Time allowed:	2 hours 30 minutes
No of qu to be answered	3
Rest of rubric	Answer Question 1 and TWO other questions
Any special requirements for resit candidates	No
Is an appendix included?	Yes
Examiners	Ioannis Patras

Question 1: THIS QUESTION IS COMPULSORY

In this question, you may assume that the Fourier Transform of a basic Gaussian signal $g(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right)$ is given by $G(\omega) = \exp\left(-\frac{1}{2}\omega^2\right)$.

(a) Using the scaling theorem, or otherwise, find the Fourier Transform of the scaled Gaussian $g_1(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}t^2\right)$.

[5 marks]

Answer:

Scaling theorem: $s(at) \leftrightarrow \frac{S(\omega / a)}{|a|}$

Here we have $at = t/\sigma$, i.e. $a = 1/\sigma$.

Notice also that we have different scale factor $\frac{1}{\sqrt{2\pi}} \to \frac{1}{\sqrt{2\pi\sigma}}$ in $g_1(t)$

Therefore

$$\begin{aligned} G_1(\omega) &= \frac{1}{\sigma} \times \frac{1}{a} G(\omega / a) = \frac{\sigma}{\sigma} \exp\left(-\tfrac{1}{2}\sigma^2\omega^2\right) \\ &= \exp\left(-\tfrac{1}{2}\sigma^2\omega^2\right) \end{aligned}$$

(b) Find the Fourier Transform of the time-shifted Gaussian $g_2(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(t - t_\mu)^2\right)$

[4 marks]

Answer:

We have $g_2(t) = g_1(t - t_\mu)$

Delay Property: $s(t-a) \leftrightarrow e^{-j\omega a} S(\omega)$

Therefore

$$
\begin{aligned}
G_2(\omega) &= G_1(\omega) e^{-j\omega t_\mu} \\
&= \exp\left(-\tfrac{1}{2}\sigma^2\omega^2 - j\omega t_\mu\right)
\end{aligned}
$$

(c) Verify that the energy $E = \int_{-\infty}^{\infty} |g_1(t)|^2 dt$ of the scaled Gaussian $g_1(t)$ is conserved by the Fourier Transform.

[10 marks]

Answer:

Calculating the energy in the time domain we get

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} |g_1(t)|^2 \, dt \\
&= \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\tfrac{1}{2\sigma^2} t^2\right) \right|^2 dt \\
&= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \exp\left(-2\tfrac{1}{2\sigma^2} t^2\right) dt \\
&= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \exp\left(-x^2\right) \sigma dx \quad \text{using} \quad t = \sigma x \\
&= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} \exp\left(-x^2\right) dx \\
&= \frac{1}{2\pi\sigma} \sqrt{\pi} \\
&= \frac{1}{2\sigma\sqrt{\pi}}
\end{aligned}
$$

Calculating from the Fourier Transform we get

$$
\begin{aligned}
E_\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|G_1(\omega)\right|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\exp(-\tfrac{1}{2}\sigma^2\omega^2)\right|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-2\tfrac{1}{2}\sigma^2\omega^2)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-x^2\right)\frac{1}{\sigma}dx \quad \text{using} \quad \omega = x/\sigma \\
&= \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\exp\left(-x^2\right)dx \\
&= \frac{1}{2\pi\sigma}\sqrt{\pi} \\
&= \frac{1}{2\sigma\sqrt{\pi}} \\
&= E
\end{aligned}
$$

as required.

(d) Show that the autocorrelation $R_{gg}(t) = g(t) * g(t)$ of the basic Gaussian signal $g(t)$ with itself, is a scaled Gaussian $g_1(t)$ with $\sigma^2 = 2$.

