Advanced Transform Methods

Fourier Transform Properties

Andy Watson

Fourier Transform Properties

- Hermitian (Conjugate) Symmetry
- Even and Odd Symmetries
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- Symmetry
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- Convolution
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Integral of even and odd functions

$$I = \int_{-\infty}^{\infty} f_e(x) f_o(x) dx$$

$$= \int_{0}^{\infty} f_e(x) f_o(x) dx + \int_{-\infty}^{0} f_e(x) f_o(x) dx$$

$$= \int_{0}^{\infty} f_e(x) f_o(x) dx + \int_{-\infty}^{0} f_e(-x) f_o(-x) d(-x)$$

$$= \int_{0}^{\infty} [f_e(x) f_o(x) + f_e(-x) f_o(-x)] dx$$

$$= \int_{0}^{\infty} [f_e(x) f_o(x) - f_e(x) f_o(x)] dx = 0$$

The integral of the product of an even and an odd function is 0.

Fourier Transform: Hermitian Symmetry

Expand the Fourier transform of a function, $s(t)$:

$$S(\omega) = \int_{-\infty}^{\infty} [s_e(t) + s_o(t)]\ [\cos(\omega t) - j\sin(\omega t)]\, dt$$

$$= \int_{-\infty}^{\infty} s_e(t)\cos(\omega t)dt + \int_{-\infty}^{\infty} s_o(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} s_e(t)\sin(\omega t)dt - j\int_{-\infty}^{\infty} s_o(t)\sin(\omega t)dt$$

$$= \int_{-\infty}^{\infty} s_e(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} s_o(t)\sin(\omega t)dt$$

If even, fourier transform is real $S_e(\omega) = \mathrm{Re}[S_e(\omega)]$

If even, fourier transform is even $S_e(\omega) = S_e(-\omega)$

If odd, fourier transform is imaginary $S_o(\omega) = \mathrm{Im}[S_o(\omega)]$

If odd, fourier transform is odd $S_o(\omega) = -S_o(-\omega)$

Hermitian or Conjugate Symmetry:

$$S(\omega) = S_e(\omega) + S_o(\omega) = S_e(-\omega) - S_o(-\omega)$$

$$= [S_e(-\omega) + S_o(-\omega)]^* = S^*(-\omega)$$

Fourier Transform Even and Odd Symmetries

Expand the Fourier transform of a function, *s*(*t*):

$$S(\omega) \;=\; \int_{-\infty}^{\infty} [\mathrm{Re}\{s(t)\} + j\,\mathrm{Im}\{s(t)\}]\ [\cos(\omega t) - j\sin(\omega t)]\,dt$$

Expanding further:

=0 if Re or Im{*s*(*t*)} is odd ↓ =0 if Re or Im{*s*(*t*)} is even ↓

$$S(\omega) = \int_{-\infty}^{\infty} \mathrm{Re}\{s(t)\}\cos(\omega t)dt + \int_{-\infty}^{\infty} \mathrm{Im}\{s(t)\}\sin(\omega t)dt \qquad \leftarrow \mathrm{Re}\{S(\omega)\}$$

$$+\,j\int_{-\infty}^{\infty} \mathrm{Im}\{s(t)\}\cos(\omega t)dt - j\int_{-\infty}^{\infty} \mathrm{Re}\{s(t)\}\sin(\omega t)dt \qquad \leftarrow \mathrm{Im}\{S(\omega)\}$$

↑ Even functions of ω ↑ Odd functions of ω

If real and even, fourier transform is real and even $\int_{-\infty}^{\infty} \mathrm{Re}\{s(t)\}\cos(\omega t)dt$

