Advanced Transform Methods

Wigner-Ville Distribution

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Wigner-Ville Distribution

STFT, Wavelet transform etc:

- Compute correlations between <u>signal</u> and <u>basis fns</u>.
- Time-freq resolution determined by basis fns.

Another approach:

- Time-frequency energy density - signal's energy density in both time and freq
- (c.f. power spectrum: energy in freq only)
- e.g. Wigner-Ville distribution

An “instantaneous power spectrum”

Recall the power spectrum:

$$P(\omega) = |S(\omega)|^2 = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau$$

where $R(\tau)$ is the autocorrelation function (acf)

$$R(\tau) = \int_{-\infty}^{\infty} s(t) s^*(t-\tau) dt = \int_{-\infty}^{\infty} s(t+\tau/2) s^*(t-\tau/2) dt$$

What happens if we use *instantaneous* autocorrelation:

$$R(t,\tau) = s(t+\tau/2) s^*(t-\tau/2)$$

instead of $R(\tau) = \int_{-\infty}^{\infty} R(t,\tau) dt$? We get:

$$WVD_s(t,\omega) = \int_{-\infty}^{\infty} s(t+\tau/2) s^*(t-\tau/2) e^{-j\omega\tau} d\tau$$

which is the *Wigner -Ville Distribution* (WVD).

Cross-WVD vs auto-WVD

Since we can define a cross - correlation,

we can also define a cross - Wigner - Ville distribution :

$$WVD_{s,g}(t,\omega) = \int_{-\infty}^{\infty} s(t+\tau/2)g^*(t-\tau/2)e^{-j\omega\tau}d\tau$$

Taking complex conjugates we find

$$WVD_{s,g}(t,\omega) = WVD^*_{g,s}(t,\omega)$$

So for the usual WVD ("auto - WVD") we have

$$WVD_s(t,\omega) = WVD_{s,s}(t,\omega) = WVD^*_s(t,\omega)$$

so the auto - WVD is always *real*.

Example: Gaussian function

Signal $s(t) = \sqrt[4]{\frac{\alpha}{\pi}} e^{-\alpha t^2/2}$ (normalized to unit energy)

For WVD, we get

$$WVDs(t,\omega) = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \exp\left\{ -\frac{\alpha}{2}\left[\left(t+\frac{\tau}{2}\right)^2 + \left(t-\frac{\tau}{2}\right)^2 \right] \right\} e^{-j\omega\tau} d\tau$$

$$= e^{-\alpha t^2} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \exp\left\{ -\frac{\alpha}{4}\tau^2 \right\} e^{-j\omega\tau} d\tau$$

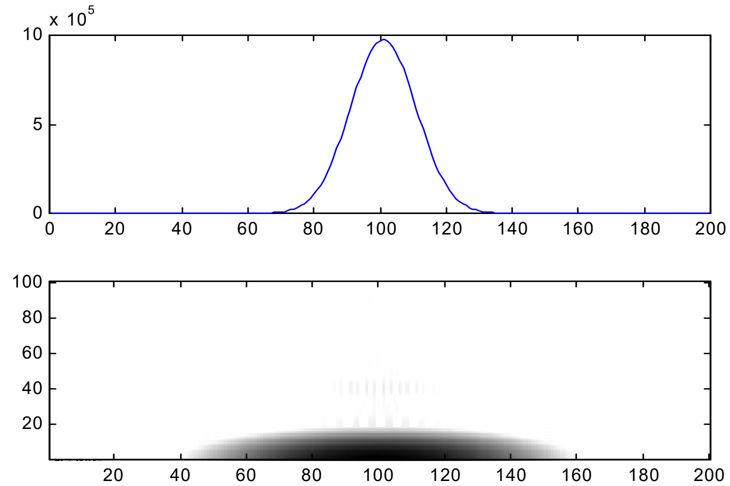
$$= 2\exp\left\{ -\left[\alpha t^2 + \frac{1}{\alpha}\omega^2 \right] \right\}$$

i.e. concentrated around $(0,0)$.

α controls spread :

"time - width": $|t| < \sqrt{\frac{1}{\alpha}}$

"freq - width": $|\omega| < \sqrt{\alpha}$



Example 2: Gaussian chirplet

Signal: $s(t) = \sqrt[4]{\frac{\alpha}{\pi}} \exp\left\{-\frac{\alpha}{2}t^2 + j\frac{\beta}{2}t^2\right\}$

Power spectrum: $|S(\omega)|^2 = \sqrt{\frac{4\pi(\alpha^2+\beta^2)}{\alpha}} \exp\left\{-\frac{\alpha}{\alpha^2+\beta^2}\omega^2\right\}$

tells us *which* freqs $s(t)$ contains, not *when*. Compare:

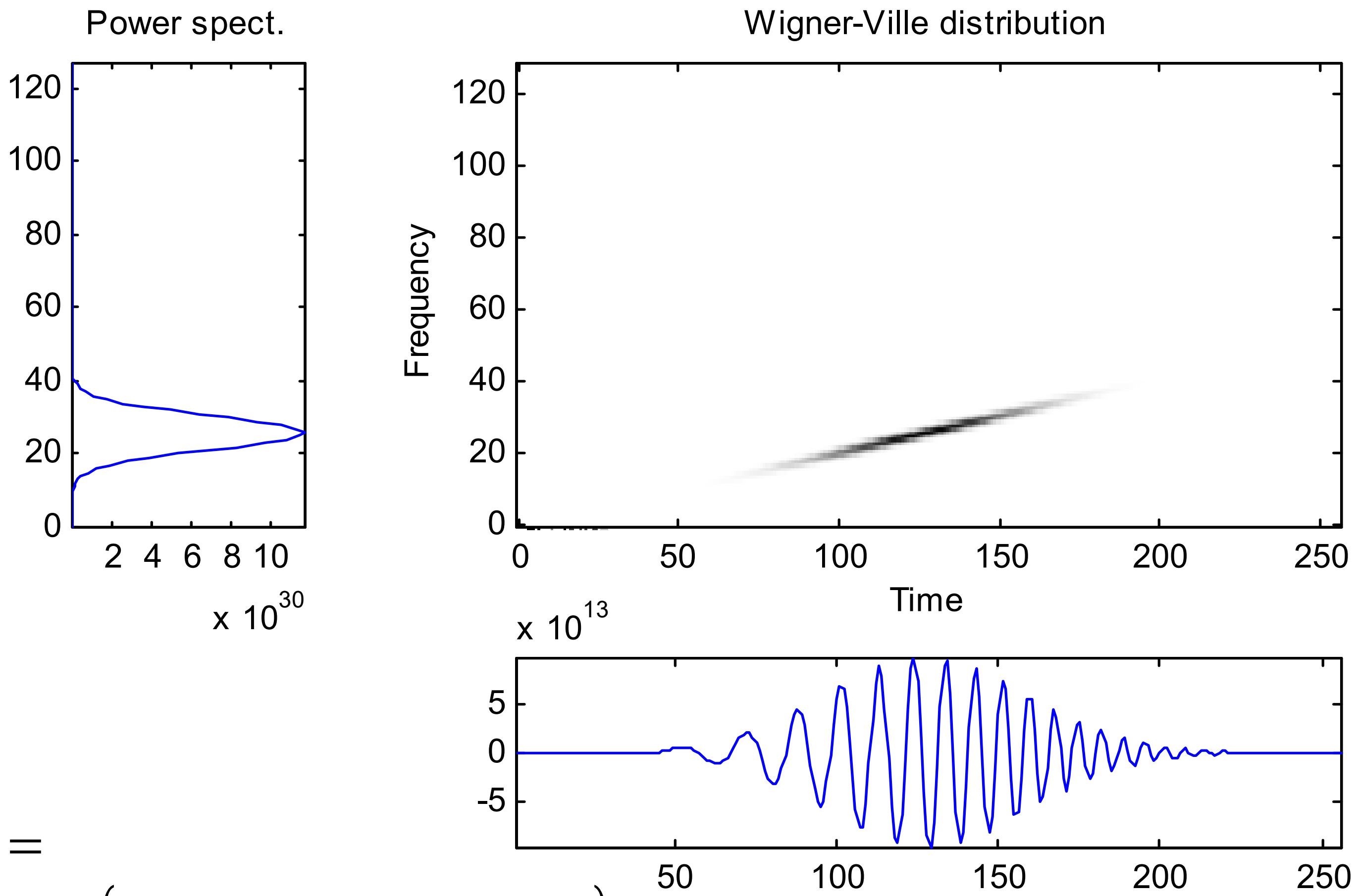
$$WVDs(t,\omega) = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}\left[\left(t+\frac{\tau}{2}\right)^2 + \left(t-\frac{\tau}{2}\right)^2\right] + \frac{j\beta}{2}\left[\left(t+\frac{\tau}{2}\right)^2 - \left(t-\frac{\tau}{2}\right)^2\right]} e^{-j\omega\tau} d\tau$$

$$= e^{-\alpha t^2} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{4}\tau^2} e^{-j(\omega-\beta t)\tau} d\tau$$

$$= 2e^{-\left[\alpha t^2 + \frac{1}{\alpha}(\omega-\beta t)^2\right]}$$

i.e. energy concentrated at $\omega = \beta t$, changing with time.

Illustration: chirplet



$$s(t) = \sqrt[4]{\frac{\alpha}{\pi}} \exp\left\{ -\frac{\alpha}{2}(t - t_0)^2 + j\frac{\beta}{2}t^2 \right\}$$

