Advanced Transform Methods

Fourier Series and Fourier Transforms

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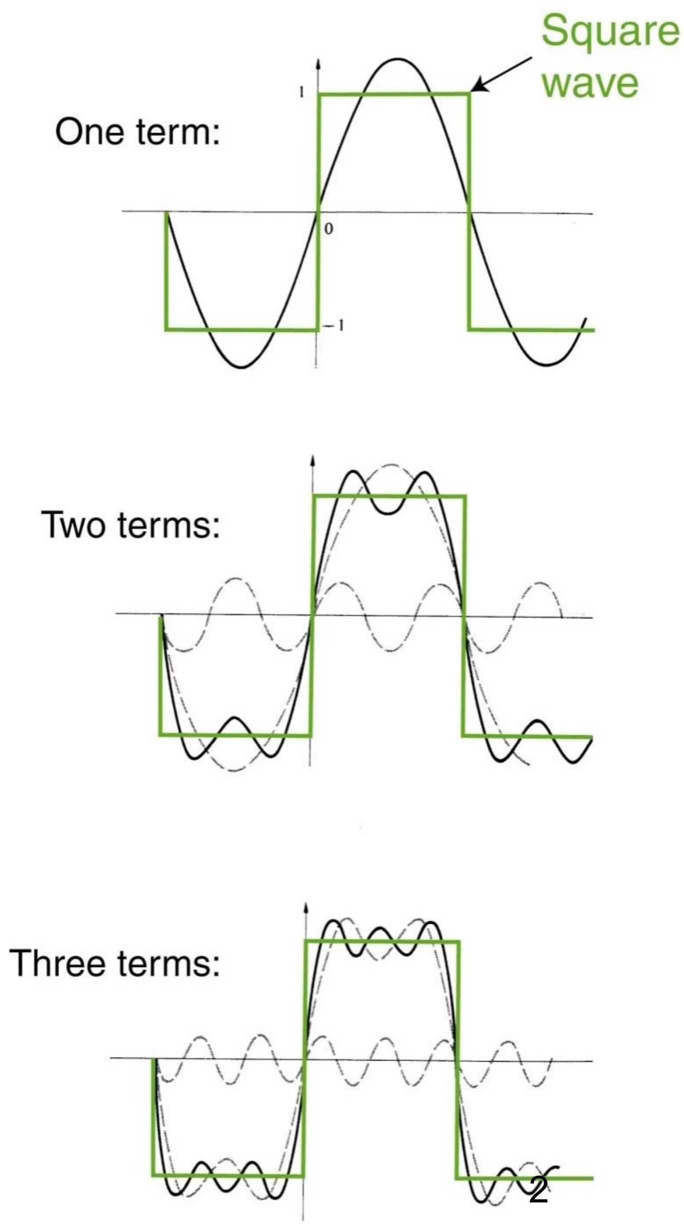
The Fourier Series & Transform

