Ch. 10 Waveguides

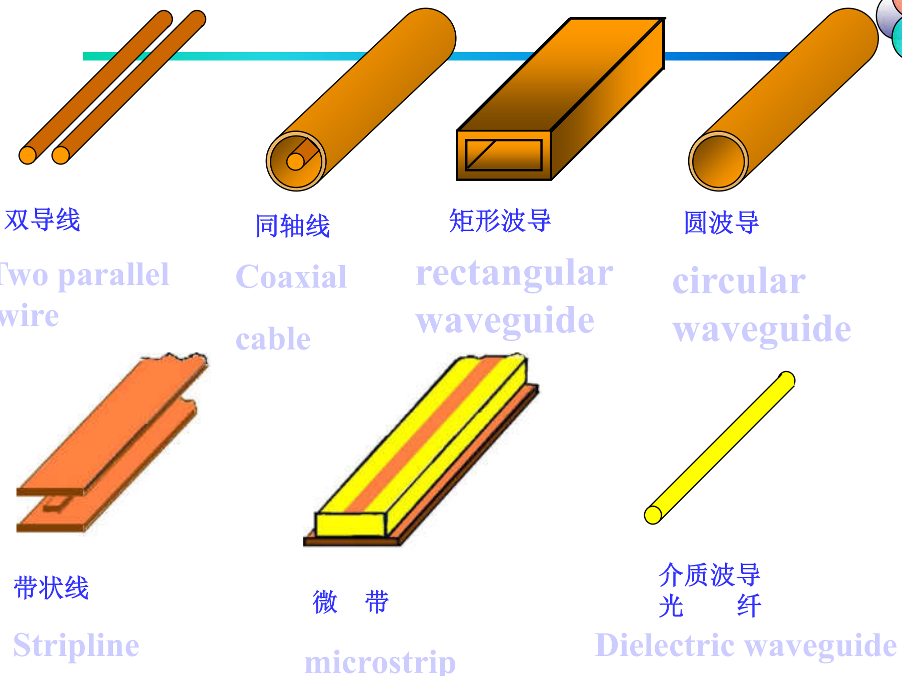
Introduction

Transmission Lines: **Guide TEM waves**, usually at low frequencies

- Parallel plates
- Parallel wires
- Coaxial lines
- Microstrip(微带)

Waveguides: **Guide TE & TM waves**, usually at high frequencies

- **Rectangular Waveguides (emphasis of this chapter)**
- Cylindrical Waveguides



3 Types of Electromagnetic Waves

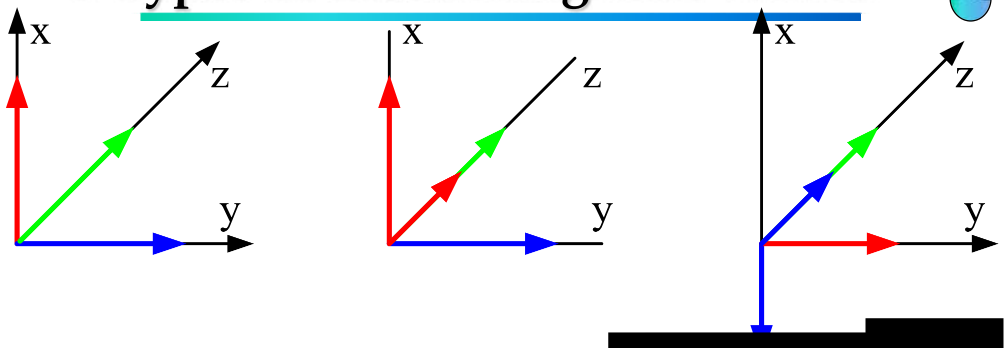
① TEM Wave: Transverse Electromagnetic Wave

② TM Wave: Transverse Magnetic Wave,
also known as E-Wave,
它具有纵向电场分量Ez,而无纵向磁场分量Hz

③ TE Wave: Transverse Electric Wave,
also known as H-Wave
它有纵向磁场分量Hz,而无纵向电场分量Ez

Transverse means there is no field component along the direction of wave propagation.

3 Types of Electromagnetic Waves



TEM TM [redacted]

- **TEM 波的情况下，传播常数β＝k**
- **TM、TE 波的情况下，传播常数β≠k，但与 k 有紧密联系β＝k_z**

Further Explanation (1)

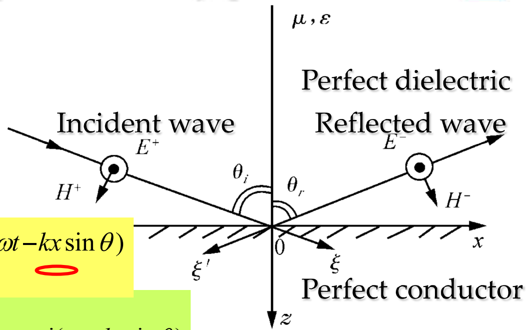
- A homogeneous plane wave （**HPW**） in **infinite media** must be a traveling **TEM** wave.
  - Its constant phase surfaces are a series of parallel planes. These constant phase planes are normal to the wave propagation direction.
  - All components of the electric & magnetic fields lie in those constant phase planes. Namely, they are transverse with respect to the wave propagation direction, resulting in the name of TEM wave.

