2.4 Fundamental Equations of Electrostatics

- divergence equation
- curl equation
- material equation

☆ Preface

Static E field is a vector field with a divergence unequal to zero.

To analyze the static E-field, we need to know

- 1 parameter
- 2 approaches
- 3 variables

1 parameter	2 approaches	3 variables	
ε	Difference equations	Source variable	ρ
$\vec{D} = \varepsilon\vec{E}$	Integral equations	Field variable 1	$\vec{E}(\vec{r})$
Material equations		Field variable 2	$\vec{D}(\vec{r})$

3 Variables

ρ **Volume density** of free charges. It's the source variable. It's the reason why static E field has divergence.

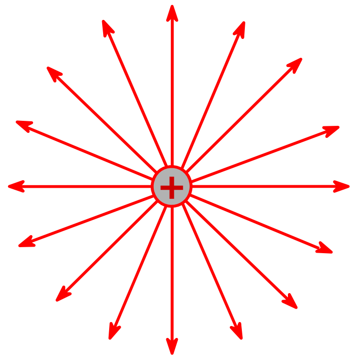
$\vec{E}(\vec{r})$ **Electric Field Intensity**, describing the action by E field on charged matter. **V/m**

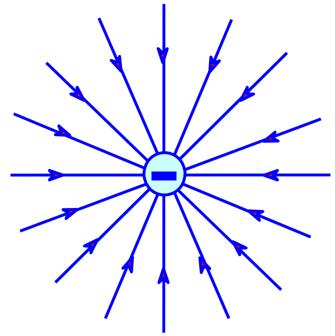
$\vec{D}(\vec{r})$ **Electric Flux Density**, or Electric Displacement

It's the electric ***flux*** per unit ***area.***

C/m²

Electric flux, =magnitude of charge, in Coulombs





Electric flux density $\vec{D}$ **C/m²** **Surface charge density**

$$\boxed{\begin{aligned}\vec{D} &= \frac{Q}{4\pi r^2}\vec{a}_R \\ \vec{D} &= \varepsilon\vec{E}\end{aligned}}$$

☆ Electrostatic Gauss's Law

$$\int_V (\nabla \bullet \vec{A}) dv = \oint_S \vec{A} \bullet d\vec{s}$$ ——静电场高斯定理

- Recall Gauss's Law
  - For a continuously differentiable vector field, the net outward flux from a closed surface equals the integral of the divergence throughout the region bounded by that surface.
  - 高斯定理：矢量场散度的体积分＝该矢量穿过包围该体积的封闭曲面的总通量
  - Now we learn **Gauss's Law in Electrostatic Case**.

☆ Div Equ. for Electrostatics

Integral form

$$\oint_S \vec{E} \bullet d\vec{S} = \sum q / \varepsilon$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\oiint \vec{D} \bullet d\vec{S} = \sum q_{fc}$$

$$\oiint \vec{D} \bullet d\vec{S} = \sum q_{fc}$$

The **net outward flux** passing through a closed surface equals to the total charge enclosed by that surface.

Please note: q refers to free charge.

Prove：see P82-83 of Guru textbook

Differential form of Div Equ.

Gauss's Law $\int_V (\nabla \bullet \vec{A}) dv = \oint_S \vec{A} \bullet d\vec{s}$

Integral form $\oint \vec{D} \bullet d\vec{S} = \sum q$

$$\int_V (\nabla \bullet \vec{D}) dv = \sum q = \int_V \rho dv$$

$$\nabla \bullet \vec{D} = \rho$$

Please note: ρ here refers to volume density of free charge.

Review the Div Equation

Div Equ: $\oint \vec{D} \bullet d\vec{S} = \sum q$ (Integral form) $\qquad \nabla \bullet \vec{D} = \rho$ (Differential form)

- Physical Meaning:
  - describing the scattering character of static E field
  - giving the relationship between E flux through a closed surface and the charges within the closed surface.
  - For integral equation:
    - E-flux through any closed surface S = charges within S
    - If 0, there is no charge within S, i.e. no source within S.
    - Flux Source of Static E Field is Free Charges.
  - For differential equation:
    - Electrostatic Div = Volume density of Q at that point
    - Div Source of Static E Field is Volume density of Free Charges.

