EBU6018
Fourier Series (Revision)

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Fourier Series (FS)

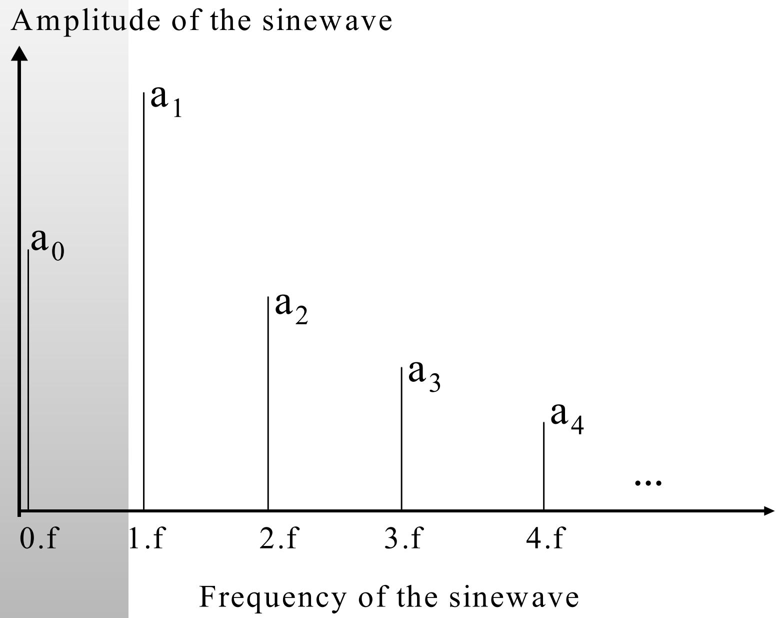
- Periodic signals can be expressed as a sum of sinusoids. The frequency spectrum can be generated by computation of the ***Fourier series***.

- The Fourier series is named after the French physicist Jean Baptiste Fourier (1768-1830), who was the first one to propose that periodic waveforms could be represented by a sum of sinusoids (or complex exponentials).

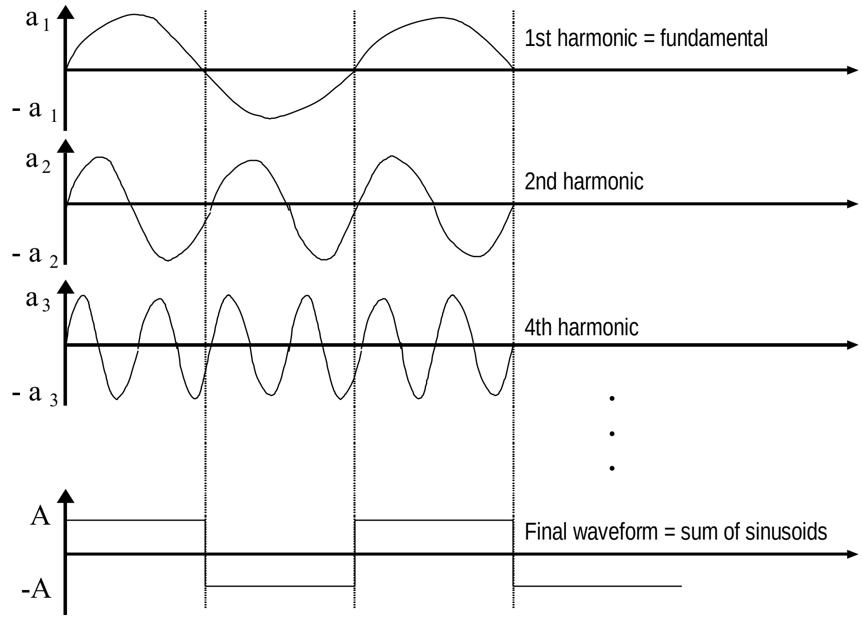
An example showing the Fourier series at work

