Queen Mary
University of London

EBU6018
Advanced Transform Methods

Week 4.4 – WVD and Uncertainty Principle

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Lecture Outline

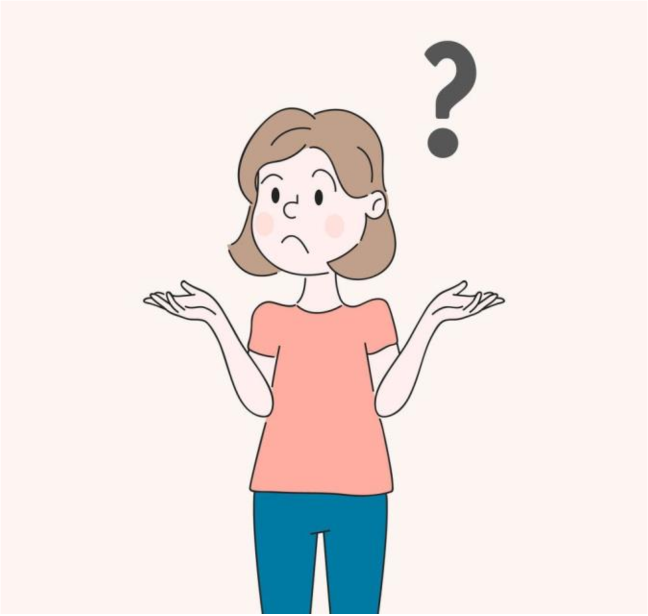
1. **Wigner-Ville Distribution**
   - Recap
   - Matlab Example
2. **Uncertainty Principle**
   - Examples
   - Fourier Transform

Wigner-Ville Distribution - Quiz

Question 1

WVD is a good at analyzing non-stationary signal.

a. True
b. False

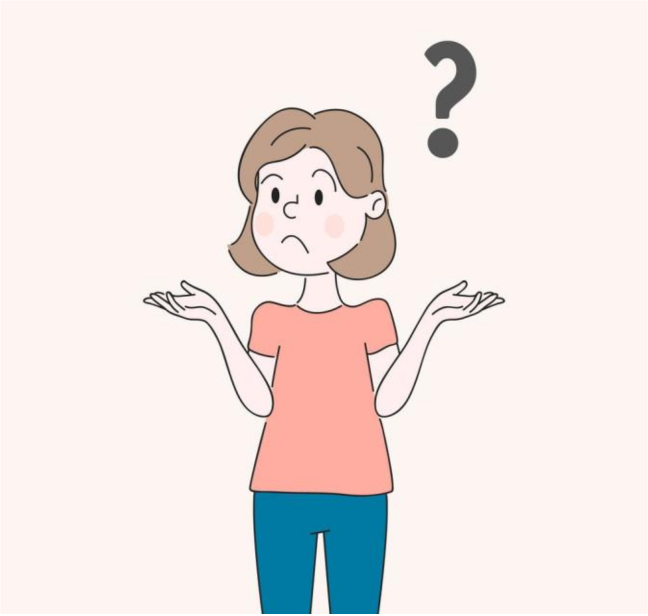


Wigner-Ville Distribution - Quiz

<u>Question 2</u>

WVD is a windowed transform.

a. True
b. False

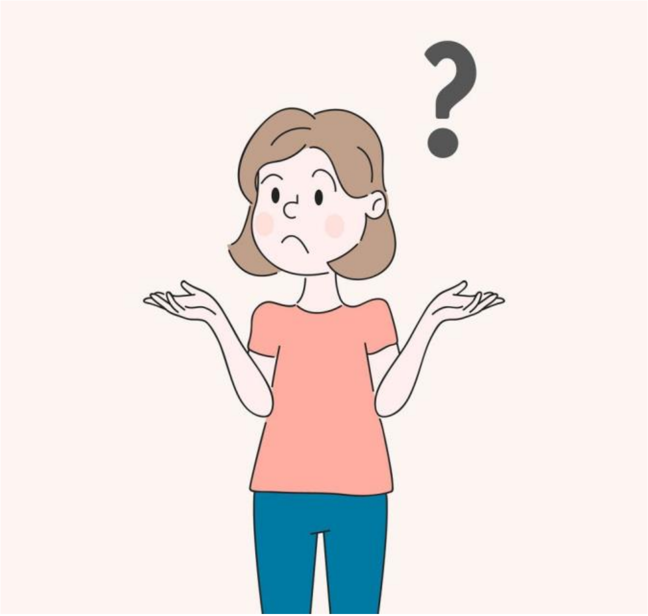


Wigner-Ville Distribution - Quiz

Question 3

WVD is good at analyzing composite signals.

