EBU6018 Advanced Transform

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Linear Algebra

- 1415B Q1cd 一些奇怪的要求证明和计算
- 1314C Q1b dual base 计算

- Kronecker Delta function:
  Define mutual orthogonality between two functions.
- 如果方阵且列向量正交，则转置就是逆矩阵。
- Matrix multiplication is distributive but not commutative(没有交换律)
- Norm:（证明 orthonormal 1213A Q1b）

Norm $\|\mathbf{a}\|$ of a vector $\mathbf{a}$ (strictly : its "2 - norm") is given by

$$\sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{\mathbf{a}^t \mathbf{a}} = \left(\sum_i a_i^2 \right)^{1/2}$$

- Frame

1. 向量集合在向量空间 V 包含更多的比空间阶数更多向量。
2. 不正交也不线性无关
3. 表示向量空间的其他任何向量

A frame is a set of vectors in vector space V that contains more vectors than the order of the space [1 mark], and that are not orthogonal or linearly independent [1 mark]. The frame vectors can be used to represent any other vector in the space [1 mark].

4. 特点：系数不唯一，向量不唯一，frame 也不组成基，只有 exact frames 才是基。
5. Tight frame example: 行比列数量多并且行之间正交

- Rank: The row and column spaces.
- Null space: the space not spanned by the rows of A, dimension is n-r.
- Non-stationary signal: frequency content varies with time.

- Linear Independent

Given a set of vectors $\{\mathbf{v}_i\}$ $i = 1,2,\ldots,k$ if we can find a set of scalars $\{c_i\}$ $i = 1,2,\ldots,k$ (excluding $c_1 = c_2 = \quad = 0$) such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \quad + c_k\mathbf{v}_k = \mathbf{0}$$

then the set of vectors is called *linearly dependent*.
If there is none, they are called *linearly independent*.

- Basis Vectors:

A set of vectors $\{\mathbf{w}_i\}$ $i = 1,2,\ldots,n$ is said to be
a *basis* for a vector space R^n if the set of vectors $\{\mathbf{w}_i\}$
(a) is linearly independen t, and
(b) spans the space R^n.
There must be exactly n vectors in the basis.
Each vecto r $\mathbf{v}$ in R^n has a unique set of *coordinate s*
$\{c_i\}$ in the basis $\{\mathbf{w}_i\}$, where $\mathbf{v} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n$

- $s = \sum_n c_n \Psi_n$ if a vector space V has basis functions $\{\psi n\}$

The advantage of $\{\psi n\}$ being orthonormal: it simplifies the calculation of the coefficients cn

$c_n = \langle s, \Psi_n \rangle$.

s can therefore be written as

$$s = \sum_j \langle s, \Psi_j \rangle \Psi_j$$
$$= \langle s, \Psi_1 \rangle \Psi_1 + \langle s, \Psi_2 \rangle \Psi_2 + \ldots.. + \langle s, \Psi_n \rangle \Psi_n$$

- Vector space

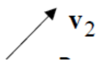
The real vector space R^n is the space of vectors
which each have n real components. The exponent n is called
the *dimension* of the space.
E.g. R^3 is our "usual" three-dimensional space.

- Span the space:

Given an n-dimensional real vector space R^n,
a set of vectors $\{\mathbf{w}_i\}$ $i = 1,2,\ldots,k$ is said to *span* the space
if every vector $\mathbf{v}$ in R^n can be written as

$$\mathbf{v} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \quad + c_k\mathbf{w}_k$$

for some set of scalars $\{c_i\}$ $i = 1,2,\ldots,k$



- Biorthogonal Bases

Answer: i) Biorthogonal bases are a pair of bases [1 mark], as follows:

If $\{\psi_n\}$ and $\{\hat{\Psi}_n\}$ are both basis vectors themselves for V [1 mark], and satisfy

[1 mark]

$$\langle \Psi_i, \hat{\Psi}_j \rangle = \delta_{ij}$$

then any $\mathbf{s}$ in V can be written as $s = \sum_{j=1}^{n} \langle s, \Psi_j \rangle \hat{\Psi}_j$ [1 mark]

ii)

・Dual bases are biorthogonal. 1617C Q1c 证明两对是对偶基

=>> Dual base and Biorthogonal base: **dual base may be linearly dependent but biorthogonal may not be.** (对偶基可能线性相关，双正交基可能不线性相关)

・Orthogonal basis & Orthonormal basis

Orthogonal basis: a set of functions whose inner product taken two at a time is zero.

Orthonormal basis: a set of functions that are orthogonal and whose magnitudes are each unit length.

・A set of orthonormal basis can span a space if:

向量相互正交 形成正交基 mutual orthogonal

向量是单位长 形成规范正交基

向量集生成唯一系数 unique coefficient

FS&FT

• 1617A Q1b 卷积性质：证明时域卷积等于频域相乘 1415B Q1e 1314B Q1c

• 1415C Q4a 用时微分性质证明一些奇怪的东西

• 1314B Q1b 用尺度性质算高斯方程的 FT，用时移性质证明其他

• 1314C Q2b 证明尺度性质

• FT 基方程在整个时域展开(spread)

• The limitation of the Fourier Transform to non-stationary signals(Why suitable for Stationary Signals): although it provides detail of the frequencies present in a signal, it does not tell when those frequencies occurred.（不能显示出什么时候频率出现）

• The limitation of stationary signals: contain a limited amount of information and so are of limited use.

