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EBU6018
Advanced Transform Methods

Tutorial: Haar Transform

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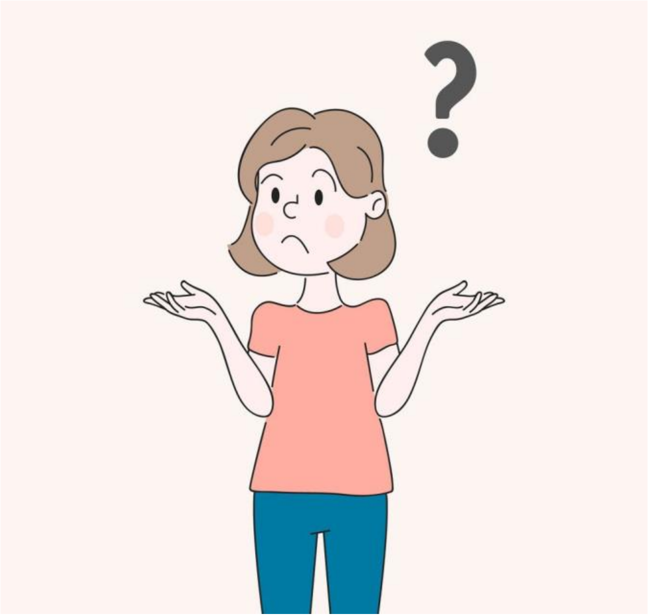
Haar Transform - Quiz

Question 1

Which of the following is true?

a. Haar transform is a non-orthogonal transform
b. Haar transform has fixed basis functions
c. Haar transform is slow
d. Haar transform is complex-valued



Haar Transform - Quiz

Question 1

Which of the following is true?

a. Haar transform is a non-orthogonal transform — The Haar transform is an orthogonal transform
b. Haar transform has fixed basis functions — Correct ! The basis functions are independent of the signal
c. Haar transform is slow — The Haar transform is fast
d. Haar transform is complex-valued — The Haar transform is real-valued

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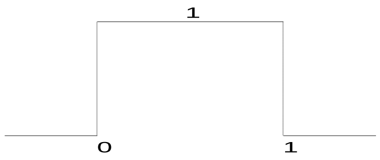
Haar Transform - Quiz

Question 2

What is the name of this function in terms of Haar transform?

a. Wavelet function
b. Transform function
c. Square function
d. Scaling function



Haar Transform - Quiz

<u>Question 2</u>

What is the name of this function in terms of Haar transform?

a. Wavelet function
b. Transform function
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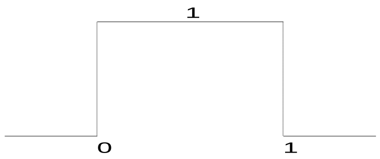
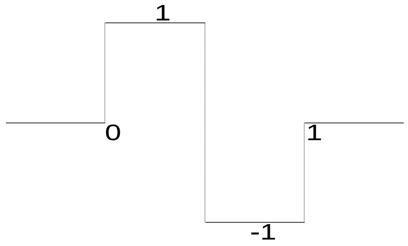


Haar Transform - Quiz

Question 3

Which one is **not** name of this function in terms of Haar transform?

a. Wavelet function
b. Mother wavelet
c. Daughter wavelet



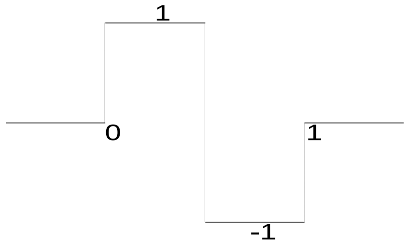
Haar Transform - Quiz

Question 3

Which one is **not** the name of this function in terms of Haar transform?

a. Wavelet function
b. Mother wavelet
c. Daughter wavelet

$$\psi(x) \equiv \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\psi_{jk}(x) \equiv \psi\left(2^j x - k\right)$$

