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EBU6018
Advanced Transform Methods

Week 3.1 – Haar Functions

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Lecture Outline

1. Haar Functions

2. Haar Transform

Haar Functions

- The Haar function was proposed in 1909 by a Hungarian mathematician Alfred Haar. It is one of the earliest transform functions proposed as an example of an orthonormal system

- Haar transform provides time-frequency information, hence is suitable for non-stationary signals
  - Compare it to Fourier transform, which only provides frequency information

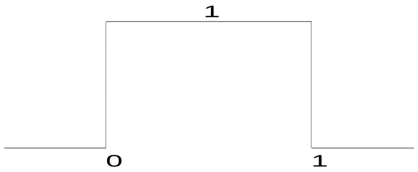
- Haar transform is the simplest of the wavelet functions, although the concept of wavelets did not exist at that time

Haar Functions

- Each wavelet function forms a set of functions, which have a SCALING FUNCTION and a WAVELET FUNCTION
- These functions can be SCALED (i.e. Compressed) and TRANSLATED, forming a family of functions
- The Scaling is DYADIC (power of 2)
- The basic wavelet function is often called the MOTHER WAVELET, and the scaled and translated functions called DAUGHTER WAVELETS
- Haar functions are all rectangular-shaped functions

Haar Function

Haar **scaling** function



$$\varphi_{00} = [1 \quad 1]$$

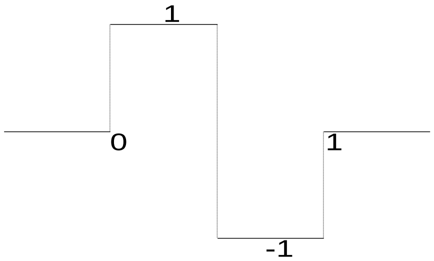
$$\varphi_{jk}(x) = \varphi\left(2^j x - k\right),$$

$$\varphi(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

- j is the scaling parameter
- k is the translation parameter

Haar Function

Haar **wavelet** function



$\psi_{00} = [1 \quad -1]$

$$\psi_{jk}(x) \equiv \psi\left(2^j x - k\right)$$

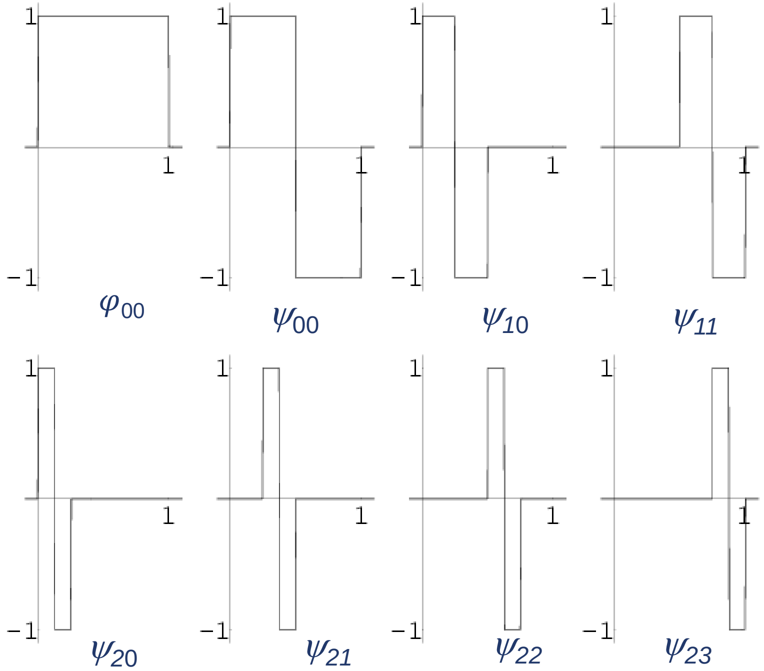
$$\psi(x) \equiv \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

