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EBU6018
Advanced Transform Methods

Week 3.2 – Wavelets

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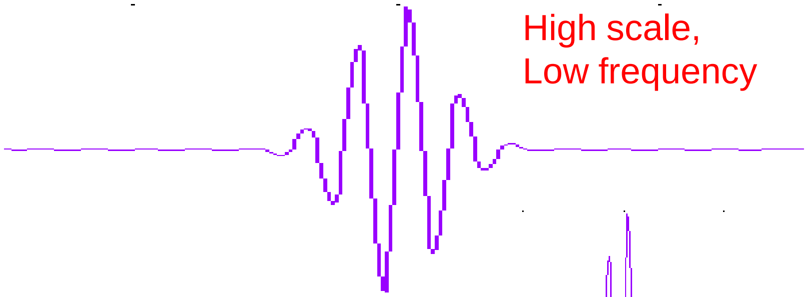
Lecture Outline

1. Wavelet Functions/Transform

2. Multi-Resolution Analysis (MRA)

The Wavelet Transform



High scale,
Low frequency

What is a "Wavelet"?

- a "small wave"

We can make a family of wavelets by:

- scaling and shifting a base or *mother* wavelet
- to create *daughter* wavelets (sometimes called *baby* wavelets)

Low (small) scale,
High frequency

Any short duration, oscillatory waveform can be used as a wavelet, but some would be more useful than others.

Same wavelet,
just scaled and time-shifted

The Wavelet Transform - Applications

- “Relatively new” (compared with Fourier) method of evaluating and processing signals
- Used with nonstationary data
- Two main applications are in feature extraction and trend analysis
- Useful in many types of applications
  1. *Pattern recognition*
     - Biotech: distinguish normal from pathological membranes
     - Biometrics: facial/corneal/fingerprint recognition
  2. *Feature extraction*
     - Metallurgy: characterization of rough surfaces
  3. *Trend detection:*
     - Finance: exploring variation of stock prices
  4. *Perfect reconstruction*
     - Communications: wireless channel signals
  5. *Video compression* – JPEG 2000

The Wavelet

- Consider scaling and translating the function Ψ(t)

$$\psi(t) \rightarrow \psi\left(\frac{t-b}{a}\right)$$

  a determines the centre frequency.

  b determines the translation.

- Time frequency centre of $\psi\left((t-b)/a\right)$

  are b (time centre)

  and $\langle\omega\rangle/a$ (frequency centre) $\langle\omega\rangle$ is mean freq of ψ

- Daughter wavelets:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

Mother Wavelet

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Continuous Wavelet Transform (CWT)

$$CWT(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} s(t)\psi^*\left(\frac{t-b}{a}\right)dt$$

Scale

Translation

$$= \int_{-\infty}^{\infty} s(t)\psi_{a,b}{}^*(t)dt = \langle s, \psi_{a,b} \rangle$$

- The continuous wavelet transform, $CWT(a,b)$ is a function of two real variables.
- Compare short-time Fourier Transform:

$$STFT(t,\omega) = \int_{-\infty}^{\infty} s(\tau)\gamma^*(\tau - t)e^{-j\omega t}d\tau$$

- Have $\psi_{a,b}{}^*(t)$ instead of $\gamma^*(\tau - t)e^{-j\omega t}$

CWT: Time-Frequency Analysis

- CWT provides a time-frequency as well as time-scale representation.

$$CWT(a,b) = TF\left(t = b, \omega = \langle\omega\rangle / a\right)$$

- We can define the Scalogram

$$SCAL(a,b) = \left|CWT(a,b)\right|^2$$

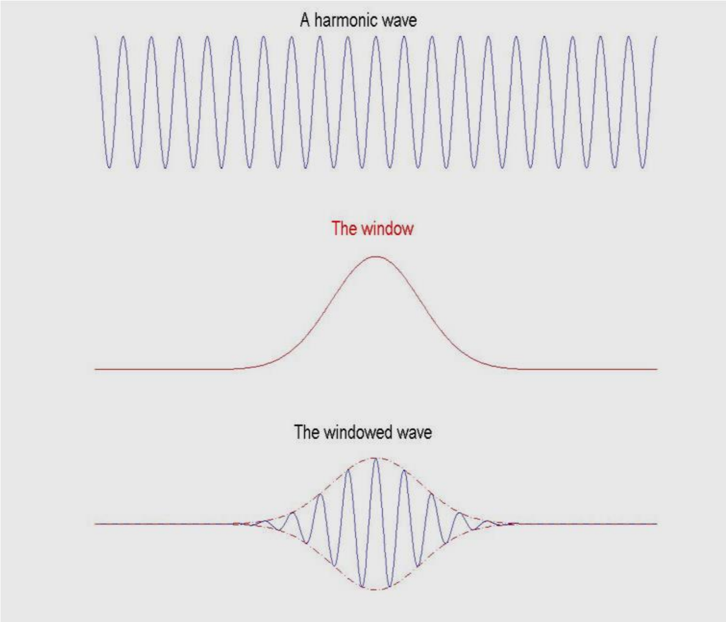
- Compare Spectrogram: $\left|STFT(t,\omega)\right|^2$

The Windowed Fourier Transform

- Harmonic wave e-jwt
(to perform the FT)

- A window γ(t)
(this will be moved across the signal)

γ(τ-t) e-jwt
(the basis function)



