SOLUTIONS

Module:	Advanced Transform Methods		
Module Code	EBU718U	Paper	C
Time allowed	2hrs	Filename	Solutions_1617_EBU718U_C
Rubric	ANSWER ALL FOUR QUESTIONS		
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Question 1.

a) Fourier Transforms are used to transform functions of time to functions of frequency.

i) Explain the type of signal for which the Fourier Transform is suitable.

ii) What is the limitation of this type of signal?

iii) State the necessary condition for a signal to have a Fourier Transform. Are signals that do not satisfy this condition suitable for use as basis functions?

[7 marks]

Answer:

i) Because the Fourier Transform gives the frequencies in a signal but not the time at which these frequencies occur [1 mark], the FT is suitable for Stationary Signals [1 mark], ie signals that have the same frequency content at all times [1 mark].

ii) Stationary signals contain a limited amount of information and so are of limited use [1 mark].

iii) To have a Fourier Transform in the ordinary sense requires a signal to be absolutely integrable, that is, to be asymptotic to zero in both directions [1 mark]

[1 mark]

Signals that do not satisfy this condition are generally not suitable for use as basis functions. [1 mark]

b) i) If a vector space V has basis functions $\{\psi_n\}$ give an expression showing how any vector s can be represented in terms of $\{\psi_n\}$.

ii) State the advantage of $\{\psi_n\}$ being orthonormal, and hence show how the expression for vector s can be rewritten.

[5 marks]

Answer: i) $s = \sum_n c_n \Psi_n$ [1 mark]

ii) The advantage of $\{\psi_n\}$ being orthonormal is that it simplifies the calculation of the coefficients c_n [1 mark].

$$c_n = \langle s, \Psi_n \rangle$$ [1 mark]

s can therefore be written as:

$$s = \sum_j \langle s, \Psi_j \rangle \Psi_j$$
$$= \langle s, \Psi_1 \rangle \Psi_1 + \langle s, \Psi_2 \rangle \Psi_2 + \ldots.. + \langle s, \Psi_n \rangle \Psi_n$$ [2 marks]

c) i) Explain what is meant by the term "Biorthogonal Bases".

ii) Dual bases are biorthogonal. Show that the following two bases are dual bases:

$$\{\psi_n\} = \{(2,0)\ ,\ (1,2)\}$$

$$\{\hat{\Psi}_n\} = \{(0.5,-0.25)\ ,\ (0,0.5)\}$$

iii) In the context of basis functions, explain what is meant by the term "Frame".

[13 marks]

Answer: i) Biorthogonal bases are a pair of bases [1 mark], as follows:

If $\{\psi_n\}$ and $\{\hat{\Psi}_n\}$ are both basis vectors themselves for V [1 mark], and satisfy [1 mark]

$$\langle \Psi_i, \hat{\Psi}_j \rangle = \delta_{ij}$$

then any **s** in V can be written as $s = \sum_{j=1}^{n} \langle s, \Psi_j \rangle \hat{\Psi}_j$ [1 mark]

ii)

$$\{\psi_n\} = \{(2,0)\ ,\ (1,2)\}$$

$$\{\hat{\psi}_n\} = \{(0.5,\ -0.25),\ (0,\ 0.5)\}$$

THEN $\langle \psi_1, \hat{\psi}_1 \rangle = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.25 \end{bmatrix} = \underline{\underline{1}}$ [1 MARK]

$\langle \psi_2, \hat{\psi}_2 \rangle = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} = \underline{\underline{1}}$ [1 MARK]

$\langle \psi_1, \hat{\psi}_2 \rangle = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} = \underline{\underline{0}}$ [1 MARK]

$\langle \psi_2, \hat{\psi}_1 \rangle = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.25 \end{bmatrix} = \underline{\underline{0}}$ [1 MARK]

FOR BIORTHOGONALITY, $\langle \psi_i, \hat{\psi}_j \rangle = \sum_k \psi_i(k)\, \hat{\psi}_j(k) = \delta_{ij}$

[1 MARK]

δ_{jk} = KRONECKER DELTA

$\therefore \{\psi_n\}$ AND $\{\hat{\psi}_n\}$ ARE DUAL BASES. [1 MARK]

iii) a frame is a set of vectors in vector space V that contains more vectors than the order of the space [1 mark], and that are not orthogonal or linearly independent [1 mark]. The frame vectors can be used to represent any other vector in the space [1 mark].

