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EBU6018
Advanced Transform Methods

Tutorial - KLT

Dr Yixuan Zou

Tutorial Example 1

- **Question:** Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

Guidance:

Step 1: Compute the covariance matrix between variable x and y

Step 2: Calculate the eigenvalues of the covariance matrix

Step 3: Calculate the eigenvectors

Step 4: Normalise the eigenvectors

x	y
-3.01	-2.67
0.18	-2.12
-6.56	-4.24
1.60	1.92
2.78	-1.20
2.03	2.14

Assuming this is a sample of a larger population

Tutorial Example 1 - Solutions

- **Question**: Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

- **Step 1**: Compute the covariance matrix R_{xy}
  - To compute the covariance matrix, we need the following quantities
    - x_{ave} = -0.50
    - y_{ave} = -1.03
    - $Var_x = \frac{\sum_1^N (x_i - x_{ave})^2}{N-1} =$ 13.00
    - $Var_y = \frac{\sum_1^N (y_i - y_{ave})^2}{N-1} =$ 6.60
    - $Cov_{x,y} = \frac{\sum_1^N (x_i - x_{ave})(y_i - y_{ave})}{N-1} =$ 7.30
    - $Cov_{y,x} = \frac{\sum_1^N (x_i - x_{ave})(y_i - y_{ave})}{N-1} =$ 7.30

x	y
-3.01	-2.67
0.18	-2.12
-6.56	-4.24
1.60	1.92
2.78	-1.20
2.03	2.14

Tutorial Example 1 - Solutions

- **<u>Question</u>**: Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

- **<u>Step 1</u>**: Compute the covariance matrix R_{xy}

  - The covariance matrix = $\begin{bmatrix} Var_x & Covar_{x,y} \\ Covar_{y,x} & Var_y \end{bmatrix}$

$$R_{xy} = \begin{bmatrix} 13.00 & 7.30 \\ 7.30 & 6.60 \end{bmatrix}$$

x	y
-3.01	-2.67
0.18	-2.12
-6.56	-4.24
1.60	1.92
2.78	-1.20
2.03	2.14

Tutorial Example 1 - Solutions

- **Question**: Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

- **Step 2**: Compute the **eigenvalues** of the covariance matrix R_{xy}

$$|R_{xy} - \lambda I| = \left| \begin{bmatrix} 13.00 & 7.30 \\ 7.30 & 6.60 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} 13.00 - \lambda & 7.30 \\ 7.30 & 6.60 - \lambda \end{vmatrix}$$

$$= 32.51 - 19.6\lambda + \lambda^2 = 0$$

So $\lambda_1 = 17.77$ and $\lambda_2 = 1.83$

x	y
-3.01	-2.67
0.18	-2.12
-6.56	-4.24
1.60	1.92
2.78	-1.20
2.03	2.14

Tutorial Example 1 - Solutions

- **Question**: Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

- **Step 3**: Compute the **eigenvectors** of the covariance matrix R_{xy}

For λ_1= 17.77, $\begin{bmatrix} -4.77 & 7.30 \\ 7.30 & -11.17 \end{bmatrix} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $\varphi_{11} = \frac{7.3}{4.77}\varphi_{12}$

Any vector that satisfies this relationship is an answer to the equation above

For λ_2= 1.83, $\begin{bmatrix} 11.17 & 7.30 \\ 7.30 & 4.77 \end{bmatrix} \begin{bmatrix} \varphi_{21} \\ \varphi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $\varphi_{21} = -\frac{7.3}{11.17}\varphi_{22}$

x	y
-3.01	-2.67
0.18	-2.12
-6.56	-4.24
1.60	1.92
2.78	-1.20
2.03	2.14

Tutorial Example 1 - Solutions

- **Question**: Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

- **Step 3**: Find the normalized **eigenvectors**

For $v_1 = \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \end{bmatrix} = \begin{bmatrix} \frac{7.3}{4.77}\varphi_{12} \\ \varphi_{12} \end{bmatrix}$

Since $|v_1|^2 = 1$, we have $\left(\frac{7.3}{4.77}\right)^2 \varphi_{12}^2 + \varphi_{12}^2 = 1$