Further Explanation (2)

- If a TEM wave in region 1 encounters the boundary between regions 1 & 2, the incident wave may be reflected into region 1 and transmitted into region 2.
- The transmitted wave in **region 2 is still a TEM wave,** but different from the incident wave in region 1.
- The superposition of the incident wave and reflected wave **in region 1 forms a TE or TM wave.**

Oblique incidence of a perpendicularly polarized wave

Along the x direction,
TE waves



$$E_y = -\mathrm{j}2E_0^+ \sin(kz\cos\theta)\mathrm{e}^{\mathrm{j}(\omega t - kx\sin\theta)}$$

$$H_z = -\mathrm{j}\frac{2E_0^+}{\eta}\sin\theta\sin(kz\cos\theta)\mathrm{e}^{\mathrm{j}(\omega t - kx\sin\theta)}$$

Along the z direction, standing wave

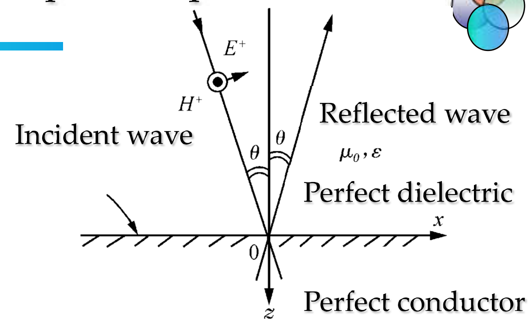
$$H_x = \frac{-2E_0^+}{\eta}\cos\theta\cos(kz\cos\theta)\mathrm{e}^{\mathrm{j}(\omega t - kx\sin\theta)}$$

Along the x direction, traveling wave

Only E lies in transverse (横向) plane, hence the name TE-wave.

Oblique incidence of a parallel polarized wave

Along the x direction,
TM waves



$$E_z = -2H_0^+\eta\sin\theta\cos(kz\cos\theta)\mathrm{e}^{\mathrm{j}(\omega t - kx\sin\theta)}$$

$$E_x = -\mathrm{j}2H_0^+\eta\cos\theta\sin(kz\cos\theta)\mathrm{e}^{\mathrm{j}(\omega t - kx\sin\theta)}$$

Along the z direction, standing wave

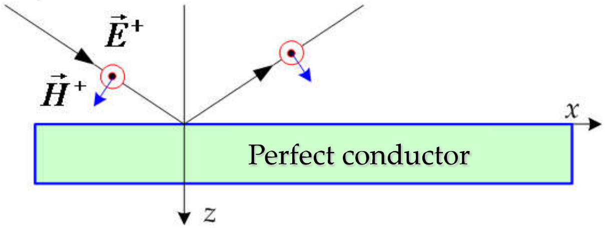
$$H_y = 2H_0^+\cos(kz\cos\theta)\mathrm{e}^{\mathrm{j}(\omega t - kx\sin\theta)}$$

Along the x direction, traveling wave

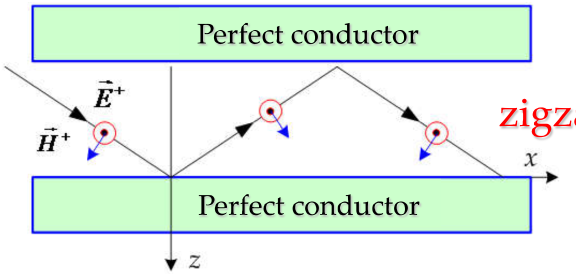
Oblique incidence of a perpendicularly polarized wave

Along the x direction,

TE waves

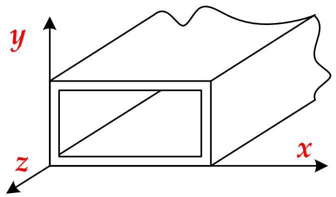


Place another conductor



zigzag

What if?



What is a waveguide?

- **Any structure having the ability to guide EM wave energy or optical energy;**
- The EM field may have a single mode or multiple modes;

Category：

1. Parallel-Plate Waveguide
2. Rectangular Waveguide
3. Cylindrical Waveguide

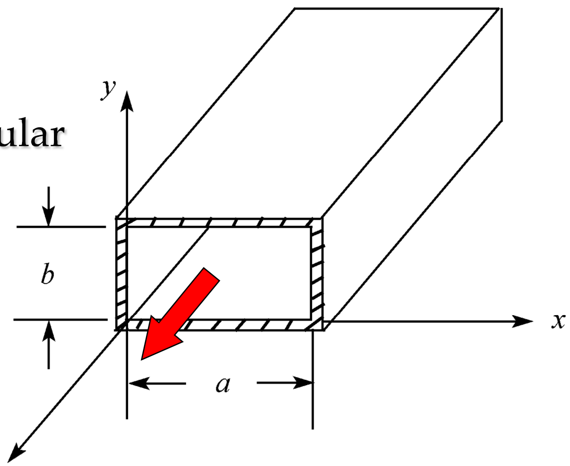
Filling：

1. hollow (air-filled)
2. dielectric

1. Wave Equations

The wave propagates along the positive z direction.

A air-filled rectangular waveguide



Solutions

Recall the solutions to the wave equations in Chapter 7:

$$E_x = E_0 e^{(j\omega t - j\beta z)} = E_0 e^{j\omega t - \gamma z} \qquad H_y = \frac{E_0}{\eta} e^{(j\omega t - j\beta z)} = \frac{E_0}{\eta} e^{j\omega t - \gamma z}$$

Likewise, all components of the EM wave have the factor $e^{j\omega t - \gamma z}$.

$$\therefore \frac{\partial}{\partial z} = -\gamma \qquad \therefore \frac{\partial^2}{\partial z^2} = \gamma^2$$

$$\gamma = \alpha + j\beta \xrightarrow{\text{lossless}} \gamma = j\beta$$

Wave equations in a perfect dielectric

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$

k has the transverse & longitudinal components

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

$$k^2 = k_x^2 + k_y^2 + \beta^2$$

$$\gamma = j\beta$$

$$k^2 = k_x^2 + k_y^2 - \gamma^2$$

E has the transverse & longitudinal components

$$k^2 = k_c^2 - \gamma^2$$

$$\boxed{k_c^2 = k_x^2 + k_y^2}$$

Cutoff

$$\frac{\partial^2 E_{x,y,z}}{\partial x^2} + \frac{\partial^2 E_{x,y,z}}{\partial y^2} + \frac{\partial^2 E_{x,y,z}}{\partial z^2} + k^2 E_{x,y,z} = 0$$

$$\nabla_T^2 E_{x,y,z} + \frac{\partial^2 E_{x,y,z}}{\partial z^2} + k^2 E_{x,y,z} = 0$$

