# § 3.10 Capacitors & Capacitance

- Categories and definitions
  1. Generally, a capacitor consists of two isolated conductors, charged by $q$ and $-q$, and with an E-potential of $U$. $C=q/U$
  2. **Self-capacitance**: an isolated conductor charged by $q$ and with E-potential $\psi$, can be of self-C. $C=q/\psi$
  3. **Distributed Capacitance**: in fact, in a system of multi-conductors with a complex distribution of charges, there exists distributed capacitances between any 2 conductors.



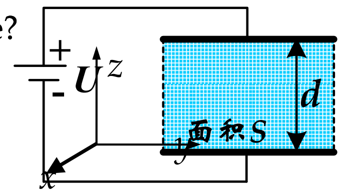
# Capacitors & Capacitance

- In fact, capacitance exists between 2 conductors of any shape adjacent to each other.
- The capacitance depends on its size and material, independent of whether it is charged or not.



# Example 1. parallel-plate capacitor

- Two parallel plates, each of area $S$, separated by a distance of $d$, with the dielectric of $\varepsilon$ between them. Please determine the capacitance?

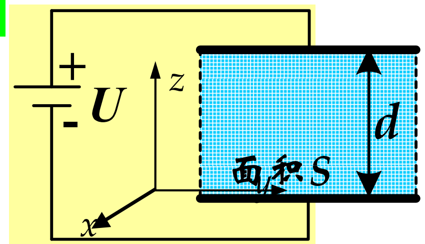



- The separation between the plates is very small compared to their other dimension, and thus we **neglect the edge effects** and assume E-field is uniformly distributed between 2 plates.
- Assume the E-potential difference $U$, and we set up the Laplace's Equ. since there is no charge between 2 plates.

$$\nabla^2\psi = \nabla^2\psi(z) = \frac{d^2\psi}{dz^2} = 0$$



$$\nabla^2\psi = \nabla^2\psi(z) = \frac{d^2\psi}{dz^2} = 0 \qquad \psi = C_1 z + C_2$$




Boundary conditions:

$$\psi\,|_{z=0} = 0 \qquad \psi\,|_{z=d} = U$$

*Dirichlet Problems*

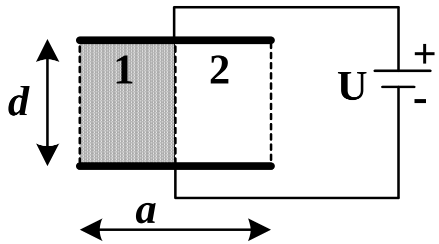
$$\psi(z) = \frac{z}{d}\cdot U \quad \Rightarrow \vec{E} = -\nabla\psi = (-U/d)\vec{e}_z \quad \Rightarrow Q = \sigma_s \bullet S = ?$$

$$\sigma_s = ?$$

$$\boxed{\therefore C = {}^{Q}\!/_{U} = \frac{S}{d}\cdot\varepsilon}$$

## Example 2.

Parallel-plate capacitor,
area of each plate --- $a \times b$
E-potential difference --- $U$




Solution 1. parallel connection of capacitors, $C=C_1+C_2$

Solution 2. via the definition, C=Q/U



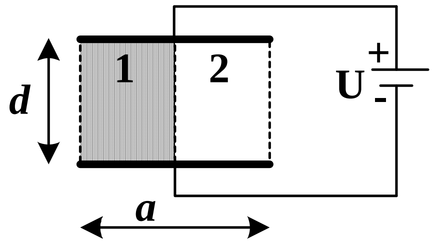
## Solution 2. via the definition, C=Q/U

$C=Q/U \qquad Q=Q_1+Q_2$

$Q_1 = \sigma_1 \cdot (\frac{a}{2} \cdot b)$

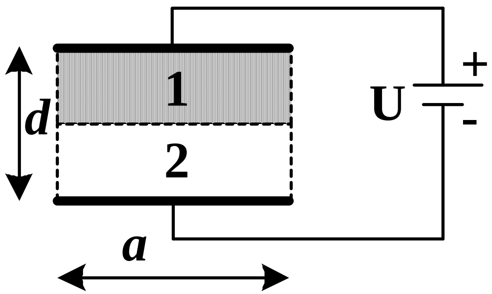
$\sigma_1 = D_{1n} = D_1 = \varepsilon_1 E_1$

$\vec{E} = ? \qquad \vec{E}_1 = \vec{E}_2\ ?$






## Homework




Please determine the capacitance.



- Expressions are more complicate in cylindrical and spherical coordinates.

