Queen Mary
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EBU6018
Advanced Transform Methods

Tutorial – Transform Matrices

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Lecture Outline

- Tutorial
  - Discrete Fourier Transform
  - Discrete Cosine Transform
  - Discrete Wavelet Transform
    - Filterbank vs. Transform Matrix
  - Comparing DFT, DCT, DWT

Lecture Outline

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk} = X(\omega)\Big|_{\omega=\frac{2\pi}{N}k}$$

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Discrete Fourier Transform - Tutorial

1. Let F_N denotes the N-point DFT matrix, state the (k,n)-th entry of F_N.

Discrete Fourier Transform - Tutorial

2. Based on the given equation, state the range of n and k

$$F_N[k,n] = {W_N}^{nk} = e^{-\frac{j2\pi nk}{N}}, \quad W_N = e^{-\frac{j2\pi}{N}}$$

Discrete Fourier Transform - Tutorial

3. Derive the
    - 2x2 DFT matrix
    - normalized 2x2 DFT matrix

Discrete Fourier Transform - Tutorial

4. Derive the
   - 4x4 DFT matrix
   - normalized 4x4 DFT matrix

Discrete Fourier Transform - Tutorial

5. Perform DFT on the given input sequence

$$S[n] = [2, -3]$$

Discrete Fourier Transform - Tutorial

6. Perform DFT on the given input sequence

$$S[n] = [1, 2, -3, -5]$$

Discrete Fourier Transform - Tutorial

7. Explain the what DFT does to the input sequence, with reference to the DFT matrix.

Lecture Outline

$$DCT[k] = c(k) \sum_{n=0}^{N-1} s[n] \cos \frac{\pi(2n+1)k}{2N}$$

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$$c(k) = \begin{cases} \sqrt{1/N} & k = 0 \\ \sqrt{2/N} & k \neq 0 \end{cases}$$

$$k = 0,1,2 \ldots N-1$$

Discrete Cosine Transform – Tutorial

2. State the assumption of DCT on the input signal

Discrete Cosine Transform – Tutorial

4. Derive the

   - normalized 2x2 DCT matrix , in terms of cosine functions

N-point DCT: $DCT[k] = c(k)\sum_{n=0}^{N-1} s[n]\cos\frac{\pi(2n+1)k}{2N}$ $\quad c(k) = \begin{cases} \sqrt{1/N} & k = 0 \\ \sqrt{2/N} & k \neq 0 \end{cases}$

$$k = 0,1,2 \ldots N-1$$

Discrete Cosine Transform – Tutorial

5. Derive the
   - normalized 4x4 DCT matrix , in terms of cosine functions

N-point DCT: $DCT[k] = c(k)\sum_{n=0}^{N-1} s[n]\cos\frac{\pi(2n+1)k}{2N}$ $c(k) = \begin{cases} \sqrt{1/N} & k = 0 \\ \sqrt{2/N} & k \neq 0 \end{cases}$

$$k = 0,1,2\ldots N-1$$

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Discrete Cosine Transform – Tutorial

6. Calculate the
    - normalized 4x4 DCT matrix, to 2 decimal points

Discrete Cosine Transform – Tutorial

7. Describe what DCT does to the input signal, with reference to the DCT matrix

Discrete Cosine Transform – Tutorial

8. Based on the given DCT output, explain what it implies on the input sequence.

$$\begin{bmatrix} 2.1 \\ 0.6 \\ -1.2 \\ 1.9 \\ -0.1 \\ 2.6 \\ -1.7 \\ 10.3 \end{bmatrix}$$

Discrete Cosine Transform – Tutorial

9. What is main application of DCT? Describe how DCT can be employed for that application.

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Discrete Wavelet Transform - Tutorial