Hence, $\left(1 + \left(\frac{7.3}{4.77}\right)^2\right) \varphi_{12}^2 = 1 \Rightarrow \varphi_{12}^2 = \frac{1}{1+\left(\frac{7.3}{4.77}\right)^2} \Longrightarrow \varphi_{12} = \sqrt{\frac{1}{1+\left(\frac{7.3}{4.77}\right)^2}}$

Finally, $\varphi_{12} = 0.5470 \Rightarrow v_1 = \begin{bmatrix} 0.8371 \\ 0.5470 \end{bmatrix}$

x	y
-3.01	-2.67
0.18	-2.12
-6.56	-4.24
1.60	1.92
2.78	-1.20
2.03	2.14

Tutorial Example 1 - Solutions

- **Question**: Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

- **Step 3**: Find the normalized **eigenvectors**

For $v_2 = \begin{bmatrix} \varphi_{21} \\ \varphi_{22} \end{bmatrix} = \begin{bmatrix} -\frac{7.3}{11.17}\varphi_{22} \\ \varphi_{22} \end{bmatrix}$

Since $|v_2|^2 = 1$, we have $\left(\frac{7.3}{11.17}\right)^2 \varphi_{22}^2 + \varphi_{22}^2 = 1$

Hence, $\left(1 + \left(\frac{7.3}{11.17}\right)^2\right) \varphi_{22}^2 = 1 \Rightarrow \varphi_{22}^2 = \frac{1}{\left(1+\left(\frac{7.3}{11.17}\right)^2\right)} \Longrightarrow \varphi_{22} = \sqrt{\frac{1}{\left(1+\left(\frac{7.3}{11.17}\right)^2\right)}}$

Finally, $\varphi_{22} = 0.8371 \Longrightarrow v_2 = \begin{bmatrix} -0.5470 \\ 0.8371 \end{bmatrix}$

x	y
-3.01	-2.67
0.18	-2.12
-6.56	-4.24
1.60	1.92
2.78	-1.20
2.03	2.14

Tutorial Example 1 - Solutions

- **Question**: Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

- **Step 3**: Find the normalized **eigenvectors**

The eigenvectors are

$$v_1 = \begin{bmatrix} 0.8371 \\ 0.5470 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -0.5470 \\ 0.8371 \end{bmatrix}$$

Note: The eigenvectors are orthogonal, i.e., $v_1^T v_2 = 0$.

x	y
-3.01	-2.67
0.18	-2.12
-6.56	-4.24
1.60	1.92
2.78	-1.20
2.03	2.14

Tutorial Example 2

- **Question**: Find the normalized eigenvectors of the following matrix:

$$R_{xy} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

Hint:

The Eigenvalues, λ, of a square matrix R_{xy} are the solutions of:

$$\left|R_{xy} - \lambda I\right| = 0$$

The Eigenvectors, v, are the solutions of:

$$\left(R_{xy} - \lambda I\right)\mathrm{v} = 0$$

Tutorial Example 2

- **Question**: Find the normalized eigenvectors of the following matrix $R_{xy} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$
- **Solution:** Compute the **eigenvalues**

$$\left|R_{xy} - \lambda I\right| = \left|\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right|$$

$$= \begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix}$$

$$= -5\lambda + \lambda^2 = 0$$

So $\lambda_1 = 5$ and $\lambda_2 = 0$

Tutorial Example 2

- **Question**: Find the normalized eigenvectors of the following matrix $R_{xy} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$
- **Solution:** Compute the **eigenvectors**

For $\lambda_1 = 5$, $\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $\varphi_{11} = -\frac{1}{2}\varphi_{12}$

For $\lambda_2 = 0$, $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \varphi_{21} \\ \varphi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $\varphi_{21} = 2\varphi_{22}$

Tutorial Example 2

- **Question**: Find the normalized eigenvectors of the following matrix $R_{xy} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