[7 marks]

Answer:

$$
\begin{aligned}
R_{gg}(t) &= \int_{-\infty}^{\infty} g(\tau)g(\tau - t)d\tau \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}\tau^2\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}(\tau - t)^2\right)d\tau \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}\tau^2\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}(\tau^2 - 2t\tau + t^2)\right)d\tau \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-(\tau^2 - t\tau + \tfrac{1}{2}t^2)\right)d\tau \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-(\tau^2 - 2(t/2)\tau + (t/2)^2 + (t/2)^2)\right)d\tau \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-(\tau - t/2)^2 - (t/2)^2)\right)d\tau \\
&= \frac{1}{2\pi}\exp\left(-(t/2)^2\right)\int_{-\infty}^{\infty}\exp\left(-(\tau - t/2)^2\right)d\tau \\
&= \frac{1}{2\pi}\exp\left(-(t/2)^2\right)\int_{-\infty}^{\infty}\exp\left(-\tau^2\right)d\tau
\end{aligned}
$$

Where the last step comes from a change of variables.

Using integral of a Gaussian: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ we get

$$
\begin{aligned}
R_{gg}(t) &= \frac{1}{2\pi}\exp\left(-(t/2)^2\right)\sqrt{\pi} \\
&= \frac{1}{2\sqrt{\pi}}\exp\left(-\frac{t^2}{2\cdot 2}\right) \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{t^2}{2\sigma^2}\right) \quad \text{with} \quad \sigma^2 = 2
\end{aligned}
$$

(e) Verify that the Fourier Transform of $R_{gg}(t)$ is the power spectrum $|G(\omega)|^2$ of the basic Gaussian $g(t)$.

[8 marks]

Answer:

Fourier Transform of R_{gg} is:

$$
\begin{aligned}
\mathrm{FT}(R_{gg})(\omega) &= G_1(\omega) \\
&= \exp(-\tfrac{1}{2}\sigma^2\omega^2) \quad \text{with} \quad \sigma^2 = 2 \\
&= \exp(-\omega^2)
\end{aligned}
$$

Power spectrum of basic Gaussian is:

$$
\begin{aligned}
|G(\omega)|^2 &= \left|\exp(-\tfrac{1}{2}\omega^2)\right|^2 \\
&= \exp(2\times -\tfrac{1}{2}\omega^2) \\
&= \exp(-\omega^2)
\end{aligned}
$$

These are the same, as required.

[Total 34 marks]

Question 2

a) Outline the procedure used in creating the Karhunen Loeve Transform of a multidimensional data set $\mathbf{X} = [\vec{x}_0, \vec{x}_1, \ldots \vec{x}_{N-1}]$ (assume that the dimensionality of each vector in the data set is Dx1). (There should be about 6 main steps- more or less depending on how you write your answer). What quantities need to be determined and how, and what vector and matrix operations need to be performed? In your answer you should mention the dimensionality of the quantities (scalars, vectors and matrices) that are calculated at the intermediate steps.

[16 marks]

Answer:

1. Find the mean vector for the input data

$$E(\vec{x}) = \frac{1}{N}\sum_{i=0}^{N-1}\vec{x}_i$$

   The mean vector is a Dx1 vector

2. Find the covariance matrix

$$\mathbf{R}_{\mathbf{XX}} = \frac{1}{N-1}\sum_{i=0}^{N-1}(\vec{x}_i - E(\vec{x}))(\vec{x}_i - E(\vec{x}))^T$$

   The covariance matrix is DxD

3. Find eigenvalues of the covariance matrix

$$\left|\mathbf{R}_{\mathbf{XX}} - \lambda\mathbf{I}\right| = 0$$

   $|\mathbf{A}|$ means "Determinant of $\mathbf{A}$"

   There are D eigenvalues. Each one is a scalar

4. Find eigenvectors of the covariance matrix

$$\left[\mathbf{R}_{\mathbf{XX}} - \lambda_i\mathbf{I}\right]\vec{\phi}_i = 0$$

   There are D eigenvectors. The dimensions of each one is Dx1.

5. Normalise the eigenvectors

$$\left\langle \vec{\phi}_i, \vec{\phi}_i \right\rangle = 1$$

6. Transform the input

$$\mathbf{Y} = \phi^T\mathbf{X}$$

   DxN matrix

b) Is the KLT an orthonormal transform? If not, why? If yes, describe which steps in the construction of the transform guarantee that it is orthonormal.

