Chapter 1. Vector Analysis

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3 Degrees & 3 Laws

1. **Gradient** of Scalar
2. Flux of vector, **Divergence**, **Gauss's Law**
3. Circulation、**Curl**, **Stokes's Law**
4. **Helmholtz Theorem**

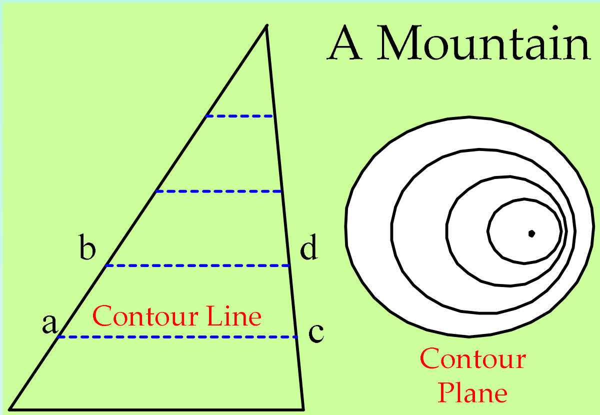
Gradient——*grad*

Divergence——*div*

Curl——*curl*

1.3 Gradient of a scalar field

- A scalar field $U(\vec{r}, t)$ four variables
- Gradient is a method of describing the space rate of change of a scalar field at a given time.
- We define the vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar as the gradient of that scalar field
- 在空间任何一点，标量场梯度的方向是该点标量场场量增加最快的方向；它的模是由该点向各个不同方向移动时场量可能有的最大增加率。

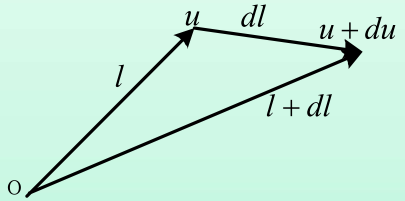


1.3 Gradient of a scalar field

$$\frac{\Delta U}{\Delta l} \Rightarrow \frac{dU}{dl}$$

数学模型：标量函数u，沿某个方向的**空间变化率**情况

The direction $\qquad d\vec{l} = \vec{a}_x dx + \vec{a}_y dy + \vec{a}_z dz$



The increment $\qquad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$

$$= \boxed{\left(\vec{a}_x \frac{\partial u}{\partial x} + \vec{a}_y \frac{\partial u}{\partial y} + \vec{a}_z \frac{\partial u}{\partial z} \right)} \bullet d\vec{l} = \boxed{\nabla u} \bullet d\vec{l} = \nabla u \bullet \vec{a}_l dl$$

$$\left(\vec{a}_x \frac{\partial u}{\partial x} + \vec{a}_y \frac{\partial u}{\partial y} + \vec{a}_z \frac{\partial u}{\partial z} \right) \bullet \left(\vec{a}_x dx + \vec{a}_y dy + \vec{a}_z dz \right) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$\boxed{\frac{\partial U}{\partial l} = (\nabla U) \bullet \vec{a}_l = \frac{dU}{dn} \vec{a}_n \bullet \vec{a}_l}$$

$$\boxed{\frac{\partial U}{\partial l} = (\nabla U)\bullet \vec{a}_l = \frac{dU}{dn}\vec{a}_n \bullet \vec{a}_l}$$

$$\boxed{\nabla U = gradU = \vec{a}_n \frac{dU}{dn}} \qquad Gradient\text{——}grad$$

The quantity and direction of a Gradient meet with the maximum increasing rate at which a scalar changes relative to especially distance.

Then, *How about the increasing rate at which a scalar changes along the other direction? 假设 l 是任意方向，n 为最大方向*

$$\frac{\partial U}{\partial l} = \frac{dU}{dn}\vec{a}_n \bullet \vec{a}_l = \frac{\partial U}{\partial n}\cos\theta_{l,n}$$

$$\left(\frac{\partial U}{\partial l}\right)_{\max} = \left(\frac{\partial U}{\partial n}\cos\theta_{l,n}\right)_{\max} = \frac{\partial U}{\partial n}$$

So the increment along a certain direction $dU = (\nabla U)\bullet d\vec{l}$

Now, we get to know the *Gradient*

$$\nabla U = gradU = \vec{a}_n \frac{dU}{dn}$$

We also need to know a **operator（算符）**

$$\nabla$$

Hamiltonian， called del

Features of Hamiltonian operator

- **w/o meaning by itself**
- **Has both vector and differential characters**
- **Only operate to function after the operator**

∇ *Hamiltonian, called del*

Features of Hamiltonian operator

- **w/o meaning by itself**
- **Has both vector and differential characters**

标量三重积 *Scalar Triple Product*

$$\vec{A} \bullet (\vec{B} \times \vec{C}) = \vec{B} \bullet (\vec{C} \times \vec{A}) = \vec{C} \bullet (\vec{A} \times \vec{B})$$

$$\boxed{\nabla \bullet (\vec{F} \times \vec{G}) = ?}$$

$$\boxed{= \vec{G} \bullet \nabla \times \vec{F} - \vec{F} \bullet \nabla \times \vec{G}}$$

Gradient in different coordinates

Cartesian Coordinates

$$\nabla = \vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}$$

Cylindrical Coordinates

$$\nabla = \vec{a}_r \frac{\partial}{\partial r} + \vec{a}_\varphi \frac{1}{r} \frac{\partial}{\partial \varphi} + \vec{a}_z \frac{\partial}{\partial z}$$

Spherical Coordinates

$$\nabla = \vec{a}_R \frac{\partial}{\partial R} + \vec{a}_\theta \frac{1}{R} \frac{\partial}{\partial \theta} + \vec{a}_\varphi \frac{1}{R \cdot \sin\theta} \frac{\partial}{\partial \varphi}$$

How to memorize it?