Time-limited or band-limited signals

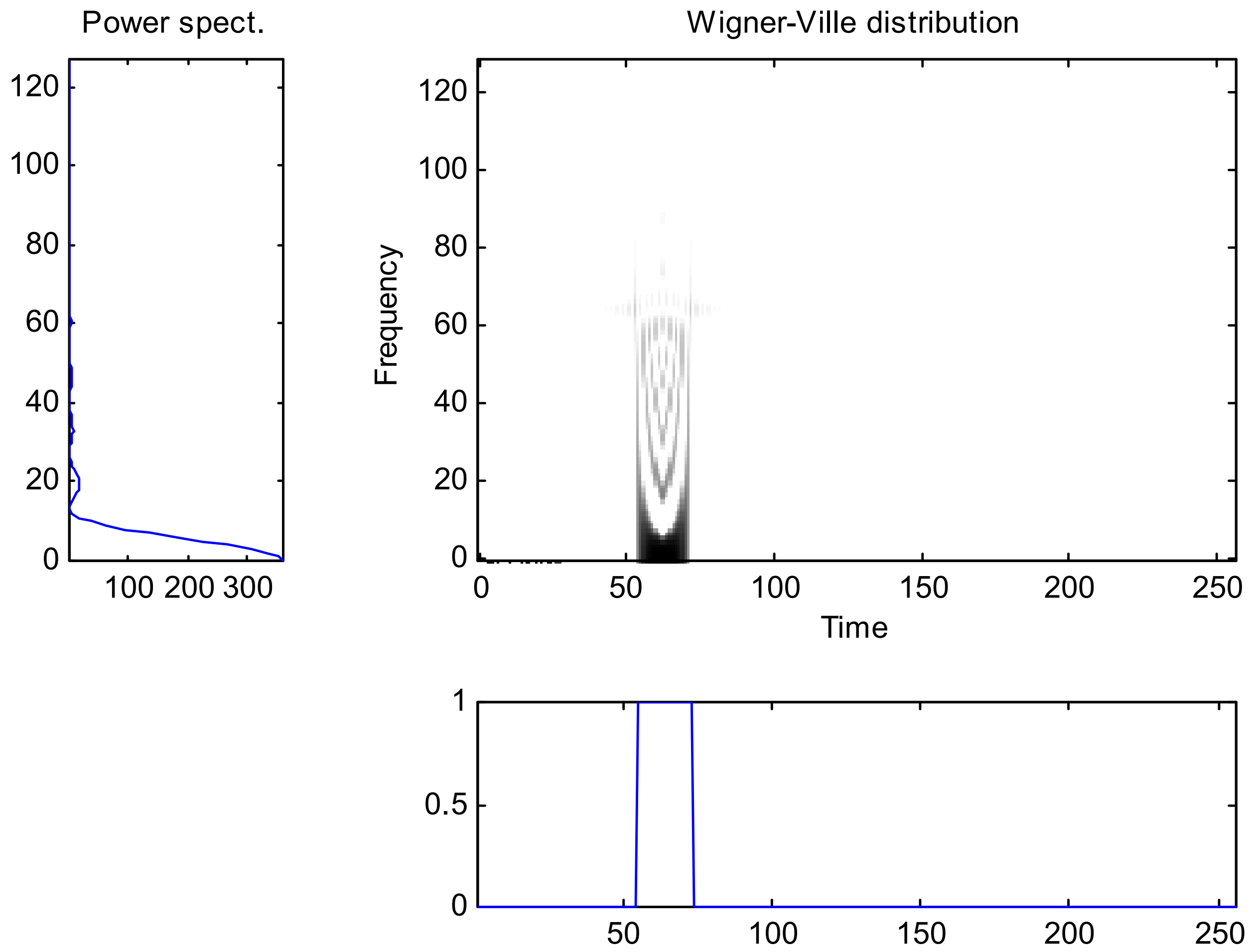
If $s(t)$ is time-limited, i.e. $s(t) = 0$ outside some interval $[t_{0,} t_1]$,
then the WVD is also time-limited, i.e.

$$WVD_s(t, \omega) = 0 \text{ for } t \notin [t_0, t_1]$$

since no value for τ can make both $s(t + \tau / 2)$ and $s(t - \tau / 2)$
non-zero if t is outside this range.

A similar result is true for freq band-limited signals.

Example: WVD of a Time-limited signal



WVD Properties:
Time Marginal Condition

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} WVD_s(t,\omega)d\omega = \int_{-\infty}^{\infty} s\left(t+\frac{\tau}{2}\right)s*\left(t-\frac{\tau}{2}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}d\omega d\tau$$

$$= \int_{-\infty}^{\infty} s\left(t+\frac{\tau}{2}\right)s*\left(t-\frac{\tau}{2}\right)\delta(\tau)d\tau$$

$$= |s(t)|^2$$

So intregral over frequency of WVD is the signal power density at time t

Compare similar result for probability densities :

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)dy$$

Frequency Marginal Condition

$$\int_{-\infty}^{\infty} WVD_s(t,\omega)dt = \int_{-\infty}^{\infty} s\left(t+\frac{\tau}{2}\right)s*\left(t-\frac{\tau}{2}\right)\int_{-\infty}^{\infty} e^{-j\omega\tau}dtd\tau$$

$$= \int_{-\infty}^{\infty} e^{-j\omega\tau}\int_{-\infty}^{\infty} s(t)s*(t-\tau)dtd\tau$$

$$= \int_{-\infty}^{\infty} e^{-j\omega\tau}R(\tau)d\tau$$

$$= |S(\omega)|^2$$

So intregral over time of WVD is the power spectral density

We also have :

$$\frac{1}{2\pi}\int_{-\infty}^{-\infty}\int_{-\infty}^{\infty} WVD_s(t,\omega)dtd\omega = \frac{1}{2\pi}\int_{-\infty}^{-\infty}|S(\omega)|^2\, d\omega = \int_{-\infty}^{\infty}|s(t)|^2\, dt$$

i.e. the WVD is unitary : the energy in $WVD_s(t,\omega)$ is equal to energy in original signal $s(t)$.

Time-shift & Freq-moduln. invariant

If the WVD of $s(t)$ is $WVD_s(t,\omega)$,

then the WVD of time-shifted signal $s_0(t) = s(t - t_0)$

is a time-shifted WVD : $WVD_{s_0}(t,\omega) = WVD_s(t - t_0, \omega)$

Further, the WVD of frequency-modulated signal

$s_1(t) = s(t)e^{j\omega_1 t}$ is a frequency-shifted WVD :

$WVD_{s_1}(t,\omega) = WVD_s(t, \omega - \omega_1)$

(Both follow immediately from the formulas for WVD)

WVD of multiple signals: Cross-terms

Wigner-Ville Distribution has many useful properties, and better resolution than STFT spectrogram. BUT

Applications are limited due to *cross-term interference.*

Consider composite signal $s(t) = s_1(t) + s_2(t)$. Then

$$WVD_s(t,\omega) = WVD_{s_1}(t,\omega) + WVD_{s_2}(t,\omega) + 2\,\mathrm{Re}\left\{WVD_{s_1,s_2}(t,\omega)\right\}$$