Haar Transform - Quiz

$$H_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

4x4 Haar transform matrix

Question 4

Given the 4x4 Haar transform matrix, which one is the 4x4 inverse Haar transform matrix?

a. A
b. B
c. C
d. D

A

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -\sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

B

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

C

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & -\sqrt{2} & 0 \\ 1 & 1 & \sqrt{2} & 0 \\ 1 & -1 & 0 & -\sqrt{2} \\ 1 & -1 & 0 & \sqrt{2} \end{bmatrix}$$

D

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & -\sqrt{2} \\ 1 & -1 & 0 & \sqrt{2} \end{bmatrix}$$

Haar Transform - Quiz

$$H_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

4x4 Haar transform matrix

Question 4

Given the 4x4 Haar transform matrix, which one is the 4x4 inverse Haar transform matrix?

a. A
b. B — Correct !
c. C
d. D

The inverse Haar matrix is the transpose of the forward Haar matrix

A

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -\sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

B

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

C

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & -\sqrt{2} & 0 \\ 1 & 1 & \sqrt{2} & 0 \\ 1 & -1 & 0 & -\sqrt{2} \\ 1 & -1 & 0 & \sqrt{2} \end{bmatrix}$$

D

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & -\sqrt{2} \\ 1 & -1 & 0 & \sqrt{2} \end{bmatrix}$$

Example 1

- Apply the Haar Transform to the 4-point input sequence:

$$S[n] = [2, 5 -3, 7]$$

Example 1 - Solution

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ -3 \\ 7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 11 \\ 3 \\ (2-5)\sqrt{2} \\ (-3-7)\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ \frac{3}{2} \\ \frac{-3}{\sqrt{2}} \\ \frac{-10}{\sqrt{2}} \end{bmatrix}$$

Example 2

$$H_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

- For the normalized 4x4 Haar matrix show that

$$H_4 H_4^T = \mathrm{I}_4$$

Example 2 - Solution

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \text{x} \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{I}_4$$

Example 3

- Perform a Haar Transform on the 4-point input sequence :

$$S[n] = [1, 2, 3, 4]$$

- Reconstruct the input sequence using the inverse Haar transform.

Example 3 - Solution

Forward Transform:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 10 \\ -4 \\ (1-2)\sqrt{2} \\ (3-4)\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{2} \\ \frac{-4}{2} \\ \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Example 3 - Solution

Inverse Transform:

$$\frac{1}{2}\begin{bmatrix}1 & 1 & \sqrt{2} & 0\\ 1 & 1 & -\sqrt{2} & 0\\ 1 & -1 & 0 & \sqrt{2}\\ 1 & -1 & 0 & -\sqrt{2}\end{bmatrix}\begin{bmatrix}5\\ -2\\ \frac{-1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}5-2-1\\ 5-2+1\\ 5+2-1\\ 5+2+1\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 3\\ 4\end{bmatrix}$$

Example 4

- Compute the **normalized** 8x8 Haar Transform Matrix

The unnormalized matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Example 4 - Solutions

- Normalize each row
  - Divide each row vector $[x_1, x_2, \ldots, x_8]$ by $\sqrt{x_1^2 + x_2^2 + \cdots + x_8^2}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{\text{Normalization}} \boxed{\frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}}$$

$* \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$

Example 5 – Part 1

- Perform Haar Transform on the 8-point input sequence:

$$[\,1, 1, 1, -1, -1, -1, 2, -2]$$

❖ Here is the 8x8 normalized Haar transform matrix:

$$\frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Example 5 - Part 1 Solutions

$$\frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{4}{\sqrt{8}} \\ \frac{2\sqrt{2}}{\sqrt{8}} \\ -\frac{2\sqrt{2}}{\sqrt{8}} \\ 0 \\ \frac{4}{\sqrt{8}} \\ 0 \\ \frac{8}{\sqrt{8}} \end{bmatrix}$$

What can you interpret from this output?