Question 2

a) The Discrete Fourier Transform (DFT) is used to obtain the frequency spectrum of a sampled signal. The input to the DFT must be a sequence of finite length.

i) Comment on the effect that the length of the sequence has on the frequency resolution of the transform.

ii) Explain what is meant by "spectral leakage", and state three ways of eliminating or reducing it.

[8 marks]

b) The definition of the Discrete Fourier Transform (DFT) is:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

Obtain the DFT of a 2 point input and represent it in signal flowgraph form.

Why does the FFT become more efficient for larger length of input sequence?

[9 marks]

c) The Karhunen-Loeve Transform (KLT) is used to decorrelate multi-dimensional input data and allow compression to be carried out with minimum error. It uses Principal Component Analysis (PCA) to do this. List the steps to perform a KLT.

[8 marks]

a) Answer:
i) The longer the sequence length the better the resolution [1 mark]. For example, suppose we have a fixed sample rate then the longer the sequence length the longer it takes for the DFT to run [1

mark]. A 1-second FT can give a resolution of 1 Hz whereas a 100ms FT can only resolve 10 Hz [1 mark].

ii) If we sample a single sinewave and the sequence contains samples from an integer number of cycles of the signal, then the imaginary part of the DFT will have zero value at all points except for the frequency of the input signal, and the real part will be all zero. [1 mark]. If the samples are not of an integer number of input cycles then all output values will be non-zero [1 mark], ie, the spectral energy is smeared across all the DFT output values.

To eliminate or reduce spectral leakage:

1. Synchronise the sample frequency to be an integer multiple of the sinewave frequency. [1 mark]
2. Increase the size of the input buffer [1 mark]
3. Apply a data window to the DFT input. [1 mark]

b)

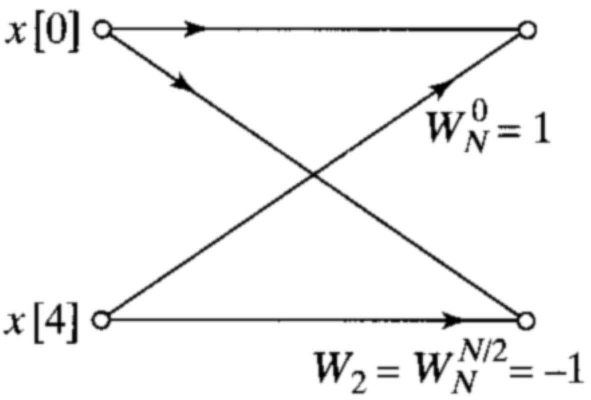
For N=2 we get

$$X[k] = \sum_{n=0}^{1} x[n] W_2^{kn} = x[0] W_2^{k0} + x[1] W_2^{k1}$$
$$= x[0] + (-1)^k x[1]$$

[2 marks]

This could be written out as $X[0] = x[0] + x[1]$ and $X[1] = x[0] - x[1]$. [2 marks]

For the signal flow graph we get the “basic butterfly”:



[3 marks]

The FFT “decimates” the input sequence in powers of 2, therefore the number of multiplications for an N-point sequence is proportional to $\log_2 N$ [2 marks].

c)

Find the mean vector for the input data
Find covariance matrix
Find the eigenvalues of the covariance matrix
Find the eigenvectors of the covariance matrix
Normalise the eigenvectors
Transform the input
Set the last row of the transformed data to zero
Invert the transform.
[8 marks: 1 for each row]

Question 3

a) i) Define the Short-time Fourier Transform (STFT) and use diagrams to explain how it is implemented.

ii) With the aid of a diagram, explain how the output from the STFT is displayed.

iii) The window length is fixed in the STFT. Explain the effect of this on the resolution.

[15 marks]

b) Wavelet transforms can be used for many applications. Briefly describe each of the following:

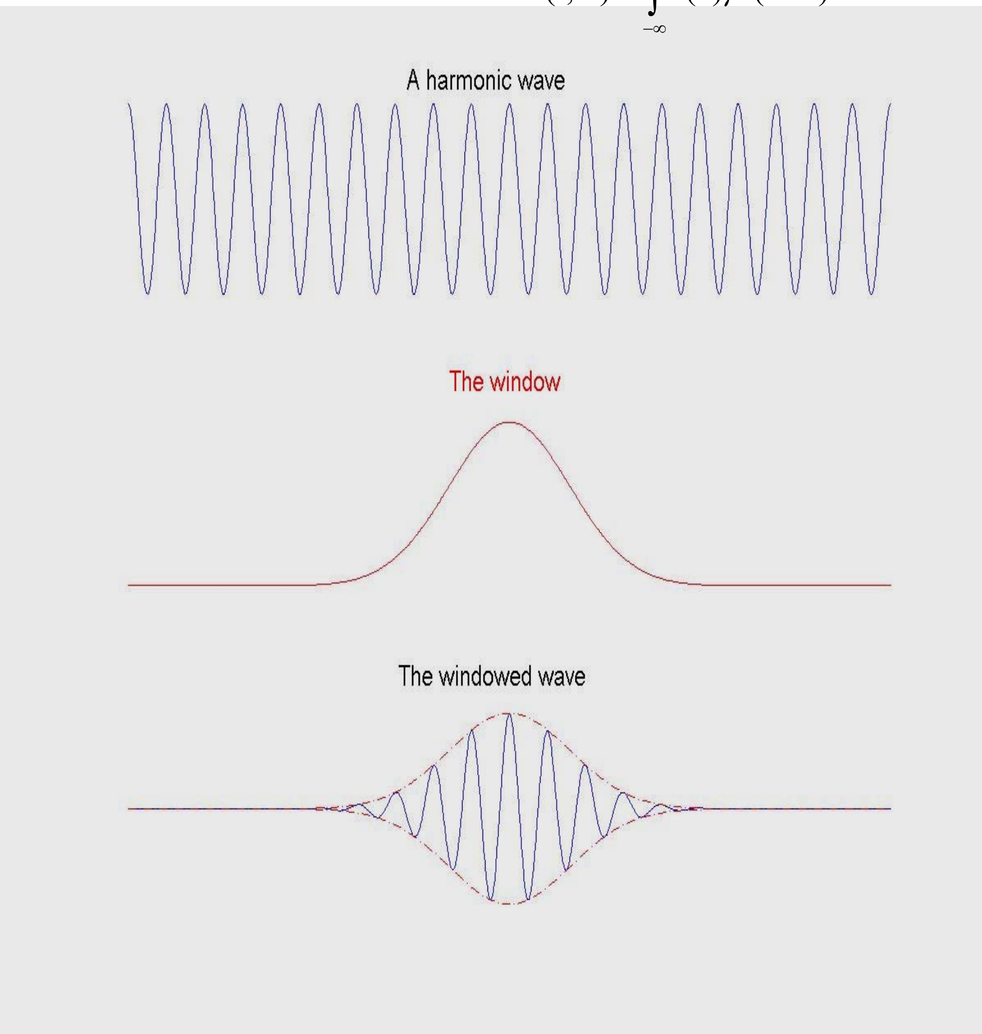
i) Resolving a signal into its component sinusoids
ii) De-noising
iii) Detecting discontinuities
iv) Detecting self-similarity
v) Compressing images.

[10 marks]

a) Answer

i) The STFT is defined as [1 mark]

$$STFT(t,\omega)=\int_{-\infty}^{\infty} s(\tau)\gamma^{*}(\tau-t)e^{-j\omega t}\,d\tau$$



[3 marks for the correct waveforms: 1 each]

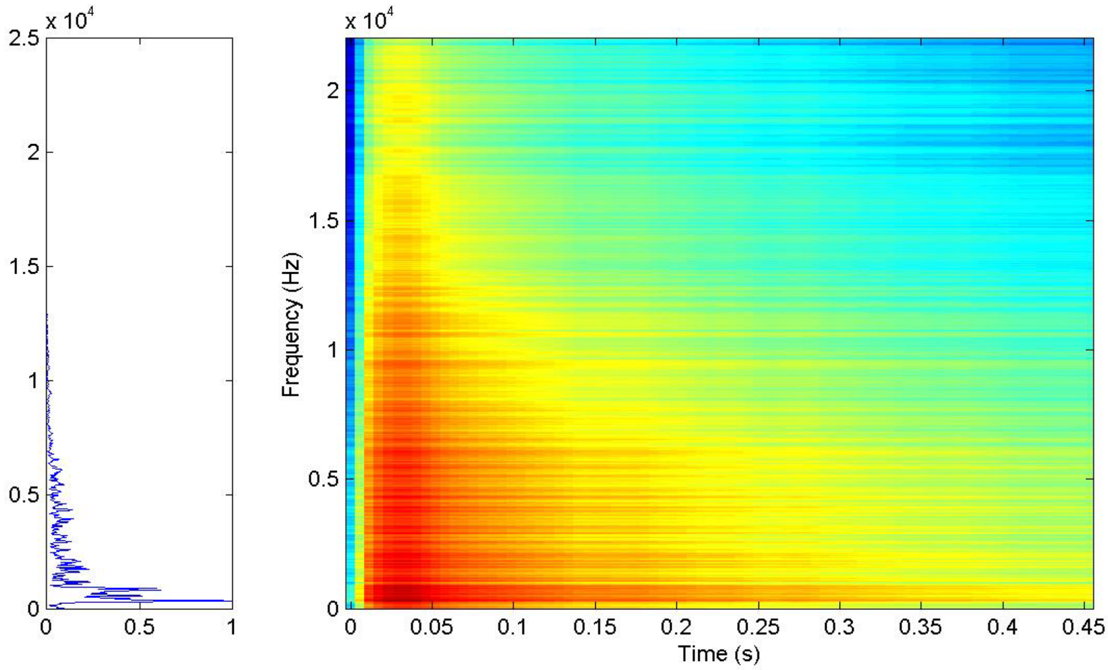
The harmonic wave in the diagram is the complex exponential of the FT [1 mark]