- j is the scaling parameter
- k is the translation parameter

- ψ_{00} is known as the Mother wavelet
- ψ_{jk} is known as the Daughter wavelet if $j, k \neq 0$

Haar Function

- Scaled and translated Haar functions



$\varphi_{00} = \varphi(x)$

$\psi_{00} = \psi(x)$

$\psi_{10} = \psi(2x)$

$\psi_{11} = \psi(2x-1)$

$\psi_{20} = \psi(4x)$

$\psi_{21} = \psi(4x-1)$

$\psi_{22} = \psi(4x-2)$

$\psi_{23} = \psi(4x-3)$

Wavelet Function:

$$\psi(x) \equiv \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_{jk}(x) \equiv \psi\left(2^j x - k\right)$$

Orthogonal and Orthonormal

- Two functions, ψ_{jk} and ψ_{lm} are **orthogonal** if their inner product is zero, given by

$$\int_0^1 \psi_{jk}(x)\psi_{lm}(x)dx = 0$$

(Here, it is assumed that both functions are defined on [0, 1])

- Two functions, ψ_{jk} and ψ_{lm} forms an **orthonormal set** if they are orthogonal and have unit norm.

$$\int_0^1 \psi_{jk}(x)\psi_{jk}(x)dx = 1$$

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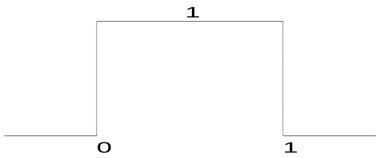
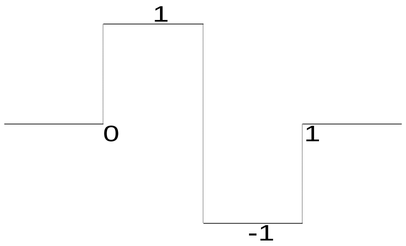
Haar Function - Orthogonal

Haar **scaling** function



$$\varphi_{00} = [1 \quad 1]$$

Haar **wavelet** function



$$\psi_{00} = [1 \quad -1]$$

- They are orthogonal: $< \varphi_{00}, \psi_{00} > = [1 \quad 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$

Haar Function – Orthogonal

- And all members of the Haar Function family are mutually orthogonal.
- Consider ψ_{10} and ψ_{11} :

$$\psi_{10} = [1, -1, 0, 0]$$
$$\psi_{11} = [0, 0, 1, -1]$$

$$< \psi_{10}, \psi_{11} > = [1, -1, 0, 0] \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = 0$$

Haar Function – Orthogonal

- And all members of the Haar Function family are mutually orthogonal.
- Example: Consider φ_{11} and ψ_{01} , show that these two Haar Functions are orthogonal

Haar Function – Orthogonal

- And all members of the Haar Function family are mutually orthogonal.
- Example: Consider φ_{11} and ψ_{01}, show that these two Haar Functions are orthogonal

$$\varphi_{11} = [0, 0, 1, 1, 0, 0, 0, 0]$$

$$\psi_{01} = [0, 0, 0, 0, 1, 1, -1, -1]$$

$$< \varphi_{11}, \psi_{01} > = [0\ 0\ 1\ 1\ 0\ 0\ 0\ 0] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = 0$$

Haar Function – Orthonormal

Normalized Haar wavelet function: $\psi_{jk}(x) = 2^{-j/2}\psi(2^j x - k)$

- For example, ψ_{10}= [1, -1, 0, 0] has a norm of $\sqrt{2}$
- So normalised, ψ_{10}= $\frac{1}{\sqrt{2}}$[1, -1, 0, 0]