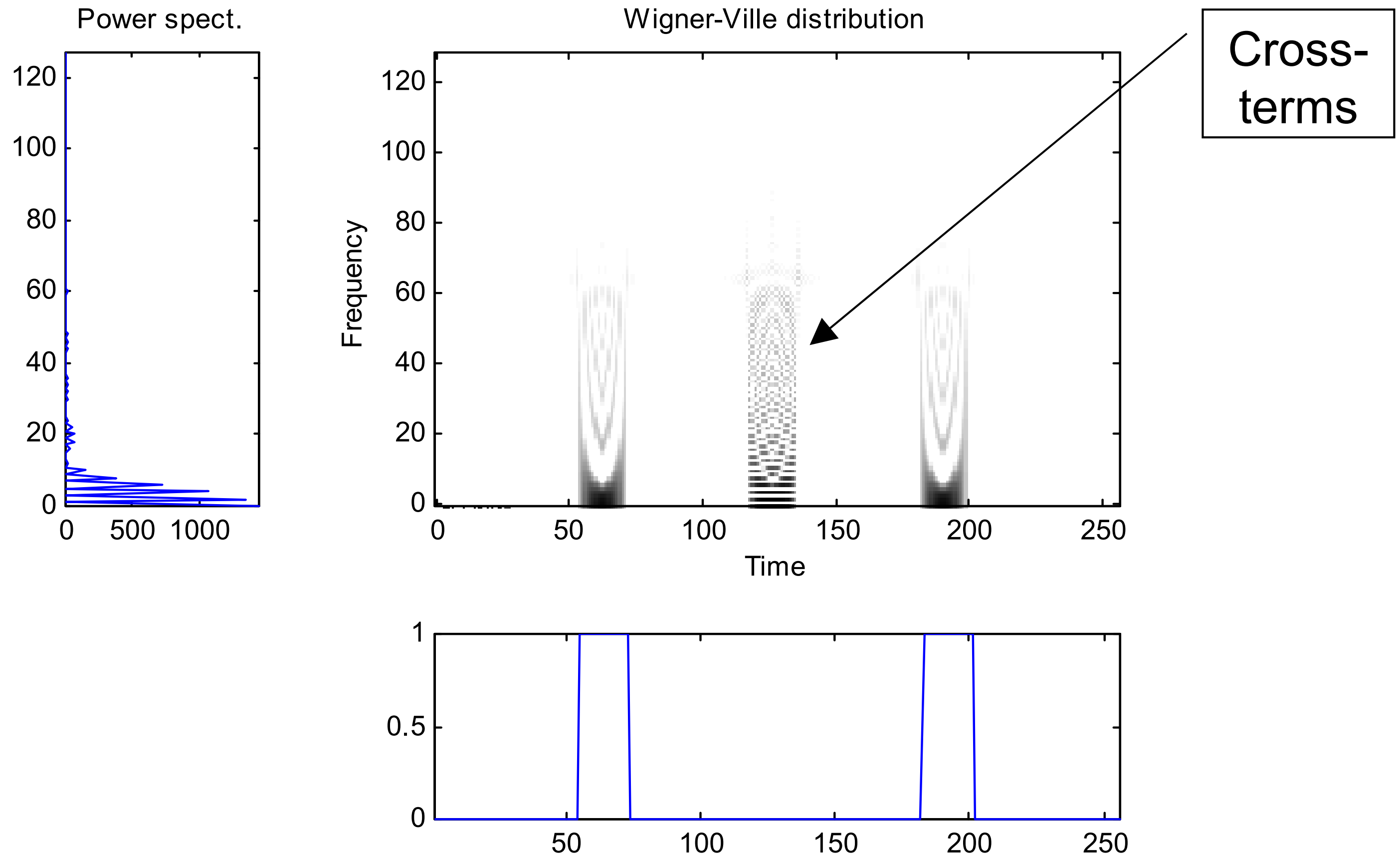
i.e. not only the sum of WVDs, but also the cross-term

