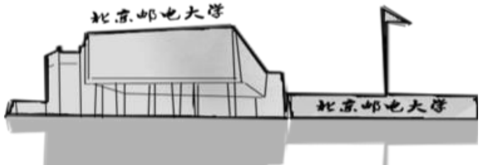
Chapter 6

Bandpass Transmission of Digital Signals

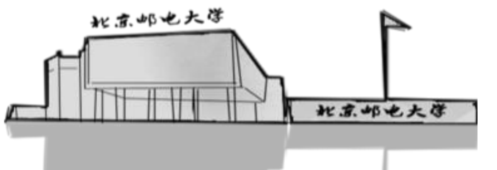
School of Information and Communication Engineering

Beijing University of Posts and Telecommunications



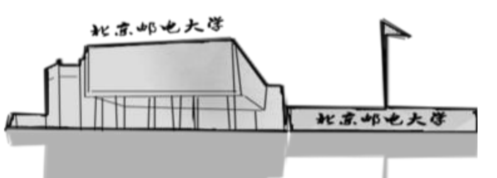
Bandpass Transmission of Digital Signals

- Introduction
- Sinusoidal carrier modulation of the binary digital signal
- Quadrature phase shift keying
- M-ary digital modulation

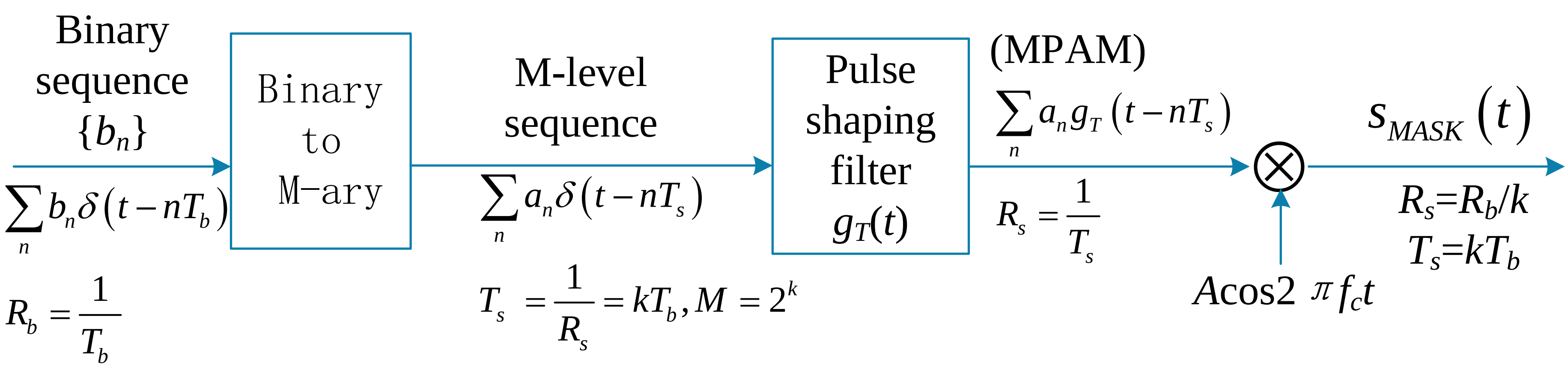


M-ary Digital Modulation

- Introduction
- Vector Representation of Digital Modulation Signals
- Statistical Decision Theory
- Optimal reception of M-ary digital modulation signals with AWGN
- **MASK**
- MPSK
- MQAM
- MFSK

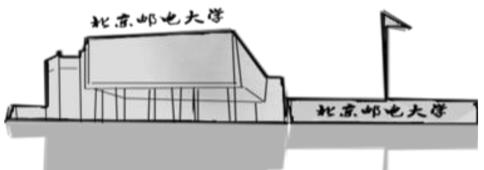


MASK Modulation



$$s_{MASK}(t) = b(t) A\cos\omega_c t = [\sum_n a_n g_T(t - nT_s)] A\cos\omega_c t$$

$$P_s(f) = \frac{A^2}{4}[P_b(f - f_c) + P_b(f + f_c)]$$ ***b* for baseband**

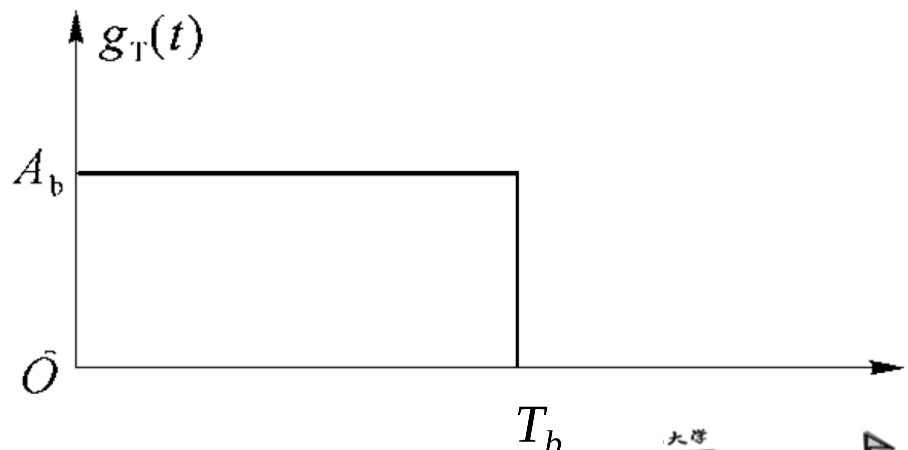


MASK

- If $g_T(t)$ is NRZ rectangle pulse，and the symbols of sequence $\{a_n\}$ are equal probability and un-correlated with $E[a_n]=0$,
- Then

$$P_b(f)=\frac{\sigma_a^2}{T_s}\left|G_T(f)\right|^2=\sigma_a^2 A_b^2 T_s \mathrm{sinc}^2(fT_s)$$

Where A_b is the amplitude of the baseband pulse $g_T(t)$.

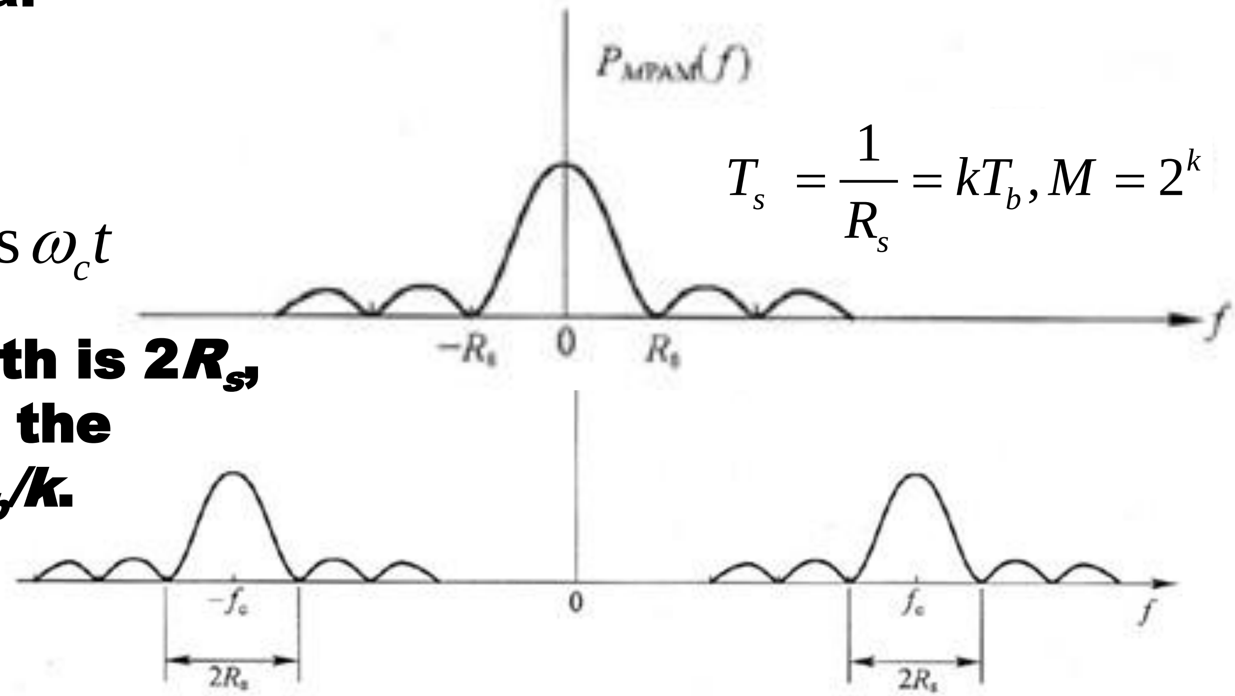


MASK

- **PSD of MASK signal**

$$b(t) = \sum_n a_n g_T(t - nT_s)$$

$$s_{MASK}(t) = b(t) A \cos \omega_c t$$



$$T_s = \frac{1}{R_s} = kT_b, M = 2^k$$

- **The main lobe width is 2*R***

- **The main lobe width is $2R_s$, it only depends on the symbol rate, $R_s = R_b/k$.**

- **Frequency efficiency**

$$\frac{R_s}{2R_s} = \frac{1}{2}\,\text{Baud/Hz} \quad \textbf{or} \quad \frac{kR_s}{2R_s} = \frac{k}{2}\,\text{bit/s/Hz}$$

- **With RRC filter, the bandwidth:**

$$2 \times \frac{R_s}{2}(1+\alpha) = \frac{R_b(1+\alpha)}{\log_2 M}$$

- **Frequency efficiency:**

$$\eta = \frac{1}{1+\alpha}(\text{Baud/ Hz}) = \frac{k}{1+\alpha}(\text{bps/ Hz})$$

MASK

● Orthogonal expansion and vector representation of MASK signal

$$s_i(t) = b_i(t)\cos\omega_c t = a_i g_T(t)\cos\omega_c t \triangleq s_i f_1(t)$$

$$0 \le t \le T_s, b_i(t) = a_i g_T(t)$$

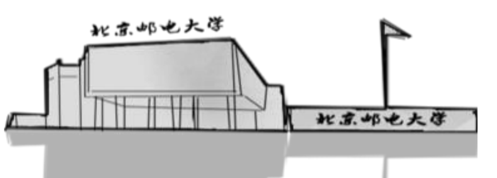
$$g_T(t) = \begin{cases} \sqrt{\dfrac{E_g}{T_s}} & 0 \leqslant t \leqslant T_s \\ 0 & \text{else} \end{cases}$$

Where: $a_i = (2i - 1 - M),$

$$a_i = -(M+1), \cdots, -3, -1, 1, 3, \ldots, M-1, i = 1, \cdots M$$

$$f_1(t) = \sqrt{\frac{2}{E_g}} g_T(t)\cos\omega_c t$$

$$\boxed{E_f = 1}$$



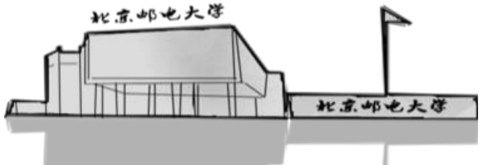
MASK

$$E_g = \int_0^{T_s} g_T^2(t)\mathrm{d}t$$

$$s_i = \int_0^{T_s} s_i(t) f_1(t)\mathrm{d}t = \sqrt{\frac{E_g}{2}} a_i \sim \text{projection of } s_i(t) \text{ to } f_1(t)$$

$$\mathbf{s_i} = [s_i], i = 1, \ldots, M$$

$$d_{mn} = \sqrt{(s_m - s_n)^2} = \sqrt{\frac{E_g}{2}} |a_m - a_n| = \sqrt{\frac{E_g}{2}} \cdot 2|m-n| = \sqrt{2E_g}\,|m-n|$$