$$\boxed{\nabla \Rightarrow \frac{d}{dl}}$$

Cartesian Coordinates

Differential Length

$$d\vec{l} = \vec{a}_x dx + \vec{a}_y dy + \vec{a}_z dz$$

$$\nabla = \vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}$$

Cylindrical Coordinates

Differential Length

$$d\vec{l} = \vec{a}_r dr + \vec{a}_\varphi (r \cdot d\varphi) + \vec{a}_z dz$$

$$\nabla = \vec{a}_r \frac{\partial}{\partial r} + \vec{a}_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi} + \vec{a}_z \frac{\partial}{\partial z}$$

Spherical Coordinates

Differential Length

$$d\vec{l} = \vec{a}_R dR + \vec{a}_\theta (R \cdot d\theta) + \vec{a}_\varphi (R \cdot \sin\theta \cdot d\varphi)$$

$$\nabla = \vec{a}_R \frac{\partial}{\partial R} + \vec{a}_\theta \frac{1}{R}\frac{\partial}{\partial \theta} + \vec{a}_\varphi \frac{1}{R \cdot \sin\theta}\frac{\partial}{\partial \varphi}$$

Example 1

Known: $V = V(R,\theta) = V_0 \cdot R \cdot \cos\theta$ and $\vec{E} = -\nabla V$

Ask: $\vec{E} = ?$

Approach 1. Direct Approach——According to definition

$$\vec{E} = -\nabla V = ?$$

$$\nabla = \vec{a}_R \frac{\partial}{\partial R} + \vec{a}_\theta \frac{1}{R}\frac{\partial}{\partial \theta} + \vec{a}_\varphi \frac{1}{R \cdot \sin\theta}\frac{\partial}{\partial \varphi}$$

$$\boxed{\vec{E} = -\nabla V = -(\vec{a}_R \cos\theta - \vec{a}_\theta \sin\theta)V_0}$$

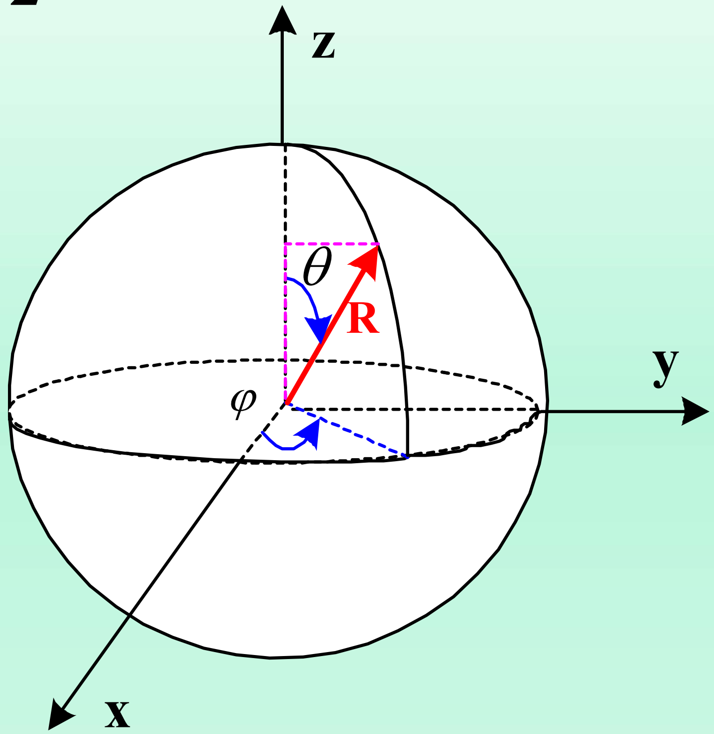
Approach 2. Comparison & Analysis

$$V = V(R,\theta) = V_0 \cdot R \cdot \cos\theta = V_0 \cdot z$$

Apply the Cartesian Coordinates!

$$\nabla = \vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}$$

$$\vec{E} = -\nabla V = -\vec{a}_z V_0$$



Answer 1.

$$\vec{E} = -\nabla V = -(\vec{a}_R \cos\theta - \vec{a}_\theta \sin\theta)V_0$$

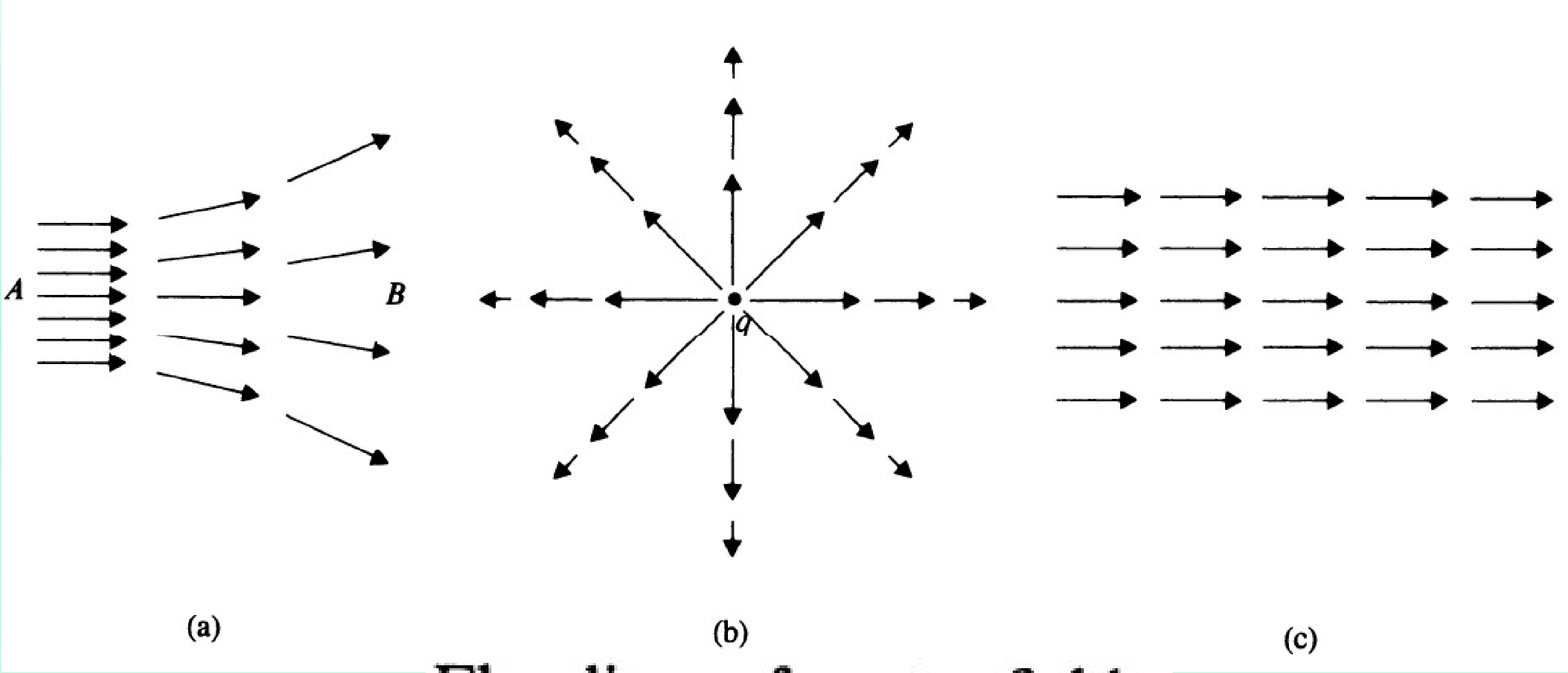
Answer 2.