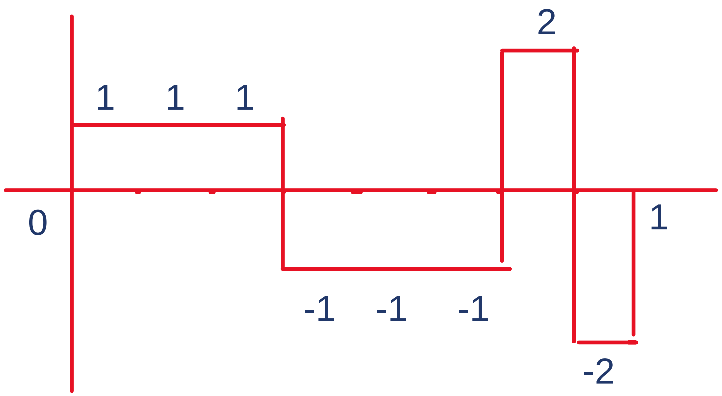
Example 5 – Part 2

- Given the input sequence and the Haar transform output. Explain the meaning of each transform coefficient in terms of the input
  - From both time and frequency prospective
  - Visualize the input may help

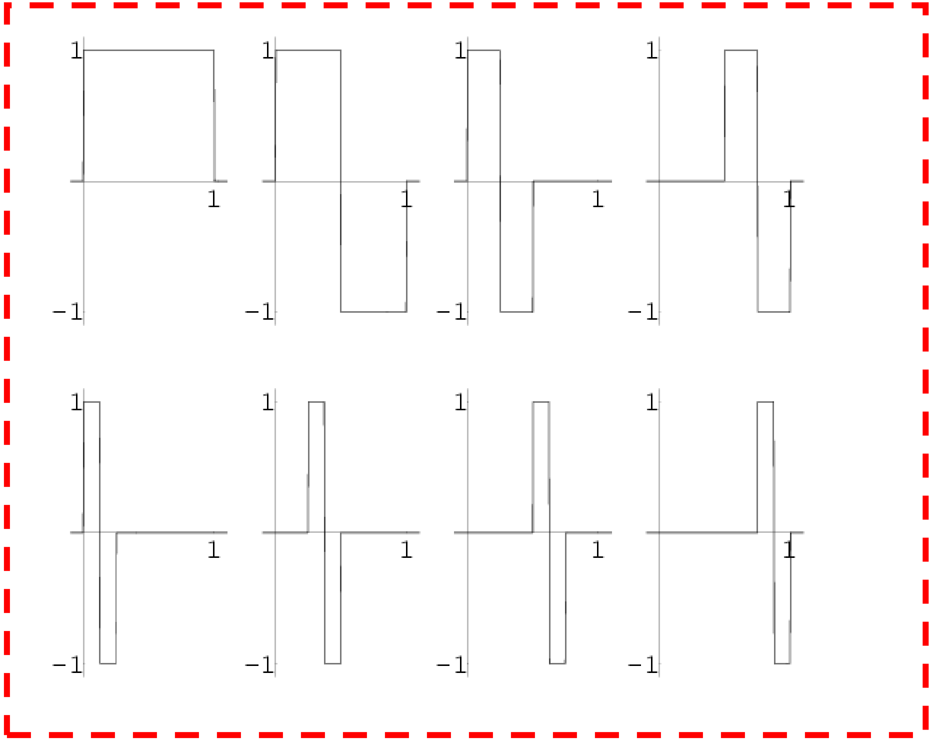
$$\frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{4}{\sqrt{8}} \\ \frac{2\sqrt{2}}{\sqrt{8}} \\ -\frac{2\sqrt{2}}{\sqrt{8}} \\ 0 \\ \frac{4}{\sqrt{8}} \\ 0 \\ \frac{8}{\sqrt{8}} \end{bmatrix}$$

