Chapter 2. Electrostatics

Stars * represent contents of self-study; Green contents are nodi

- **Contents**
  - Coulomb's law, superposition principle, and Electric field intensity
  - Fundamental Equs. (Gauss laws and curl equation)
  - Electric Potential
  - Electric Dipole
  - Dielectric Materials *
  - Boundary Conditions
  - Capacitors and capacitance
  - Force and Energy in E Field *

```
  ___       ___       ___       ___
 (' v' )   (' v' )   (' v' )   (' v' )
((    ))  ((    ))  ((    ))  ((    ))
-/-"--"----/-"--"----/-"--"----/-"--"--
```

Concept of Static Electric Fields

- Electrostatics, or Static Electric Fields
  - All charges are **fixed in space**;
  - All charge densities are **constant in time**;
  - The charge is **the only source** of the electric field.

The charge is a scalar and its distribution forms a scalar field.

☆ Distribution of the Charge---field

Density

unit

1. ***V**olume Density* $$\rho_V(\vec{r}) = \lim_{\Delta v \to 0}\left(\frac{\Delta q}{\Delta v}\right)$$ C/m^3

2. ***S**urface Density* $$\sigma_S(\vec{r}) = \lim_{\Delta S \to 0}\left(\frac{\Delta q}{\Delta S}\right)$$ C/m^2

3. ***L**ine Density* $$\rho_l(\vec{r}) = \lim_{\Delta l \to 0}\left(\frac{\Delta q}{\Delta l}\right)$$ C/m

☆ The methods to study Electrostatics

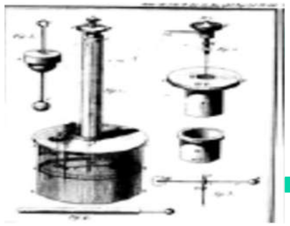
Experimental laws → Field quantity → Fundamental Equs.

Experimental laws
1. Coulomb’s laws
2. Superposition laws

Field quantity
Electric field intensity

Fundamental Equs.
Guass laws
Curl equation

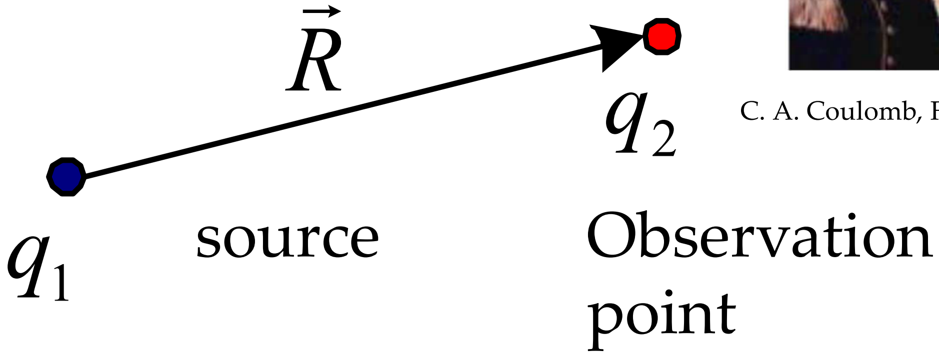
Helmholtz Theorem



1. ☆ *Coulomb's Law*

C. A. Coulomb, France

$$\boxed{\vec{F}_{12} = \frac{q_1 \cdot q_2}{4\pi\varepsilon_0 \cdot R^2}\vec{a}_R}$$



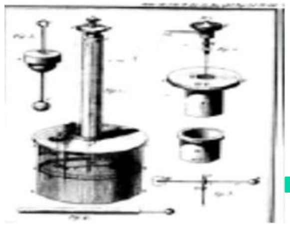
$$\varepsilon_0 \approx \frac{1}{36\pi \times 10^9} \approx 8.85 \times 10^{-12} (F/m)$$

ε_0 refers to ***the Dielectric Constant*** *in free space*

Two **static point** charges q_1 and q_2 in the space, with distance R between them,

The electro-static force by q_1 on q_2 is in direct proportion to their charges, and in inverse proportion to R squared.

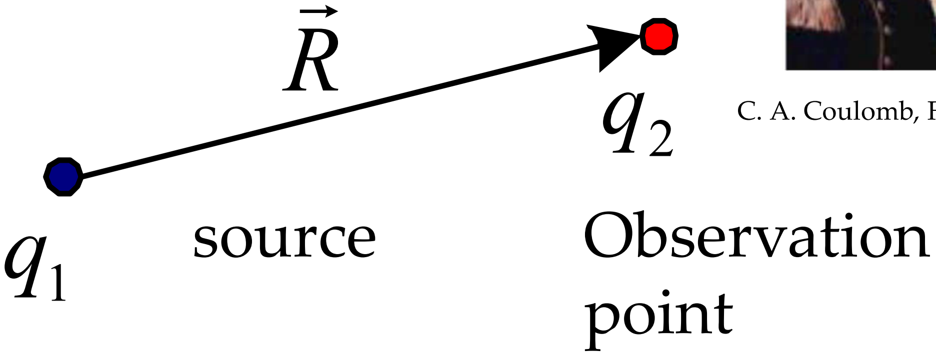
The direction of the force is from q_1 to q_2



1. ☆ *Coulomb's Law*

C. A. Coulomb, France

$$\boxed{\vec{F}_{12} = \frac{q_1 \cdot q_2}{4\pi\varepsilon_0 \cdot R^2}\vec{a}_R}$$



Is that a fully experiment law? w/o any postulate?

1. If the **size** of charges are near with the **distance** between them, the charge can not be assumed as point charges

Coulombs's law has been verified to hold for distance 10^{-14} meter

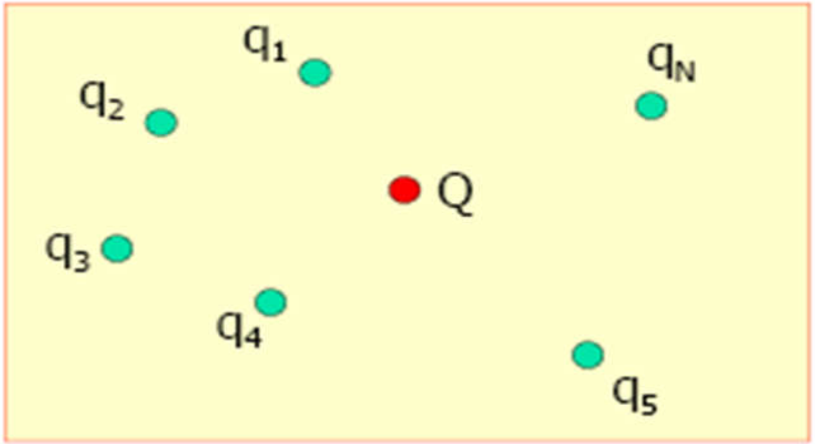
2. Exactly Inverse-square relation with distance? Experiment measurement can not be infinite accuracy. (2.000001~1.999999)