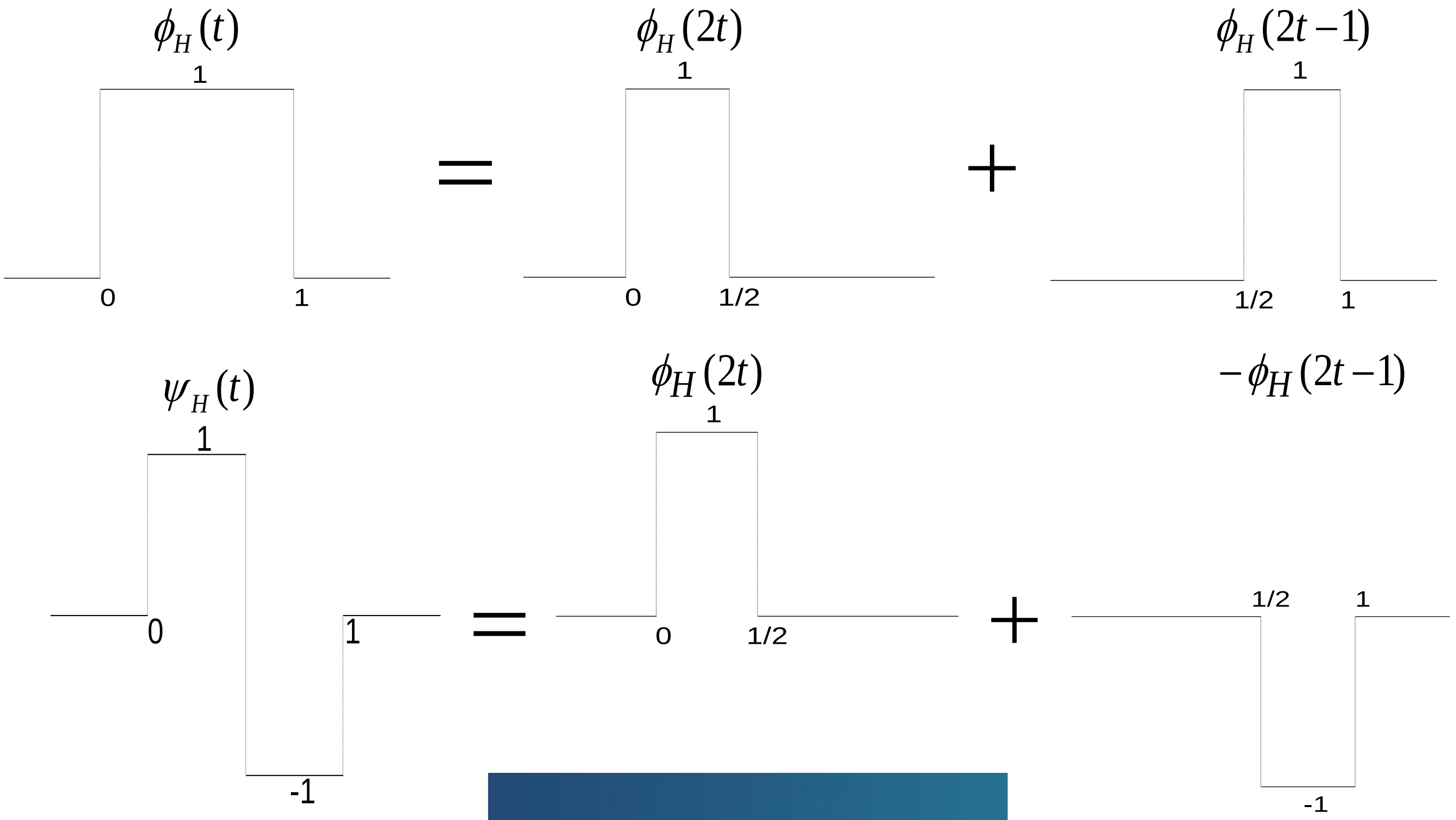
Haar Function – Two-Scale Relations

- A useful property of Haar Functions is that both the Scaling Function and the Wavelet Function can be constructed from scaled and translated versions of the Scaling Function only.

- This property can be used when Haar Functions are applied to Wavelet Transforms.

