

§ 5.2 Magnetic Vector Potential



In § 5.1 we got $\vec{B} = \nabla \times \left[\frac{\mu_0 I}{4\pi} \oint_C \frac{d\vec{l}'}{R} \right] = \nabla \times \vec{A}$
Magnetic Vector Potential

Unit: Wb/m (Weber/m)

We call it **M-vector potential**, but not **M potential** directly.
 The reason is that there still exists **M-scalar potential**.

According to ...?, we know that **the divergence** has to be known to determine \vec{A}

$$\nabla \times \vec{A} = \vec{B} \quad \text{By definition}$$

$$\nabla \cdot \vec{B} = 0 \rightarrow \nabla \cdot \vec{A} = 0$$

Coulomb's Gauge 库仑规范



Let's go on.

—— Vector Poisson Equation

$$\begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{B} = \mu_0 \vec{J} \end{cases} \xrightarrow{\text{red arrow}} \nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} \quad \text{blue arrow}$$

Using the formula of **Vector Product**

$$\nabla \times (\nabla \times \vec{a}) = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$$

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\therefore \nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A})$$

$$\nabla \cdot \vec{A} = 0 \quad \therefore \nabla^2 \vec{A} = -\nabla \times (\nabla \times \vec{A}) = -\mu_0 \vec{J}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \quad \text{—— Vector Poisson Equation}$$

Discussions $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

A vector Poisson equation may be decomposed into several scalar Poisson equations.

$$\begin{cases} \nabla^2 A_x = -\mu_0 J_x \\ \nabla^2 A_y = -\mu_0 J_y \\ \nabla^2 A_z = -\mu_0 J_z \end{cases} \quad \nabla^2 ? = - **$$

$$\text{Scalar Poisson Equation} \quad \nabla^2 \psi = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \psi = -\frac{\rho}{\epsilon_0} \Rightarrow \psi = \frac{1}{4\pi\epsilon_0} \int_{V_{\text{source}}} \left(\frac{\rho}{R_{\text{source-spot}}} \right) dV$$

Similarly, we get

$$A_* = \frac{\mu_0}{4\pi} \int_{V_{\text{source}}} \left(\frac{J_*}{R_{\text{source-spot}}} \right) dV$$

$$\text{Compose:} \Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \int_{V_{\text{source}}} \left(\frac{\vec{J}}{R_{\text{source-spot}}} \right) dV$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla \psi$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{V_{\text{source}}} \frac{\vec{J}_V dV}{R_{\text{source-spot}}}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{S_{\text{source}}} \frac{\vec{J}_S dS}{R_{\text{source-spot}}}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi} \oint_{C_{\text{source}}} \frac{d\vec{l}}{R_{\text{source-spot}}}$$

$$d\vec{A} = \frac{\mu_0 I d\vec{l}}{4\pi R_{\text{source-spot}}}$$

Reasons to introduce this vector:

To make the algebra simpler:

- Coincide with the current in direction
- Be linear to the current element sometimes
- 二阶偏微分方程常可分解成标量泊松方程形式

Its direction?

Example 1. current in finite thin wire

→ Determine M-flux Density around.

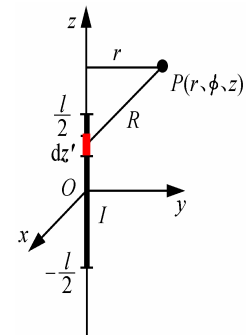
$$d\vec{A} = \frac{\mu_0 I d\vec{l}}{4\pi R_{\text{source-spot}}}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$R = \sqrt{(z - z')^2 + r^2}$$

$$A_z = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{\mu_0 I dz'}{4\pi \sqrt{(z - z')^2 + r^2}}$$

$$= \frac{\mu_0 I}{4\pi} \ln \left[\frac{\sqrt{(l/2 - z)^2 + r^2} + (l/2 - z)}{\sqrt{(l/2 + z)^2 + r^2} - (l/2 + z)} \right]$$



$$A_z = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{\mu_0 I dz'}{4\pi \sqrt{(z-z')^2 + r^2}}$$

$$= \frac{\mu_0 I}{4\pi} \ln \left[\frac{\sqrt{(l/2 - z)^2 + r^2} + (l/2 - z)}{\sqrt{(l/2 + z)^2 + r^2} - (l/2 + z)} \right]$$

$$\vec{B} = \nabla \times \vec{A} = -\vec{e}_\phi \frac{\partial A_z}{\partial r}$$

$$= \frac{\mu_0 I}{4\pi} \left[\frac{l/2 - z}{\sqrt{(l/2 - z)^2 + r^2}} + \frac{l/2 + z}{\sqrt{(l/2 + z)^2 + r^2}} \right] \vec{e}_\phi$$

$$\vec{A}(r) = \vec{a}_z \frac{\mu_0 I}{4\pi} \ln \left[\frac{\sqrt{(l/2 - z)^2 + r^2} + (l/2 - z)}{\sqrt{(l/2 + z)^2 + r^2} - (l/2 + z)} \right]$$