MASK

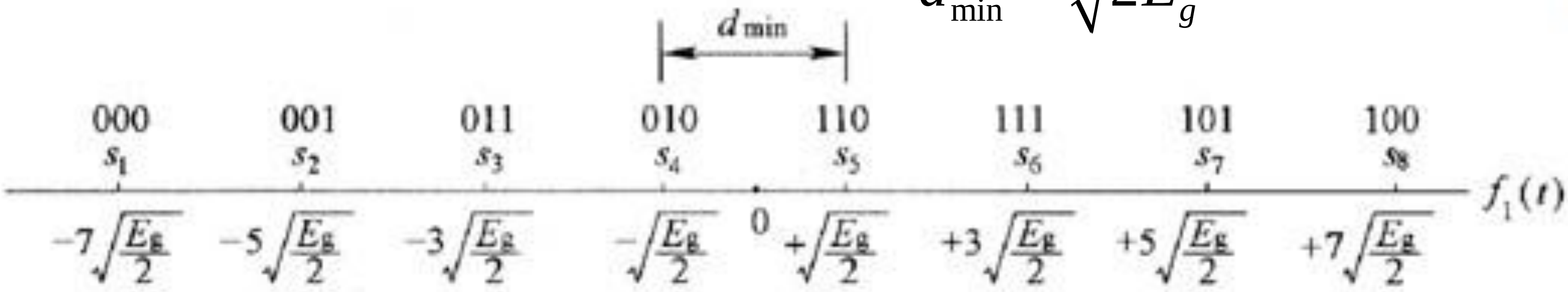
- **Orthogonal expansion and vector representation of MASK signal**

  - **Example: Constellation of 8ASK**

$$a_i = \pm 1, \pm 3, \pm 5, \pm 7$$

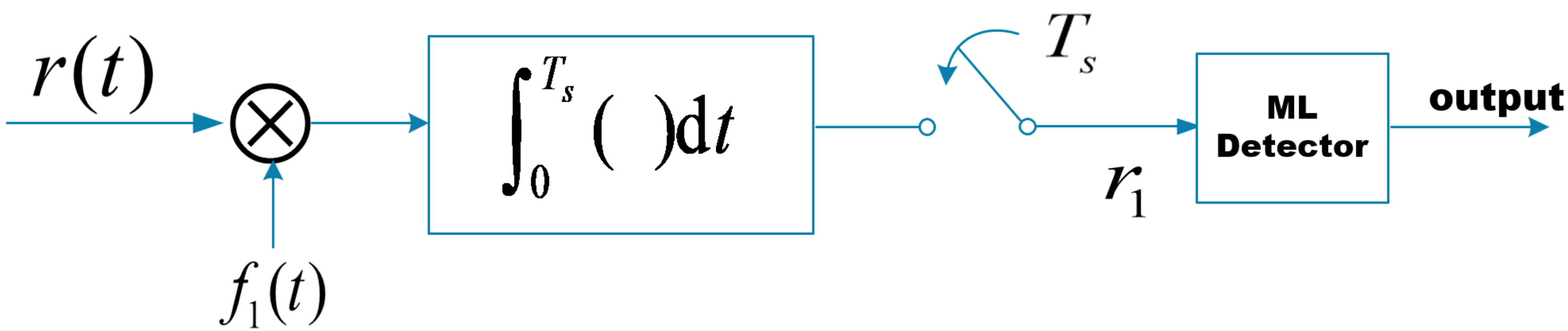
$$s_i = \sqrt{\frac{E_g}{2}} a_i, i = 1, 2, \ldots, 8$$

$$d_{\min} = \sqrt{2E_g}$$



MASK

- **Optimal reception of MASK signal**



ML criterion with equal a priori probability:

$$r_1 = \int_0^{T_s} r(t) f_1(t)\mathrm{d}t = \int_0^{T_s} [s_i(t) + n_w(t)] f_1(t)\mathrm{d}t$$

$$\boxed{E_f = 1}$$

$$= \int_0^{T_s} [s_i f_1(t) + n_w(t)] f_1(t)\mathrm{d}t = s_i + n, i = 1, \ldots, M$$

$$\hat{s} = \arg\max_{s_i} p(r_1 \mid s_i)$$

MASK

- **Error decision probability**

$$n = \int_0^{T_s} n_w(t) f_1(t) dt$$

$$E\left[r_1 \middle| s_i\right] = s_i + n$$

$$E(n) = E[n = \int_0^{T_s} n_w(t) f_1(t) dt] = 0$$

$$D\left[r_1 \middle| s_i\right] = \frac{N_0}{2}$$

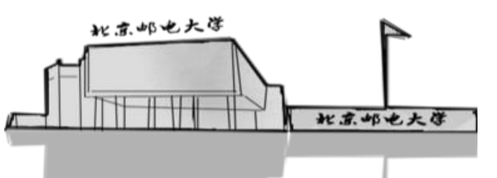
$$D(n) = E[\int_0^{T_s}\int_0^{T_s} n_w(t) n_w(z) f_1(t) f_1(z) \mathrm{d}t\mathrm{d}z] - E^2(n)$$

$$= \int_0^{T_s}\int_0^{T_s} E[n_w(t) n_w(z)] f_1(t) f_1(z) \mathrm{d}t\mathrm{d}z$$

$$\boxed{E_f = 1}$$

$$= \int_0^{T_s}\int_0^{T_s} \frac{N_0}{2} \delta(t-z) f_1(t) f_1(z) \mathrm{d}t\mathrm{d}z = \frac{N_0}{2}$$

$$r_1 = s_i + n \Rightarrow p(r_1 \mid s_i) = \frac{1}{\sqrt{\pi N_0}} \exp[-\frac{(r_1 - s_i)^2}{N_0}]$$

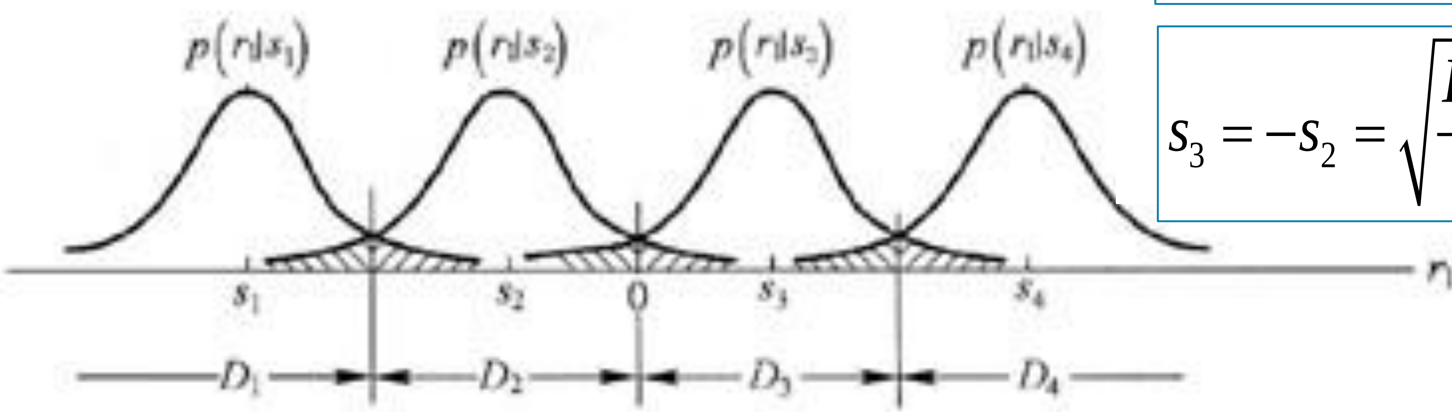


MASK

- ***M*=4**

$$d_{\min} = \sqrt{2E_g}$$

$$s_3 = -s_2 = \sqrt{\frac{E_g}{2}}, E_2 = E_3 = \frac{E_g}{2}$$

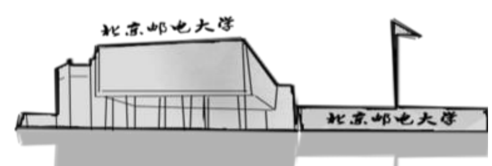


The error decision probabilities of s_2 and s_3 are the same as P_e of BPSK.

$$\frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{d_{\min}^2}{4N_0}}\right) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_g}{2N_0}}\right) \triangleq P_e$$

$$P_M = P(s_1)P_e + P(s_2)\cdot 2P_e + P(s_3)\cdot 2P_e + P(s_4)P_e \overset{P(s_i)=\frac{1}{4}}{=} \frac{3}{2}P_e$$

$$= \frac{1}{4}\times 3\times(2P_e) = \frac{3}{4}\left[2Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)\right]$$



MASK

- **Generally, for** $M = 2^k$

$$d_{\min} = \sqrt{2E_g}$$

$$P_M = \frac{(M-1)}{M}\left[2Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)\right] = \frac{2(M-1)}{M}Q\left(\sqrt{\frac{E_g}{N_0}}\right)$$

$$s_i = \sqrt{\frac{E_g}{2}}a_i$$

$$a_i = 2i-1-M$$

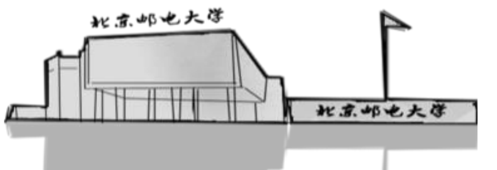
- **The energy of the *i*th MASK signal**

$$E_i = |s_i|^2 = \frac{E_g a_i^2}{2}, i = 1, \ldots, M$$

Average symbol energy

$$E_{av} = \frac{1}{M}\sum_{i=1}^{M} E_i = \frac{E_g}{2M}\sum_{i=1}^{M}(2i-1-M)^2 = \frac{(M^2-1)}{6}E_g$$

$$E_g = \frac{d_{\min}^2}{2} = \frac{6E_b \log_2 M}{M^2-1}$$



MASK

Average bit energy:

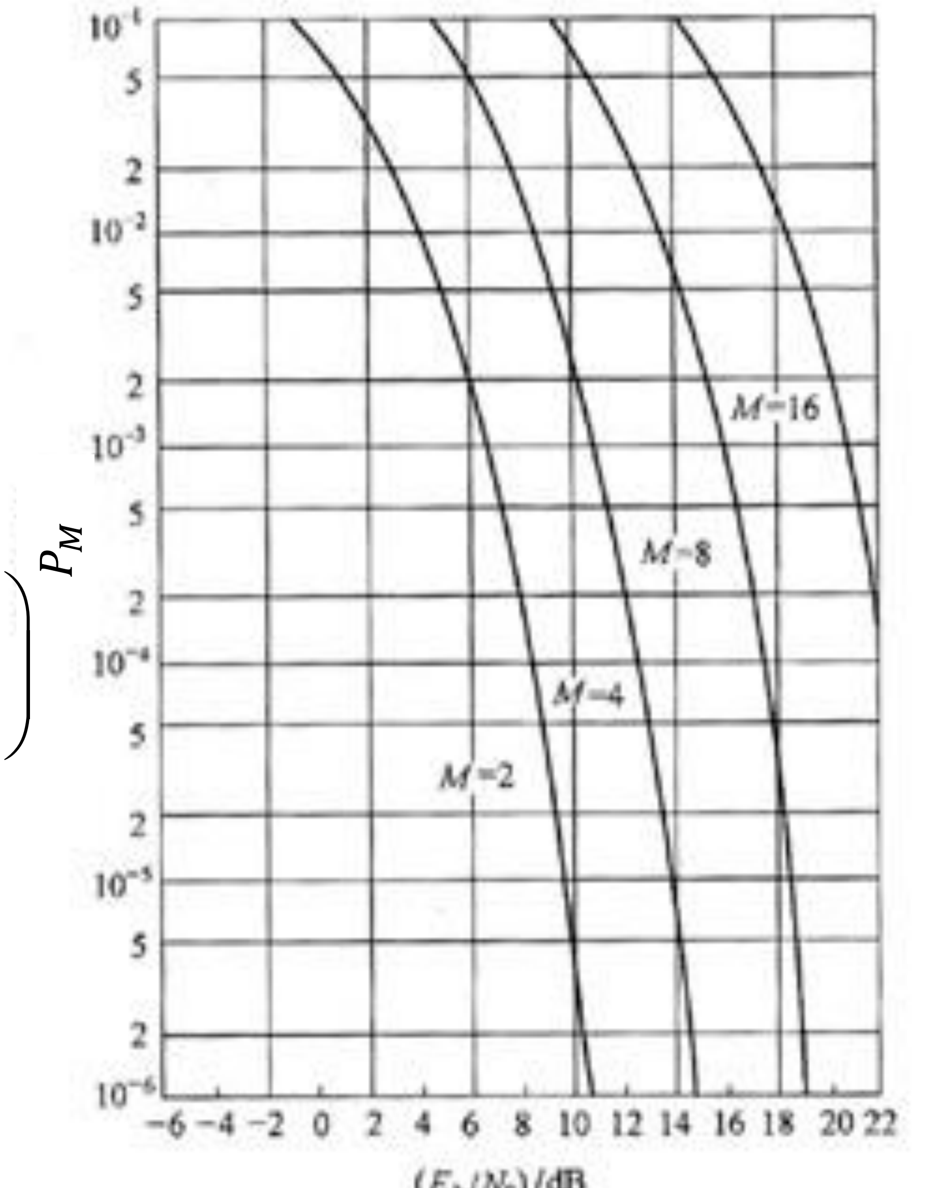
$$E_b = \frac{E_{av}}{\log_2 M} = \frac{(M^2 - 1)E_g}{6\log_2 M}$$

- **SER P_M: with given SNR, M↑, P_M↑**

$$\therefore P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2 M}{M^2 - 1} \cdot \frac{E_b}{N_0}}\right)$$

- **BER P_b: with gray coding, Pb is approximately calculated as**

$$P_b \approx \frac{P_M}{\log_2 M}$$



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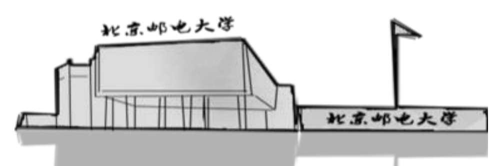


MPSK

● MPSK Modulation

$$s_i(t) = g_T(t)\cos[2\pi f_c t + \frac{2\pi(i-1)}{M}]$$

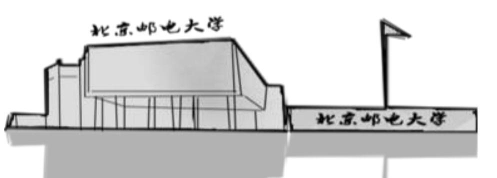
$$= g_T(t)[a_{i_c}\cos\omega_c t - a_{i_s}\sin\omega_c t], i = 1,2,\ldots M, 0 \le t \le T_s$$

$$\begin{cases} a_{i_c} = \cos\dfrac{2\pi(i-1)}{M} = \cos\theta_i \\ a_{i_s} = \sin\dfrac{2\pi(i-1)}{M} = \sin\theta_i \end{cases}, \quad a_{i_c}^2 + a_{i_s}^2 = 1$$

- **the MPSK symbols have the same symbol energy**

$$E_s = \int_0^{T_s} s_i^2(t)\mathrm{d}t = \int_0^{T_s}\frac{1}{2}g_T^2(t)\mathrm{d}t = \frac{E_g}{2}, i = 1,2,\ldots,M$$

***E_g* is the symbol energy of** $g_T(t)$

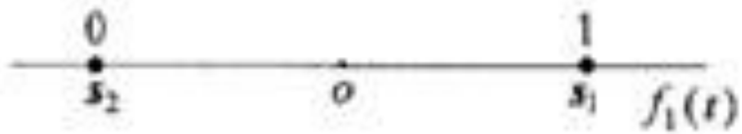


MPSK

The vector representation of MPSK signal

$$s_i(t) = g_T(t)[a_{i_c} \cos \omega_c t - a_{i_s} \sin \omega_c t],$$

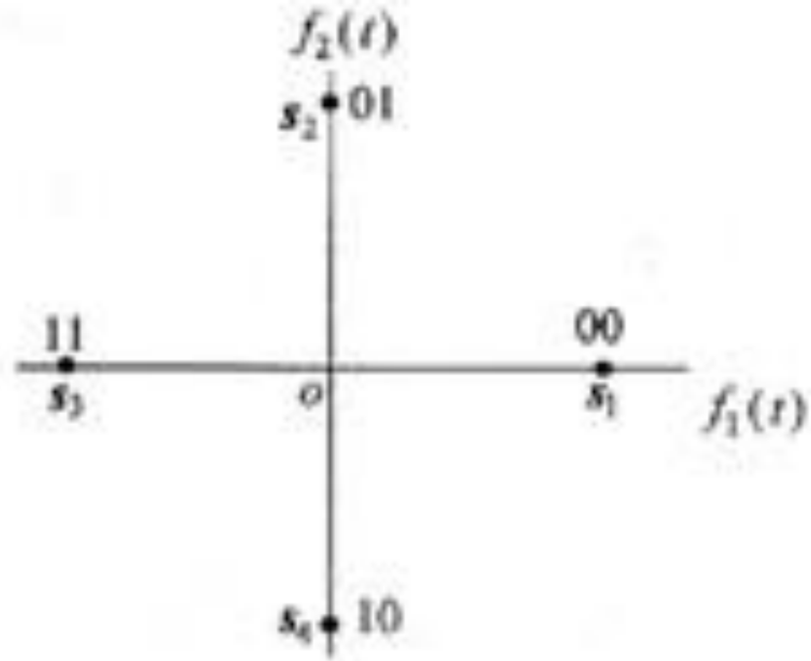
$$0 \le t \le T_s$$



$$f_1(t) = \sqrt{2/E_g}\, g_T(t) \cos 2\pi f_c t$$

$$f_2(t) = -\sqrt{2/E_g}\, g_T(t) \sin 2\pi f_c t$$

$$\longrightarrow \quad s_i(t) = s_{i1} f_1(t) + s_{i2} f_2(t)$$



MPSK

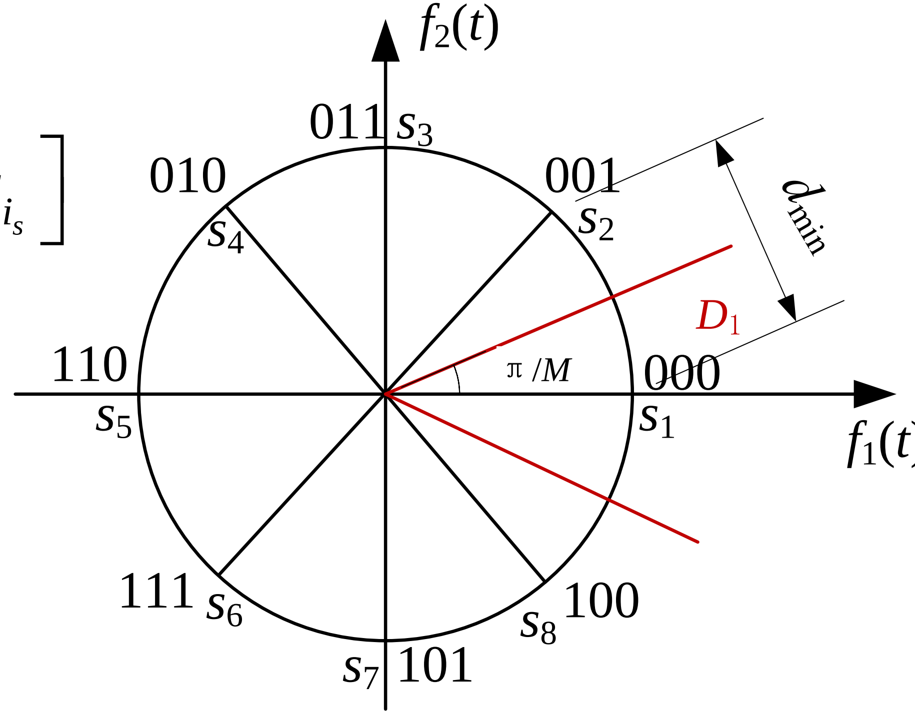
The vector representation of MPSK signal

$$s_{i1} = \int_0^{T_s} s_i(t) f_1(t)dt = \sqrt{E_s} a_{i_c}, \quad a_{i_c}^2 + a_{i_s}^2 = 1$$

$$s_{i2} = \int_0^{T_s} s_i(t) f_2(t)dt = \sqrt{E_s} a_{i_s}$$

$$\Rightarrow \mathrm{s} = \left[s_{i1}, \ \ s_{i2} \right] = \left[\sqrt{E_s} a_{i_c}, \sqrt{E_s} a_{i_s} \right]$$

$$d_{\min} = 2\sqrt{E_s} \sin \frac{\pi}{M}$$



MPSK

The generation of MPSK signal

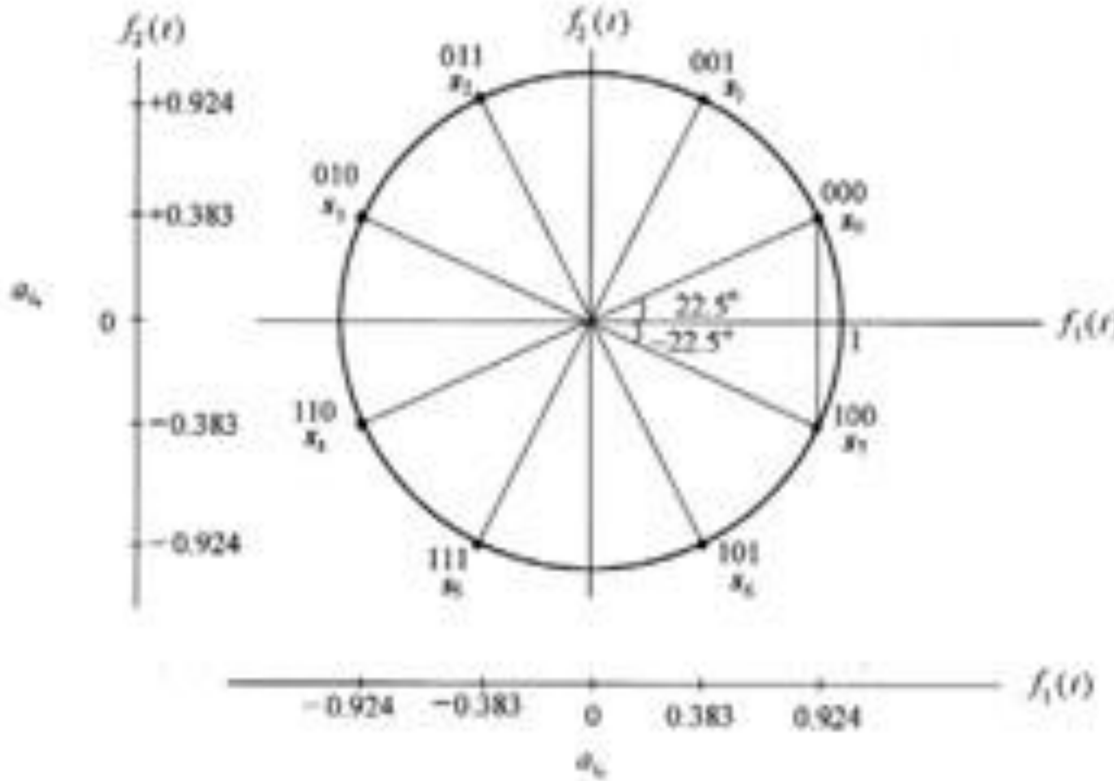
$$s_i(t) = g_T(t)[a_{i_c} \cos \omega_c t - a_{i_s} \sin \omega_c t]$$