- In wavelet transform theory this property is called the “Two-Scale Relations”

Haar Function – Two-Scale Relations



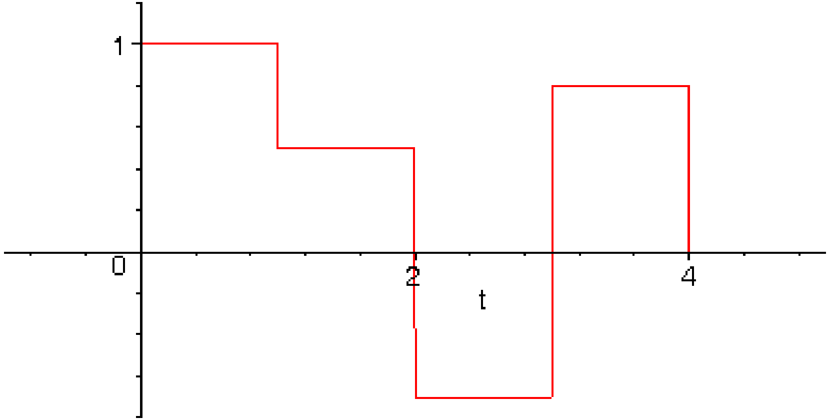
Haar Function – Linear Piecewise Approximation

- Another useful property of Haar Functions is that they can be used to approximate any continuous real function by linear combinations of scaled and translated members of the family of Haar Functions.

- This principle is illustrated on the following slide:

Haar Function – Linear Piecewise Approximation

- Suppose we have samples of a continuous signal.
- These could be represented by a “Linear Piecewise Approximation” by joining the samples with horizontal lines.
- This approximation can be synthesised by combinations of Haar Functions.
- (“Analyse means “take apart”. “Synthesise means “put together”.)
- An approximation of a signal can be obtained by joining its samples.



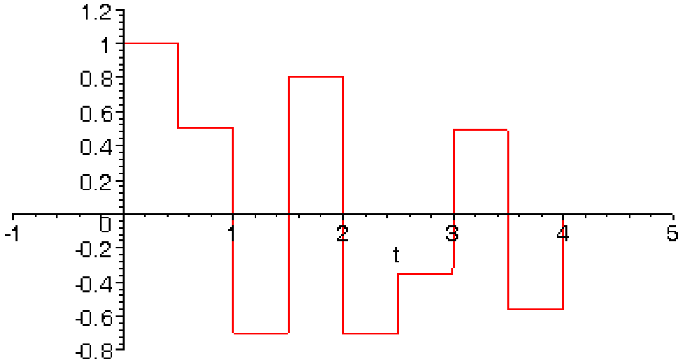
Using just $\varphi(x)$:

$\varphi_{00}+0.5\varphi_{01}-0.5\varphi_{02}+0.75\varphi_{03}$

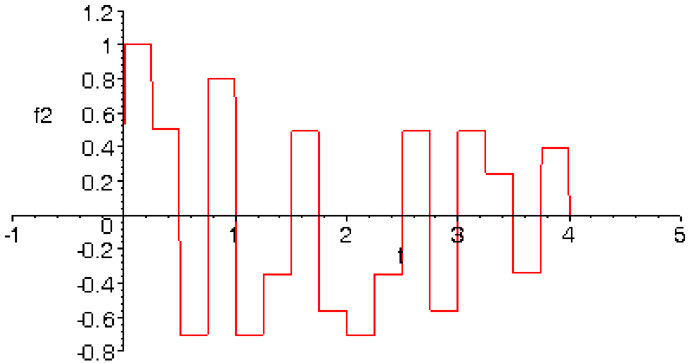
Haar Function – Linear Piecewise Approximation

- Other examples of the principle:

Using Ψ_{j0} and φ



Using Ψ_{j1} , Ψ_{j0} and φ



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Haar Function – Linear Piecewise Approximation

- Example: Using scaled and translated Haar Functions, construct the function given by:

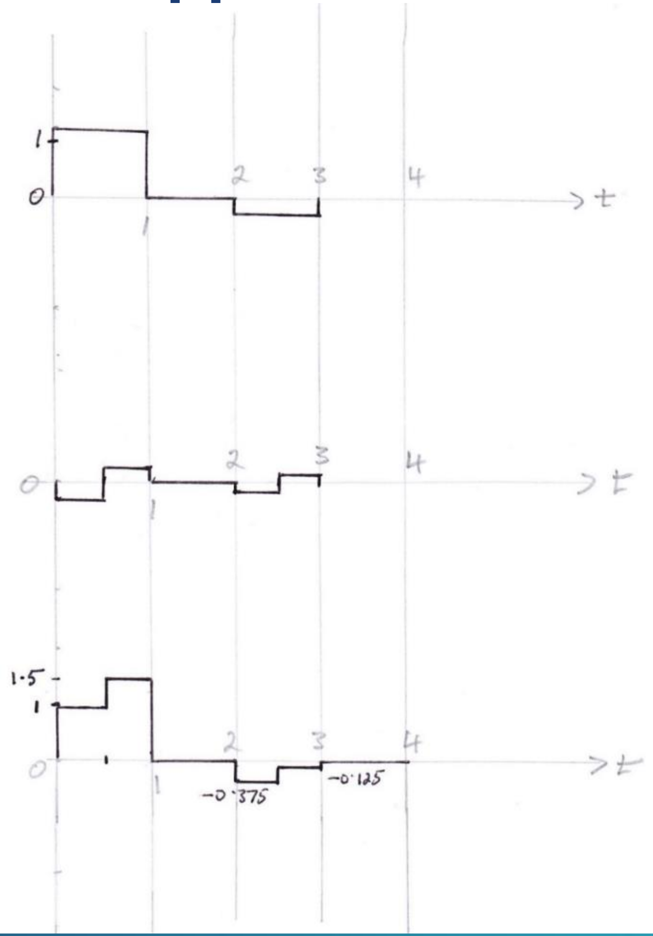
$$1.25\varphi_{0,0}(t)-0.25\varphi_{0,2}(t)-0.25\psi_{0,0}(t)-0.125\psi_{0,2}(t)$$

Haar Function – Linear Piecewise Approximation

$1.25\varphi_{0,0}(t) - 0.25\varphi_{0,2}(t)$

$-0.25\psi_{0,0}(t) - 0.125\psi_{0,2}(t)$

Sum



Haar Function – Summary

Haar Functions are a relatively simple set of functions with some useful properties:

- They are orthogonal (and can be normalised)
- They can be used to approximate any continuous real function
- Both the Scaling Function and the Wavelet Function can be derived from the Scaling Function only

How do we apply them?

Haar Transform Matrix

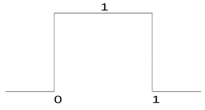
- These two functions can be written as the 2x2 Haar matrix:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- All Haar Matrices need to be normalized before applying to signal. The **normalized 2x2 Haar matrix** is

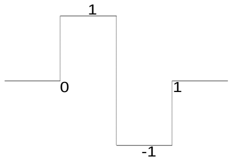
$$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Haar **scaling** function



$$\varphi_{00} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Haar **wavelet** function



$$\psi_{00} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Haar Transform Matrix

- 4x4 Haar matrix:

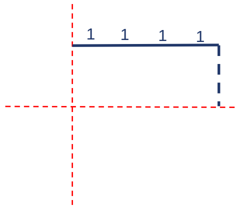
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{array}{c} \varphi_{00} \\ \psi_0 \\ \psi_{1,0} \\ \psi_{1,1} \end{array}$$