a. True
b. False

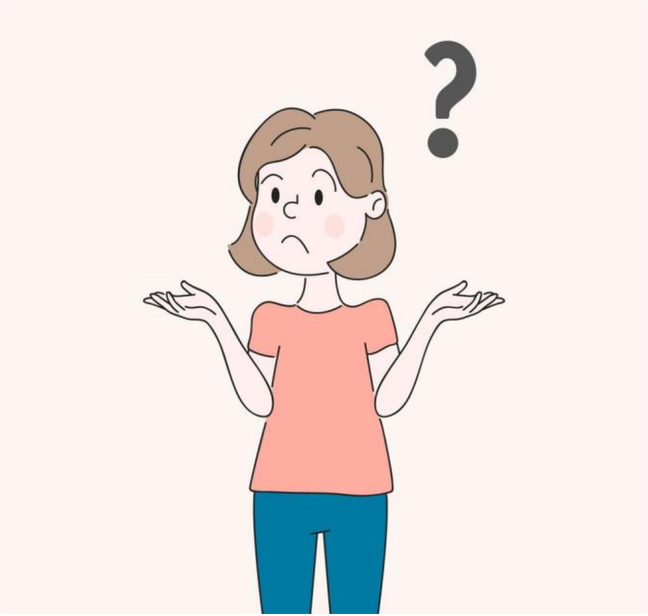


Wigner-Ville Distribution - Quiz

<u>Question 4</u>

The cross-terms in WVD can be eliminated by smoothing.

a. True
b. False



Wigner-Ville Distribution - History

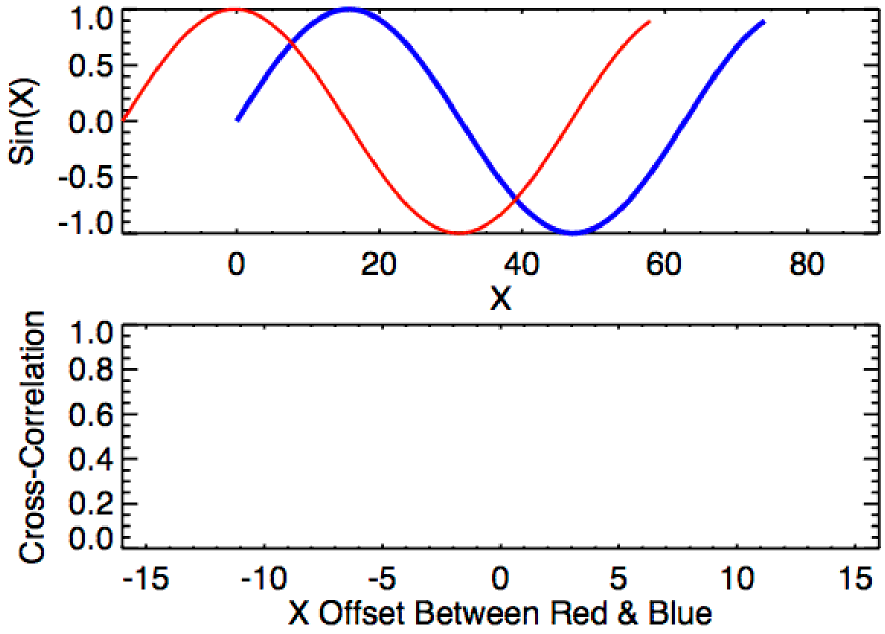
- First proposed in 1932 by Eugene Wigner in the field of quantum mechanics
  - Eugene Wigner later won the Nobel prize in Physics

- In 1948, J. Ville reformulated WVD in terms of time-frequency energy of a signal
  - Position and Momentum are the Physics pair of Time and Frequency. They are both constrained by the Uncertainty Principle

Wigner-Ville Distribution – How does it work?

