EBU6018
Advanced Transform Methods

Discrete Fourier Transform_2
Fast Fourier Transform

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Fast Fourier Transform (FFT)

- What is the FFT?
  - A collection of “tricks” that exploit the symmetry of the DFT calculation to make its execution much faster
  - Speedup increases with DFT size

- This lecture: outline the basic workings of the simplest formulation, the radix-2 decimation-in-time algorithm.
- Radix-2 means that the number of elements is a power of 2 (4, 8, 16, 32, etc)
- Decimation-in-time means that the input function is sampled in the time domain.

Acknowledgement: adapted from R M Stern (CMU)

Introduction, continued

- Some dates:
  - ~1880 - algorithm first described by Gauss
  - 1965 - algorithm rediscovered (not for the first time) by Cooley and Tukey

- FFT Revolutionized digital signal processing from 1960s

- E.g. in 1967 8192-point DFT on mainframe IBM 7094:
  - ~30 minutes using conventional techniques
  - ~5 seconds using FFTs

Measures of computational efficiency

- Could consider
  - Number of additions
  - Number of multiplications
  - Amount of memory required
  - Scalability and regularity

- Focus most on number of multiplications
  - More costly than additions for fixed-point processors
  - Same cost as additions for floating-point processors, but number of operations is comparable

Computational Cost of Discrete-Time Operation

- In general, the output of a system (for example a transform) is the convolution of the input with the impulse response of the system
- So, the convolution of an *N*-point input with an *M*-point unit sample response ….
- Direct convolution: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

Number of multiplies ≈ *MN*

For $N >> M$ the computation is $O(N^2)$

Computational Cost of Discrete-Time Operation

- Convolution of an *N*-point input with an *M*-point unit sample response ….

  Replace the convolution by multiplying the transforms of the two functions then getting the inverse transform of the product.

- Using transforms directly: $$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$
  - Computation of *N*-point DFTs requires N^2 multiplies
  - Each convolution (two direct transforms plus an inverse transform) requires three DFTs

For $N >> M$ the computation is approx $3N^2$ $O(N^2)$

So, no time is saved replacing convolution by DFT, but can now use FFT to be faster

Cooley-Tukey decimation-in-time algorithm

- Consider DFT algorithm for an integer power of 2,

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \qquad W_N = e^{-j2\pi/N}$$

Take alternate values of the input sequence.

- Create separate sums for even and odd values of *n:*

$$X[k] = \sum_{n\ even} x[n] W_N^{nk} + \sum_{n\ odd} x[n] W_N^{nk}$$

Note different sign in twiddle factor in this lecture
– common in FFT texts

- Letting $n = 2r$ for *n* even and $n = 2r+1$ for *n* odd, we get

$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k}$$

Cooley-Tukey decimation in time algorithm

- Splitting indices in time, we have obtained

$$X[k]=\sum_{r=0}^{(N/2)-1} x[2r]W_N{}^{2rk}+\sum_{r=0}^{(N/2)-1} x[2r+1]W_N{}^{(2r+1)k}$$

- But $\quad W_N^2=e^{-j2\pi 2/N}=e^{-j2\pi/(N/2)}=W_{N/2}$

- and $\quad W_N^{2rk}W_N^k=W_N^k W_{N/2}^{rk}$

So:

$$X[k]=\underbrace{\sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{rk}}_{N/2\text{-point DFT of } x[2r]}+W_N^k\underbrace{\sum_{r=0}^{(N/2)-1} x[2r+1]W_{N/2}^{rk}}_{N/2\text{-point DFT of } x[2r+1]}$$

Savings so far …

- We have split the DFT computation into two halves:

$$X[k]=\sum_{k=0}^{N-1} x[n]W_N{}^{nk}$$

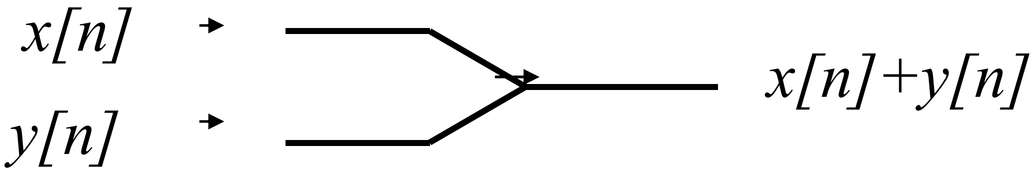
$$=\sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{rk}+W_N^k\sum_{r=0}^{(N/2)-1} x[2r+1]W_{N/2}^{rk}$$