What the Fourier Transform is all about

Fourier Cosine Series for even functions

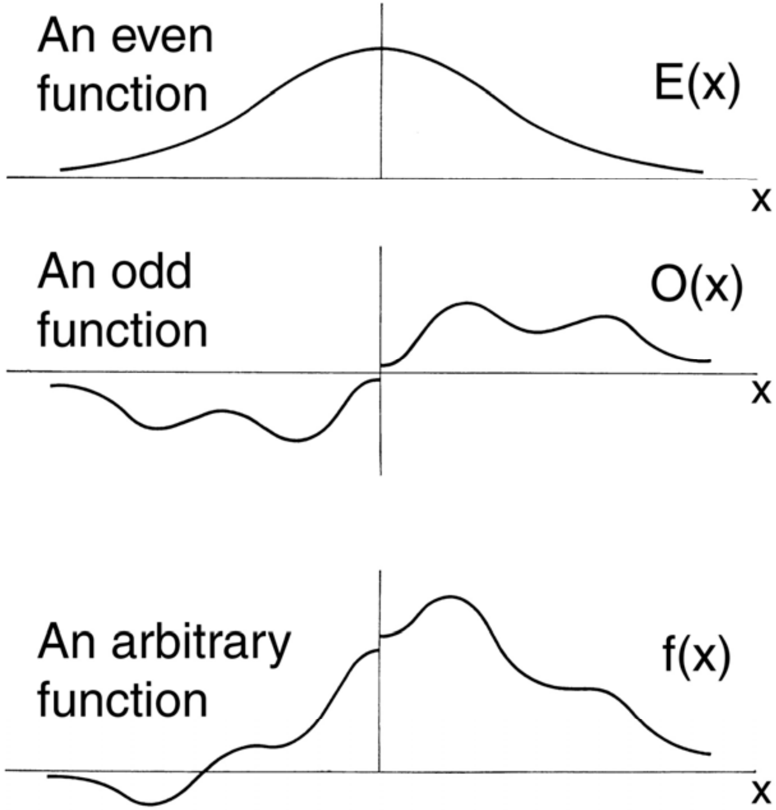
Fourier Sine Series for odd functions

The continuous limit: the Fourier transform (and its inverse)



Here, we write a square wave as a sum of sine waves.

Any function can be written as the sum of an even and an odd function



$$E(x) \equiv [f(x) + f(-x)]/2$$

$$O(x) \equiv [f(x) - f(-x)]/2$$

$$\Downarrow$$

$$f(x) = E(x) + O(x)$$

Fourier Cosine Series and Sine Series

cos(*mt*) is an even function (for all *m*), so we can write an even function *f(t)* as

$$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt)$$

$\{F_m;\ m=0, 1, \ldots\}$ is a set of coefficients that define the series.

sin(*mt*) is an odd function (for all *m*), So we can write any odd function as

$$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m \sin(mt)$$

$\{F'_m;\ m=0, 1, \ldots\}$ is a set of coefficients that define the series.
Only worry about the function f(t) over the interval (–π,π).

Finding the coefficients in a Fourier Cosine Series

Fourier Cosine Series: $f(t) = \dfrac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt)$

Multiply by cos(*m’t*), where *m’* is another integer, and integrate:

$$\int_{-\pi}^{\pi} f(t)\cos(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty}\int_{-\pi}^{\pi} F_m \cos(mt)\cos(m't)\,dt$$

But: $$\int_{-\pi}^{\pi} \cos(mt)\,\cos(m't)dt = \begin{cases} \pi & if\ \ m = m' \\ 0 & if\ \ m \neq m' \end{cases} \equiv \pi\,\delta_{m,m'}$$

Since $$\cos(mt)\,\cos(m't) = \left[\cos(mt - m't) + \cos(mt + m't)\right]$$

So: $$\int_{-\pi}^{\pi} f(t)\cos(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m\,\pi\,\delta_{m,m'}$$ ← only m’ = m term contributes

Drop ‘ from the *m*: $$\boxed{F_m = \int_{-\pi}^{\pi} f(t)\cos(mt)\,dt}$$ ← yields coefficients for any *f(t)*!

Finding coefficients in a Fourier Sine Series

To find F_m, multiply each side by sin($m't$), where m' is another integer, and integrate:

But: $$\int_{-\pi}^{\pi} f(t)\sin(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty}\int_{-\pi}^{\pi} F'_m \sin(mt)\sin(m't)\,dt$$

So: $$\int_{-\pi}^{\pi} \sin(mt)\sin(m't)dt = \begin{cases} \pi & if\ \ m = m' \\ 0 & if\ \ m \neq m' \end{cases} \equiv \pi\,\delta_{m,m'}$$

Since $$\sin(mt)\sin(m't) = \left[\cos(mt - m't) - \cos(mt + m't)\right]$$

$$\int_{-\pi}^{\pi} f(t)\sin(m't)\,dt = \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m\,\pi\,\delta_{m,m'}$$ ← only the m' = m term contributes