The window is the γ function in the definition [1 mark]. It is translated across the signal to be transformed [1 mark]. The window can be any shape, but careful choice of shape minimises problems and reduces redundancy [1 mark].

s(t) is the signal to be transformed [1 mark]

The STFT is the integral of the product of these waveforms [1 mark].

ii) The output from the STFT is displayed on a spectrogram [1 mark]. This is a 3D plot of signal energy against time and frequency [1 mark]. (Note: students are NOT required to sketch the 3D plot.)



iii) The Heisenberg Uncertainty Principle states that the product of uncertainty in time and uncertainty in frequency is a constant [1 mark]. Because the window length is fixed the uncertainty in time is constant and therefore the uncertainty in frequency is also fixed [1 mark]. So the resolution is constant for all time and all frequency [1 mark].

b) i) Resolving a signal into its frequency components: multiresolution analysis (MRA) separates a waveform into its course and fine detail till only the average value is left [1 mark]. The detail that has been separated out are the frequencies in the signal with the highest ones being separated out first [1 mark]

De-noising: if all the noise is high frequency it can be removed by compression or by discarding the first levels of detail in the MRA [1 mark]. However if the noise is broadband or has different components at different frequencies then denoising can be carried out by identifying the features in the signal that we are looking for and ignoring the rest. This can be achieved by using a wavelet function of the same shape as the features t be identified [1 mark]

Detecting discontinuities: if the discontinuity is between bursts of different frequencies, then at some level of decomposition the discontinuity will be obvious because the higher frequency will disappear from the coarse level [1 mark]. It will also detect an abrupt change in the rate of change [1 mark].

Detecting self-similarity: wavelets can be used to detect fractal signals (ie similar to itself at different scales) then the wavelet coefficients will also be similar at different scales [1 mark]. This generates a set of characteristic lines on the scalogram [1 mark]

Compression of images: This is carried out by performing the MRA and then setting a threshold value [1 mark]. values lower than the threshold value are discarded resulting in compression [1 mark].

Question 4

a) The Short-time Fourier Transform (STFT) and Wavelet Transform (WT) obtain signal information in both time and frequency by correlating the signal with a set of basis functions.

The Wigner-Ville Distribution (WVD) is an instantaneous autocorrelation function that obtains the signal's energy density in time and frequency.

i) State the advantage of the WVD over both the STFT and WT and explain its disadvantage. Illustrate this disadvantage by showing the WVD spectrogram for a signal that comprises two pulses separated in time.

ii) Briefly discuss how the disadvantage of the WVD can be addressed.

[11 marks]

b) Briefly explain the Heisenberg Uncertainty Principle and discuss its implications in the design of transforms for time-frequency analysis.

