Advanced Transform Methods

Multiresolution

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Tackling the Redundancy Problem

- CWT(a,b) encodes more than enough information to reconstruct the signal. A sampled version is sufficient.

- Let us sample in a dyadic grid: $a = 2^{-m} \quad b = n2^{-m}$

$$CWT(2^{-m}, n2^{-m}) = \frac{1}{\sqrt{|2^{-m}|}} \int_{-\infty}^{\infty} s(t)\psi^* \left(\frac{t - n2^{-m}}{2^{-m}} \right) dt$$

$$d_{m,n} = CWT(2^{-m}, n2^{-m}) = \int_{-\infty}^{\infty} s(t)\psi^*_{m,n} dt$$

where $\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n)$

- If the set of $\psi_{m,n}$ forms a frame, we can recover

$$s(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_{m,n}\hat{\psi}_{m,n}(t)$$

where $\hat{\psi}_{m,n}(t)$ forms the dual frame

Orthonormal Frame

For the orthonormal frame, we have $\hat{\psi}_{m,n}(t) = \psi_{m,n}(t)$

so we get the *wavelet series* (transform and inverse):

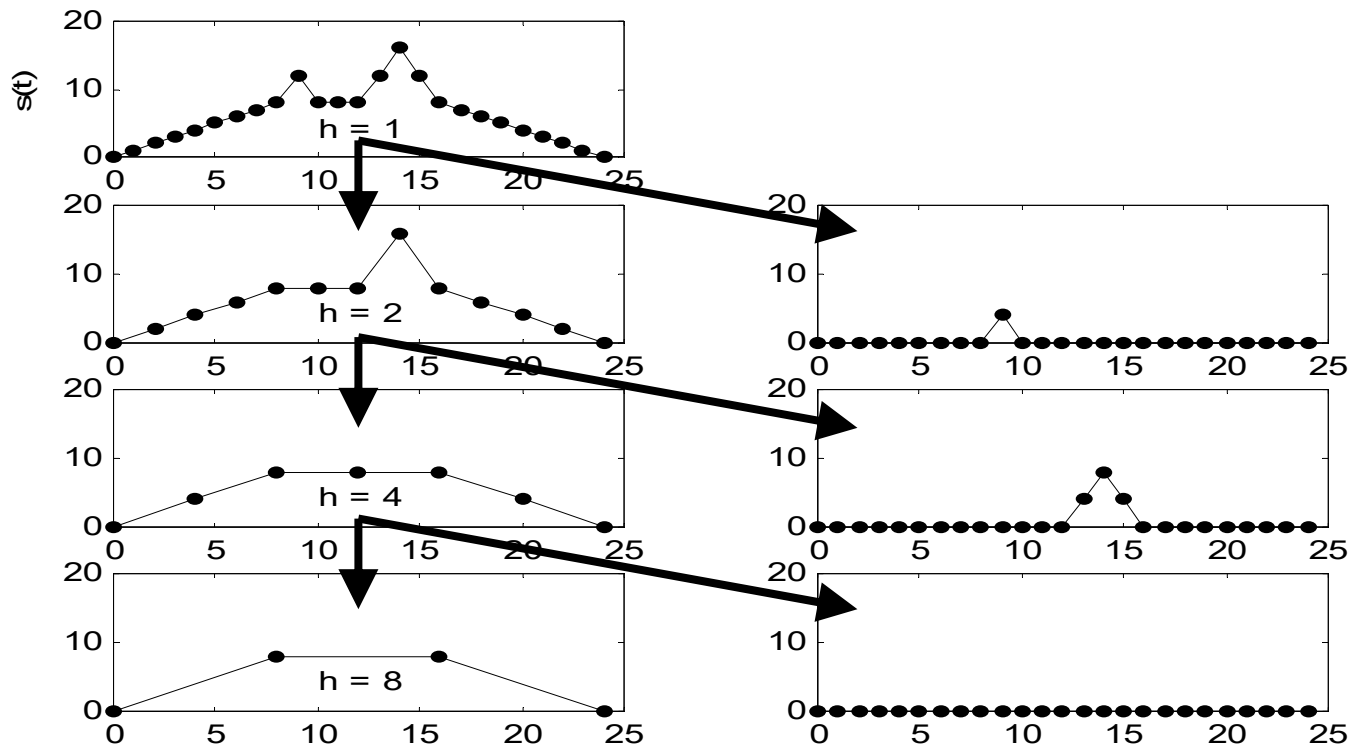$$d_{m,n} = \int_{-\infty}^{\infty} s(t) \psi^*_{m,n} dt$$