- **Solution:** Find the normalized **eigenvectors**

  For $v_1 = \begin{bmatrix} \varphi_{11} \\ \varphi_{12} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\varphi_{12} \\ \varphi_{12} \end{bmatrix}$

  Since $|v_1|^2 = 1$, we have $\frac{1}{4}\varphi_{12}^2 + \varphi_{12}^2 = 1$

  Hence, $\frac{5}{4}\varphi_{12}^2 = 1 \Rightarrow \varphi_{12}^2 = \frac{4}{5} \Longrightarrow \varphi_{12} = \frac{2}{\sqrt{5}}$

  Finally, $v_1 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

Tutorial Example 2

- **Question**: Find the normalized eigenvectors of the following matrix $R_{xy} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

- **Solution:** Find the normalized **eigenvectors**

For $v_2 = \begin{bmatrix} \varphi_{21} \\ \varphi_{22} \end{bmatrix} = \begin{bmatrix} 2\varphi_{22} \\ \varphi_{22} \end{bmatrix}$

Since $|v_2|^2 = 1$, we have $4\varphi_{22}^2 + \varphi_{22}^2 = 1$

Hence, $5\varphi_{22}^2 = 1 \Rightarrow \varphi_{22}^2 = \frac{1}{5} \Longrightarrow \varphi_{22} = \frac{1}{\sqrt{5}}$

Finally, $v_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

Tutorial Example 2

- **Question**: Find the normalized eigenvectors of the following matrix:

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

Solutions:

$$\lambda_1 = 5,\ v_1 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$\lambda_2 = 0,\ v_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

Tutorial Example 3

- **<u>Question</u>**: Find the eigenvalues of the following 3x3 matrix:

$$A = \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix}$$

<u>Hint:</u>

The Eigenvalues, λ, of a square matrix A are the solutions of:

$$|A - \lambda I| = 0$$

Tutorial Example 3 - Solutions

- **Question**: Find the eigenvectors of the following matrix:

$$A = \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix}$$

- **Solutions**:

$$|A - \lambda I| = \left| \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} 4-\lambda & 6 & 10 \\ 3 & 10-\lambda & 13 \\ -2 & -6 & -8-\lambda \end{vmatrix}$$

$$= (4-\lambda)\begin{vmatrix} 10-\lambda & 13 \\ -6 & -8-\lambda \end{vmatrix} - 6\begin{vmatrix} 3 & 13 \\ -2 & -8-\lambda \end{vmatrix} + 10\begin{vmatrix} 3 & 10-\lambda \\ -2 & -6 \end{vmatrix}$$

$$= (4-\lambda)[(10-\lambda)(-8-\lambda) - 13(-6)]$$

$$-6[3(-8-\lambda) + 2 * 13] + 10[3 * (-6) + 2(10-\lambda)]$$

Tutorial Example 3 - Solutions

- **<u>Question</u>**: Find the eigenvectors of the following matrix:

$$A = \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix}$$

- **<u>Solutions</u>**:

$$|A - \lambda I| = (4-\lambda)[(10-\lambda)(-8-\lambda) - 13(-6)]$$
$$-6[3(-8-\lambda) + 2*13] + 10[3*(-6) + 2(10-\lambda)]$$

...

$$= -\lambda^3 + 6\lambda^2 - 8\lambda = 0$$

$$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 8\lambda = 0$$

$$\Rightarrow \quad \lambda(\lambda^2 - 6\lambda + 8) = 0 \qquad \Rightarrow \lambda_1 = 0 \quad \Rightarrow$$

Indicates that matrix A is singular and non-invertible

$$\Rightarrow \quad \lambda^2 - 6\lambda + 8 = 0$$

$$\Rightarrow \quad \lambda_2 = 2, \lambda_3 = 4$$

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Summary

- Covariance matrix
  - Refers to the measure of the directional relationship between two random variables.
  - Always symmetrical, so the eigenvalues will be real and the eigenvectors will be orthogonal.
- Eigenvalues/Eigenvectors
  - Frequently used in matrix decomposition for dimension reduction
  - Equation of eigenvalue: $|A - \lambda I| = 0$
  - Equation of eigenvector: $(A - \lambda I)v = 0$
- Calculating the Eigenvector Matrix is a relatively computation-intensive process.
  - This is a disadvantage of the Karhunen-Loeve Transform, which is based on multivariable statistics.