[6 marks]

Answer:

It isan orthonormal transform. The eigenvalue decomposition guarantees that the eigenvectors are orthogonal. The normalisation step (step 5) guarantees that they are normalised.

c) What is the main reason that PCA is used?

[6 marks]

Answer:

It is optimal in the MSE sense. That is, it is the linear transform that minimises the Mean Square reconstruction Error

d) What are some of the differences between PCA and other transforms such as the DCT and Fourier transform? Why are other transforms often preferred over the KLT even when KLT can, theoretically, achieve better results?

[5 marks]

Answer:

The main difference is that the basis vectors of the KLT are not fixed but depend on the data. In contrast the basis vectors for the DCT transform (cosines) and for the fourier transform (cosines and sines, or complex exponentials) are fixed. For this reason the KLT requires that the basis vectors are sent when it is used for compression.

Other transforms are preferred over the KLT because KLT is slow, it requires the transmition of the basis vectors and it is nonseperable.

[Total 33 marks]

Question 3

The Discrete Cosine Transform (DCT) is given by $DCT[k] = c(k)\sum_{n=0}^{N-1} s[n]\cos\frac{\pi(2n+1)k}{2N}$ where

$$c(k) = \begin{cases} \sqrt{1/N} & k = 0 \\ \sqrt{2/N} & k \neq 0 \end{cases}$$

a) What are the basis functions for the DCT?

[7 marks]

Answer:

For an N-point transform based on a basis vector set $\{\psi_k\}$, we can write

$$S[k] = \langle s, \psi_k \rangle = \sum_{n=0}^{N-1} s[n]\psi_k^*[n].$$

Hence for the DCT transform $\psi_k[n] = \left[c(k)\cos\frac{\pi(2n+1)k}{2N}\right]^* = c(k)\cos\frac{\pi(2n+1)k}{2N}$ with $c(k)$ as above.

b) Show that the DCT for N=4 is orthonormal. [Hint: Simplify the expressions using the identities given in the Appendix.]

[14 marks]

Answer:

(ii) For DCT, $\psi_k[n] = c(k)\cos\frac{\pi(2n+1)k}{2N}$ i.e.

$$\psi_0 = \frac{1}{2}(1,1,1,1) \text{ and } \psi_k = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi k}{8}, \cos\frac{3\pi k}{8}, \cos\frac{5\pi k}{8}, \cos\frac{7\pi k}{8}\right) \text{ for } k \neq 0$$

Writing these out, we get

$$\psi_1 = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, \cos\frac{5\pi}{8}, \cos\frac{7\pi}{8}\right) = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, -\cos\frac{3\pi}{8}, -\cos\frac{\pi}{8}\right)$$

$$\psi_2 = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4}, \cos\frac{3\pi}{4}, \cos\frac{5\pi}{4}, \cos\frac{7\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{2}(1, -1, -1, 1)$$

$$\psi_3 = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{8}, \cos\frac{9\pi}{8}, \cos\frac{15\pi}{8}, \cos\frac{21\pi}{8}\right) = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{8}, -\cos\frac{\pi}{8}, \cos\frac{\pi}{8}, -\cos\frac{3\pi}{8}\right)$$

For DCT we get

$$\langle\psi_0, \psi_1\rangle = \frac{1}{2\sqrt{2}}\left(1.\cos\frac{\pi}{8} + 1.\cos\frac{3\pi}{8} + 1.-\cos\frac{3\pi}{8} + 1.-\cos\frac{\pi}{8}\right) = 0$$

$$\langle\psi_0, \psi_2\rangle = \frac{1}{4}(1.1 + 1.-1 + 1.-1 + 1.1) = 0$$

$$\langle\psi_0, \psi_3\rangle = \frac{1}{2\sqrt{2}}\left(1.\cos\frac{3\pi}{8} + 1.-\cos\frac{\pi}{8} + 1.\cos\frac{\pi}{8} + 1.-\cos\frac{3\pi}{8}\right) = 0$$

$$\langle\psi_1, \psi_2\rangle = \frac{1}{2}\left(\cos\frac{\pi}{8}.1 + \cos\frac{3\pi}{8}.-1 + -\cos\frac{3\pi}{8}.-1 + -\cos\frac{\pi}{8}.1\right) = 0$$