Example 1. Calculate $\vec{D}$

- A spherical region (radius a) is full of free charges, for which the volume density is $\rho(\vec{r})=\rho_0(1-r^2/a^2)$. Please calculate $\vec{D}$.
- Analysis:
  - spherical region---point symmetry---spherical coordinates
  - Treat the fields inside and outside the sphere respectively.
- Solution 1. via *Electrostatic Gauss's Law* $\oint_S \vec{E}\bullet d\vec{S}=\frac{1}{\varepsilon_0}\int_V \rho dV$

$$\oint_S \vec{E}\bullet d\vec{S}=\oint_S E_R\cdot dS=E_R\cdot(4\pi r^2)$$

$$\int_V \rho dV=\int_0^r \rho(r)\cdot(4\pi R^2)dR=\begin{cases} ? & \text{inside sphere } (r\le a) \\ ? & \text{outside sphere } (r>a)\end{cases}$$

$$\vec{E}=?\Rightarrow \vec{D}=\varepsilon_0\vec{E}=?$$

- Solution 2. via fundamental equations

$$\nabla\bullet\vec{D}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2D)=\rho=\begin{cases}\rho(r) & \text{inside sphere} \\ 0 & \text{outside sphere}\end{cases}$$

- Boundary conditions are applied to determine the integral constant.
  - When $r=a$ …
  - When $r=\infty$ …
  - We will learn to apply the boundary conditions later on.

Example 2. Calculate Charge Distribution

- E-intensity in space is known as follows. Please determine the charge distribution.

$$\vec{\boldsymbol{E}}=\vec{a}_rE_0(r/a)^2 \qquad 0<r<a$$
$$\vec{\boldsymbol{E}}=\vec{a}_rE_0(a/r)^2 \qquad r>a$$

- Analysis:
  - Due to spherical symmetry, E has only radial component;
  - Apply div equ in differential form;

$$\vec{E}=?\Rightarrow\vec{D}=\varepsilon_0\vec{E}=?$$
$$\nabla\bullet\vec{D}=\rho$$

- Please check after the class time that the results for Example 2 are

$$\rho=\varepsilon_0\nabla\cdot\vec{\boldsymbol{E}}=\frac{4\varepsilon_0E_0r}{a^2} \qquad 0<r<a$$
$$\rho=0 \qquad r>a$$

☆ Electrostatic Gauss's Law

$$\oint_S \vec{E} \bullet d\vec{S} = \frac{1}{\varepsilon}\int_V \rho dV = \frac{Q}{\varepsilon}$$

Kernel of this law:

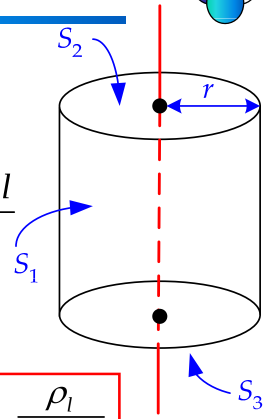
1. on Left Side: Net outward flux of E from a closed surface
2. on Right Side: Total charges within the closed surface over ε

It is significantly useful for
——solution to E Intensity in symmetrical cases.

Example 3. Infinite Line Charges

Solution 3. Indirect Solution via Gauss's Law

- Axial Symmetry——construct a cylindrical surface, in unit height, with line charges as the axis, and r as the radius.



$$\oint_S \vec{E} \bullet d\vec{S} = \int_{S1} \vec{E} \bullet d\vec{S} + \int_{S2} \vec{E} \bullet d\vec{S} + \int_{S3} \vec{E} \bullet d\vec{S} = \frac{\rho_l \cdot l}{\varepsilon_0}$$

Since the E field has only radial component,

$$\oint_S \vec{E} \bullet d\vec{S} = \int_{S1} \vec{E} \bullet d\vec{S} + 0 + 0$$

$$= 2\pi r E_r = \frac{\rho_l}{\varepsilon_0}$$

$$\therefore \boxed{\vec{E} = \vec{a}_r E_r = \vec{a}_r \frac{\rho_l}{2\pi r \varepsilon_0}}$$

Please note this tip.

When the charge distribution is symmetrical,
——Try *E-Gauss's Law*!