- The **normalized 4x4 Haar matrix:**

$$H_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

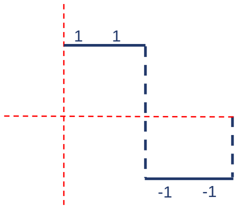
- Normalize each row of the matrix

Haar **scaling** function



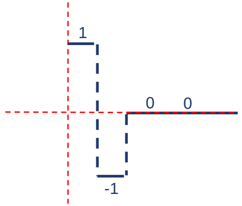
φ_{00} = [1 1 1 1]

Haar **wavelet** function



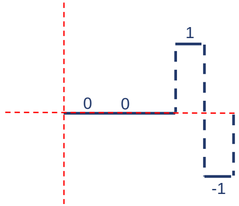
ψ_{00} = [1 1 -1 -1]

Scaled **wavelet** function



ψ_{10} = [1 -1 0 0]

Scaled **wavelet** function



ψ_{11} = [0 0 1 -1]

Haar Transform Matrix

- Unnormalized **8x8 Haar matrix**:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
=
\begin{matrix}
\varphi_{00} \\ \psi_{00} \\ \psi_{1,0} \\ \psi_{1,1} \\ \psi_{2,0} \\ \psi_{2,1} \\ \psi_{2,2} \\ \psi_{2,3}
\end{matrix}
$$

- The 1st row measures the **smoothed** value = $\sqrt{N}$ x mean
- The 2nd row measures a **low frequency component**
- The 3rd and 4th rows measure the **moderate frequency component**
- The 5-8th rows measure the **high frequency component**

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Haar Transform Matrix

- Normalized **8x8 Haar matrix**:

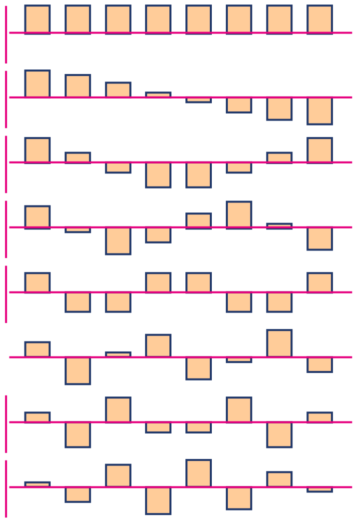
$$H_8 = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Matrix only contains real numbers and lots of zeros → low computational complexity

Haar Transform Matrix – Compare to DCT

- In a DCT matrix, the coefficients in each row of the transform matrix are the amplitudes of 8 samples of a cosine wave. The first row is a cosine wave of 0Hz (DC), then the frequencies of each cosine wave are increasing (AC).

$$\Psi = \frac{1}{2}\begin{bmatrix} .71 & .71 & .71 & .71 & .71 & .71 & .71 & .71 \\ .98 & .83 & .56 & .19 & -.19 & -.56 & -.83 & -.98 \\ .92 & .38 & -.38 & -.92 & -.92 & -.38 & .38 & .92 \\ .83 & -.19 & -.98 & -.56 & .56 & .98 & .19 & -.83 \\ .71 & -.71 & -.71 & .71 & .71 & -.71 & -.71 & .71 \\ .56 & -.98 & .19 & .83 & -.83 & -.19 & .98 & -.56 \\ .38 & -.92 & .92 & -.38 & -.38 & .92 & -.92 & .38 \\ .19 & -.56 & .83 & -.98 & .98 & -.83 & .56 & -.19 \end{bmatrix}$$