Changing the window width does not change the frequency of the oscillation

During the window, we cannot localise the frequencies in time, that is delta t

$$\Delta_t \Delta_\omega \geq 1/2$$

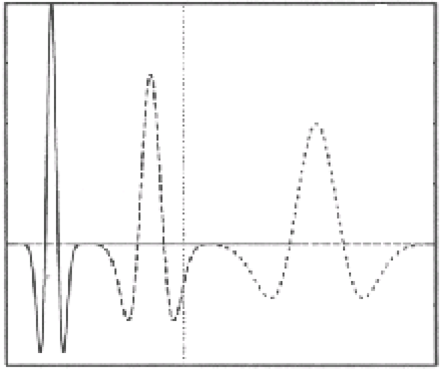
CWT versus STFT

CWT: Variable time-frequency resolution

$$CWT(a,b) = \langle s, \psi_{a,b} \rangle$$

Scale   Translation

Different width (scale);
Same no of cycles



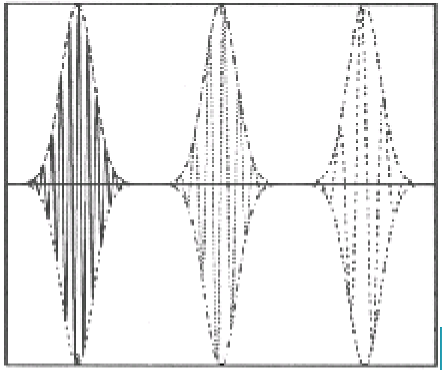
Scale and frequency are inversely proportional,
the smaller the scale the higher the frequency.
Going from large scale to small scale is equivalent to zooming in.

STFT: Constant time-frequency resolution

$$STFT(u,b) = \langle s, g_{u,b} \rangle$$

Frequency   Translation

Different window width;
Different no of cycles
of frequency.
For variable time-frequency
resolution would need to
change width and frequency
simultaneously



Scaling of a signal

Consider time-scaling a signal: $r(t) = s(t/\alpha)$ *

This changes Fourier Transform: $R(\omega) = \alpha S(\alpha\omega)$ *

So changes energy: $E_r = \int_{-\infty}^{\infty} \left|s(t/\alpha)\right|^2 dt = \int_{-\infty}^{\infty} \left|s(\tau)\right|^2 d(\tau\alpha) = \alpha E$

New centre freq:

$$\langle\omega\rangle_R = \frac{1}{2\pi E_R} \int_{-\infty}^{\infty} \omega \left|R(\omega)\right|^2 d\omega$$

$$= \frac{1}{2\pi\alpha E} \int_{-\infty}^{\infty} \omega \left|\alpha S(\alpha\omega)\right|^2 d\omega \qquad R(\omega)=\alpha S(\alpha\omega)$$

$$= \frac{1}{2\pi\alpha E} \int_{-\infty}^{\infty} \frac{\Omega}{\alpha} \left|\alpha S(\Omega)\right|^2 d\,\frac{\Omega}{\alpha} \qquad \Omega=\alpha\omega$$

$$= \frac{1}{2\pi\alpha E} \int_{-\infty}^{\infty} \Omega \left|S(\Omega)\right|^2 d\Omega \;=\; \frac{\langle\omega\rangle}{\alpha}$$

Scaled centre freq

Scaling of a signal

New frequency width:

$$\Delta_\omega^2(R) = \frac{1}{2\pi E_R}\int_{-\infty}^{\infty}\omega^2 |R(\omega)|^2\, d\omega - \langle\omega\rangle_R{}^2$$

$$= \frac{1}{2\pi\alpha E}\int_{-\infty}^{\infty}\omega^2 |\alpha S(\alpha\omega)|^2\, d\omega - (\langle\omega\rangle/\alpha)^2$$

$$= \frac{1}{2\pi\alpha E}\int_{-\infty}^{\infty}(\Omega/\alpha)^2 |\alpha S(\Omega)|^2\, d\frac{\Omega}{\alpha} - \langle\omega\rangle^2/\alpha^2$$

$$= \frac{1}{2\pi\alpha^2 E}\int_{-\infty}^{\infty}\Omega^2 |S(\Omega)|^2\, d\Omega - \langle\omega\rangle^2/\alpha^2$$

$$= \frac{\Delta_\omega^2(S)}{\alpha^2}$$

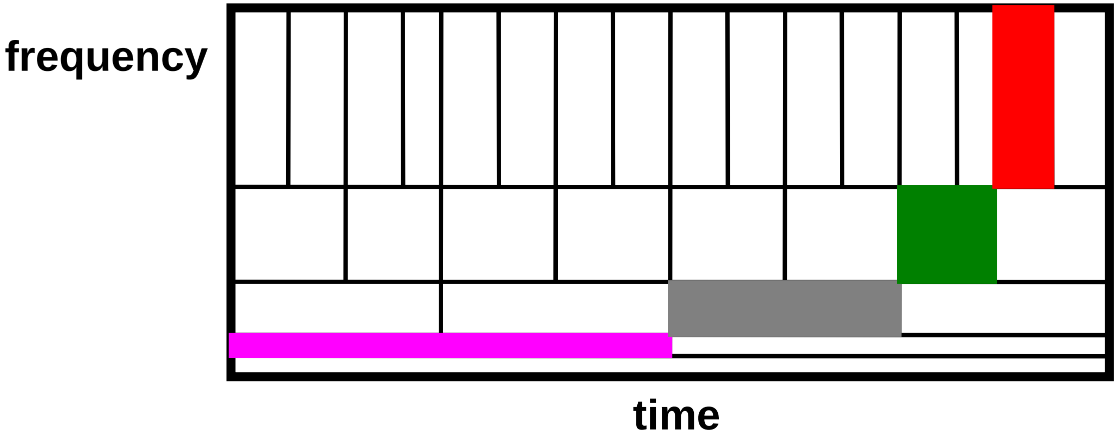
Scaled frequency resolution

Partition of the time-frequency plane

- High scale (low frequency)
  - large window size, better frequency resolution
- Low scale (high frequency)
  - small window size, better time resolution.

