SOLUTIONS

Module:	Advanced Transform Methods		
Module Code	EBU6018	Paper	A
Time allowed	2hrs	Filename	**Solutions_201920_EBU6018_A**
Rubric	ANSWER ALL FOUR QUESTIONS		
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Question 1.

(a) The matrix A is: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

i) Find an orthogonal basis of the NULL space of A. [10 marks]
ii) Find the RANK of A. [2 marks]
iii) Find an orthonormal basis of the ROW space of A. [5 marks]

Answer:

1920 Q 1 (a)
A

i) NULL SPACE CONSISTS OF THE SOLUTIONS OF

$$Ax = 0 \qquad [1 \text{ MARK}]$$

LET $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ [1 MARK]

THEN $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ [1 MARK]

GIVING $\left.\begin{matrix} x_1 = -x_3 \\ x_2 = 0 \end{matrix}\right\}$ [2 MARKS]

THEN $x = \begin{bmatrix} -x_3 \\ 0 \\ x_2 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ [1 MARK]

SO $x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ IS A BASIS FOR NULL SPACE OF A
[1 MARK]

FOR ORTHONORMALITY, FIND NORM OF x [1 MARK]

$\|x\| = \sqrt{2}$ [1 MARK]

SO ORTHONORMAL BASIS REQUIRED IS $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ [1 MARK]

ii) A HAS 2 NON-ZERO ROWS [1 MARK]
SO RANK IS 2 [1 MARK]

iii) FOR ROW SPACE OF A, NON-ZERO ROWS ARE A BASIS
[1 MARK]

SO BASIS IS $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ [1 MARK]

DOT PRODUCT IS 0, $\therefore$ ORTHOGONAL [1 MARK]

NORMS ARE $\sqrt{2}$ AND 1 RESPECTIVELY. [1 MARK]

SO ORTHONORMAL BASIS FOR ROW SPACE OF A IS:

$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ [1 MARK]

(b) An FFT is a fast algorithm for implementing a DFT.

i) Estimate the approximate number of computations that are required to perform the FFT of an 8-point sequence. [2 marks]

ii) One FFT structure is radix-2 decimation-in-time. Illustrate this FFT structure using the following 8-point sequence:

S[n] = [2, 6, 3, 9, 7, 4, 1, 11]

[6 marks]

Answer: i) number of computations is $N\log_2 N$ [1 mark]
So no of computations = 8x3 = 24. [1 mark]

S[n] = [7, 3, -5, 2, 6, 4, -1, 8]

STEP 1: [7, -5, 6, -1][3, 2, 4, 8]

STEP 2: [7, 6][-5, -1][3, 4][2, 8]

STEP 3: [7][6][-5][-1][3][4][2][8]

[6 marks: 2 for each step]

Question 2.

(a) With the aid of a suitable diagram, briefly explain the application of Linear Transform Coding (LTC) to the processing of images. [10 marks]

Answer:

(a)

Divide image (or signal) into P blocks of N pixels (samples). [1 mark]

The kth block is now an N-dimensional vector: [1 mark]

$$\mathbf{x}_k = (x_{1,k}, x_{2,k}, \ldots, x_{N,k})^T$$

The image (signal) is now a sequence of vectors $\{\mathbf{x}_k\}$.[1*mark*]

We now transform each $\mathbf{x}$ by multiplying by a linear matrix

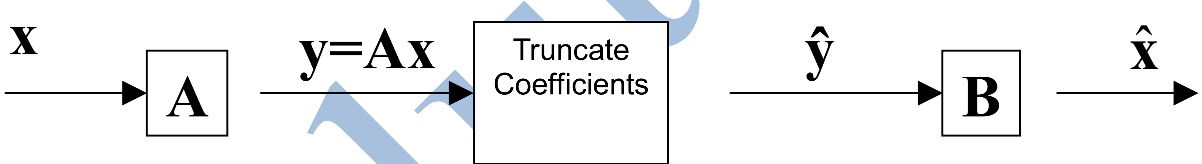
$\mathbf{y} = \mathbf{A}\mathbf{x}$[1*mark*]

transmit the first M coefficients $\hat{\mathbf{y}} = (y_1, \ldots, y_M)^T$, [1 mark]

discarding the remaining $N - M$ coeffs $y_{M+1} \cdots y_N$[1*mark*]

