Advanced Transform Methods

Heisenberg Uncertainty Principle

Andy Watson

Heisenberg Uncertainty Principle

$$\Delta_t \Delta_\omega \geq 1/2$$

Time Convolution

The Fourier transform of the convolution is the product of the transforms.

$$\int_{-\infty}^{\infty} \left[s_1(t) * s_2(t) \right] e^{-j\omega t} dt = S_1(\omega) S_2(\omega)$$

So the convolution of two signals is the inverse Fourier transform of the product of the transforms.

$$\int_{-\infty}^{\infty} s_1(\tau) s_2(t-\tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_1(\omega) S_2(\omega) e^{j\omega t} d\omega$$

Frequency Convolution

To find the inverse Fourier transform of the convolution $S_1(\omega) * S_2(\omega) \equiv \mathrm{S}(\omega)$

$$\mathrm{s}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[S_1(\omega) * S_2(\omega)\right]e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega t}\left[\int_{-\infty}^{\infty}S_1(\alpha)S_2(\omega-\alpha)d\alpha\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{j(x+\alpha)t}S_1(\alpha)S_2(x)d\alpha dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}S_1(\alpha)e^{j\alpha t}d\alpha\int_{-\infty}^{\infty}e^{jxt}S_2(x)dx$$

$$= 2\pi s_1(t)s_2(t)$$

Inverse Fourier Transform of the convolution of 2 Fourier transforms is 2π times the product of the signals.

→

The convolution of 2 Fourier transforms is 2π times the Fourier transform of the product of the signals

$$\int_{-\infty}^{\infty}S_1(\alpha)S_2(\omega-\alpha)d\alpha = 2\pi\int_{-\infty}^{\infty}s_1(t)s_2(t)e^{-j\omega t}dt$$

Parseval's Formula

From the time convolution formula,

$$\int_{-\infty}^{\infty} s_1(\tau)s_2(t-\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_1(\omega)S_2(\omega)e^{j\omega t}d\omega$$

So

$$\int_{-\infty}^{\infty} s_1(t)s_2(-t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_1(\omega)S_2(\omega)d\omega$$

If $s_2^*(t) = f(-t)$ then the Hermitian property, $s^*(t) \leftrightarrow S^*(-\omega)$ tells us

$$S_2^*(-\omega) = F(-\omega)$$

$$S_2(\omega) = F^*(\omega)$$

So

$$\int_{-\infty}^{\infty} s_1(t)f^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_1(\omega)F^*(\omega)d\omega$$

Energy

Parseval's formula states,

$$\int_{-\infty}^{\infty} s(t) f^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) F^*(\omega) d\omega$$