Dropping ' from the m:

$$\boxed{F'_m = \int_{-\pi}^{\pi} f(t)\sin(mt)\,dt}$$

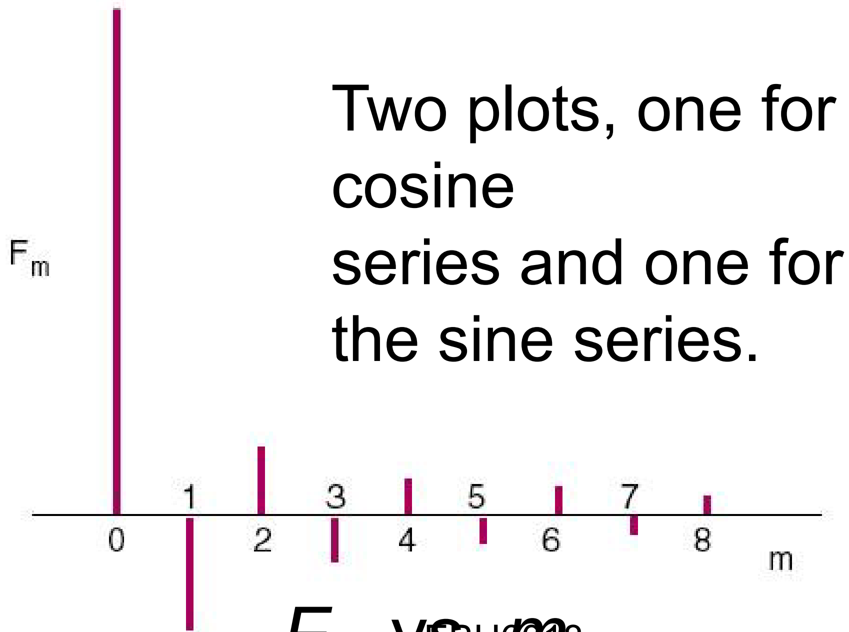
← yields the coefficients for any f(t)!

Fourier Series

$$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt) + \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m \sin(mt)$$

even component odd component

where $F_m = \int f(t)\cos(mt)\,dt$ and $F'_m = \int f(t)\sin(mt)\,dt$



Two plots, one for cosine
series and one for
the sine series.

F_m vs. m

Fourier Transform

Define a function that has both cosine and sine series coefficients, with the sine series coefficients distinguished by making them imaginary:

$$F(m) \equiv F_m - j\,F'_m = \int f(t)\cos(mt)\,dt - \;i \int f(t)\sin(mt)\,dt$$

Allow *f(t)* to range from $-\infty$ to ∞, and redefine *m* to be the frequency, which we now call ω:

$$\boxed{F(\omega) \;=\; \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt}$$

The Fourier Transform

$F(\omega)$ is the Fourier Transform of $f(t)$.

Contains equivalent information to that in $f(t)$.

$f(t)$ lives in the time domain, and $F(\omega)$ in frequency domain.

The Inverse Fourier Transform

Recall our formula for the Fourier Series of f(t):

$$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt) + \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m \sin(mt)$$

Transform sums to integrals from –∞ to ∞, and replace with $F(\omega)$

$$\boxed{f(t) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)\,d\omega}$$

Inverse Fourier Transform

We can transform to frequency domain and back.

There are different definitions of these transforms. The 2π can occur in several places.

Advanced Transform Methods

Fourier Notation

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Types

For the Fourier transform there are three notations:

- continuous-time continuous-frequency
- discrete-time continuous-frequency
- discrete-time discrete-frequency.

Continuous-time discrete-frequency is a special case that arises as the Fourier Series.

Definitions

- **Continuous-time continuous-frequency**

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-j\omega t} dt \qquad s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{j\omega t} d\omega$$

- **Discrete-time continuous-frequency**

$$\tilde{S}(\theta) = \sum_{m=-\infty}^{\infty} s[m] e^{-j\theta m} \qquad s[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{S}(\theta) e^{j\theta m} d\theta$$

- **Discrete-time discrete-frequency**

$$\tilde{S}[n] = \sum_{m=0}^{M-1} \tilde{s}[m] e^{-2\pi nm/N} \quad \tilde{s}[m] = \frac{1}{M}\sum_{n=0}^{N-1} \tilde{S}[n] e^{2\pi nm/N}$$

The notations used are of minor importance in most situations, but the placement of the 2π term is significant, because it changes the form of other resultant equations by a constant factor.