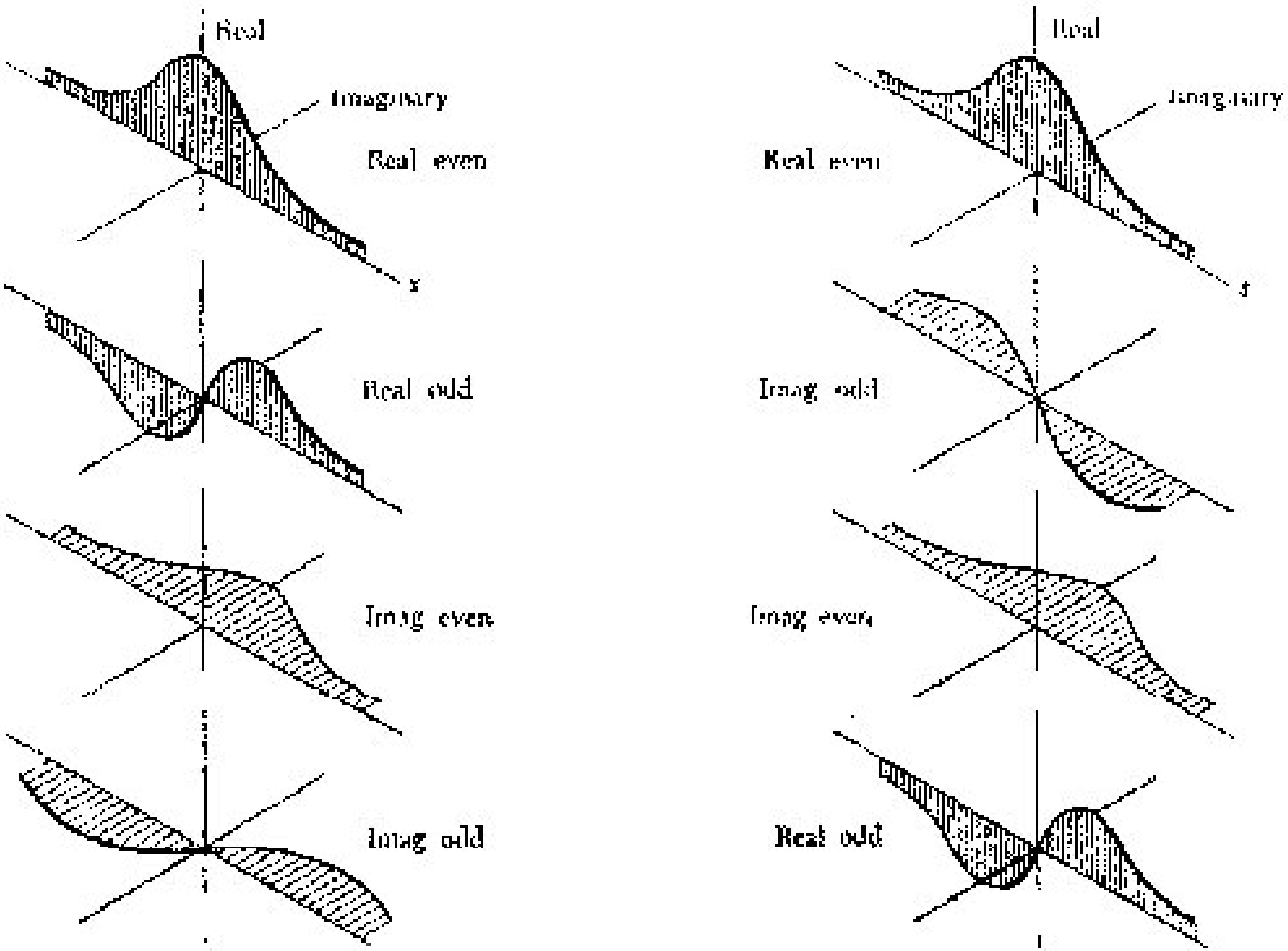
If real and odd, fourier transform is imaginary and odd $-j\int_{-\infty}^{\infty} \mathrm{Re}\{s(t)\}\sin(\omega t)dt$

If imaginary and even, fourier transform is imaginary and even $j\int_{-\infty}^{\infty} \mathrm{Im}\{s(t)\}\cos(\omega t)dt$

If imaginary and odd, fourier transform is real and odd $\int_{-\infty}^{\infty} \mathrm{Im}\{s(t)\}\sin(\omega t)dt$

Fourier Transform Symmetry Examples I

$s(t)$ $S(\omega)$



Linearity

If $s_1(t)$ has Fourier transform $S_1(w)$ and $s_2(t)$ has Fourier transform $S_2(w)$, then