If *s*=*f*,

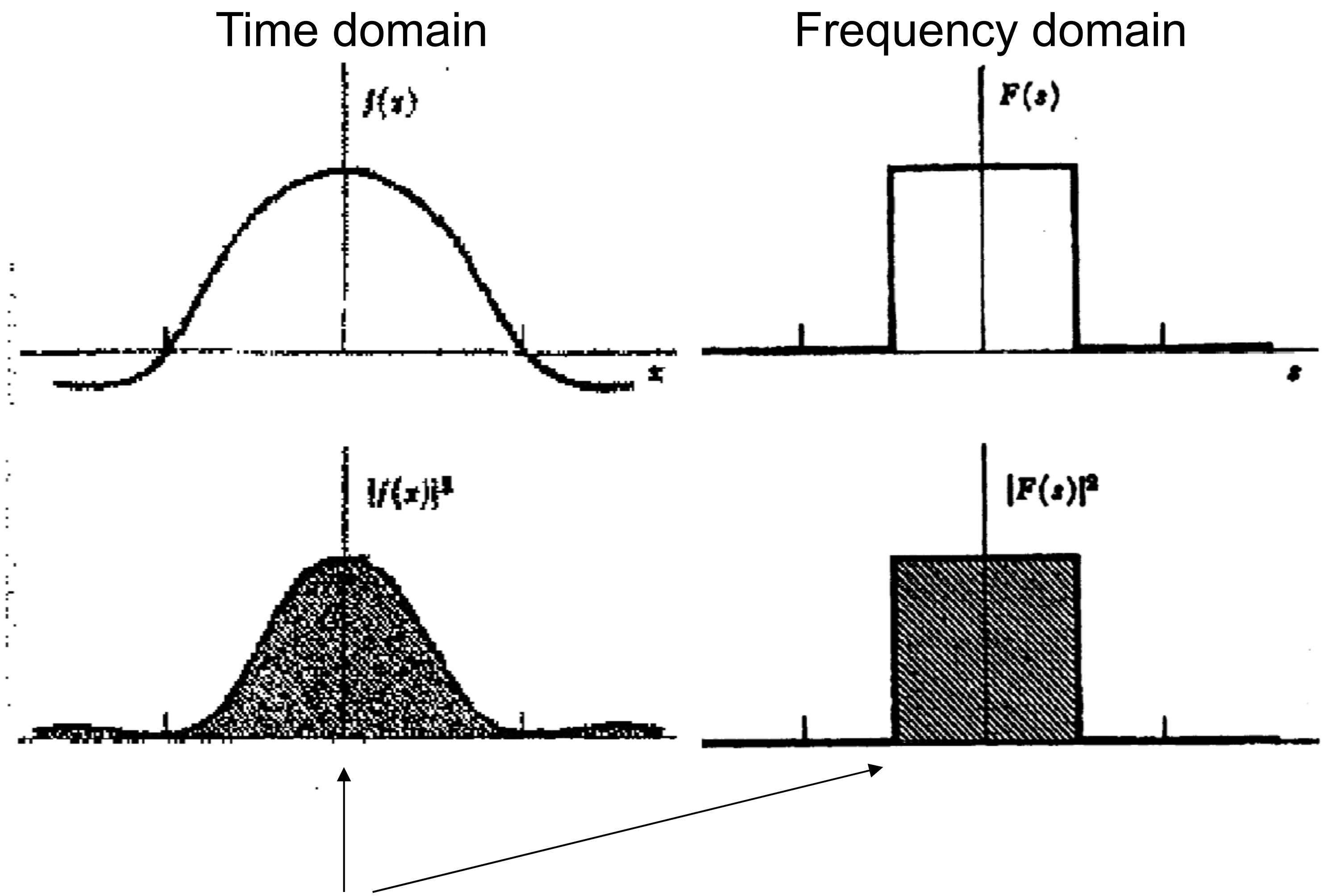
$$\int_{-\infty}^{\infty} |s(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 d\omega$$

We can define energy in both time and frequency domains

$$E = \|s(t)\|^2 = \int_{-\infty}^{\infty} |s(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 d\omega$$

Energy is conserved

Parseval's Theorem in action



The two shaded areas (i.e., measures of the signal energy) are the same.

Mean Time and Time Duration

$$E = \int_{-\infty}^{\infty} |s(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 \, d\omega$$

- The signal's gravitational centres in the time domain.
  - Mean Time $$\langle t \rangle = \frac{1}{E} \int_{-\infty}^{\infty} t |s(t)|^2 \, dt$$
- The signal's energy spread in the time domain
  - Time Duration Δ_t $$\Delta_t^2 = \frac{1}{E} \int_{-\infty}^{\infty} \left(t - \langle t \rangle\right)^2 |s(t)|^2 \, dt$$

Mean Frequency and Frequency Bandwidth

$$E = \int_{-\infty}^{\infty} |s(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 \, d\omega$$

- The signal's gravitational centre in the frequency domain.
  - Mean Frequency $\langle \omega \rangle = \frac{1}{2\pi E} \int_{-\infty}^{\infty} \omega |S(\omega)|^2 \, d\omega$
- The signal's energy spread in the frequency domain.
  - Frequency Bandwidth Δ_ω

$$\Delta_\omega^2 = \frac{1}{2\pi E} \int_{-\infty}^{\infty} \left(\omega - \langle \omega \rangle\right)^2 |S(\omega)|^2 \, d\omega$$

Duration and Bandwidth

– Time Duration

$$\Delta_t^{\ 2} = \frac{1}{E}\int_{-\infty}^{\infty}(t^2 - 2t\langle t\rangle + \langle t\rangle^2)\left|s(t)\right|^2 dt$$

$$= \frac{1}{E}\int_{-\infty}^{\infty} t^2\left|s(t)\right|^2 dt - \frac{2\langle t\rangle}{E}\int_{-\infty}^{\infty}\left|s(t)\right|^2 tdt + \frac{\langle t\rangle^2}{E}\int_{-\infty}^{\infty}\left|s(t)\right|^2 dt$$

$$= \frac{1}{E}\int_{-\infty}^{\infty} t^2\left|s(t)\right|^2 dt - 2\langle t\rangle\langle t\rangle + \langle t\rangle^2 = \frac{1}{E}\int_{-\infty}^{\infty} t^2\left|s(t)\right|^2 dt - \langle t\rangle^2$$

