Advanced Transform Methods

Fast Fourier Transform (FFT)

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Fast Fourier Transform (FFT)

- What is the FFT?
  - A collection of “tricks” that exploit the symmetry of the DFT calculation to make its execution much faster
  - Speedup increases with DFT size

- This lecture: outline the basic workings of the simplest formulation, the radix-2 decimation-in-time algorithm

Introduction, continued

- Some dates:
  - ~1880 - algorithm first described by Gauss
  - 1965 - algorithm rediscovered (not for the first time) by Cooley and Tukey

- FFT Revolutionized digital signal processing from 1960s

- E.g. in 1967 8192-point DFT on mainframe IBM 7094:
  - ~30 minutes using conventional techniques
  - ~5 seconds using FFTs

Measures of computational efficiency

- Could consider
  - Number of additions
  - Number of multiplications
  - Amount of memory required
  - Scalability and regularity

- Focus most on number of multiplications
  - More costly than additions for fixed-point processors
  - Same cost as additions for floating-point processors, but number of operations is comparable

Comput. Cost of Discrete-Time Filtering

Convolution of an *N*-point input with an *M*-point unit sample response ….

- Direct convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  – Number of multiplies ≈ *MN*

Comput. Cost of Discrete-Time Filtering

Convolution of an *N*-point input with an *M*-point unit sample response ….

- Using transforms directly:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

  - Computation of *N*-point DFTs requires N^2 multiplies
  - Each convolution (two direct transforms plus an inverse transform) requires three DFTs of length *N+M-1*

$$3(N+M-1)^2 + (N+M-1)$$

For $N >> M$ the computation is $O(N^2)$

Cooley-Tukey decimation-in-time algorithm

- Consider DFT algorithm for an integer power of 2, $N = 2^{\nu}$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N{}^{nk} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \qquad W_N = e^{-j2\pi/N}$$

- Create separate sums for even and odd values of *n:*

$$X[k] = \sum_{n\,even} x[n] W_N{}^{nk} + \sum_{n\,odd} x[n] W_N{}^{nk}$$

- Letting $n = 2r$ for *n* even and $n = 2r+1$ for *n* odd, we get

$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r] W_N{}^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N{}^{(2r+1)k}$$

$\boxed{\text{Note different sign in twiddle factor in this lecture – common in FFT texts}}$

Cooley-Tukey decimation in time algorithm

- Splitting indices in time, we have obtained

$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k}$$

- But $\quad W_N^2 = e^{-j2\pi 2/N} = e^{-j2\pi/(N/2)} = W_{N/2}$

  and $\quad W_N^{2rk} W_N^k = W_N^k W_{N/2}^{rk}$

So: $$X[k] = \underbrace{\sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk}}_{} + W_N^k \underbrace{\sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk}}_{}$$

N/2-point DFT of *x[2r]* *N/2*-point DFT of *x[2r+1]*

Savings so far …

- We have split the DFT computation into two halves:

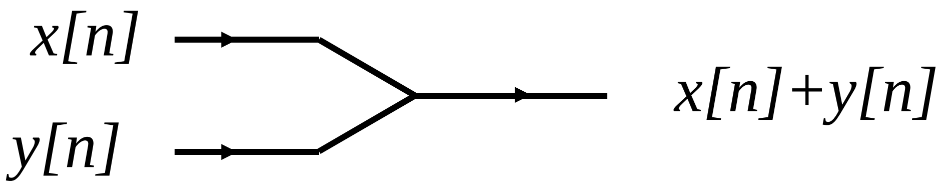
$$X[k] = \sum_{k=0}^{N-1} x[n] W_N{}^{nk}$$

$$= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk}$$

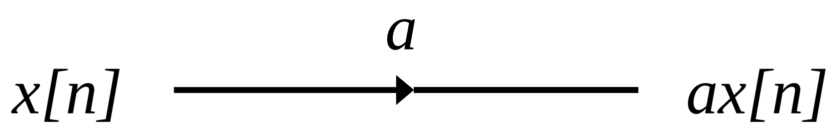
- Have we gained anything? Consider the nominal number of multiplications for $N = 8$
  – Original form produces $8^2 = 64$ multiplications
  – New form produces $2(4^2) + 8 = 40$ multiplications
  – So we're already ahead ….. Let's keep going!!