$a \cdot s_1(t) + b \cdot s_2(t)$ has Fourier transform

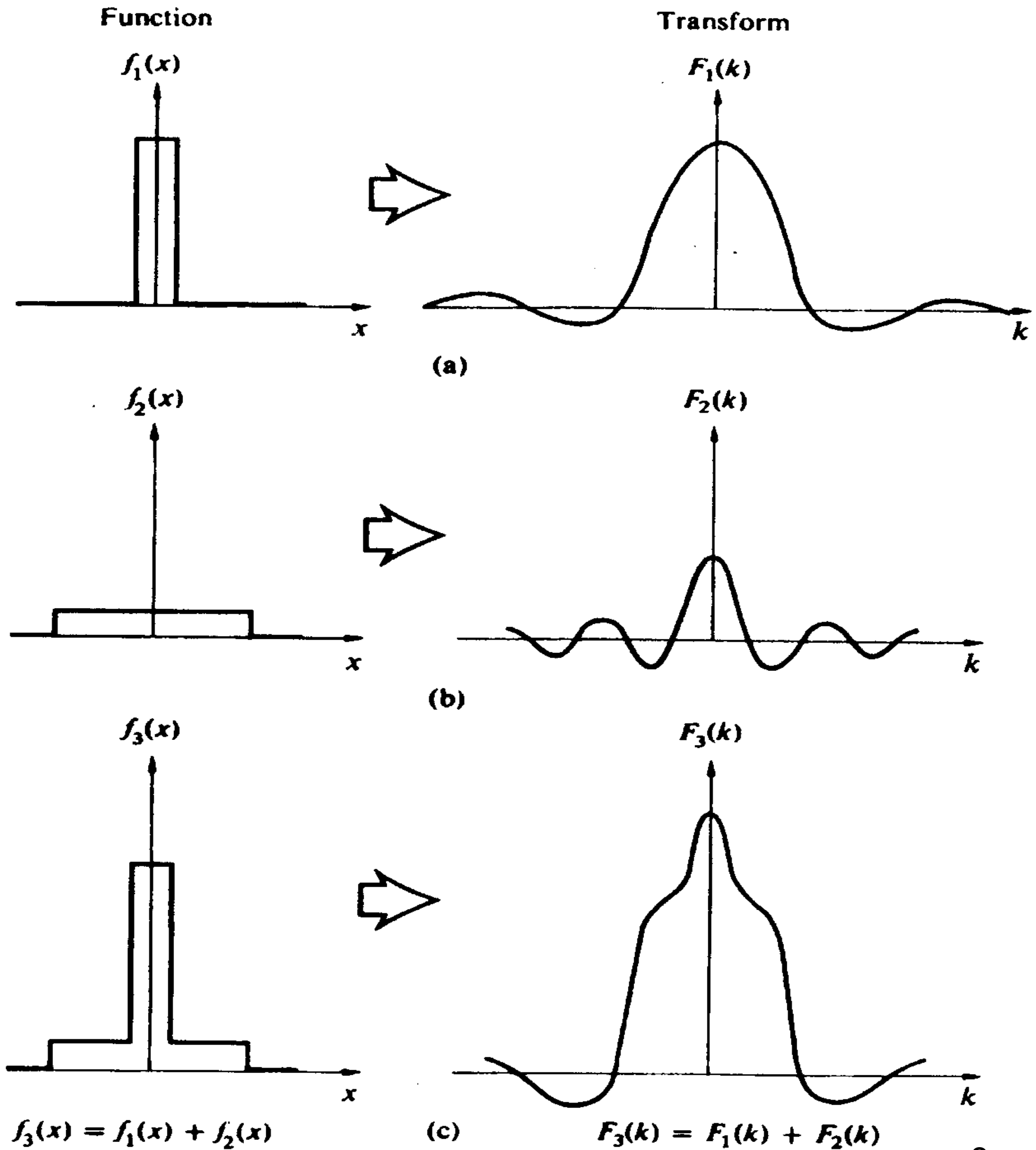
$$\int_{-\infty}^{\infty} [a \cdot s_1(t) + b \cdot s_2(t)] e^{-j\omega t} dt$$

$$= a \int_{-\infty}^{\infty} s_1(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} s_2(t) e^{-j\omega t} dt$$

$$= a \cdot S_1(\omega) + b \cdot S_2(\omega)$$

Similarly for the inverse Fourier transform

The Fourier Transform of a sum of two functions is the sum of the Fourier Transforms of each function



Similarity

If $s(t)$ has Fourier transform $S(w)$ then $s(a \cdot t)$ has Fourier transform $\dfrac{S(\omega / a)}{|a|}$

$a > 0 \rightarrow$

$$\int_{-\infty}^{\infty} s(a \cdot t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} s(x) e^{-j\omega x/a} d(x/a)$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} s(x) e^{-j\omega x/a} dx = \frac{S(\omega / a)}{a}$$