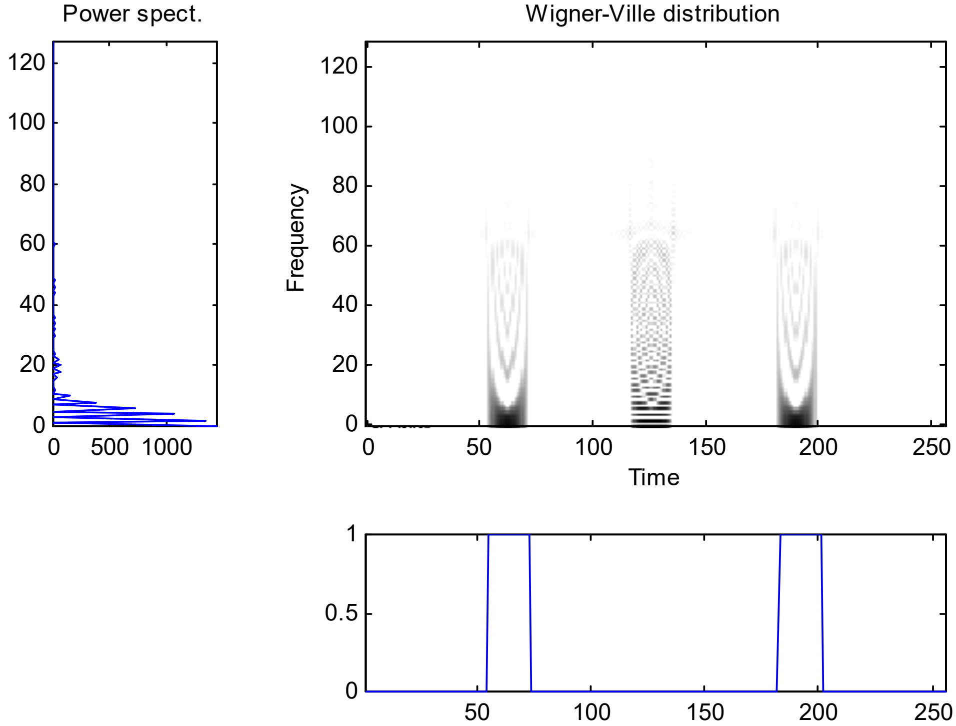
[10 marks]

c) The Short Time Fourier Transform and the Wavelet Transform are both used for time-frequency analysis. Compare the form of "window" functions that are used in these two transforms (refer to the definitions of the two transforms in your answer).

[4 marks]

Answer:
a) i) The time-frequency resolution of the WVD is better than that of both the STFT and WT [1 mark]. However, because the WVD is defined by a correlation function, if the signal is composite (ie the sum of separate signals) then in addition to the autocorrelation there will be cross-correlation

terms [1 mark]. This can be illustrated



The diagram (at bottom) shows two pulses, one at about 50ms and the other at about 200ms [1 mark].
The plot top left shows the signal's power density [1 mark].
The larger diagram shows the WVD spectrogram [1 mark]. Because the pulses are time-bounded the WVD is also time-bounded (a property of the WVD) [1 mark]. The spectrogram shows the cross-term midway between the two pulses. [1 mark]

ii) The crossterms can be filtered to remove them,[1 mark] but doing so degrades the resolution of the distribution [1 mark]. Because the cross-terms oscillate, they can be removed by low-pass filtering [1 mark]. So there is a trade-off between smoothing the cross-terms and reduced resolution [1 mark].

b) Simply, the Heisenberg Uncertainty Principle in the context of time-frequency analysis states that it is not possible to know accurately both the time location and frequency value at that time [1 mark]. There is an uncertainty in both time and frequency, and the product of the uncertainties is a constant [1 mark]. If Δ_t is the uncertainty in time and Δ_ω is the uncertainty in frequency, it can be shown that (although the value of the constant depends on the definition of the Fourier Transform that is used in its proof):

$\Delta_t \Delta_\omega \geq 1/2$ [1 mark]

In time-frequency analysis, we use a moving window to investigate the frequency content of a signal at different time locations [1 mark]. The longer the duration of the window the more accurately the frequency content is known but the less accurately the time location is known [1 mark]. The shorter the duration of the window the more accurately time location is known but the less accurately the frequency [1 mark]. The shape of the window function affects the accuracy of the results [1 mark]. We need a function that dies away quickly and also has a limited frequency content [1 mark]. It can be shown that the only function minimising the product of Δ_t and Δ_ω is a Gaussian function

$g(t) = e^{-at^2}$ [1 mark]

The Gabor STFT uses a Gaussian function as its window [1 mark].

c) In the STFT, the Fourier transform is carried out by multiplying the signal by a window function and a sinusoid (the complex exponential) [1 mark].

$$STFT(t,\omega) = \int_{-\infty}^{\infty} s(\tau)\gamma^*(\tau - t)e^{-j\omega t}d\tau$$ [1 mark]

In the wavelet transform, the sinusoid is not present

$$CWT(a,b) = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty} s(t)\psi^*\left(\frac{t-b}{a}\right)dt$$ [1 mark]

$$= \int_{-\infty}^{\infty} s(t)\psi_{a,b}{}^*(t)dt = < s, \psi_{a,b} >$$

This means that the wavelet function must be oscillatory [1 mark].