$$\langle\psi_1, \psi_3\rangle = \frac{1}{2}\left(\cos\frac{\pi}{8}.\cos\frac{3\pi}{8} + \cos\frac{3\pi}{8}.-\cos\frac{\pi}{8} + -\cos\frac{3\pi}{8}.\cos\frac{\pi}{8} + -\cos\frac{\pi}{8}.\cos\frac{3\pi}{8}\right) = 0$$

$$\langle\psi_2, \psi_3\rangle = \frac{1}{2\sqrt{2}}\left(1.\cos\frac{3\pi}{8} + -1.-\cos\frac{\pi}{8} + 1.\cos\frac{\pi}{8} + -1.\cos\frac{3\pi}{8}\right) = 0$$

Hence for the 4-point DCT we have $\langle\psi_k, \psi_l\rangle = 0$ for $k \neq l$ which is also *orthogonal*.

We also have

$$\langle \psi_0, \psi_0 \rangle = \frac{1}{4}(1+1+1+1) = 1$$

$$\langle \psi_1, \psi_1 \rangle = \frac{1}{2}\left(2\left(\cos\frac{\pi}{8}\right)^2 + 2\left(\cos\frac{3\pi}{8}\right)^2\right) = \frac{1}{2}\left(1+\cos\frac{\pi}{4}+1+\cos\frac{3\pi}{4}\right) = \frac{1}{2}\left(1+\cos\frac{\pi}{4}+1-\cos\frac{\pi}{4}\right) = 1$$

$$\langle \psi_2, \psi_2 \rangle = \frac{1}{4}(1+1+1+1) = 1$$

$$\langle \psi_3, \psi_3 \rangle = \frac{1}{2}\left(2\left(\cos\frac{\pi}{8}\right)^2 + 2\left(\cos\frac{3\pi}{8}\right)^2\right) = \frac{1}{2}\left(1+\cos\frac{\pi}{4}+1+\cos\frac{3\pi}{4}\right) = \frac{1}{2}\left(1+\cos\frac{\pi}{4}+1-\cos\frac{\pi}{4}\right) = 1$$

Hense the orthonormality conditions are satisfied

d) What is the discrete cosine transform of s=(0,0,0,1) for N=4?

[6 marks]

Answer:

For an N-point transform based on a basis vector set $\{\psi_k\}$, we can write

$$S[k] = \langle s, \psi_k \rangle = \sum_{n=0}^{N-1} s[n]\psi_k^*[n]$$

Therefore,

$$S[0] = \langle s, \psi_0 \rangle = \frac{1}{2}(0.1+0.1+0.1+1.1) = \frac{1}{2}$$

$$S[1] = \langle s, \psi_1 \rangle = \frac{1}{\sqrt{2}}\left(0.\cos\frac{\pi}{8}+0.\cos\frac{3\pi}{8}-0.\cos\frac{3\pi}{8}-1.\cos\frac{\pi}{8}\right) = -\frac{1}{\sqrt{2}}\cos\frac{\pi}{8}$$

$$S[2] = \langle s, \psi_2 \rangle = \frac{1}{2}(0.1-0.1-0.1+1.1) = \frac{1}{2}$$

$$S[3] = \langle s, \psi_3 \rangle = \frac{1}{\sqrt{2}}\left(0.\cos\frac{3\pi}{8}-0.\cos\frac{\pi}{8}+0.\cos\frac{\pi}{8}-1.\cos\frac{3\pi}{8}\right) = -\frac{1}{\sqrt{2}}\cos\frac{3\pi}{8}$$

e) Compare the discrete cosine transform and the discrete Fourier transform. More specifically, describe how the expected inputs and outputs might differ, what domains they transform a signal into, and for which types of applications they might be used.

[6 marks]

Answer:

(ii) For an N-point transform based on a basis vector set $\{\psi_k\}$, we can write

They are very similar. They both transform to frequency representations. They are both separable, and form orthonormal bases. However, the DCT has real input and real output, whereas the DFT has complex input, complex output. The DCT is mainly used for image compression, whereas the Fourier transform is used much more generally.