$WVD_{s_1,s_2}(t,\omega)$

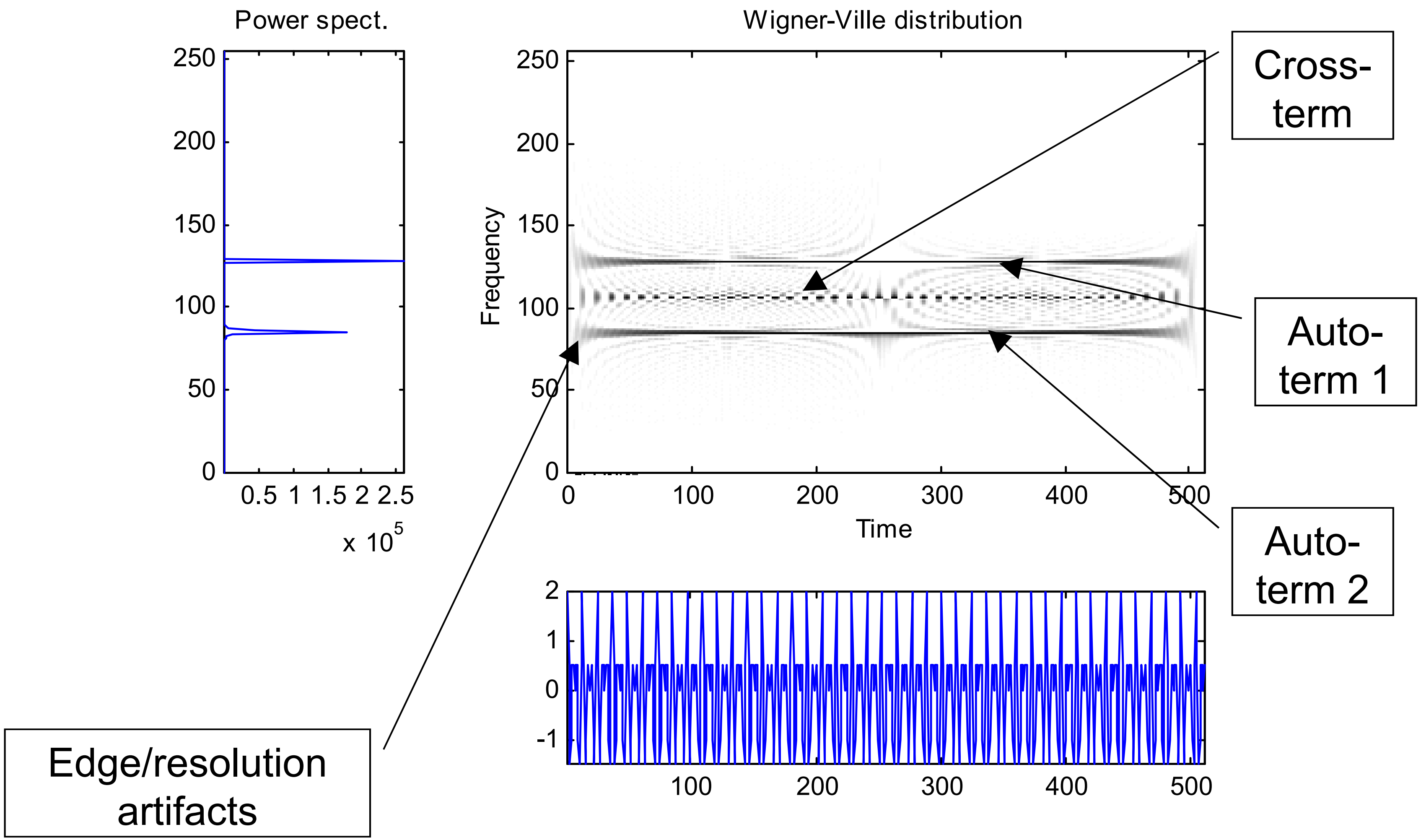
Also, cross-term is included at double the magnitude of the auto-terms, so often obscures useful patterns.

Example of cross-terms

WVD gives cross-terms for all but simplest signals, e.g.:



Example: sum of two sinusoids



Example: sum of sinusoids (cont)

If $s(t) = \exp(j\omega_0 t)$ then

$$WVD_s(t,\omega) = \int_{-\infty}^{\infty} \exp\left\{j\omega_0\left(t + \tfrac{t}{2} - t + \tfrac{t}{2}\right)\right\} e^{-j\omega t}\, d\tau = 2\pi\delta(\omega - \omega_0)$$

i.e. the WVD is a "ridge" along frequency ω_0.

Now let $s(t) = \exp(j\omega_1 t) + \exp(j\omega_2 t)$. The power spectrum is

$$|S(\omega)|^2 = 2\pi\delta(\omega_1) + 2\pi\delta(\omega_2)$$

while the WVD is

$$WVD_s(t,\omega) = 2\pi\delta(\omega - \omega_1) + 2\pi\delta(\omega - \omega_2) + 4\pi\delta(\omega - \omega_\mu)\cos(\omega_d t)$$

where $\omega_\mu = \frac{\omega_1 + \omega_2}{2}$ and $\omega_d = \omega_1 - \omega_2$.

Example (cont): cross-term

Get a large cross-term $4\pi\delta(\omega-\omega_{\mu})\cos(\omega_d t)$

which varies as $\cos\left((\omega_1-\omega_2)t\right)$ at ω_d,

mid-way between the auto-terms.