$$\Delta_t \Delta_\omega \geq 1/2$$

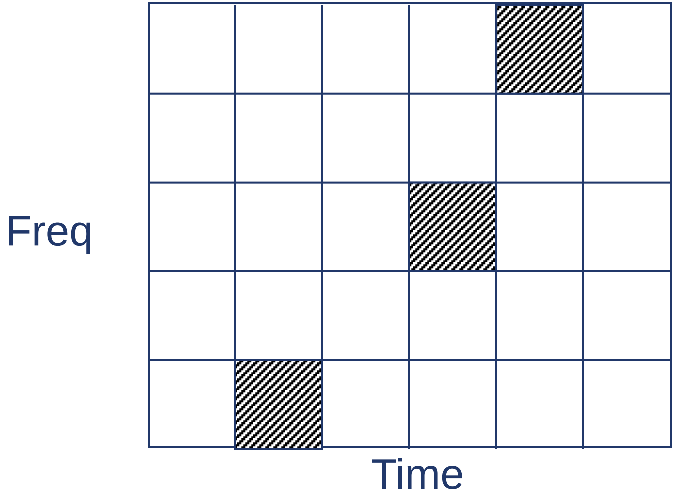
Constant



- High scale, low frequency/wide "window", wavelet equivalent to large scale not detailed map, e.g. map of China
- Low scale, high frequency/narrow "window", wavelet equivalent to small scale detailed map, e.g. map of Beijing.

Partition of the time-frequency plane

➢ FT: Equal time and frequency resolution



$$\Delta_t \Delta_\omega \geq 1/2$$

Constant

The window width is fixed for the STFT, that is delta t

In the WT it is variable by scaling.

Advantage of Wavelet Transform

- In time-frequency analysis we are correlating the signal being transformed with the basis function in both time and frequency.

- With the STFT we change the window width but not the basis frequency (although we could change that simultaneously independently but that would be difficult).

- With the Wavelet Transform, changing only the scale changes both window width and basis frequency simultaneously.

Inverse CWT: The Admissability Criterion

- We can construct an Inverse FT to reconstruct *s*(*t*)
  Can we do the same for CWT?
- Yes: provided that the *Admissibility Condition* is satisfied:

$$C_\Psi = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$$

where $\Psi(\omega) = \int_{-\infty}^{\infty} \psi(t)\, e^{-j\omega t}\, dt$ is the Fourier Transform of $\psi(t)$

Reconstruction: $$s(t) = \frac{1}{C_\Psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^2} CWT(a,b) \psi_{a,b}(t) da db$$

Wavelet function energy in the frequency domain must be compact and decay "quickly"

Inverse CWT: The Admissability Criterion

- Square of the Fourier transform must decay faster than 1/*w*.
- Admissibility is measure of signal's band-limitedness.
- Admissibility implies **zero average**:

$$\Psi(0)=\int_{-\infty}^{\infty}\psi(t)\,\mathrm{e}^{-j0t}\,dt=\int_{-\infty}^{\infty}\psi(t)dt=0$$

because otherwise $\dfrac{|\Psi(\omega)|^2}{|\omega|}\rightarrow\infty$ as $\omega\rightarrow 0$

If not zero mean then the average value will integrate to ∞

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Comparison of STFT and CWT

- **Similarities:**
  - signal is multiplied by a function, and the transform is computed separately for different segments of signals.
  - can be written in inner product form

$$STFT(b,\omega)=\left\langle s(t),\gamma(t-b)e^{j\omega t}\right\rangle \qquad CWT(b,a)=\left\langle s(t),\frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)\right\rangle$$

  - Time-frequency window area remains constant.
- **Difference:**
  - Fixed time duration and freq bandwidths of $\gamma(t)$
  - Variable time duration and bandwidth of $\psi(t)$

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Comparison of Bases

- **Fourier Transform**
  - Basis is global (across all time)
  - Sinusoids with frequencies in arithmetic progression
- **Gabor Transform (STFT)**
  - Basis is local (in time)
  - Sinusoid times Gaussian
  - Fixed-width Gaussian “window”
- **Wavelet Transform**
  - Basis is local (in time)
  - Frequencies in geometric progression (this applies to all transforms performed using dyadic scaling)
  - Basis has constant shape independent of scale

Problems with CWT

Redundancy (because continuous)

- Basis functions for CWT are shifted and scaled versions of each other. Usually do not form an orthonormal base.

Infinite solution space

- The result holds an infinite number of wavelets:
  hard to solve and hard to find the desired results out of the transformed data.

Efficiency

- Most transforms cannot be solved analytically. Solutions have to be calculated numerically: time-consuming.
  Must find efficient algorithms.

Solution?