~The sum of two M/2 ASK signals

- **Example: 8PSK**:

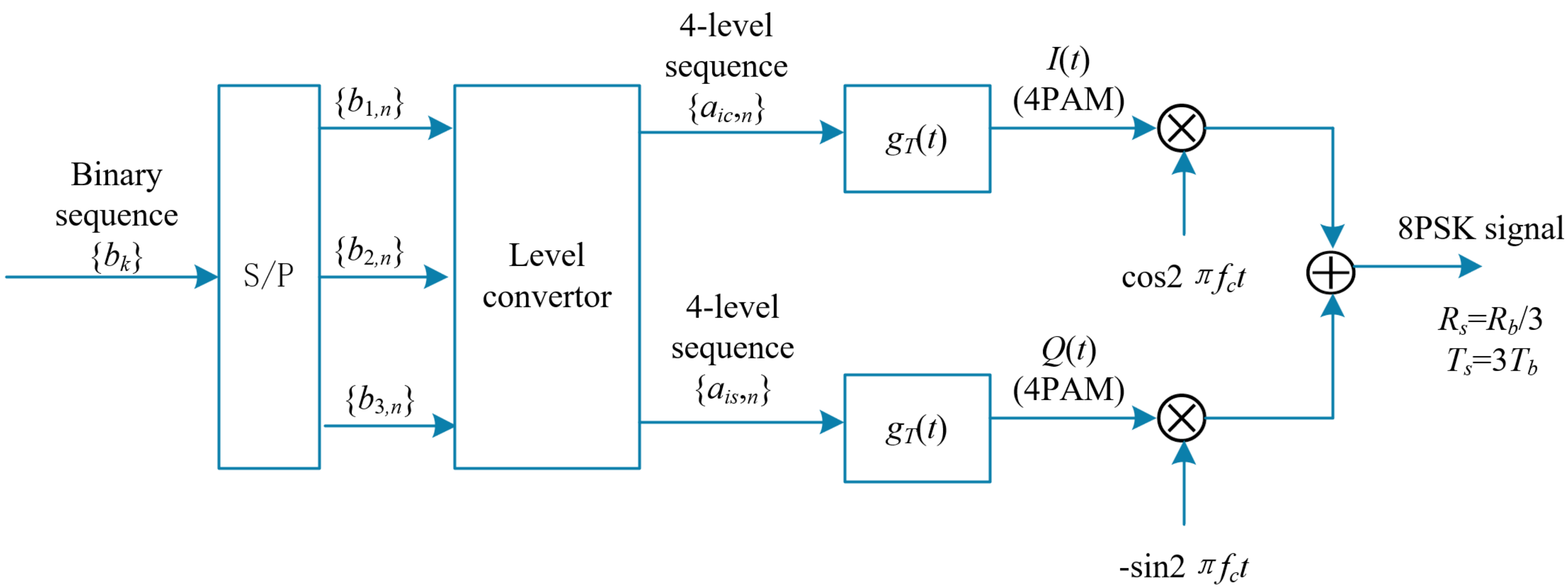
$$\theta_i = \frac{2i-1}{8}\pi, i = 1, \ldots, 8$$

$$\begin{cases} a_{i_c} = \cos \theta_i \\ a_{i_s} = \sin \theta_i \end{cases}$$



MPSK

● Modulation of 8PSK signal



MPSK

Average PSD of MPSK signal

$g_T(t)$ ~ NRZ rectangle pulse

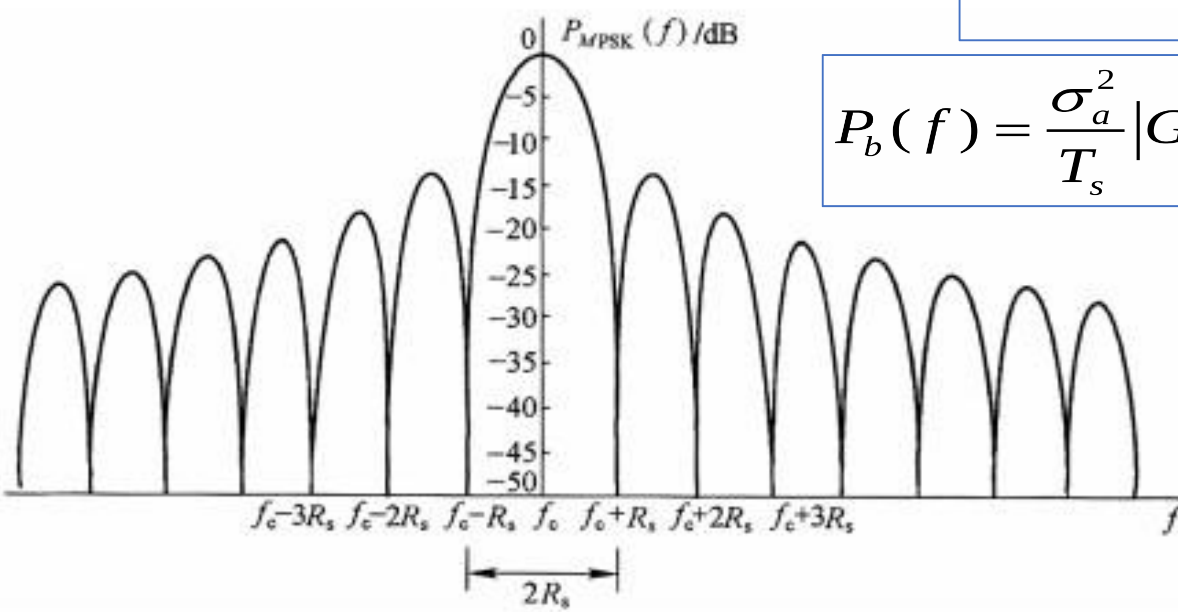
$\{a_n\}$ ~ bipolar equal probability and un-correlated sequence

$$P_{MPSK} = \frac{E_s}{2}\left\{\sin c^2[(f - f_c)T_s] + \sin c^2[(f + f_c)T_s]\right\}$$

$$\boxed{P_s(f) = \frac{A^2}{4}[P_b(f - f_c) + P_b(f + f_c)]}$$

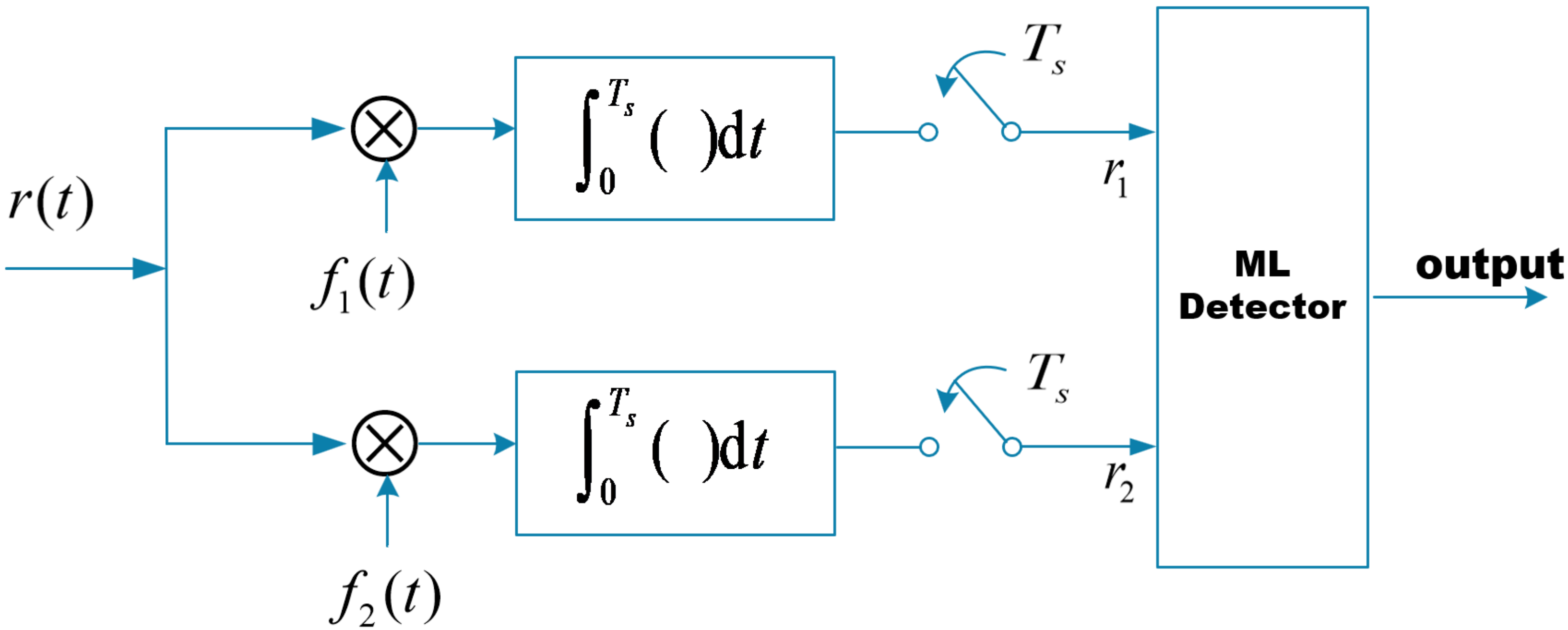
$$\boxed{P_b(f) = \frac{\sigma_a^2}{T_s}|G_T(f)|^2 = \sigma_a^2 A_b^2 T_s \text{sinc}^2(fT_s)}$$