• DTFT 结果方程连续

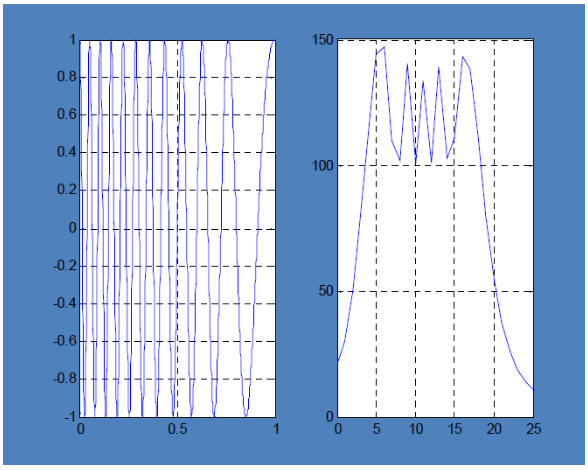
• FT necessary condition:

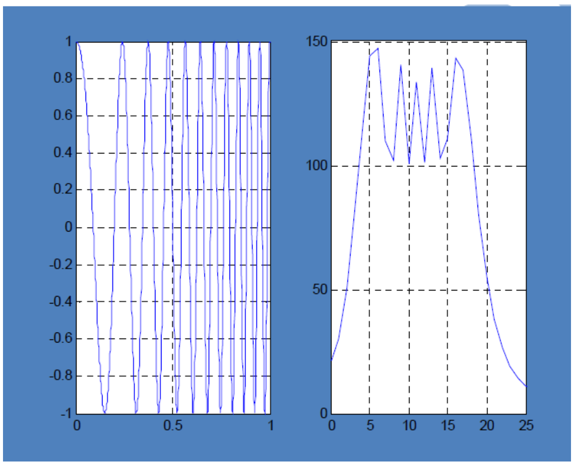
x(t) is "well behaved" if:

1. the signal x(t) has a finite number of discontinuities, maxima, and minima within any finite interval of time.
2. if x(t) is absolutely integrable $\int_{t=-\infty}^{t=\infty} |x(t)|dt < \infty$

Signals that do not satisfy this condition are generally not suitable for use as basis functions.

•Using a chirp signal as an example to illustrate the limitation of non-stationary signal for FT.(WVD)





How this limitation can be overcome that FT does not tell when those frequencies occurred.

This limitation can be overcome by applying a **window that translates across the signal**, while obtaining the Fourier Transform within the window. **This localises the frequencies in time.**

• Shift property:

If $s(t)$ has Fourier transform $S(w)$ then $s(t)e^{j\omega_0 t}$ has Fourier transform

$$\int_{-\infty}^{\infty} s(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} s(t)e^{-j(\omega-\omega_0)t}dt$$

$$= S(\omega-\omega_0)$$

Thus multiplication by $e^{j\omega_0 t}$ shifts the frequency spectrum of $s(t)$ so that it is centred on the point $\omega=\omega_0$ in the frequency domain (MODULATION).

- Hermitian Symmetry

Fourier Transform: Hermitian Symmetry

Expand the Fourier transform of a function, $s(t)$:

$$S(\omega)=\int_{-\infty}^{\infty} [s_e(t)+s_o(t)]\ [\cos(\omega t)-j\sin(\omega t)]\,dt$$

$$=\int_{-\infty}^{\infty} s_e(t)\cos(\omega t)dt+\int_{-\infty}^{\infty} s_o(t)\cos(\omega t)dt-j\int_{-\infty}^{\infty} s_e(t)\sin(\omega t)dt-j\int_{-\infty}^{\infty} s_o(t)\sin(\omega t)dt$$

$$=\int_{-\infty}^{\infty} s_e(t)\cos(\omega t)dt-j\int_{-\infty}^{\infty} s_o(t)\sin(\omega t)dt$$

If even, fourier transform is real $S_e(\omega)=\mathrm{Re}[S_e(\omega)]$

If even, fourier transform is even $S_e(\omega)=S_e(-\omega)$

If odd, fourier transform is imaginary $S_o(\omega)=\mathrm{Im}[S_o(\omega)]$

If odd, fourier transform is odd $S_o(\omega)=-S_o(-\omega)$

Hermitian or Conjugate Symmetry:

$$S(\omega)=S_e(\omega)+S_o(\omega)=S_e(-\omega)-S_o(-\omega)$$

$$=[S_e(-\omega)+S_o(-\omega)]^*=S^*(-\omega)$$

4

$-\infty$ $-\infty$

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Time Differentiation

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}\,d\omega$$

$$f'(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)\,j\omega e^{j\omega t}\,d\omega$$

$$f'(t)\leftrightarrow F(\omega)\,j\omega=j\omega F(\omega)$$

$$F[f'(t)]=j\omega F(\omega)$$

Sample

• 1313B Q2b 1213BQ2cd

• **Sampling in the time domain is convolution with comb of delta functions in the frequency domain.**

• Aliasing can be prevented by LPF before sampling plus sampling at a sufficiently high rate.

• Sample at Nyquist rate and there is no gap between band-pass spectra, and we need a LPF to extract the baseband signal. A practical filter at n6dB per octave where n is order of a filter.

