Advanced Transform Methods

The Gabor Transform

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Revision: Bases and Frames

- A set of vectors or functions $\{\psi_n\}$ spans a vector space if any element of that space can be expressed as a linear combination of members of that set

$$s = \sum_n c_n \psi_n$$

- $\{\psi_n\}$ is a *basis set* if the c_n are unique.
- The set is an orthogonal basis if $n \neq m \Rightarrow \langle \psi_n, \psi_m \rangle = 0$
- The set is an orthonormal basis if $n = m \Rightarrow \langle \psi_n, \psi_m \rangle = 1$

Dual basis

- A basis set $\{\hat{\psi}_i\}$ is said to be the dual basis of $\{\psi_i\}$ if the biorthogonality condition

$$\left\langle \psi_i, \hat{\psi}_j \right\rangle = \sum_k \psi_i(k)\hat{\psi}_j(k) = \delta_{ij}$$

As a transform,
the dual basis set $\{\hat{\psi}_j\}$ is the
inverse of the basis set $\{\psi_i\}$

is satisfied.

- Example:

$$\{\psi_i\} = \{\ (1,0),(1,1)\}$$

$$\{\hat{\psi}_i\} = \{\ (1,-1),(0,1)\}$$

$$\left\langle \psi_1, \hat{\psi}_1 \right\rangle = \sum_k (1,0)\cdot(1,-1) = 1$$

$$\left\langle \psi_1, \hat{\psi}_1 \right\rangle = \sum_k (1,0)\cdot(1,-1) = 1$$

$$\left\langle \psi_1, \hat{\psi}_2 \right\rangle = \sum_k (1,0)\cdot(0,1) = 0$$

$$\left\langle \psi_2, \hat{\psi}_1 \right\rangle = \sum_k (1,1)\cdot(1,-1) = 0$$

$$\left\langle \psi_2, \hat{\psi}_2 \right\rangle = \sum_k (1,1)\cdot(0,1) = 1$$

Frames

Analysis $c_n = \langle s, \psi_n \rangle = \int_{-\infty}^{\infty} s(t)\psi_n^*(t)dt$ (or $\sum_m s[m]\psi_n^*[m]$)

So instead of s, we have the coefficients $\{c_n\}$.

What might happen to squared norm $E_1 = \sum_n |c_n|^2 = \sum_n |\langle s, \psi_n \rangle|^2$

compared to the original squared norm $E = \|s\|^2 = \int_{-\infty}^{\infty} |s(t)|^2 dt$?

If the squared norm of the transform of s is

upper and lower bounded relative to $\|s\|^2$, we have a *frame* :

$$A\|s\|^2 \le \sum_n |\langle s, \psi_n \rangle|^2 \le B\|s\|^2 \qquad 0 < A \le B < \infty$$

Implications :

If $\|s\|^2 > 0$, then $\langle s, \psi_n \rangle > 0$ for at least one of the ψ_n

If $\|s\|^2 < \infty$, then $\sum_n |\langle s, \psi_n \rangle|^2 < \infty$

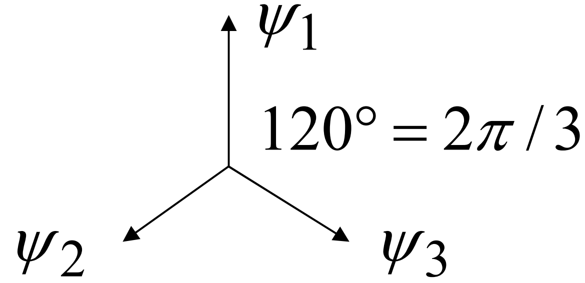
Nothing "gets lost" or "blows up"

Example: Tight frame

$$\psi_1 = (0,1)$$

$$\psi_2 = (-\sqrt{3}/2, -1/2)$$

$$\psi_3 = (\sqrt{3}/2, -1/2)$$



For 2-D vector $s = (s_1, s_2)$, with $\|s\|^2 = s_1^2 + s_2^2$

have $\langle s, \psi_1 \rangle = s_2, \quad \langle s, \psi_2 \rangle = -\frac{\sqrt{3}}{2} s_1 - \frac{1}{2} s_2, \quad \langle s, \psi_3 \rangle = \frac{\sqrt{3}}{2} s_1 - \frac{1}{2} s_2$