$a < 0 \rightarrow$

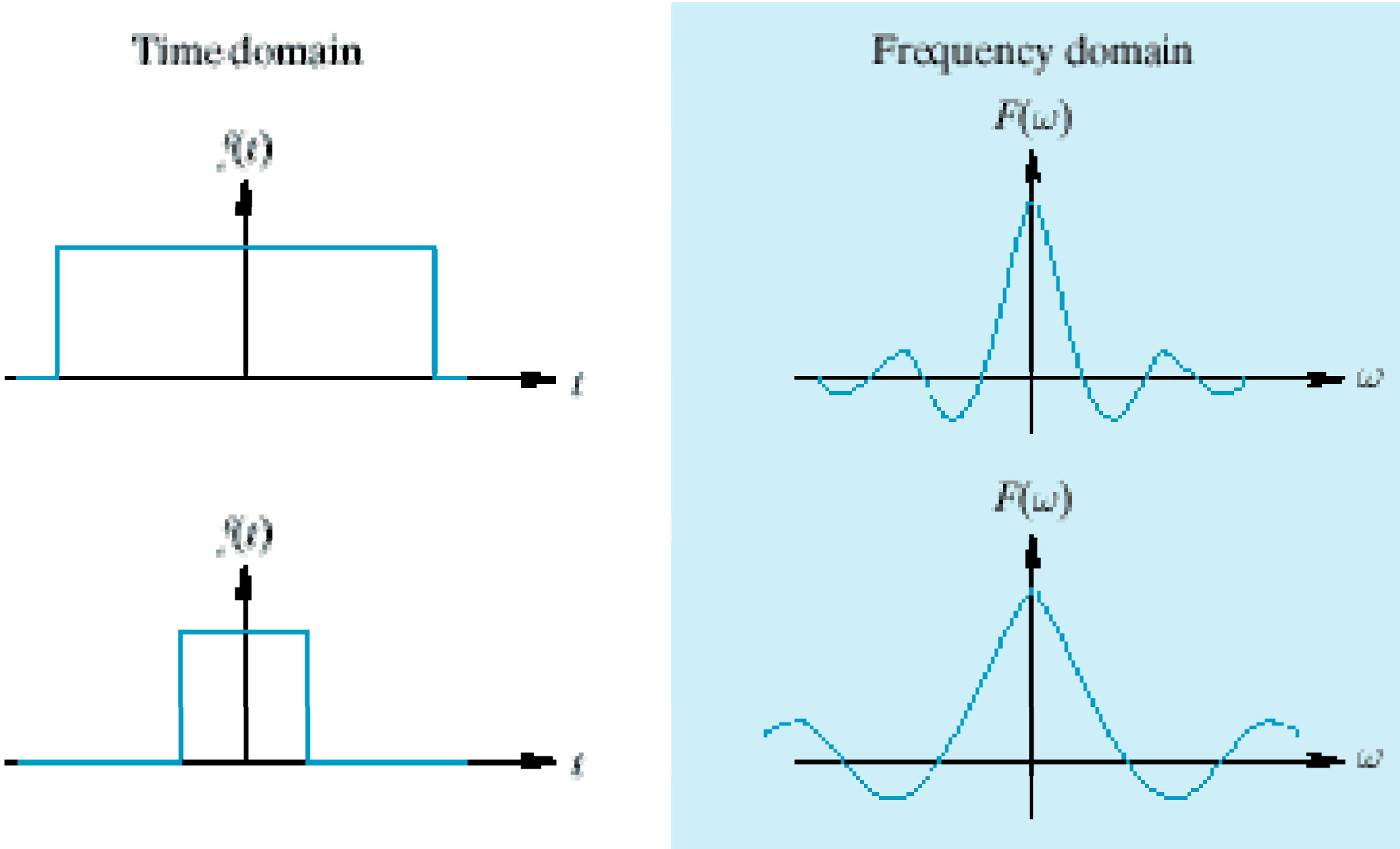
$$\int_{-\infty}^{\infty} s(a \cdot t) e^{-j\omega t} dt$$

$$= \int_{\infty}^{-\infty} s(x) e^{-j\omega x/a} d(x/a)$$

$$= \frac{-1}{a} \int_{-\infty}^{\infty} s(x) e^{-j\omega x/a} dx = \frac{-S(\omega / a)}{a}$$

Also known as the Scaling Theorem

Scaling Example



Transform pairs for a rectangular pulse for two pulse durations.

Compressing the pulse in the time domain causes the transform to spread out in the frequency domain, or vice versa. “reciprocal spreading property”

Reducing the time-domain duration of a waveform causes its spectrum to spread out in the frequency domain, dictating a larger system bandwidth.