- Compares the information in the signal with its own information at other times and frequencies
  - Known as auto-correlation

Auto-correlation is the correlation of a signal with a delayed copy of itself as a function of delay

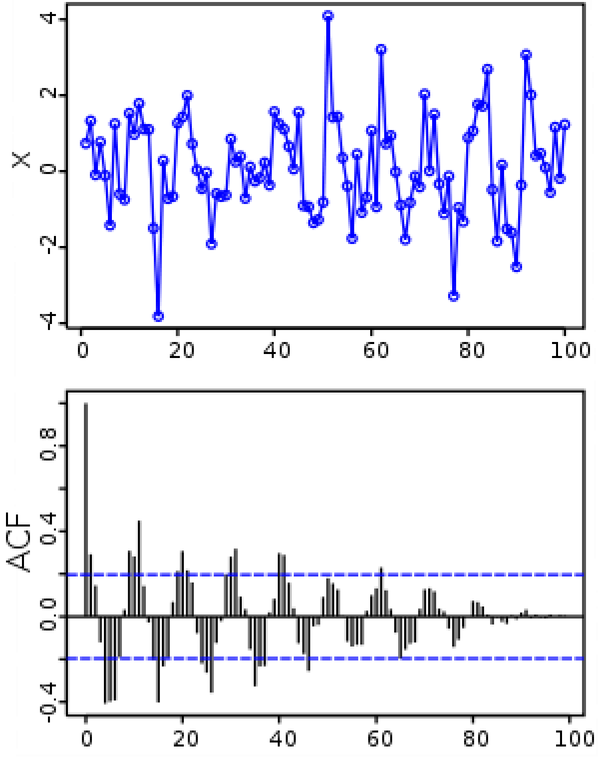


Source: Patrick McCauley, for From Quarks to Quasars

Wigner-Ville Distribution – How does it work?

- Compares the information in the signal with its own information at other times and frequencies
  - Known as auto-correlation

The analysis of autocorrelation is a mathematical tool for finding repeating patterns, such as the presence of a periodic signal obscured by noise



Source: Wikipedia, 'Autocorrelation'

Wigner-Ville Distribution – How does it work?

- Power spectrum of a signal is the Fourier Transform of its auto-correlation function

$$P(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega t} \mathrm{d}\tau \qquad R(\tau) = \int s(t)s(t+\tau)dt$$

- Auto-/Cross-WVD is the Fourier Transform of the instantaneous auto-/cross-correlation function

Auto-WVD: $$WVD_s(t,\omega) = \int_{-\infty}^{\infty} R(t,\tau)\, e^{-j\omega t} \mathrm{d}\tau \qquad \mathrm{R}(\mathrm{t},\ \tau) = \mathrm{s}(\mathrm{t}+\tau/2)\mathrm{s}^*(\mathrm{t}-\tau/2)$$