http://www.falstad.com/fourier/

Introduction to Fourier Series

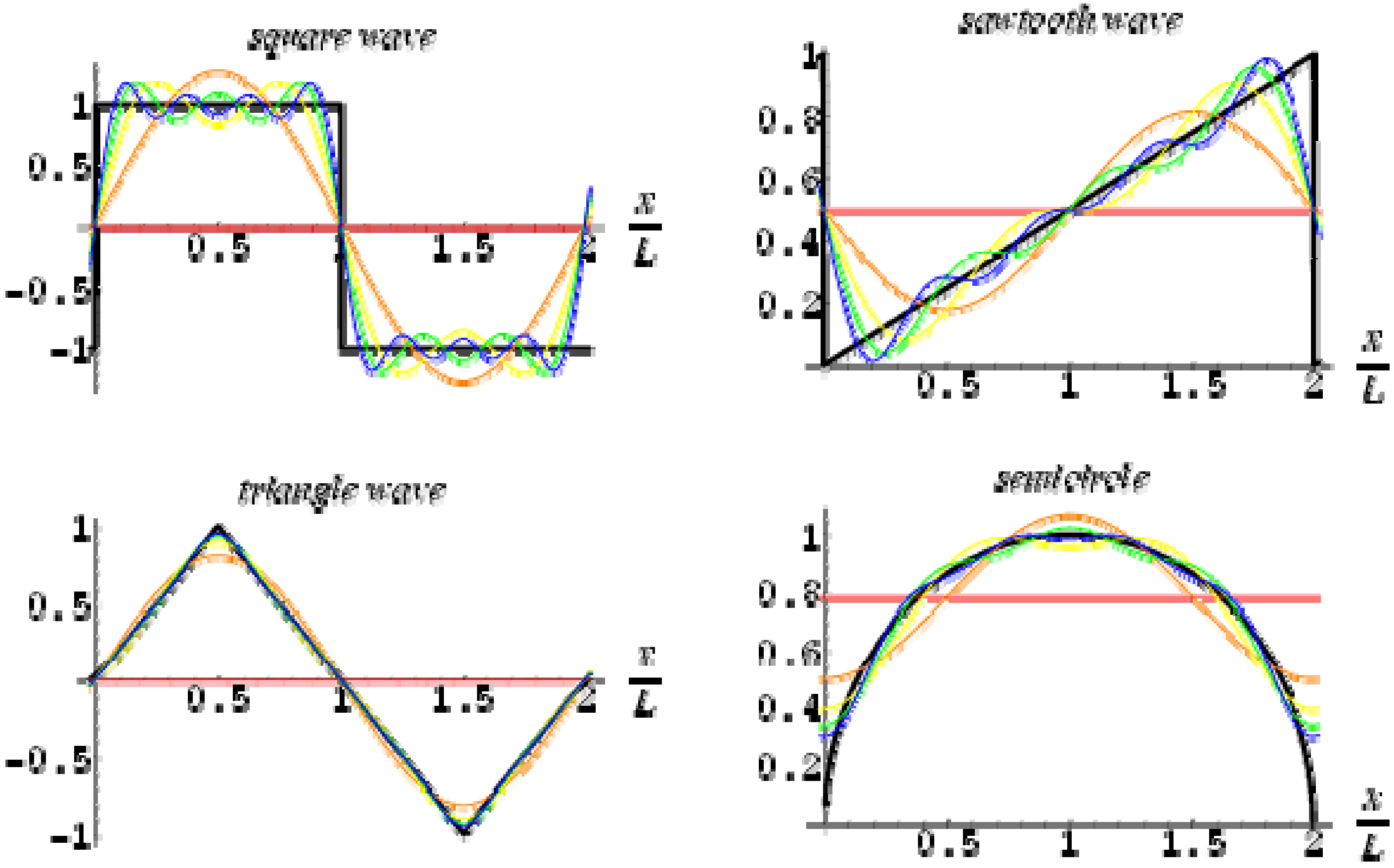


This diagram represents the frequency domain



This diagram represents the time domain (but those sinewaves do not give that time signal)

Fourier Representation



Trigonometric FS - 1

A periodic signal, x(t), whose period is T, can be represented by the appropriate sum of sin and cos components:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} b_n.\sin(n.\omega.t) \qquad (1)$$

a_0 is the ***mean value***, or ***zero frequency*** term.