Kernel of *E-Gauss's Law*:

(1) Find a closed surface $(\vec{S})$

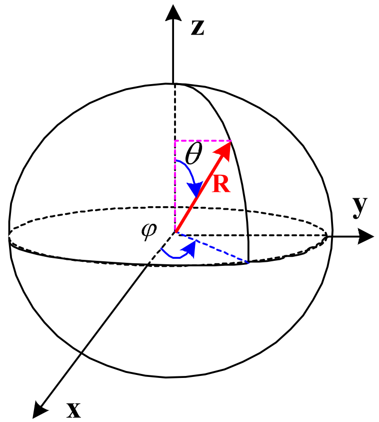
(2) The quantity of $\vec{E}$ *on the surface is constant.*

$$\oint_S \vec{E} \bullet d\vec{S} = \frac{Q}{\varepsilon}$$

Example 4. Spherical Charges

Example 3.9 P85 in textbook

- Conductor ball in space, with charge of Q, radius of a, Try to calculate the E Intensity inside and outside the ball.
- **Popular Solution**:
  - Surface charge density is
    $$\sigma_s = Q/4\pi a^2$$
  - Differential surface element is
    $$ds = ad\theta \cdot a\sin\theta d\varphi$$
  - Then we get the differential charge element dq and apply vector sum
  - We must be very careful of the direction in integral.



Advanced Solution

Due to symmetrical distribution

We apply *E-Gauss's Law*

$$\boxed{\oint_S \vec{E}\bullet d\vec{S} = \frac{1}{\varepsilon_0}\int_V \rho dV = \frac{Q}{\varepsilon_0}}$$

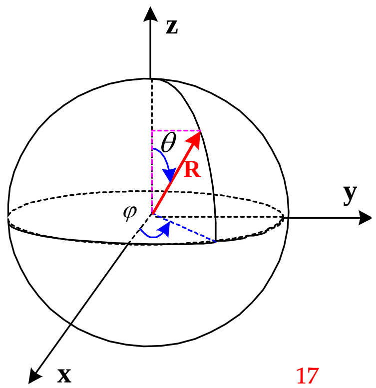
Inside the ball ($r<a$): $\because \frac{1}{\varepsilon_0}\int_V \rho dV = 0 \quad \therefore \vec{E} = 0$

Outside the ball ($r>a$):

$$\because \frac{1}{\varepsilon_0}\int_V \rho dV = ? = \frac{Q}{\varepsilon_0}$$

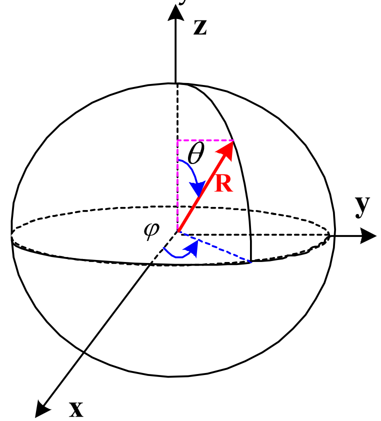
$$\oint_S \vec{E}\bullet d\vec{S} = E\cdot(4\pi\cdot r^2)$$

$$\boxed{\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0\cdot r^2}\cdot Q\vec{a}_R}$$



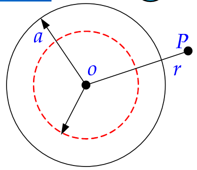
Example 5. Spheri-form Charges

- **A ball** in space full of charge, with volume charge density of ρ_0, radius of a. Try to calculate the E Intensity inside and outside the ball.
- **Popular Solution:**
  - Volume charge density is ???
  - Differential volume element is
  - Then we get the differential charge element dq and apply vector sum
  - We must be very careful of the direction in integral.
- **Simple Solution:**
  - Via *E-Gauss's Law*



- r>=a, E Intensity is similar to that in Example 4.

$$\boxed{\vec{E} = \vec{a}_R\frac{Q}{4\pi\varepsilon_0\cdot r^2} = \vec{a}_R\frac{\rho_0 4\pi a^3/3}{4\pi\varepsilon_0\cdot r^2} = \frac{\rho_0}{3\varepsilon_0}\cdot\frac{a^3}{r^2}}$$



- r<a:
  - Construct a inner ball with radius r
  - According to E-Gauss's Law, the charges in the inner ball contribute to E(r).

$$\oiint_S \vec{E}\bullet d\vec{S} = E\cdot(4\pi\cdot r^2)$$

in direction of $\vec{a}_R$

$$Q' = (\frac{4}{3}\pi r^3)\times\rho_0$$

$$E = \frac{1}{4\pi r^2}\times\frac{Q'}{\varepsilon_0} = \frac{1}{4\pi r^2}\times\frac{1}{\varepsilon_0}\times\frac{4}{3}\pi\cdot r^3 = \frac{\rho_0}{3\varepsilon_0}r$$