$$\vec{E} = -\nabla V = -\vec{a}_z V_0$$

Both are correct!!

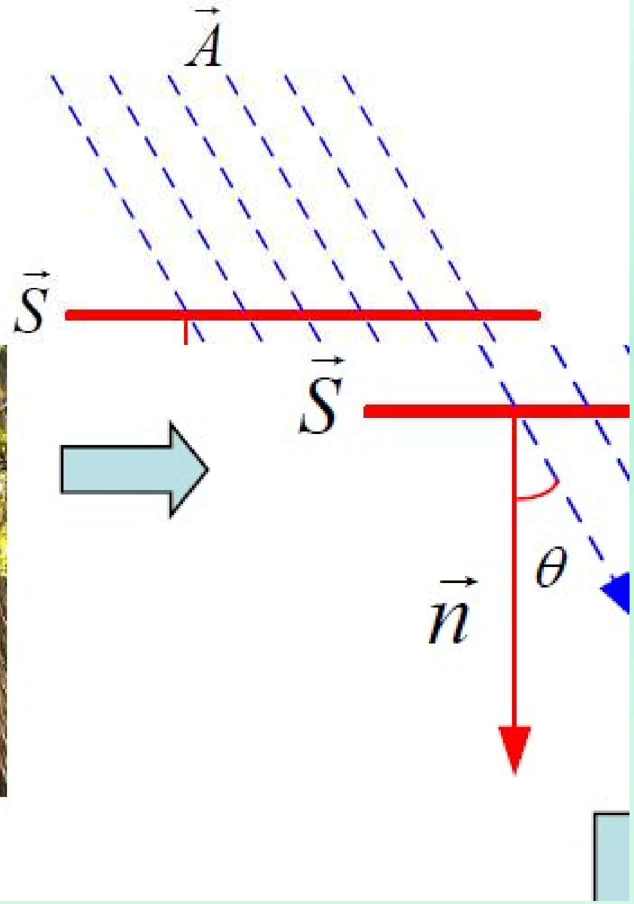
Divergence of a vector filed

Flux lines: directed lines or curves, with density is magnitude of vector, direction is the direction of vector at each point



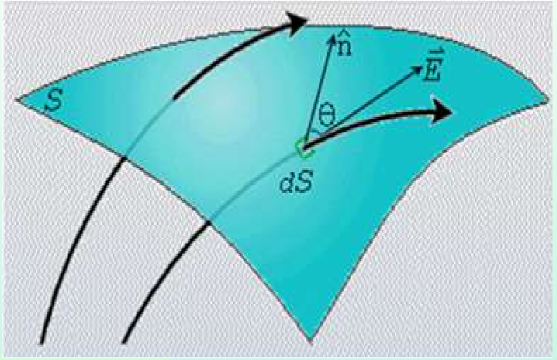
Flux lines of vector fields.

Flux (通量)



1.4 Flux, Divergence and Gauss's Law

Flux (通量) $\psi = \int_s \vec{A} \cdot d\vec{S}$



The outward flow of the vector field $\vec{A}$ through the surface $\vec{S}$.

$$\psi = \oint_S \vec{A} \bullet d\vec{S}$$

gives the ***net outward flow*** of flux of the vector field $\vec{A}$ from closed surface S (or from volume V).

矢量场沿某一有向曲面的面积分称为该矢量通过该面的**通量**。

The net outward flux of a vector A through a surface bounding a volume indicates the presence of a source, this can be called

flow source

Divergence

散 度，空间某一点矢量场的发散特性

检验空间每一点是否是 flow source

Divergence is a vector's *net outward flow per unit volume*.