Signal flowgraph notation

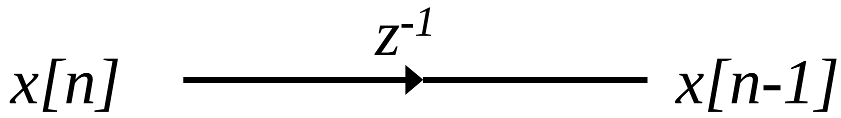
- In generalizing this formulation, it is most convenient to adopt a graphic approach …
- Signal flowgraph notation describes the three basic DSP operations:
  - Addition



  - Mult by a constant

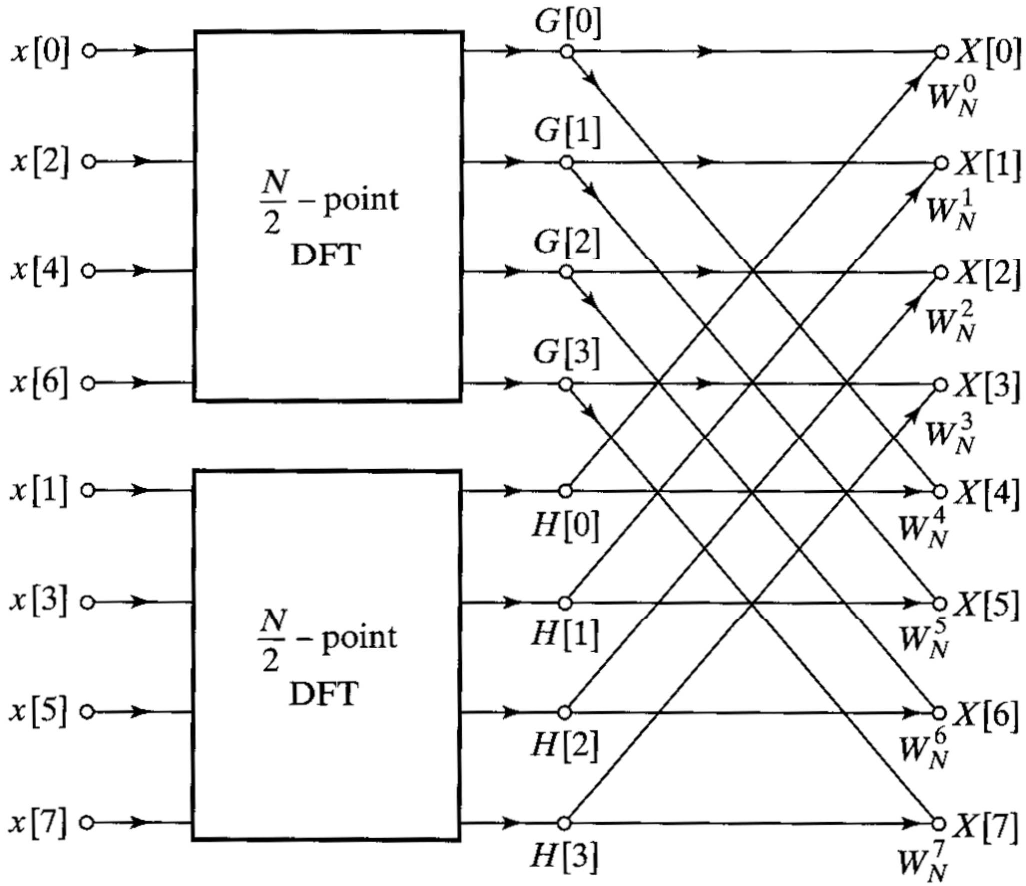


  - Delay



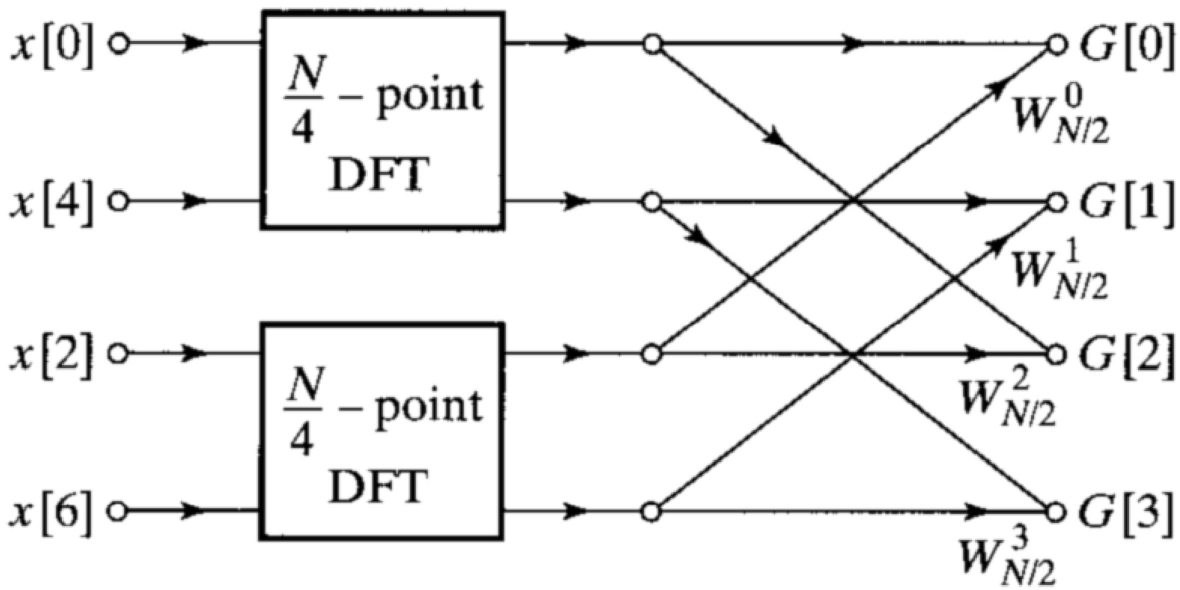
Signal flowgraph representation of 8-point DFT

- Recall that the DFT is now of the form $X[k] = G[k] + W_N^k H[k]$
- The DFT in (partial) flowgraph notation:

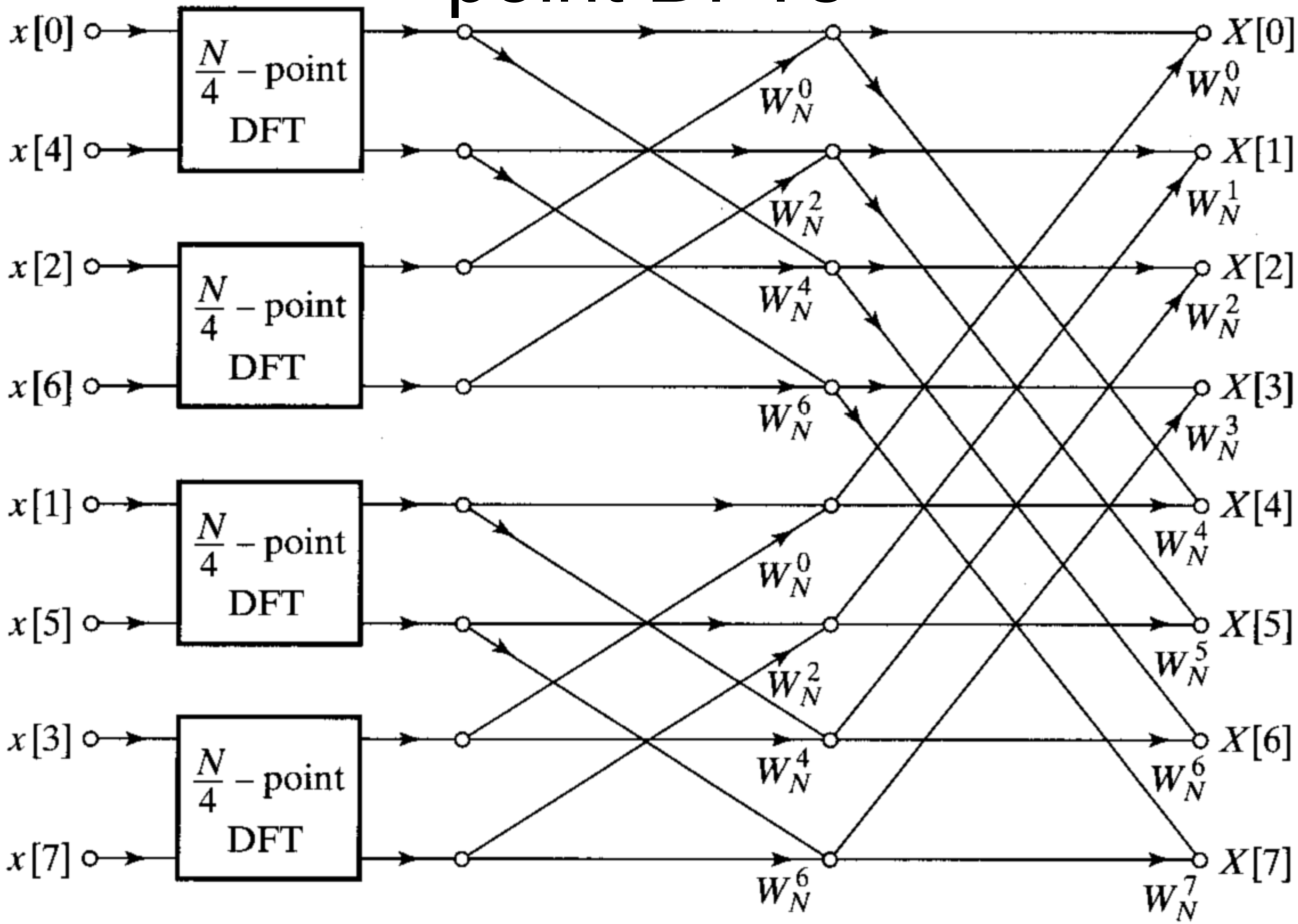


Continuing with the decomposition …

- So why not break up into additional DFTs?
- Let's take the upper 4-point DFT and break it up into two 2-point DFTs:



The complete decomposition into 2-point DFTs



Now let's take a closer look at the 2-point DFT

- The expression for the 2-point DFT is:

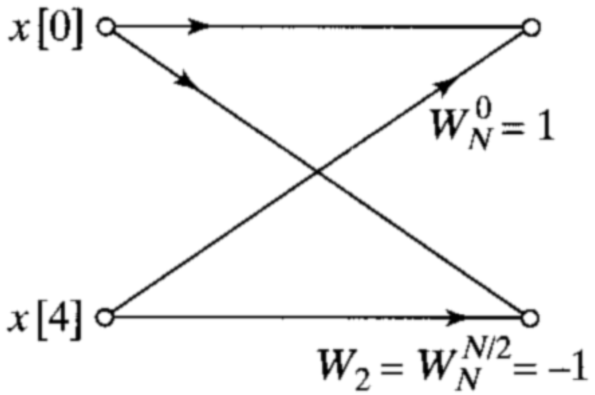
$$X[k] = \sum_{n=0}^{1} x[n]W_2^{nk} = \sum_{n=0}^{1} x[n]e^{-j2\pi nk/2}$$