$\nabla^2 \vec{E} + k^2 \vec{E} = 0$ Wave Equations

$$\nabla_T^2 E_{x,y,z} + \frac{\partial^2 E_{x,y,z}}{\partial z^2} + k^2 E_{x,y,z} = 0$$

$$\frac{\partial^2}{\partial z^2} = \gamma^2$$

$$\nabla_T^2 E_{x,y,z} + \gamma^2 E_{x,y,z} + k^2 E_{x,y,z} = 0$$

$$\nabla_T^2 E_{x,y,z} + k_c^2 E_{x,y,z} = 0$$

$$\boxed{k^2 = k_c^2 - \gamma^2}$$

$$\nabla_T^2 E_{x,y,z} + k_c^2 E_{x,y,z} = 0$$

$$\nabla_T^2 E_x + k_c^2 E_x = 0$$

$$\nabla_T^2 E_y + k_c^2 E_y = 0$$

$$\nabla_T^2 E_z + k_c^2 E_z = 0$$

Wave Equations

$$\nabla_T^2 E_{x,y,z} + k_c^2 E_{x,y,z} = 0$$

$$k^2 = k_c^2 - \gamma^2$$

$$\nabla_T^2 E_x + k_c^2 E_x = 0$$

$$\nabla_T^2 E_y + k_c^2 E_y = 0$$

$$\boxed{\nabla_T^2 E_z + k_c^2 E_z = 0}$$

H has the similar wave equations.

EM wave: E_x E_y E_z H_x H_y H_z

TEM: E_x E_y H_x H_y

TE: E_x E_y H_x H_y H_z

TM: E_x E_y E_z H_x H_y

Solutions ---- TM wave (1)

TM wave : E_x E_y E_z H_x H_y

Solve E_z $$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0$$

Using the technique of separation of variables:

$$E_z(x,y,z,t) = X \cdot Y \cdot e^{j\omega t - \gamma z}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \qquad X = A\cos(k_x x) + B\sin(k_x x)$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2 \qquad Y = C\cos(k_y y) + D\sin(k_y y)$$

$$k_c^2 = k_x^2 + k_y^2$$

Solutions ---- TM wave (2)

$E_z(x,y,z,t)=X\cdot Y\cdot e^{j\omega t-\gamma z}$

$X=A\cos(k_x x)+B\sin(k_x x)$

$Y=C\cos(k_y y)+D\sin(k_y y)$

$E_z=[A\cos(k_x x)+B\sin(k_x x)]\cdot[C\cos(k_y y)+D\sin(k_y y)]\, e^{j\omega t-\gamma z}$

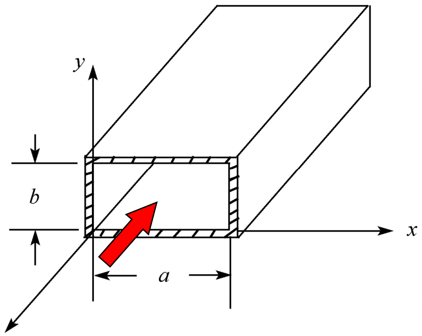
Apply the boundary conditions on the 4 walls: $E_t=0$

(1) $x=0,\ y\in[0,b],\ E_z=0$ ➔ $A=0$

(2) $x=a,\ y\in[0,b],\ E_z=0$ ➔ $k_x=m\pi/a$

(3) $y=b,\ x\in[0,a],\ E_z=0$ ➔ $k_y=n\pi/b$

(4) $y=0,\ x\in[0,a],\ E_z=0$ ➔ $C=0$

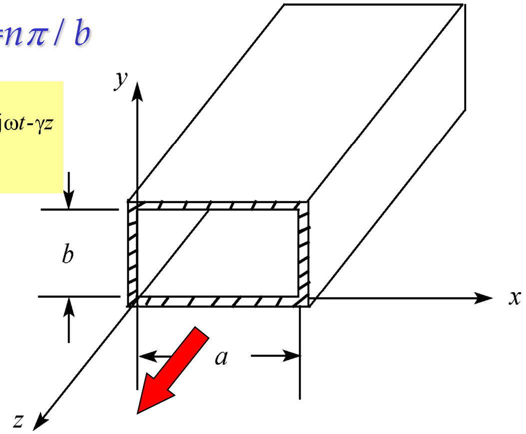


Solutions ---- TM wave (3)

$E_z=[A\cos(k_x x)+B\sin(k_x x)]\cdot[C\cos(k_y y)+D\sin(k_y y)]\, e^{j\omega t-\gamma z}$

$A=0 \quad C=0 \quad k_x=m\pi/a \quad k_y=n\pi/b$

$$E_z=B_0\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j\omega t-\gamma z}$$



Express the other components in terms of the z components

$$\nabla\times\vec{H}=j\omega\varepsilon\vec{E}$$

$$\nabla\times\vec{E}=-j\omega\mu\vec{H}$$

$\partial/\partial z=-\gamma$

$\partial/\partial t=j\omega$

$$\left.\begin{aligned}&\frac{\partial H_z}{\partial y}+\gamma H_y=j\omega\varepsilon E_x\\&-\gamma H_x-\frac{\partial H_z}{\partial x}=j\omega\varepsilon E_y\\&\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=j\omega\varepsilon E_z\end{aligned}\right\}$$

$$\left.\begin{aligned}&\frac{\partial E_z}{\partial y}+\gamma E_y=-j\omega\mu H_x\\&-\gamma E_x-\frac{\partial E_z}{\partial x}=-j\omega\mu H_y\\&\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-j\omega\mu H_z\end{aligned}\right\}$$

