3.31 A cylindrical conductor of radius a is enclosed by another cylindrical conductor of radius b to form a cylindrical capacitor. The permittivity of the medium is ϵ. Obtain an expression for the capacitance per unit length using equation (3.74). If the length of the capacitor is L, what is its total capacitance?

Exercise 3.31 $\quad D_\rho = \frac{a}{\rho}\rho_s \Rightarrow E_\rho = \frac{a\rho_s}{\rho\epsilon}$. $V_{ab} = -\int_b^a \frac{a\rho_s}{\epsilon\rho}\,d\rho = \frac{a\rho_s}{\epsilon}\ln(b/a)$

$$C = \frac{Q}{V_{ab}} = \frac{(2\pi a L \rho_s)\,\epsilon}{a\rho_s \ln(b/a)} = \frac{2\pi\epsilon L}{\ln(b/a)}$$

3.37 A homogeneous dielectric medium fills the region between two concentric spherical shells of radii a and b. The inner shell is held at a potential of V_0, and the outer shell is grounded. Compute (a) the potential distribution, (b) the electric field intensity, (c) the electric flux density, (d) the surface charge density on the inner surface, (e) the capacitance, and (f) the total energy stored in the system.

Exercise 3.37 $\quad \nabla^2 V = 0 \Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \Rightarrow V = -\frac{C_1}{r} + C_2$

at $r=b$, $V=0 \Rightarrow C_2 = \frac{C_1}{b}$. Thus, $V = -C_1\left[\frac{1}{r} - \frac{1}{b}\right]$

at $r=a$, $V=V_0 \Rightarrow C_1 = \frac{V_0}{\frac{1}{b}-\frac{1}{a}}$. Finally, $V = \frac{V_0}{\frac{1}{a}-\frac{1}{b}}\left[\frac{1}{r}-\frac{1}{b}\right]$

$\vec{E} = -\nabla V = -\frac{C_1}{r^2}\vec{a}_r = \frac{ab\,V_0}{(b-a)\,r^2}\vec{a}_r \qquad D_r = \frac{\epsilon V_0\,ab}{(b-a)\,r^2} \qquad \rho_{sa} = \frac{\epsilon V_0\,ab}{(b-a)a^2},\ \rho_{sb} = \frac{-\epsilon V_0 ab}{(b-a)b^2}$

$Q_a = \frac{\epsilon V_0\,ab}{(b-a)\,a^2}\cdot 4\pi a^2 = \frac{4\pi\epsilon V_0 ab}{(b-a)}$, $C = \frac{Q_a}{V_0} = \frac{4\pi\epsilon ab}{b-a}$, $W = \frac{1}{2}CV_0^2 = \frac{2\pi\epsilon ab}{b-a}V_0^2$

3.38 The space between the conductors in a coaxial cable is filled with two concentric layers of dielectric, as shown in Figure 3.46. Determine (a) the potential function in each medium, (b) the $\vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$ fields in each region, (c) the charge distribution on the inner conductor, and (d) the capacitance per unit length. Show that the capacitance is equivalent to that of two capacitors connected in series.

Exercise 3.38

L = Length

$\nabla^2 V = 0 \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial r}\right) = 0$

Region-1: $V_1 = k_1 \ln \rho + k_2$

at $\rho = a$, $V_1 = V_0 \Rightarrow k_2 = V_0 - k_1 \ln a$

$V_1 = V_0 + k_1 \ln(\rho/a)$

$E_{1\rho} = -\frac{\partial V_1}{\partial \rho} = -\frac{k_1}{\rho}$, $D_{1\rho} = -\frac{\epsilon_1 k_1}{\rho}$

at $\rho = c$, $V_1 = V_2$ and $D_{1\rho} = D_{2\rho}$

Thus, $V_0 + k_1 \ln\left(\frac{c}{a}\right) = k_3 \ln(c/b)$

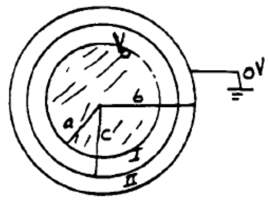
and $\epsilon_1 k_1 = \epsilon_2 k_3$

$k_1 = \frac{-\epsilon_2 V_0}{M}$, $k_3 = \frac{-\epsilon_1 V_0}{M}$

where $M = \epsilon_1 \ln(b/c) + \epsilon_2 \ln(c/a)$

$\rho_s|_{\rho=a} = D_{1\rho}|_{\rho=a} = -\frac{\epsilon_1 k_1}{a} = \frac{\epsilon_1 \epsilon_2 V_0}{a M}$