- *Multiresolution Analysis*

Additional Notes about Wavelets

The Haar Function is a simple function that satisfies the criteria to be a wavelet:

- Short duration
- Oscillatory
- Mean value zero
- It forms a family of orthogonal functions making it useful in performing transforms
- However, its energy is not concentrated near the centre (in both time and frequency….its FT is a sinc function) so it has a limited number of applications

Wavelet Functions

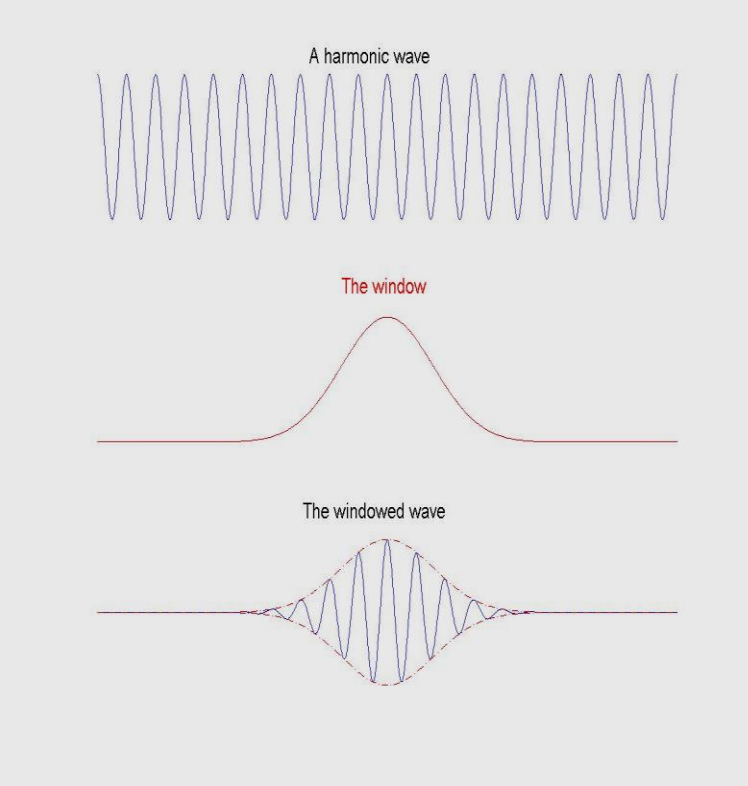
There are many different wavelet functions, some designed for use in specific applications, and more are continually being designed.

Examples are:

- **Daubechies wavelet**. Family of orthogonal wavelets called "dbk", when k=1 we have the Haar function.
- **Morlet wavelet**. Based on the Gabor Transform
- CDF wavelet (Cohen-Daubechies-Feauveau). Biorthorgonal wavelets used in JPEG2000 compression standard
- Coiflet. Have many applications
- Etc………

Morlet Wavelet

This is a wavelet composed of a complex exponential multiplied by a Gaussian Window.



The Gabor Short-Time Fourier Transform uses this with a fixed window width.

For Wavelet Transform it is scaled and translated

It is used in music applications as it can capture short bursts of alternating music notes, and in other applications.

Not orthogonal

Applications of Wavelet Transforms

The applications of wavelet transforms are almost endless.

General areas of application include:

- Signal processing (eg filtering)
- Feature extraction
- Trend analysis (eg financial data)
- Analysis of medical images (eg MRI scans) and physiological signals (eg ECG, EEG)
- Climatology (weather patterns, etc)
- Speech recognition
- Computer graphics
- Physics (astrophysics, seismic geophysics, optics……)

Specific Examples of Application

Fingerprint recognition. Used for digital storage (to replace paper methods) allowing high quality ease of transmission between law-enforcement agencies. The features of a fingerprint are unique to each individual. Features include directional information, edge parameters, centre area)

Image Compression. By removing redundant data to represent the information by a smaller number of bits (eg JPEG2000 image compression and MPEG-4 video coding)

Specific Examples of Application

Image Denoising. To remove additive noise while retaining as much of the original signal's features and characteristics as possible. This can be done by thresholding or by using the fact that noise is poorly correlated across scale in the wavelet transform.

Face Recognition. Could use the location of facial features (eyes, ears, nose, mouth) or markers such as hair colour, etc)

Image Fusion. Multiple images are decomposed then reconstructed to enhance the resulting image (eg in medical imaging and robotics)

Tackling the Redundancy Problem

- Because the CWT(a,b) is a continuous transform, it encodes more than enough information to reconstruct the signal. A sampled version is sufficient.
- Instead of moving the wavelet continuously, let's move it in discrete steps that do not overlap and leave no gaps.
- a is scaling factor. b is translation term.
- Let us sample in a dyadic (power of 2) grid: $a = 2^{-m}$ $b = n2^{-m}$

$$CWT(2^{-m}, n2^{-m}) = \frac{1}{\sqrt{\left|2^{-m}\right|}} \int_{-\infty}^{\infty} s(t)\psi^*\left(\frac{t - n2^{-m}}{2^{-m}}\right) dt$$

$$d_{m,n} = CWT(2^{-m}, n2^{-m}) = \int_{-\infty}^{\infty} s(t)\psi^*{}_{m,n} dt$$

where $\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n)$ Definition of the CWT with unit energy

Discrete (Sampled) Transform

If the set of functions $\psi_{m,n}$ forms a frame, then we can recover s(t)

$$s(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_{m,n} \hat{\psi}_{m,n}(t)$$

where $\hat{\psi}_{m,n}(t)$ forms the dual frame

Compare with FS: sum of 2 sets of coefficients of orthogonal functions

For the orthonormal frame, we have $\hat{\psi}_{m,n}(t) = \psi_{m,n}(t)$

So we get the *wavelet series* (transform and inverse):