$$s(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_{m,n} \psi_{m,n}(t)$$

So: wavelet series is not redundant

Remaining problem: How to design orthogonal wavelets?
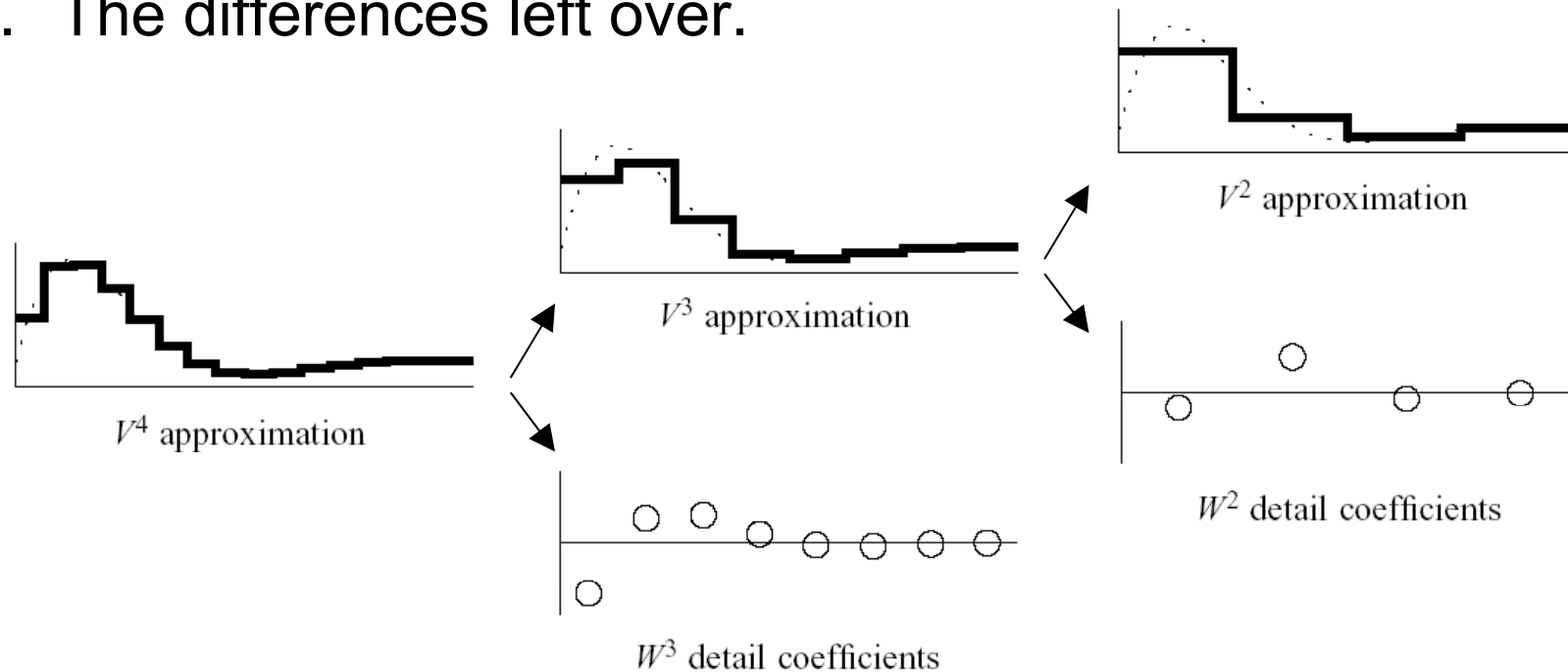
Multiresolution Analysis (MRA)

- Objective: To analyze a complicated function by dividing it into several simpler ones and studying them separately.