• The ideal LPF cut-off frequency is at half the sampling rate.

DFT&FFT

・1415A 计算 Q1 b,d 为啥 IDFT 用 FFT 硬件单元 1415C Q1e
・1617A Q2a FFT 过程
・1617C 两点 DFT 1415C Q1b 1314C Q2c
・1415C Q1c DFT 的另一种写法
・1415C Q1d DFT 复数乘用卷积计算和 Twiddle Factors 复杂度

・DFT is accurate if the sequence is periodic。
・DIF FFT is the transpose of DIF FFT

・Bit reversal:抽样的信号顺序 bit of index 与自然数顺序(natural number order)相反。as each position is the reverse of the binary value of the original position.举例：8 points signal: the order of indexed are 000,100,010,110…111, which natural number are 000,001,010,011…111. In the context of DFT, the term means in the way that DFT derived（DFT 得出 input 和 output is bit reversed order）

・The FFT "decimates" the input sequence in powers of 2, therefore the number of multiplications for an N-point sequence is proportional to log2N. (reduce multiplication)

・The Discrete Fourier Transform (DFT) is used to obtain the frequency spectrum of a sampled signal. The input to the DFT must be a sequence of finite length.

・The effect that the length of the sequence has on the frequency resolution of the transform.
The longer the sequence length the better the resolution [越长解析度越好]. For example, suppose we have a fixed sample rate then the longer the sequence length the longer it takes for the DFT to run [序列越长 DFT 运行时间越长在固定采样率下]. A 1-second FT can give a resolution of 1 Hz whereas a 100ms FT can only resolve 10 Hz [一秒 FT 可以给 1Hz 的解析度，然而 100ms 只能解析 10Hz].(f0=1/T, f0 是最小频率间隔, T 是窗的时间长度)

・Spectral leakage and elimination:
If we sample a single sinewave and the sequence contains samples from an integer number of cycles of the signal, then the imaginary part of the DFT will have zero value at all points except for the frequency of the input signal, and the real part will be all zero. [采样 sin 和序列整数个周期，DFT 后的点虚部为 0 除去有输入序列的地方]. If the samples are not of an integer number of input cycles then all output values will be non-zero [如果不就非 0 了], ie, the spectral energy is smeared across all the DFT output values.

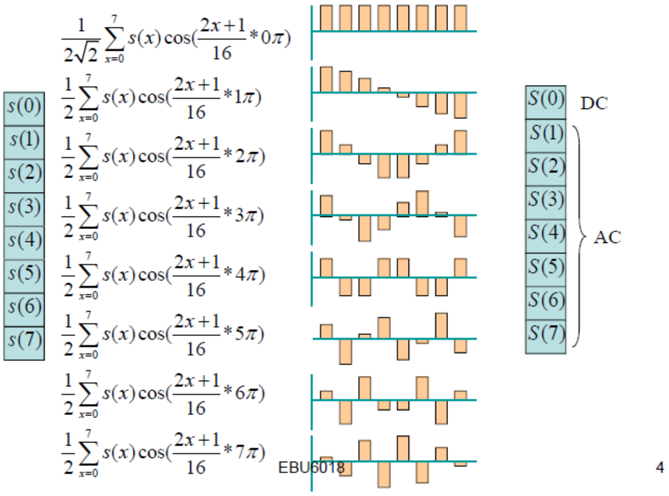
・To eliminate or reduce spectral leakage:
1. Synchronise the sample frequency to be an **integer multiple** of the sinewave frequency. [同步采样频率为正弦波的整数倍]
2. Increase the size of the input buffer [增加输入缓存]
3. Apply a data window to the DFT input. [应用 DFT 输入数据窗口]

DCT

- 1415A Q4,abc DCT 计算
- 1415C Q2b DCT 计算
- 1314B Q3b DCT 计算

- DCT real input and real output. It's reversable and separable.

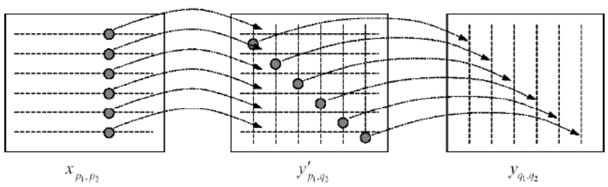
- **Why DCT for compression? Data decorrelation and data independent basis functions and Fast implementation.**
- DCT process: The DCT correlates the input data （关联输入数据）with a series of cosine functions of increasing frequency, starting with dc （和一系列 cos 函数增加频率）. These discrete cosine functions are the basis functions. The basis functions for an 8-point DCT are as below. Correlation gives a large value of output when there is a similarity between the input data distribution and a particular basis function (关联数据后如果输入数据分布和特定基方程相似，会给出较大值). Therefore the output sequence from the DCT is related to the frequency of the distribution of data in the input sequence （DCT 频率分布和输入序列有关）.



- 1D-DCT to 2D-DCT

Separable Transforms

May be implemented by applying the one dimensional transform first to the rows of the image and then to its columns (note that changing the application order does not change the result).