1. What are the definitions of wavelets and a wavelet family?

Discrete Wavelet Transform - Tutorial

2. State the unnormalized 8x8 Haar matrix.

Discrete Wavelet Transform - Tutorial

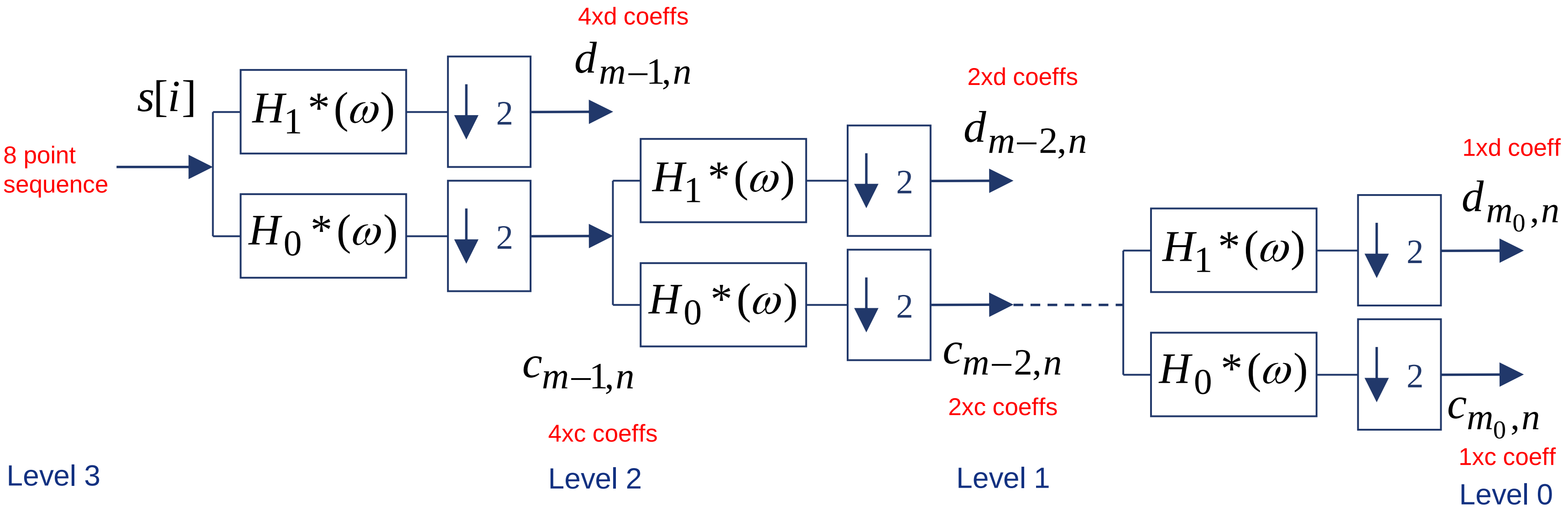
3. Determine the output of the unnormalised 8x8 Haar Transform for the input sequence:

S[n] = [1, 1, 1, 1, 10, 10, 10, 10]

s(t):

NOTE: I am using the UNNORMALISED matrix because I am looking for any significant feature in the data. Its exact transform is not required for this. The unnormalised arithmetic is simpler.

Discrete Wavelet Transform - Tutorial



- The 8x8 Haar Transform Matrix is performing 3 levels of decomposition.
- Input is at level 3, then decomposing to level 2, then to level 1 then to level 0.
- d coefficients are detail coefficients

Discrete Wavelet Transform - Tutorial

$$H_8 = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}}_{\text{8x8 Haar Transform Matrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} = \underbrace{\begin{bmatrix} c_{0,0} \\ d_{0,0} \\ d_{1,0} \\ d_{1,1} \\ d_{2,0} \\ d_{2,1} \\ d_{2,2} \\ d_{2,3} \end{bmatrix}}_{\text{Filterbank coef.}} \quad \begin{matrix} \varphi_0(\mathrm{t}) \\ \psi_0(\mathrm{t}) \\ \psi_{1,0}(\mathrm{t}) \\ \psi_{1,1}(\mathrm{t}) \\ \psi_{2,0}(\mathrm{t}) \\ \psi_{2,1}(\mathrm{t}) \\ \psi_{2,2}(\mathrm{t}) \\ \psi_{2,3}(\mathrm{t}) \end{matrix}$$

Notes on Haar Transform

- To perform a Haar Transform (or in general any wavelet transform) we would use the normalised filters in a filterbank and the normalised functions in a matrix.