Example 5 – Part 2 Solutions

- The input can be plotted as



➢ Compare the shape to the 8-point Haar functions



Example 5 – Part 2 Solutions

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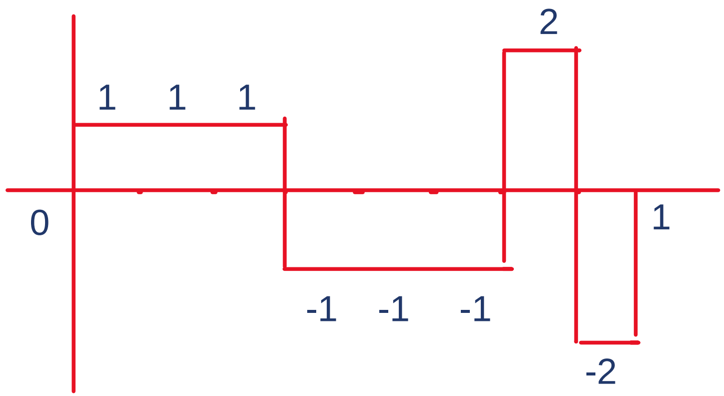
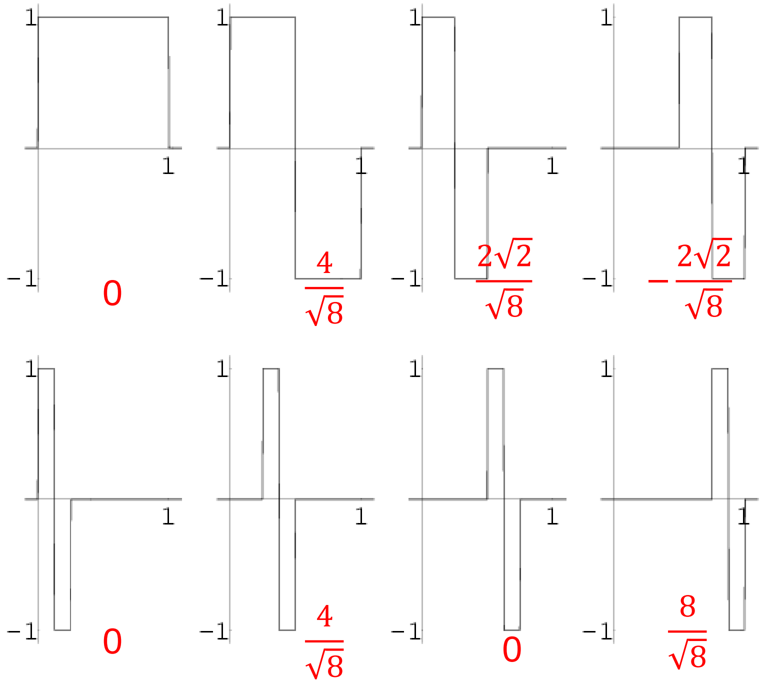
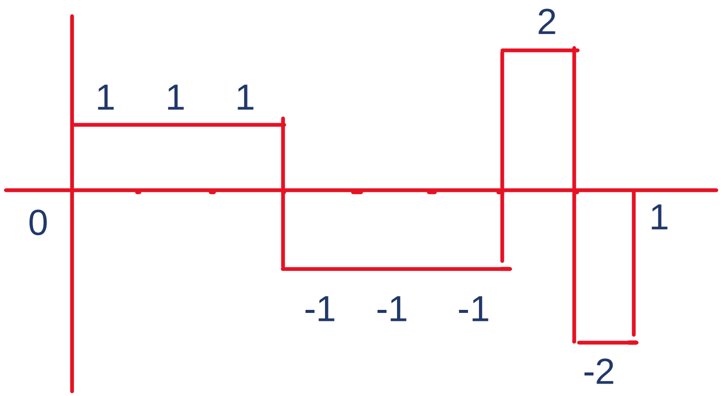