The Coulombs's law can be understand as a law of nature, discovered by Coulomb, and verified by his limited accuracy experiment

2. ☆ *Principle of Superposition （叠加原理）*

The total force F, acting on a point charge Q due to a system of N point charges is the vector sum of forces exerted individually by each charge on Q

Coulombs's force obeys the principle of superposition

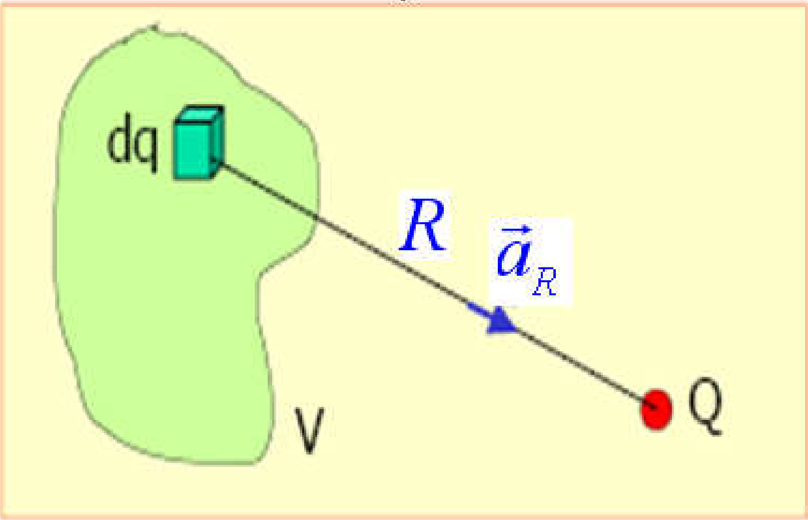


➢**For scattered charges**

$$\vec{F}_Q = \sum_{i=1}^{N} \vec{F}_i = \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{R_i^2} \vec{a}_{R,i}$$

➢**For chargers distribution**

$$\vec{F}_Q = \frac{Q}{4\pi\varepsilon_0} \int_V \frac{dq}{R^2} \vec{a}_R$$



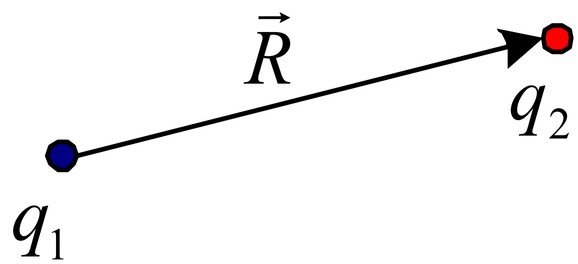
3. ☆ *Electric Field Intensity（电场强度）*

With Coulomb's laws and the principle of superposition, we can compute the forces among stationary charges,
Why we need define **Electric Field intensity**

The problem:

if one charge is moved toward the other,

1. ***The force experienced by the charges must change immediately according coulomb's law***
2. ***Basing on the theory of relativity, it take some time to transfer the change information***



There must exist an extra entity besides two charges, electric field

3. ☆ *Electric Field Intensity（电场强度）*

$$\vec{E} = \lim_{q_{test} \to 0} \vec{F} \Big/ q_{test}$$

Unit: $N/C, or \quad {}^{V}\!/_{m}$

Electric Field Intensity is defined as the electro-static force per unit charge.

Why q *test*→0 ?

- **Electric field intensity for point charge:**

$$\vec{E}(\vec{R}, q) = \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{R^2} \vec{a}_R$$

- **Electric field intensity satisfies superposition principle**

$$\vec{E} = \sum_{i=1}^{N} \vec{E}_i$$

```
  ____      ____      ____      ____
 (’ v’ )   (’ v’ )   (’ v’ )   (’ v’ )
((    ))  ((    ))  ((    ))  ((    ))
-/-”--”----/-”--”----/-”--”----/-”--”--
```

E Intensity in more Complicate cases

For distributed charges → Vector integral

$$d\vec{E}(\vec{r}) = \frac{dq}{4\pi\varepsilon_0} \cdot \frac{\vec{a}_{R(?)}}{R(?)^2}$$

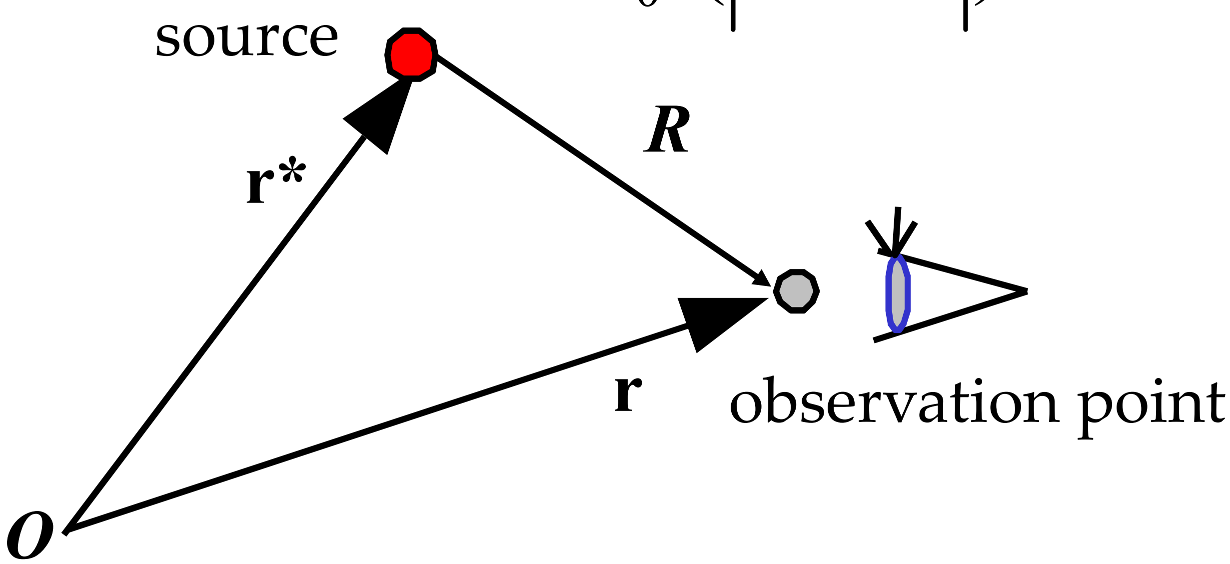
Differential element $dq = ?$

$\rho_V / \sigma_s / \rho_l \qquad dq = \rho_V dV \text{ or } \sigma_s ds \text{ or } \rho_l dl$