Express the other components in terms of the z components

$$\left.\begin{aligned}&\frac{\partial H_z}{\partial y}+\gamma H_y=j\omega\varepsilon E_x\\&-\gamma H_x-\frac{\partial H_z}{\partial x}=j\omega\varepsilon E_y\\&\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=j\omega\varepsilon E_z\end{aligned}\right\}$$

$$\left.\begin{aligned}&\frac{\partial E_z}{\partial y}+\gamma E_y=-j\omega\mu H_x\\&-\gamma E_x-\frac{\partial E_z}{\partial x}=-j\omega\mu H_y\\&\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-j\omega\mu H_z\end{aligned}\right\}$$

$$\left.\begin{aligned}E_x&=-\frac{1}{k_c^2}\left(\gamma\frac{\partial E_z}{\partial x}+j\omega\mu\frac{\partial H_z}{\partial y}\right)\\E_y&=\frac{1}{k_c^2}\left(-\gamma\frac{\partial E_z}{\partial y}+j\omega\mu\frac{\partial H_z}{\partial x}\right)\\H_x&=\frac{1}{k_c^2}\left(j\omega\varepsilon\frac{\partial E_z}{\partial y}-\gamma\frac{\partial H_z}{\partial x}\right)\\H_y&=-\frac{1}{k_c^2}\left(j\omega\varepsilon\frac{\partial E_z}{\partial x}+\gamma\frac{\partial H_z}{\partial y}\right)\end{aligned}\right\}$$

Solutions ---- TM wave (4)

TM wave: E_x E_y E_z H_x H_y

$$E_z = B_0 \sin\left(\frac{m\pi n\pi}{a} x\right) \sin\left(\frac{}{b} y\right) e^{j\omega t-\gamma z}$$

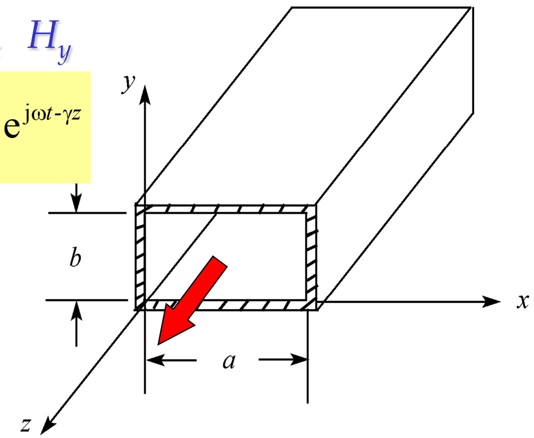
$$H_z = 0$$

$$E_x = -\frac{1}{k_c^2}\left(\gamma\frac{\partial E_z}{\partial x} + j\omega\mu\frac{\partial H_z}{\partial y}\right)$$

$$H_x = \frac{1}{k_c^2}\left(j\omega\varepsilon\frac{\partial E_z}{\partial y} - \gamma\frac{\partial H_z}{\partial x}\right)$$

$$E_y = \frac{1}{k_c^2}\left(-\gamma\frac{\partial E_z}{\partial y} + j\omega\mu\frac{\partial H_z}{\partial x}\right)$$

$$H_y = -\frac{1}{k_c^2}\left(j\omega\varepsilon\frac{\partial E_z}{\partial x} + \gamma\frac{\partial H_z}{\partial y}\right)$$



$$E_z = B_o \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$E_x = -j\frac{\beta}{k_c^2}\frac{m\pi}{a}B_o\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

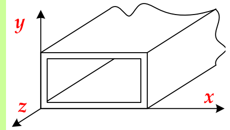
$$E_y = -j\frac{\beta}{k_c^2}\frac{n\pi}{b}B_o\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$H_x = j\frac{\omega\varepsilon}{k_c^2}\frac{n\pi}{b}B_o\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$H_y = -j\frac{\omega\varepsilon}{k_c^2}\frac{m\pi}{a}B_o\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$H_z = 0$$

$$k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$



TM wave

Neither m nor n is zero!

TM 波的 传输特点

✦1，电磁波的相位仅与变量 z 有关，而振幅与 x，y 有关。

在*Z*方向上为行波，在 *X* 及 *Y* 方向上形成驻波

✦2，z 等于常数的平面为波面，但振幅与 x, y 有关

TM波为非均匀的平面波

✦3，当 m 或 n 为零时，上述各个分量均为零，因此 m 及 n 应为非零的整数。

m 及 n 的物理意义：m 为宽壁上的半个驻波的数目， n 为窄壁上半个驻波的数目

✦4，由于 m 及 n 为多值，因此场结构均具有多种模式。 m 及 n 的每一种组合构成一种模式，以TM_{mn}表示。 例如 TM_{11}表示 m = 1, n = 1 的场结构，具有这种场结构的波称为TM_{11}波。

✦5，数值大的 m 及 n 模式称为高次模，数值小的称为低次模。 **矩形波导中TM波的最低模式是TM11波。Dominant mode**

Solutions ---- TE wave

TE wave: E_x E_y H_x H_y H_z

Solve H_z $$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k_c^2 H_z = 0$$

Using the technique of separation of variables:

$$H_z(x,y,z,t) = X\cdot Y\cdot e^{j\omega t-\lambda z}$$

$$\boxed{H_z = [A\cos(k_x x) + B\sin(k_x x)]\cdot[C\cos(k_y y) + D\sin(k_y y)]\, e^{j\omega t-\lambda z}}$$