Symbol Conventions

- A continuous function is typically given as *s*(*t*).
- Functions of frequencies usually use the same symbol (in this case *s*), but capitalised(S).
- We use the tilde symbol, ~, to indicate that a function is periodic $\tilde{S}$.
- For discrete-time, we use the convention $s[m\Delta t] = s(t)$
- A discrete time function is typically given as s[m], where it is assumed that the sampling interval $\Delta t = 1$. The Fourier Transform of a continuous function is given as $S(\omega)$. The discrete-time Fourier transform is then given as $\tilde{S}(\theta)$ where $\theta = \omega\Delta t$. When the frequency values are also discrete, as is the case when they are computed on digital computers, then the discrete-time discrete-frequency Fourier transform is given as $\tilde{S}[n]$.
- Thus we have 3 notations for the Fourier transform and its inverse, depending on the discrete or continuous nature of the functions.

Alternatives

The notation used is not symmetric, since the 2π term is confined to the inverse transform. Other books might use other notations, where the 2π is put in the exponent,

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-2\pi j \omega t} dt \qquad s(t) = \int_{-\infty}^{\infty} S(\omega) e^{2\pi j \omega t} d\omega$$

or shared between the terms,

$$S(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s(t) e^{-j\omega t} dt \qquad s(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S(\omega) e^{j\omega t} d\omega$$

Alternatives

- Finally, the minus exponent may appear in the Fourier transform or in the inverse transform, as follows:

$$S(\omega)=\int_{-\infty}^{\infty} s(t)e^{j\omega t}dt \qquad s(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{-j\omega t}d\omega$$

Notes

- Each alternative gives similar modifications for the discrete cases.
- It doesn’t really matter which notation is used, but it must be consistent.
- The properties are the same.
- With the possible exception of a constant term in certain formulas (Parseval’s formula, energy formula, the Uncertainty Principle, the Convolution Theorem, and the Fourier transform of the derivative of a function) all may have slightly different forms depending on the notation used.

Some functions do not have Fourier transforms

The condition for the existence of a given $F(\omega)$ is:

$$\int_{-\infty}^{\infty} \left| f(t) \right| \, dt \; < \; \infty$$

Functions that do not asymptote to zero in both the $+\infty$ and $-\infty$ directions generally do not have Fourier transforms.

When we talk about a basis for functions, we generally mean for integrable functions that have this property.

Fourier Transform

$$S(\omega) = \int_{-\infty}^{\infty} s(t)\, e^{-j\omega t}\, dt$$

Fourier Transform

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{j\omega t}\, d\omega$$

Inverse Fourier Transform

- For the Fourier transform there are 3 notations,
  - Continuous-time, continuous-frequency (FT)
  - Discrete-time, continuous-frequency (DTFT)
  - Discrete-time, discrete-frequency (DFT).

- Continuous-time discrete-frequency is the FS.
- Notations used are of minor importance in most situations, but the placement of the 2π term is significant, because it changes the form of other resultant equations by a constant factor.

Fourier Transform Examples

Fourier Transform

$$S(\omega) = \int_{-\infty}^{\infty} s(t)\, e^{-j\omega t}\, dt$$

Fourier Transform

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{j\omega t}\, d\omega$$

Inverse Fourier Transform

$$S[n] = \sum_{m=0}^{M-1} s[m]\, W_N^{-nm}$$
$$= \sum_{m=0}^{M-1} s[m]\, e^{-2\pi nm/N} \qquad n = 0,1,2 \ldots N-1$$

Discrete Fourier Transform

$$s[m] = \frac{1}{M} \sum_{n=0}^{N-1} S[n]\, W_N^{nm}$$
$$= \frac{1}{M} \sum_{n=0}^{N-1} S[n]\, e^{2\pi nm/N} \qquad m = 0,1,2 \ldots M-1$$

Inverse Discrete Fourier Transform

The delta function

Kronecker delta is a function of 2 integers $\delta_{m,n} \equiv \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}$

Dirac delta is a function of a real variable $\delta(t) \equiv \begin{cases} \infty \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}$