Integrating both sides of eqn (1), between = -T/2 and T/2 :

$$\int_{-T/2}^{T/2} x(t)\,dt = \int_{-T/2}^{T/2} a_0 + \int_{-T/2}^{T/2} \left[\sum_{n=1}^{\infty} a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} b_n.\sin(n.\omega.t) \right] dt$$

Trigonometric FS - 2

$$\int_{-T/2}^{T/2} x(t)\,dt = \int_{-T/2}^{T/2} a_0 + \int_{-T/2}^{T/2} \cancel{\left[\sum_{n=1}^{\infty} a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} b_n.\sin(n.\omega.t) \right] dt}$$

$$\int_{-T/2}^{T/2} x(t)\,dt = \int_{-T/2}^{T/2} a_0 \; dt = a_0.T$$

$$a_0 = 1/T \int_{-T/2}^{T/2} x(t)\,dt$$

Trigonometric FS - 3

To find a formula for an it is necessary to multiply both sides of eqn(1) by cos(m.ω.t) and then integrate over the same limits:

$$\int_{-T/2}^{T/2} x(t)\cos(m.\omega.t)\ dt = \int_{-T/2}^{T/2} a_0.\cos(m.\omega.t)\ +$$

$$+ \int_{-T/2}^{T/2} [\ \underbrace{\sum_{n=1}^{\infty} \cos(m.\omega.t).a_n.\cos(n.\omega.t)}_{\text{the “cos.cos” terms}} + \underbrace{\sum_{n=1}^{\infty} \cos(m.\omega.t)\ b_n.\sin(n.\omega.t)}_{\text{the “cos.sin” terms}}\ ]dt$$

•Using the appropriate trig identities we can see that the cos.sin terms all produce
cos(A).sin(B) = ½ (sin(A+B) +/- sin(A-B))
odd waveforms which all disappear under integration.
•The cos.cos terms produce:
cos(A).cos(B) = ½ (cos(A+B) +/- cos(A-B))
which will not necessarily disappear under integration:

Trigonometric FS - 4

$$\int_{-T/2}^{T/2} \sum_{n=1}^{\infty} \qquad \underline{\cos(m.\omega.t)\ .\ a_n.\cos(n.\omega.t)}$$

$$a_n\ \tfrac{1}{2}\ (\ \cos((m+n).\omega.t)\ \ +\cos((m-n).\omega.t)\ )$$

HOWEVER, we are integrating over -T/2 $\rightarrow$ +T/2 and this represents an integer number of cycles of the sinusoid, whatever the value of 'm' and 'n'. BUT when m=n, we have a non-zero term after integration:

$$\int_{-T/2}^{T/2} x(t).\cos(m.\omega.t)\ \ dt\ = \int_{-T/2}^{T/2} \cancel{a_0.\ \cos(m.\omega.t)}\ + \int_{-T/2}^{T/2} a_n.\ \ \tfrac{1}{2}\ \ \cos((0).\omega.t)\ )$$

$$+ \int_{-T/2}^{T/2} [\ \cancel{\sum_{n=1}^{\infty} \cos(m.\omega.t).a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} \cos(m.\omega.t)\ b_n.\sin(n.\omega.t)}\ ]dt$$

$$\int_{-T/2}^{T/2} x(t)\ \cos(m.\omega.t)\ \ dt\ \ =\ \ (a_n./2)\ \Big|\ t\ \Big|_{-T/2}^{T/2} \qquad =\ \ a_n\ .\ T\ /2$$

Trigonometric FS - 5

BUT m=n, so:

$$\int_{-T/2}^{T/2} x(t)\cos(n.\omega.t)\ dt = a_n./2\ |\ t\ |_{-T/2}^{T/2} = a_n\ .\ T/2$$

$$a_n = 2/T \int_{-T/2}^{T/2} x(t).\cos(n.\omega.t)\ dt$$

And by similar reasoning:

$$b_n = 2/T \int_{-T/2}^{T/2} x(t).\sin(n.\omega.t)\ dt$$

Trigonometric Fourier Series – Cosine-with-phase form

The trigonometric Fourier series given by equ (1) can also be written in the cosine-with-phase form

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_k) \qquad -\infty < t < \infty$$

$$A_n = \sqrt{a_n^2 + b_n^2} \quad , n = 1,2,....$$

$$\theta_n = \begin{cases} \tan^{-1}(-\dfrac{b_n}{a_n}) \, , & n = 1,2,..., when \;\; a_n \geq 0 \\ \pi + \tan^{-1}(-\dfrac{b_n}{a_n}) \, , & n = 1,2,..., when \;\; a_n < 0 \end{cases}$$

Trigonometric Fourier Series – Dirichlet conditions

Fourier believed that any periodic signal could be expressed as a sum of sinusoids. However, this turned out not to be the case, although virtually all periodic signals arising in engineering do have a Fourier series representation. In particular, a periodic signal x(t) has a Fourier series if it satisfies the following *Dirichlet conditions*:

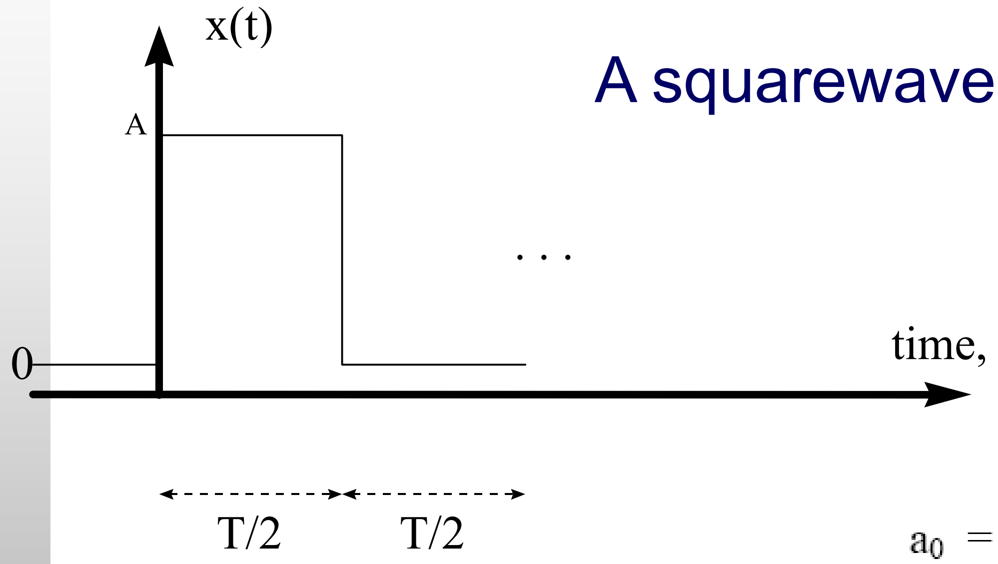
1. x(t) is absolutely integrable over any period; that is

$$\int_{a}^{a+T} |x(t)| dt < \infty \qquad for\ \ any\ \ a$$

2. x(t) has only a finite number of maxima and minima over any period.

3. x(t) has only a finite number of discontinuities over any period.

Application of the FS Example 1



A squarewave

$$a_0 = 1/T \int_{-T/2}^{T/2} x(t)\,dt$$

$$= 1/T \int_{0}^{T/2} A\,dt + \int_{T/2}^{T} 0\,dt$$

$$= 1/T\,|A.t|_{0}^{T/2}$$

$$= A/2$$

Application of the FS Example 1

$$a_n = 2/T \int_{-T/2}^{T/2} x(t).\cos(n.\omega.t)\, dt \quad = 2/T \int_{0}^{T/2} A.\cos(n.\omega.t)\, dt + \int_{T/2}^{T} 0\, dt$$

$$= 2A/T \left| \sin(n.\omega.t) \;/\; (n.\omega) \right|_{0}^{T/2}$$

$$= A/n\pi\, [\sin(n.\pi)] = 0$$

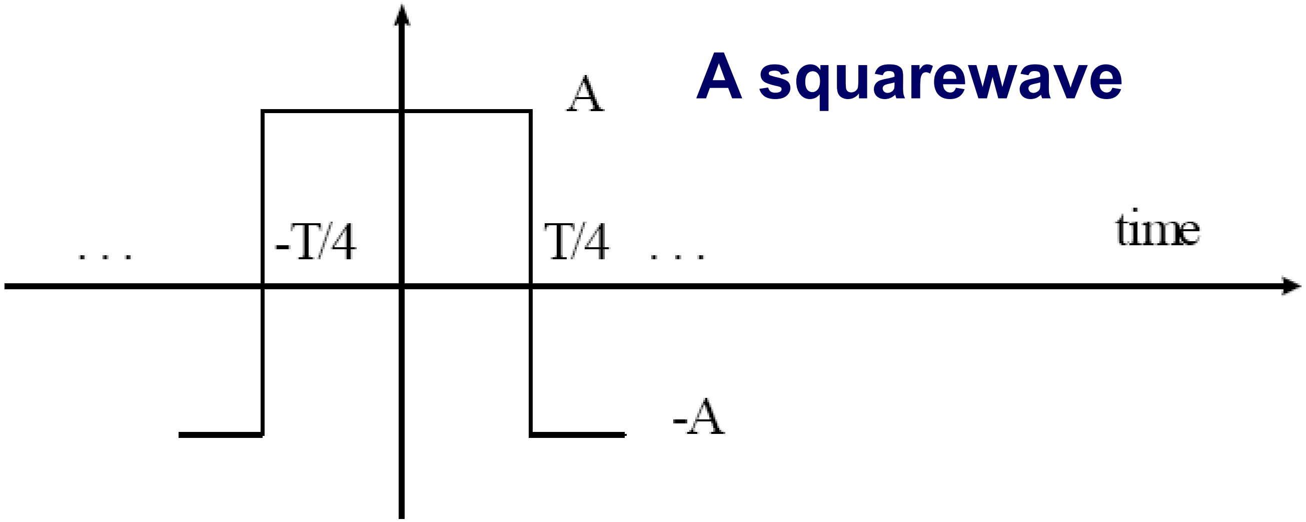
$$b_n = 2/T \int_{-T/2}^{T/2} x(t).\sin(n.\omega.t)\, dt$$