If the source does not stay at the origin point,

$$\vec{E}(\vec{R}, q_1) = \frac{q_1}{4\pi\varepsilon_0}\frac{1}{R^2}$$

$$\vec{E}(\vec{r}-\vec{r}^{\,*}; q_1) = \frac{q_1}{4\pi\varepsilon_0}\frac{1}{\left(\left|\vec{r}-\vec{r}^{\,*}\right|\right)^2}$$



It is one of main method to calculate E field intensity

For scattered charges → Vector sum

$$\vec{E} = \sum_{i=1}^{N} \vec{E}_i$$

For distributed charges → Vector integral

$$\vec{E} = \int d\vec{E}(\vec{r}) = \int \frac{dq}{4\pi\varepsilon_0} \cdot \frac{\vec{a}_{R(?)}}{R(?)^2}$$

Differential element $dq = ?$

$$\rho_V / \sigma_s / \rho_l \qquad dq = \rho_V dV \text{ or } \sigma_s ds \text{ or } \rho_l dl$$

Example 1. Line Charges in Length of 2L

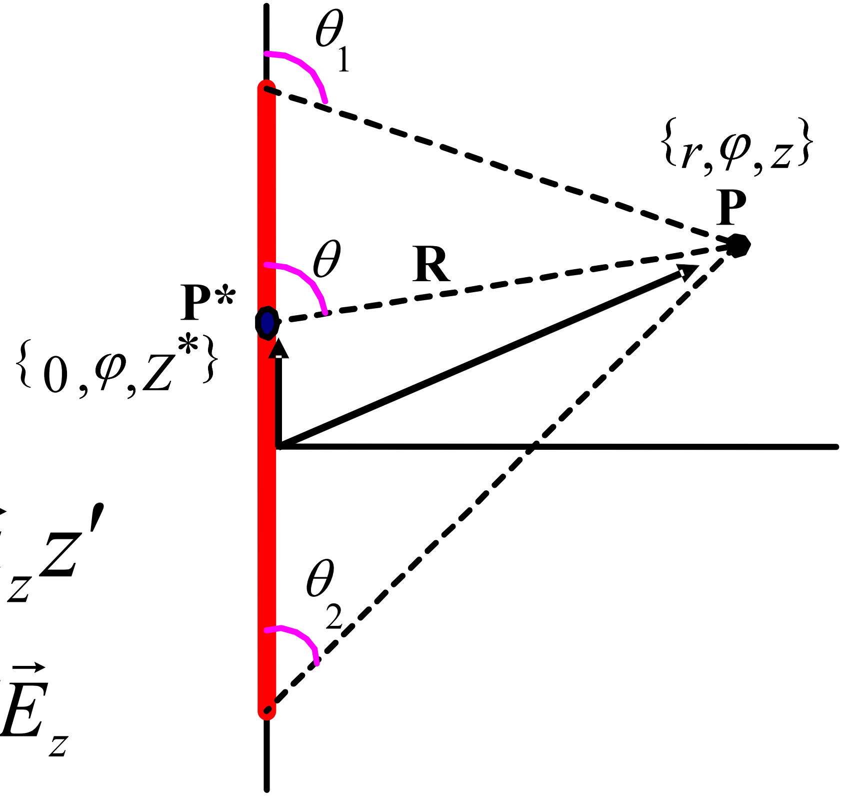
Analysis:

(1) Due to axial symmetry, $\vec{E}$ has only E_r & E_z components.

(2) Select Cylindrical Coordinates

Direct Solution with E define

$$d\vec{E}(\vec{R}) = \frac{dq}{4\pi\varepsilon_0}\frac{\vec{R}}{R^3}$$



$$\vec{R} = \vec{P} - \vec{P}' = (\vec{a}_r r + \vec{a}_z z) - \vec{a}_z z'$$

$$d\vec{E}(\vec{R}) = \frac{dq}{4\pi\varepsilon_0}\frac{\vec{R}}{R^3} = d\vec{E}_r + d\vec{E}_z$$

$$\boxed{R = \frac{r}{\sin\theta}} \qquad \boxed{dq = \rho_l dz'}$$

Example 1. Line Charges in Length of 2*L*

E *at point P:*

$$E_r = \int_{-l}^{l} \frac{\rho_l \sin\theta \mathrm{d}z'}{4\pi\varepsilon_0 R^2}$$

$$E_z = \int_{-l}^{l} \frac{\rho_l \cos\theta \mathrm{d}z'}{4\pi\varepsilon_0 R^2}$$

$$\boxed{z' = z - R\cos\theta = z - r/\tan\theta, \mathrm{d}z' = \frac{r\mathrm{d}\theta}{\sin^2\theta}}$$

We get

$$E_r = \int_{-l}^{l} \frac{\rho_l \sin\theta \mathrm{d}z'}{4\pi\varepsilon_0 R^2} = \frac{\rho_l}{4\pi\varepsilon_0 r}(\cos\theta_2 - \cos\theta_1)$$

$$E_z = \int_{-l}^{l} \frac{\rho_l \cos\theta \mathrm{d}z'}{4\pi\varepsilon_0 R^2} = \frac{\rho_l}{4\pi\varepsilon_0 r}(\sin\theta_1 - \sin\theta_2)$$