The Shift Property

If $s(t)$ has Fourier transform $S(w)$ then
$s(t)e^{j\omega_0 t}$ has Fourier transform

$$\int_{-\infty}^{\infty} s(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} s(t)e^{-j(\omega-\omega_0)t}dt$$

$$= S(\omega - \omega_0)$$

Thus multiplication by $e^{j\omega_0 t}$ shifts the frequency spectrum of $s(t)$ so that it is centred on the point $\omega = \omega_0$ in the frequency domain (MODULATION).

The Delay Property

If $s(t)$ has Fourier transform $S(w)$ then the delayed version $s(t-a)$, has Fourier transform

$$\int_{-\infty}^{\infty} s(t-a)e^{-j\omega t}dt = \int_{-\infty}^{\infty} s(t)e^{-j\omega(t+a)}dt$$

$$= e^{-j\omega a}\int_{-\infty}^{\infty} s(t)e^{-j\omega t}dt = e^{-j\omega a}S(\omega)$$

Thus implies that delaying a signal by a time a causes its Fourier transform to be multiplied by $e^{-j\omega a}$.

Time-Differentiation Property

If $s\,(t)$ has Fourier transform $S\,(\omega)$ then its derivative $s'(t)$ has Fourier transform $j\ \omega\, S\,(\omega)$

Inverse Fourier Transform $$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, \mathrm{e}^{j\omega t}\ d\omega$$

So, $$s'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[S(\omega)\, \mathrm{e}^{j\omega t}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega S(\omega)\, \mathrm{e}^{j\omega t}\, d\omega$$

Differentiation in the time domain is equivalent to algebraic multiplication in the frequency domain.

Integration Property

One expects that integrating $s(t)$ in the time domain corresponds to dividing $S(\omega)$ by $j\omega$ in the frequency domain.

But

- Integrating a waveform may produce a constant offset or dc component.
- The Fourier transform of a dc component is an impulse at 0 in the frequency domain.

The **integration property** is

$$S\left[\int_{-\infty}^{t} s(t)dt\right] = \frac{S(\omega)}{j\omega} + \pi S(0)\delta(\omega)$$

Symmetry

If $s(t)$ has Fourier transform $S(\omega)$ then $S(t)$ has Fourier transform $2\pi s(-\omega)$

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{j\omega t}\, d\omega$$ Replace ω with y

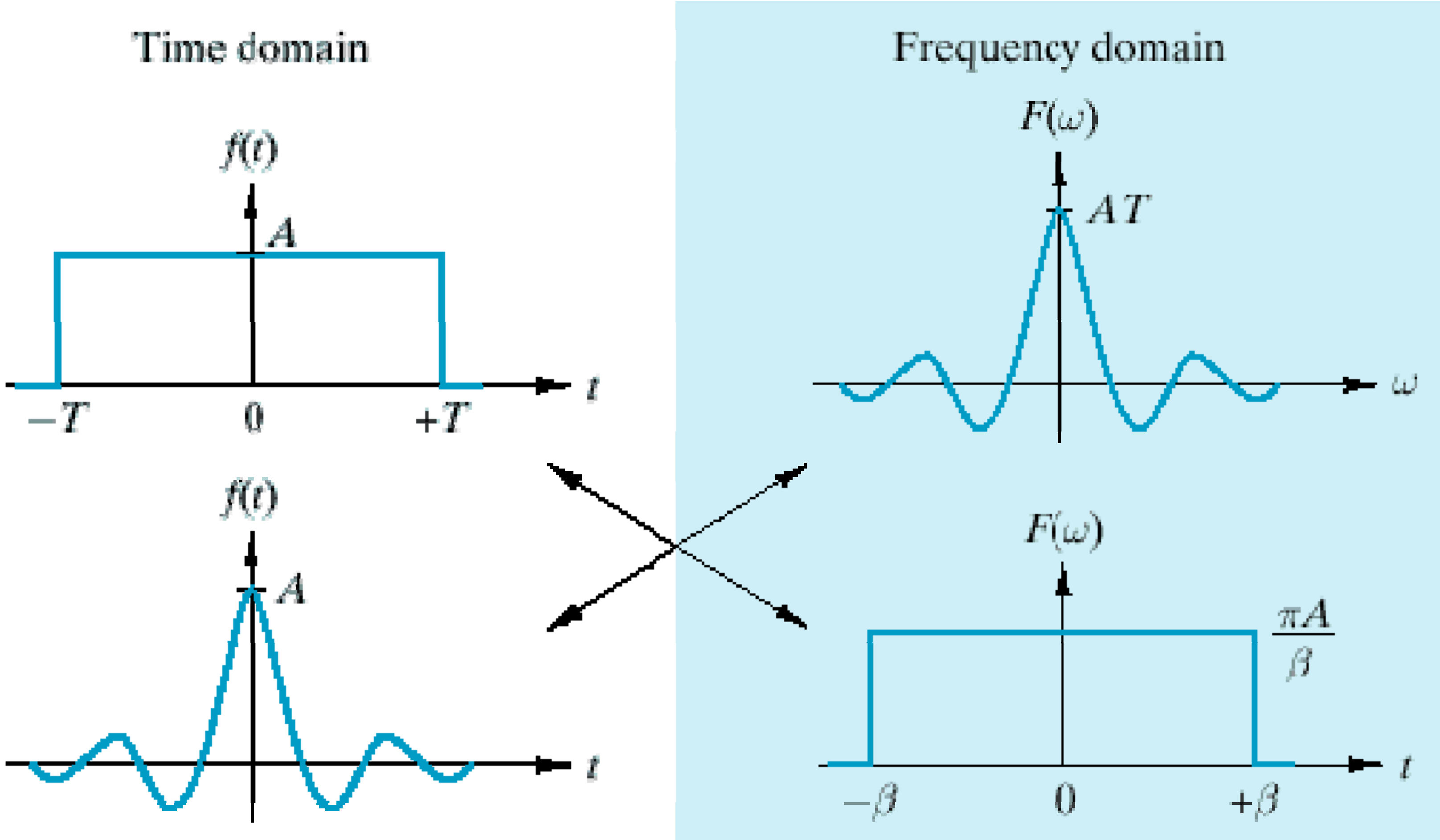
$$2\pi s(t) = \int_{-\infty}^{\infty} S(y)\, e^{jyt}\, dy$$ Replace t with $-\omega$