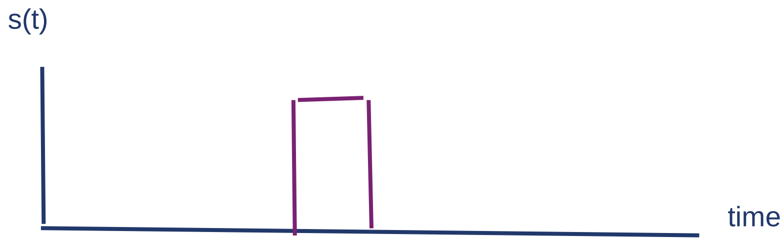
- For the examples given in the lecture on filterbanks using the Haar functions, the filter is given as $H = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$ because when multiplied by the $\sqrt{2}$ of the recursive equation then the filter is normalised.

- If the normalised filter is given, then there is no need to use the $\sqrt{2}$ of the recursive equation.

Discrete Wavelet Transform - Tutorial

4. Determine the output of the unnormalised 8x8 Haar Transform for the input sequence:

$$S[n] = [1, 1, 1, 10, 10, 1, 1, 1]$$



NOTE: I am using the UNNORMALISED matrix because I am looking for any significant feature in the data. Its exact transform is not required for this. The unnormalised arithmetic is simpler.

Discrete Wavelet Transform – Notes

- The resolution is not good in this example because for an 8-point sequence we have only three frequencies of the wavelet function.
- That is, only two changes of scale.
- So the scaled wavelet is not sufficiently narrow to identify narrow changes in the input signal.
- In practice, the input sequence would be much larger and we would use the relevant dimension of matrix giving more changes of scale and so much narrower functions.
- Also in practice, if we want to identify features of different shape we would use wavelet functions of a similar shape.

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  - **Comparing DFT, DCT, DWT**

Transform Methods – Exercise 1

For a 4-point input sequence

$$S[n] = [6, 3, -2.5, 7]$$

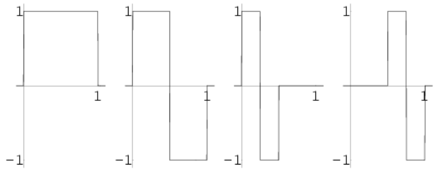
Determine the output from each of the following discrete transforms:

a. DFT

b. DCT

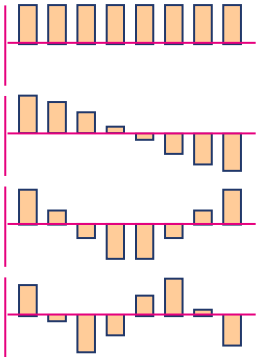
c. DWT, using a Haar wavelet function.

Transform Methods – Exercise 1 Solution



Input: S[n] = [6, 3, -2.5, 7]

$$\text{DFT} = \begin{bmatrix} 6.75 \\ 4.25 + 2i \\ -3.25 \\ 4.25 - 2i \end{bmatrix} \qquad \text{DCT} = \begin{bmatrix} 6.75 \\ 0.835 \\ 6.25 \\ -3.845 \end{bmatrix}$$



$$\text{DHT} = \begin{bmatrix} 6.75 \\ 2.25 \\ 2.12 \\ -6.72 \end{bmatrix}$$

- The first output of all three methods are the same. i.e. the smoothed value, divide by $\sqrt{N}$ to obtain mean
- DFT is the only one that produces complex outputs

- Three transform methods are designed for different purposes
  - DFT – frequency spectrum as a function of time
  - DCT – rate of change in the input
  - DHT – identify short duration features

DFT/DCT/DHT – Tutorial

- State the applications of DFT, DCT, and DWT

Notes on Comparing DFT/DCT/DHT

- DFT is the only transform method (among DFT,DCT,DHT) that outputs complex numbers
- DFT and DCT both assume the input to be periodic
- DCT further assumes the input to be even
- If the above assumption is not met, the output does not accurately represent the frequency information of the input signal that we sampled from
- The outputs of DFT and DCT on even functions are the same
- DHT (or any wavelet transform) is the only transform method (among DFT,DCT,DHT) that can perform time-frequency analysis

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