Dirac delta function properties

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt = f(a)$$

Useful identities:

$$\int_{-\infty}^{\infty} \mathrm{e}^{\pm j\omega t}\, dt = 2\pi\, \delta(\omega)$$

$$\int_{-\infty}^{\infty} \mathrm{e}^{\pm j(\omega-\omega')t}\, dt = 2\pi\, \delta(\omega-\omega')$$

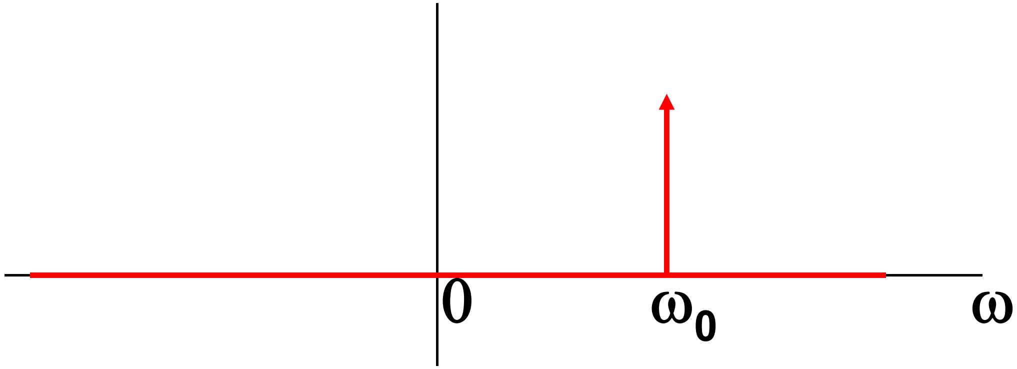
The Fourier transform of $e^{j\omega_0 t}$

$$s(t) = e^{j\omega_0 t}$$

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \mathrm{e}^{-j[\omega-\omega_0]t}\, dt = 2\pi\delta(\omega-\omega_0)$$



The function $\mathrm{e}^{j\omega_0 t}$ is the essential component of Fourier analysis. It is a pure frequency.

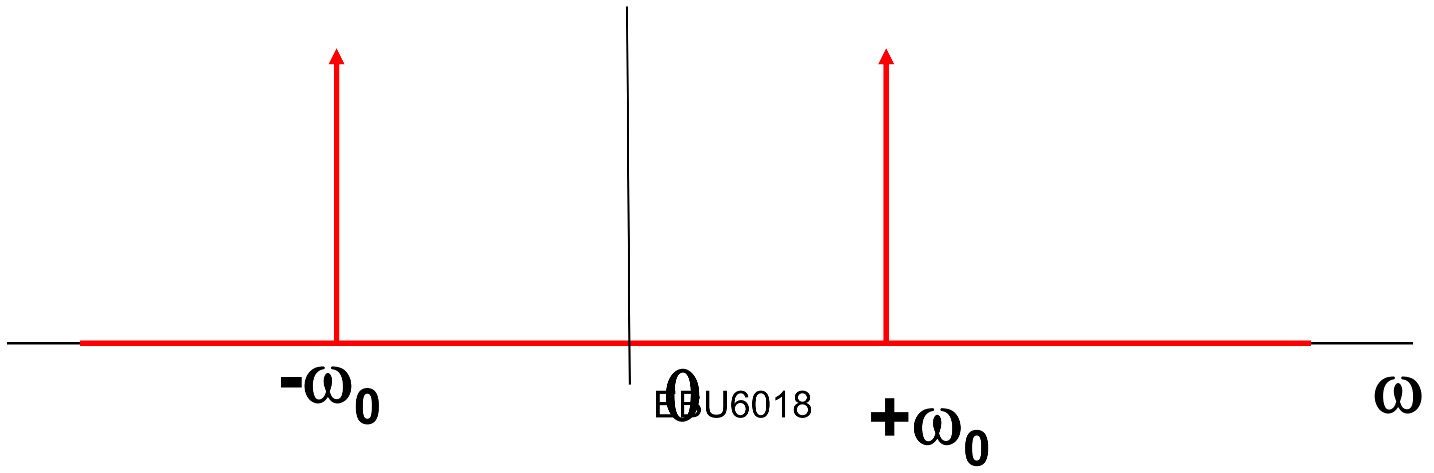
The Fourier transform of cos(ω_0t)

$$S(\omega) = \int_{-\infty}^{\infty} \cos(\omega_0 t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[e^{j\omega_0 t} + e^{-j\omega_0 t} \right] e^{-j\omega t}\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-j[\omega-\omega_0]t}\, dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j[\omega+\omega_0]t}\, dt$$