$$\boxed{E_s = \frac{A^2}{2}E_g, \quad E_g = \sigma_a^2 A_b^2 T_s}$$



MPSK

● Optimal reception of MPSK signal



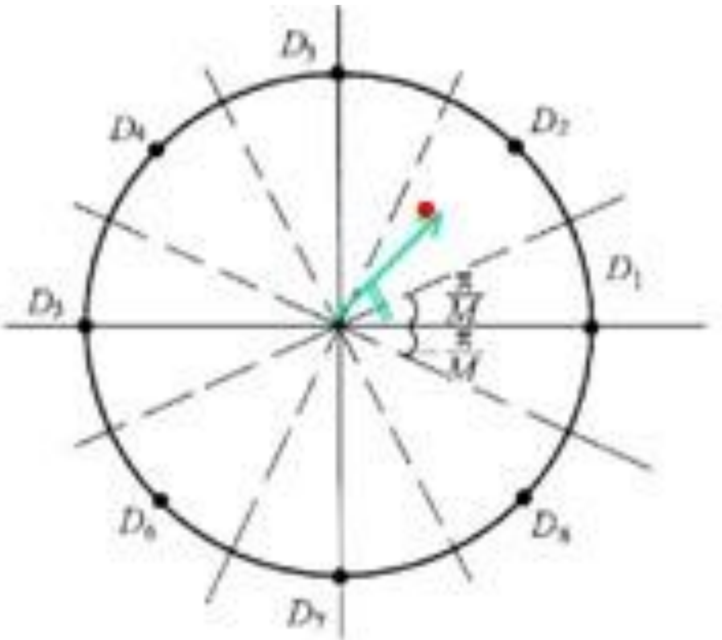
$$r(t) = s_i(t) + n_w(t)$$

$$r = s_i + n = [r_1, r_2] = [\sqrt{E_s}\, a_{i_c} + n_1, \sqrt{E_s}\, a_{i_s} + n_2]$$

$$\theta_r = \arctan\frac{r_2}{r_1}$$

MPSK

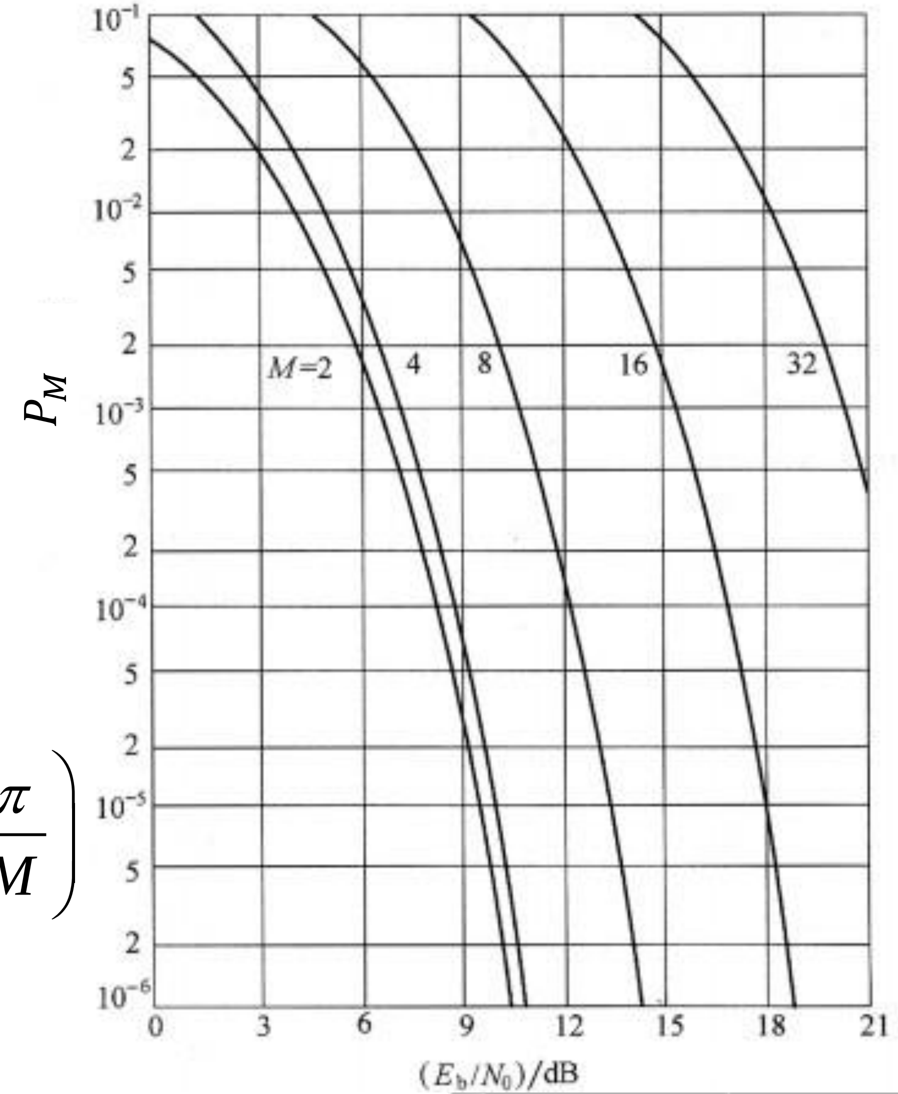
SER of MPSK signal



$$P(e \mid s_1) = 1 - \int_{-\pi/M}^{\pi/M} p(\theta_r \mid s_1) d\theta_r$$

$$P_M = \sum_{i=1}^{M} P(s_i) P(e \mid s_1) < erfc\left(\sqrt{\frac{E_s}{N_0}} \sin\frac{\pi}{M}\right)$$

$$P_b \approx \frac{1}{\log_2 M} P_M = \frac{1}{K} P_M$$



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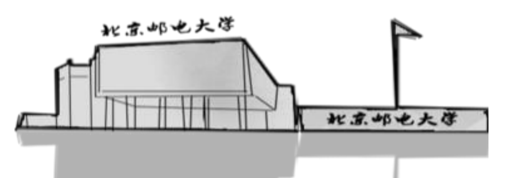


MQAM

MQAM Modulation

$$s_{QAM}(t) = a_{i_c} g_T(t)\cos\omega_c t - a_{i_s} g_T(t)\sin\omega_c t, i = 1, 2, \ldots, M, 0 \le t \le T_s$$

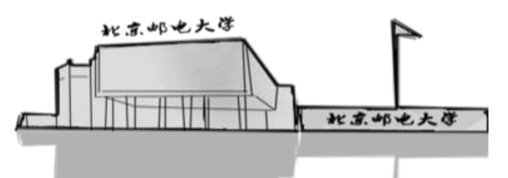
$$= \mathrm{Re}[V_i e^{j\theta_i} g_T(t) e^{j\omega_c t}]$$

where $\begin{cases} V_i = \sqrt{a_{i_c}^2 + a_{i_s}^2} \\ \theta_i = \arctan \dfrac{a_{i_s}}{a_{i_c}} \end{cases}$

$$f_1(t) = \sqrt{\frac{2}{E_g}} g_T(t)\cos\omega_c t$$

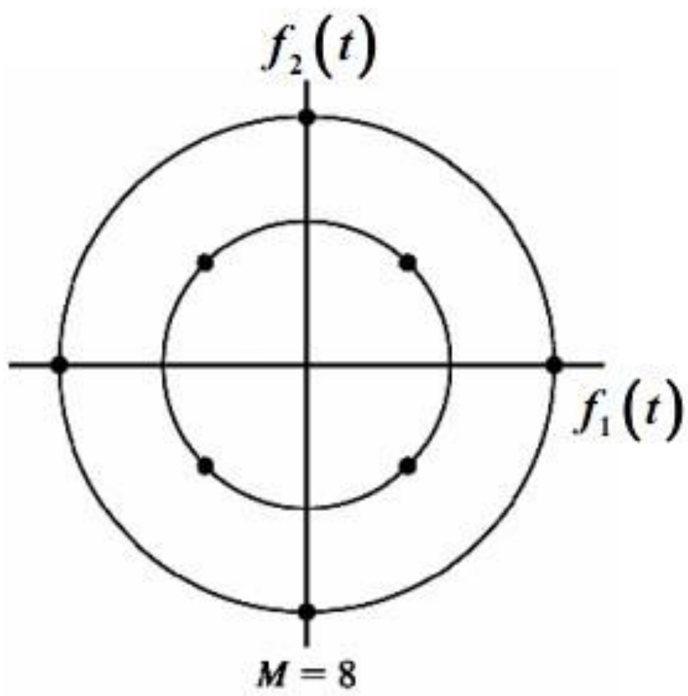
$$f_2(t) = -\sqrt{\frac{2}{E_g}} g_T(t)\sin\omega_c t$$

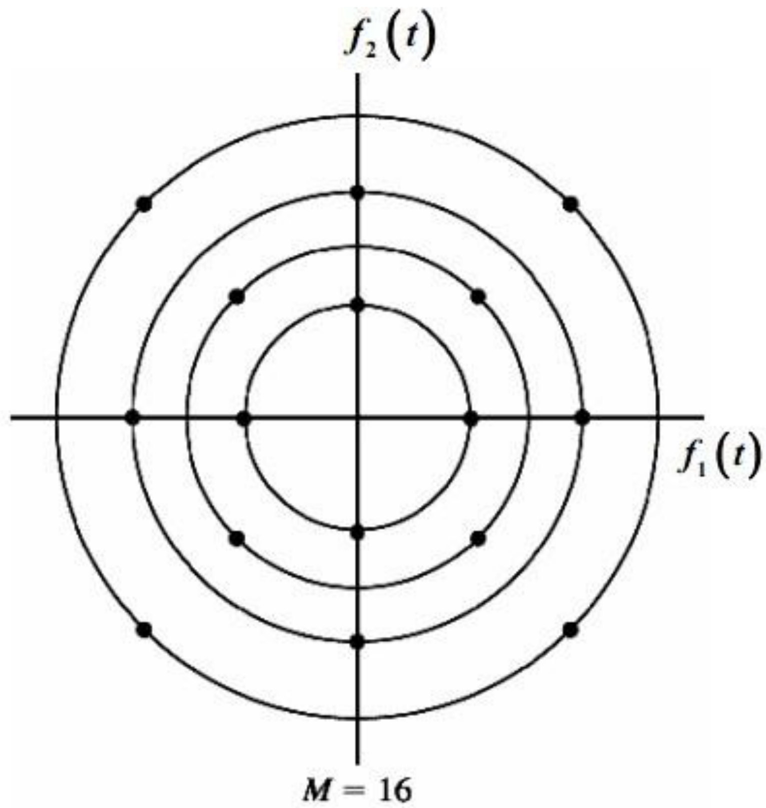
$$\Rightarrow \quad \mathrm{s}_i = [s_{i1}, \quad s_{i2}] = [\sqrt{\frac{E_g}{2}} a_{i_c}, \sqrt{\frac{E_g}{2}} a_{i_s}]$$