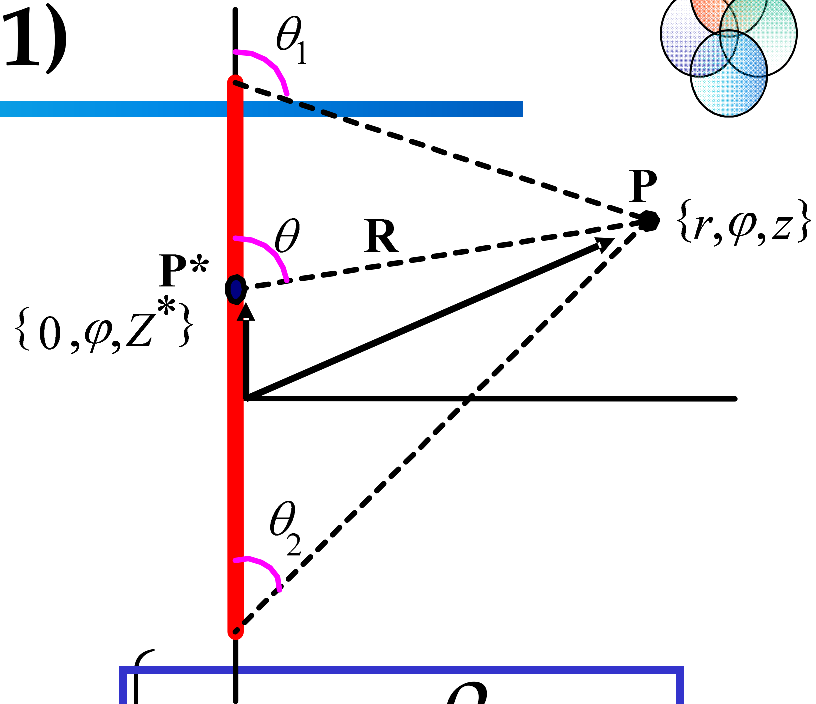
Discussion (1)

- For infinite line charges, or *L>>R*

$$\vec{E} = \{E_r, 0, E_z\}$$

$$E_r = \frac{\rho_l}{4\pi\varepsilon_0 r}\cdot(\cos\theta_2 - \cos\theta_1)$$

$$E_z = \frac{\rho_l}{4\pi\varepsilon_0 r}\cdot(\sin\theta_1 - \sin\theta_2)$$



$$\because l \to \infty \qquad \therefore \begin{cases} \theta_1 \to \pi \\ \theta_2 \to 0 \end{cases} \qquad \Rightarrow \begin{cases} \boxed{E_r = \dfrac{\rho_l}{2\pi\varepsilon_0 r}} \\ E_z \equiv 0 \end{cases}$$

The E Intensity exists only in radial direction.

Discussion (2)

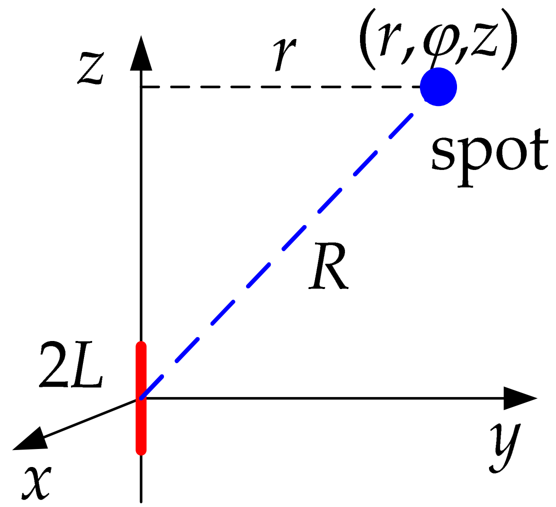
- For short line charges, i.e. $L \ll R$

$$E_r = \frac{\rho_l}{4\pi\varepsilon_0 r}\cdot\left(\cos\theta_2 - \cos\theta_1\right) = \frac{\rho_l}{4\pi\varepsilon_0 r}\cdot\left(\frac{L+z}{\sqrt{(L+z)^2+r^2}} - \frac{z-L}{\sqrt{(L-z)^2+r^2}}\right)$$

$$E_z = \frac{\rho_l}{4\pi\varepsilon_0 r}\cdot\left(\sin\theta_1 - \sin\theta_2\right) = \frac{\rho_l}{4\pi\varepsilon_0 r}\cdot\left(\frac{r}{\sqrt{(L-z)^2+r^2}} - \frac{r}{\sqrt{(L+z)^2+r^2}}\right)$$

$$\vec{E} = \vec{a}_r E_r + \vec{a}_z E_z \approx \vec{a}_R \frac{q}{4\pi\varepsilon_0 R^2}$$

Equivalent to that of a point charge.



Charged bodies are envisioned as point charges as long as their sizes are much less than the distance between them, or the distance from them to the calculating spot.

```
()" "()   ()*"()
(  `o`)   (`o`  )
(")(")    (")(")
```

——Now, let’s go on.

How to study *Electric Field Intensity* ?

Experimental laws → Field quantity → Fundamental Equs.

Experimental laws	Field quantity	Fundamental Equs.
1. Coulomb's laws 2. Superposition laws	Electric field intensity	Gauss laws Curl equation

Helmholtz Theorem: ——亥姆霍兹公理

$$\nabla \bullet \vec{A}$$

$$\nabla \times \vec{A}$$

Boundary conditions

Gauss laws

$$\int_V (\nabla \bullet \vec{A})dV = \oint_S \vec{A} \bullet d\vec{S}$$

$$\int_S (\nabla \times \vec{A}) \bullet d\vec{S} = \oint_C \vec{A} \bullet d\vec{l}$$

Curl equation

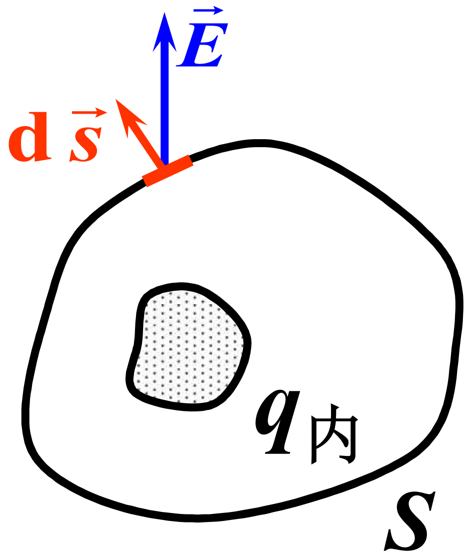
4. ☆ Electrostatic Gauss's Law（高斯定理）

$$\Phi_{\mathrm{e}} = \oint_S \vec{E} \cdot \mathrm{d}\vec{S} = \frac{1}{\varepsilon_0} \sum_{i=1}^{n} q_i$$

J. C. F. Gauss, Germany

- **Gauss's law:**

The **net electric flux** emanating from a closed surface is numerically equals to the **sum of charge** inside the closed surface over ε_0