$$2\pi s(-\omega) = \int_{-\infty}^{\infty} S(y)\, e^{-jy\omega}\, dy$$ Replace y with t

$$2\pi s(-\omega) = \int_{-\infty}^{\infty} S(t)\, e^{-jt\omega}\, dt$$

So if $s(t)$ and $S(w)$ form a Fourier transform pair then $S(t)$ and $2\pi s(-\omega)$ form another Fourier transform pair.

Transform Pairs Example



The sinc function transforms into the rectangle function, and vice versa.

Convolution Definition

The convolution of two functions *f* and *g* is

$$f(u) * g(u) \equiv (f * g)(u) \equiv \int_{-\infty}^{\infty} f(x) g(u-x) dx$$

It expresses the amount of overlap of a function *g* as it is shifted over another function *f*.

It "blends" one function with another.

In a sense, $(f * g)(u)$ is the sum of all the terms $f(x)g(y)$ where $x+y=u$.

Convolution Theorem

$$f(u)*g(u)\equiv(f*g)(u)\equiv\int_{-\infty}^{\infty}f(x)g(u-x)dx$$

If $s_1(t)$ has Fourier transform $S_1(w)$ and $s_2(t)$ has Fourier transform $S_2(w)$, then the Fourier transform of the convolution $s_1(t)*s_2(t)$ is given by

$$\int_{-\infty}^{\infty}\left[s_1(t)*s_2(t)\right]e^{-j\omega t}dt=\int_{-\infty}^{\infty}e^{-j\omega t}\left[\int_{-\infty}^{\infty}s_1(\tau)s_2(t-\tau)d\tau\right]dt$$

$$=\int_{-\infty}^{\infty}s_1(\tau)\left[\int_{-\infty}^{\infty}s_2(t-\tau)e^{-j\omega t}dt\right]d\tau=\int_{-\infty}^{\infty}s_1(\tau)\left[\int_{-\infty}^{\infty}s_2(x)e^{-j\omega(x+\tau)}dx\right]d\tau$$

$$=\int_{-\infty}^{\infty}s_1(\tau)e^{-j\omega\tau}d\tau\int_{-\infty}^{\infty}s_2(x)e^{-j\omega x}dx=S_1(\omega)S_2(\omega)$$

The Fourier transform of the convolution is the product of the transforms.

TABLE 3.1 Properties of the Fourier Transform

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right)\ a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega)\ n = 1, 2, \ldots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)\ \ \omega_0$ real
Multiplication by $\sin(\omega_0 t)$	$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)]$
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega)\ n = 1, 2, \ldots$
Integration in the time domain	$\int_{-\infty}^{t} x(\lambda)\, d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)\, d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\, d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$