Tutorial: Linear Algebra Solutions

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Question 1

Consider the pair of vectors:

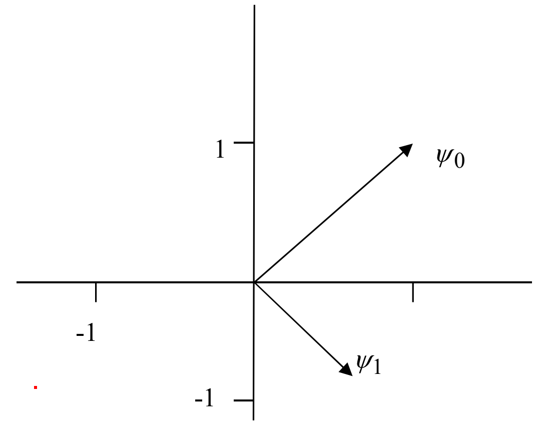
$$\psi_0 = (1,1) \qquad \psi_1 = \sqrt{\tfrac{1}{2}}(1,-1)$$

By calculating relevant inner products and norms, identify whether or not these vectors form an orthogonal or an orthonormal set.
Sketch these two vectors on a diagram to confirm your answer

Question 1: Solution

$$\langle \psi_0, \psi_1 \rangle = \sum_{n=0}^{1} \psi_0[n]\psi_1^*[n]$$
$$= 1 \times \sqrt{\tfrac{1}{2}}\,1 + 1 \times \sqrt{\tfrac{1}{2}}\,(-1)$$
$$= 0$$

Hence they are orthogonal.
Now for the norms we find:



$$\left\|\psi_0\right\|^2 = \langle \psi_0, \psi_0 \rangle$$
$$= \sum_{n=0}^{1} \psi_0[n]\psi_0^*[n]$$
$$= 1 \times 1 + 1 \times 1$$
$$= 2$$
$$\left\|\psi_0\right\| = \sqrt{2} \neq 1$$

$$\langle \psi_1, \psi_1 \rangle = \sum_{n=0}^{1} \psi_1[n]\psi_1^*[n]$$
$$= \left(\sqrt{\tfrac{1}{2}}\right)^2 (1)^2 + \left(\sqrt{\tfrac{1}{2}}(-1)\right)^2$$
$$= \tfrac{1}{2} + \tfrac{1}{2} = 1$$
$$\left\|\psi_1\right\| = 1$$

Therefore only ONE vector has unit norm, the other does not. Hence this set is *orthogonal* but NOT *orthonormal*.

Question 2

Two sets of functions are given by:

$$\{\Psi_1\}=[(2,0),(a,2)]$$

$$\{\Psi_2\}=[(a,-1/8),(0,b)]$$

State the condition required for these two sets to be a Dual Basis and determine the corresponding values of a and b.

Kronecker Delta Function

$$\left\langle \Psi_i, \hat{\Psi}_j \right\rangle = \sum_k \Psi_i(k)\hat{\Psi}_j(k) = \delta_{ij}$$

This is the condition for two sets of vectors (or functions) to be BI-ORTHOGONAL

If it is satisfied, then the two sets of vectors are called a DUAL BASIS.

If the vectors are ORTHONORMAL, then using them is simplified.

Question 2 Solution

The Kronecker Delta Function must be satisfied

$$\left\langle \Psi_i, \hat{\Psi}_j \right\rangle = \sum_k \Psi_i(k)\hat{\Psi}_j(k) = \delta_{ij}$$

$$\{\Psi_1\} = \{(2,0),\ (a,2)\}$$

$$\{\Psi_2\} = \{(a, -\tfrac{1}{8}),\ (0,b)\}$$

$$\langle \Psi_1, \Psi_2 \rangle = [2,0]\begin{bmatrix} a \\ -1/8 \end{bmatrix} = 2a, \quad \therefore\ \underline{\underline{a = 0.5}}$$

$$[2,0]\begin{bmatrix} 0 \\ b \end{bmatrix} = \underline{\underline{0}}$$

$$[a,2]\begin{bmatrix} a \\ -1/8 \end{bmatrix} = a^2 - \frac{2}{8} = 0.25 - 0.25$$
$$= \underline{\underline{0}}$$

$$[a,2]\begin{bmatrix} 0 \\ b \end{bmatrix} = 2b \quad \therefore\ \underline{\underline{b = 0.5}}$$

Question 3

The matrix A is:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Find an orthonormal basis of the NULL space of A.
- Find the RANK of A.
- Find an orthonormal basis of the ROW space of A.

Null Space; Orthogonal Subspaces

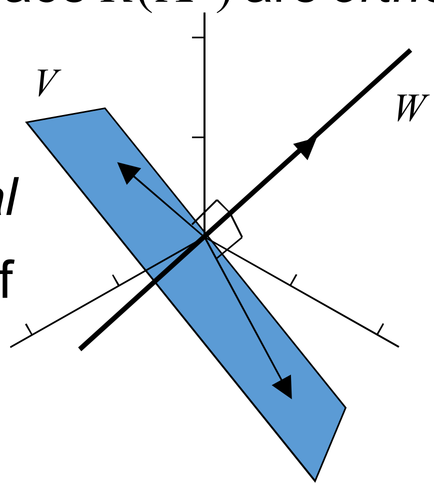
The *nullspace* $N(\mathbf{A})$ of $\mathbf{A}$ is the space *not* spanned by the rows of $\mathbf{A}$. This has dimension $n-r$.