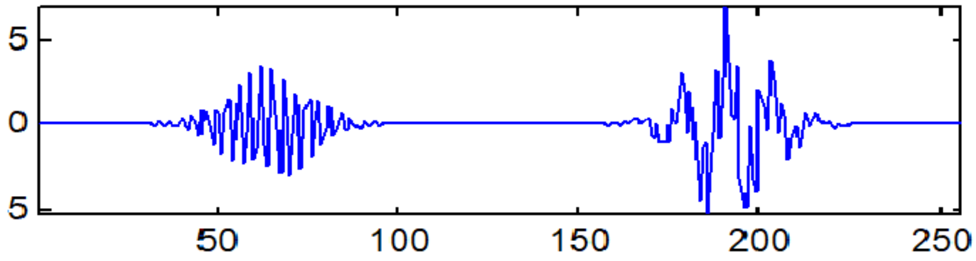


Haar Transform Matrix – Compare to DCT

- We see from the previous slide that rows of the Haar Matrix correspond to increasing frequency in a similar way to the DCT.
- However, in addition, the rows of the Haar Matrix also contain translation, that is, shifting in time.
- Remember that performing a transform is a similar process to correlation.
- We can then see that if we use the Haar Matrix to perform a transform we can then find short duration signals of different frequencies located at different times in a signal……Feature Extraction

Haar Transform Matrix – Feature Extraction

Non-stationary signal containing features of interest:



- ➢ Performing a Haar Transform on this signal could identify those features and the times they occurred.
- ➢ However, the correlation would not be very good because the shape of the Haar Functions and the shape of the features are different.
- ➢ So having functions of similar shape to the features would be useful…..hence a type of function called Wavelets

Haar Transform - Applications

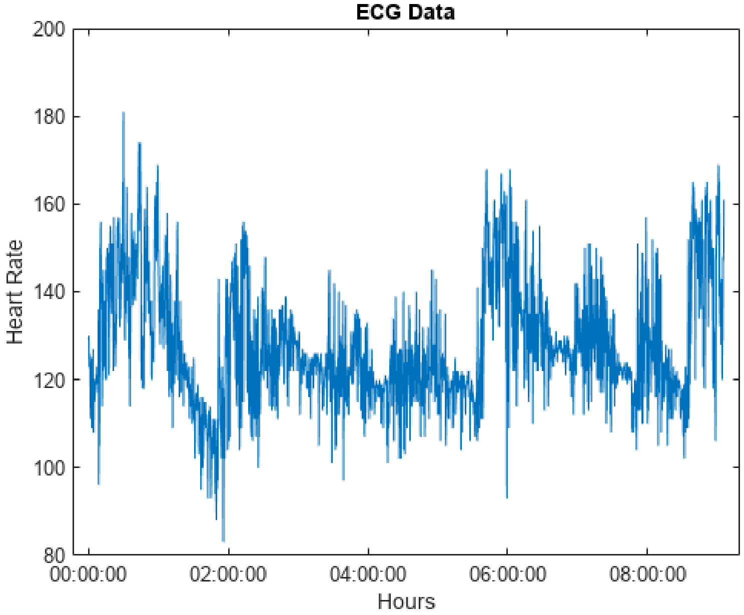
- Haar transform is beneficial for image/signal compression
  - However, Haar transform is a lossy data compression
  - The advantage of Haar transform is that it can achieve high compression ratio versus lossless techniques

- Haar transform is also good for denoising
  - When signal is received after transmission over some distance, it is often distorted by noise.

- General procedure: After Haar transform is performed, a thresholding is used, i.e., any values of the transformed signal lie below the noise/compression threshold is set to 0. Then, inverse Haar transform is performed to obtain the denoised/compressed signal

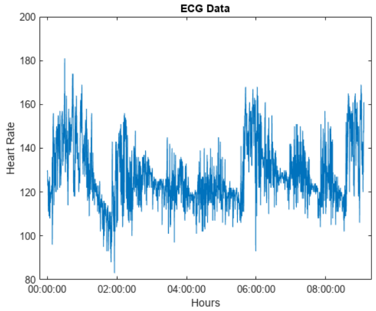
Haar Transform – Example

- Heart rate signal:
  - Sampled once every 16 seconds
  - Total duration is almost 10 hours ≈ 36,000 seconds
  - Total $2048 = 2^{11}$ samples
    - Max level 11 Haar Transform
  - Integer-valued heart rates
    - Discontinuity in data. Suitable for Haar Transform