While the auto-terms are $+$ve, this oscillates $+$ve & -ve

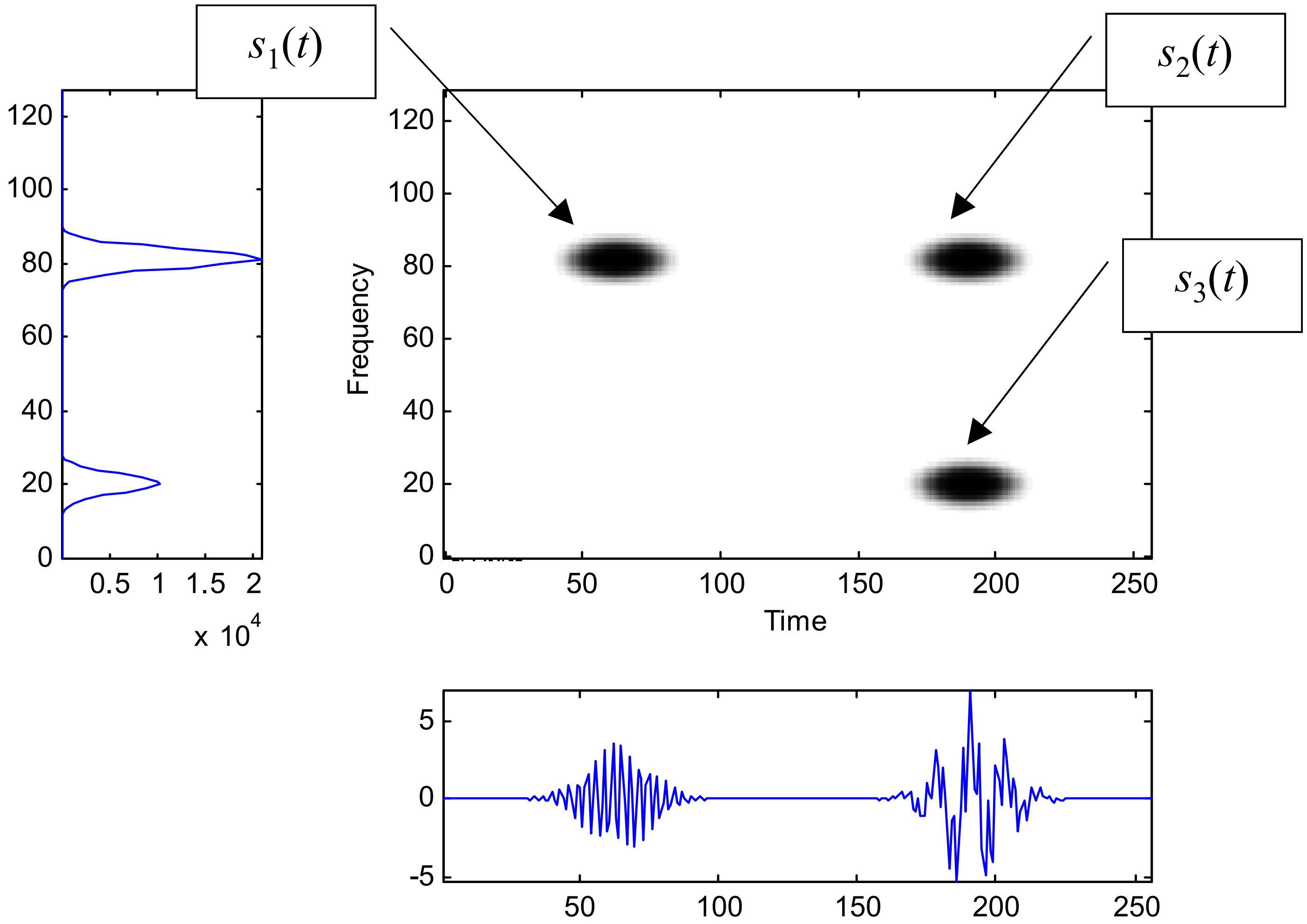
Average of the cross-term is zero:

$$\int_{-\infty}^{\infty} 4\pi\delta(\omega-\omega_{\mu})\cos(\omega_d t)dt = 0 \qquad \omega_d \neq 0$$

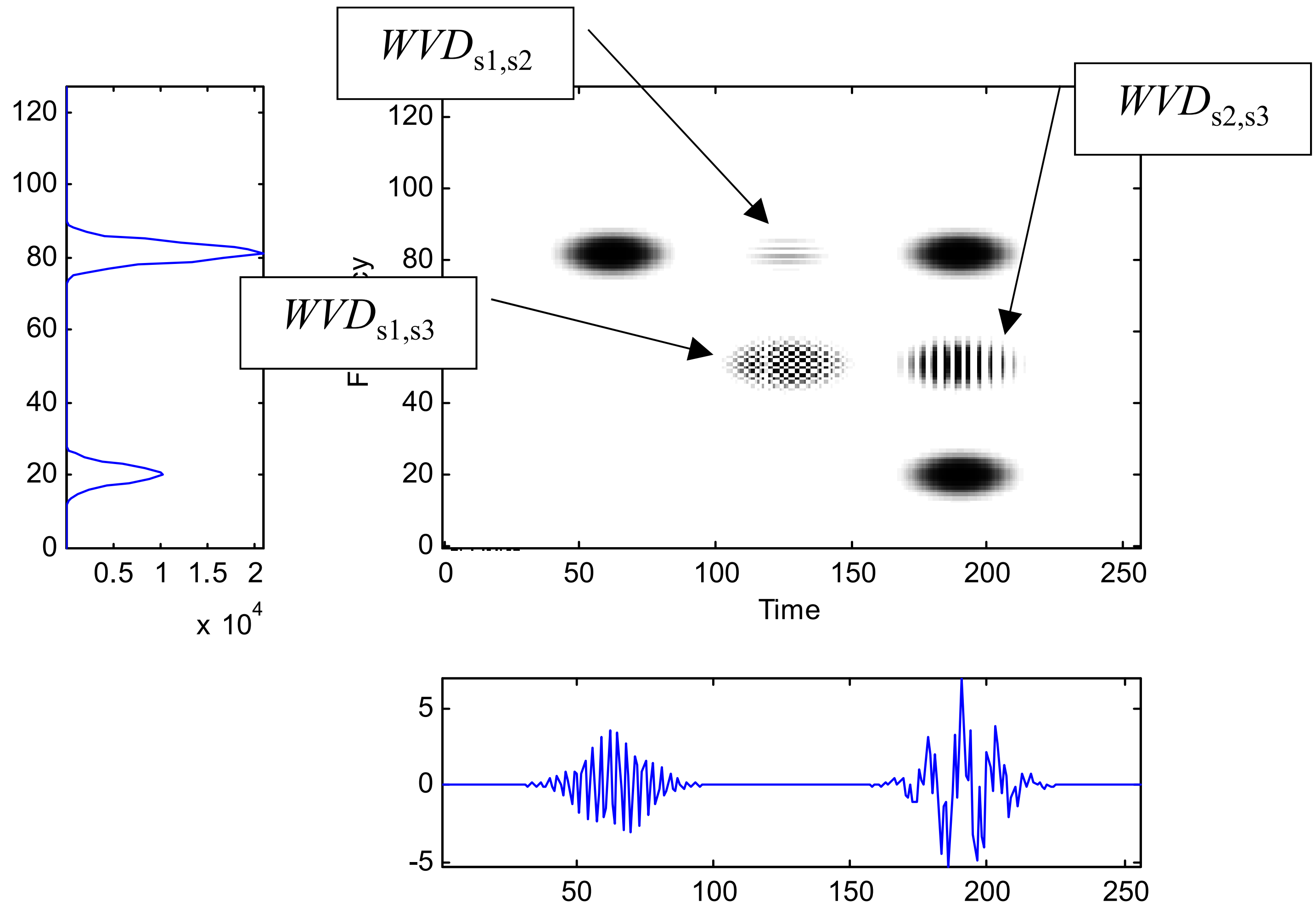
This suggests we may be able to remove these