- Evaluating for $k = 0,1$ we obtain

$$X[0] = x[0] + x[1]$$

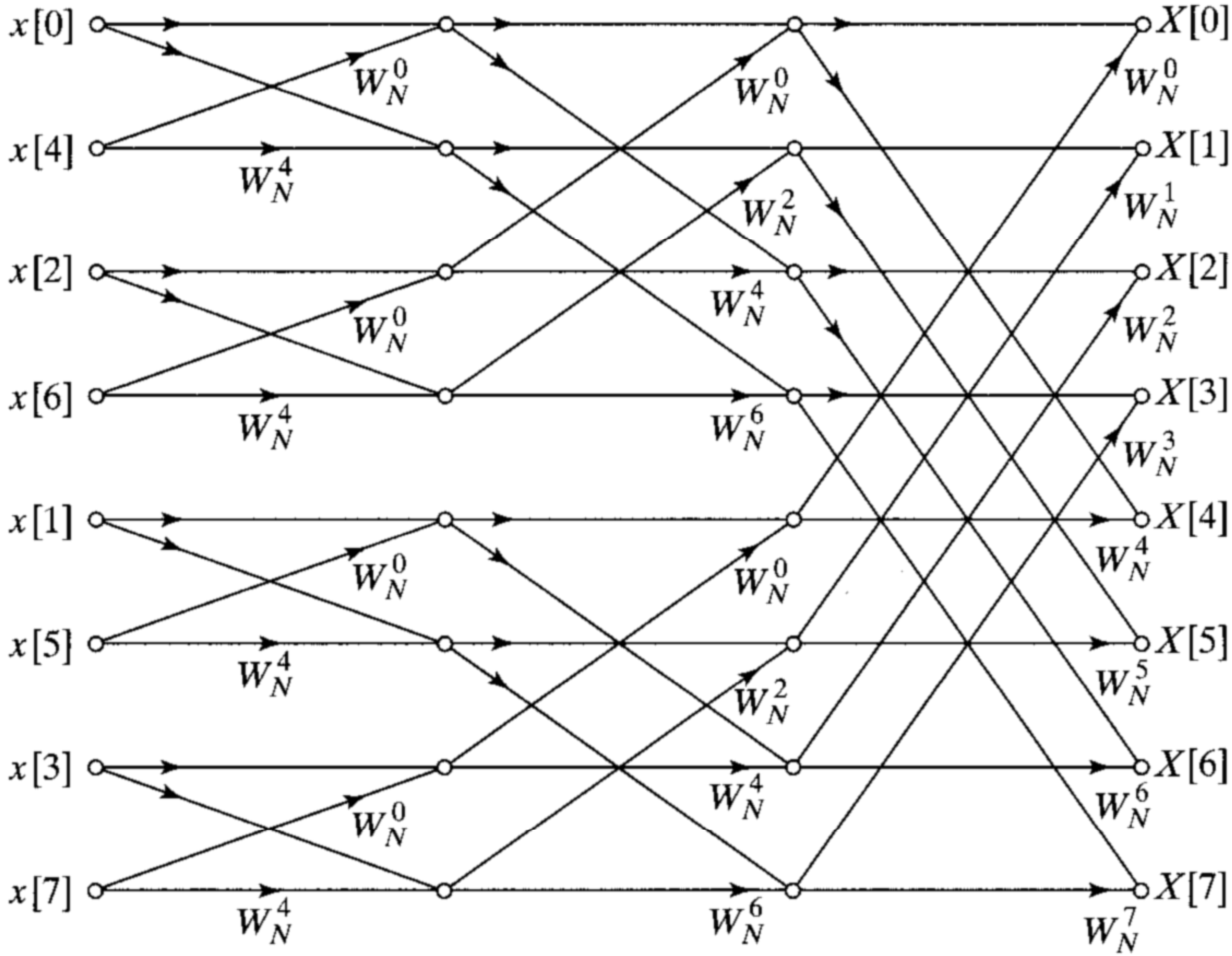
$$X[1] = x[0] + e^{-j2\pi 1/2}x[1] = x[0] - x[1]$$

which in signal flowgraph notation looks like ...

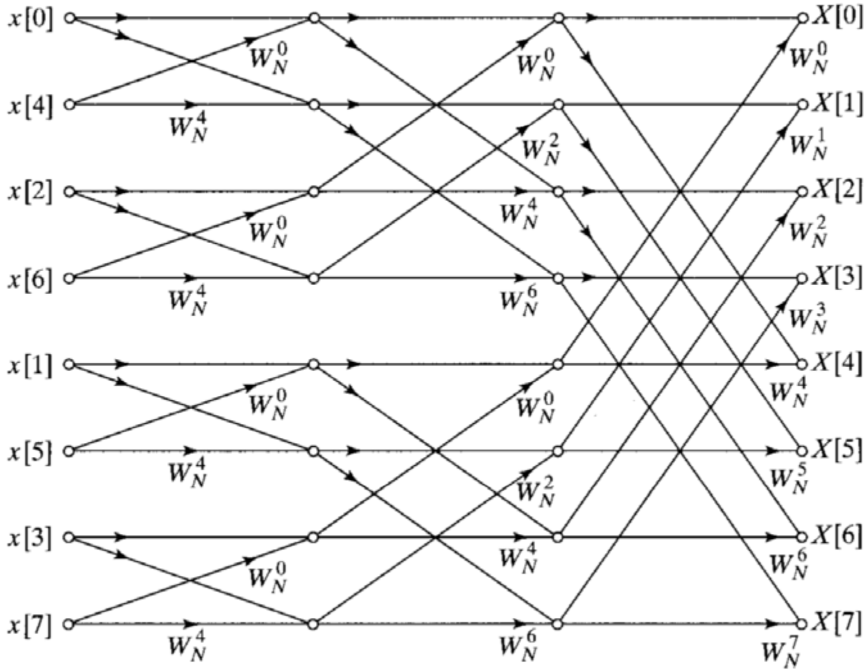


This topology is called the basic "butterfly"