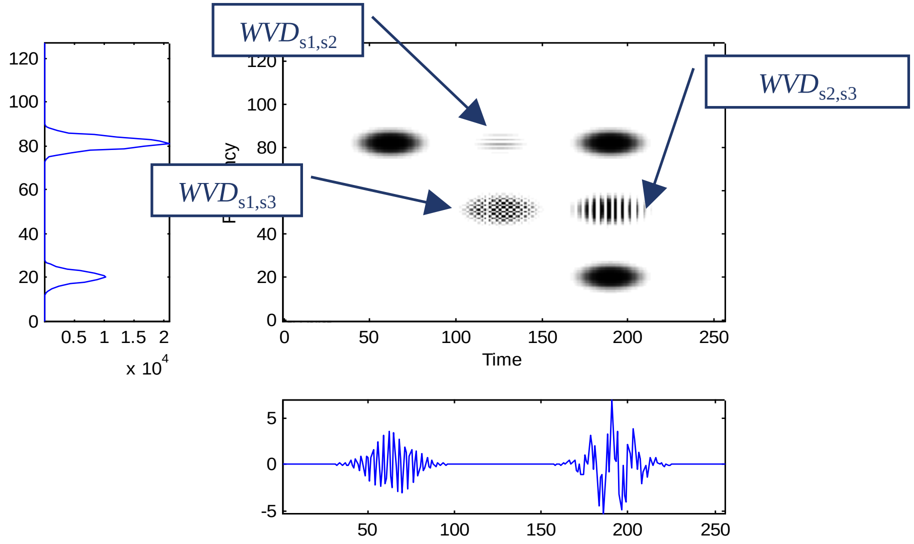
Cross-WVD: $$WVD_{s,g}(t,\omega) = \int_{-\infty}^{\infty} R_{s,g}(t,\tau)\, e^{-j\omega t} \mathrm{d}\tau \qquad \mathrm{R}_{\mathrm{s,g}}(\mathrm{t},\tau) = s\left(t+\frac{\tau}{2}\right) g^*(t-\tau/2)$$

Wigner-Ville Distribution – Properties

Cross-term

- If $s(t) = s_1(t) + s_2(t)$. Then

$$WVD_s(t,\omega) = WVD_{s_1}(t,\omega) + WVD_{s_2}(t,\omega) + 2\,\mathrm{Re}\{WVD_{s_1,s_2}(t,\omega)\}$$



Wigner-Ville Distribution – Properties

- For any real or complex signal, its auto-WVD is always real-valued
  - See p10 for a simple proof
  - The cross-WVD between two signals is not always real

- WVD is invariant to time and frequency shifts
  - Time invariant: $WVD_{s_0}(t,\omega) = WVD_s(t-t_0,\omega)$
  - Frequency invariant: $WVD_{s_1}(t,\omega) = WVD_s(t,\omega-\omega_1)$
  - Both follows immediately from the equation

WVD – Time Invariant Derivation

$$WVD_s(t,\omega) = \int_{-\infty}^{\infty} R(t,\tau)\, e^{-j\omega t} \mathrm{d}\tau \qquad \mathrm{R(t,\ \tau) = s(t+\tau/2)s^*(t-\tau/2)}$$

$$WVD_{s-t_0}(t,\omega) = \int_{-\infty}^{\infty} R_{s-t_0}(t,\tau)\, e^{-j\omega t} \mathrm{d}\tau \qquad \mathrm{R_{s-t_0}(t,\ \tau) = s(t+\tau/2-t_0)s^*(t-\tau/2-t_0) = R(t,\ \tau-2t_0)}$$