$$d_{m,n} = \int_{-\infty}^{\infty} s(t) \psi^{*}_{m,n} dt$$

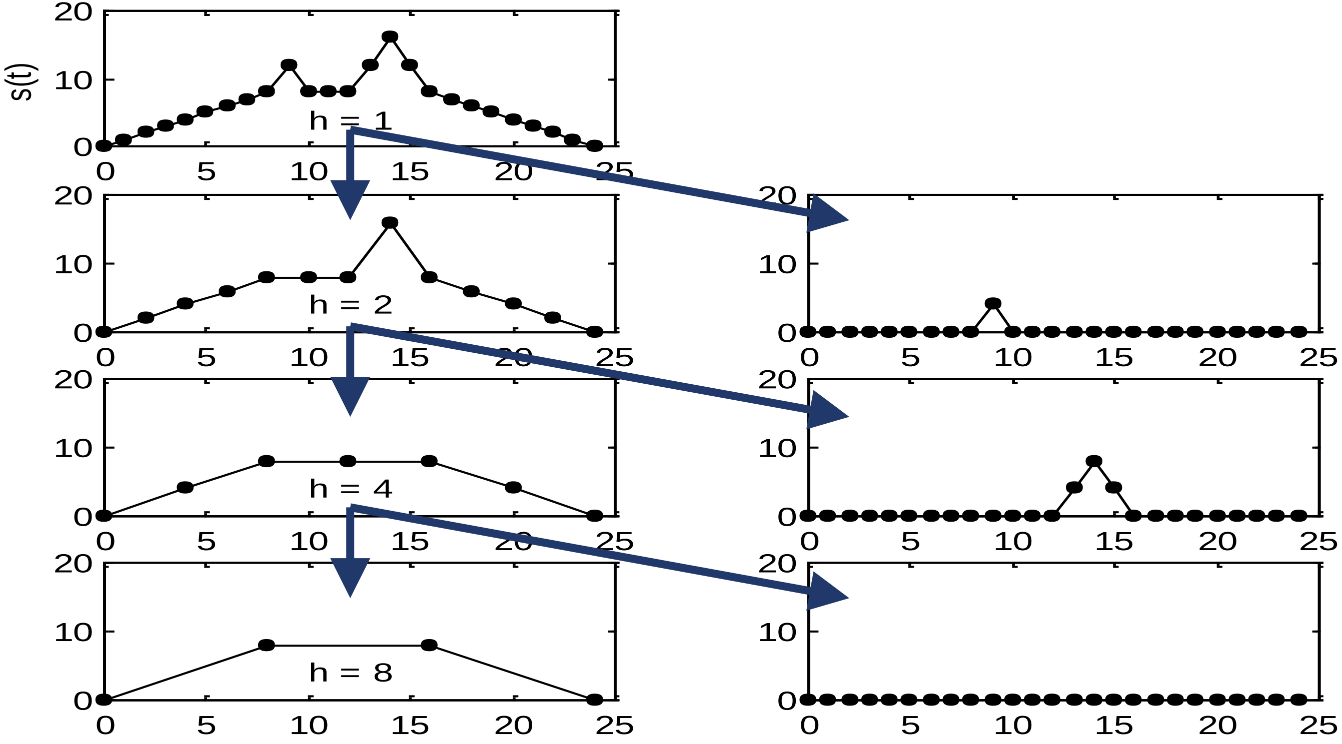
$$s(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_{m,n} \psi_{m,n}(t)$$

So: wavelet series is not redundant (finite number of coefficients)

A continuous transform would have an infinite number of coefficients

Multiresolution Analysis (MRA)

- Objective: To analyze a complicated function by dividing it into several simpler ones and studying them separately.