Two subspaces V and W are orthogonal if every vector $\mathbf{v}$ in V is orthogonal to every vector $\mathbf{w}$ in W.

I.e. we must have $\mathbf{v}^t\mathbf{w} = 0$ for all $\mathbf{v} \in V$, $\mathbf{w} \in W$.

So, the nullspace $N(\mathbf{A})$ and row space $R(\mathbf{A}^t)$ are *orthogonal*.

Example: 2-d subspace V (plane)
is orthogonal to 1-d subspace W (line)



In the diagram, W is the *orthogonal complement* $V^{\perp}$ of V (the space of all vectors orthogonal to V).

Question 3 Solution 1

i) NULL SPACE OF A CONSISTS OF THE SOLUTIONS OF $\underline{\underline{Ax = 0}}$

LET $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

SO, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

GIVES $x_1 = -x_3$, $x_2 = 0$

SO $x = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = \underline{\underline{x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}}$

Question 3 Solution 2

SO, $x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ IS A BASIS FOR THE NULL SPACE OF A

NORM OF $x = \sqrt{\langle x, x \rangle} = \sqrt{2}$.

SO, ORTHONORMAL BASIS IS

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Question 3 Solution 3

ii) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

2 NON - ZERO ROWS.

$\therefore$ RANK = 2

Question 3 Solution 4

iii) FOR ROW SPACE OF A,

NON ZERO ROWS ARE A BASIS.

SO, BASIS IS $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right.$

DOT PRODUCT IS 0, $\therefore$ ORTHOGONAL.

NORMS ARE $\sqrt{2}$ AND 1

SO, ORTHONORMAL BASIS FOR ROW SPACE

OF A IS $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Note:
Because the rows are orthogonal,
we can find an orthonormal basis.

However, although we could not find
an orthonormal basis if the rows
had not been orthogonal, it would be
possible for non-orthogonal rows to
be a basis if the row vectors are linearly
independent and span the space.

Question 4

Vectors v_1 and v_2 span the space V.

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Do v_1 and v_2 form an orthonormal basis for V?
If they do not, then find an orthonormal basis for V.

Question 4 Solution

$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad v_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\langle v_1, v_2 \rangle = [2 \quad 1] \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \underline{\underline{0}}$

$\langle v_1, v_1 \rangle = [2 \quad 1] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \underline{\underline{5}}$

$\langle v_2, v_2 \rangle = [1 \quad -2] \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \underline{5}$

$\therefore$ ORTHOGONAL BUT NOT ORTHONORMAL.

NORMS ARE $\sqrt{5}$.

$\therefore \; w_1 = \frac{1}{\sqrt{\cancel{2}}\,5} v_1$ AND $w_2 = \frac{1}{\sqrt{\cancel{2}}\,5} v_2$

ARE AN ORTHONORMAL BASIS FOR V.

Question 5 Part i)

$$v_1 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix},\;\; v_2 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},\;\; v_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Show that these three vectors are an orthonormal basis for R^3

Question 5 Part i) Solution 1

$$\langle v_1, v_1 \rangle = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{18} + \frac{1}{18} + \frac{16}{18} = \underline{1}$$

$$\langle v_2, v_2 \rangle = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = \underline{1}$$

$$\langle v_3, v_3 \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{2} + \frac{1}{2} + 0 = \underline{1}$$

Question 5 Part ii) Solution 2

$$\langle v_1, v_2 \rangle = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \frac{2}{9\sqrt{2}} + \frac{2}{9\sqrt{2}} - \frac{4}{9\sqrt{2}} = \underline{\underline{0}}$$

$$\langle v_1, v_3 \rangle = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-4}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \frac{1}{3 \times 2} - \frac{1}{3 \times 2} + 0 = \underline{\underline{0}}$$

$$\langle v_2, v_3 \rangle = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} = \underline{\underline{0}}$$

Question 5 Part ii)

If vector $x = [1 \quad 1 \quad 1]'$

write x as a linear combination of v1, v2 and v3.

Properties of Orthonormal Bases

- If $\{\psi_n\}$ constitutes a basis for V, then any vector or function in V can be written as

$$s = \sum_n c_n \Psi_n$$

- However, c_n may be difficult to compute. If $\{\psi_n\}$ form an orthonormal basis, this difficulty is eliminated, since then

$$\boxed{c_n = \langle s, \Psi_n \rangle}$$

- Thus if $\{\psi_n\}$ is a set of orthonormal basis for V, then any ***s*** in V can be written as

$$s = \sum_j \langle s, \Psi_j \rangle \Psi_j$$

$$= \langle s, \Psi_1 \rangle \Psi_1 + \langle s, \Psi_2 \rangle \Psi_2 + \ldots\ldots + \langle s, \Psi_n \rangle \Psi_n$$

Question 5 Part ii) Solution 1

$$x = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$c_1 = x^T v_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} = \frac{-2}{3\sqrt{2}}$$

$$c_2 = x^T v_2 = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \frac{5}{3}$$

$$c_3 = x^T v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0$$

Question 5 Part ii) Solution 2

$$\therefore\ x = \frac{-2}{3\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} + \frac{5}{3} \cdot \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 0 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \frac{-1}{9} \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} + \frac{5}{9} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$