$$= \int_{-\infty}^{\infty} R\,(t,\tau-2t_0)\, e^{-j\omega t} \mathrm{d}\tau$$

$$= \int_{-\infty}^{\infty} R(t,\tau)\, e^{-j\omega t} \mathrm{d}\tau$$

$$= WVD_s(t,\omega)$$

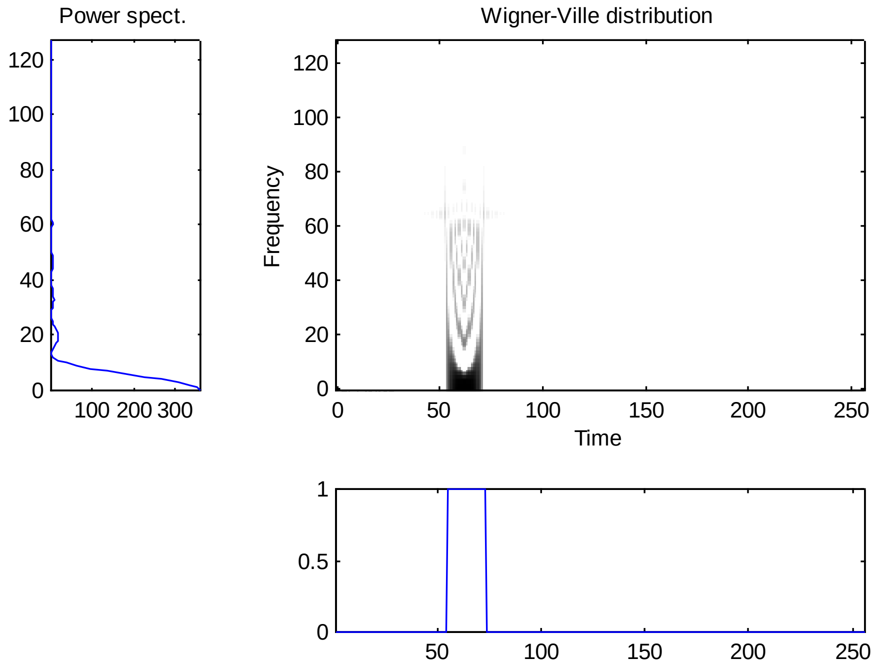
Wigner-Ville Distribution – Properties

- Time marginal condition
  - $\frac{1}{2\pi}\int_{-\infty}^{\infty} WVD_s(t,\omega)d\omega = |s(t)|^2$
  - Integral over frequency of WVD is the signal power density at time t
- Frequency marginal condition
  - $\int_{-\infty}^{\infty} WVD_s(t,\omega)dt = |S(\omega)|^2$
  - Integral over time of WVD is the power spectral density

Wigner-Ville Distribution – Properties

- WVD has the exact same time interval as the original time-limited signal

- WVD has the exact same frequency interval as the original frequency-band limited signal



Wigner-Ville Distribution - Pros

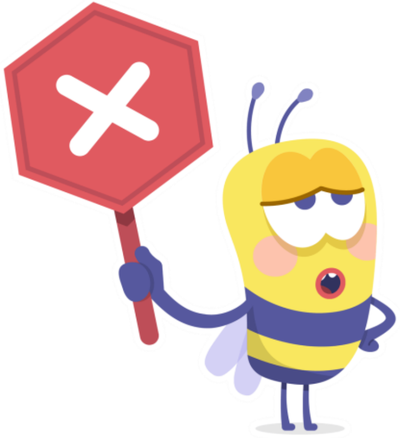
- In contrast to Fourier transform, WVD analyzes the signal from both time and frequency prospective.
  - Hence, WVD is beneficial when the signal is non-stationary

- WVD provides the highest possible time-frequency resolution which is mathematically possible within the limitations of the uncertainty principle

Wigner-Ville Distribution - Con

- If a signal is composed of several signal components, WVD will result in large cross-terms between every pair of signal components
  - 2 components ⇒ 1 cross-term
  - 3 components ⇒ 3 cross-terms
  - 4 components ⇒ 6 cross-terms
  - …
  - N components ⇒ $(N-1)!$ cross-terms
  - Moreover, the magnitudes of the cross-terms are always larger than the auto-terms

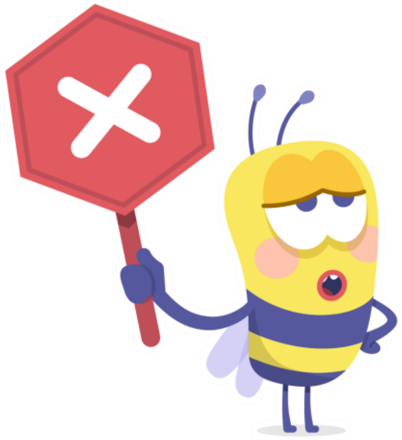
More cross-terms than auto-terms!



Wigner-Ville Distribution - Con

- The cross-terms can be suppressed by smoothing techniques
  - E.g., use a windowed version of the WVD, called the Pseudo-WVD (PWVD)
  - Similar to frequency smoothing or low-pass filtering

- BUT, smoothing reduces the time-frequency resolution



Wigner-Ville Distribution - Matlab

- Aim
  - To illustrate the WVD on all sorts of signals
    - Stationary signal e.g. sine wave
    - Non-stationary signal e.g. chirp
    - Modulated chirp
    - Composite signal (2 sub-signals, 3 sub-signals, 4 sub-signals)
  - To compare the time-frequency resolution of WVD vs. STFT
  - To illustrate the cross-terms of WVD

