Advanced Transform Methods

Short-Time Fourier Transform

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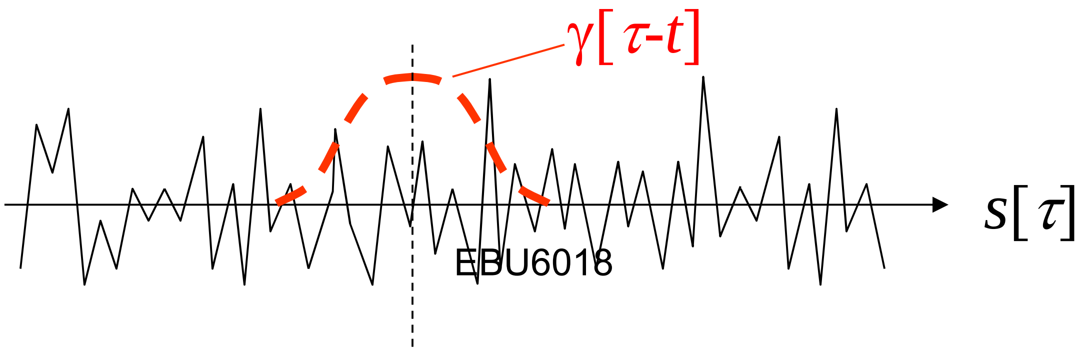
Limitations of Fourier Transform

- The basis functions (complex sinusoids) are spread over the entire time domain.
- Loses all time information since integration is performed over all times.
- Difficult to discriminate signals that have the same frequency but occuring at different times.
- Signals suitable for Fourier transform are time invariant.
- Applications suitable for the Fourier transform are those which concern frequency only.

Short-Time Fourier Transform

- Often desirable to have an estimate of the input signal spectrum for a short interval, especially for nonstationary signals. Want to see changes in spectrum with time.
- Consider finding a spectral "snapshot" by calculating the Fourier transform of a short interval of the signal
- If we assume the signal is stationary in certain time slots (window) (quasi-stationary), we can perform a Fourier transform on this part of signal and obtain frequency information as well as time information.
- This is the "Short Time Fourier Transform"



Features of STFT

- Unlike FT, STFT will give the information of frequency at different time intervals.
- How could we choose the width of window function?
- One can’t know what frequency components exist at what instances of times. What one can know are the time intervals in which certain band of frequencies exist.
- Uncertainty Principle still holds- The product of time resolution and frequency resolution is always greater than a minimum value.
- Narrow window means good time resolution but poor frequency resolution and wide window means poor time resolution but good frequency resolution.
- For a given window, time resolution is fixed.