$$\begin{aligned}
\sum_{n=1}^{3} \left| \langle s, \psi_n \rangle \right|^2 &= s_2^2 + \left(-\frac{\sqrt{3}}{2} s_1 - \frac{1}{2} s_2 \right)^2 + \left(+\frac{\sqrt{3}}{2} s_1 - \frac{1}{2} s_2 \right)^2 \\
&= s_2^2 + \left(\frac{3}{4} s_1^2 + 2\frac{\sqrt{3}}{4} s_1 s_2 + \frac{1}{4} s_2^2 \right) + \left(\frac{3}{4} s_1^2 - 2\frac{\sqrt{3}}{4} s_1 s_2 + \frac{1}{4} s_2^2 \right) \\
&= 2\frac{3}{4} s_1^2 + (s_2^2 + 2\frac{1}{4} s_2^2) \\
&= \frac{3}{2}(s_1^2 + s_2^2) = \frac{3}{2}\|s\|^2
\end{aligned}$$

I.e. tight frame with $A = B = \frac{3}{2}$

Gabor Expansion

- Altered version of the STFT that uses a Gaussian function as the window.
- Represent a signal in 2 dimensions, with time and frequency as coordinates.
- Expand signal as a series of elementary functions.
- Constructed from a single building block by
  - translation
  - modulation (translation in the frequency domain)
- Transform sampled at regular intervals.
- Intervals are labelled T in time and Ω in frequency.
- Transform still defines the signal.
- The transform divides the signal into regular windows

$$s(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{m,n} h_{m,n}(t)$$

$$h_{m,n}(t) = h(t - mT) e^{jn\Omega t}$$

Choice of elementary function

- The Gabor expansion holds for almost any function h.
- The only requirement is that $T\Omega \leq 2\pi$
  - Oversampling $T\Omega < 2\pi$
  - Critical sampling $T\Omega = 2\pi$
  - Undersampling $T\Omega > 2\pi$

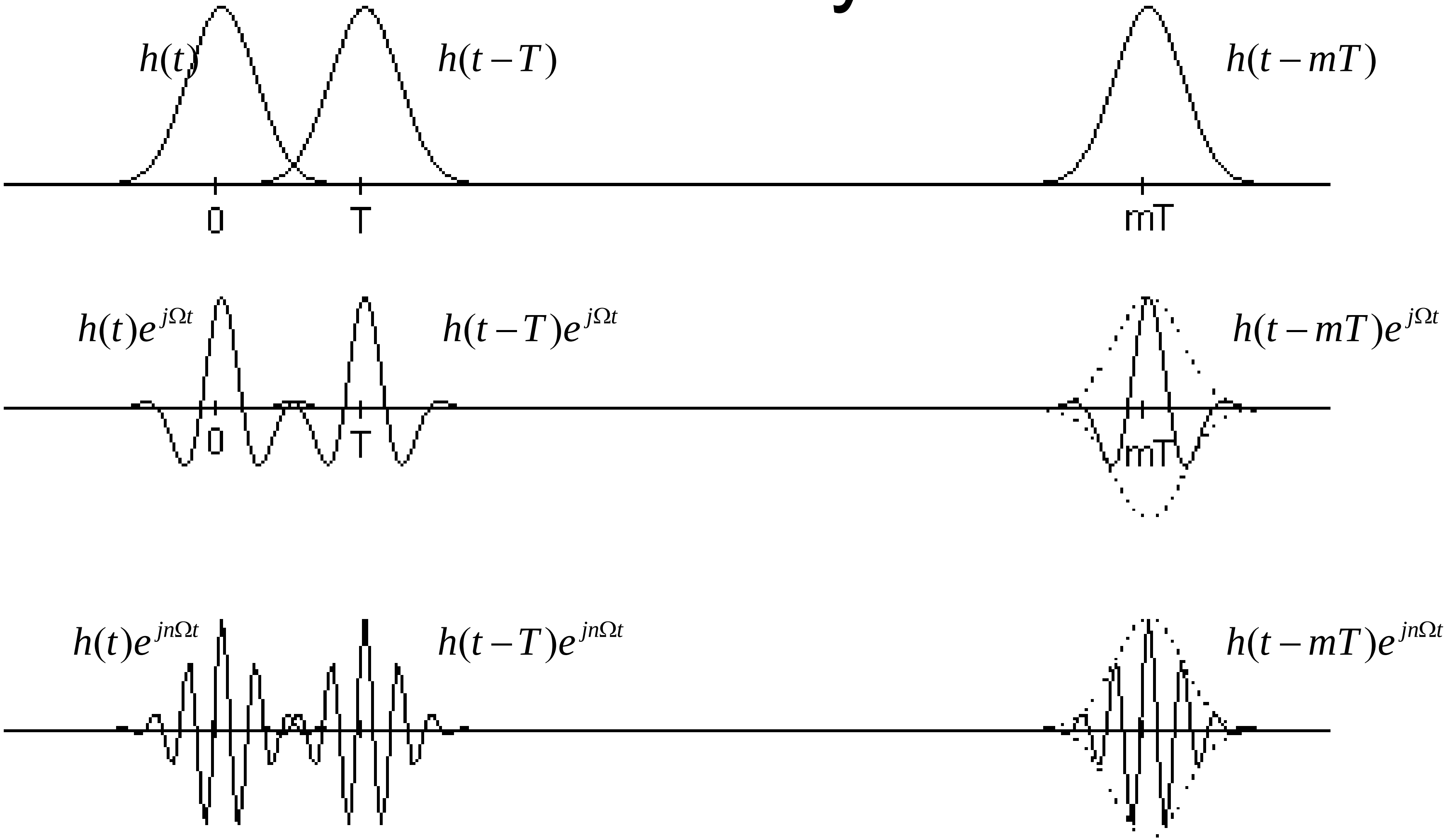
- The most useful window function is a Gaussian window:

$$h(t) = g(t) = \sqrt[4]{\alpha / \pi}\, e^{-\alpha t^2/2} \qquad \Delta t \Delta \omega = \frac{1}{\sqrt{\alpha}} \frac{\sqrt{\alpha}}{2} = 1/2$$

  - Optimally concentrated in the time frequency domain.
- This does not produce orthogonal basis functions.
- This function gives the best resolution

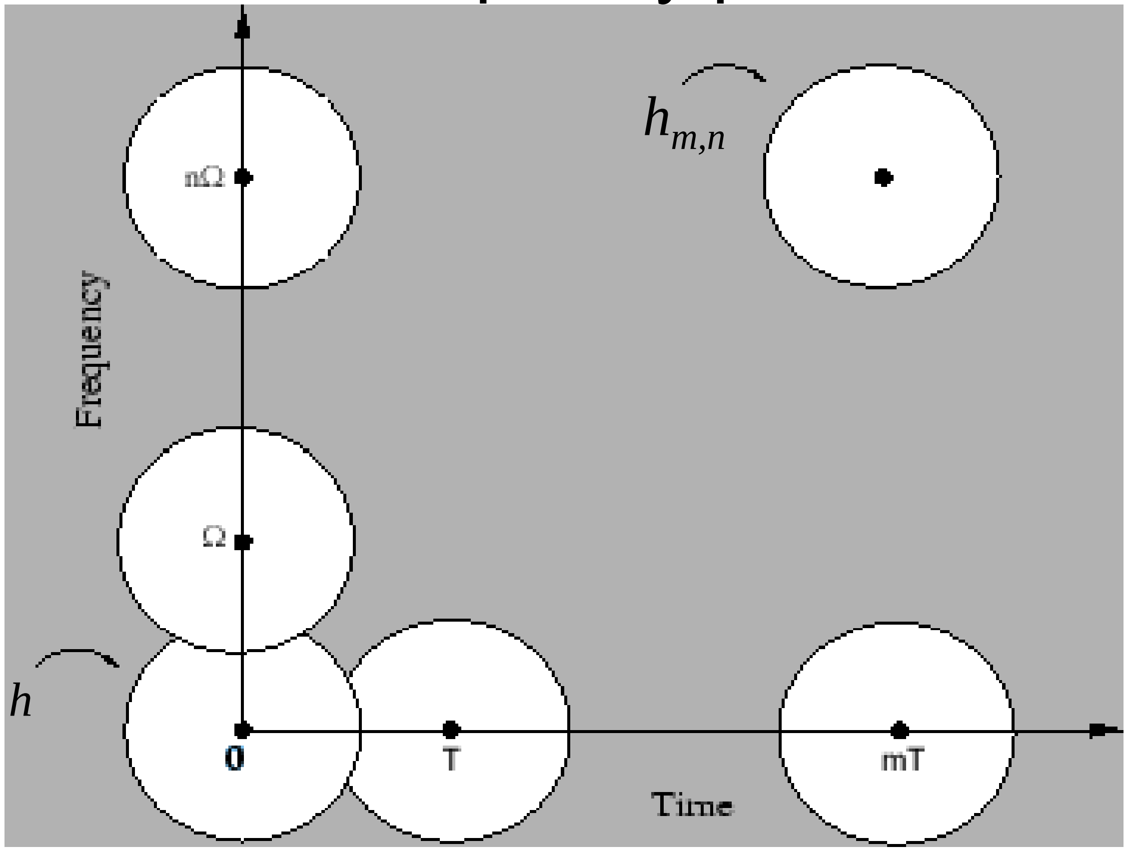
Unlike Fourier transform, Gabor transform is not an orthogonal basis (and often not a tight frame).
Finding the dual functions is typically not straightforward!