The complete 8-point decimation-in-time FFT



Number of multiplies for N-point FFTs

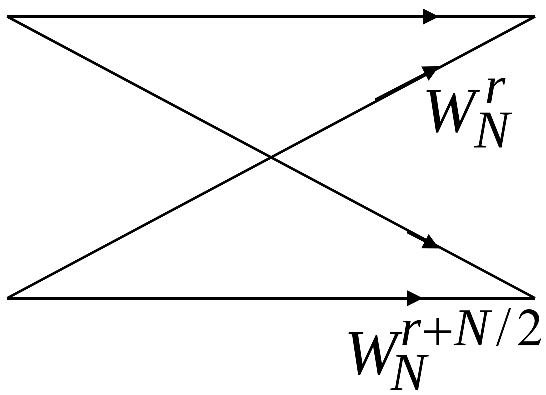


- Let $N = 2^{\nu}$ where $\nu = \log_2(N)$
- ($\log_2(N)$ columns)($N/2$ butterflys/column)(2 mults/butterfly)

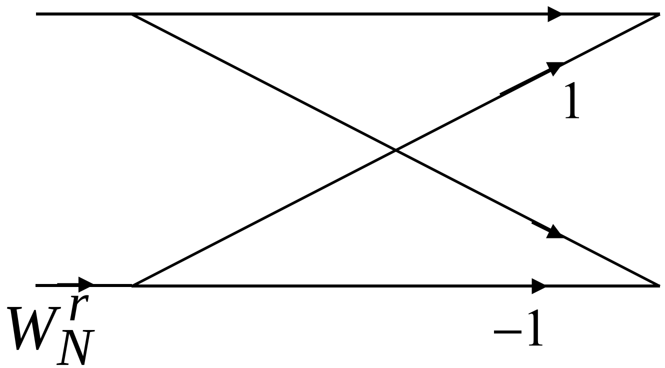
  or ~ $N \log_2(N)$ multiplies

Additional timesavers: reducing multiplications in the basic butterfly

- As we derived it, the basic butterfly is of the form

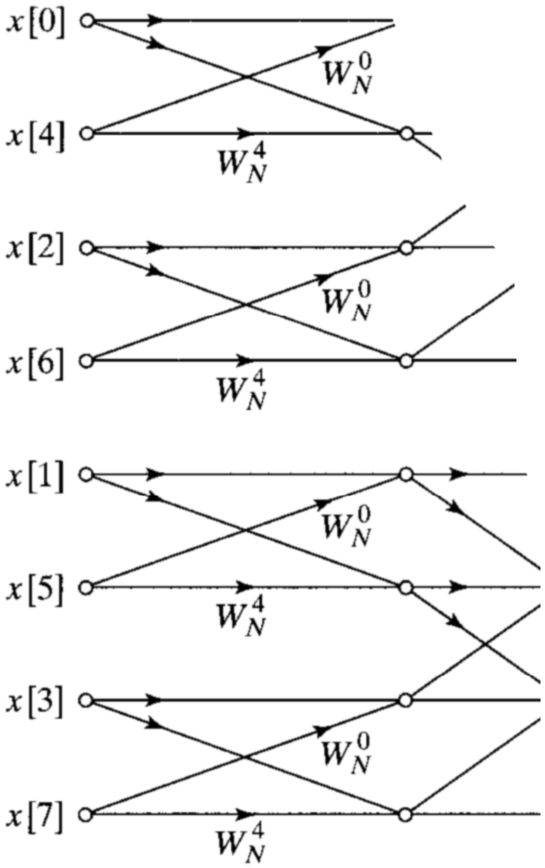


- Since $W_N^{N/2} = -1$ we can reduce computation by 2 by premultiplying by W_N^r



Bit reversal of the input

- Recall the first stages of the 8-point FFT:



Consider the binary representation of the indices of the input:

0	000
4	100
2	010
6	110
1	001
5	101
3	011
7	111

If these binary indices are time reversed, we get the binary sequence representing
0,1,2,3,4,5,6,7

Hence the indices of the FFT inputs are said to be in bit-reversed order

Some comments on bit reversal

- This implementation of FFT: input is bit reversed, output is in natural order
- Some other implementations: input in natural order, output bit reversed
- Sometimes convenient to implement filtering applications by
  – Use FFTs with input in natural order, output in bit-reversed order
  – Multiply frequency coefficients together (in bit-reversed order)
  – Use inverse FFTs with input in bit-reversed order, output in natural order
- Computing in this fashion means we never have to compute bit reversal explicitly