– Frequency Bandwidth

$$\Delta_\omega^{\ 2} = \frac{1}{2\pi E}\int_{-\infty}^{\infty}(\omega^2 - 2\omega\langle\omega\rangle + \langle\omega\rangle^2)\left|S(\omega)\right|^2 d\omega$$

$$= \frac{1}{2\pi E}\int_{-\infty}^{\infty}\omega^2\left|S(\omega)\right|^2 d\omega - \frac{2\langle\omega\rangle}{2\pi E}\int_{-\infty}^{\infty}\omega\left|S(\omega)\right|^2 d\omega + \frac{\langle\omega\rangle^2}{2\pi E}\int_{-\infty}^{\infty}\left|S(\omega)\right|^2 d\omega$$

$$= \frac{1}{2\pi E}\int_{-\infty}^{\infty}\omega^2\left|S(\omega)\right|^2 d\omega - 2\langle\omega\rangle^2 + \frac{\langle\omega\rangle^2}{E}E = \frac{1}{2\pi E}\int_{-\infty}^{\infty}\omega^2\left|S(\omega)\right|^2 d\omega - \langle\omega\rangle^2$$

Normalised in Time and Frequency

– Given a signal $s(t)$, can one find a signal with the same energy, bandwidth and duration but normalised such that the mean frequency and the mean time of the signal are both set to 0?

– Yes:

$$r(t) = e^{-jt\langle\omega\rangle} s(t + \langle t \rangle)$$

– That is, a shift left in time and a shift left in frequency. The shift in frequency is obtained by modulation in time.

Uncertainty Principle

To show :

if $\sqrt{t}s(t) \to 0$ as $|t| \to \infty$ (i.e. $s(t)$ decays fast enough)

and the signal has unit energy :

$$E = \int_{-\infty}^{\infty} \left|s(t)\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|S(\omega)\right|^2 d\omega = 1$$

then

$$\Delta_t \Delta_\omega \geq \frac{1}{2}$$

where equality holds when $s(t)$ is a Gaussian, $s(t) = Ae^{-\alpha t^2}$

We will assume that $\langle t \rangle = 0$ and $\langle \omega \rangle = 0$. (Normalised)

{ NB Will do for simple case of real $s(t) = s^*(t)$ }

Squared time width is given by : $\Delta_t^2 = \int_{-\infty}^{\infty} t^2 |s(t)|^2 \, dt$

and sq. frequency width by : $\Delta_\omega^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |S(\omega)|^2 \, d\omega$

with their product as :

$$\Delta_t^2 \Delta_\omega^2 = \int_{-\infty}^{\infty} t^2 |s(t)|^2 \, dt \, \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |S(\omega)|^2 \, d\omega$$

From the time derivative property, we know

if $\quad h(t) = \frac{d}{dt} s(t) \quad$ then $\quad H(\omega) = j\omega S(\omega)$

So using this in the frequency width we get

$$\Delta_\omega^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |S(\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |j\omega S(\omega)|^2 \, d\omega$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega$$

So we have $$\Delta_\omega^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|H(\omega)\right|^2 d\omega$$

But from Parseval we have the conservation of energy :

$$\int_{-\infty}^{\infty} \left|h(t)\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|H(\omega)\right|^2 d\omega$$

so $\Delta_\omega^2 = \int_{-\infty}^{\infty} \left|h(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|\frac{d}{dt} s(t)\right|^2 dt$ since $h(t) = \frac{d}{dt} s(t)$

Therefore for the original product we get

$$\Delta_t^2 \Delta_\omega^2 = \int_{-\infty}^{\infty} t^2 \left|s(t)\right|^2 dt \int_{-\infty}^{\infty} \left|\frac{d}{dt} s(t)\right|^2 dt$$

So we have $$\Delta_t^2 \Delta_\omega^2 = \int_{-\infty}^{\infty} t^2 \left|s(t)\right|^2 dt \int_{-\infty}^{\infty} \left|\frac{d}{dt} s(t)\right|^2 dt$$