Apply the boundary conditions on the 4 walls: $E_t = 0$

$$H_z = A_o\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j\omega t-\lambda z}$$

$$E_z = 0$$

$$E_x = -\frac{1}{k_c^2}\left(\gamma\frac{\partial E_z}{\partial x} + j\omega\mu\frac{\partial H_z}{\partial y}\right)$$

$$H_x = \frac{1}{k_c^2}\left(j\omega\varepsilon\frac{\partial E_z}{\partial y} - \gamma\frac{\partial H_z}{\partial x}\right)$$

$$E_y = \frac{1}{k_c^2}\left(-\gamma\frac{\partial E_z}{\partial y} + j\omega\mu\frac{\partial H_z}{\partial x}\right)$$

$$H_y = -\frac{1}{k_c^2}\left(j\omega\varepsilon\frac{\partial E_z}{\partial x} + \gamma\frac{\partial H_z}{\partial y}\right)$$

$$H_z = A_o \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

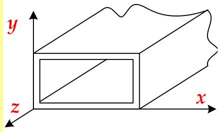
$$H_x = j\frac{\beta}{k_c^2}\frac{m\pi}{a}A_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$H_y = j\frac{\beta}{k_c^2}\frac{n\pi}{b}A_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$E_x = j\frac{\omega\mu}{k_c^2}\frac{n\pi}{b}A_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$E_y = -j\frac{\omega\mu}{k_c^2}\frac{m\pi}{a}A_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$E_z = 0$$



TE wave

$$k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

Mode

$$E_z = B_o \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$E_x = -j\frac{\beta}{k_c^2}\frac{m\pi}{a}B_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

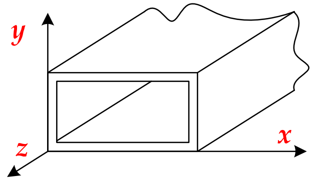
$$E_y = -j\frac{\beta}{k_c^2}\frac{n\pi}{b}B_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$H_x = j\frac{\omega\varepsilon}{k_c^2}\frac{n\pi}{b}B_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$H_y = -j\frac{\omega\varepsilon}{k_c^2}\frac{m\pi}{a}B_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$H_z = 0$$

For given m & n, the mode of the EM wave is called TM_{mn} mode



m & n indicates the numbers of maximum field values along the x & y directions, respectively, e.g. TM_{32}

Mode

$$H_z = A_o \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$H_x = j\frac{\beta}{k_c^2}\frac{m\pi}{a}A_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

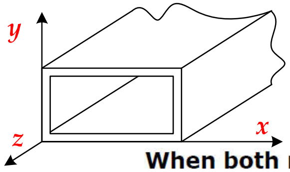
$$H_y = j\frac{\beta}{k_c^2}\frac{n\pi}{b}A_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$E_x = j\frac{\omega\mu}{k_c^2}\frac{n\pi}{b}A_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$E_y = -j\frac{\omega\mu}{k_c^2}\frac{m\pi}{a}A_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$E_z = 0$$

For given m & n, the mode of the EM wave is called TE_{mn} mode



When both m and n are zero, all other field components but H_z will disappear, so TE_{00} cannot exist in a waveguide.

TE_{10} is the lowest-order mode (a>b).

2. Parameters of TE & TM waves

- The Propagating Mode
  - The EM energy is transmitted.

$$k^2 = k_x^2 + k_y^2 + \beta^2$$

$$k^2 = k_x^2 + k_y^2 - \gamma^2$$

- In other words,
  - The EM energy cannot be transmitted in any mode in a waveguide;
  - The EM energy cannot be transmitted in any frequency in a waveguide;

$E(x,y,z,t)=E_0\cdot e^{j\omega t-\gamma z}$

- The Cutoff Frequency: f_c
  - When the propagation constant $\gamma^2 \geq 0$, the wave cannot be transmitted.
  - When the frequency (wavelength) of the EM field is above (below) a certain frequency (wavelength), the wave can be transmitted.

$$k^2 = k_c^2 - \gamma^2$$

$$\gamma^2 = k_c^2 - k^2 \ge 0$$

$$\beta^2 = -\gamma^2 \le 0$$

The wave is said to be evanescent. 逐渐消失

$$\therefore k_c^2 \le k^2 = \omega^2 \mu\varepsilon = (2\pi f)^2 \mu\varepsilon$$

$$\therefore k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \le (2\pi f)^2 \mu\varepsilon$$

The cutoff frequency

$$f_c = \frac{1}{2\sqrt{\varepsilon\mu}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{v}{\dfrac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}} = \frac{v}{\lambda_c}$$

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

$\lambda < \lambda_c$ or $f > f_c$, the wave can propagate.

v is phase velocity in the unbounded medium

$$k^2 = k_c^2 - \gamma^2 = k_c^2 + \beta^2$$

$$j\beta = \gamma$$

$$\beta^2 = k^2 - k_c^2$$

$$\boldsymbol{k}^2 - \beta^2 = \boldsymbol{k}_x^2 + \boldsymbol{k}_y^2 \Rightarrow \beta = \sqrt{\omega^2 \varepsilon\mu - \left(\frac{\boldsymbol{m}\pi}{\boldsymbol{a}}\right)^2 - \left(\frac{\boldsymbol{n}\pi}{\boldsymbol{b}}\right)^2}$$

$$k^2 = \omega^2 \mu\varepsilon = (2\pi f)^2 \mu\varepsilon$$

$$\frac{k^2}{f^2} = (2\pi)^2 \mu\varepsilon$$

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = f_c^2 (2\pi)^2 \varepsilon\mu$$

$$f_c = \frac{1}{2\sqrt{\varepsilon\mu}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = k^2 \frac{f_c^2}{f^2}$$

$$\beta^2 = k^2\left(1 - \frac{f_c^2}{f^2}\right) = k^2\left(1 - \frac{\lambda^2}{\lambda_c^2}\right)$$