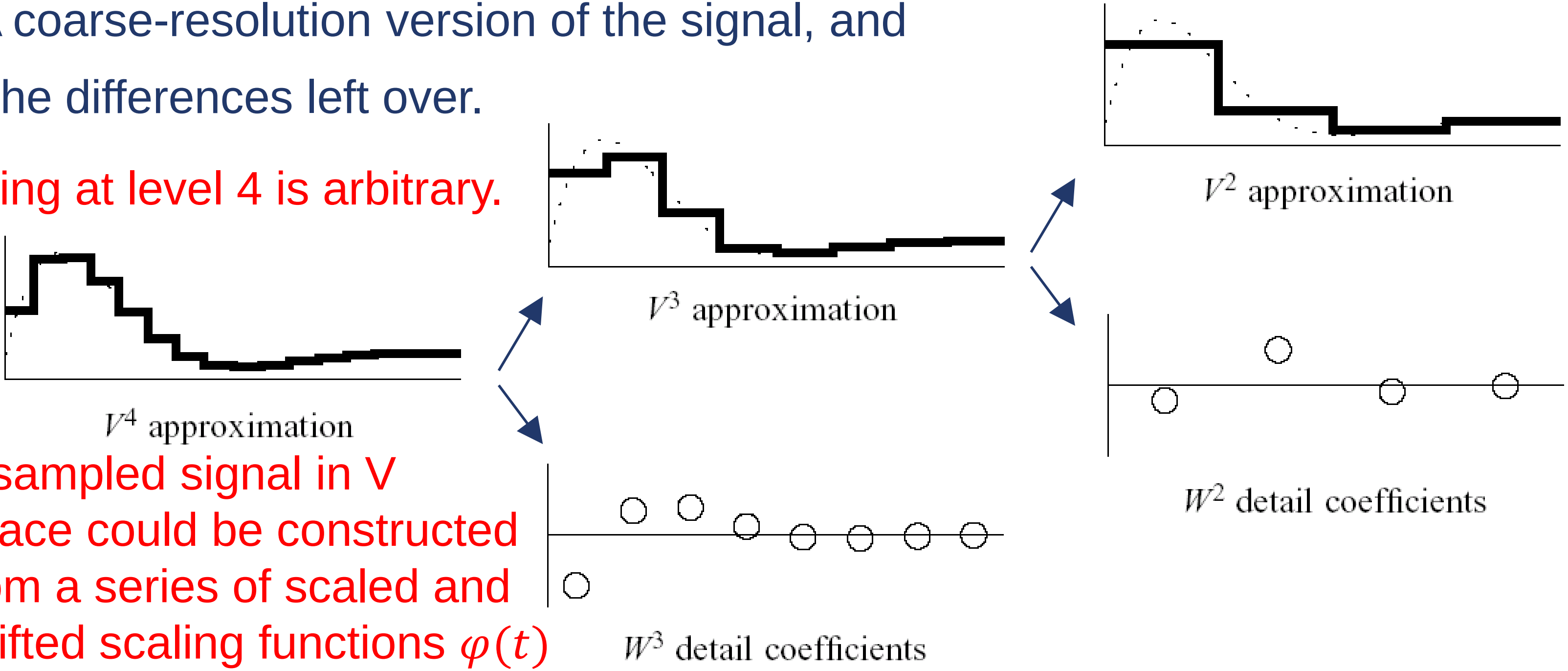


Piecewise Approximation

Basic Concept: Decompose a fine-resolution signal into

1. A coarse-resolution version of the signal, and
2. The differences left over.

Starting at level 4 is arbitrary.



A sampled signal in V space could be constructed from a series of scaled and shifted scaling functions $\varphi(t)$

So: what about wavelets? Theory coming next...

Multiresolution Analysis (MRA)

- Starting at level 4 is arbitrary.
- If we remove the fine detail from the signal in space V^4
- and put that in space W^3 then the approximate signal left in V^3 and the detail in W^3 have no elements in common.
- So the signal in V space and the signal in W space at a given level of decomposition are orthogonal.
- Similarly for V^2 and W^2, etc.

Multiresolution Analysis (MRA)

With the Haar Function we saw that the scaling function can be constructed from a series of scaled and translated scaling functions. That is:

A *scaling function* $\phi(t)$ generates a nested subsequence of subspaces $\{V_j\}$,

$$\{0\} \leftarrow \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

and satisfies a *dilation* (refinement) equation

$$\phi(t) = \sum_k p_k \phi(at - k) \qquad \text{for integer } k$$

for some $a > 0$ and coefficients $\{p_k\}$

We consider $a = 2$ that corresponds to octave-scales, that is, dyadic.

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Spaces of functions

The space V_0 is generated by $\{\phi(t-k)\}$ for integers k

In general, V_j is generated by $\{\phi(2^j t-k)\}$

Then:

$$V_n = \bigoplus_{j=-\infty}^{n-1} W_j \qquad \text{(orthorgonal sum of subspaces).}$$

$$V_{n+1} = \bigoplus_{j=-\infty}^{n} W_j = \bigoplus_{j=-\infty}^{n-1} W_j \oplus W_n = V_n \oplus W_n$$

Subspace W_n is the *orthogonal complementary subspace* of V_n in V_{n+1}.

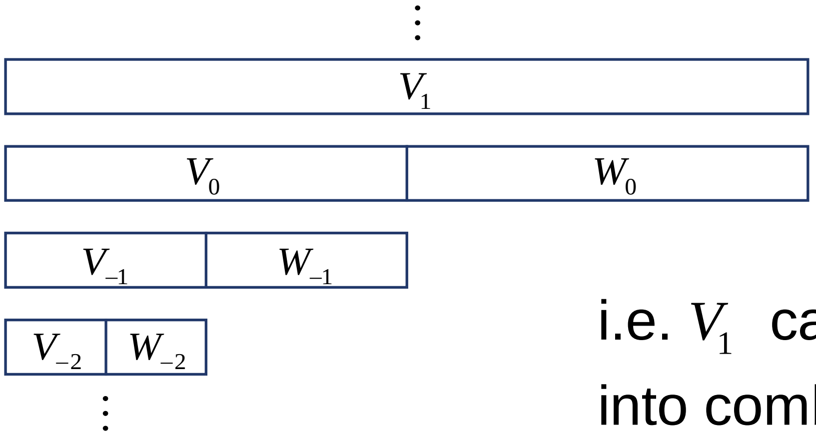
That is, we can reconstruct the original signal by adding together V and W at lower levels. (We could also provide compression by omitting the coefficients at low levels.)

Multiresolution Analysis (MRA)

Subspaces $\{V_j\}$ are nested

while subspaces $\{W_j\}$ are mutually orthogonal. Consequently,

$$\begin{cases} V_j \cap V_l = V_j & l > j; \\ W_j \cap W_l = \{0\} & j \neq l; \\ V_j \cap W_l = \{0\} & j \leq l. \end{cases}$$



i.e. V_1 can be decomposed

into combination of V_0 and W_0, etc.

Multiresolution Analysis (MRA)

If the Scaling function and the Wavelet function are orthogonal, as they are for the Haar Function, then we can say that because V and W are orthogonal:

Subspaces $\{W_j\}$ are generated by *wavelets* $\psi(t)$

And $\{V_j\}$ is generated by scaling functions $\phi(t)$

In other words, any $f_j(t) \in V_j$

can be written as

$$f_j(t) = \sum_k c_k^j \phi(2^j t - k),$$

Sum of scaling functions.

Linear Piecewise Approximation.

And……

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Multiresolution Analysis (MRA)

….. any function $g_j(t) \in W_j$

can be written as $g_j(t) = \sum_k d_k^j \psi(2^j t - k)$, Sum of wavelet functions

for some coefficients

Furthermore, $\{c_k^j\}_{k \in z}, \ \{d_k^j\}_{k \in z}$

$$f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$$

$$f(t) \in V_j \Leftrightarrow f\left(t + \frac{1}{2^j}\right) \in V_j$$

Multiresolution Analysis (MRA)

Subspaces $\{V_j\}$ are called "approximation subspaces"

and $\{W_j\}$ are the "wavelet subspaces".