If x[n] is 8-point, then we now have two 4-point

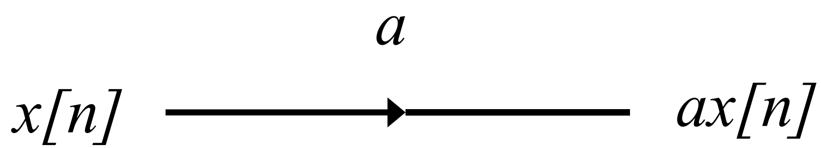
- Have we gained anything? Consider the nominal number of multiplications for $N=8$
  - Original form produces $8^2=64$ multiplications
  - New form produces $2(4^2)+8=40$ multiplications
  - So we're already ahead ….. Let's keep going!!

Signal flowgraph notation

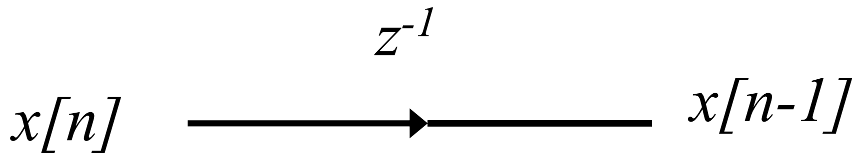
- In generalizing this formulation, it is most convenient to adopt a graphic approach …
- Signal flowgraph notation describes the three basic DSP operations:
  - Addition