[Total 33 marks]

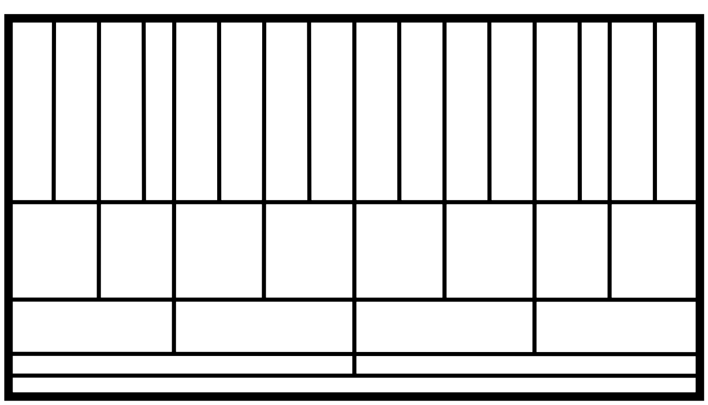
Question 4

(a) With the aid of a diagram, compare the time-frequency tiling of the Wavelet Transform (WT) with that of the short-time Fourier transform (STFT). Suggest possible reasons why this time-frequency tiling might make the WT more suitable than the STFT for the analysis of some real-world signals.

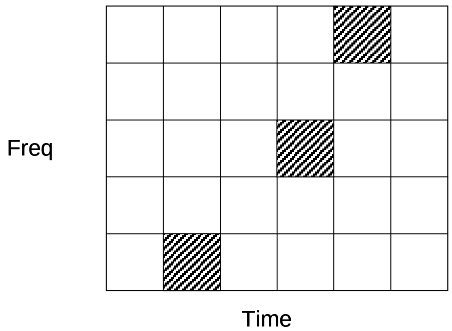
[6 marks]

Answer:

Time-frequency tiling of wavelet transform:



Compare t-f tiling of STFT:



For STFT, have constant time and frequency resolution at all frequencies.

For WT, have variable time and frequency resolution, depending on centre frequency. At low freq, have fine freq resolution, but coarse time resolution.

At high freq, have coarse freq resolution, but fine time resolution.

Several possible justifications here, including:

(i) Many real-world signals caused by energy impact with decaying vibrations: energy decays faster at high frequencies than low frequencies, so need higher time resolution at high frequency

(ii) WT analysis corresponds to a constant-Q filter ($Q = 2\Delta\omega / \omega_0$), corresponding to physical resonators of similar complexity at different frequencies.

(iii) For musical signals, freq resolution is a constant proportion of an octave (or: constant resolution as measured by semitones).

(b) In the wavelet transform, the scaling function coefficients $c_{m,n}$ and wavelet series coefficients $d_{m,n}$ can be calculated recursively according to the following equations:

$$c_{m-1,n} = \sqrt{2}\sum_i h_0[i-2n]c_{m,i}$$
$$d_{m-1,n} = \sqrt{2}\sum_i h_1[i-2n]c_{m,i}$$

Explain how this can be interpreted in terms of filtering and downsampling, and hence leads to the concept of an *analysis filterbank*. Sketch a diagram to illustrate this filterbank.

[5 marks]

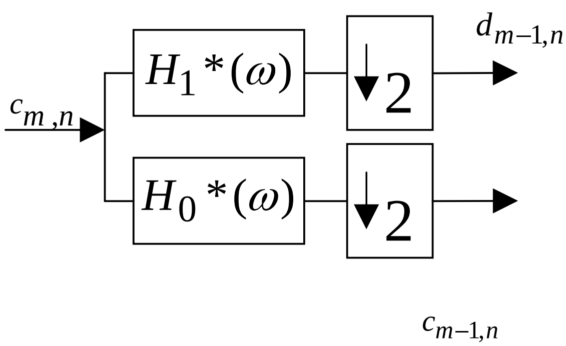
Answer:

If $c_{m,i}$ are the scaling function coefficients at level *m*, we can calculate the scaling function coefficients and wavelet series coefficients $d_{m-1,n}$ at level *m*-1 recursively from these.