The phase constant β

$$\beta^2 = k^2\left(1 - \frac{f_c^2}{f^2}\right) = k^2\left(1 - \frac{\lambda^2}{\lambda_c^2}\right)$$

$$\beta = k\sqrt{1 - (\lambda/\lambda_c)^2}$$

The Waveguide wavelength: $\lambda_g = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{k\sqrt{1-(\lambda/\lambda_c)^2}} = \dfrac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}}$

The phase velocity

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{k\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{v}{\sqrt{1-(\lambda/\lambda_c)^2}}$$

The group velocity v_g

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{d\beta/d\omega}$$

$$2\pi = k\lambda = k_c\lambda_c$$

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{\left(\frac{\omega}{v}\right)^2 - \left(\frac{2\pi}{\lambda_c}\right)^2}$$

$$v_g = v\sqrt{1-(\lambda/\lambda_c)^2}$$

Thus, a waveguide acts as a dispersive medium.

Discussion:

$$v_p = \frac{\omega}{\beta} = \frac{v}{\sqrt{1-(\lambda/\lambda_c)^2}} \qquad v_g = v\sqrt{1-(\frac{\lambda}{\lambda_c})^2} \qquad \beta = k\sqrt{1-(\frac{\lambda}{\lambda_c})^2}$$

When $\lambda<\lambda_c$,

β is real, v_p & v_g are also real, TE & TM waves can propagate.

When $\lambda=\lambda_c$, $e^{j(\omega t-\beta z)}$

$\beta=0$, $v_g=0$, $v_p=\infty$, the wave is evanescent. 电磁波刚好截止

When $\lambda>\lambda_c$, $e^{j(\omega t-\beta z)}=e^{j\omega t}\cdot e^{-|\beta|z}$

β, v_p & v_g are imaginary, the wave is evanescent.

Wave impedance for a TM wave: $Z_{W(TM)} = \frac{E_T}{H_T} = \frac{\gamma}{j\omega\varepsilon}$

$$E_x = -j\frac{\beta}{k_c^2}\frac{m\pi}{a}B_o\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$H_y = -j\frac{\omega\varepsilon}{k_c^2}\frac{m\pi}{a}B_o\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$E_y = -j\frac{\beta}{k_c^2}\frac{n\pi}{b}B_o\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$H_x = j\frac{\omega\varepsilon}{k_c^2}\frac{n\pi}{b}B_o\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)}$$

$$\left.\begin{aligned} j\omega\varepsilon E_x &= \gamma H_y \\ j\omega\varepsilon E_y &= -\gamma H_x \end{aligned}\right\}$$

$$\vec{E}_T = \vec{e}_x E_x + \vec{e}_y E_y = \vec{e}_x\left(\frac{\gamma}{j\omega\varepsilon}H_y\right) + \vec{e}_y\left(\frac{-\gamma}{j\omega\varepsilon}H_x\right) = \frac{\gamma}{j\omega\varepsilon}\left[\vec{e}_x H_y - \vec{e}_y H_x\right]$$

$$Z_{W(TM)} = \frac{\gamma}{j\omega\varepsilon} = \frac{j\beta}{j\omega\varepsilon} = \frac{\beta}{\omega\varepsilon} \qquad Z_{W(TM)} = \eta\sqrt{1-(\lambda/\lambda_c)^2}$$

Likewise, $Z_{W(TE)} = \frac{j\omega\mu}{\gamma} = \frac{\omega\mu}{\beta}$ $\qquad Z_{W(TE)} = \frac{\eta}{\sqrt{1-(\lambda/\lambda_c)^2}}$

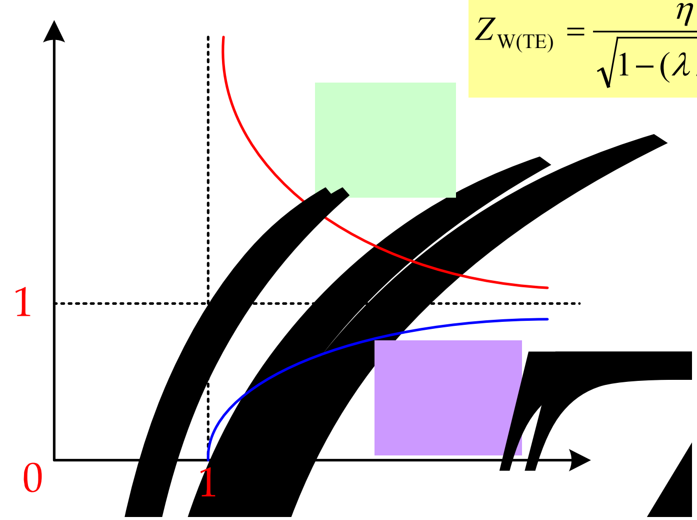
Recall $Z_{TEM} = \eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{k}{\omega\varepsilon} = \frac{\omega\mu}{k}$

$$\beta = k\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}$$

Wave Impedance

$$Z_{W(TM)} = \eta\sqrt{1-(\lambda/\lambda_c)^2} = \eta\sqrt{1-(f_c/f)^2}$$

$$Z_{W(TE)} = \frac{\eta}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{\eta}{\sqrt{1-(f_c/f)^2}}$$



Discussion:

$$Z_{W(TM)} = \eta\sqrt{1-(\lambda/\lambda_c)^2} \qquad Z_{W(TE)} = \frac{\eta}{\sqrt{1-(\lambda/\lambda_c)^2}}$$

When $\lambda<\lambda_c$, TE & TM waves can propagate.

$Z_{W(TM)}$ & $Z_{W(TE)}$ are both real.

When $\lambda=\lambda_c$, the wave is evanescent.

$Z_{W(TM)}=0$, $Z_{W(TE)}=\infty$

When $\lambda>\lambda_c$, the wave is evanescent.

