Rank, Kernel and Nullspace of a Matrix

Andy Watson

Kernel of a Matrix

A set of linear simultaneous equations can be represented by a matrix equation

For example

$$
\begin{aligned}
a_1x_1 + a_2x_2 + a_3x_3 &= b_1 \\
a_4x_1 + a_5x_2 + a_6x_3 &= b_2 \\
a_7x_1 + a_8x_2 + a_9x_3 &= b_3
\end{aligned}
$$

Can be written as [A][x] = [b]

The Kernel of matrix A is the solution of [A][x] = [0]

Kernel = Nullspace

The kernel is called the nullspace of [A]

If A is an m x n matrix (m rows x n columns) then:

Dimension(ker[A]) + rank[A] = n

This is called the “rank-nullity theorem”

Basis for Nullspace

For example, in the linear algebra tutorial, [A] is a 2 x 3 matrix,

[A] = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and the solution of [A][x] = [0] is [x] = $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

So the kernel (nullspace) is a one-dimensional space whose basis is the single vector [x] = $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Now, dimension(ker[A]) + rank[A] = n. In this example, 1 + 2 = 3.

Dim(ker[A]) = n – r (where, for [A], n is the number of columns and r is the rank).

Null Space; Orthogonal Subspaces

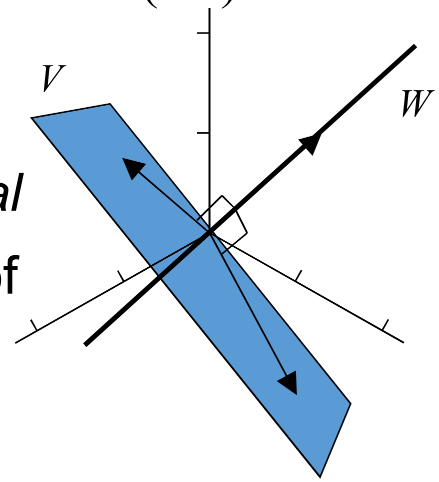
The *nullspace* $N(\mathbf{A})$ of $\mathbf{A}$ is the space *not* spanned by the rows of $\mathbf{A}$. This has dimension $n-r$.

Two subspaces V and W are orthogonal if every vector $\mathbf{v}$ in V is orthogonal to every vector $\mathbf{w}$ in W.

I.e. we must have $\mathbf{v}^t\mathbf{w}=0$ for all $\mathbf{v}\in V$, $\mathbf{w}\in W$.

So, the nullspace $N(\mathbf{A})$ and row space $R(\mathbf{A}^t)$ are *orthogonal*.

Example: 2-d subspace V (plane)
is orthogonal to 1-d subspace W (line)



In the diagram, W is the *orthogonal complement* $V^{\perp}$ of V (the space of all vectors orthogonal to V).

Note:

- We could apply row reduction (for example by Gaussian elimination or Gauss-Jordan elimination, giving the reduced row echelon form or an upper triangular matrix).
- This does not change the kernel of the matrix or the rank of the matrix.
- The upper triangular matrix gives the rank of the matrix as the number of nonzero rows.

Example

If matrix [A] = $\begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix}$,

i) Calculate the basis for the nullspace of [A]
ii) Show that the basis for the nullspace of [A] is orthogonal to the row vectors of [A].

Example answer

Basis for nullspace of [A] = $\begin{bmatrix} -1 \\ -26 \\ 16 \end{bmatrix}$

[2 3 5]. $\begin{bmatrix} -1 \\ -26 \\ 16 \end{bmatrix}$ = 0

[-4 2 3]. $\begin{bmatrix} -1 \\ -26 \\ 16 \end{bmatrix}$ = 0

Rank of [A] = 2, nullity of [A] = 1, dimension of [A] = 3 columns

2 + 1 = 3