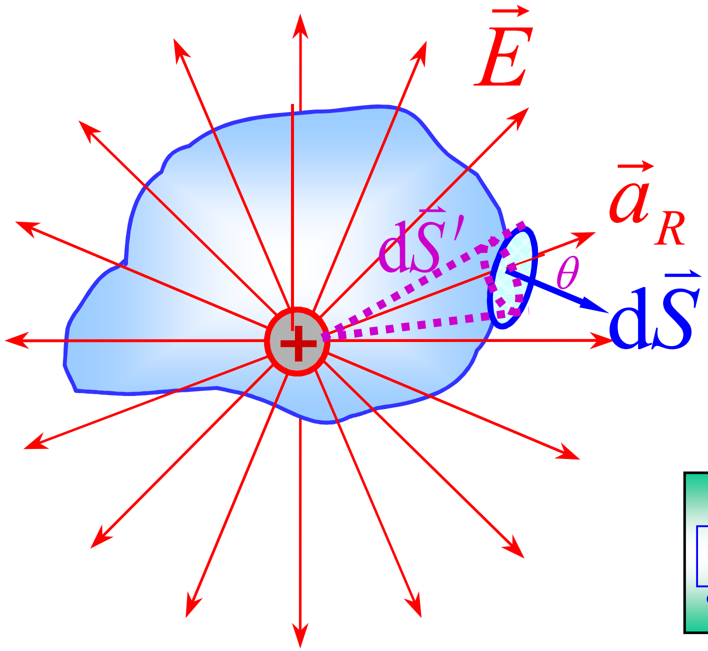


4. ☆ Electrostatic Gauss's Law（高斯定理）

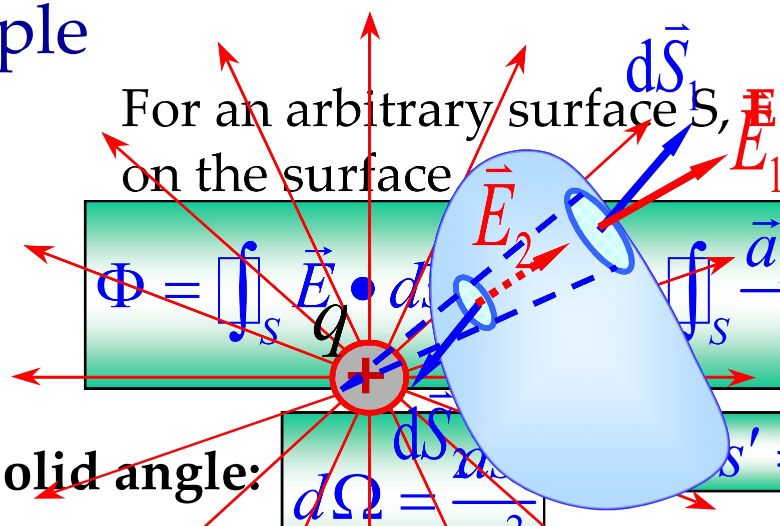
How to get Gauss's law?

- Coulomb's law
- Superposition principle

→ **Gauss's law**



For an arbitrary surface S, E at point P on the surface



$$\Phi = \oint_S \vec{E}\bullet d\vec{s} \quad \cdots \quad \oint_S \frac{\vec{a}_R \bullet \vec{a}_s ds}{R^2}$$

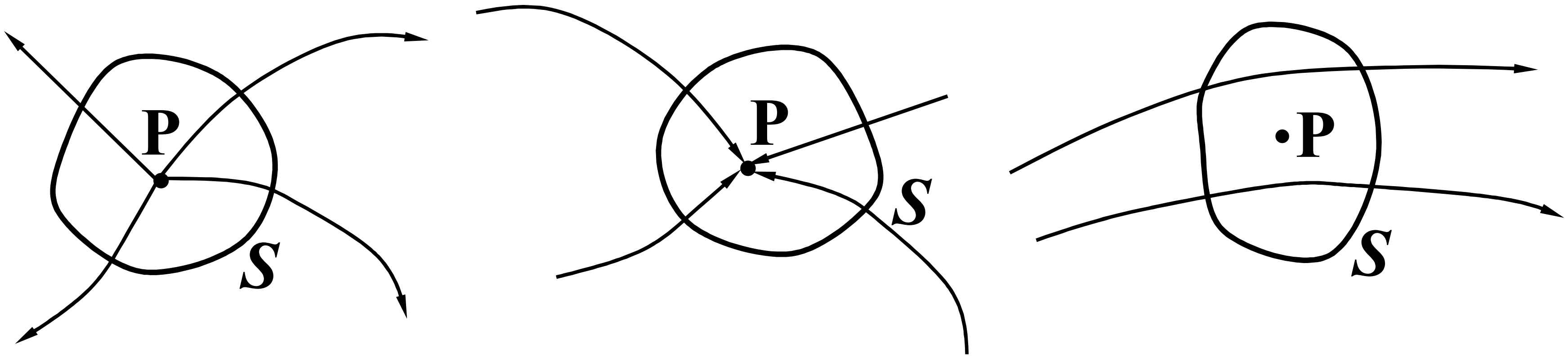
Solid angle: $d\Omega = \frac{ds'}{R^2}$ $\quad ds' = ds\cos\theta$

$$\oint_S \frac{\vec{a}_R \bullet \vec{a}_s ds}{R^2} = \oint_S \frac{\cos\theta ds}{R^2} = \oint_S \frac{ds'}{R^2} = \oint_S d\Omega = 4\pi$$

$$\Phi = \oint_S \vec{E}\bullet d\vec{s} = \frac{Q}{4\pi\varepsilon_0}4\pi = \frac{Q}{\varepsilon_0}$$

4. ☆ Electrostatic Gauss's Law（高斯定理）

- **Discuss Gauss's law:**
  1. **The charge is the source of electrostatics;**
  2. **The line of electrostatic is from positive charge, end at negative charge, continually at W/O charge point;**
  3. **Electric flux only relates with the charges inside the enclosed surface.**
  4. **Although E are generated by all charges in space,**