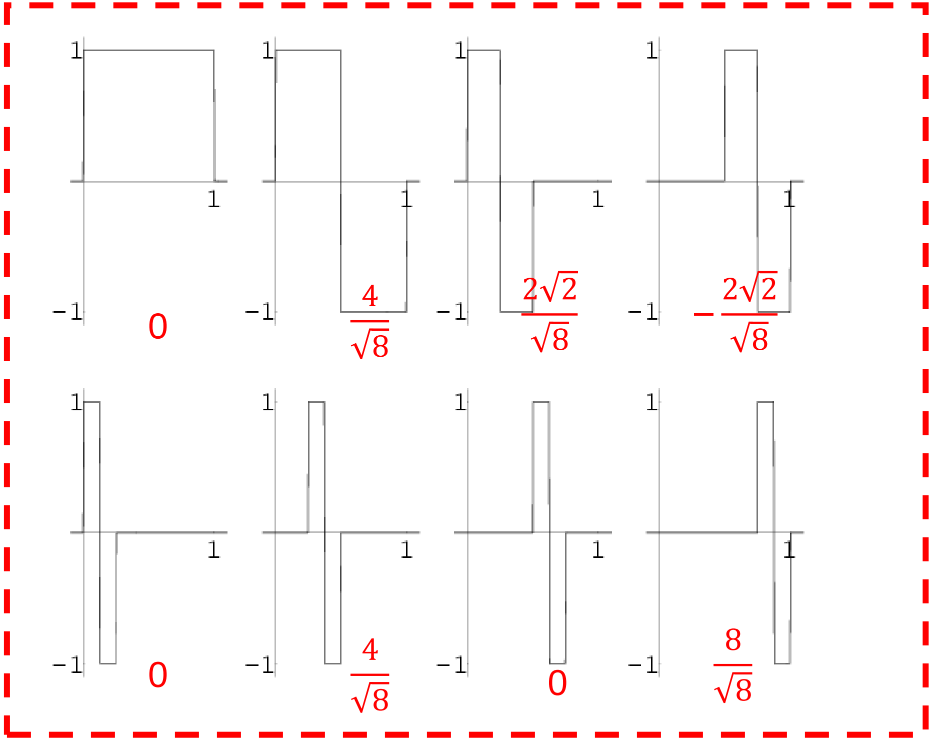


- ➢ Compare the shape to the 8-point Haar functions
- ➢ Compare the transform coefficients of the functions

Example 5 – Part 2 Solutions

- The input can be plotted as





➢ Generally speaking, each coefficient describes how similar the input and the Haar function are.