- The Matlab script will be uploaded to QMplus after the session

Lecture Outline

1. Wigner-Ville Distribution
   - Recap
   - Matlab Example

2. Uncertainty Principle
   - Examples
   - Fourier Transform

Uncertainty Principle

- If I ask you what note is this, how confident are you with your answer?

  a. 100%
  b. 80%
  c. 50%
  d. 30%
  e. 0%

Uncertainty Principle

- If I ask you what note is this, how confident are you with your answer?

  a. 100%
  b. 80%
  c. 50%
  d. 30%
  e. 0%

Uncertainty Principle

As we listen to the note for a longer duration

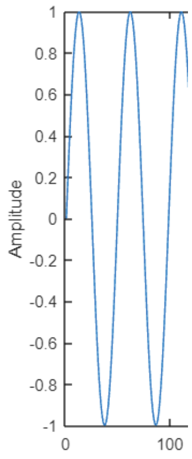
↓

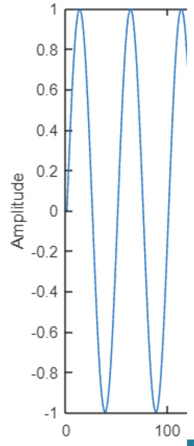
The more confident we are at our answers

Uncertainty Principle

- Do you think these two waves have the same frequency?

  a. Yes
  b. No
  c. Not sure





Uncertainty Principle

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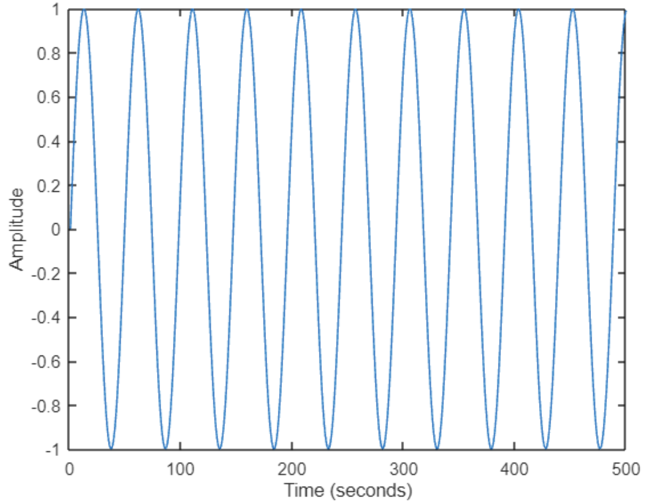

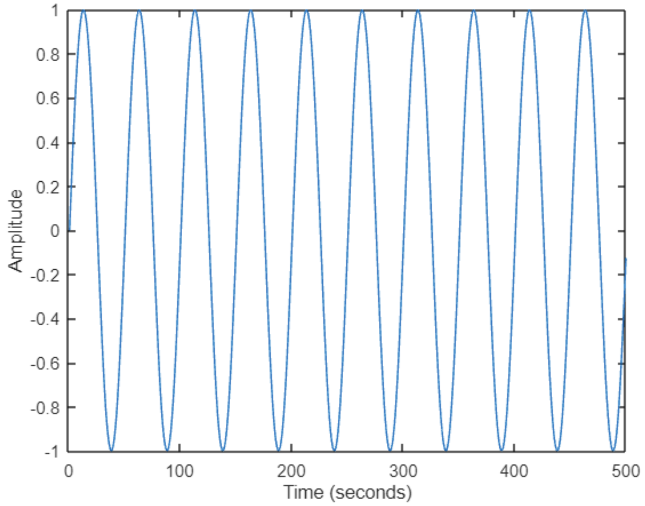

Uncertainty Principle

As we observe the signal for a longer time duration

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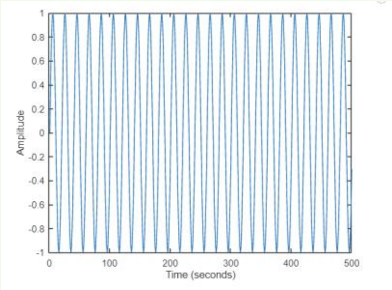
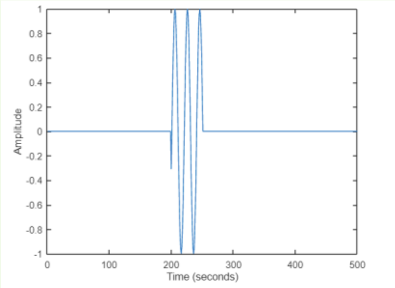
The uncertainty in frequency decreases