4. ☆ Electrostatic Gauss's Law（高斯定理）

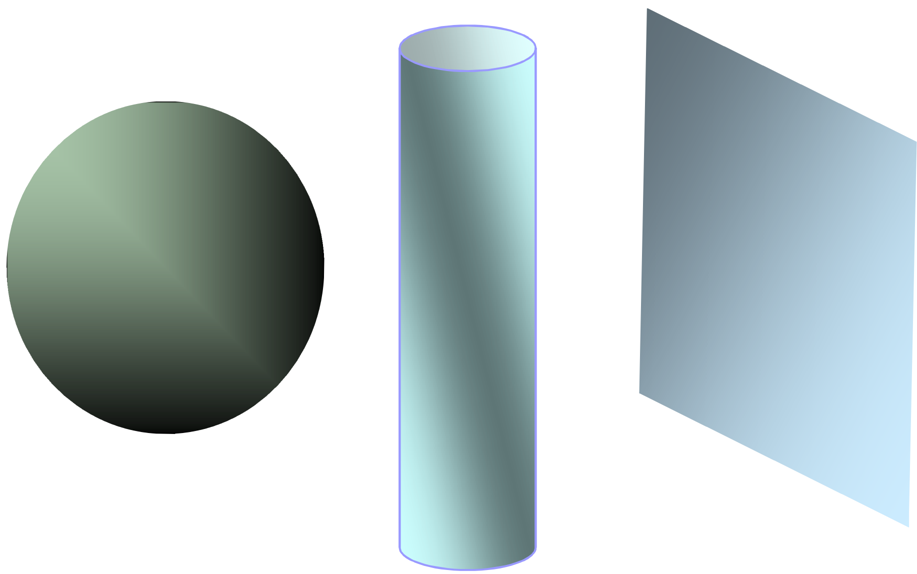
$$\oint_S \vec{E} \bullet d\vec{S} = \frac{1}{\varepsilon_0}\sum_{i=1}^{n} q_i = \frac{1}{\varepsilon_0}\int_V \rho dV$$

It is significantly useful for

——solution to E Intensity in symmetrical cases.

Symmetrical system:

1. **Spherical symmetrical**
2. **Cylindrical symmetrical**
3. **Surface symmetrical**



4. ☆ Electrostatic Gauss's Law（高斯定理）

When the charge distribution is symmetrical,

——Try *E-Gauss's Law*!

$$\oint_S \vec{E} \bullet d\vec{S} = \frac{1}{\varepsilon_0}\sum_{i=1}^{n} q_i = \frac{1}{\varepsilon_0}\int_V \rho dV$$

The tip of *E-Gauss's Law*:

(1) Find a closed surface $(\vec{S})$

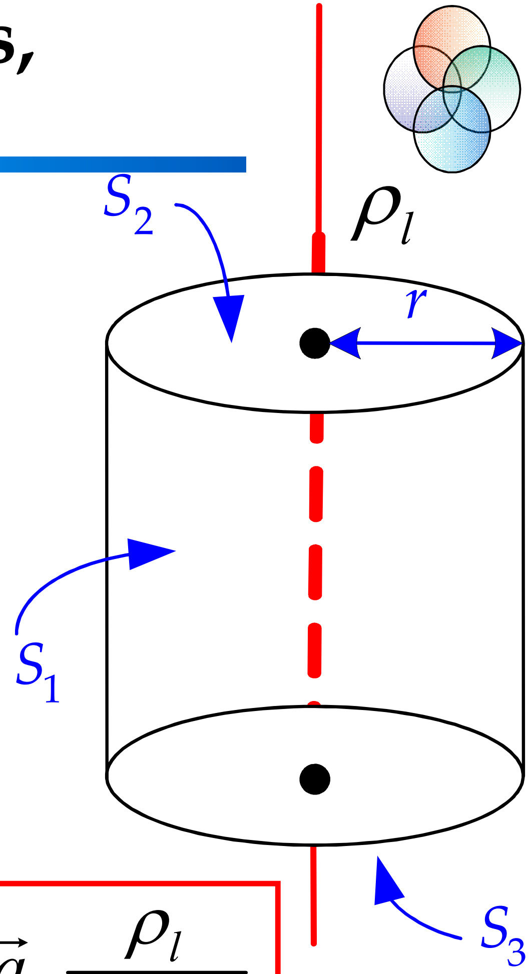
(2) The quantity of $\vec{E}$ *on the surface is constant.*

(^_^)

Example --->>> in next page

Example : Infinite Line Charges, determine E

- Axial Symmetry——construct a cylindrical surface, in height l, with line charges as the axis, and r as the radius.



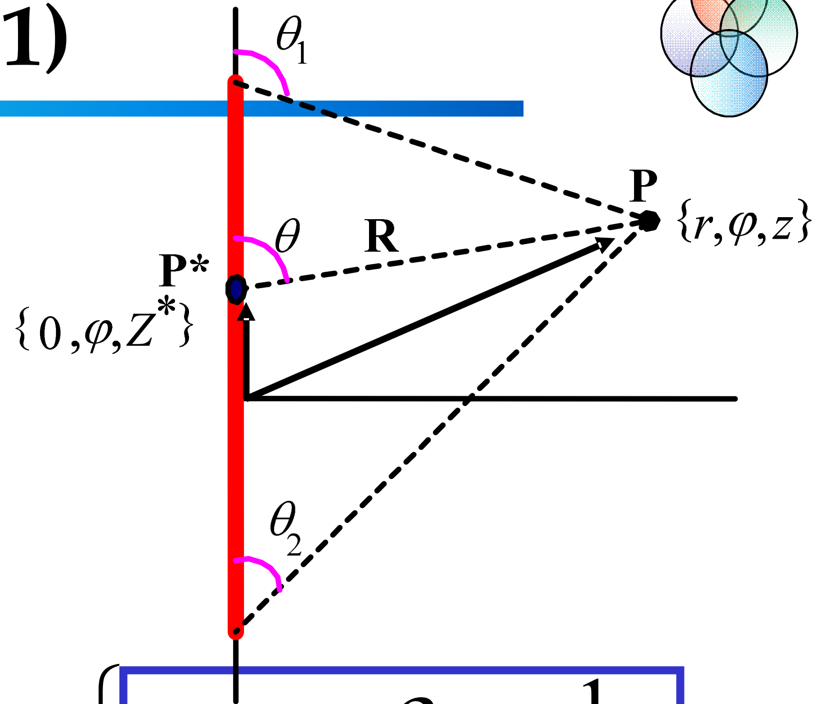
$$\oint_S \vec{E} \bullet d\vec{S} = \int_{S1} \vec{E} \bullet d\vec{S} + \int_{S2} \vec{E} \bullet d\vec{S} + \int_{S3} \vec{E} \bullet d\vec{S} = \frac{\rho_l \cdot l}{\varepsilon_0}$$

Since the E field has only radial component,

$$\therefore \quad \oint_S \vec{E} \bullet d\vec{S} = \int_{S1} \vec{E} \bullet d\vec{S} + 0 + 0$$

$$= 2\pi r l E_r = \frac{\rho_l \cdot l}{\varepsilon_0} \quad \therefore \boxed{\vec{E} = \vec{a}_r E_r = \vec{a}_r \frac{\rho_l}{2\pi r \varepsilon_0}}$$

Discussion (1)



- For infinite line charges, or *L>>R*

$$\vec{E} = \{E_r, 0, E_z\}$$

$$E_r = \frac{\rho_l}{4\pi\varepsilon_0 r}\cdot(\cos\theta_2 - \cos\theta_1)$$

$$E_z = \frac{\rho_l}{4\pi\varepsilon_0 r}\cdot(\sin\theta_1 - \sin\theta_2)$$

$$\because l \to \infty \qquad \therefore \begin{cases}\theta_1 \to \pi \\ \theta_2 \to 0\end{cases} \Rightarrow \begin{cases}\boxed{E_r = \dfrac{\rho_l}{2\pi\varepsilon_0}\cdot\dfrac{1}{r}} \\ E_z \equiv 0\end{cases}$$

The E Intensity exists only in radial direction.