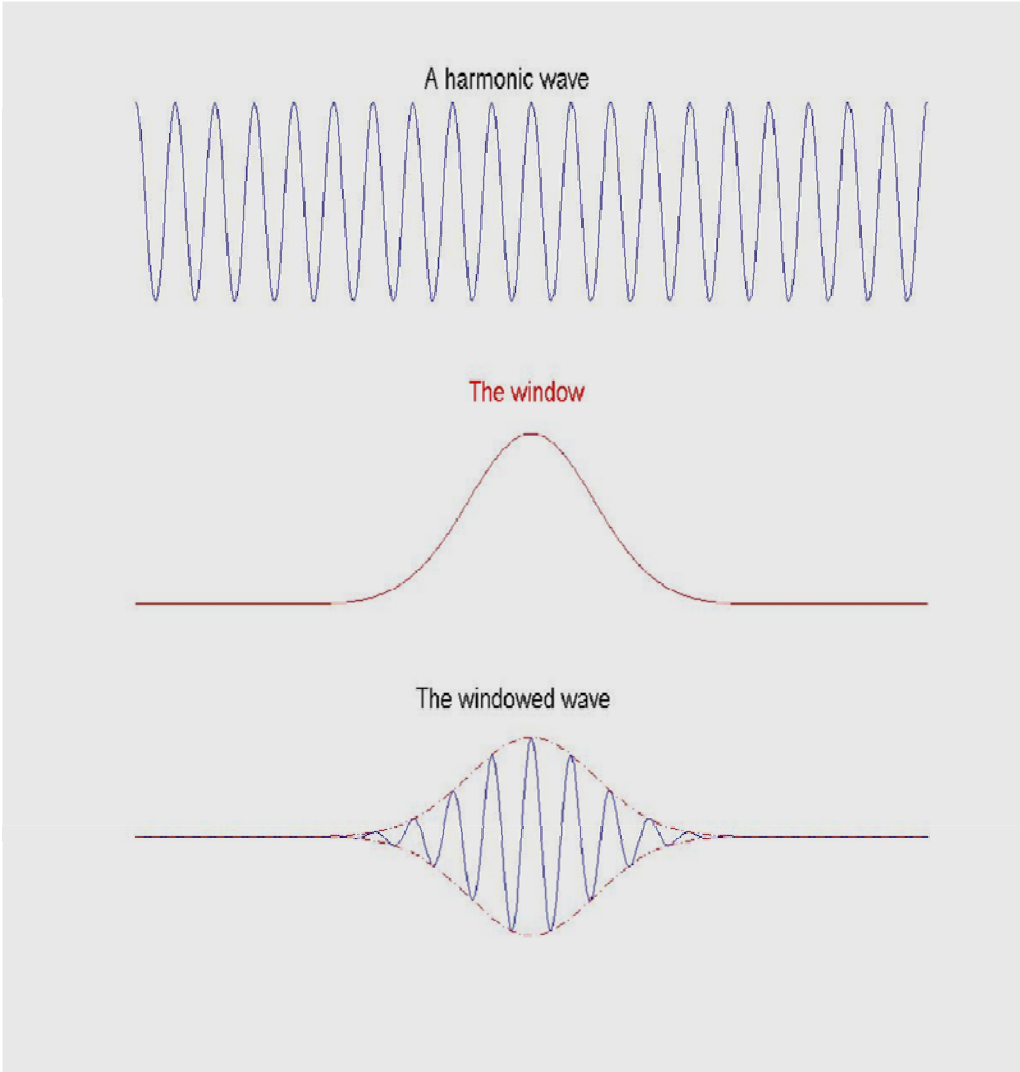
The Windowed Fourier Transform

- **Harmonic wave e^{jwt}**

- **A window $\gamma(t)$**

- **A windowed wave**

  $\gamma(\tau\text{-}t)\, e^{j\omega\tau}$



Time-Frequency Window

Windowing a function $s(\tau)$ near $\tau = t$:

$$s_b(\tau) \equiv s(\tau)\gamma^*(\tau - t)$$

where $\gamma(t)$ is a suitable time-window, such as $\gamma(t) = \chi_{[0,1)}(t)$
with

$$\chi_{[0,1)}(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t < 0, t > 1 \end{cases}$$

$s_b(\tau)$ **contains the information of the original function $s(\tau)$ within the time-window**

$$[t + \tau_0 - \Delta_\tau / 2, t + \tau_0 + \Delta_\tau / 2]$$

where τ_0 is the center and Δ_τ is the time duration of the window function $\gamma(\tau)$.

Time-Frequency Window

In general, for a windowing function $\gamma(t)$

Center $$\langle t \rangle_{\gamma} \equiv \frac{1}{\|\gamma\|^2} \int_{-\infty}^{\infty} t\,|\gamma(t)|^2\, dt$$

Radius $$\Delta_{\gamma} \equiv \frac{1}{\|\gamma\|} \left[\int_{-\infty}^{\infty} (t - \langle t \rangle_{\gamma})^2\, |\gamma(t)|^2\, dt \right]^{1/2}$$

Width $$= 2\Delta_{\gamma}$$

$\|W\|$ **is the norm of $W(t)$ defined as**

$$\|W(t)\|^2 = \langle W, W \rangle = \int_{-\infty}^{\infty} |W(t)|^2\, dt$$

Short-Time Fourier Transform (STFT)

The Short-time Fourier transform $STFT(t,\omega)$ of a function $s(t)$ with respect to a window function γ evaluated at a point (t,ω) in the t-ω plane is defined as

$$STFT(t,\omega) = \int_{-\infty}^{\infty} s(\tau)\gamma^*(\tau - t)e^{-j\omega t}d\tau = \int_{-\infty}^{\infty} s(\tau)\gamma_{t,\omega}{}^*(\tau)d\tau$$

$$s(t)\gamma^*(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} STFT_\gamma(\tau - t,\omega)e^{j\omega\tau}d\omega$$

The STFT $STFT(t_0,\omega_0)$ provides *local spectral information* of the function $s(t)$ around the point t_0. More precisely, it offers information in the time-frequency window

$$[t + t_0 - \Delta_t/2, t + t_0 + \Delta_t/2] \times [\omega + \omega_0 - \Delta_\omega/2, \omega + \omega_0 + \Delta_\omega/2]$$

That is, it characterises the signal's behaviour in the vicinity of t_0, ω_0 in the time frequency domain.

Short Time Fourier transform

- A window function $\gamma(t)$ is introduced.
- The transform is

$$STFT(t,\omega) = \int_{-\infty}^{\infty} s(\tau)\gamma^*(\tau - t)e^{-j\omega t}d\tau$$

- Compare with the Fourier transform $S(\omega) = \int_{-\infty}^{\infty} s(t)\,\mathrm{e}^{-j\omega t}\,dt$

- For each time *t*, there is a Fourier transform. We obtain a time-frequency representation of the signal.
- Typically, one computes and plots the **Spectrogram**

$$\left|STFT(t,\omega)\right|^2$$

Spectrogram Example