$Z_{W(TM)}$ & $Z_{W(TE)}$ are both imaginary; $Z_{W(TM)}$ is capacitive, $Z_{W(TE)}$ is inductive.

3. The lowest-order mode & single-mode propagation

The lowest-order mode is the one with the maximum cutoff wavelength.

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}}$$

When $\lambda<\lambda_c$, the wave can propagate.

The cutoff wavelengths of TE_{mn} modes

When $a = 2b$,

$\lambda_{c(10)} = 2a$ for TE_{10}

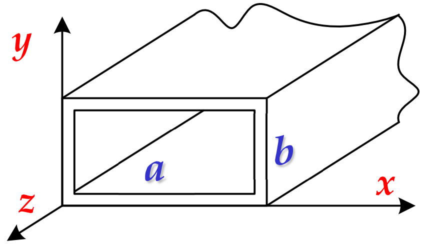
$\lambda_{c(20)} = a$ for TE_{20}

$\lambda_{c(01)} = 2b = a$ for TE_{01}

$\lambda_{c(11)} = 2a/\sqrt{5}$ for TE_{11}

$\lambda_{c(30)} = 2a/3$ for TE_{30}

$\lambda_{c(02)} = b = a/2$ for TE_{02}

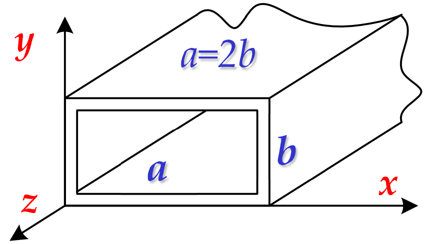


$$\lambda_c = \frac{2\pi}{k_c} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}}$$

The maximum cutoff wavelength of TE waves

$\lambda_{c(10)} = 2a$

TE_{10} mode is the lowest-order mode in all TE modes.



Likewise, for TM waves

$\lambda_{c(11)} = 2a/\sqrt{5}$

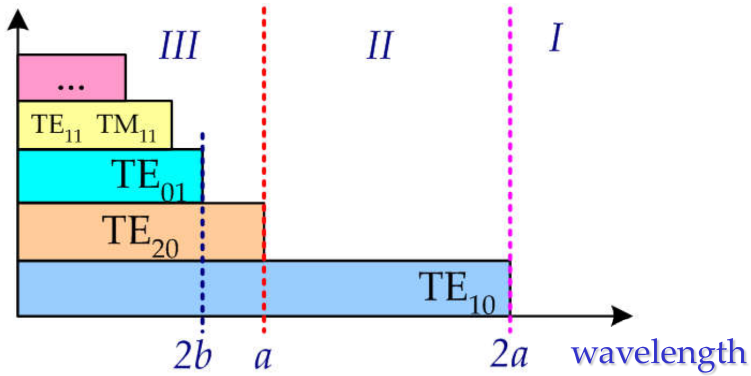
TM_{11} mode is the lowest-order mode in all TM modes.

Therefore,

When $a=2b$, the lowest-order mode is TE_{10}.

$\lambda_{c(10)} = 2a$

The distribution of cutoff wavelength



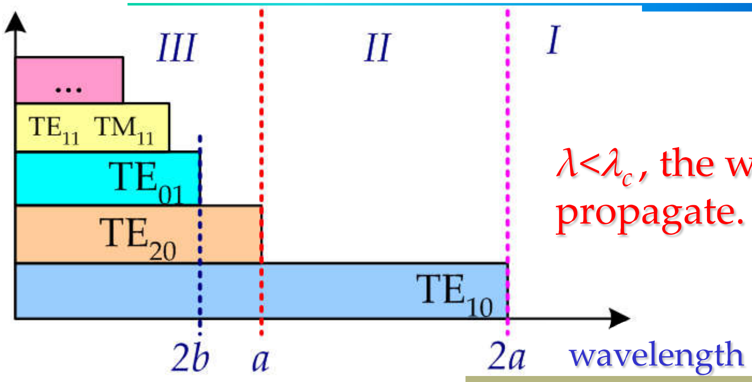
When $\lambda<\lambda_c$, the wave can propagate.

Region I： cutoff——evanescent

Region II： single-mode ——only one mode can exist

Region III： multi-mode——more than one mode can exist

The conditions of single-mode propagation

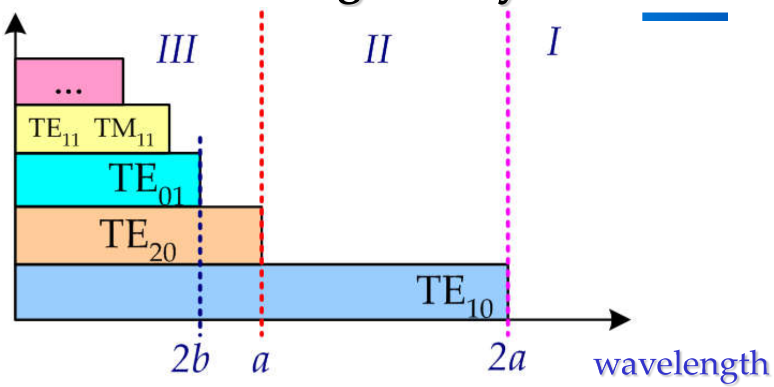


$\lambda<\lambda_c$, the wave can propagate.

$$\begin{cases} 2a>\lambda>a \\ \lambda/2<a<\lambda \ \ (a\geq 2b) \end{cases}$$

TE_{10} is the only propagating mode, called the dominant mode.

Mode degeneracy

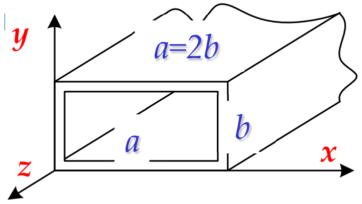


Mode degeneracy: two modes have the same cutoff wavelength.