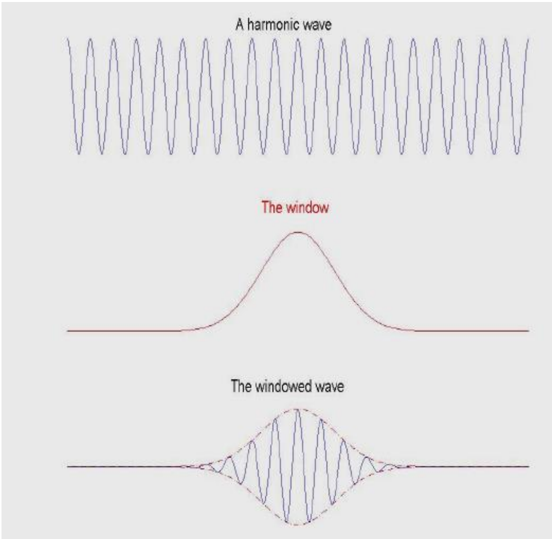
STFT

Windowed FT: 一个窗口平移在被转换的信号上，频率信息就在时间上 localised, FT 就在窗口里被 truncated 的部分。

STFT Process:

i) The STFT is defined as

$$STFT(t,\omega) = \int_{-\infty}^{\infty} s(\tau)\gamma^{*}(\tau - t)e^{-j\omega t}d\tau$$



The harmonic wave in the diagram is the complex exponential of the FT [FT 复数指数幂]
The window is the γ function in the definition [γ 是窗函数]. It is translated across the signal to be transformed [可平移到被转换的信号]. The window can be any shape, but careful choice of shape minimises problems and reduces redundancy [可以是任何形状，注意选取时注意最小化冗余].
s(t) is the signal to be transformed [s(t)是被转换的信号] The STFT is the integral of the product of these waveforms [这些波乘积的积分 STFT].

1. 选取窗口有限长；2. 把窗口放在 signal t=0; 3. 用信号截断(truncate)窗口；4. 计算被截断信号的 FT；5. 增量地（Incrementally）移动窗口；6. 重复步骤三

选取 STFT 形状和窗口宽度注意的问题：
形状：如果是矩形窗—avoiding overlap at edges or 移动窗口的时候 allowing gaps；并且还有不连续的问题 =>Gaussian and Hamming preferred.在恢复变换的时候也有帮助
宽度：我们需要 narrow enough to guarantee the signal within the window is stationary. 太小了频率解析度不好=>trade off time localisation & frequency localisation because Uncertainty Principle limits.

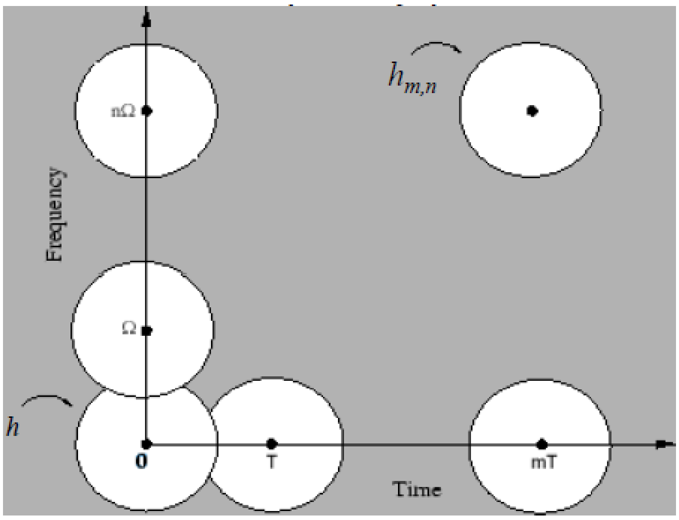
• Output: The output from the STFT is displayed on a **spectrogram** [1 mark]. This is a 3D plot of signal energy against time and frequency [3D 信号能量图对比时间和频率].

• The effect of fixed window in STFT on the resolution:
The Heisenberg Uncertainty Principle states that the product of uncertainty in time and uncertainty

in frequency is a constant [Heisenberg 定理：时间和频率不确定度相乘为定值]. Because the window length is fixed the uncertainty in time is constant and therefore the uncertainty in frequency is also fixed [STFT 窗函数大小确定即时间不确定度确定，频率不确定也确定了]. So the resolution is constant for all time and all frequency [所以分辨率固定了].

Gabor Transform

Use diagram to explain Gabor Transform.



组成：Constructed from blocking block by translation and modulation (translation by frequency domain)
抽样：Sampled at regular intervals.
区间标识和圈圈得来： T are labelled in time, omega in frequency. h m,n is obtained by shifting h along T-F plane.
如果 h 和他的 FT 在原点(centred at the origin), h m,n is centred at (mT , nW) in T-Ff plan.
每一个 h m,n 占有中心地点在 T-F plane.
选取合适的转移频率, h m,n 占有整个 F-T plane.