5. ☆ Divergence equation （散度方程）

Divergence equation

$$\int_V (\nabla \bullet \vec{A}) dv = \oint_S \vec{A} \bullet d\vec{s}$$

Gauss's Law of **E**

$$\oiint_S \vec{E} \cdot \mathrm{d}\vec{S} = \frac{1}{\varepsilon_0} \sum_{i=1}^{n} q_i$$

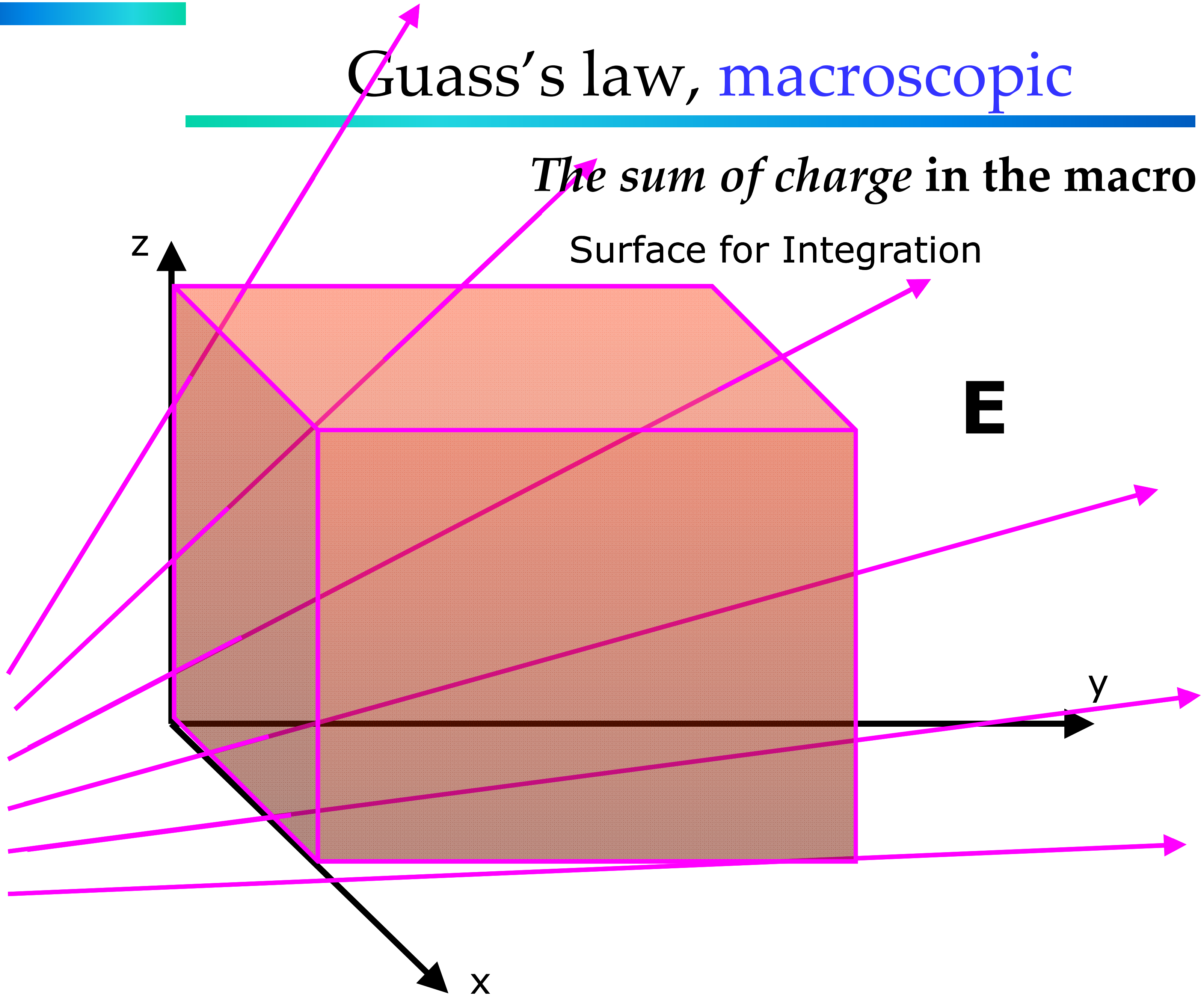
$$\int_V \nabla \bullet \vec{E} dv = \frac{1}{\varepsilon_0} \sum_{i=1}^{n} q_i = \frac{1}{\varepsilon_0} \int_V \rho \, dv$$

$$\nabla \bullet \vec{E} = \rho / \varepsilon_0$$

Please note: ρ here refers to volume density of all charges.