Karhunen Loève Transform (KLT) – Procedures

$$\boxed{\mathbf{X} = [\vec{x}_0, \vec{x}_1, \ldots \vec{x}_{N-1}]}$$

1. Find mean vector for input data $E(\mathbf{X}) = \frac{1}{N}\sum_{i=0}^{N-1} \vec{x}_i$
2. Find covariance matrix $\mathbf{R}_{\mathbf{XX}} = \frac{1}{N-1}\sum_{i=0}^{N-1} (\vec{x}_i - E(\vec{x}))(\vec{x}_i - E(\vec{x}))^T$
3. Find eigenvalues of the covariance matrix $|\mathbf{R}_{\mathbf{XX}} - \lambda \mathbf{I}| = 0$
4. Find eigenvectors of the covariance matrix $(\mathbf{R}_{\mathbf{XX}} - \lambda_i \mathbf{I})\vec{\varphi}_i = 0$
5. Normalise the eigenvectors $\vec{\varphi}^*{}_i = \frac{\vec{\varphi}_i}{|\vec{\varphi}_i|}$ so that $\langle \vec{\varphi}_i, \vec{\varphi}_i \rangle = 1$
6. Transform the input $\mathbf{Y} = \varphi^T \mathbf{X}$, where $\varphi^T = [\vec{\varphi}^*{}_1, \vec{\varphi}^*{}_2, \ldots]$

Karhunen Loève Transform (KLT) – Tutorial Question 1

- Find the KLT of the given dataset (sampled from a population):

a	b
-1	0
-2	-1
0	2
0	-1
2	4

Karhunen Loève Transform (KLT) – Tutorial Solution

- Step 1: Find the mean vector

$$E(X) = \begin{bmatrix} -0.2 \\ 0.8 \end{bmatrix} \qquad X = \begin{bmatrix} -1 & -2 & 0 & 0 & 2 \\ 0 & -1 & 2 & -1 & 4 \end{bmatrix}$$

- Step 2: Find covariance matrix

$$var_a = \frac{1}{N-1}\sum_{0}^{N-1}(a_i - a_{mean})^2 = \frac{1}{4}[(-1+0.2)^2 + (-2+0.2)^2 + 0.2^2 + 0.2^2 + (2+0.2)^2] = 2.2$$

$$var_b = \frac{1}{N-1}\sum_{0}^{N-1}(b_i - b_{mean})^2 = \frac{1}{4}[0.8^2 + (-1-0.8)^2 + (2-0.8)^2 + (-1-0.8)^2 + (4-0.8)^2] = 4.7$$

$$cov_{a,b} = \frac{1}{N-1}\sum_{0}^{N-1}(a_i - a_{mean})(b_i - b_{mean})$$

$$= \frac{1}{4}[(-1+0.2)(-0.8) + (-2+0.2)(-1-0.8) + 0.2(2-0.8) + 0.2(-1-0.8) + (2+0.2)(4-0.8)]$$

$$= 2.7$$

$$R_{ab} = \begin{bmatrix} 2.2 & 2.7 \\ 2.7 & 4.7 \end{bmatrix}$$

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Karhunen Loève Transform (KLT) – Tutorial Solution

- Step 3: Find the eigenvalues of the covariance matrix

$$|R_{ab} - \lambda \mathbf{I}| = 0$$

$$\left|\begin{bmatrix} 2.2 & 2.7 \\ 2.7 & 4.7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 2.7 - \lambda & 2.7 \\ 2.7 & 4.7 - \lambda \end{bmatrix}\right| = 0$$