Gabor most useful window:

- The most useful window function is a Gaussian window:

$$h(t) = g(t) = \sqrt[4]{\alpha / \pi}\, e^{-\alpha t^2/2} \qquad \Delta t \Delta \omega = \frac{1}{\sqrt{\alpha}} \frac{\sqrt{\alpha}}{2} = 1/2$$

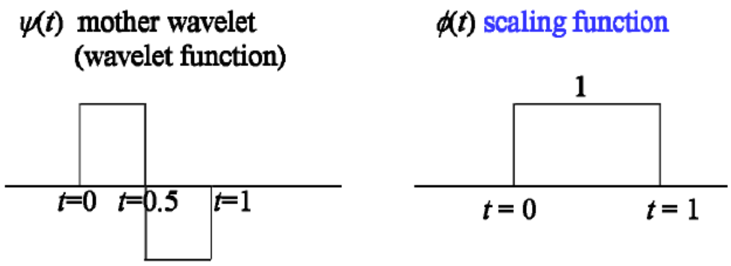
  – Optimally concentrated in the time frequency domain.
- This does not produce orthogonal basis functions.
- This function gives the best resolution

(好处与坏处)

• Gabor Transform for three ranges of sampling rate
Oversampling: Frame exist allowing 最好的 T-F localisation.
Critical sampling: Frame and orthonormal bases exist but T-F localisation is not good.
Under sampling: Gabor transform incomplete because frames do not exist.

Harr Functions

• 1415C Q3b
• 1314C Q4b
• filters for Haar basis:
Haar LPF: h0[0]=h0[1]=1/2 and Haar HPF: h1[0]=1/2, h1[1]=-1/2
• Haar functions are orthonormal and have limited applications.



Wavelet Transforms

• 1213A Q3c 证明 wavelet 不冗余

• Father wavelet: scaling function—MRA gives coarse detail, mother wavelet: wavelet function—MRA gives fine detail.

Daughter wavelets: $$\psi_{a,b}(t)=\frac{1}{\sqrt{a}}\underbrace{\psi}_{\text{Mother Wavelet}}\left(\frac{t-b}{a}\right)$$

• 可以快或慢变化的信号进行时频分析，震荡（oscillatory）with zero mean，可以检测信号不连续，wavelet 可以不正交
• Wavelet Admissibility condition: The wavelet average value is 0.
• The Continuous Wavelet Transform suffers from three problems:
Design of the wavelet functions: Orthonormal basis functions are difficult to find. [规范正交方程可太难找了，不好设计小波方程]
Redundancy: The result holds an infinite number of wavelets and so it is difficult to extract the required information from the transformed data. [冗余，太多小波很难提取我们想要的信息]
Efficiency: Most transforms cannot be solved analytically and require time-consuming numerical analysis. [效率很多变换都不能分析着去解决，还得数值分析]

• Discrete transform is not redundant, so it used to solve CWT problem by sampling. **1617a Q3c is process.**

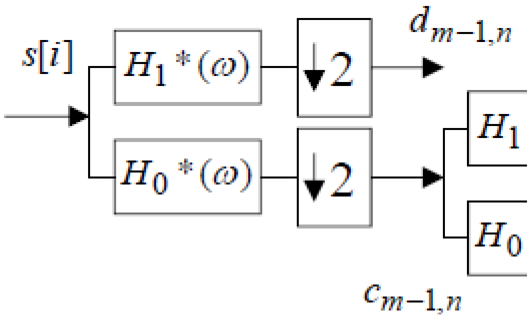
Can be reversed condition: Admissibility Condition

$$C_\Psi=\int_{-\infty}^{\infty}\frac{|\Psi(\omega)|^2}{|\omega|}d\omega<\infty$$

∞

Square of the Fourier transform must decay faster than 1/w.
Admissibility implies zero average.

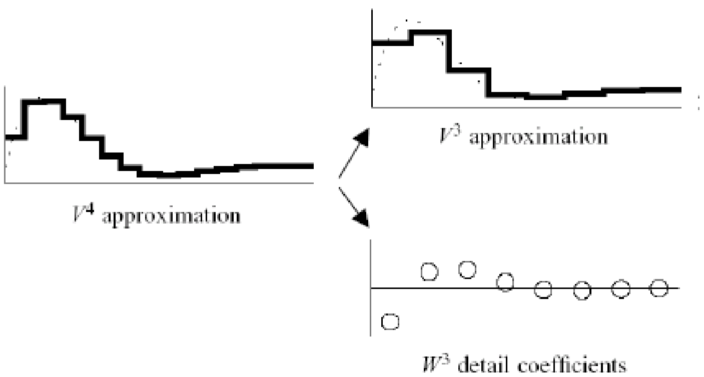
• One stage of a dyadic wavelet transform and process



Dyadic：序列长度为 power of 2
The inputs to stage are coarse details from the previous stage. H0 LPF to obtain the next level of coarse detail; H1 HPF to obtain the next level of fine detail. Down sampling is carried out so that each level there is half numbers of values of each as at the previous stage. (Down sampling 让每一个 level 数量比上一个阶段要少一半) If the sequence is not the power of 2, it can be zero padded to make it so.

Multiresolution Analysis (MRA)(Compression)

Definition: MRA is to decompose a fine-resolution signal into a coarse-resolution version of the signal, and the difference left over. It's a fast wavelet transform.
Piecewise approximation:



• 1617A Q4 全过程
• **Filter bank** 1415A Q3a,b 分解和恢复过程
• 1415B Q3c 计算
• HPF from wavelet function, LPF from scaling function.
• High pass synthesis filter is derived from Low pass analysis filter, and vice-versa.