Piecewise Approximation

Basic Concept: Decompose a fine-resolution signal into

1. A coarse-resolution version of the signal, and

2. The differences left over.

V^4 approximation

V^3 approximation

V^2 approximation

W^3 detail coefficients

W^2 detail coefficients

So: what about wavelets? Theory coming next...

Multiresolution Analysis

A *scaling function* $\phi(t)$ generates a nested subsequence of subspaces $\{V_j\}$,

$$\{0\} \leftarrow \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

and satisfies a *dilation* (refinement) equation

$$\phi(t) = \sum_k p_k \phi(at - k) \qquad \text{for integer } k$$

for some $a > 0$ and coefficients $\{p_k\}$

We will consider $a = 2$ that corresponds to octave-scales.

Spaces of functions

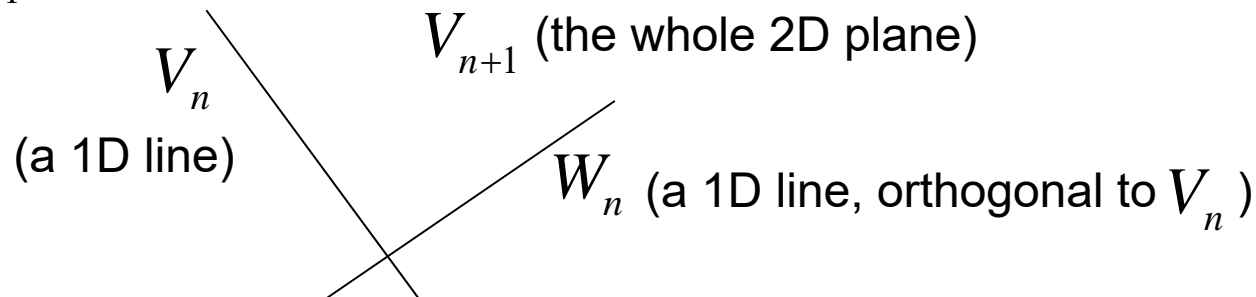The space V_0 is generated by $\{\phi(t-k)\}$ for integers k

In general, V_j is generated by $\{\phi(2^j t - k)\}$

Then:
$$V_n = \bigoplus_{j=-\infty}^{n-1} W_j \qquad \text{(orthorgonal sum of subspaces)}.$$

$$V_{n+1} = \bigoplus_{j=-\infty}^{n} W_j = \bigoplus_{j=-\infty}^{n-1} W_j \oplus W_n = V_n \oplus W_n$$

Subspace W_n is the *orthogonal complementary subspace* of V_n in V_{n+1}.

Conceptual idea:

V_n (a 1D line)

V_{n+1} (the whole 2D plane)

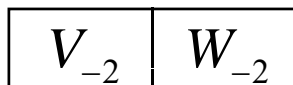W_n (a 1D line, orthogonal to V_n)

Multiresolution Analysis (cont)

Subspaces $\{V_j\}$ are nested

while subspaces $\{W_j\}$ are mutually orthogonal.

Consequently,

$$\begin{cases} V_j \cap V_l = V_j & l > j; \\ W_j \cap W_l = \{0\} & j \neq l; \\ V_j \cap W_l = \{0\} & j \leq l. \end{cases}$$

$\vdots$

V_1

V_0	W_0

V_{-1}	W_{-1}

i.e. V_1 can be decomposed

V_{-2}	W_{-2}

into combination of V_0 and W_0, etc.

$\vdots$

Multiresolution Analysis (cont.)

Subspaces $\{W_j\}$ are generated by *wavelets* $\psi(t)$

While $\{V_j\}$ is generated by scaling functions $\phi(t)$

In other words, any $f_j(t) \in V_j$

can be written as

$$f_j(t) = \sum_k c_k^j \phi(2^j t - k),$$

and any function $g_j(t) \in W_j$

can be written as

$$g_j(t) = \sum_k d_k^j \psi(2^j t - k),$$

for some coefficients $\{c_k^j\}_{k \in z}, \ \{d_k^j\}_{k \in z}$

Furthermore,

$$f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$$

$$f(t) \in V_j \Leftrightarrow f\left(t + \frac{1}{2^j}\right) \in V_j$$

Multiresolution Analysis (cont.)