The *Schwarz Inequality* states: $\left\|\psi_1\right\|^2 \left\|\psi_2\right\|^2 \geq \left|\left\langle \psi_1, \psi_2 \right\rangle\right|^2$ i.e.

$$\int_{-\infty}^{\infty} \left|f(t)\right|^2 dt \cdot \int_{-\infty}^{\infty} \left|g(t)\right|^2 dt \geq \left|\int_{-\infty}^{\infty} f(t) g(t) dt\right|^2$$

which we can use with $f(t) = ts(t)$ and $g(t) = \frac{d}{dt} s(t)$ to get

$$\Delta_t^2 \Delta_\omega^2 = \int_{-\infty}^{\infty} t^2 \left|s(t)\right|^2 dt \int_{-\infty}^{\infty} \left|\frac{d}{dt} s(t)\right|^2 dt \qquad \geq \qquad \left|\int_{-\infty}^{\infty} ts(t) \frac{d}{dt} s(t) dt\right|^2$$

{Remember using real $s(t)$ }

So far we have $\Delta_t^2 \Delta_\omega^2 \ \geq \ \left(\int_{-\infty}^{\infty} ts(t) \frac{d}{dt} s(t) dt \right)^2$

$\{s(t)$ etc are real$\}$

Now, try differentiating $s(t)^2$

$$\frac{d}{dt} s(t)^2 = 2s(t) \frac{d}{dt} s(t)$$

so inserting this we get

$$\Delta_t^2 \Delta_\omega^2 \ \geq \ \left(\int_{-\infty}^{\infty} ts(t) \frac{d}{dt} s(t) dt \right)^2$$

$$= \left(\frac{1}{2} \int_{-\infty}^{\infty} t \frac{d}{dt} s(t)^2 dt \right)^2$$

So far we have $\Delta_t^2 \Delta_\omega^2 \;\geq\; \left(\frac{1}{2} \int_{-\infty}^{\infty} t \, \frac{d}{dt} s(t)^2 \, dt \right)^2$

Now remember *integration by parts* :

$$\int_a^b u dv = [uv]_a^b - \int_a^b v du$$

and use $u = t, \quad v = s^2(t), \quad du = dt, \quad dv = \frac{d}{dt} s^2(t) dt$

Then $\frac{1}{2} \int_{-\infty}^{\infty} t \frac{d}{dt} s(t)^2 \, dt = \frac{1}{2} [ts(t)^2]_{-\infty}^{\infty} - \frac{1}{2} \int_{-\infty}^{\infty} s(t)^2 \, dt$

But we specified that $\sqrt{t} s(t) \to 0$ as $|t| \to \infty$, so

$$[ts(t)^2]_{-\infty}^{\infty} = \infty s(\infty)^2 - (-\infty) s(-\infty)^2 = 0 - 0 = 0$$

So we now have

$$\Delta_t^2 \Delta_\omega^2 \geq \left(-\frac{1}{2} \int_{-\infty}^{\infty} s^2(t) dt \right)^2$$

So far we have

$$\Delta_t^2 \Delta_\omega^2 \ge \left(-\tfrac{1}{2} \int_{-\infty}^{\infty} s^2(t)\,dt \right)^2$$

But $\int_{-\infty}^{\infty} s^2(t)\,dt = 1$ since it is the energy of the signal,

so $\Delta_t^2 \Delta_\omega^2 \ge \left(-\tfrac{1}{2} \times 1 \right)^2$ i.e. $\underline{\underline{\Delta_t \Delta_\omega \ge \frac{1}{2}}}$

Which was what we wanted.

Finally, Schwartz' inequality is exact (an equality) when the two functions ψ_1 and ψ_2 are colinear, $\psi_1 = c\psi_2$

With our functions $\psi_1 = ts(t)$ and $\psi_2 = \frac{d}{dt} s(t)$

this happens for Gaussian $s(t) = Ae^{-\alpha t^2}$,

$$\psi_2 = \frac{d}{dt} s(t) = -2\alpha t A e^{-\alpha t^2} = -2\alpha t s(t) = -2\alpha \psi_1$$