MQAM

- **Constellation of MQAM signal**



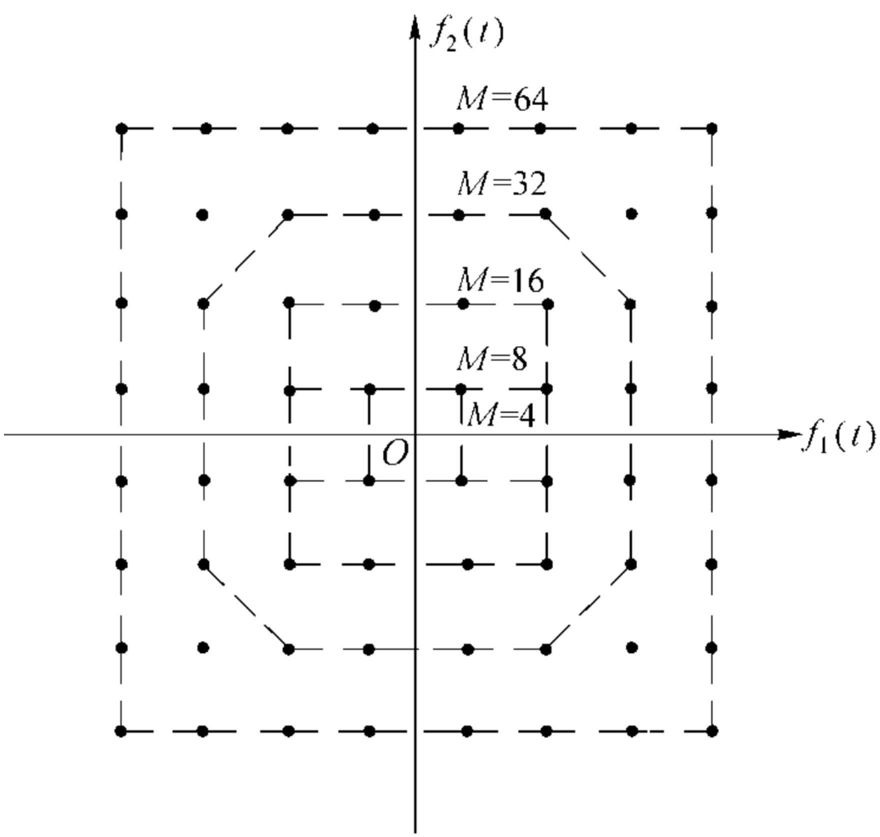


Circular constellation

MQAM

● Rectangular constellation of MQAM signal

When *k* is even, MQAM signal is the sum of two independent orthogonal $M^{1/2}$ ASK signals. $k=\log_2 M$.



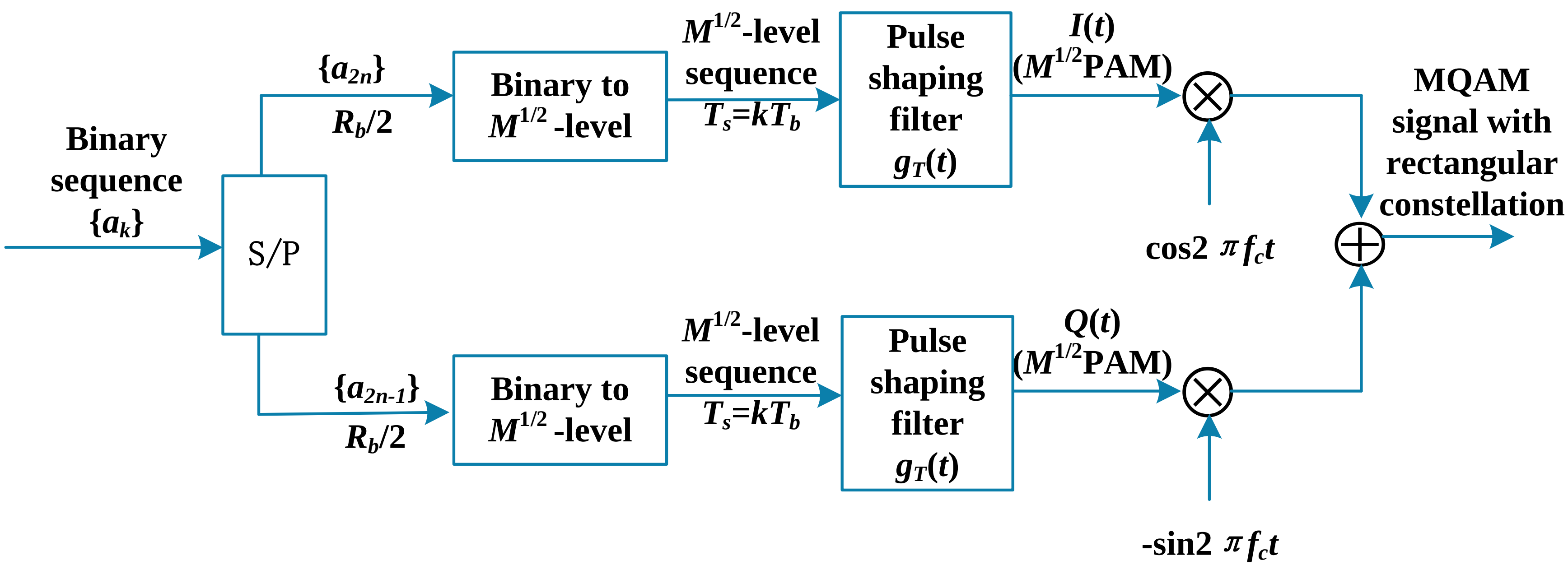
According to MASK Modulation

$$d_{\min} = \sqrt{2E_g} = \sqrt{\frac{6E_b \log_2 M}{M-1}}$$

MQAM

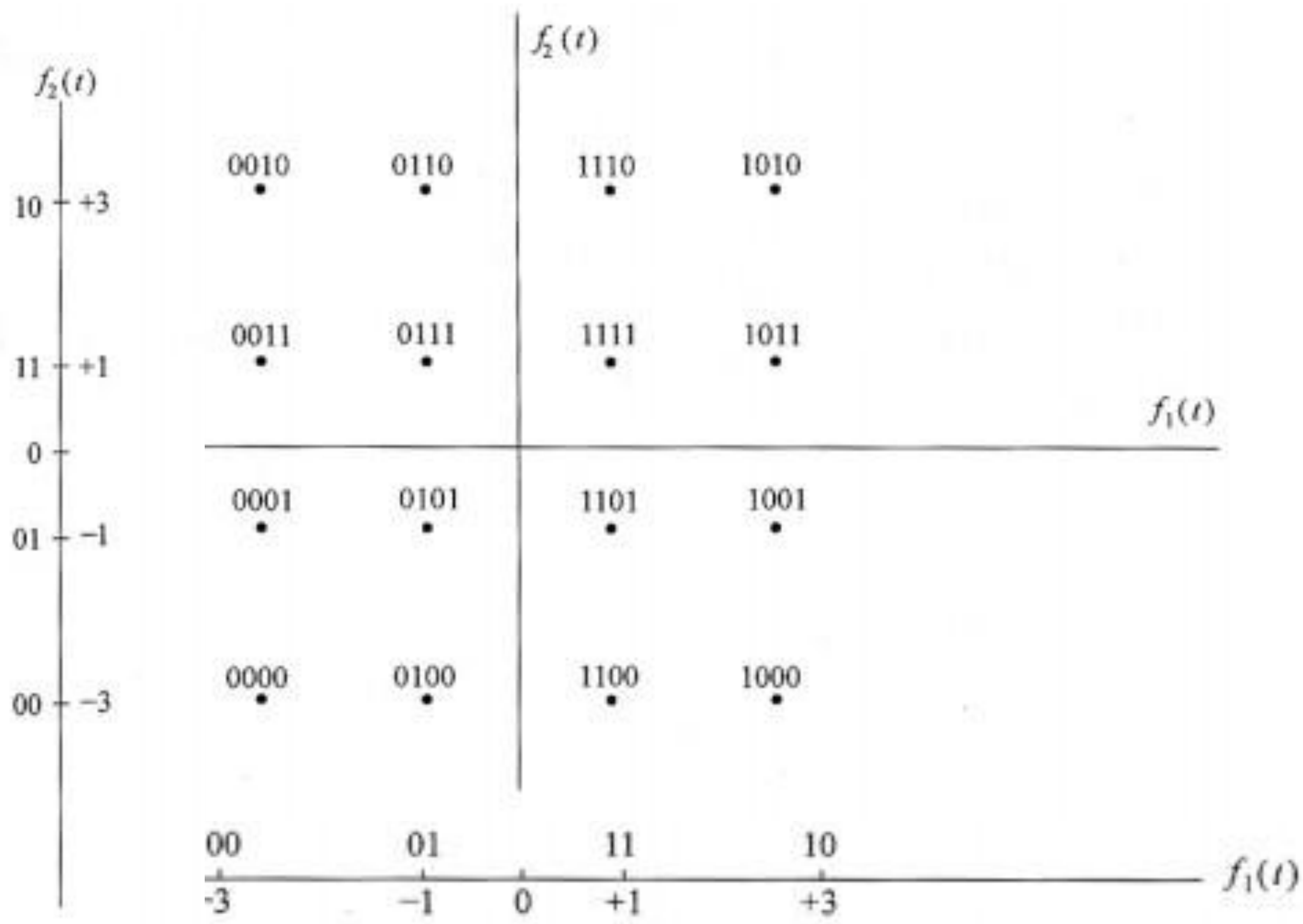
- **The generation of MQAM signal with rectangular constellation**

$$s_{QAM}(t) = a_{i_c} g_T(t)\cos\omega_c t - a_{i_s} g_T(t)\sin\omega_c t,$$
$$i = 1, 2, \ldots, M, 0 \le t \le T_s$$



MQAM

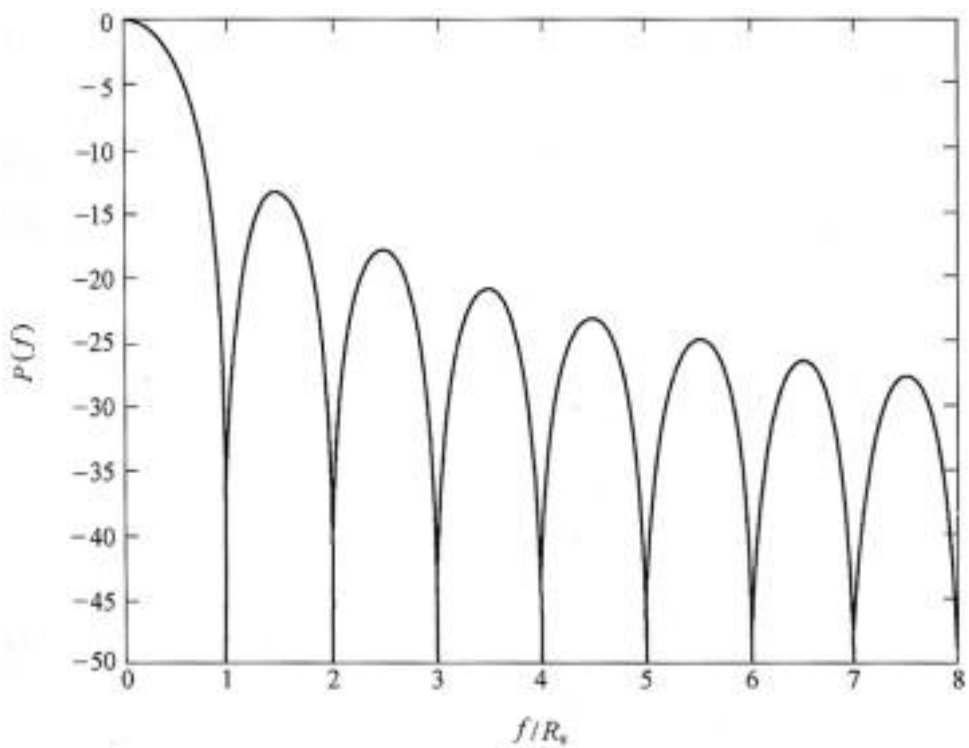
Constellation of 16QAM signal



MQAM

Average PSD of MQAM signal

The PSD of MQAM signal is the sum of the PSDs of in-phase signal and quadrature signal.