So it is the volume **density of net outward flow of a vector.**

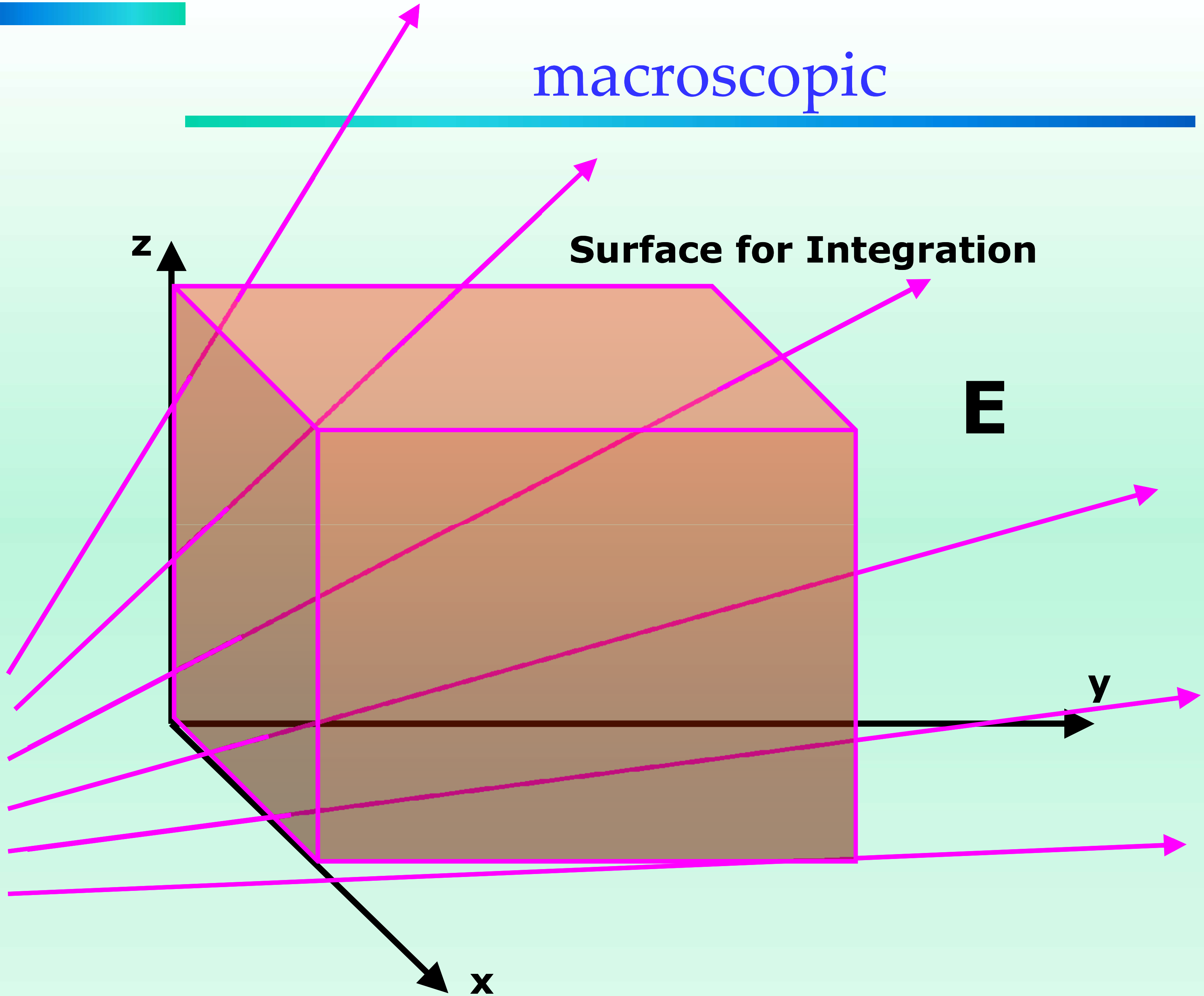
散度定义：单位体积的净流散通量

$$div\vec{A} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{A} \bullet d\vec{s}}{\Delta V}$$

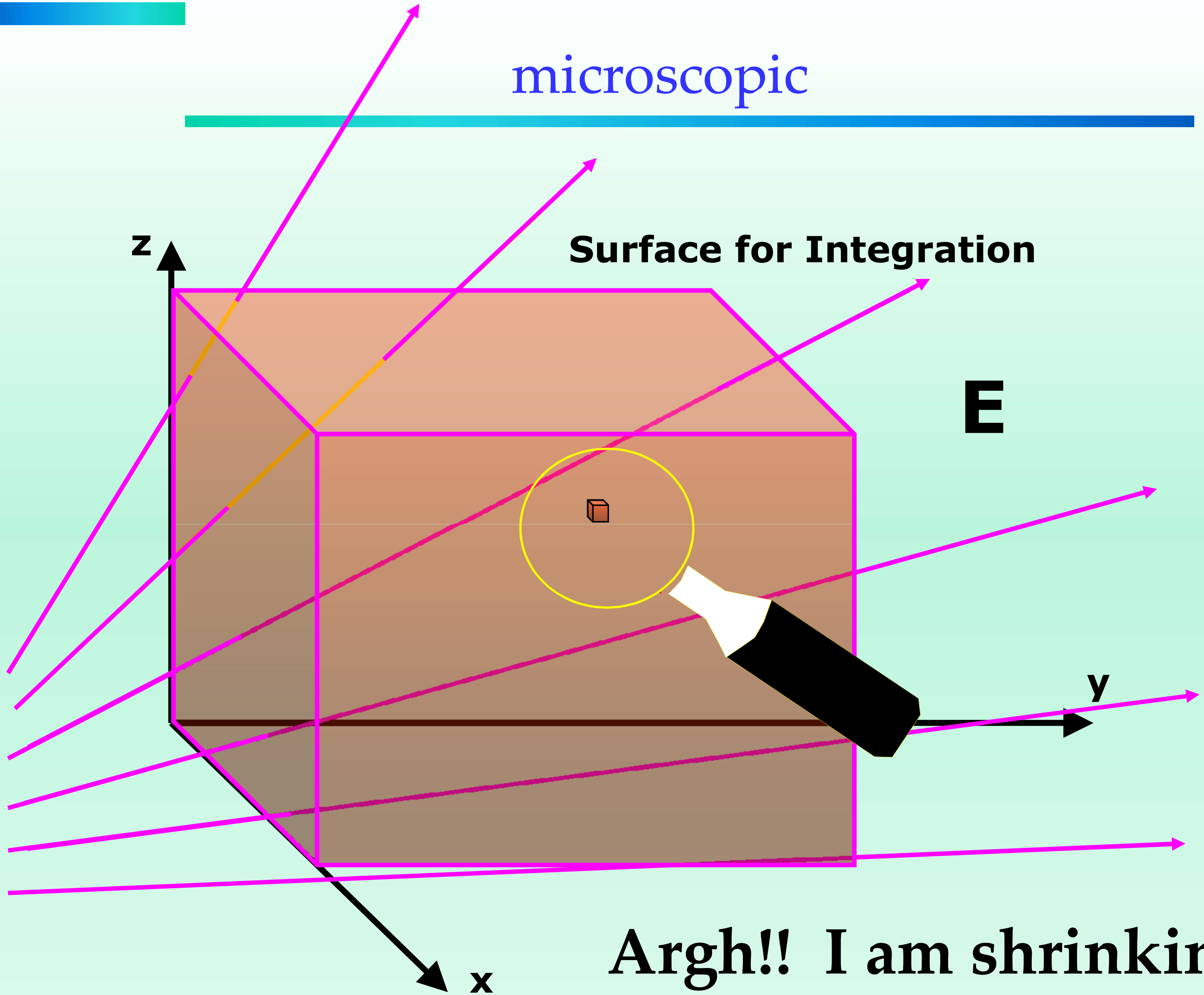
Div. is the net outward flow of flux of $\vec{A}$ from per unit volume.