by *smoothing* (see later.)

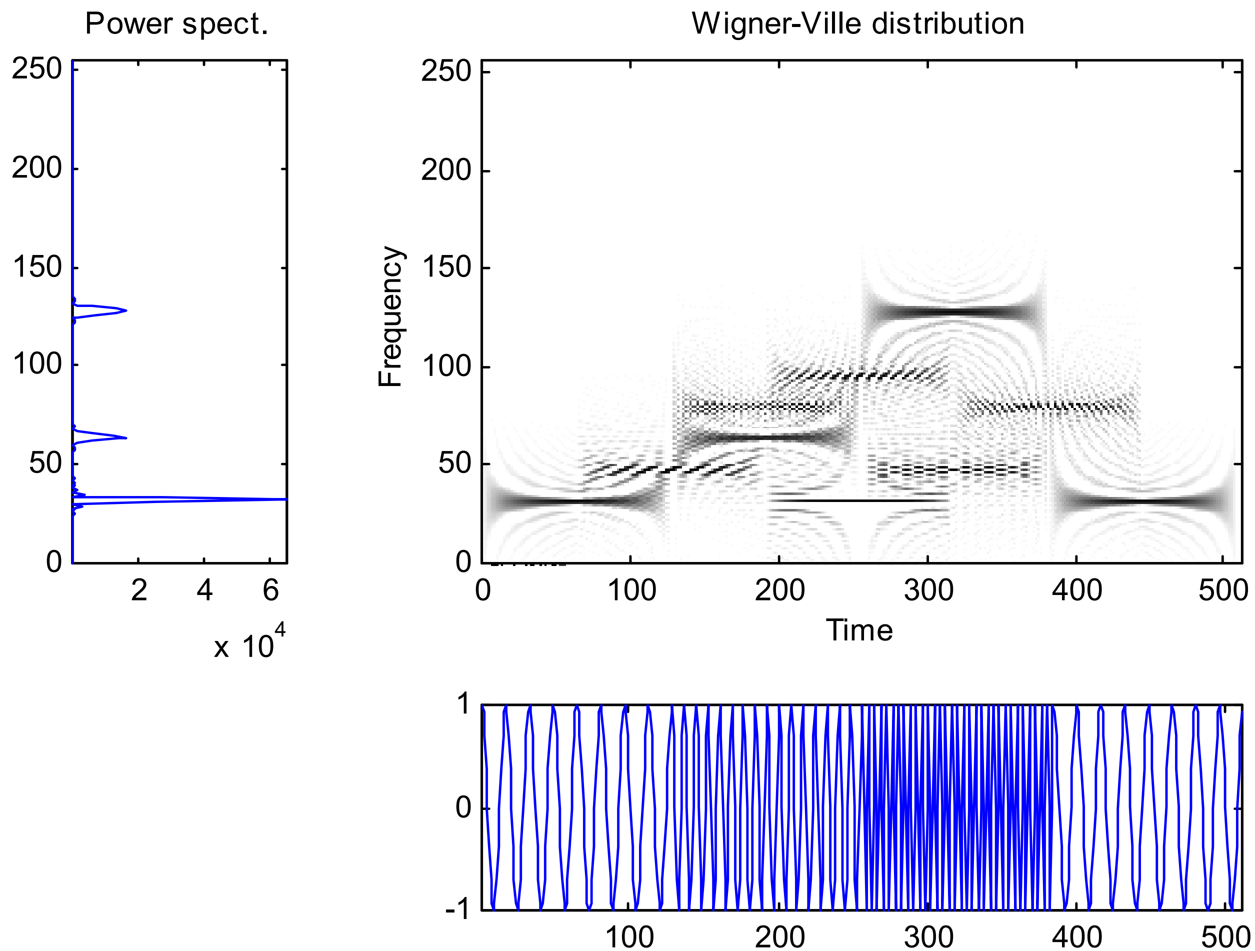
Example: 3-tone test signal



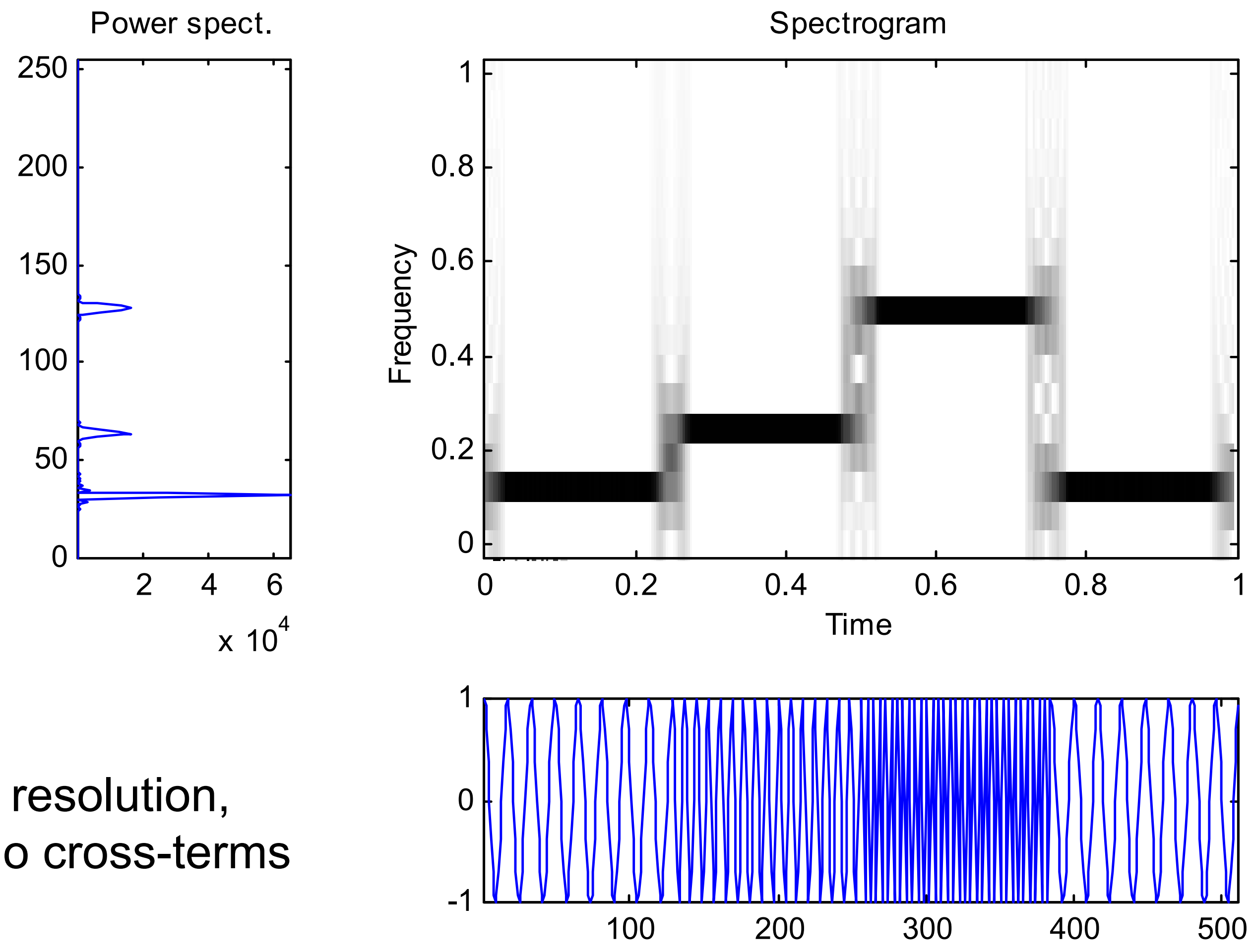
Wigner-Ville Distribution



Another example: Frequency pulses



Compare spectrogram



Less resolution,
but no cross-terms

Wigner-Ville Distribution

Since the cross-terms WVD are usually strongly oscillating,

try removing them by using 2D low-pass filtering,

to give a "smoothed Wigner-Ville distribution" (SWVD):

$$SWVD_s(t,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y) WVD_s(t-x,\omega-y)\,dxdy$$

where $\phi(x,y)$ is a 2D low-pass filter.

Example: 2D Gaussian

$$\phi(x,y) = e^{-\alpha t^2 - \beta\omega^2} \qquad \alpha,\beta > 0$$

We have a trade-off:

more smoothing $\rightarrow$ less cross-terms, BUT

more smoothing $\rightarrow$ reduced resolution

STFT Spectrogram from WVD

In general, WVD can have - ve values.

But if $\alpha\beta \geq 1$ in the Gausian 2D filter,

then the smoothed WVD will be non - negative.

Special case : when $\alpha\beta = 1$ in $\phi(x, y) = e^{-\alpha t^2 - \beta\omega^2}$