$$(2.7 - \lambda)(4.7 - \lambda) - 2.7^2 = 0$$

$$12.69 - 7.4\lambda + \lambda^2 - 7.29 = 0$$

$$5.4 - 7.4\lambda + \lambda^2 = 0$$

$$\lambda_1 = 0.4747, \lambda_2 = 6.4253$$

$$\lambda_1 = 0.4747, \lambda_2 = 6.4253$$

$$(\mathbf{R}_{ab} - \lambda_i \mathbf{I})\vec{\varphi}_i = 0$$

$$\left(\begin{bmatrix} 2.2 & 2.7 \\ 2.7 & 4.7 \end{bmatrix} - \begin{bmatrix} 0.4747 & 0 \\ 0 & 0.4747 \end{bmatrix}\right)\varphi_1 = 0$$

$$\begin{bmatrix} 1.7253 & 2.7 \\ 2.7 & 4.2253 \end{bmatrix}\begin{bmatrix} \varphi_{1,1} \\ \varphi_{1,2} \end{bmatrix} = 0$$

$$\varphi_{1,1} = -\frac{2.7}{1.7253}\varphi_{1,2}$$

Karhunen Loève Transform (KLT) – Tutorial Solution

- Step 3: Find the eigenvalues of the covariance matrix

$$|R_{ab} - \lambda \mathbf{I}| = 0$$

$$\left| \begin{bmatrix} 2.2 & 2.7 \\ 2.7 & 4.7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 2.7 - \lambda & 2.7 \\ 2.7 & 4.7 - \lambda \end{bmatrix} \right| = 0$$

$$(2.7 - \lambda)(4.7 - \lambda) - 2.7^2 = 0$$

$$12.69 - 7.4\lambda + \lambda^2 - 7.29 = 0$$

$$5.4 - 7.4\lambda + \lambda^2 = 0$$

$$\lambda_1 = 0.4747, \lambda_2 = 6.4253$$

$$\lambda_1 = 0.4747, \lambda_2 = 6.4253$$

$$(\mathbf{R}_{ab} - \lambda_i \mathbf{I})\vec{\varphi}_i = 0$$

$$\left(\begin{bmatrix} 2.2 & 2.7 \\ 2.7 & 4.7 \end{bmatrix} - \begin{bmatrix} 6.4253 & 0 \\ 0 & 6.4253 \end{bmatrix} \right) \varphi_2 = 0$$

$$\begin{bmatrix} -4.2253 & 2.7 \\ 2.7 & -1.7253 \end{bmatrix} \begin{bmatrix} \varphi_{2,1} \\ \varphi_{2,2} \end{bmatrix} = 0$$

$$\varphi_{2,1} = \frac{2.7}{4.2253} \varphi_{2,2}$$

Karhunen Loève Transform (KLT) – Tutorial Solution

- Step 5: Normalize the eigenvectors

From last step, we got $\varphi_{1,1} = -\frac{2.7}{1.7253}\varphi_{1,2}$ and $\varphi_{2,1} = \frac{2.7}{4.2253}\varphi_{2,2}$

Using the fact that $\varphi_{1,1}^2 + \varphi_{1,2}^2 = 1$, we get $\left(\frac{2.7}{1.7253}\right)^2 \varphi_{1,2}^2 + \varphi_{1,2}^2 = 1.$

Hence, $\varphi_{1,2} = 0.5385, \varphi_{1,1} = -0.8427 \Rightarrow \varphi_1 = \begin{bmatrix} -0.8427 \\ 0.5385 \end{bmatrix}$

Similarly, since $\varphi_{2,1}^2 + \varphi_{2,2}^2 = 1$, we get $\left(\frac{2.7}{4.2253}\right)^2 \varphi_{2,2}^2 + \varphi_{2,2}^2 = 1.$

Hence, $\varphi_{2,2} = 0.8427, \varphi_{2,1} = 0.5385 \Rightarrow \varphi_2 = \begin{bmatrix} 0.5385 \\ 0.8427 \end{bmatrix}$

Karhunen Loève Transform (KLT) – Tutorial Solution

- <u>Step 6</u>: Transform the data

$$\varphi_1 = \begin{bmatrix} -0.8427 \\ 0.5385 \end{bmatrix}, \varphi_2 = \begin{bmatrix} 0.5385 \\ 0.8427 \end{bmatrix}$$