- Divergence is a scalar.
- Divergence is a micro parameter.
- Divergence is the outward flux of a vector at a specific point of the space.
- 描述矢量场在空间某一点上在其平行方向上的变化关系

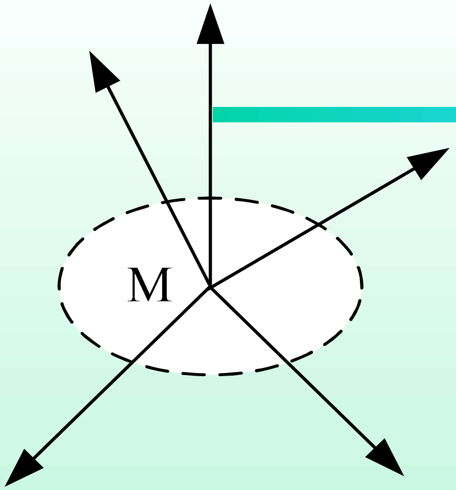
macroscopic



microscopic

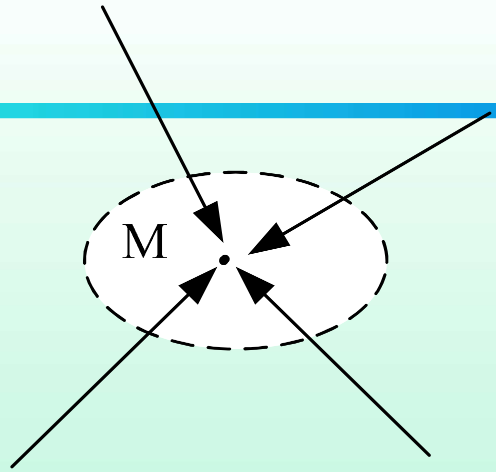


Argh!! I am shrinking!!!



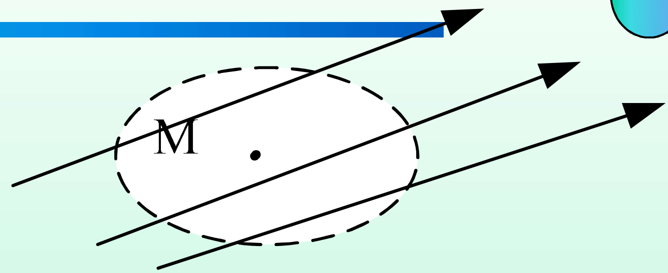
$$div\vec{A} > 0$$

a source point



$$div\vec{A} < 0$$

a sink point



$$div\vec{A} = 0$$

a common point

If the Div everywhere is zero, the field is called a continuous field, or a solenoidal field（管形场、无散场）

Divergence in **Cartesian Coordinates**

$$div\vec{A} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{A} \bullet d\vec{s}}{\Delta V}$$

分别计算矢量通过六个面的通量

$\vec{S}_x(\mathrm{x, y, z})$的面元： $\Delta\vec{\mathrm{S}}_x(\mathrm{x, y, z}) = \Delta y \Delta z(-\vec{a}_x)$

$\vec{S}_x(\mathrm{x, y, z})$的通量： $(\mathrm{A}_x\vec{a}_x)\bullet(\Delta\vec{\mathrm{S}}_x) = \text{-}\mathrm{A}_x\Delta y\Delta z$

$\vec{S}_x(\mathrm{x}+\Delta x, \mathrm{y, z})$的面元： $\Delta\vec{\mathrm{S}}_x(\mathrm{x}+\Delta x, \mathrm{y, z}) = \Delta y \Delta z(\vec{a}_x)$

$$\vec{A}_x(\mathrm{x}+\Delta x, \mathrm{y, z}) = (A_x + \Delta A_x)\vec{a}_x = (A_x + \frac{\partial A_x}{\partial x}\Delta x)\vec{a}_x$$