Gabor Elementary Functions



$h_{m,n}$ are shifted and modulated copies of a single building block h.
T denotes the time shift parameter.
Ω denotes the frequency shift parameter.
Each $h_{m,n}$ has an envelope of the shape of h.
Only the real part of the functions is shown.

Time-frequency plane



- $h_{m,n}$ are obtained by shifting h along a lattice in the *time-frequency plane*.
- If h and its Fourier transform are centred at the origin, then $h_{m,n}$ is centred at (mT,nW) in the time-frequency plane.
- Each $h_{m,n}$ essentially occupies a certain area in the time-frequency plane.
- Each expansion coefficient $c_{m,n}$, associated to a certain area of the time-frequency plane via $h_{m,n}$ represents one *quantum of information*.
- For properly chosen shift parameters the $h_{m,n}$ cover the time-frequency plane.

Gabor coefficients

- $c_{m,n}$ are the Gabor coefficients. They can be found from:

$$c_{m,n} = \int_{-\infty}^{\infty} s(t)\gamma_{m,n}{}^{*}(t)dt = STFT[mT, n\Omega] = \left\langle s, \gamma_{m,n} \right\rangle$$

- As long as $\gamma(t)$ and $h(t)$ constitute a dual basis
- Reconstruction / synthesis window $\gamma(t)$ is not necessarily the same as the analysis window $h(t)$.
- The windows are only identical if the functions are orthogonal.
- A system $g(t)$ is a Gabor frame if

$$A\|s\|^2 \leq \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left| \left\langle f, g_{m,n} \right\rangle \right|^2 \leq B\|s\|^2$$

- For a Gabor frame

$$f = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left\langle f, g_{m,n} \right\rangle \gamma_{m,n} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left\langle f, \gamma_{m,n} \right\rangle g_{m,n}$$

- $\gamma(t)$ and $h(t)$ are a Gabor frame

Gabor frame properties

- Gabor frame operator

$$S = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left\langle f, g_{m,n} \right\rangle g_{m,n}$$

- Commutes with translation

$$S(t - mT) = S(t) - mT$$

- Commutes with modulation

$$S(te^{jn\Omega t}) = S(t)e^{jn\Omega t}$$

- So the dual frame can be found through translation and modulation as well.

$$\gamma_{m,n} = S^{-1} g_{m,n} = S^{-1} g(t - mT)e^{jn\Omega t} = \gamma(t - mT)e^{jn\Omega t}$$

- Elements of a frame are in general linearly dependent.
- Many choices for the coefficients $c_{m,n}$ and even different choices of γ are possible.
- Coefficients determined by the dual frame are the most economical ones

Similar to Nyquist density for sampling and reconstruction of bandlimited functions in Shannon's Sampling Theorem.