The signals $h_0[\cdots]$ and $h_1[\cdots]$ are (time-reversed) low-pass and high-pass filters.

Steps of 1 in *n* correspond to steps of 2 in *i*, corresponding to downsampling by a factor of 2 from *n* to *i*.

Therefore these equations represent filtering followed by downsampling, as shown in the diagram:



(c) The process described in part (b) above can be reversed using the recursive equation $c_{m,n} = \sqrt{2}\left(\sum_i h_0[n-2i]c_{m-1,i} + \sum_i h_1[n-2i]d_{m-1,i}\right)$. Explain how this leads to a corresponding reconstruction filterbank using upsampling and filtering, and sketch a diagram of the resulting filterbank.

[5 marks]

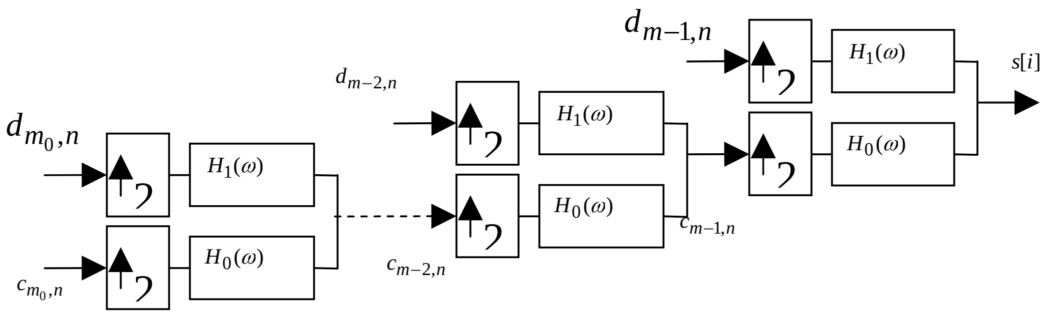
Answer:

The $c_{m,n}$ at level *m* are reconstructed from the $c_{m-1,n}$ and $d_{m-1,n}$ at level *n*-1.

The filters $h_k[n-2i]$ advance every second sample – i.e. *n* advances 2 for every advance of 1 in *i*. This therefore corresponds to upsampling.

The filters are not time-reversed (c.f. answer to (c) above) so the filters have non-conjugate frequency response.

The process is summarized (several steps this time) in the following diagram:



(d) Suppose we have a Haar wavelet transform analysis filterbank which uses a low-pass filter $h_0[0] = h_0[1] = \frac{1}{2}$ and a high-pass filter $h_1[0] = \frac{1}{2}$, $h_1[1] = -\frac{1}{2}$.

Using these filters, calculate the Haar wavelet transform for a sampled signal $s[n] = [1, 3, 0, -1]$ after 1 and 2 stages of the transform filterbank, with the initial setting $c_{2,n} = s[n]$ for $n = 0...3$.

Calculate the inverse wavelet transform of your result using the resynthesis filterbank, to confirm that the result of the inverse transform is the original $s[n]$.

[12 marks]

Answer:

Start with the signal is the finest resolution coefficient,

$$s[n] = [1, 3, 0, -1] = c_{2,i} \quad \text{or} \quad h_0[i-2n] = \begin{cases} \frac{1}{2} & \text{if } i = 2n \text{ or } 2n+1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\begin{aligned} c_{m-1,n} &= \sqrt{2} \cdot \tfrac{1}{2}(c_{m,2n} + c_{m,2n+1}) \\ &= \tfrac{1}{\sqrt{2}}(c_{m,2n} + c_{m,2n+1}) \end{aligned}$$

similarly

$$d_{m-1,n} = \tfrac{1}{\sqrt{2}}(c_{m,2n} - c_{m,2n+1})$$

First level:

$$c_{1,0} = \tfrac{1}{\sqrt{2}}(1+3) = 4/\sqrt{2}$$
$$c_{1,1} = \tfrac{1}{\sqrt{2}}(0+(-1)) = -1/\sqrt{2}$$
$$d_{1,0} = \tfrac{1}{\sqrt{2}}(1-3) = -2/\sqrt{2}$$
$$d_{1,1} = \tfrac{1}{\sqrt{2}}(0-(-1)) = 1/\sqrt{2}$$

Hence the first level of the wavelet transform is $\frac{1}{\sqrt{2}}(4, -1, -2, 1)$.