We then reconstruct the image block using another matrix

$\hat{\mathbf{x}} = \mathbf{B}\hat{\mathbf{y}}$[1*mark*]



[3 marks for diagram, 1 for each block]

(b) i) Briefly explain the use of the Karhunen-Loeve Transform (KLT) in the compression of images. [8 marks]

iii) List and state the advantage and THREE (3) disadvantages of the KLT for the compression of images, and THREE (3) advantages of using the Discrete Cosine Transform (DCT). [7 marks]

Answer:

i) Uses Principal Component Analysis (PCA) Multivariate statistics. [1 mark]

Finds a projection of the observations onto orthogonal axes contained in the space defined by the original variables. [1 mark]

Correlated variables transformed into uncorrelated variables. [1 mark]
Ordered by reducing variability. [1 mark]
Computes compact, optimal description of data set. [1 mark]
Rotates data so that maximum variabilities projected onto the axes. [1 mark]
Rotates existing axes to new positions in the space defined by the original variables. [1 mark]

Uses PCA is to reduce dimensionality of a data set while retaining as much information as is possible. [1mark]

ii)

Advantage: KLT maximises the coding gain, i.e. maximises the SNR after a given level of compression. [1 mark]

Disadvantages: [3 marks, 1 mark for any three of the following list]

- Estimate of correlation can be unwieldy
- Solution of eigenvector decomposition is computationally intensive (i.e. slow)
- Calculation of forward and inverse transforms is $O(MN)$ for each image block
- Transmission of data-dependent basis **A** is required
- The technique is linear, therefore any non-linear correlation between variables will not be captured.

Comparison: [3 marks: 1 mark for any three of the following list]

DCT has fixed basis functions.
a good approximation to KLT for typical images.
needs no eigenvalue decomposition.
transform is $O(N \log N)$.

Question 3.

(a) Compare Short-time Fourier Transform (STFT) ($\gamma(t)$) and Continuous Wavelet Transform (CWT) ($\psi(t)$) in terms of

i) similarities and
ii) differences. [5 marks]

Answer:

i) Similarities [3 marks, 1 mark each]
- Signal is multiplied by a function, and the transform is computed separately for different segments of signals. [1mark]
- Both can be written in inner product form

$$STFT(b,\omega)=\left\langle s(t),\gamma(t-b)e^{j\omega t}\right\rangle \quad CWT(b,a)=\left\langle s(t),\frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)\right\rangle$$ [1mark]

- Time-frequency window area remains constant [1mark]

ii) Differences [2 marks, 1 mark each]
- Fixed time duration and freq bandwidths of $\gamma(t)$ [1 mark]
- Variable time duration and bandwidth of $\psi(t)$ [1 mark]

b) Describe the problems with Continuous Wavelet Transform (CWT) [5 marks]

Answer:

Described 1 problem: [1 mark]
Described 2 problems: [3 marks]
Described 3 problems: [5 marks]

1. Redundancy
: Basis functions for CWT are shifted and scaled versions off each other. Cannot form a very orthonormal base.

2. Infinite solution space
: The result holds an infinite number of wavelets: hard to solve and hard to find the desired results out of the transformed data.