At any level j, V_j contains the smooth part

and W_j contains the "details" of the original function.

$$j \uparrow \Rightarrow f_j(t), \text{ a finer approximation of } f(t),$$

$$j \downarrow \Rightarrow f_j(t), \text{ a coarser approximation of } f(t).$$

Large j means greater scaling so more compression of the mother wavelet so the daughter wavelet is higher frequency.

Two-Scale Relations

Since $\phi(t) \in V_0 \subset V_1,$ and $\psi(t) \in W_0 \subset V_1;$

we should be able to write $\phi(t)$ and $\psi(t)$ in terms of the bases that generate V_1 .

In other words, there exists two sequences $\{p_k\}, \{q_k\}$

such that

$$\phi(t) = \sum_k p_k \phi(2t-k),$$

$$\psi(t) = \sum_k q_k \phi(2t-k).$$

Both are a function of scaling functions

In general, for any j , the relationship between V_j, W_j with V_{j+1} is governed by

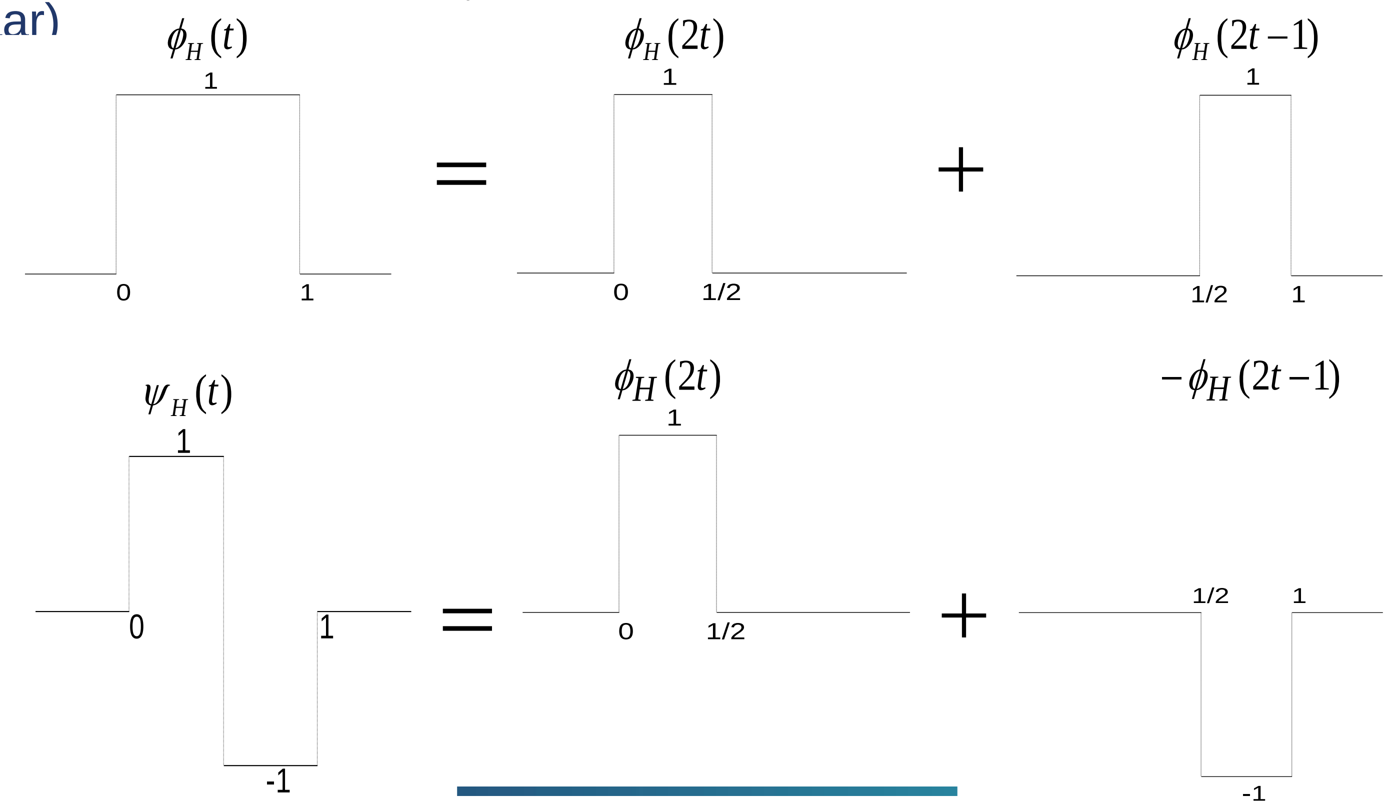
$$\phi(2^j t) = \sum_k p_k \phi(2^{j+1} t - k),$$

$$\psi(2^j t) = \sum_k q_k \phi(2^{j+1} t - k).$$

Two-Scale Relations

Example: (H: Haar)

$$p_0 = p_1 = 1, \quad p_k = 0 \text{ for integer } k \text{ (except } 1, 0)$$



$$q_0 = 1, \; q_1 = -1, \quad q_k = 0 \text{ for integer } k \text{ (except } 1, 0)$$

Two-Scale Relations

What does the two-scale relations really mean?

- Imagine you have a signal, like a piece of music or an image.
- The two-scale relations in wavelet theory are like looking at that signal at different levels of detail or resolution.
- It's like zooming in and out to see both the big picture and the small details.
- These relations help us understand how the wavelet function at one level relates to the wavelet function at a different, usually smaller, level.
- This is crucial for breaking down the signal into different components and analyzing it effectively.