Second level:

$$c_{0,0} = \tfrac{1}{\sqrt{2}}.\tfrac{1}{\sqrt{2}}(4+(-1)) = 3/2$$
$$d_{0,0} = \tfrac{1}{\sqrt{2}}.\tfrac{1}{\sqrt{2}}(4-(-1)) = 5/2$$

Hence the second level of the wavelet transform is $\frac{1}{2}(3, 5, -2\sqrt{2}, \sqrt{2})$

(the d wavelet coefficients in the right hand half are unchanged).

For the resynthesis, we have

$$c_{m,n} = \sqrt{2}\left(\sum_i h_0[n-2i]c_{m-1,i} + \sum_i h_1[n-2i]d_{m-1,i}\right)$$

i.e. with the Haar filters we have

$c_{m,n} = \sqrt{2}\left(\tfrac{1}{2}c_{m-1,n/2} + \tfrac{1}{2}d_{m-1,n/2}\right)$ for n even, or

$c_{m,n} = \sqrt{2}\left(\tfrac{1}{2}c_{m-1,(n-1)/2} - \tfrac{1}{2}d_{m-1,(n-1)/2}\right)$ for n odd.

From level 0 to level 1:

$$c_{1,0} = \sqrt{2}\left(\tfrac{1}{2}c_{0,0} + \tfrac{1}{2}d_{0,0}\right) = \sqrt{2}\left(\tfrac{1}{2}3/2 + \tfrac{1}{2}5/2\right) = \sqrt{2}(8/4) = 4/\sqrt{2}$$
$$c_{1,1} = \sqrt{2}\left(\tfrac{1}{2}c_{0,0} - \tfrac{1}{2}d_{0,0}\right) = \sqrt{2}\left(\tfrac{1}{2}3/2 - \tfrac{1}{2}5/2\right) = \sqrt{2}(-2/4) = -1/\sqrt{2}$$

The d coefficients in the 2nd half of the WT are unchanged, so we have

$\frac{1}{\sqrt{2}}(4, -1, -2, 1)$ which is the same as the first stage transform in the analysis direction.

From level 1 to level 2 we have

$$c_{2,0} = \sqrt{2}\left(\tfrac{1}{2}c_{1,0} + \tfrac{1}{2}d_{1,0}\right) = \sqrt{2}\left(\tfrac{1}{2}4/\sqrt{2} + \tfrac{1}{2}(-2/\sqrt{2})\right) = \frac{1}{2}(4-2) = 1$$
$$c_{2,1} = \sqrt{2}\left(\tfrac{1}{2}c_{1,0} - \tfrac{1}{2}d_{1,0}\right) = \sqrt{2}\left(\tfrac{1}{2}4/\sqrt{2} - \tfrac{1}{2}(-2/\sqrt{2})\right) = \frac{1}{2}(4+2) = 3$$
$$c_{2,2} = \sqrt{2}\left(\tfrac{1}{2}c_{1,1} + \tfrac{1}{2}d_{1,1}\right) = \sqrt{2}\left(\tfrac{1}{2}.-1/\sqrt{2} + \tfrac{1}{2}1/\sqrt{2}\right) = \frac{1}{2}(-1+1) = 0$$
$$c_{2,3} = \sqrt{2}\left(\tfrac{1}{2}c_{1,1} - \tfrac{1}{2}d_{1,1}\right) = \sqrt{2}\left(\tfrac{1}{2}.-1/\sqrt{2} - \tfrac{1}{2}1/\sqrt{2}\right) = \frac{1}{2}(-1-1) = -1$$

so for the inverse transformed signal we have $[1, 3, 0, -1]$ which is our original signal successfully recovered.