3. Efficiency
: Most transforms cannot be solved analytically. Solutions must be calculated numerically: time-consuming.

(c) Describe multi resolution analysis in terms of its
i) objective
ii) concept with a diagram describing piecewise approximation of a signal. [5 marks]

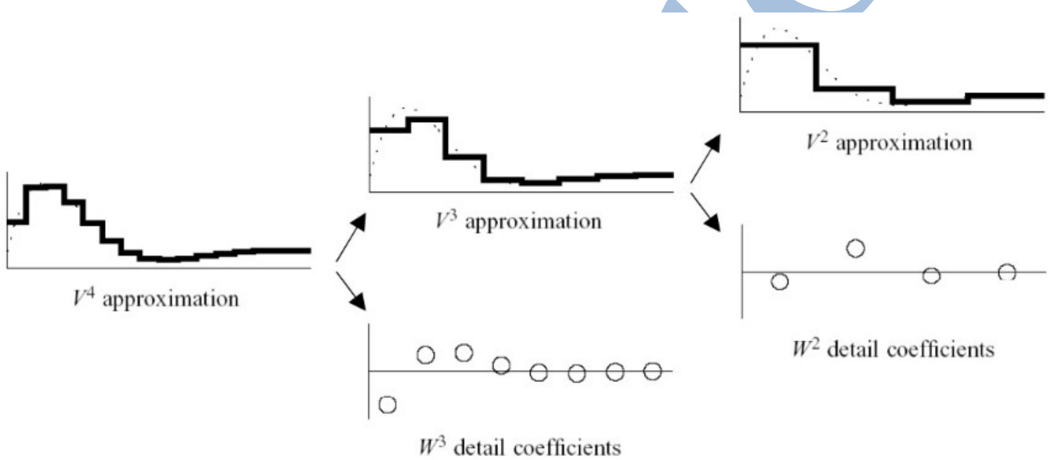
Answer:

i) objective [2 marks]
: To analyze a complicated function by dividing it into several simpler ones and studying them separately.

ii) concept with a diagram describing piecewise approximation of a signal [3marks]

[1 mark] for the description
[2 marks] for the diagram

- Decompose a fine-resolution signal into a coarse-resolution version of the signal and the differences left over. [1 mark]



d) Apply the transform defined by

$$x_{n-1,i} = (x_{n,2i} + x_{n,2i+1})/2$$
$$d_{n-1,i} = (x_{n,2i} - x_{n,2i+1})/2$$

to the sequence

$$[x_{n,i}] = [3,4,2,3,4,2,2,3] ,$$

where i = 0…..7, is the index position in the sequence, and n is the level. The next level is $n-1$.

At each level, calculate the sequences for $x_{n-1,i}$ and $d_{n-1,i}$

Continue till no further levels are possible.

i) State the significance of the first element in the final level

ii) Has any information lost in the process?

iii) Describe how this process could be used to compress the data. [10 marks]

Answer:

Applying the transform:

n=3 [3.0, 4.0, 2.0, 3.0, 4.0, 2.0, 2.0, 3.0],

n=2 [3.5, 2.5, 3.0, 2.5, -0.5, -0.5, 1.0, -0.5]

n=1 [3.0, 2.75, 0.5, 0.25, -0.5, -0.5, 1.0, -0.5]

n=0 [2.875, 0.125, 0.5, 0.25, -0.5, -0.5, 1.0, -0.5] [6 marks: 2 for each row]

i) The first element is the average of all the elements in the original sequence [1 mark].

ii) No information has been lost [1 mark]

iii) Because most of the values in the final level are small, potentially fewer bits would be required to store it [1 mark]. Where there are zeroes, they do not need to be stored, although the positions of the other values would need to be stored [1 mark]. Small values could be replaced by zeroes without significant loss of detail [1 mark], this can be done by applying a threshold value, the bigger the threshold the greater the loss of detail [1 mark].

Question 4.