$$= 2/T \int_{0}^{T/2} A.\sin(n.\omega.t)\, dt + \int_{T/2}^{T} 0\, dt$$

$$= 2A/T \left| -\cos(n.\omega.t) \;/\; (n.\omega) \right|_{0}^{T/2}$$

$$= A/n\pi\, [\, 1 - \cos(n.\pi)\,]$$

Application of the FS
Example 2



A squarewave

$a0 = 0$ by inspection

$$a_n = 2/T \int_{-T/4}^{3T/4} x(t).\cos(n.\omega.t)\, dt$$

Application of the FS Example 2

$$= 2/T \int_{-T/4}^{T/4} A.\cos(n.\omega.t)\,dt + \int_{T/4}^{3T/4} -A.\cos(n.\omega.t)\,dt$$

$$= 2A/T \left| \sin(n\omega t) / n\omega \right|_{-T/4}^{T/4} - 2A/T \left| \sin(n\omega t) / n\omega \right|_{T/4}^{3T/4}$$

$$= 2A/nT\omega \; [\, \sin(n\omega T/4) - \sin(n\omega(-T)/4) - \sin(3n\omega T/4) + \sin(n\omega T/4) \,]$$

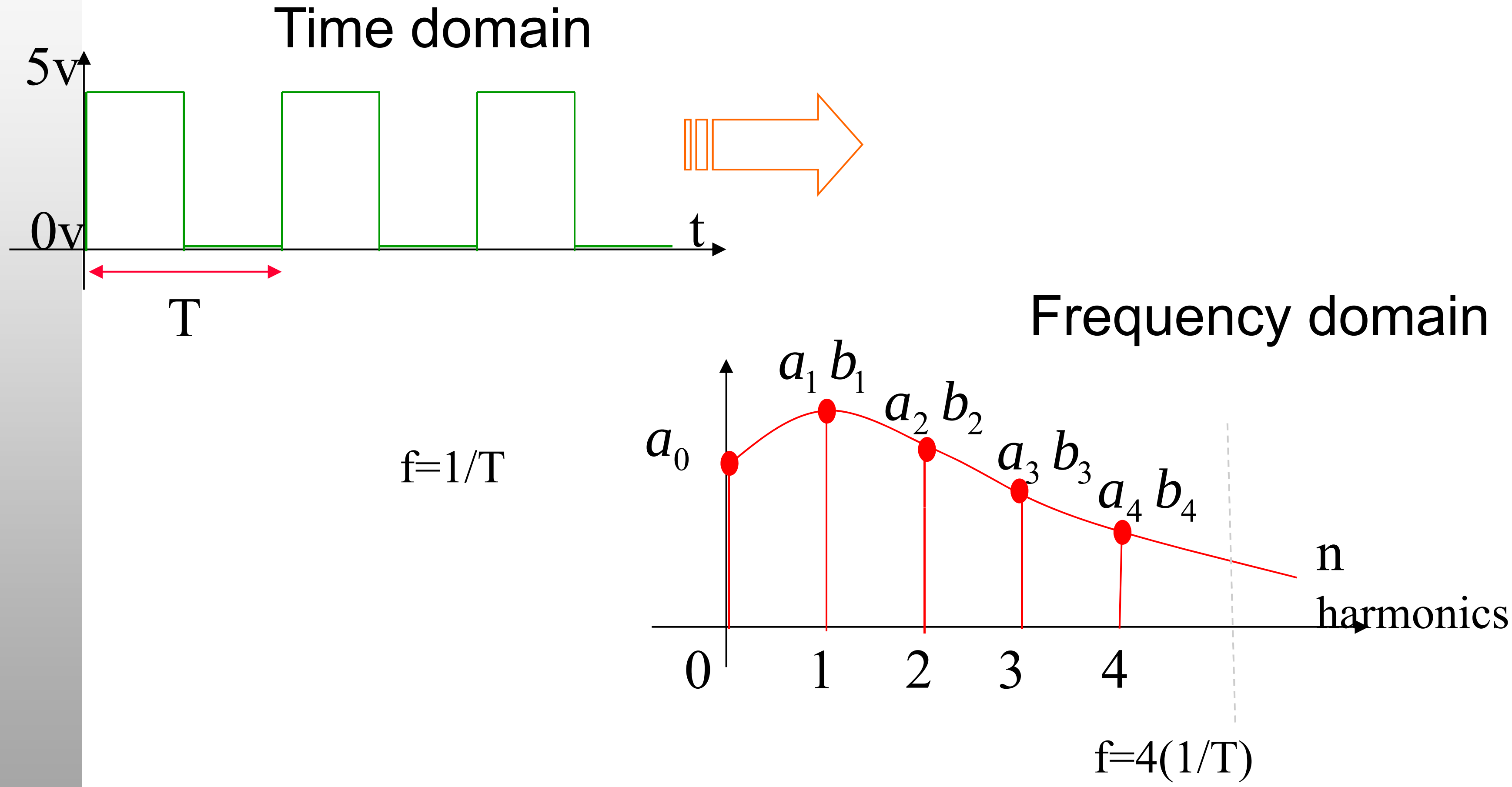
but $\omega T = (2\pi f).(1/f) = 2\pi$, $\sin(-A) = -\sin(A)$ and $\sin(3n2\pi/4) = -\sin(n\pi/2)$ therefore:

$$= 2A/n2\pi \; [\, \sin(n2\pi/4) - \sin(-n2\pi/4) - \sin(3n2\pi/4) + \sin(n2\pi/4) \,]$$