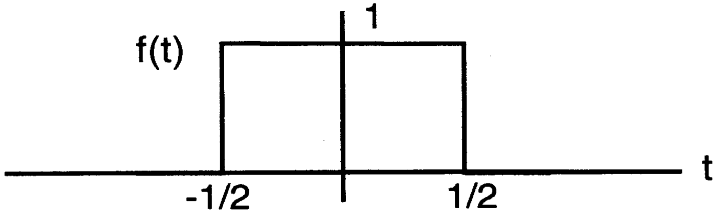
$$= \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

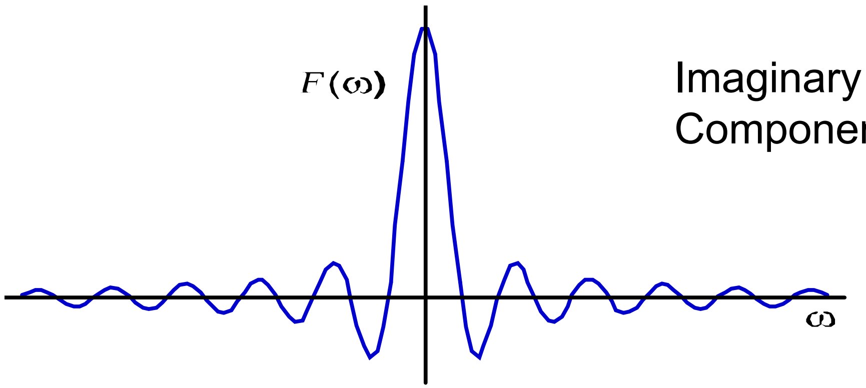


Fourier Transform of a rectangle function

$$S(\omega) = \int_{-1/2}^{1/2} e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-1/2}^{1/2} = \frac{e^{-j\omega/2} - e^{j\omega/2}}{-j\omega}$$

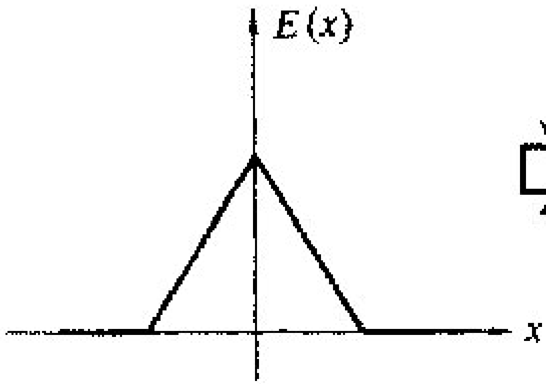
$$= \frac{1}{\omega/2} \frac{e^{j\omega/2} - e^{-j\omega/2}}{2j} = \frac{\sin(\omega/2)}{\omega/2} = \mathrm{sinc}(\omega/2)$$