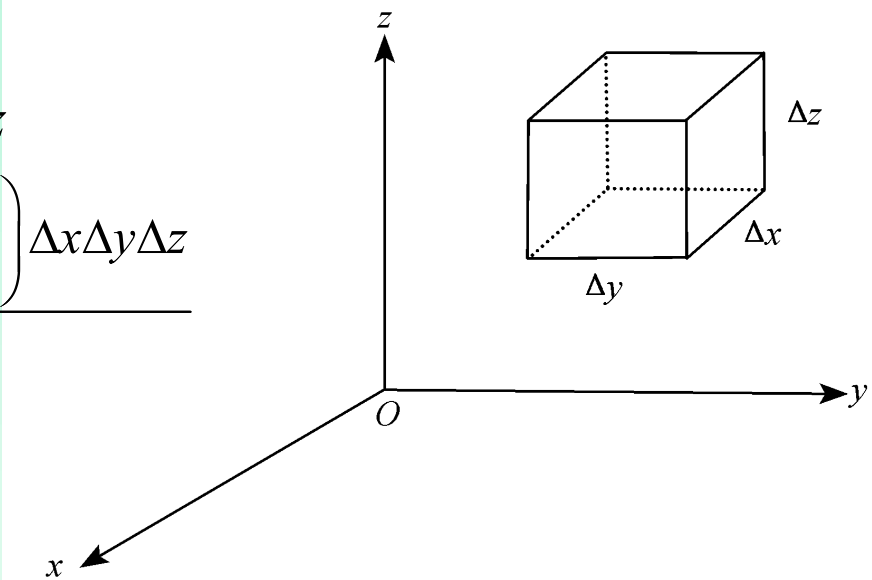
$\vec{S}_{x+\Delta x}$的通量： $\vec{A}_x(\mathrm{x}+\Delta x, \mathrm{y, z})\bullet\Delta\vec{\mathrm{S}}_x(\mathrm{x}+\Delta x, \mathrm{y, z}) = (A_x + \frac{\partial A_x}{\partial x}\Delta x)\Delta y\Delta z$

$\vec{S}_{x+\Delta x}$和$\vec{S}_x$平面的净通量：

$$(A_x + \frac{\partial A_x}{\partial x}\Delta x)\Delta y\Delta z\text{-}A_x\Delta y\Delta z = \frac{\partial A_x}{\partial x}\Delta x\Delta y\Delta z$$

$$\mathrm{div}\vec{\mathrm{A}} = \lim_{\Delta V \to 0}\frac{\oint_S \vec{\mathrm{A}}\cdot \mathrm{d}\vec{\boldsymbol{S}}}{\Delta V} = \lim_{\Delta V \to 0}\frac{\left(\frac{\partial \mathrm{A}_x}{\partial x} + \frac{\partial \mathrm{A}_y}{\partial y} + \frac{\partial \mathrm{A}_z}{\partial z}\right)\Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z}$$

$$= \frac{\partial \mathrm{A}_x}{\partial x} + \frac{\partial \mathrm{A}_y}{\partial y} + \frac{\partial \mathrm{A}_z}{\partial z}$$



Divergence in Different Coordinates

Cartesian Coordinates

$$\boxed{\nabla \bullet \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

Cylindrical Coordinates

$$\nabla \bullet \vec{A} = \frac{1}{r} \cdot \frac{\partial}{\partial r}(r \cdot A_r) + \frac{1}{r} \cdot \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

Spherical Coordinates

$$\boxed{\nabla \bullet \vec{A} = \frac{1}{R^2} \cdot \frac{\partial}{\partial R}(R^2 \cdot A_R) + \frac{1}{R \cdot \sin\theta} \cdot \frac{\partial}{\partial \theta}(A_\theta \cdot \sin\theta) + \frac{1}{R \cdot \sin\theta} \cdot \frac{\partial A_\varphi}{\partial \varphi}}$$

You Have to Remember Div **in Cartesian Coordinates**

$$\nabla = \vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}$$

$$\nabla \bullet ? = (\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}) \bullet ?$$

$$\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

$$\nabla \bullet \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence Theorem

Inspect the definition of Divergence

先看散度定义：单位体积的净流通量

$$div\vec{A} = \lim_{\Delta V \to 0}\left(\frac{\oint_S \vec{A} \bullet d\vec{s}}{\Delta V}\right) = \nabla \cdot \vec{A}$$

$$div\vec{A} = \nabla \bullet \vec{A}$$

So

$$\boxed{\int_V (\nabla \bullet \vec{A})dv = \oint_S \vec{A} \bullet d\vec{s}}$$

It is so called divergence theorem, or *Gauss's Law.*

$$\int_V (\nabla \bullet \vec{A})dv = \oint_S \vec{A} \bullet d\vec{s}$$

- Divergence Theorem
  - For a continuously differentiable vector field, the *net outward flux* from a closed surface equals *the integral of the divergence* throughout the region bounded by that surface.
  - 散度定理：某一区域内连续可微的矢量场，空间每一点上净通量的体密度之和等于整个区域的净通量
  - **It is powerful** when we need to ***convert a closed surface integral into an equivalent volume integral.***
  - And vice versa.

Curl of a vector filed

Divergence of a vector filed： **flow source** **in vector field direction**

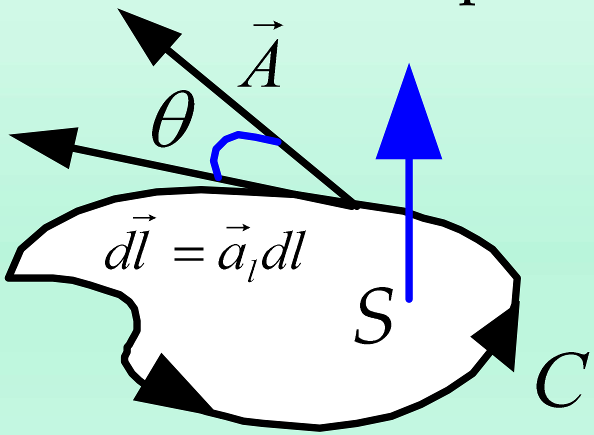
Curl of the vector field: **vortex (漩涡) source** **in perpendicular to vector field direction**

1.5 Circulation, Curl and Stokes' Law

Circulation（环量）:

Magnitude: the line integral of a vector along a closed path.

e.g. the circulation of $\vec{A}$ with aspect to C is $\oint_C \vec{A} \bullet d\vec{l}$