Winger-Ville Distribution

• 1415B Q4a 维纳分布时移频移性质 b 中间项证明是实部 c（1415C Q4c）计算高斯信号的 WVD【课件】
• 1213B Q3b Chirp WVD illustration
• 1415C Q4b 又是一个奇怪的证明

• The Wigner-Ville Distribution (WVD) is an instantaneous autocorrelation function that obtains the signal's energy density in time and frequency.
• If the input bounded in time, output will also be bounded.
• If a signal is shifted in time, then the output will also be shifted in time.

$$WVD_s(t,\omega) = \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)e^{-j\omega\tau}d\tau$$

which is the *Wigner - Ville Distribution* (WVD).

we can also define a cross - Wigner - Ville distribution :

$$WVD_{s,g}(t,\omega) = \int_{-\infty}^{\infty} s(t+\tau/2)g^*(t-\tau/2)e^{-j\omega\tau}d\tau$$

then the WVD of time - shifted signal $s_0(t) = s(t-t_0)$

time - shifted WVD : $WVD_{s_0}(t,\omega) = WVD_s(t-t_0,\omega)$

$s_1(t) = s(t)e^{j\omega_1 t}$ is a frequency - shifted WVD :

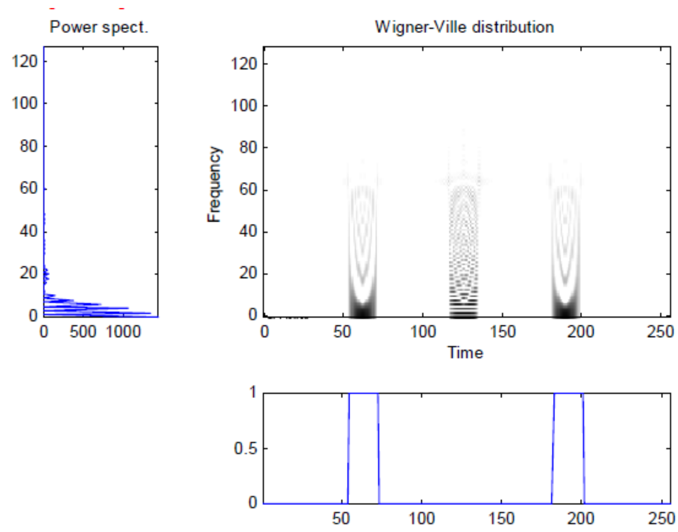
$$WVD_{s_1}(t,\omega) = WVD_s(t,\omega-\omega_1)$$

Consider composite signal $s(t) = s_1(t) + s_2(t)$. Then

$$WVD_s(t,\omega) = WVD_{s_1}(t,\omega) + WVD_{s_2}(t,\omega) + 2\,\mathrm{Re}\{WVD_{s_1,s_2}(t,\omega)\}$$

• State the advantage of the WVD over both the STFT and WT and explain its disadvantage.
The time-frequency resolution of the WVD is better than that of both the STFT and WT [WVD 时频分辨率比 STFT 和 WT 好]. (High resolution both in time and frequency) [NOT frequency analysis]
However, because the WVD is defined by a correlation function, if the signal is composite (ie the sum of separate signals) then in addition to the autocorrelation there will be cross-correlation terms. [如果信号是复合的 composite，就是分离信号的组合，将会生成 cross-correlation 互相关部分]

• Use illustration to explain cross term.



The diagram (at bottom) shows two pulses, one at about 50ms and the other at about 200ms [两个冲击在 50 和 200ms].
The plot top left shows the signal's power density [左上 power density].
The larger diagram shows the WVD spectrogram [WVD spectrogram]. Because the pulses are time-bounded the WVD is also time-bounded (a property of the WVD) [时域信号 bounded, WVD 也 bounded]. The spectrogram shows the cross-term midway between the two pulses. [形成了 cross-term]

- How the disadvantage(cross-term) of the WVD can be addressed.

The cross-terms can be filtered to remove them, [可以通过滤波器] but doing so degrades the resolution of the distribution [降低了 WVD 的分辨率]. Because the cross-terms oscillate, they can be removed by low-pass filtering [因为 cross-terms 是震荡的所以通过 LPF 过滤掉]. So there is a trade-off between smoothing the cross-terms and reduced resolution [trade-off 在 cross-terms 和 resolution].

Karhunen-Loeve Transform (KLT)

• KLT Basis functions are eigenvectors of the covariance matrix RXX of the input signal.

- This set of basis vectors is **not fixed**
- Basis vectors depend on the data set
- They are **orthonormal—linear transform**

• Karhunen-Loeve Transform (KLT) compression is an example of Linear Transform Coding (LTC).

• The process of LTC：

The image or signal to be compressed is divided into N blocks of pixels or samples. [分为 N 块]. Each block is then an N-dimensional vector, so we have a sequence of vectors [每一块形成 N 维矩阵]. Each vector, x, is then transformed by multiplying by a linear matrix A [乘线性矩阵 A]. We then discard the less significant values from the transformed data [删去不重要值] and multiply by another matrix B to reconstruct the image or signal, x̂ [乘 B 矩阵重构].