$Q_a = 2\pi a L \, \rho_s|_{\rho=a} = \frac{2\pi \epsilon_1 \epsilon_2 V_0 L}{M}$



Region-2: $V_2 = k_3 \ln \rho + k_4$

at $\rho = b$, $V_2 = 0 \Rightarrow k_4 = -k_3 \ln b$

$V_2 = k_3 \ln(\rho/b)$

$E_{2\rho} = -\frac{\partial V_2}{\partial \rho} = -\frac{k_3}{\rho}$, $D_{2\rho} = -\frac{\epsilon_2 k_3}{\rho}$

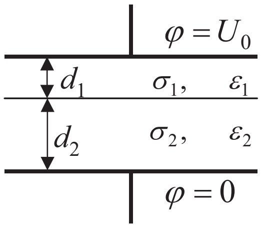
$C = \frac{Q_a}{V_0} = \frac{2\pi \epsilon_1 \epsilon_2 L}{\epsilon_1 \ln(b/c) + \epsilon_2 \ln(c/a)}$

$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$ where

$C_1 = \frac{2\pi \epsilon_1 L}{\ln(c/a)}$, $C_2 = \frac{2\pi \epsilon_2 L}{\ln(b/a)}$

C_1 and C_2 are connected in series.

4 在平行板电容器的两极板之间，填充两理想媒质，如图所示。若在电极之间外加电压U_0，求：（1）两种介质片中的 *E*；（2）每种介质片上的电压；（3）上、下极板和介质分界面上的自由电荷面密度。



解：

（1）首先求解电场

方法 1：利用无源理想介质的边界条件$D_{1n} = D_{2n}$求$\boldsymbol{E}$

对于题中的平行板电容器，有

$$U_0 = E_1 d_1 + E_2 d_2$$

由无源理想介质的边界条件$D_{1n} = D_{2n}$，有

$$\varepsilon_1 E_1 = \varepsilon_2 E_2$$

联立上面两式，可解得介质 1 和介质 2 中的电场分别为

$$E_1 = \frac{\varepsilon_2 U_0}{\varepsilon_2 d_1 + \varepsilon_1 d_2}，\quad E_2 = \frac{\varepsilon_1 U_0}{\varepsilon_2 d_1 + \varepsilon_1 d_2}$$

相应的，介质 1 和介质 2 中的电流为

$$D_1 = D_2 = \frac{\varepsilon_1 \varepsilon_2 U_0}{\varepsilon_2 d_1 + \varepsilon_1 d_2}$$

（2）每介质片上的电压为

$$U_1 = E_1 d_1 = \frac{d_1 \varepsilon_2 U_0}{\varepsilon_2 d_1 + \varepsilon_1 d_2}，\ U_2 = E_2 d_2 = \frac{d_2 \varepsilon_1 U_0}{\varepsilon_2 d_1 + \varepsilon_1 d_2}$$

（3）各极板和介质分界面上的自由电荷面密度

上极板为 $\rho_{S1} = D_1 = \varepsilon_1 E_1 = \dfrac{\varepsilon_1 \varepsilon_2 U_0}{\varepsilon_2 d_1 + \varepsilon_1 d_2}$

下极板为 $\rho_{S2} = -D_2 = -\varepsilon_2 E_2 = \dfrac{-\varepsilon_2 \varepsilon_1 U_0}{\varepsilon_2 d_1 + \varepsilon_1 d_2}$

介质分界面上为 $\rho_S = D_2 - D_1 = \dfrac{\left(\varepsilon_2 \varepsilon_1 - \varepsilon_1 \varepsilon_2\right) U_0}{\varepsilon_2 d_1 + \varepsilon_1 d_2}$