For example, TE_{11} and TM_{11} are the degenerate mode.

4. The dominant mode: TE_{10}

$$\left.\begin{aligned}
H_z &= A_o \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)} \\
H_x &= j\frac{\beta}{k_c^2}\frac{m\pi}{a}A_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)} \\
H_y &= j\frac{\beta}{k_c^2}\frac{n\pi}{b}A_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)} \\
E_x &= j\frac{\omega\mu}{k_c^2}\frac{n\pi}{b}A_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)} \\
E_y &= -j\frac{\omega\mu}{k_c^2}\frac{m\pi}{a}A_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{j(\omega t-\beta z)} \\
E_z &= 0
\end{aligned}\right\}\quad \begin{aligned} m&=1 \\ n&=0 \end{aligned}$$

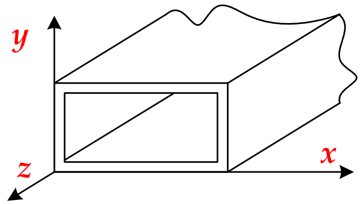


TE_{10} Mode

$$E_y = -j\frac{\omega\mu}{k_c^2}\frac{\pi\pi}{a}A_o \sin\left(\frac{\ }{a}x\right)e^{j(\omega t-\beta z)}$$

$$H_x = j\frac{\beta}{k_c^2}\frac{\pi\pi}{a}A_o \sin\left(\frac{\ }{a}x\right)e^{j(\omega t-\beta z)}$$

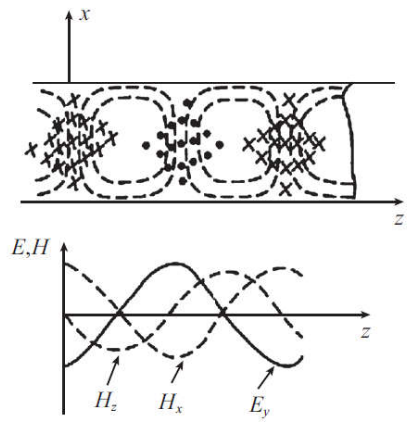
$$H_z = A_o \cos\left(\frac{\pi}{a}x\right)e^{j(\omega t-\beta z)}$$



$$E_y = -j\frac{\omega\mu}{k_c^2}\frac{\pi\pi}{a}A_o \sin\left(\frac{\ }{a}x\right)e^{j(\omega t-\beta z)}$$

$$H_x = j\frac{\beta}{k_c^2}\frac{\pi\pi}{a}A_o \sin\left(\frac{\ }{a}x\right)e^{j(\omega t-\beta z)}$$

$$H_z = A_o \cos\left(\frac{\pi}{a}x\right)e^{j(\omega t-\beta z)}$$



(b) Distribution of the electric and magnetic field for TE_{10} wave in the xOz plane

Field plots

Parameters of TE_{10} Mode

The cutoff wavelength:

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} = 2a$$

The cutoff frequency:

$$f_c = \frac{v}{\lambda_c} = \frac{1}{2a}\frac{c}{\sqrt{\varepsilon_r\mu_r}}$$

The phase constant:

$$\beta = k\sqrt{1-(\lambda/\lambda_c)^2} = k\sqrt{1-(\lambda/2a)^2}$$

The waveguide wavelength:

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1-(\lambda/2a)^2}}$$

The phase velocity:

$$v_p = \frac{\omega}{\beta} == \frac{v}{\sqrt{1-(\lambda/2a)^2}}$$

The group velocity:

$$v_g = v\sqrt{1-(\lambda/\lambda_c)^2} = v\sqrt{1-(\lambda/2a)^2}$$

The wave impedance:

$$Z_{W(TE)} = \frac{\eta}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{\eta}{\sqrt{1-(\lambda/2a)^2}}$$

The transmission of EM energy

$$P = \ldots = \frac{1}{2 \cdot Z}\int_S \left|\vec{E}\right|^2 dS$$

$$TE10 \quad \left|E_y\right| = \ldots = \omega\mu \cdot \left(\frac{a}{\pi}\right) \cdot H_{zm} \cdot \left|\sin\left(\frac{\pi}{a} \cdot x\right)\right|^2$$

$$P_{TE10} = \frac{1}{2 \cdot Z_{TE}}\int_S \left|\vec{E}\right|^2 dS = \frac{ab}{4 \cdot Z_{TE}}\left(\omega\mu \cdot \left(\frac{a}{\pi}\right) \cdot H_{zm}\right)^2$$

Homework

- Exercises 10.1, 10.4, 10.6, 10.7

例 若内充空气的矩形波导尺寸为 $\lambda < a < 2\lambda$ ，工作频率为3GHz。如果要求工作频率至少高于主模TE_{10}波的截止频率的20%，且至少低于TE_{01}波的截止频率的20%。试求：①波导尺寸a及b；②根据所设计的波导，计算工作波长，相速，波导波长及波阻抗。

解 ① TE_{10}波的截止波长 $\lambda_c = 2a$

对应的截止频率 $f_c = c/\lambda_c = c/2a$ 。

TE_{01}波的截止波长 $\lambda_c = 2b$ ，

对应的截止频率 $f_c = c/2b$ ，

$$3\times10^9 \ge c/2a\times1.2 \qquad 3\times10^9 \le c/2b\times0.8$$

$$\Rightarrow a \ge 0.06\text{m} \qquad b \le 0.04\text{m}$$

取 $a = 0.06\text{m}$ $b = 0.04\text{m}$

② 工作波长，相速，波导波长及波阻抗分别

$$\lambda = c/f = 0.1\text{m}$$

$$v_p = \frac{c}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}} = 5.42\times10^3$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}} = 0.182$$

$$Z_{TE_{10}} = \frac{Z}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}} = 682(\Omega)$$