❖ Each coef. has a more specific meaning

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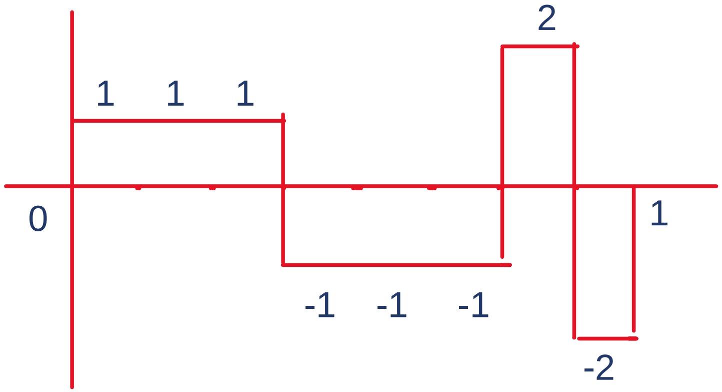
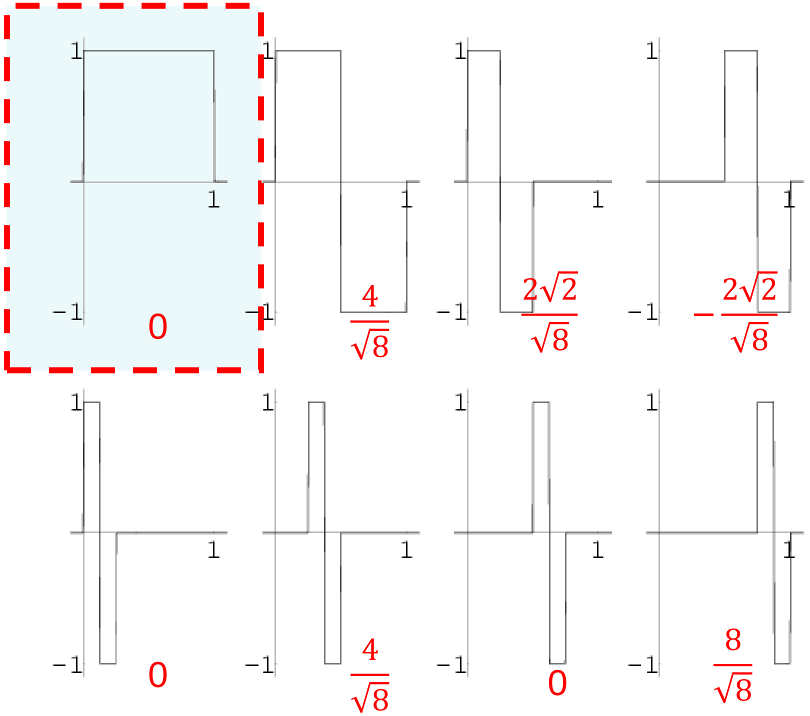
Example 5 – Part 2 Solutions