So, the transform matrix is $\varphi = \begin{bmatrix} -0.8427 & 0.5385 \\ 0.5385 & 0.8427 \end{bmatrix}$

Hence, the output is $\mathrm{Y} = \varphi^T X = \begin{bmatrix} -0.8427 & 0.5385 \\ 0.5385 & 0.8427 \end{bmatrix} \begin{bmatrix} -1 & -2 & 0 & 0 & 2 \\ 0 & -1 & 2 & -1 & 4 \end{bmatrix}$

$$\mathrm{Y} = \begin{bmatrix} 0.8427 & 1.1468 & 1.0769 & -0.5385 & 0.4685 \\ -0.5385 & -1.9196 & 1.6853 & -0.8427 & 4.4475 \end{bmatrix}$$

Karhunen Loève Transform (KLT) – Tutorial Question 2

- Find the covariance matrix of the given dataset (sampled from a population):

a	b	c
-1	0	2
-2	-1	4
0	2	0
0	-1	-2
2	4	0

Pop quiz

What is the dimension of this covariance matrix?

a. 2 x 2
b. 2 x 3
c. 3 x 3
d. 3 x 2
e. 5 x 5

* Same data a and b as in Question 1.
* You can directly use the solutions of Question 1 to save some work.

$$R_{ab} = \begin{bmatrix} 2.2 & 2.7 \\ 2.7 & 4.7 \end{bmatrix}$$

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Karhunen Loève Transform (KLT) – Tutorial Question 2

- Find the covariance matrix of the given dataset (sampled from a population):

a	b	c
-1	0	2
-2	-1	4
0	2	0
0	-1	-2
2	4	0

Pop quiz

What is the dimension of this covariance matrix?

a. 2 x 2
b. 2 x 3
c. 3 x 3
d. 3 x 2
e. 5 x 5

1. Covariance matrices are always square matrices
2. The dimension of a covariance matrix is N x N, where N is the number of variables

* Same data a and b as in Question 1.
* You can directly use the solutions of Question 1 to save some work.

$$R_{ab} = \begin{bmatrix} 2.2 & 2.7 \\ 2.7 & 4.7 \end{bmatrix}$$

Karhunen Loève Transform (KLT) – Tutorial Solution

- Step 1: Find the mean vector

$$E(X) = \begin{bmatrix} -0.2 \\ 0.8 \\ 0.8 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & -2 & 0 & 0 & 2 \\ 0 & -1 & 2 & -1 & 4 \\ 2 & 4 & 0 & -2 & 0 \end{bmatrix}$$

- Step 2: Find covariance matrix

In Question 1, we have $var_a = 2, var_b = 4.7, cov_{a,b} = 2.7$
Only need to compute $var_c =$, $cov_{a,c} =$, $cov_{b,c} =$

$$var_c = \frac{1}{N-1}\sum_{0}^{N-1}(c_i - c_{mean})^2 = \frac{1}{4}[(2-0.8)^2 + (4-0.8)^2 + 0.8^2 + (-2-0.8)^2 + 0.8^2] = 5.2$$

$$cov_{a,c} = \frac{1}{N-1}\sum_{0}^{N-1}(a_i - a_{mean})(c_i - c_{mean})$$

$$= \frac{1}{4}[(-1+0.2)(2-0.8) + (-2+0.2)(4-0.8) + 0.2(-0.8) + 0.2(-2-0.8) + (2+0.2)(-0.8)]$$

$$= -2.3$$

$$cov_{b,c} = \frac{1}{N-1}\sum_{0}^{N-1}(b_i - b_{mean})(c_i - c_{mean})$$

$$= \frac{1}{4}[(-0.8)(2-0.8) + (-1-0.8)(4-0.8) + (2-0.8)(-0.8) + (-1-0.8)(-2-0.8) + (4-0.8)(-0.8)]$$

$$= -1.3$$

$$R_{a,b,c} = \begin{bmatrix} 2 & 2.7 & -2.3 \\ 2.7 & 4.7 & -1.3 \\ -2.3 & -1.3 & 5.2 \end{bmatrix}$$

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