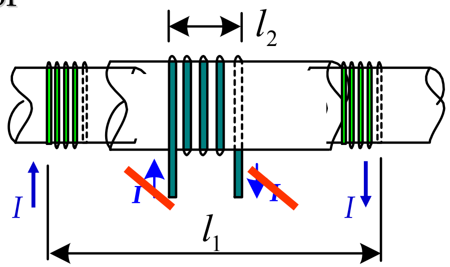
(4) Get Ψ_{12}

(5) According to definition: $M_{12}=\Psi_{12}/I_1$

(6) $M_L=M_{12}=M_{21}$

Example 1. mutual inductance between 2 coaxial solenoids

Both of radius a, total turns of N_1 & N_2, in lengths of l_1 & l_2, $l_1>>l_2$, please determine M.



(1) Assume I_1 in C1 and get B_1

(2) and (3) Get Φ_{12} by C1 through C2

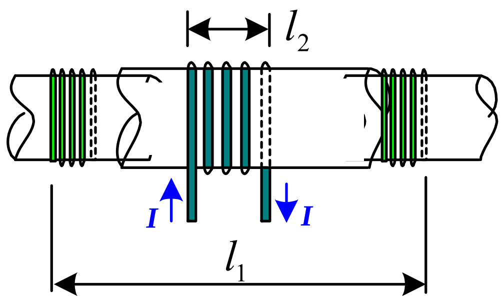
(4) Get Ψ_{12}

$$B_1 = \mu_0 \cdot \frac{N_1}{l_1} \cdot I_1 \qquad \Phi_{12} = B \cdot \left(\pi \cdot a^2\right) \qquad \Psi_{12} = N_2 \cdot \left[B \cdot \left(\pi \cdot a^2\right)\right]$$

(5) According to definition: $M_{12}=\Psi_{12}/I_1$

(6) $M_L=M_{12}=M_{21}$

$$M_{12} = \frac{\Psi_{12}}{I_1} = \frac{\mu_0}{l_1} \cdot \left[N_1 \cdot N_2 \cdot \left(\pi \cdot a^2\right)\right]$$

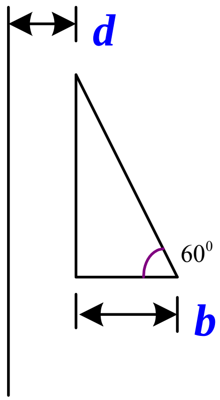


Example 2 Mutual inductance between two loops

An infinite straight line and a triangular loop

$$M_{12} = M_{21}$$

Which shall be selected as l_1 ?



For infinite wire l_1 $\quad \vec{B}_1 = \vec{a}_\varphi \dfrac{\mu_0 I_1}{2\pi \cdot r}$

M-flux through l_2 $\quad \Phi_{12} = \int_{S_2} \vec{B}_1 \bullet d\vec{S}$

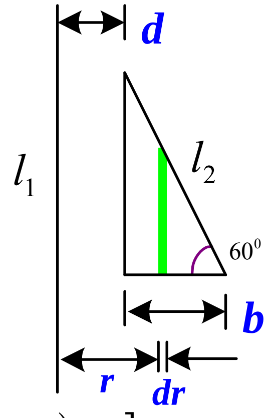
$$d\vec{S} = \vec{a}_\varphi \left[\left((b+d) - r \right) \cdot tg60^\circ \right] \cdot dr$$

Linkage through l_2

$$\Psi_{12} = 1 \cdot \Phi_{12} = \ldots = \int_d^{b+d} \ldots$$

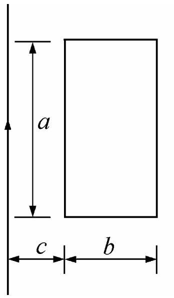
Mutual inductance

$$M_L = M_{12} = \ldots = \frac{\sqrt{3}\mu_0}{2\pi} \cdot \left[(b+d)\ln\left(1+\tfrac{b}{d}\right) - b \right]$$

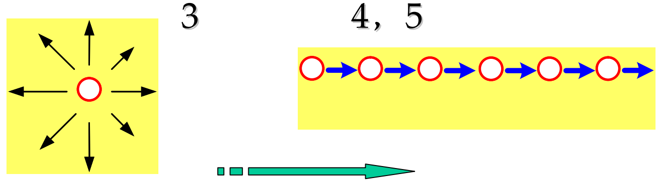


homework

- 一根长直导线与一个边长为a、b的矩形线圈在同一平面内,线圈宽边与直导线平行,如图所示。求线圈与直导线的互感。
- 空气绝缘的同轴线,其内导体半径为a,外导体内半径为b,通过电流为I,设外导体厚度很薄,其中的储能可忽略不计。求单位长度的电感。



Up to now, we have gone so long and so far.



q