- The input can be plotted as

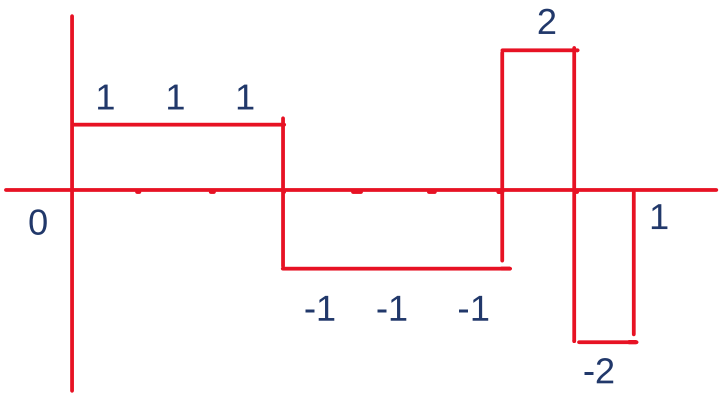


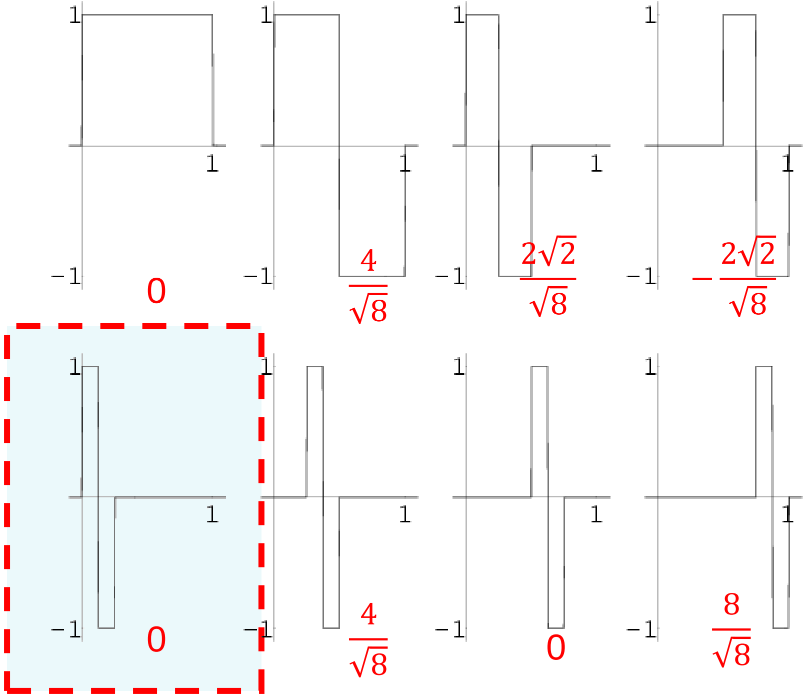


- More specifically:
  - The 1st coef. Indicates that the input has a mean of zero

Example 5 – Part 2 Solutions

- The input can be plotted as



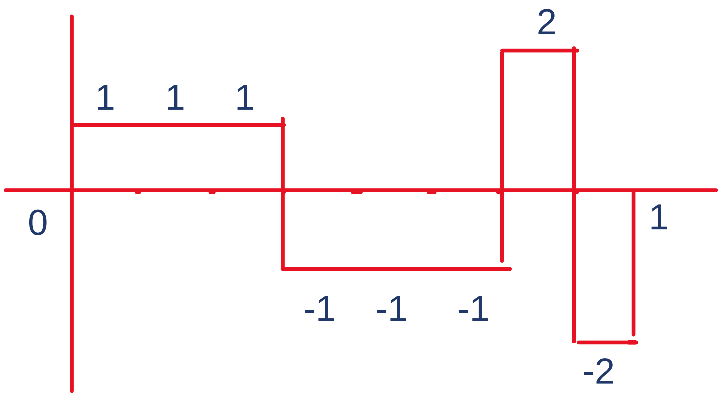


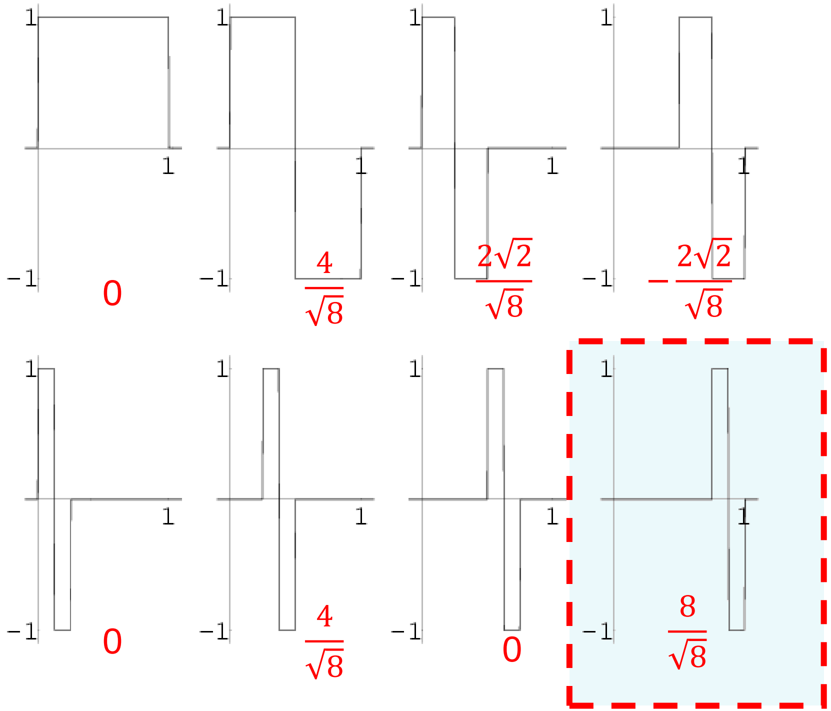
➢ More specifically:

❖ The 5th coef. Indicates that the input has no high-frequency component during $t = [0, \frac{2}{8}]$