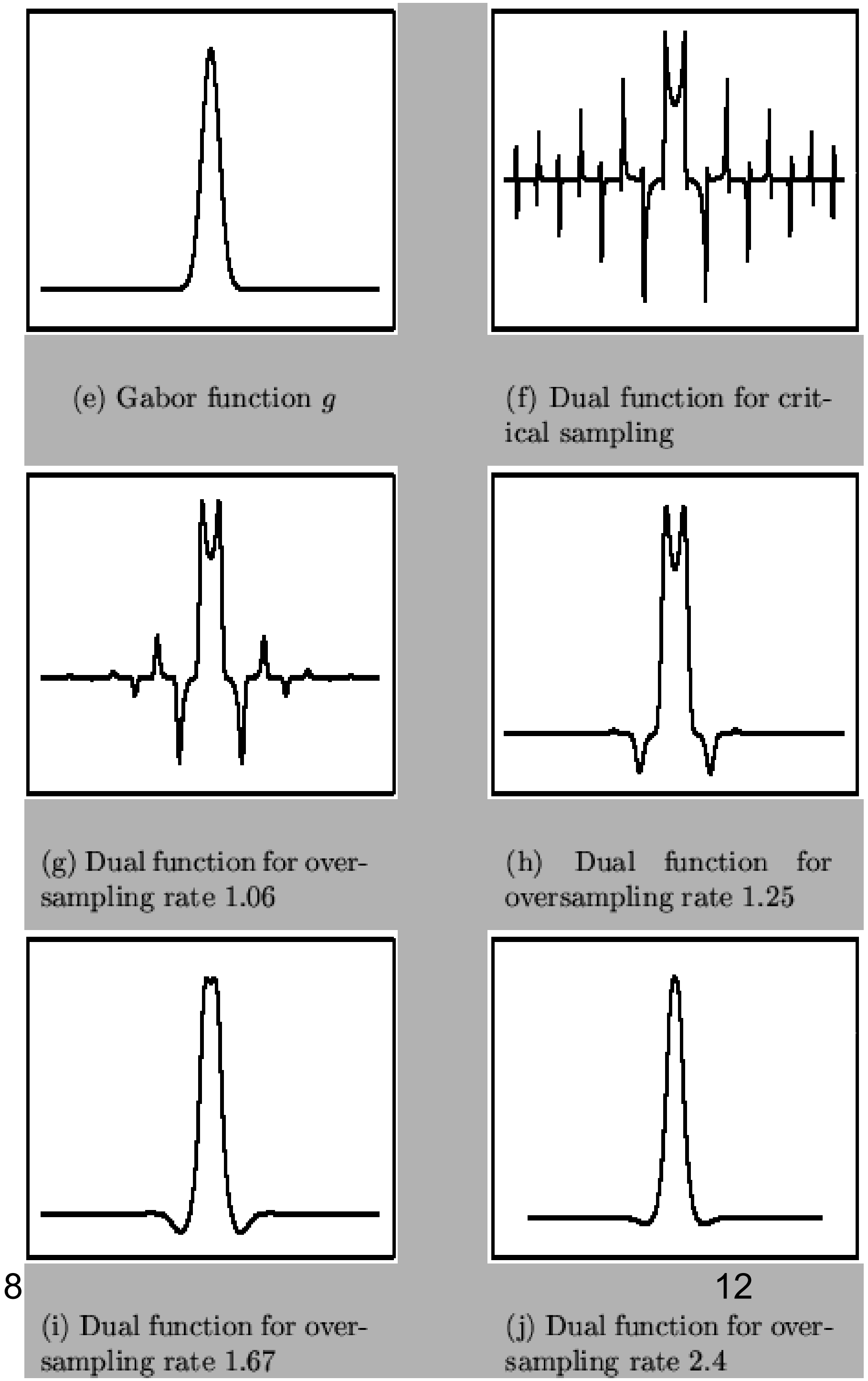
Classify Gabor systems according to the corresponding sampling density of the time-frequency lattice:

❑***Oversampling-*** **Frames with excellent time-frequency localization properties exist**

- ➢**Gaussian with appropriate oversampling rate**

❑***critical sampling-*** **Frames and orthonormal bases possible, but without good time-frequency localization**

❑***Undersampling-*** **Any Gabor family will be incomplete. Cannot have a frame.**



(e) Gabor function g

(f) Dual function for critical sampling

(g) Dual function for oversampling rate 1.06

(h) Dual function for oversampling rate 1.25

(i) Dual function for oversampling rate 1.67

(j) Dual function for oversampling rate 2.4

Gabor summary

- If we want to use a Gaussian function as the reconstruction function, we need to find the window function $\gamma(t)$.
- Discrete form is difficult to find, but possible
- No fast transform
- *But…* Lots of applications
    - **Speech signal analysis:**
    - **Time-Varying Spectral Estimation:**
    - **Representation and Identification of Linear Systems:**
    - **Digital Communication:**
    - **Image representation and biological vision**
- Active area of research