With rectangular baseband pulse

$$B = 2R_s = \frac{2R_b}{\log_2 M}$$

$$\eta = \frac{R_b}{B} = \frac{R_b}{\frac{2R_b}{\log_2 M}} = \frac{\log_2 M}{2} \text{ bit/s/Hz}$$

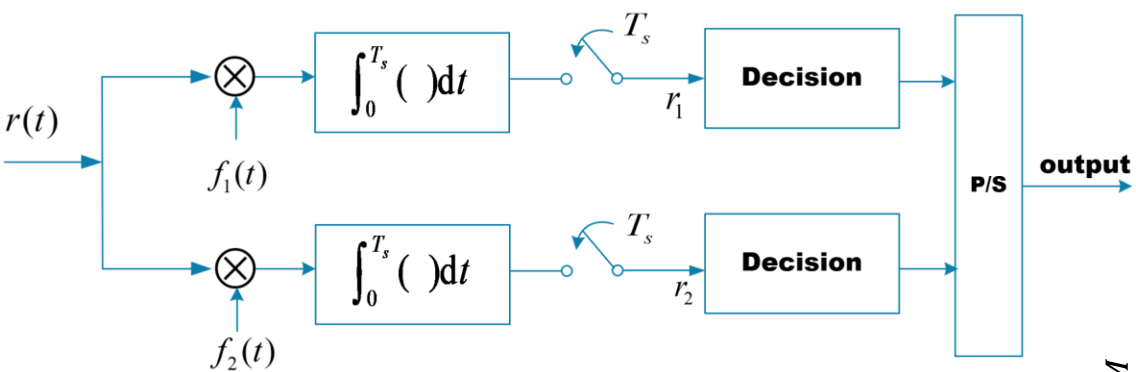
With RC pulse

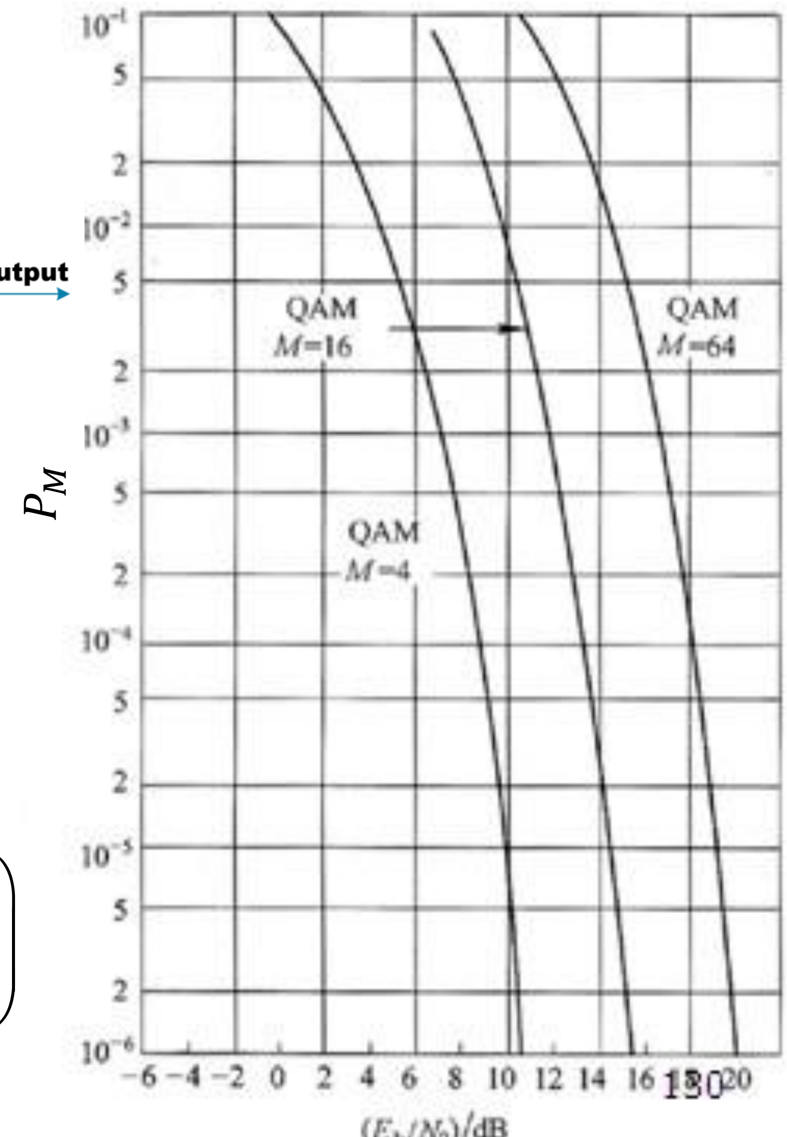
$$B = (1+\alpha)R_s = \frac{(1+\alpha)R_b}{\log_2 M}$$

$$\eta = \frac{R_b}{B} = \frac{\log_2 M}{1+\alpha} \text{ bit/s/Hz}$$

MQAM

- **Optimal reception of MQAM signal with rectangular constellation**





- **SER**

$$P_c = (1 - P_{\sqrt{M}})^2$$

where

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{E_b}{N_0}}\right)$$

$$P_M = 1 - P_c = 1 - (1 - P_{\sqrt{M}})^2$$

MQAM

- **Performance comparison between MQAM and MPSK signal**

$$P_{M-PSK} \simeq 2Q\left(\sqrt{2\frac{E_{av}}{N_0}\cdot \sin^2\frac{\pi}{M}}\right)$$

$$P_{M-QAM} \simeq 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3}{M-1}\cdot\frac{E_{av}}{N_0}}\right)$$

$$\Re_M = \frac{3/(M-1)}{2\sin^2 \pi / M}$$

~ ratio of required SNRs with given SER

- M=4: $\Re_M = 1$
- M>4: $\Re_M > 1$

M	$10\lg \Re_M$ (dB)
8	1.65
16	4.20
32	7.02
64	9.95

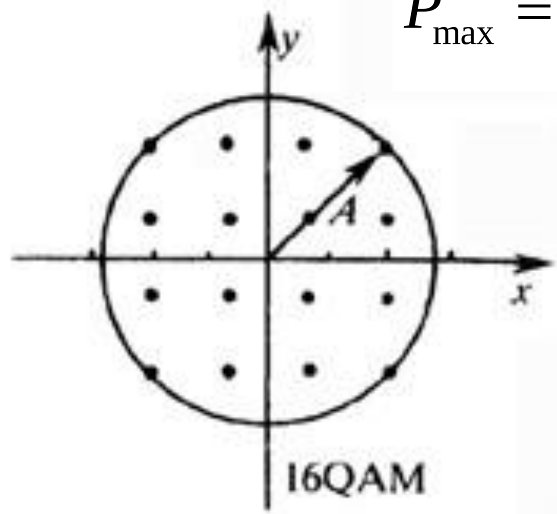
MQAM outperforms MPSK.

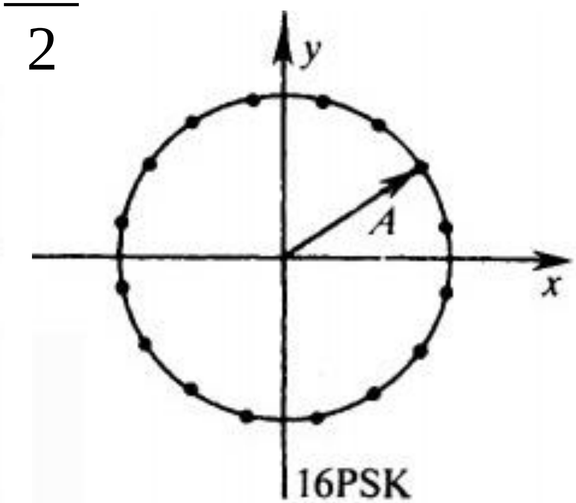
MQAM

● 16QAM and 16PSK

$$P_{\max} = \frac{A^2}{2}$$

$$d_{\min-QAM} = \frac{\sqrt{2}A}{\sqrt{M}-1} = 0.47A$$



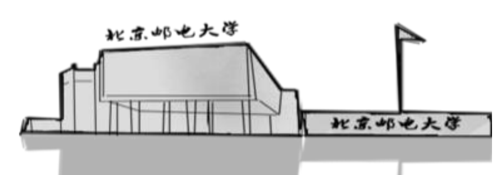


$$\xi_{QAM} = \frac{P_{\max}}{P_{QAM}} = \frac{\sqrt{M}(\sqrt{M}-1)^2}{2\sum_{i=1}^{\sqrt{M}/2}(2i-1)^2} = 1.8$$

$$d_{\min-PSK} \approx 2A\sin\frac{\pi}{16} = 0.39A$$

$$\xi_{PSK} = \frac{P_{\max}}{P_{PSK}} = 1$$

$$P_{QAM} = P_{PSK} \Longrightarrow \frac{d^2_{\min-\mathrm{QAM}}}{d^2_{\min-\mathrm{PSK}}} = 2.62 = 4.19\mathrm{dB}$$



M-ary Digital Modulation

- Introduction
- Vector Representation of Digital Modulation Signals
- Statistical Decision Theory
- Optimal reception of M-ary digital modulation signals with AWGN
- MASK
- MPSK
- MQAM
- MFSK

MFSK

- **MFSK signal**

$$s_{i-MFSK}(t)=\sqrt{\frac{2E_s}{T_s}}\cos\left[2\pi f_c t+2\pi i\Delta f t\right]=\mathrm{Re}\left[v_i(t)e^{j2\pi f_c t}\right],$$

$$i=1,2,\ldots,M,0\le t\le T_s,$$

- **Symbol energy**

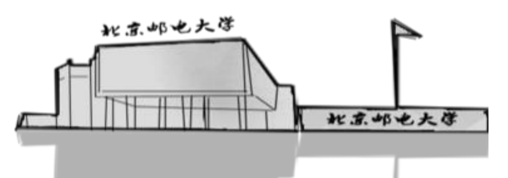
$$E_s=\int_{-\infty}^{\infty}s_i^2(t)dt$$