### In Cylindrical Coordinates

$$\nabla^2 u(r,\varphi,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}$$

### In Spherical Coordinates

$$\nabla^2 u(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

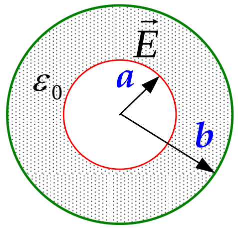
## Example 3. spherical capacitor

- Formed by a metallic ball and a concentric metallic sphere.
- Please determine the capacitance.
- In general, the first step is to assume $Q$ or $U$.

$$\vec{E} = ?$$

$$U = \int_{?}^{?} \vec{E} \cdot d\vec{r}$$

$$C = Q / U$$




Refer to Example 3.20 in
textbook page 110



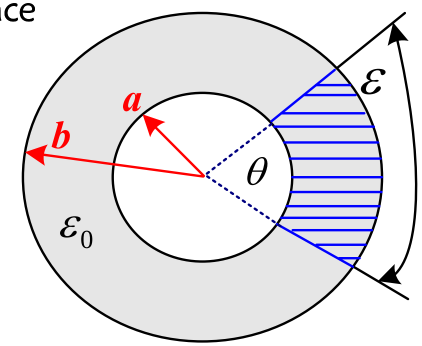
## Example 4. special coaxial lines

Please calculate the capacitance per unit length.

Analysis:

(1) How many approaches are there to calculate a capacitance?

(2) Is there symmetry? What coordinates shall we choose?






(1) Assume 0 E-potential to be at infinite

(2) Assume the boundary conditions: $\psi\,|_{r=a} = U \qquad \psi\,|_{r=b} = 0$

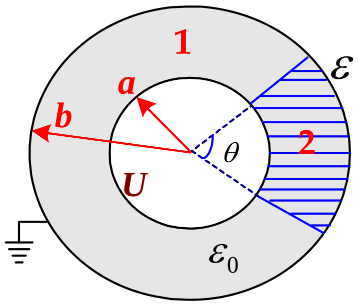
(3) Present Laplace's Equ.

(4) Due to axial symmetry, E-potential depends on only r.

$$\nabla^2 \psi_1 = 0$$

$$\psi_1 = \psi_1(r) = A \ln r + B$$

$$\psi_2 = \psi_2(r) = E \ln r + D$$



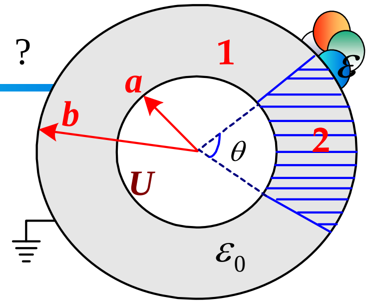

Apply boundary conditions

$$\psi_1 = \psi_2 = \frac{U}{\ln \frac{b}{a}} \ln \frac{b}{r}$$



$$\psi_1 = \psi_2 \qquad \vec{E}_1 = \vec{E}_2 \ ? \qquad \vec{D}_1 = \vec{D}_2 \ ?$$

$$\therefore \vec{E} = -\nabla \psi = \vec{a}_r \frac{U}{\ln \frac{b}{a}} \cdot \frac{1}{r}$$




Charges on the inner line per meter

$$Q = \sigma \cdot S = ?$$

$$Q = D_1[a(2\pi - \theta)] + D_2[a\theta]$$

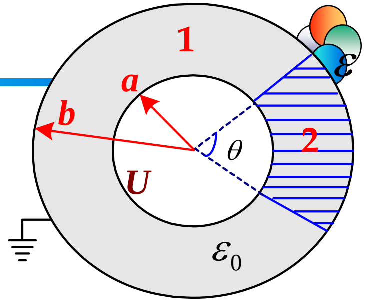
$$= \varepsilon_0 E_1 a(2\pi - \theta) + \varepsilon E_2 a\theta$$

$$= aE[\varepsilon_0(2\pi - \theta) + \varepsilon\theta] = \frac{U}{\ln \dfrac{b}{a}}[\varepsilon_0(2\pi - \theta) + \varepsilon\theta]$$

$$\psi_1 = \psi_2 = U$$

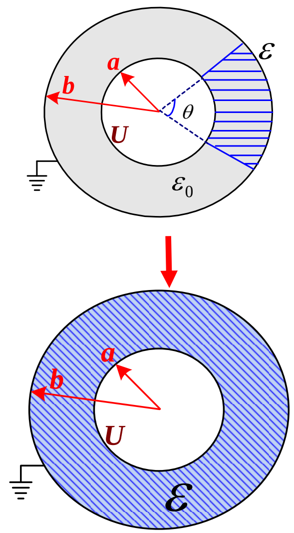
$$Q = \cdots$$

$$C = \frac{Q}{U} = \frac{\varepsilon\theta + \varepsilon_0(2\pi - \theta)}{\ln(\frac{b}{a})}$$






## Capacitance of Common Coaxial Lines




$$C = \frac{Q}{U} = \frac{\varepsilon\theta + \varepsilon_0(2\pi - \theta)}{\ln(\frac{b}{a})}$$

$$\theta \to 2\pi$$

$$C = \frac{Q}{U} = \frac{2\pi \cdot \varepsilon}{\ln(\frac{b}{a})}$$



## Homework

```
  (’v’)     (’v’)     (’v’)     (’v’)
 ((   ))   ((   ))   ((   ))   ((   ))
-/-”--”----/-”--”----/-”--”----/-”--”--
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**Exercises:**

**3.16,3.18,3.31**

**3.37 3.38**