Subspaces $\{V_j\}$ are called "approximation subspaces" and $\{W_j\}$ are the "wavelet subspaces".

At any level j, V_j contains the smooth part and W_j contains the "details" of the original function.

$$j \uparrow \Rightarrow f_j(t), \text{ a finer approximation of } f(t),$$

$$j \downarrow \Rightarrow f_j(t), \text{ a coarser approximation of } f(t).$$

Two-Scale Relations

Since $\phi(t) \in V_0 \subset V_1,$ and $\psi(t) \in W_0 \subset V_1;$

we should be able to write $\phi(t)$ and $\psi(t)$ in terms of the bases that generate V_1.

In other words, there exists two sequences $\{p_k\}, \{q_k\}$

such that

$$\phi(t) = \sum_k p_k \phi(2t - k),$$
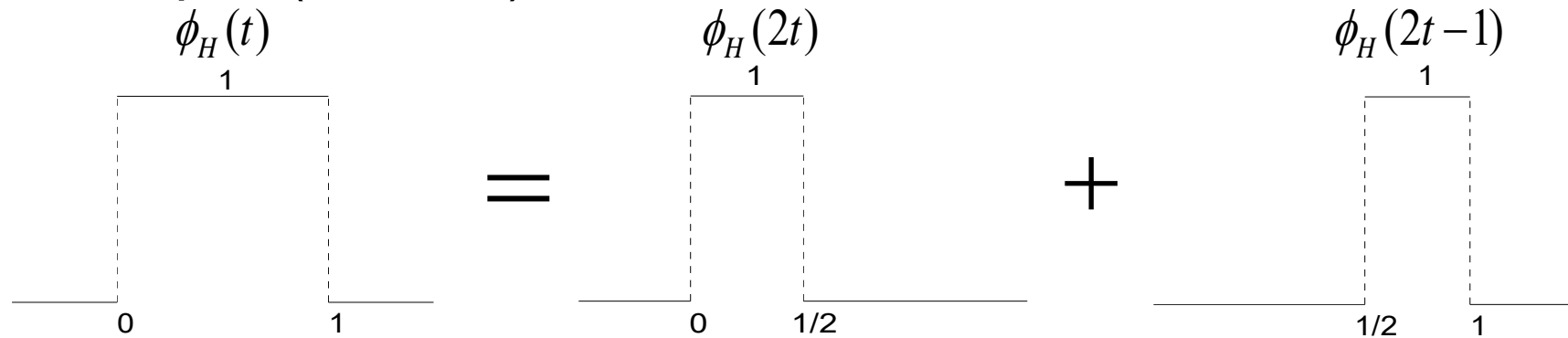
$$\psi(t) = \sum_k q_k \phi(2t - k).$$

In general, for any j, the relationship between V_j, W_j with V_{j+1} is governed by