Directions of the surface and the closed path match ***Right-Hand Rule.***

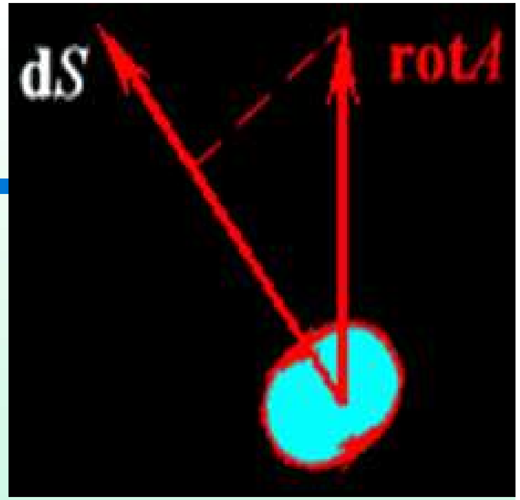
Curl (or Rotation) ——面环流密度

Curl Definition

$$curl\vec{A} = \vec{e}_{(curl\vec{A})}\left|curl\vec{A}\right|$$

$$\left|curl\vec{A}\right| = \max[\lim_{\Delta S \to 0}\left(\frac{\oint_c \vec{A}\cdot d\vec{l}}{\Delta S}\right)]$$

$$\vec{e} \quad = \vec{e} \quad when \ \left|curl\vec{A}\right|$$



The curl of a vector A, is a vector those magnitude is the maximum net circulation of A per unit area as the area tends to zero, and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum.

It is a micro-parameter, or a distributed parameter.

Curl in Cartesian Coordinates

$$\vec{A}\times\vec{B} = \vec{a}_{AB}(A\cdot B\cdot\sin\theta_{AB}) = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{B} = \vec{a}_x B_x + \vec{a}_y B_y + \vec{a}_z B_z$$

$$\nabla = \vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}$$

$$\nabla\times\vec{B} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A}\times\vec{B}=(?)\times(?)=\vec{a}_x(A_yB_z-?)+\vec{a}_y(A_zB_x-?)+\vec{a}_z(A_xB_y-?)$$

$$\nabla=\vec{a}_x\frac{\partial}{\partial x}+\vec{a}_y\frac{\partial}{\partial y}+\vec{a}_z\frac{\partial}{\partial z}$$

$$\nabla\times\vec{B}=\vec{a}_x(\frac{\partial}{\partial y}B_z-\frac{\partial}{\partial z}B_y)+\vec{a}_y(\frac{\partial}{\partial z}B_x-?B_z)+\vec{a}_z(\frac{\partial}{\partial x}B_y-?B_x)$$

Stokes' Law

From the definition of the curl ——斯托克斯定理

$$\vec{a}_n \cdot rot\vec{A} = \vec{a}_n \cdot curl\vec{A} = \lim_{\Delta S \to 0}\left(\frac{\oint_C \vec{A} \bullet d\vec{l}}{\Delta S}\right) = (\nabla \times \vec{A})_n$$

$$\int_S (\nabla \times \vec{A}) \bullet d\vec{S} = \oint_C \vec{A} \bullet d\vec{l}$$

The integral of a curl of a vector over an area equals to the line integral of that vector along the boundary of that area.

矢量场旋度的面积分=该矢量沿包围该表面的封闭曲线的积分

小结： 谈谈梯度、散度和旋度

- 梯度：描述标量场的空间变化率，自身是矢量
- 散度：描述矢量场自身方向的空间变化率，自身是标量
  - 表征场的发散特性
  - 散度为零==>无源场、管形场==>静磁场
- 旋度：描述矢量场垂直方向的空间变化率，自身是矢量
  - 表征场的旋转特性
  - 旋度为零==>无旋场、保守场==>静电场

Formula for Gradient

$$\nabla C = 0, \quad \text{where } C \text{ is a contant.}$$

$$\nabla (Cu) = C\nabla u, \quad \text{where } C \text{ is a constant.}$$

$$\nabla (u \pm v) = \nabla u \pm \nabla v$$

$$\nabla f(u,v) = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v$$

Formula for Divergence

$$\nabla \bullet \vec{C} = 0, \quad \vec{C} \text{ is a constant vector.}$$

$$\nabla \bullet (C\vec{F}) = C\nabla \bullet \vec{F}, \quad C \text{ is a constant.}$$

$$\nabla \bullet (\vec{F} \pm \vec{G}) = \nabla \bullet \vec{F} \pm \nabla \bullet \vec{G}$$

$$\nabla \bullet (u\vec{F}) = u\nabla \bullet \vec{F} + \vec{F} \bullet \nabla u, \quad u \text{ is a scalar function.}$$

Formula for Curl

$\nabla \times \vec{C} = 0, \quad \vec{C}$ is a constant vector.

$\nabla \times (C\vec{F}) = C\nabla \times \vec{F}, \quad C$ is a constant.

$\nabla \times (\vec{F} \pm \vec{G}) = \nabla \times \vec{F} \pm \nabla \times \vec{G}$

$\nabla \times (u\vec{F}) = u\nabla \times \vec{F} + \nabla u \times \vec{F}, \quad u$ is a scalar function.

$$\boxed{\nabla \bullet (\vec{F} \times \vec{G}) = \vec{G} \bullet \nabla \times \vec{F} - \vec{F} \bullet \nabla \times \vec{G}}$$