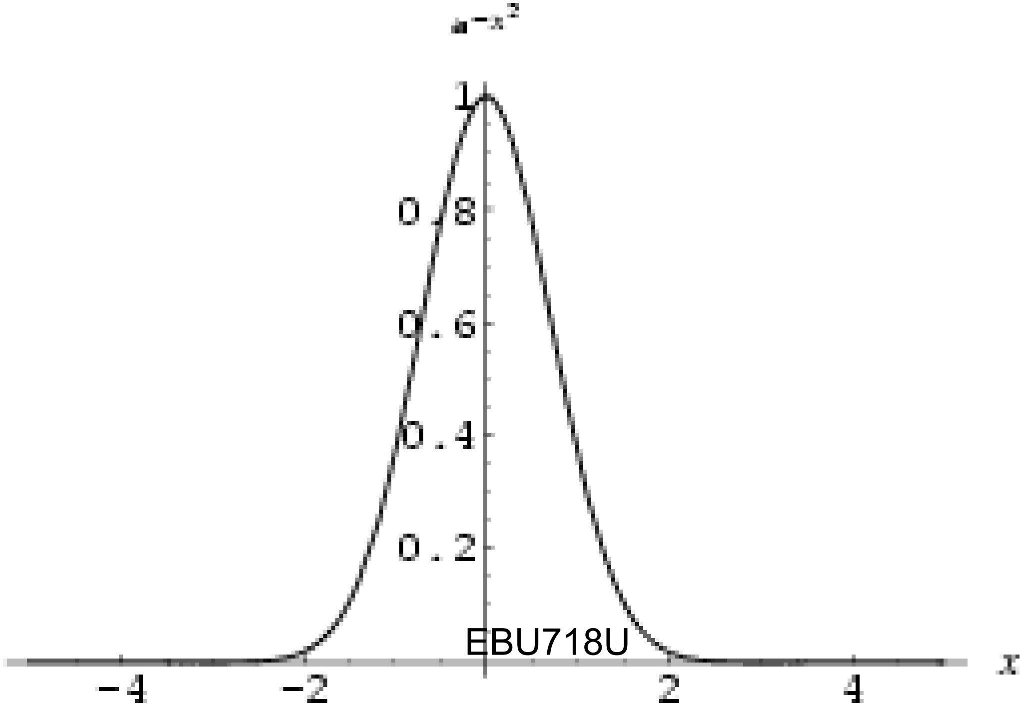
So $\Delta_t \Delta_\omega = \frac{1}{2}$ for a Gaussian

Uncertainty Principle

- We can easily find signals normalised in both time and frequency simultaneously, but it is not possible to *localise* a function in time and frequency simultaneously.
- Different definitions of the Fourier Transform yield different versions of the Uncertainty Principle.
- The Time-Bandwidth Product $\Delta_t \Delta_\omega$ is a measure of the pulse complexity
- A signal fixed in time has infinite bandwidth
- A signal fixed in frequency has infinite duration
- Discovered by Heisenberg and Applied to quantum mechanics

Special Properties of the Gaussian

- Simple form $$g(t) = e^{-at^2}$$
- General form $$g(t) = ae^{jbt}e^{-c(t-d)^2}$$
- Normalised form $$g(t) = \sqrt[4]{\frac{\alpha}{\pi}}\, e^{-\frac{\alpha}{2}t^2}$$



Uncertainty Principle and Gaussian functions

$$\frac{d}{dt}e^{-at^2} = -2ate^{-at^2}$$

$$E^2\Delta_t^2\Delta_\omega^2 = \left(\int_{-\infty}^{\infty} t^2 \left|e^{-at^2}\right|^2 dt\right)\left(\int_{-\infty}^{\infty} \left|\frac{d}{dt}e^{-at^2}\right|^2 dt\right)$$

$$= \left(\int_{-\infty}^{\infty} t^2 \left(e^{-at^2}\right)^2 dt\right)\left(\int_{-\infty}^{\infty} \left(-2ate^{-at^2}\right)^2 dt\right)$$

$$= \left(\int_{-\infty}^{\infty} -2at^2e^{-at^2}e^{-at^2} dt\right)\left(\int_{-\infty}^{\infty} -2at^2e^{-at^2}e^{-at^2} dt\right)$$

$$= \left(\int_{-\infty}^{\infty} te^{-at^2}\frac{d}{dt}e^{-at^2} dt\right)\left(\int_{-\infty}^{\infty} te^{-at^2}\frac{d}{dt}e^{-at^2} dt\right)$$

In fact, the Gaussian is the *only* function that gives equality in the uncertainty relationship.