Uncertainty Principle – Fourier Transform

Less certain about the signal's value in time

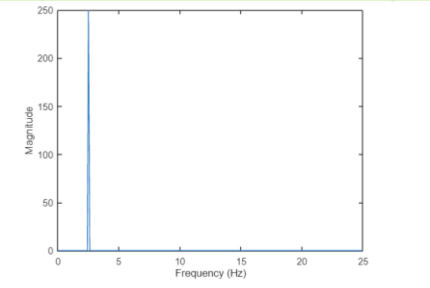
Time Domain

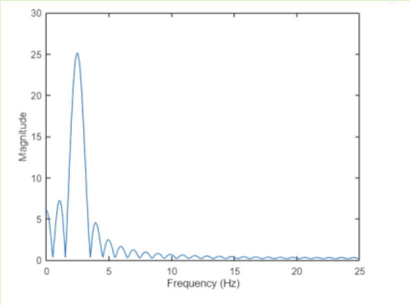




Fourier Transform

Frequency Domain

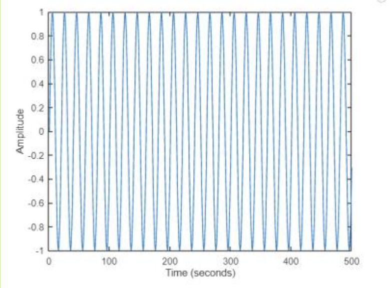


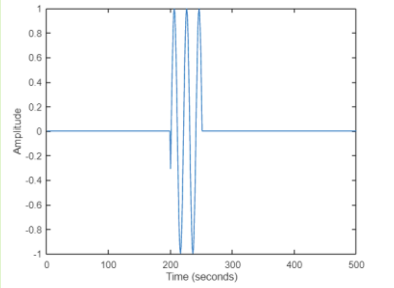


More certain about the frequency

Uncertainty Principle – Fourier Transform

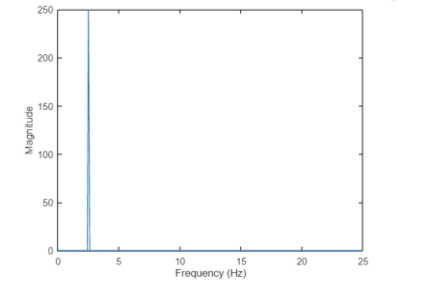
Time Domain

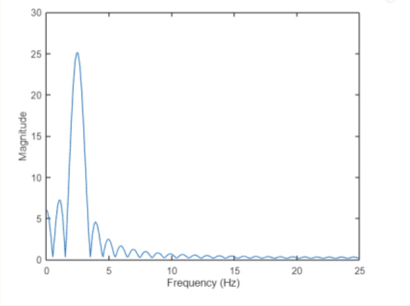




Fourier Transform

Frequency Domain





More certain about the signal in time

Less certain about the frequency

Uncertainty Principle - Summary

- An inverse relationship between the certainty in time and certainty in frequency
- We denote the uncertainty in time and frequency as Δt and $\Delta\omega$, respectively

- Uncertainty Principle:

$$\Delta t \Delta\omega \geq \frac{1}{2}$$

  - The product of uncertainty in time and uncertainty in frequency is a constant

- Different definitions of the Fourier Transform yield different versions of the Uncertainty Principle.

Uncertainty Principle - Summary

- Uncertainty Principle:

$$\Delta t \Delta \omega \geq \frac{1}{2}$$

  - The product of uncertainty in time and uncertainty in frequency is a constant

- It explains the trade-off of the uncertainty between
  - Time and frequency → Signal processing
  - Distance and Velocity → Radar
  - Position and Momentum → Quantum Mechanics

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