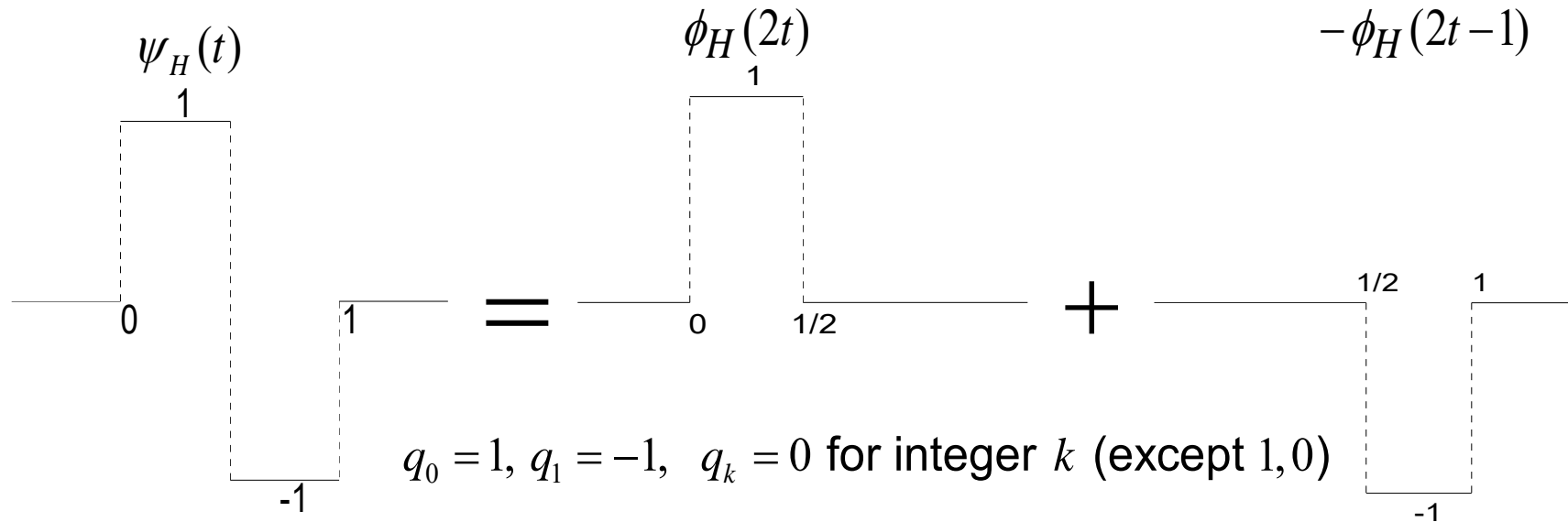$$\phi(2^j t) = \sum_k p_k \phi(2^{j+1} t - k),$$

$$\psi(2^j t) = \sum_k q_k \phi(2^{j+1} t - k).$$

Two-Scale Relations (cont.)

Example: (H: Haar)

$$\phi_H(t) \qquad \phi_H(2t) \qquad \phi_H(2t-1)$$

$$=\qquad +$$

$$p_0 = p_1 = 1, \quad p_k = 0 \text{ for integer } k \text{ (except } 1,0)$$

$$\psi_H(t) \qquad \phi_H(2t) \qquad -\phi_H(2t-1)$$

$$=\qquad +$$

$$q_0 = 1, \ q_1 = -1, \quad q_k = 0 \text{ for integer } k \text{ (except } 1,0)$$

Decomposition Relation

Since $V_1 = V_0 + W_0$ and $\phi(2t),\ \phi(2t-1) \in V_1$, there exist two pairs of sequences