• The KLT minimises the mean squared error between x and x̂ by decorrelating the input data [减少均方误差通过解除输入数据的关系, PCA] and **discarding** the set with the least variance. [删去最小方差的集合]

KLT process:

1. Find mean vector for input data $\mathbf{X} = [\vec{x}_0, \vec{x}_1, \ldots \vec{x}_{N-1}]$
$$E(\vec{x}) = \frac{1}{N}\sum_{i=0}^{N-1}\vec{x}_i$$
2. Find covariance matrix
$$\mathbf{R}_{\mathbf{XX}} = \frac{1}{N-1}\sum_{i=0}^{N-1}(\vec{x}_i - E(\vec{x}))(\vec{x}_i - E(\vec{x}))^T$$
3. Find eigenvalues of the covariance matrix
$$|\mathbf{R}_{\mathbf{XX}} - \lambda \mathbf{I}| = 0$$
$|\mathbf{A}|$ means "Determinant of $\mathbf{A}$"
4. Find eigenvectors of the covariance matrix
$$[\mathbf{R}_{\mathbf{XX}} - \lambda_i \mathbf{I}]\vec{\phi}_i = 0$$
5. Normalise the eigenvectors
$$\langle \vec{\phi}_i, \vec{\phi}_i \rangle = 1$$
6. Transform the input
$$\mathbf{Y} = \phi^T \mathbf{X}$$
7. *To check*, find covariance matrix of $\mathbf{Y}$
$$\mathbf{R}_{\mathbf{YY}} = \frac{1}{N-1}\sum_{i=0}^{N-1}(\vec{y}_i - E[\vec{y}])(\vec{y}_i - E[\vec{y}])^T$$
8. *Optional,*
   1. set last row vector(s) of $\mathbf{Y}$ to 0
   $$\mathbf{Y} \rightarrow \mathbf{Y}'$$
   2. Inverse transform this
   $$\mathbf{X}' = \phi \mathbf{Y}'$$

• Principal Components Analysis (PCA):

Main use of PCA is to **reduce dimensionality** of a data set while retaining as much information as is possible.

Finds a projection of the observations onto orthogonal axes (basis functions).

Do **NOT** compress **multi-dimensional** data.

Correlated variables transformed into uncorrelated variables ordered by reducing variability.

Uncorrelated variables are linear combinations of original variables.

Non-linear correlation between variables are lost.

• Why KLT?

KLT is theoretically optimal in the MSE sense (MSE: Mean Square Error), i.e. it is a linear transform that minimises the MSE between an original set of data and a set constructed from a compressed version of data by ensuring the data with largest variance is retained during compression. [在均方误差的角度是最优化的，因为这是一个线性变换去减少原始数据和重构后的压缩数据的均方误差利用最大数据方差保留]

• KLT drawback:

Intensive computation[estimation of correlation is unwieldy(估算相关性不便处理); solution of eigenvector decomposition is slow(特征向量分解太慢)]

Transmission of data depends on the basis vectors, they are not fixed and need to be sent with the data.(传输数据取决于基然而基并不固定并且和数据一起发送)

Technique is linear, so non-linear correlation between variables is lost.（PCA 过程是线性的，变量非线性关系丢失）

Uncertainty Principle

• A signal fixed in time has infinite bandwidth.

• Explain the Heisenberg Uncertainty Principle: [乘积值与窗口方程形状有关关]

The Heisenberg Uncertainty Principle in the context of time-frequency analysis states that it is not possible to know accurately both the time location and frequency value at that time [时频分析说这是不可能得到精确值在时间定位和和该时间点的频率值]. There is an uncertainty in both time and frequency, and the product of the uncertainties is a constant [不确定值乘积是定值]. If Δt is the uncertainty in time and Δω is the uncertainty in frequency, it can be shown that (although the value of the constant depends on the definition of the Fourier Transform that is used in its proof):

$$\Delta_t \Delta_\omega \geq 1/2$$

• H's Uncertainty Principle implications

In time-frequency analysis, we use a moving window to investigate the frequency content of a signal at different time locations [用移动的窗来在不同时间定位确定频率内容]. The longer the duration of the window the more accurately the frequency content is known but the less accurately the time location is known [窗口持续时间越长频率越精确但是时间定位就不精确了]. The shorter the duration of the window the more accurately time location is known but the less accurately the frequency [与上句话相反]. The shape of the window function affects the accuracy of the results [窗口形状也影响精确度]. We need a function that dies away quickly and also has a limited frequency content [我们需要消失的快并且有限的频率内容]. It can be shown that the only function minimising the product of Δt and Δω is a Gaussian function $g(t) = e^{-at^2}$. The Gabor STFT uses a Gaussian function as its window. [Gabor Transform 用高斯方程作为窗]

Summary& Others

• Lossless compression is where fewer bits are required to store the data but no information is discarded. [无信息丢失]. This is achieved by eliminating redundancy [消除冗余]

• Lossy compression results in a loss of information [信息丢失]. This is achieved by discarding marginally important information [消除边界重要信息].

• **Sub-band filtering is used to decompose a signal into a set of band-limited components.**分解信号成有限带宽的部分(1314B Q4)

• Continuous transforms are impractical because there is redundant function, involve infinite time and cannot be carried out digitally.(信息冗余，时间无限，不能数字化)

• Compare Dual basis, a Biorthogonal Basis and a Frame &1314C Q1c

先说规范正交向量：如果空间可以由规范正交向量张成，然后空间里的任何向量就可以由他们唯一线性组合。这些向量叫做 basis

如果找不到规范正交基，就得找两组 vectors 他们互相正交 mutually orthogonal

Dual 可能线性相关，但是 biorthogonal may not.