then $\phi(x, y)$ is actually a WVD of a Gaussian function.

The STFT spectrogram is a smoothed WVD, with

the WVD of the analysis function $\gamma(t)$ doing the smoothing :

$$|STFT_S(t,\omega)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} WVD_\gamma(x, y) WVD_S(t - x, \omega - y) dxdy$$

(Quan p163)

Wavelet Scalogram from WVD

The scalogram (square of the wavelet transform) can be written in terms of the WVD :

$$SCAL_s(a,b) = |CWT_s(a,b)|^2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} WVD_s(x,y)\,WVD_\psi\left(\frac{x-b}{a}, ay\right)dxdy$$

where $WVD_s(x,y)$ is the WVD of the signal $s(t)$ and $WVD_\psi\left(\frac{x-b}{a}, ay\right)$ is the WVD of the mother wavelet $\psi(t)$.

This operation is known as *affine correlation*.

From WVD to …?

$$
\begin{tikzcd}
{|s(t)|^2} & & {|S(\omega)|^2} \\
& {WVD_s(t,\omega)} \arrow[ul, "\text{Integrate over } \omega"'] \arrow[ur, "\text{Integrate over } t"] \arrow[dl, "\text{Affine correlation}"'] \arrow[dr, "\text{Convolution}"] & \\
{|CWT_s(a,b)|^2} & & {|STFT_s(t,\omega)|^2}
\end{tikzcd}
$$