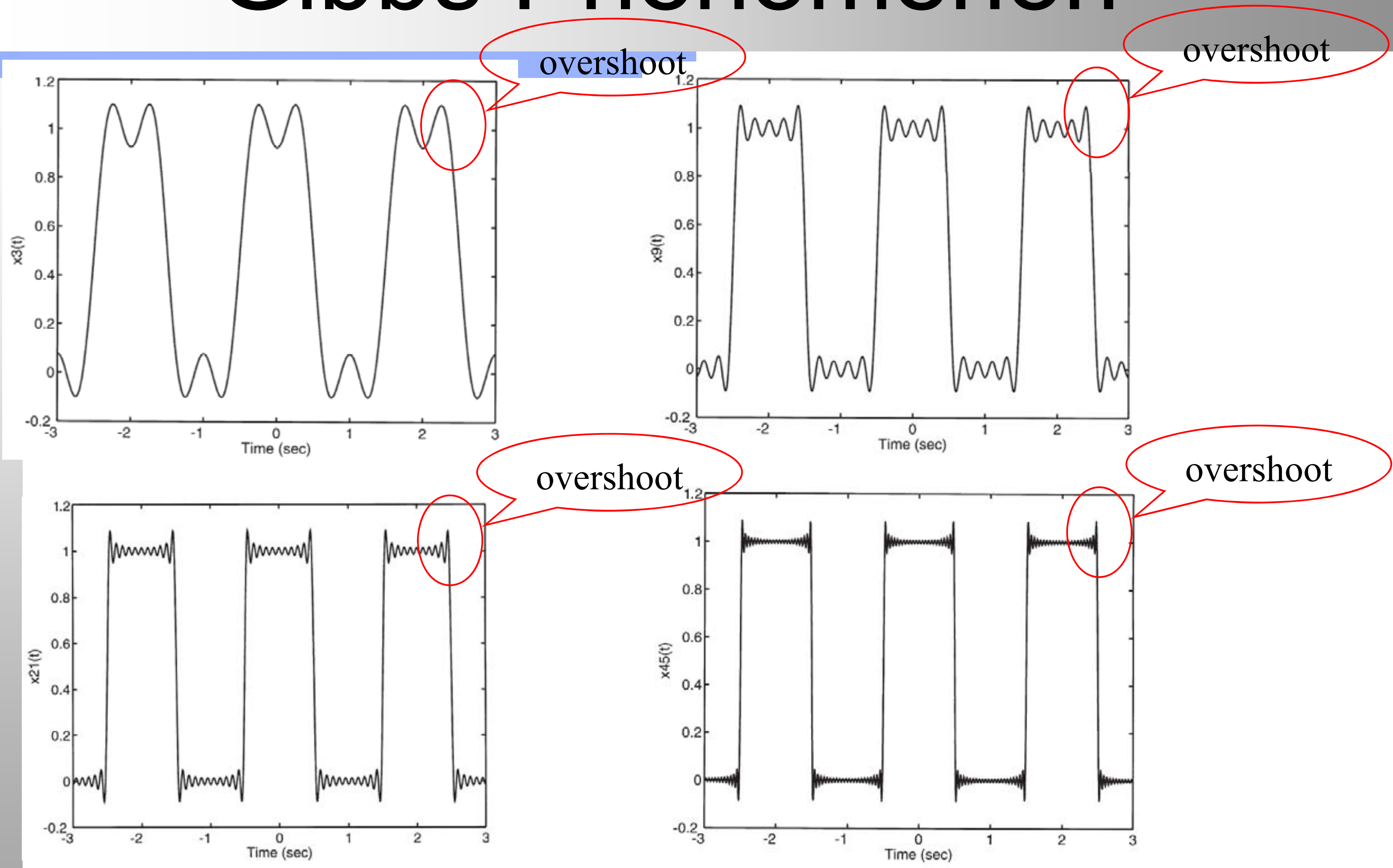
$$= 4A/n\pi[\, \sin(n\pi/2) \,]$$

$b_n = 0$ by inspection

Example: transform the signal to line spectra



Gibbs Phenomenon



Gibbs Phenomenon

- **The overshoot at the corners is still present even in the limit as N approaches to infinity. This characteristic was first discovered by Josiah Willard Gibbs (1893-1903), and this overshoot is referred to as the *Gibbs phenomenon*.**
- **Now let x(t) be an arbitrary periodic signal. As a consequence of the Gibbs phenomenon, the Fourier series representation of x(t) is not actually equal to the true value of x(t) at any points where x(t) is discontinuous.**
- **If x(t) is discontinuous at $t = t_1$, the Fourier series representation is off by approximately 9% at t_1^+ and t_1^-.**

The exponential form of the Fourier Series

- **Let's recall the original form of Fourier series:**

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} b_n.\sin(n.\omega.t)$$

- **In order to reduce the amount of 'writing out' the Fourier series, an exponential form can be expressed as:**