Two null identities

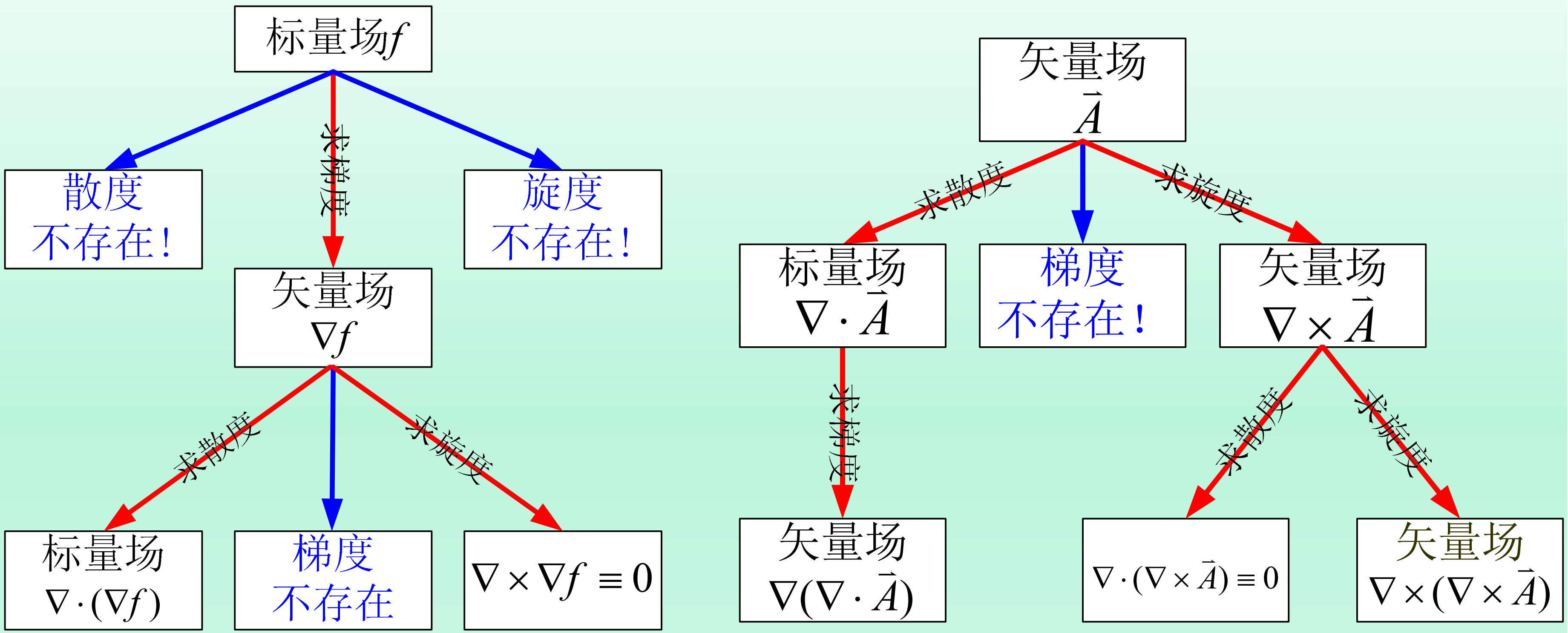
$$\text{div}(\text{rot}\vec{A}) = \nabla \cdot (\nabla \times \vec{A}) \equiv 0$$

$$\text{rot}(\text{grad}f) = \nabla \times (\nabla f) \equiv 0$$

Conclusion---1： if $\nabla \cdot \vec{F} = 0$， there exists a vector $\vec{G}$, satisfy $\vec{F} = \nabla \times \vec{G}$

Conclusion---2： if $\nabla \times \vec{F} = 0$， there exists a scalar f, satisfy $\vec{F} = \nabla f$

Compare scalar with vector field



2.6 Helmholtz Theorem

1. A scalar field, can be determinable by its Gradient

一个标量场唯一的由其梯度决定

That is $\vec{A} = \nabla f \Rightarrow f$ 。

Prove： if $\vec{A} = \nabla f_1$ and $\vec{A} = \nabla f_2$ ， we can construct a scalar filed $u = f_1 - f_2$ ， then

$$\nabla u = \nabla(f_1 - f_2) = \vec{A} - \vec{A} = 0$$

$$\Rightarrow u = C$$

$$\Rightarrow f_1 = f_2 + C$$

1.6 Helmholtz Theorem

2. A vector field, can be determinable by its Div. and Curl

A vector $\vec{A}(\vec{r})$, can be described by the sum of a W/O curl component and a W/O Divergence component,

satsify

$$\vec{A} = \vec{A}_1 + \vec{A}_2 \qquad \nabla\times\vec{A}_1 = 0, \nabla\cdot\vec{A}_1 = \rho;$$

$$\nabla\cdot\vec{A}_2 = 0, \nabla\times\vec{A}_2 = \vec{J}$$

Hence, for a vector field, we have

$$\nabla\cdot\vec{A} = \nabla\cdot(\vec{A}_1 + \vec{A}_2) = \nabla\cdot\vec{A}_1 = \rho$$

$$\nabla\times\vec{A} = \nabla\times(\vec{A}_1 + \vec{A}_2) = \nabla\times\vec{A}_2 = \vec{J}$$

Helmholtz Theorem ——亥姆霍兹定理(公理)

$$\vec{F} = \text{Sum of Two Special Vector Field}$$

$$= \vec{X}_{\text{no div}} + \vec{Y}_{\text{no curl}} \qquad = \nabla \times \vec{A} + (-\nabla U)$$

In limited region, any vector field can be uniquely determined by its curl, divergence and the boundary conditions.

The boundary conditions here refers to the distribution of the vector on the surface of the limited region.

Helmholtz Theorem is an idea, a way, and a clue that is applied throughout almost all chapters of this course.

A Summary of 3 Laws

Gauss's Law $$\int_V (\nabla \bullet \vec{A})dv = \oint_S \vec{A} \bullet d\vec{s}$$

Stokes's Law $$\int_S (\nabla \times \vec{A}) \bullet d\vec{S} = \oint_C \vec{A} \bullet d\vec{l}$$

Helmholtz Theorem $$\vec{F} = -\nabla U + \nabla \times \vec{A}$$

Homework

(焦老师英文版教材)

- E1.2, E1.3, E1.4, E1.9