☆ Curl Equ. for Electrostatics

$$\oint_C \vec{E}\bullet d\vec{l} = 0 \qquad\qquad \nabla\times\vec{E} = 0$$

Integral Form — Differential form

E Intensity for point charge: $\vec{E}(\vec{R},q_1) = \frac{q_1}{4\pi\varepsilon_0}\cdot\frac{1}{R^2}\vec{a}_R$

In spherical coordinates: $\nabla = \vec{a}_R\frac{\partial}{\partial R} + \vec{a}_\theta\frac{1}{R}\frac{\partial}{\partial\theta} + \vec{a}_\varphi\frac{1}{R\cdot\sin\theta}\frac{\partial}{\partial\varphi}$

Note that $\nabla\left(\frac{1}{R}\right) = -\vec{a}_R\frac{1}{R^2}$

We obtain $\vec{E}(\vec{R},q_1) = -\frac{q_1}{4\pi\varepsilon_0}\nabla(\frac{1}{R})$

$$\nabla\times\vec{E} = \nabla\times(-\nabla U)\equiv 0$$

☆ Curl Equ. for Electrostatics

$$\oint_C \vec{E} \bullet d\vec{l} = 0$$

Integral Form

$$\nabla \times \vec{E} = 0$$

Differential form

- First of all, they are **valid only for static E** field, but not any type of E field;
- Integral form
  - It tells us **electrostatic circulation is zero.**
  - *C* refers to a certain closed curve
  - Directions of *C* and corresponding surface obey Rule of Right Hand;
- Differential form
  - It tells us **electrostatic curl is zero**,
  - no mater whether there is charge at that spot or not.

$\oint_C \vec{E} \bullet d\vec{l} = 0$

Derivation of the Curl Equ in Integral Form:

- According to ***General Physics*** in 1st year, E-force will do no work when moving a point charge from spot A to spot A, regardless its specific path.
- The work by E-force is similar to that by gravity.
- This kind of field is called a conservative field.
- Hence the Curl Equ. of Electrostatic Field in integral form.
- Detailed description is found in textbook pp. 86-87.

$\boxed{\nabla \times \vec{E} = 0}$

Describing the field at a certain point in space.

True for both cases whether there is charge at that point or not.

Question

The electric field intensity for a certain electric field is given as

$$\vec{E} = \vec{e}_x(yz - 2x) + \vec{e}_y xz + \vec{e}_z xy$$

Whether the field is conservative?
And why?

Review the Curl Equ.

$$\oint_C \vec{E} \bullet d\vec{l} = 0$$

Integral Form

$$\nabla \times \vec{E} = 0$$

Differential form

- Physical meaning:
  - Static E-field is a conservative or W/O rotational field.
  - Work by this field in moving a charge depends only on the endpoints, independent of specific path.
  - Integral form implies electrostatic circulation along any closed path is ZERO.
  - Differential form implies there exists no curl source for static E-field.

☆ Fundamental Equations

Integral form | Difference form

1. Gauss's Law in space

Div Equ. $\oint \vec{D} \bullet d\vec{S} = \sum q$ $\qquad \nabla \bullet \vec{D} = \rho$

2. Conversation law for Electrostatics

Curl Equ. $\oint \vec{E} \bullet d\vec{l} = 0$ $\qquad \nabla \times \vec{E} = 0$

3. Material Equ. $\vec{D} = \varepsilon \vec{E}$

Why do we present the same idea in 2 different forms?

- The integral form is useful to explain the significance of an equation;
- The differential form is convenient for performing mathematical operation.

☆ conclusions

Static E field is a vector field with a divergence unequal to zero.

To analyze the static E-field, we need to know

- 1 parameter
- 2 approaches
- 3 variables

1 parameter	2 approaches	3 variables	
ε	Difference equations	Source variable	ρ
$\vec{D} = \varepsilon \vec{E}$		Field variable 1	$\vec{E}(\vec{r})$
Material equations	Integral equations	Field variable 2	$\vec{D}(\vec{r})$

☆ Fundamental Equations

Integral form | Difference form

1. Gauss's Law in space

Div Equ. $\oint \vec{D} \bullet d\vec{S} = \sum q$ $\qquad \nabla \bullet \vec{D} = \rho$

2. Conversation law for Electrostatics

Curl Equ. $\oint \vec{E} \bullet d\vec{l} = 0$ $\qquad \nabla \times \vec{E} = 0$

3. Material Equ. $\boxed{\vec{D} = \varepsilon \vec{E}}$

Left for latter hours.