(a) A Haar wavelet transform is implemented using an analysis filterbank using normalised low-pass and high-pass filters:

$$h_0 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \quad h_1 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$

Calculate the Haar Transform for 2 levels of decomposition for the input sequence:
$s[n] = [2, 0, 2, 4]$. [10 marks]

Answer:
[5 marks] 1st level

$$\left[\frac{2+0}{\sqrt{2}}, \frac{2+4}{\sqrt{2}}; \frac{2-0}{\sqrt{2}}, \frac{2-4}{\sqrt{2}}\right]$$

$$= \left[\frac{2}{\sqrt{2}}, \frac{6}{\sqrt{2}}; \frac{2}{\sqrt{2}}, \frac{-2}{\sqrt{2}}\right]$$

[5 marks] 2nd level

$$\left[\frac{2+6}{\sqrt{2}\cdot\sqrt{2}}, \frac{2-6}{\sqrt{2}\cdot\sqrt{2}}; \frac{2-2}{\sqrt{2}\cdot\sqrt{2}}, \frac{2+2}{\sqrt{2}\cdot\sqrt{2}}\right]$$

$$= \left[\frac{8}{2}, \frac{-4}{2}; 0, \frac{4}{2}\right] = [4, -2; 0, 2]$$

(b) Filter banks are used to implement wavelet transforms.

The analysis low pass filter is referred to as H_0. and the high pass filter as H_1.

The synthesis low pass filter is referred to as γ_0 and the high pass filter as γ_1.

For orthogonal analysis filters:

$$H_1(z) = (-z)^{-N} H_0(-z^{-1})$$

And for the synthesis filters

$$\gamma_0(z) = H_1(-z)$$

$$\gamma_1(z) = -H_0(-z)$$

Daubechies wavelets are orthogonal. For a Daubechies 2nd order wavelet, determine

i) $H_1[n]$,

ii) $\gamma_0[n]$,

iii) $\gamma_1[n]$,

in terms of the lowpass filter $H_0[n] = [0.4830, 0.8365, 0.2241, \quad 0.1294]$. [9 marks]

Answer:

$H_0[n] = [H_0[0], H_0[1], H_0[2], H_0[3]]$

$H_1[n] = [H_0[3], -H_0[2], H_0[1], -H_0[0]] = [0.1294, -0.2241, 0.8365, -0.4830]$ [3 marks]

$\gamma_0[n] = [H_0[3], H_0[2], H_0[1], H_0[0]] = [-0.1294, 0.2241, 0.8365, 0.4830]$ [3 marks]

$\gamma_1[n] = [-H_0[0], H_0[1], -H_0[2], H_0[3]] = [-0.4830, 0.8365, -0.2241, -0.1294]$ [3 marks]

c) In the wavelet transform, the scaling function coefficients $c_{m,n}$ and wavelet series coefficients $d_{m,n}$ can be calculated recursively according to the following equations:

$$c_{m-1,n} = \sqrt{2} \sum_i h_0[i-2n] c_{m,i}$$
$$d_{m-1,n} = \sqrt{2} \sum_i h_1[i-2n] c_{m,i}$$

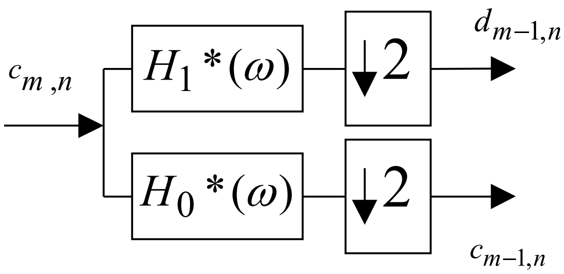
Explain how this can be interpreted in terms of filtering and downsampling, and hence leads to the concept of an *analysis filterbank*. Sketch a diagram to illustrate this filterbank. [6 marks]

Answer:

If $c_{m,i}$ are the scaling function coefficients at level *m*, we can calculate the scaling function coefficients and wavelet series coefficients $d_{m-1,n}$ at level *m*-1 recursively from these.
[2 marks]

The signals h_0 and h_1 are (time-reversed) low-pass and high-pass filters. [1 mark]
Steps of 1 in *n* correspond to steps of 2 in *i*, corresponding to downsampling by a factor of 2 from *n* to *i*. [1 mark]

Therefore these equations represent filtering followed by downsampling, as shown in the diagram:



[2 marks for diagram]