$$(\{a_{2k}\},\ \{b_{2k}\})$$

such that

$$\phi(2t) = \sum_k \{a_{2k}\phi(t-k) + b_{2k}\psi(t-k)\};$$

$$\phi(2t-1) = \sum_k \{a_{2k-1}\phi(t-k) + b_{2k-1}\psi(t-k)\}.$$

Combining these two relations, we have

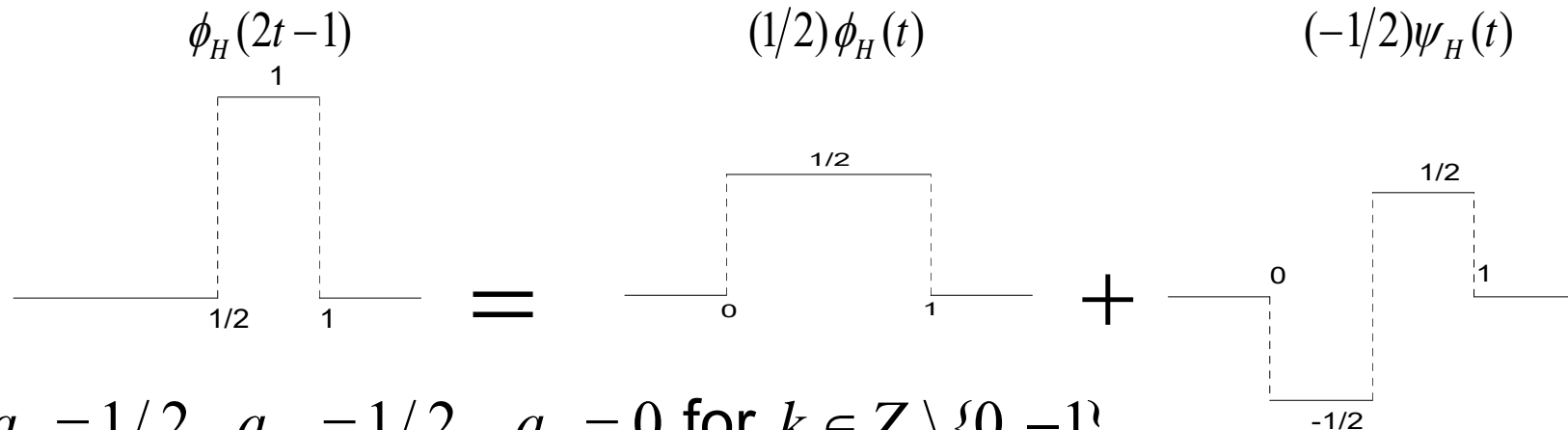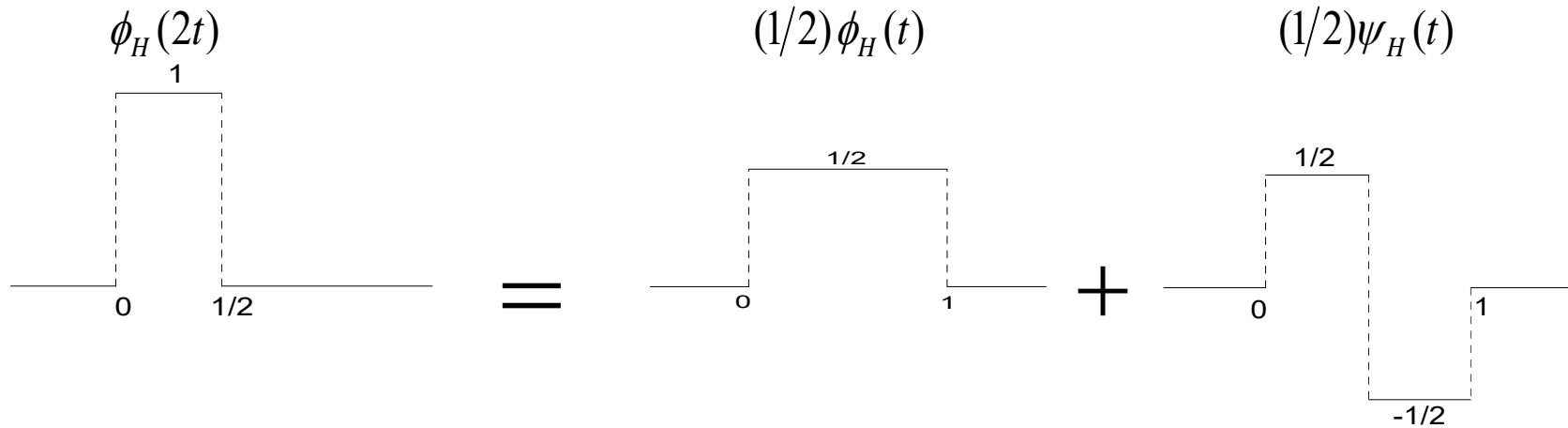$$\phi(2t-l) = \sum_k \{a_{2k-l}\phi(t-k) + b_{2k-l}\psi(t-k)\}.$$

for all integer l

In general, we have

$$\phi(2^{j+1}t-l) = \sum_k \{a_{2k-l}\phi(2^j t-k) + b_{2k-l}\psi(2^j t-k)\}.$$

Decomposition Relation (cont.)

Example: (H: Haar)

$$\phi_H(2t) \qquad (1/2)\phi_H(t) \qquad (1/2)\psi_H(t)$$

$$\phi_H(2t-1) \qquad (1/2)\phi_H(t) \qquad (-1/2)\psi_H(t)$$

$$a_0 = 1/2, \ a_{-1} = 1/2, \ a_k = 0 \text{ for } k \in Z \setminus \{0, -1\}$$

$$b_0 = 1/2, b_{-1} = -1/2, \ b_k = 0 \text{ for } k \in Z \setminus \{0, -1\}$$