If $l \gg (r^2 + z^2)^{\frac{1}{2}}$

$$A_z \approx \frac{\mu_0 I}{4\pi} \ln \left(\frac{\sqrt{(l/2)^2 + r^2} + l/2}{\sqrt{(l/2)^2 + r^2} - l/2} \right)$$

$$\approx \frac{\mu_0 I}{4\pi} \ln \left(\frac{l}{r} \right)^2 = \frac{\mu_0 I}{2\pi} \ln \left(\frac{l}{r} \right)$$

$$A_z \approx \frac{\mu_0 I}{2\pi} \ln \left(\frac{l}{r} \right)$$

If the wire is infinite in length, $l \rightarrow \infty$

Assume a reference point P , and the distance from P to the axis is r_0 .

$$\vec{A} = \vec{a}_z \frac{\mu_0 I}{2\pi} \ln \frac{r_0}{r}$$

In practice r_0 is always assumed to be **1**.

Example 2. parallel double lines

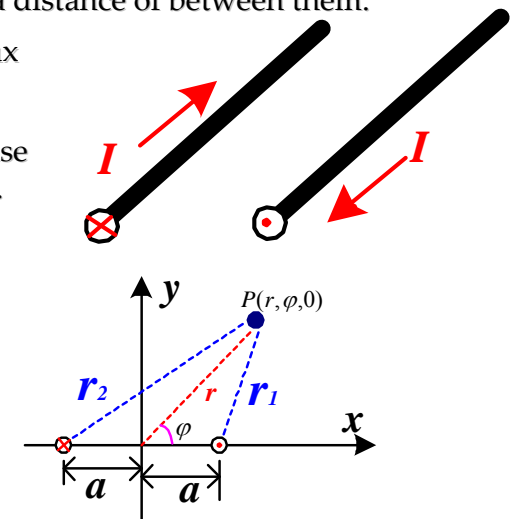
Parallel double lines with a distance of between them.

Please determine the M-flux density in space.

In this example we try to use M-vector potential around.

Analysis:

1. Direction of \vec{A}
2. Direction of \vec{B}
3. The coordinates?



From example 1, we know that around infinite thin wire ...

$$\vec{A}_1 = \vec{a}_z I_1 \frac{\mu_0}{2\pi} \cdot \ln \frac{1}{r}$$

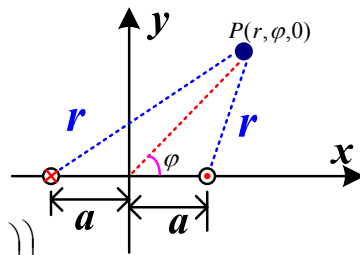
At a certain point in space P

$$\vec{A} = \vec{A}_1 + \vec{A}_2$$

$$= (+\vec{a}_z I) \left(\frac{\mu_0}{2\pi} \cdot \ln \left(\frac{1}{r_1} \right) \right) + (-\vec{a}_z I) \left(\frac{\mu_0}{2\pi} \cdot \ln \left(\frac{1}{r_2} \right) \right)$$

$$\therefore \vec{A} = \vec{a}_z \left(\frac{\mu_0 \cdot I}{2\pi} \cdot \ln \left(\frac{r_2}{r_1} \right) \right) = \dots \ln \left(\frac{a^2 + r^2 + 2ar \cdot \cos \varphi}{a^2 + r^2 - 2ar \cdot \cos \varphi} \right)$$

$$\vec{B} = \nabla \times \vec{A} = ?$$



$$\nabla \times (u \vec{A}) = \nabla u \times \vec{A} + u \nabla \times \vec{A}$$

$$\vec{A} = \vec{a}_z \left(\frac{\mu_0 \cdot I}{2\pi} \cdot \ln \frac{a^2 + r^2 + 2ar \cdot \cos \varphi}{a^2 + r^2 - 2ar \cdot \cos \varphi} \right)$$

$$\vec{B} = \nabla \times \vec{A} = \nabla \times (A \vec{a}_z) = \nabla A \times \vec{a}_z + 0$$

$$= \dots = \left(\vec{a}_r \frac{\partial A}{\partial r} + \vec{a}_\varphi \frac{1}{r} \frac{\partial A}{\partial \varphi} \right) \times \vec{a}_z = ?$$

Please finish this example after class.

Homework

$$\frac{(\vec{r} \cdot \vec{v})}{((\vec{r} \cdot \vec{v}))} = \frac{(\vec{r} \cdot \vec{v})}{((\vec{r} \cdot \vec{v}))} = \frac{(\vec{r} \cdot \vec{v})}{((\vec{r} \cdot \vec{v}))} = \frac{(\vec{r} \cdot \vec{v})}{((\vec{r} \cdot \vec{v}))}$$

➡ Exercises: 5.10, 5.12, 5.14, 5.15