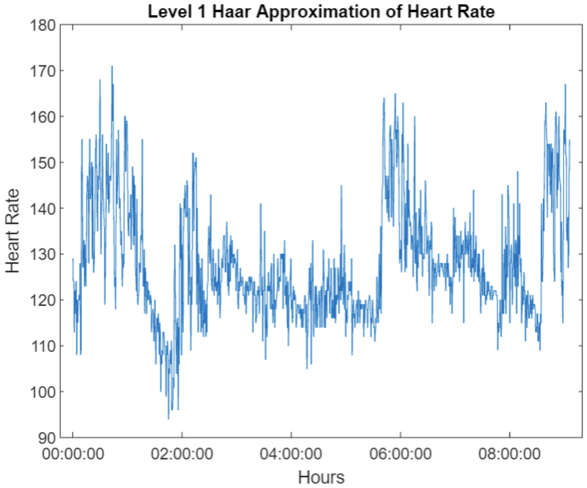


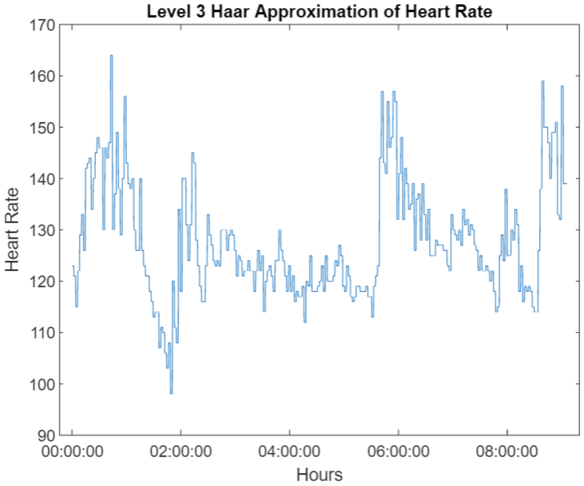
Haar Transform - Example

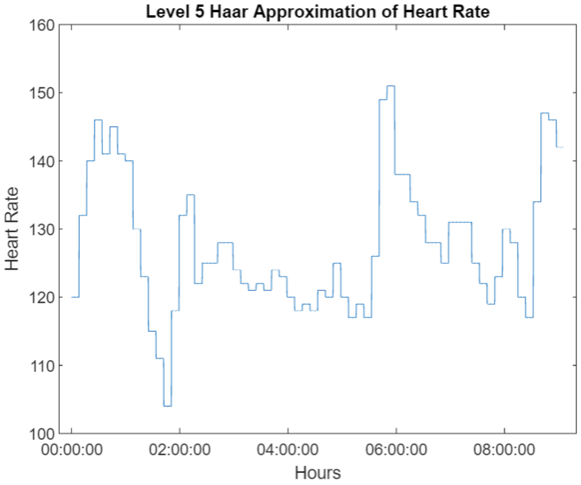
Original signal:



- Perform Level 11 forward Haar Transform
- Treat lower level coefs. as 0, then perform inverse Haar Transform







More levels of transform coefficients are discarded

Inverse Haar Transform Matrix

- The Haar transform matrix is real and orthogonal. Thus, the inverse Haar transform can be derived as its transpose:

$$H^{-1} = H^T$$

- For example, the normalized 4x4 inverse Haar transform matrix is derived as

$$H_4^{-1} = H_4^T = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

Inverse Haar Transform Matrix

- You can check this equation by evaluating HH^T, which gives you the identity matrix:

$$HH^T = I$$

$$H_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \qquad H_4^{-1} = H_4^T = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

Summary

- Haar functions are the simplest wavelet functions
- The system of Haar functions consists of the scaled and translated versions of the scaling function and the wavelet function.
- Haar Transform is fast because the matrix contains many zero terms and it is real (no complex terms).
- The inverse Haar matrix is the transpose of the forward Haar matrix
- Haar matrices need to be normalized before applied to the signal

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