(e) Given scaling function coefficients defined by $c_{m,n} = \int_{-\infty}^{\infty} s(t)\phi^*{}_{m,n}(t)dt$ where $\phi_{m,n}(t) = 2^{m/2}\phi(2^m t - n)$, and the dilation equation $\phi(t/2) = 2\sum_n h_0[n]\phi(t-n)$, verity the equation $c_{m-1,n} = \sqrt{2}\sum_i h_0[i-2n]c_{m,i}$ used in part (b) above holds.

[5 marks]

Answer:

Starting from the definition of the scaling function coefficient at level *m*-1, we get

$$\begin{aligned}
c_{m-1,n} &= \int_{-\infty}^{\infty} s(t)\phi^*{}_{m-1,n}(t)dt \\
&= 2^{(m-1)/2}\int_{-\infty}^{\infty} s(t)\phi^*\left(\tfrac{2^m t - 2n}{2}\right)dt \\
&= 2^{(m-1)/2}\int_{-\infty}^{\infty} s(t)2\sum_i h_0[i]\phi^*(2^m t - 2n - i)dt \\
&= \sqrt{2}\sum_i h_0[i]\int_{-\infty}^{\infty} s(t)\phi^*{}_{m,2n+i}(t)dt \\
&= \sqrt{2}\sum_i h_0[i]c_{m,2n+i}
\end{aligned}$$

i.e., substituting $i \to i - 2n$ we get

$$c_{m-1,n} = \sqrt{2}\sum_i h_0[i-2n]c_{m,i}$$

as required.

[Total 33 marks]

End of Paper - An Appendix of 1 pages follows

Appendix of Standard Formulae

Discrete Fourier Transform:

DFT: $X[k]=\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$ Inverse DFT: $x[n]=\frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j2\pi kn/N}$

Discrete Cosine Transform:

$DCT[k]=c(k)\sum_{n=0}^{N-1} s[n]\cos\frac{\pi(2n+1)k}{2N}$ where $c(k)=\begin{cases}\sqrt{1/N} & \text{if } k=0\\ \sqrt{2/N} & \text{otherwise}\end{cases}$

$\cos(\pi\pm x)=-\cos x=-\cos(-x)$

$\cos^2(x)=(1+\cos(2x))/2$

$\cos\frac{\pi}{8}=-\cos\frac{7\pi}{8}=-\cos\frac{9\pi}{8}=\cos\frac{15\pi}{8}$

$\cos\frac{3\pi}{8}=-\cos\frac{5\pi}{8}=-\cos\frac{21\pi}{8}$

$\cos\frac{\pi}{4}=-\cos\frac{3\pi}{4}=-\cos\frac{5\pi}{4}=\cos\frac{7\pi}{4}=1/\sqrt{2}$

Scaling Theorem: If $s(t)\leftrightarrow S(\omega)$, then $s(at)\leftrightarrow\frac{S(\omega/a)}{|a|}$

Delay Property: $s(t-a)\leftrightarrow e^{-j\omega a}S(\omega)$

Integral of a Gaussian: $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$

Parseval's formula: $\int_{-\infty}^{\infty} s(t)h^*(t)dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)H^*(\omega)d\omega$

Signal energy: $\|s\|=\langle s,s\rangle=\int_{-\infty}^{\infty} s(t)s^*(t)dt$

Signal inner product: $\langle s,\psi\rangle=\int_{-\infty}^{\infty} s(t)\psi^*(t)dt$ Vector inner product: $\langle \mathbf{s},\mathbf{v}\rangle=\sum_n s_n v_n^*$

Wavelet filterbank coefficients: $$\begin{aligned} c_{m-1,n} &= \sqrt{2}\sum_i h_0[i-2n]c_{m,i}\\ d_{m-1,n} &= \sqrt{2}\sum_i h_1[i-2n]c_{m,i}\end{aligned}$$

Tranform with orthonormal basis functions $\{\psi_\kappa\}$: $s=\sum_k S[k]\psi_k$, with $S[k]=\langle s,\psi_k\rangle=\sum_n s[n]\psi^*(t)$