Frame：如果 Dual 和 Biorthogonal 都不成了，就 Frames 上场 frame 他们既不正交也不线性无关 数量比空间维数多

• Compare the form of “window” functions that are used in STFT and WT.

c) In the STFT, the Fourier transform is carried out by multiplying the signal by a window function and a sinusoid (the complex exponential) [1 mark].

$$STFT(t,\omega)=\int_{-\infty}^{\infty} s(\tau)\gamma^*(\tau-t)e^{-j\omega t}d\tau \qquad \text{[1 mark]}$$

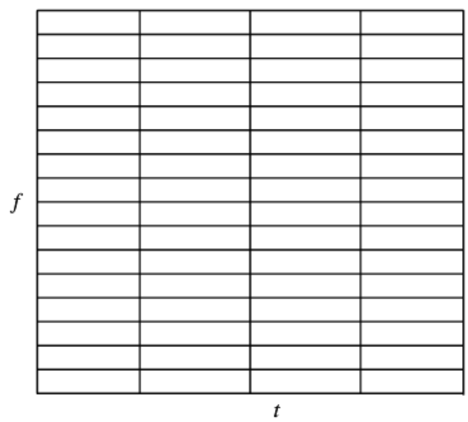
In the wavelet transform, the sinusoid is not present

$$CWT(a,b)=\frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty} s(t)\psi^*\left(\frac{t-b}{a}\right)dt \qquad \text{[1 mark]}$$

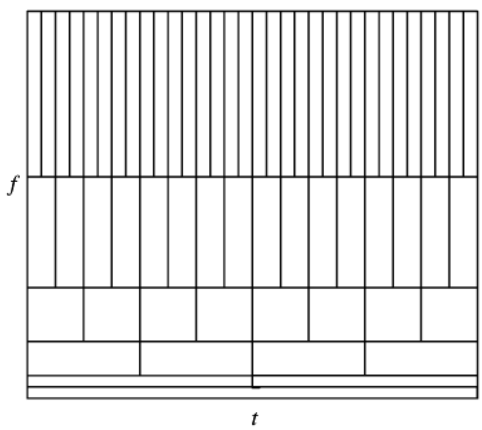
$$=\int_{-\infty}^{\infty} s(t)\psi_{a,b}^*(t)dt=<s,\psi_{a,b}>$$

This means that the wavelet function must be oscillatory [1 mark].

• Time frequency tilling.



信号STFT的时频分辨率



信号DWT的时频分辨率

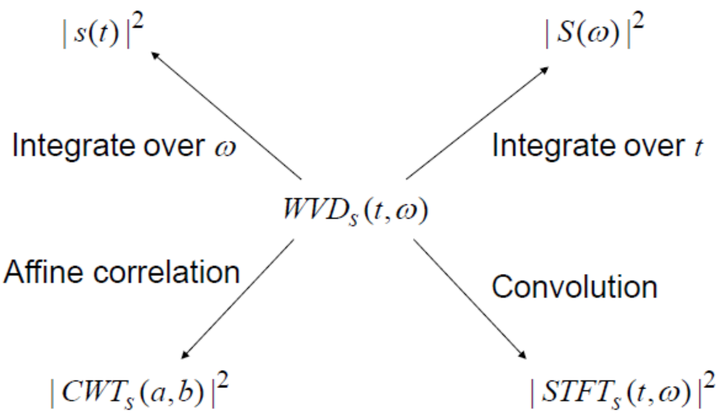
Note: For STFT, have constant time and frequency resolution at all frequencies.

For WT, variable time and frequency resolution depend on centre frequency. At low frequency, it has fine frequency resolution, but coarse time resolution. At high frequency, it has coarse frequency resolution, but fine time resolution.

• WT is more suitable than STFT in real world:

Many real world signals caused by energy impact with decaying vibrations(能量影响导致震动逐渐消失的信号): energy decays faster at high frequencies than low frequencies. （能量在高频小时更快）so we need higher time resolution in high frequency(在时间高分辨率) For musical signal, frequency resolution is constant proportion as measured by semitones.

• WVD-STFT-WT-Power Density



a) The definition of the Fourier Transform is $X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t)dt$

The definition of the Short-time Fourier Transform is $STFT(t,\omega) = \int_{-\infty}^{\infty} s(\tau)\gamma^*(\tau - t)e^{-j\omega t}d\tau$

The definition of the Wavelet Transform is

$$CWT(a,b) = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty} s(t)\psi^*\left(\frac{t-b}{a}\right)dt$$
$$= \int_{-\infty}^{\infty} s(t)\psi_{a,b}^*(t)dt = \langle s, \psi_{a,b} \rangle$$

	FT	STFT	CWT
Kind of decomposition	frequency	time-frequency	time-scale
Analysing function	infinitely long sines and cosines（无限长）	wave limited in time, multiplied by **sinusoidal oscillations(**the complex exponential**)**. Window size is fixed, but frequency inside the window varies	a wave limited in time with a fixed number of oscillations. Wavelet is compressed or dilated to change the widow size and change the scale at which one looks at the signal. Number of **oscillation**s is fixed so the frequency of the wavelet changes as

			scale changes.
Variable	frequency	frequency and the position of the window	scale and the position of the wavelet
Information obtained from the transform	frequencies that make up the signal	time and frequency, the smaller the window the better the time information and vice-versa.	time and frequency but small wavelets give good time information but poor frequency information. Vice versa for large wavelets.
Types of signal for which suited	stationary signals	quasi-stationary signals (stationary at the scale of the window)	nonstationary signals