- **Normalized Correlation Coefficient**

$$\rho_{mk}=\frac{1}{E_s}\int_{-\infty}^{\infty}s_m(t)s_k(t)dt=\frac{2}{T_s}\int_0^{T_s}\left[\cos 2\pi(f_c+k\Delta f)t\bullet\cos 2\pi(f_c+m\Delta f)\right]dt$$

$$=\mathrm{sinc}\left[2(m-k)\Delta f T\right]$$

$$\Delta f=\frac{1}{2T_s},\rho_{mk}=0 \longrightarrow$$ **Orthogonal MFSK**



MFSK

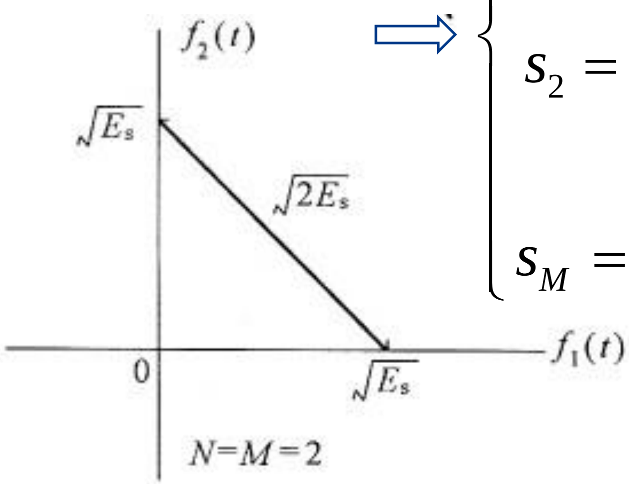
The vector representation of MFSK signal

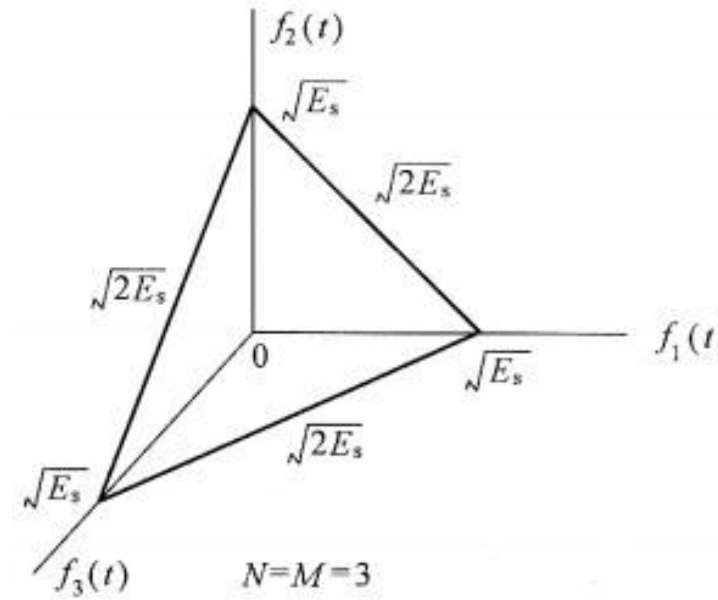
$$s_i(t) = \sqrt{E_s}\, f_i(t) = \sum_{k=1}^{M} s_{ik} f_k(t), i = 1, \ldots, M$$

where

$$f_i(t) = \sqrt{2/T_s}\cos 2\pi(f_c + i\Delta f)t, i = 1, \ldots, M,\quad s_{ik} = \int_0^{T_s} s_i(t) f_k(t) dt$$

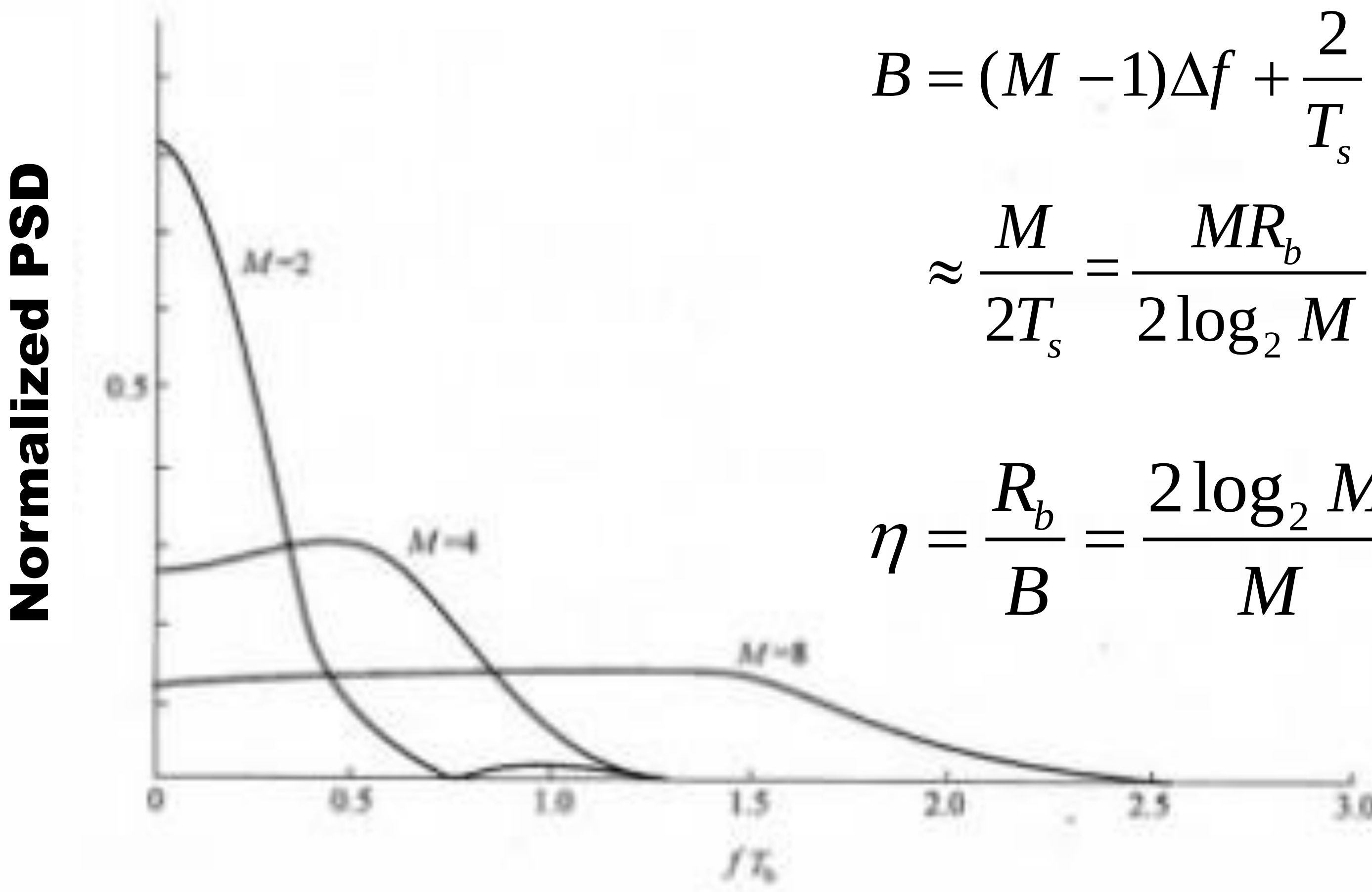
$$\Rightarrow \begin{cases} s_1 = (s_{11}, s_{12}, 0, \ldots, s_{1M}) \\ \quad = (\sqrt{E_s}, 0, 0, \ldots, 0) \\ s_2 = (0, \sqrt{E_s}, 0, \ldots, 0) \\ \quad \vdots \\ s_M = (0, 0, 0, \ldots, \sqrt{E_s}) \end{cases} \Rightarrow d_{kn} = \sqrt{2E_s}, \forall k, n$$





MFSK

Average PSD of MFSK signal



$$B=(M-1)\Delta f+\frac{2}{T_s}=\frac{M+3}{2T_s}$$

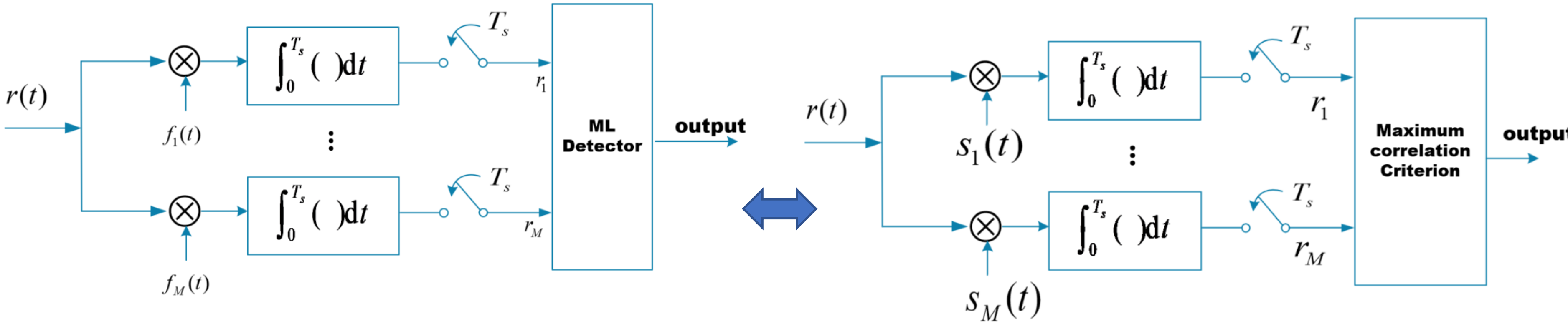
$$\approx\frac{M}{2T_s}=\frac{MR_b}{2\log_2 M}$$

$$\eta=\frac{R_b}{B}=\frac{2\log_2 M}{M}\,bit/s/Hz$$

Beijing University of Posts and Telecommunications

MFSK

- **The optimal reception of orthogonal MFSK signal**



$$
\begin{aligned}
d_i^2 &= \| \boldsymbol{r} - \boldsymbol{s}_i \|^2 \\
&= \| \boldsymbol{r} \|^2 + \| \boldsymbol{s}_i \|^2 - 2\boldsymbol{r} \cdot \boldsymbol{s}_i \qquad i = 1, 2, \cdots, M \\
&= \| \boldsymbol{r} \|^2 + E_s - 2 r_i \sqrt{E_s}
\end{aligned}
$$

The minimal d_i must correspond to the maximal r_i.

MFSK

SER analysis of MFSK reception

- **Suppose s_1 transmitted. Let event A_i denote that the noise of the ith branch is higher than the signal plus noise of the first branch.**

$$P_M = P\left(e \mid s_1\right) \qquad P(A_i) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_s}{2N_0}}\right)$$

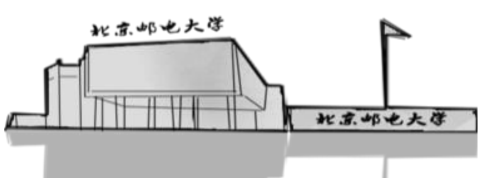
$$P(e \mid s_1) = P(A_2 \cup A_3 \cup \cdots \cup A_M)$$

- **The upper bound of P_M**

$$P_M = P(e \mid s_1) \leqslant P(A_2) + P(A_3) \cdots + P(A_M)$$

$$= \frac{M-1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_s}{2N_0}}\right)$$

$$= \frac{M-1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b \cdot \log_2 M}{2N_0}}\right)$$



MFSK

● SER analysis of MFSK reception

- **The increase in M results in a higher bandwidth and lower SER, creating a tradeoff between efficiency and reliability.**

$$\text{when} \quad M \to \infty,$$

$$\text{to make sure} \quad P_M \to 0,$$

$$\text{must have} \quad \frac{E_b}{N_0} > \ln 2 (-1.6\text{dB})$$

- **Average BER P_b**

with optimal reception

$$P_b = \frac{M}{2(M-1)} P_M$$

$$\xrightarrow[\textbf{large M}]{\textbf{with a}} \quad P_b \approx \frac{P_M}{2}$$