Decomposition Relation

Since $V_1 = V_0 + W_0$ and $\phi(2t),\ \phi(2t-1) \in V_1$, there exist two pairs of sequences $(\{a_{2k}\}, \{b_{2k}\})$ such that

$$\phi(2t) = \sum_k \{a_{2k}\phi(t-k) + b_{2k}\psi(t-k)\};$$

$$\phi(2t-1) = \sum_k \{a_{2k-1}\phi(t-k) + b_{2k-1}\psi(t-k)\}.$$

Combining these two relations, we have $\phi(2t-l) = \sum_k \{a_{2k-l}\phi(t-k) + b_{2k-l}\psi(t-k)\}.$

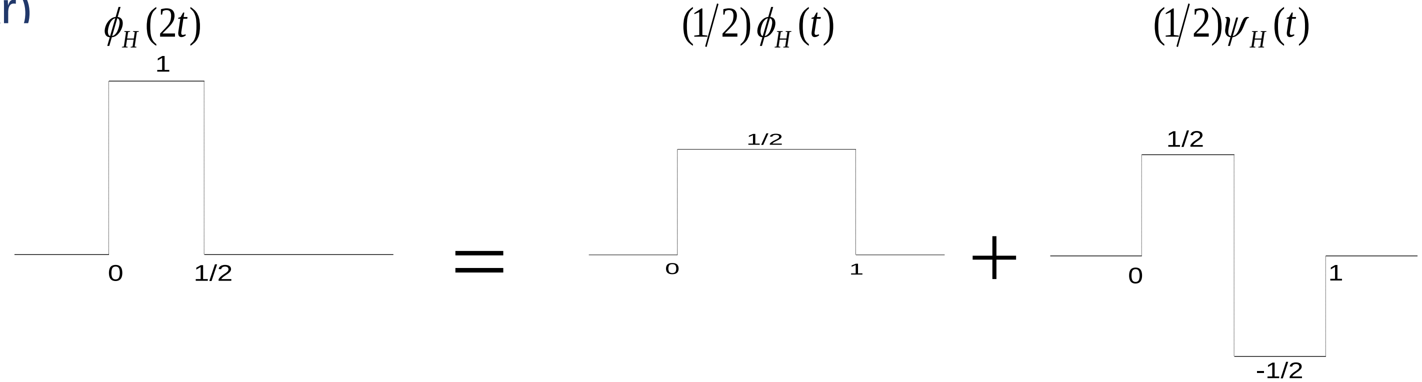
for all integer l

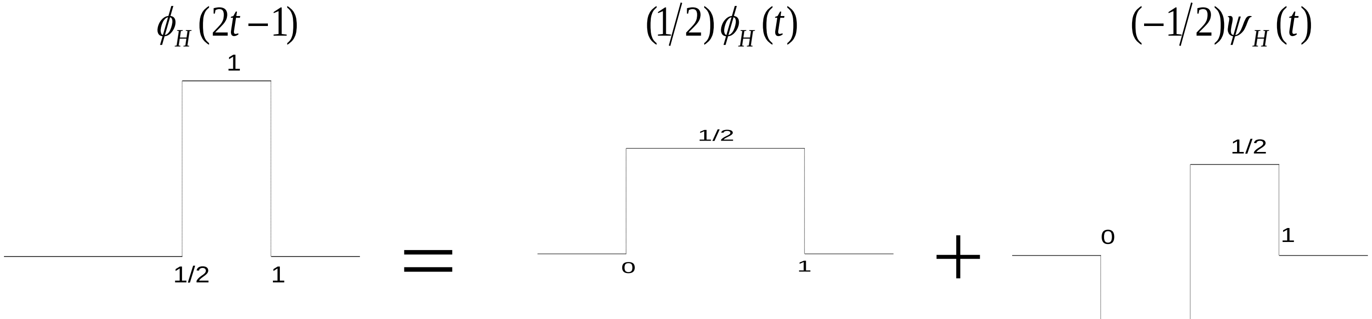
In general, we have

$$\phi(2^{j+1}t-l) = \sum_k \{a_{2k-l}\phi(2^j t-k) + b_{2k-l}\psi(2^j t-k)\}.$$

Decomposition Relation

Example: (H: Haar)





$$a_0 = 1/2,\ \ a_{-1} = 1/2,\ \ \ a_k = 0 \text{ for } k \in Z \setminus \{0,-1\}$$

$$b_0 = 1/2,\ b_{-1} = -1/2,\ \ \ b_k = 0 \text{ for } k \in Z \setminus \{0,-1\}$$

Decomposition Relation

What does the decomposition relations really mean?

- Now, let's say you want to break down your signal into different parts to understand it better.
- The decomposition relation in the Discrete Wavelet Transform (DWT) is like a recipe for doing this decomposition.
- It tells you how to split your signal into approximation (low-frequency) and detail (high-frequency) components.
- The **approximation part** gives you the overall shape or trend of the signal.
- The **detail part** highlights the fine details or sudden changes in the signal.
- So, the decomposition relation guides the process of breaking down a signal into these meaningful components, helping you understand it in a more detailed and organized way.

Summary

We have seen that a series of sampled values of a continuous signal can be divided into two sequences:

- One which is approximations
- The other fine detail
- These two sequences are orthogonal
- The approximations can be derived as a series of scaled and shifted scaling functions, each with a coefficient (same as a linear piecewise approximation)
- The fine detail can be derived as a series of scaled and shifted wavelet functions, each with a coefficient
- These two sequences can be recombined to form the original sequence

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