$$a_n.\cos(n.\omega.t) = (a_n/2) . [e^{jn\omega t} + e^{-jn\omega t}]$$

$$b_n.\sin(n.\omega.t) = (b_n/2j) . [e^{jn\omega t} - e^{-jn\omega t}]$$

$$a_n.\cos(n.\omega.t) + b_n.\sin(n.\omega.t) = (a_n/2) [e^{jn\omega t} + e^{-jn\omega t}] + (b_n/2j) [e^{jn\omega t} - e^{-jn\omega t}]$$

$$= X_n . e^{jn\omega t} + X_{-n} . e^{-jn\omega t}$$

where: $X_n = ½ (a_n - j\, b_n) \quad n \neq 0$

$X_{-n} = ½ (a_n + j\, b_n) \quad n \neq 0$

So the original Fourier Series can be written out as: $$x(t) = \sum_{n=-\infty}^{\infty} X_n. e^{jn\omega t}$$

Where we have defined: $X_0 = a_0$

Summary of the Fourier Series

- **Three forms**
  - **Original (sine and cosine components)**

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} b_n.\sin(n.\omega.t)$$

  - **Cosine-with-phase form**

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_k) \qquad -\infty < t < \infty$$

  - **Exponential form**

$$x(t) = \sum_{n=-\infty}^{\infty} X_n.\, e^{jn\omega t}$$

- **Dirichlet conditions**
- **Gibbs Phenomenon**