A Gaussian transforms to a Gaussian

$$\int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \cos(\omega t) e^{-at^2} dt - j \int_{-\infty}^{\infty} \sin(\omega t) e^{-at^2} dt$$

$$= \int_{-\infty}^{\infty} \cos(\omega t) e^{-at^2} dt = \sqrt{\frac{\pi}{a}} e^{-\omega^2/4a}$$

The narrower a Gaussian is in one domain, the broader it is in the other domain.

Gaussian and Convolution- The Central Limit Theorem

The Central Limit Theorem says:

The convolution of the convolution of the convolution etc. of any signal approaches a Gaussian.

Mathematically,

$$f(x) * f(x) * f(x) * f(x) * \ldots * f(x) \rightarrow e^{(-x/a)2}$$

or: $$f(x)^{*n} \rightarrow exp[(-x/a)^2]$$

The Central Limit Theorem is why everything has a Gaussian distribution.

The Central Limit Theorem for a square function, $\Pi(x)$

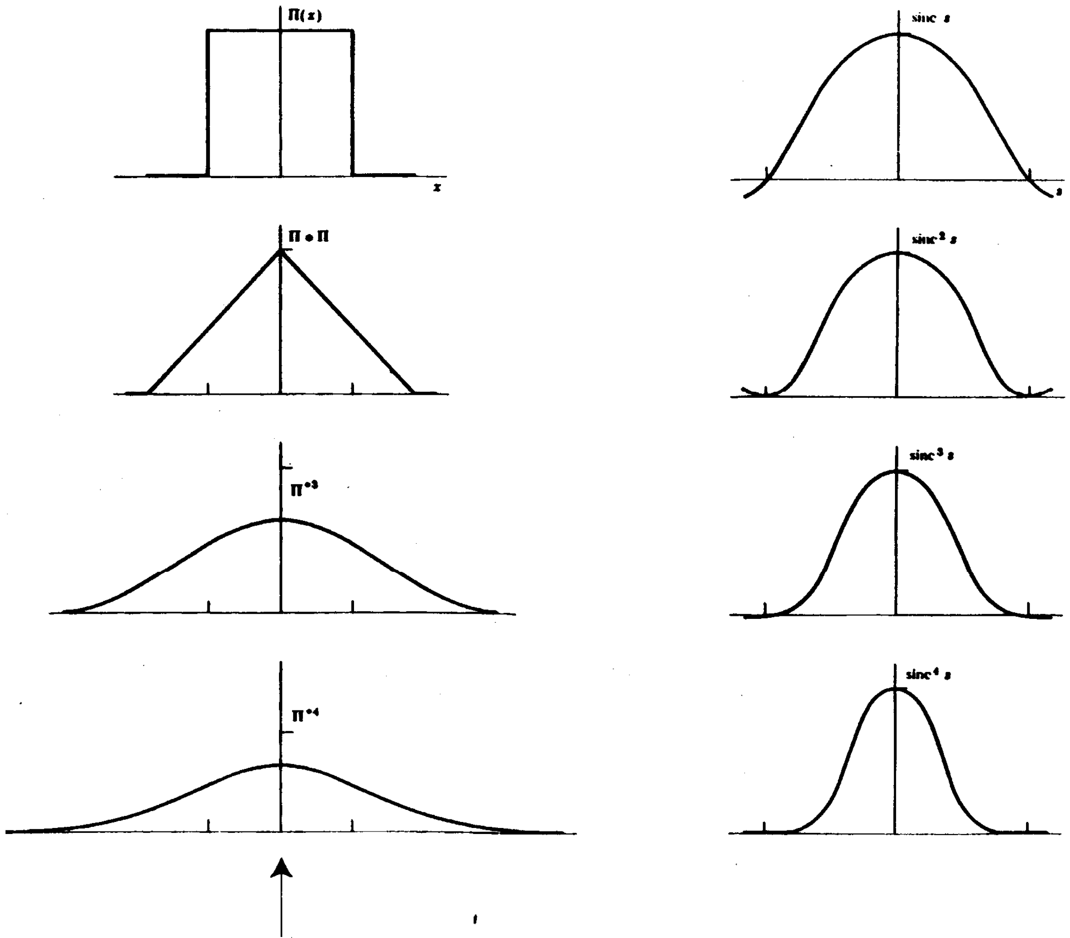
$\Pi(x)^{*n}$ | $\mathrm{sinc}(x)^n$

n = 1

n = 2

n = 3

n = 4



Note that $P(x)^{*4}$ already looks like a Gaussian!