Example 5 – Part 2 Solutions

- The input can be plotted as



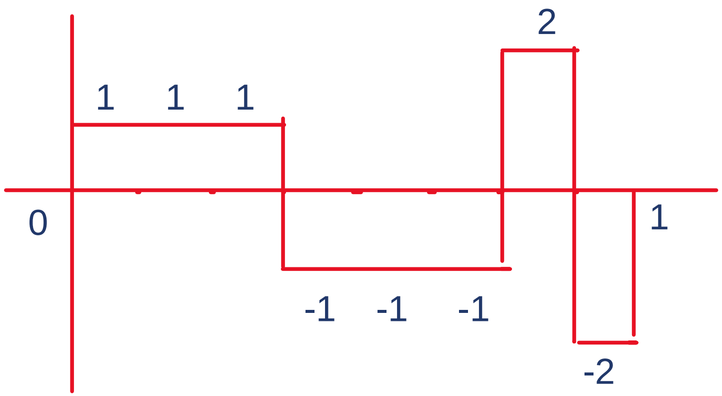


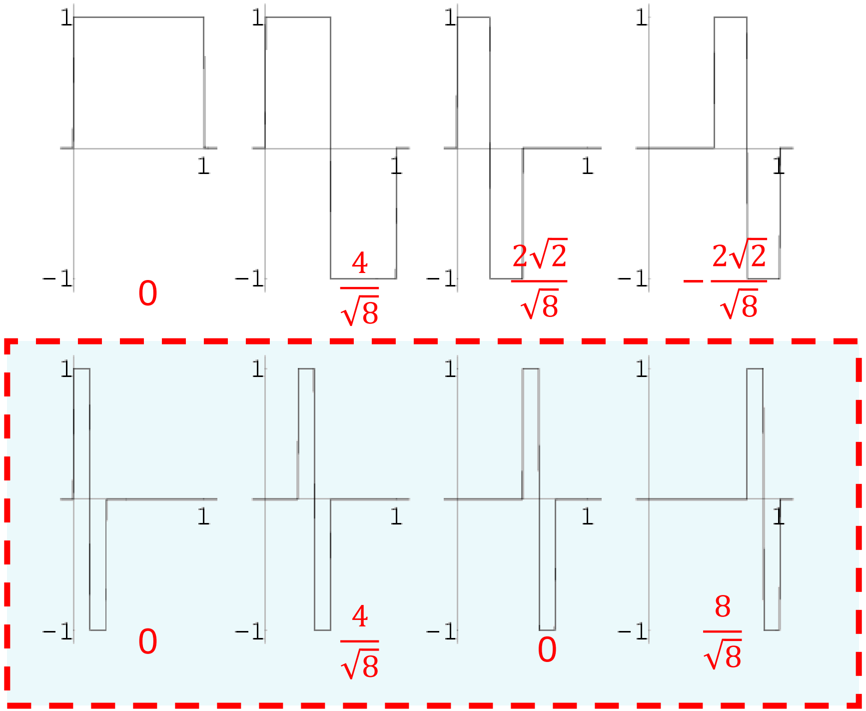
➢ More specifically:

❖ The 8th coef. Indicates that the input has some high-frequency component during time 6-8

Example 5 – Part 2 Solutions

- The input can be plotted as





- More specifically:
  - The magnitude of the 8th coef. Is the largest, which indicates the signal has the highest frequency during that period among all time slots

Summary

- We have seen that a Haar Matrix can be constructed to perform Haar Transforms directly.
- The **Haar Transform is fast** because the matrix contains many zero terms and it is real (no complex terms).
- It can be used to **identify frequency components** in the signal to be analysed (fine detail).
- It can be used to **identify the trends in the input data** (approximations).
- It can be used for **compression** by reducing or eliminating the coefficients corresponding to high frequencies in the signal and then inverting the transform.

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