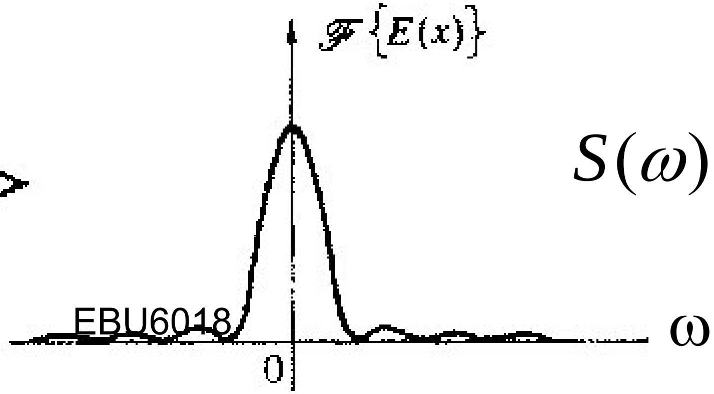




Imaginary Component =0

Fourier Transform of triangle function





$$S(\omega) = \mathrm{sinc}^2(\omega/2)$$

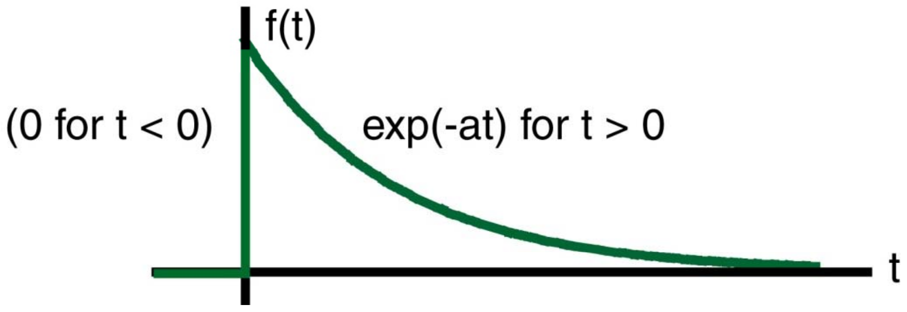
Fourier Transform of decaying exponential e^{-at}, $t>0$

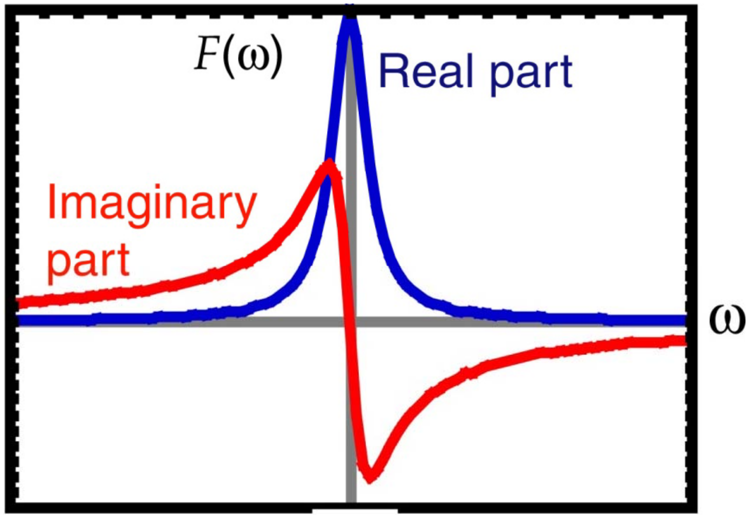
$$S(\omega) = \int_0^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-[a+j\omega]t}\, dt = \frac{-1}{a+j\omega} e^{-[a+j\omega]t}\Big|_0^{+}$$

$$= \frac{-1}{a+j\omega}[0-1] = \frac{(a-j\omega)}{(a+j\omega)(a-j\omega)}$$

$$= \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}$$





Computing the Inverse Transform

Find the waveform, corresponding to the rectangular transform

$$S(\omega) = \begin{cases} \dfrac{\pi A}{\beta} & |\omega| \le \beta \\ 0 & |\omega| > \beta \end{cases}$$

SOLUTION: Since $S(\omega)$=0 everywhere except from ω=-β to ω=+β, the inverse transform takes the form

$$s(t) = \frac{1}{2\pi} \int_{-\beta}^{\beta} \frac{\pi A}{\beta} \mathrm{e}^{j\omega t}\, d\omega$$

$$= \frac{A\mathrm{e}^{j\omega t}}{2\beta jt}\Bigg]_{-\beta}^{\beta} = \frac{A}{\beta t} \frac{\mathrm{e}^{j\beta t} - \mathrm{e}^{-j\beta t}}{2j}$$

$$= \frac{\sin(\beta t)}{\beta t} = \mathrm{sinc}(\beta t)$$