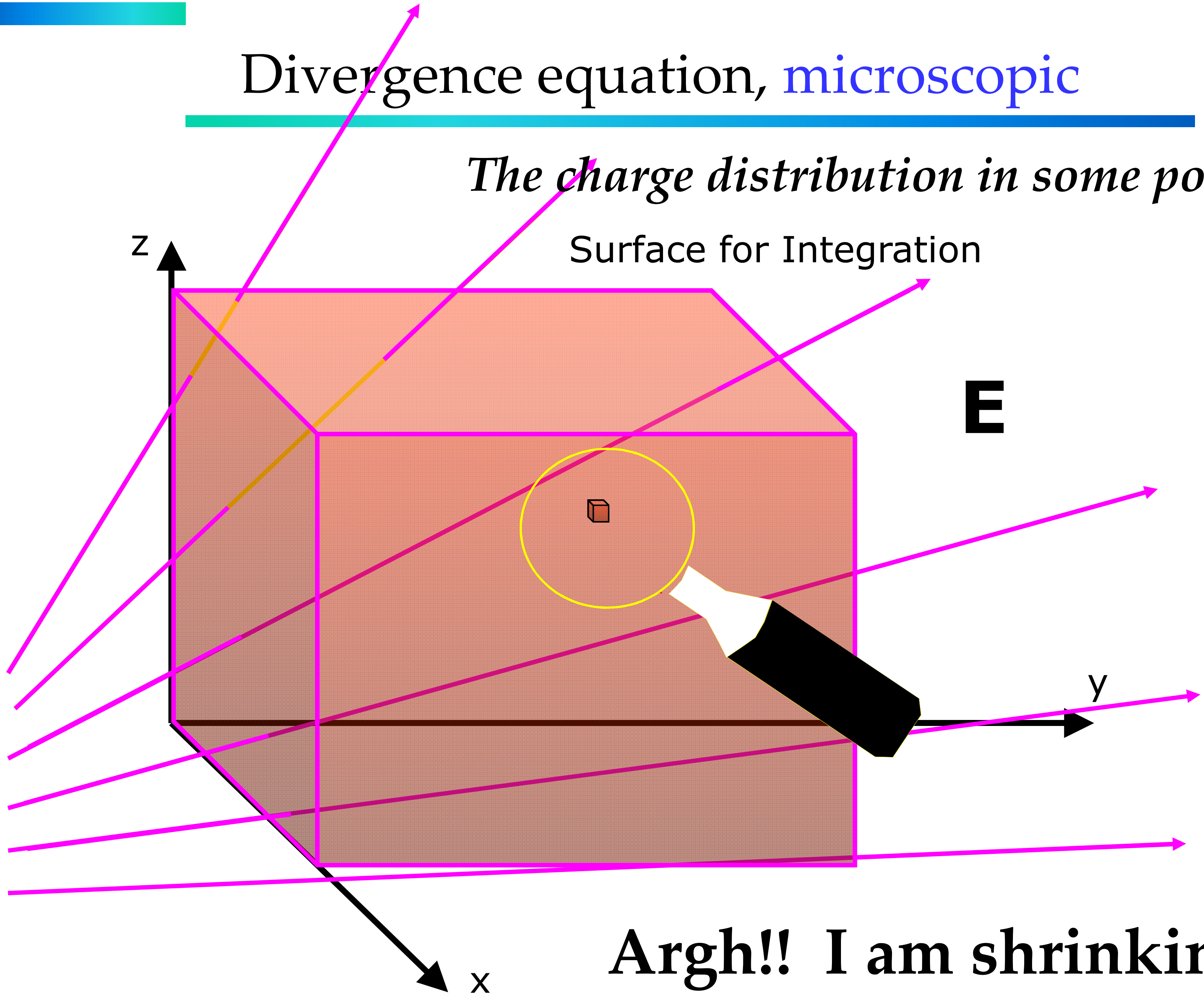
Guass's law, macroscopic

The sum of charge **in the macro region**



Divergence equation, microscopic

The charge distribution in some point



Argh!! I am shrinking!!!

☆ Solve the charge distribution with divergence eq.

$$\nabla \bullet \vec{E} = \rho / \varepsilon_0$$

- E-intensity in space is known as follows. Please determine the charge distribution.

$$\vec{\boldsymbol{E}} = \vec{a}_r E_0 (r/a)^2 \qquad 0 < r < a$$

$$\vec{\boldsymbol{E}} = \vec{a}_r E_0 (a/r)^2 \qquad r > a$$

- Analysis:
  - Due to spherical symmetry, E has only radial component;
  - Apply div equ in differential form;

☆ Solve the charge distribution with divergence eq.

$$\vec{\boldsymbol{E}} = \vec{a}_r E_0 (r/a)^2 \qquad 0 < r < a$$

Spherical Coordinates

$$\vec{\boldsymbol{E}} = \vec{a}_r E_0 (a/r)^2 \qquad r > a$$

$$\boxed{\nabla \bullet \vec{A} = \frac{1}{R^2} \cdot \frac{\partial}{\partial R}(R^2 \cdot A_R) + \frac{1}{R \cdot \sin\theta} \cdot \frac{\partial}{\partial \theta}(A_\theta \cdot \sin\theta) + \frac{1}{R \cdot \sin\theta} \cdot \frac{\partial A_\varphi}{\partial \varphi}}$$

$$\nabla \cdot \boldsymbol{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \begin{cases} \dfrac{4rE_0}{a^2} & 0 < r < a \\ 0 & r > a \end{cases}$$

$$\boxed{\begin{array}{ll} \rho = \varepsilon_0 \nabla \cdot \boldsymbol{E} = \dfrac{4\varepsilon_0 E_0 r}{a^2} & 0 < r < a \\ \rho = 0 & r > a \end{array}}$$

(^_^)

--------------->>> in next page

☆ Summary

1. Coulomb's laws

$$\vec{F}_{12} = \frac{q_1 \cdot q_2}{4\pi\varepsilon_0 \cdot R^2}\vec{a}_R$$

2. Superposition laws

3. Electric field intensity

Guass' law

$$\oint_S \vec{E} \bullet d\vec{S} = \frac{1}{\varepsilon_0}\sum_{i=1}^{n} q_i$$

Integral form

Div Equ:

$$\nabla \bullet \vec{E} = \rho / \varepsilon_0$$

Differential form

Review Guass's Law and the Div Equation

$$\oiint_S \vec{E} \bullet d\vec{S} = \frac{1}{\varepsilon_0} \sum_{i=1}^{n} q_i \qquad \nabla \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$$

Integral form — Differential form

- **Physical Meaning:**
  - describing the **scattering character** of static E field
  - **For integral equation:**
    - E-flux through any closed surface *S* = charges within *S*
    - Flux Source of Static E Field is Charges.
  - **For differential equation:**
    - Electrostatics Div = Volume density of *Q* at that point
    - Div Source of Static E Field is Volume density of Charges.

Please note this tip.

When the charge distribution is symmetrical,
——Try *E-Gauss's Law*!

Kernel of *E-Gauss's Law*:

(1) Find a closed surface $(\vec{S})$

(2) The quantity of $\vec{E}$ *on the surface is constant.*

$$\oint_S \vec{E} \bullet d\vec{S} = \frac{1}{\varepsilon_0} \sum_{i=1}^{n} q_i$$

Homework-Guru

- **Exercises: 3.4, 3.7, 3.8 , 3.10**
- **Problems: 3.21**