  - Mult by a constant



  - Delay

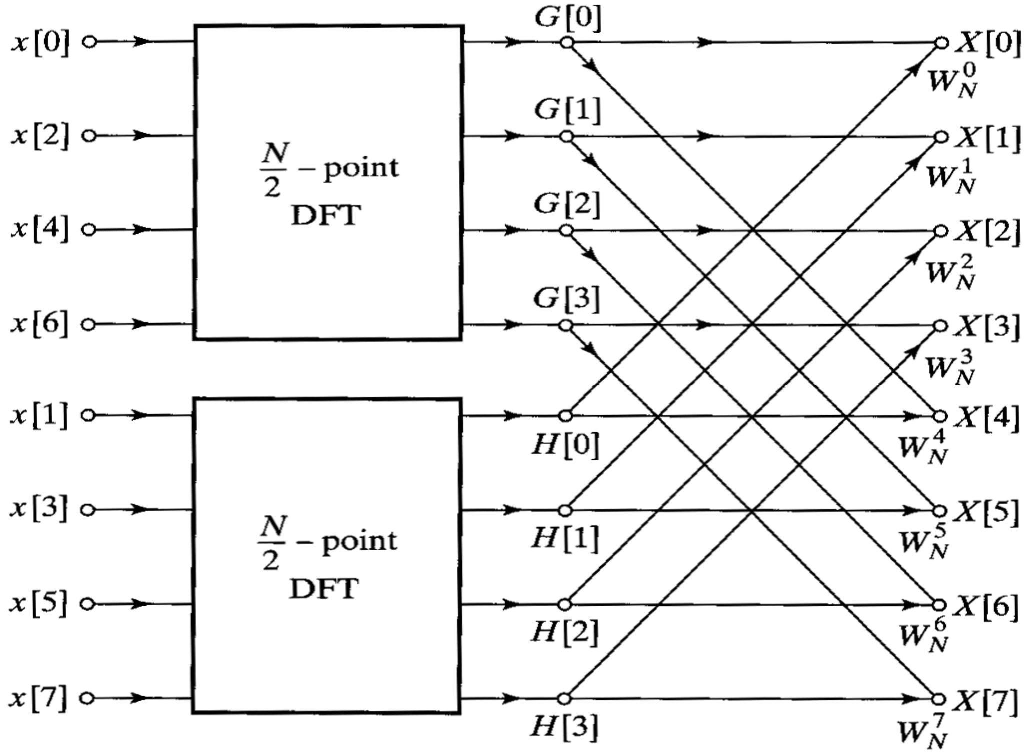


Signal flowgraph representation of 8-point DFT

- Recall that the DFT is now of the form

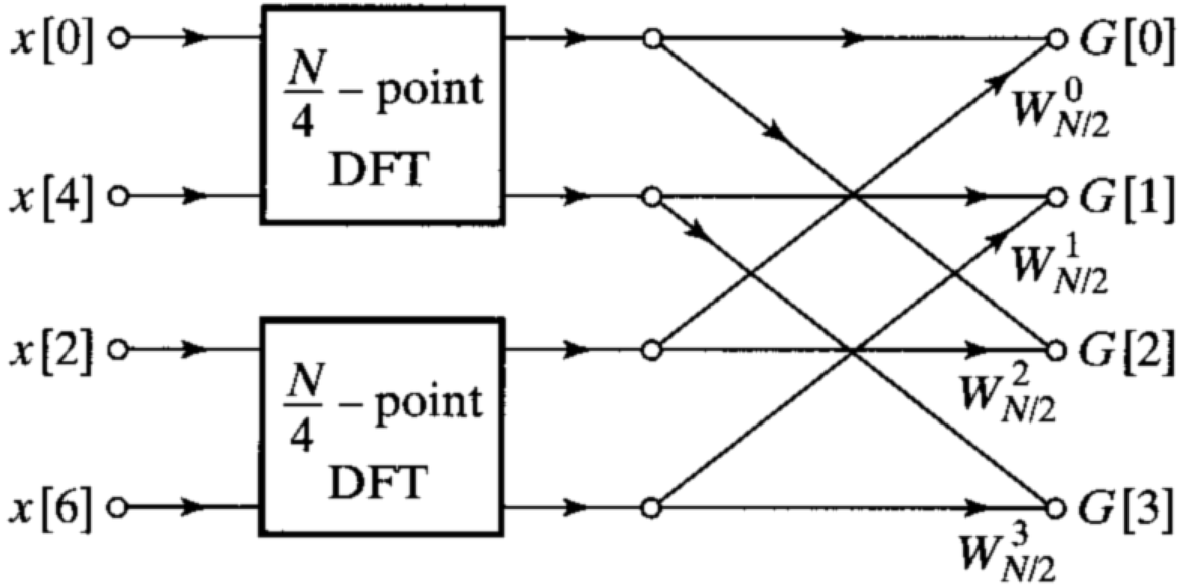
$$X[k] = G[k] + W_N^k H[k]$$

- The DFT in (partial) flowgraph notation:

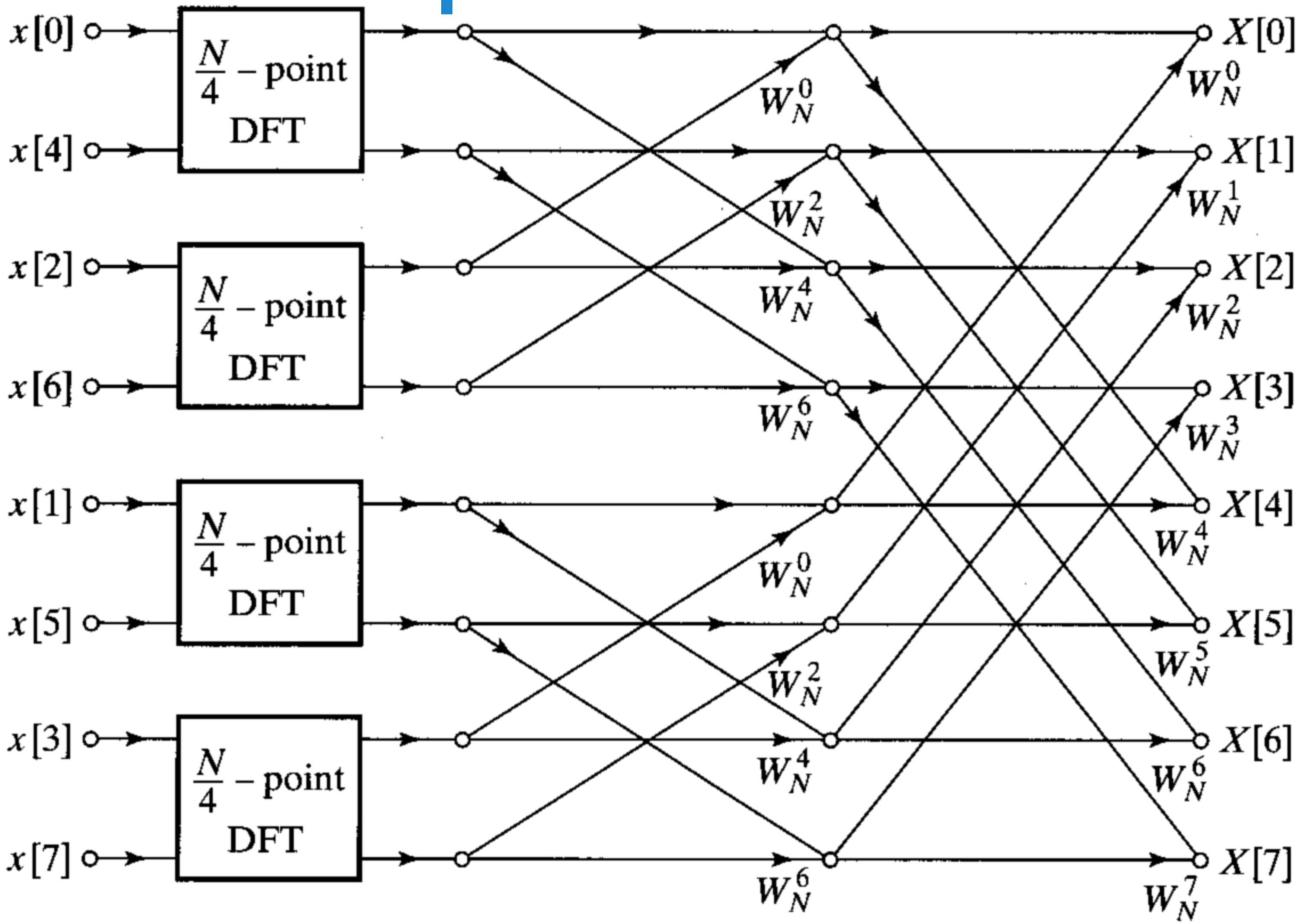


Continuing with the decomposition

- So why not break up into additional DFTs?
- Let’s take the upper 4-point DFT and break it up into two 2-point DFTs:



The complete decomposition into 2-point DFTs



Now let's take a closer look at the 2-point DFT

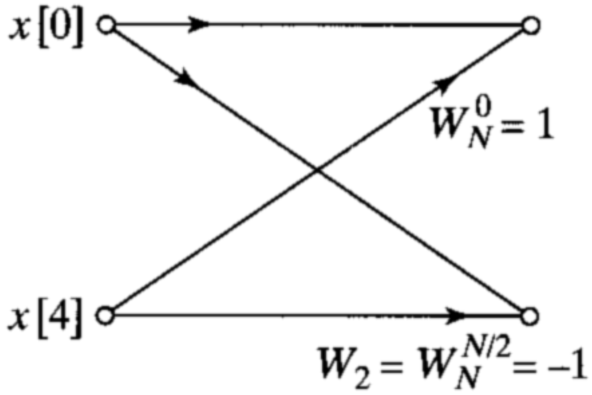
- The expression for the 2-point DFT is:

$$X[k] = \sum_{n=0}^{1} x[n] W_2^{nk} = \sum_{n=0}^{1} x[n] e^{-j2\pi nk/2}$$

- Evaluating for $k = 0,1$ we obtain

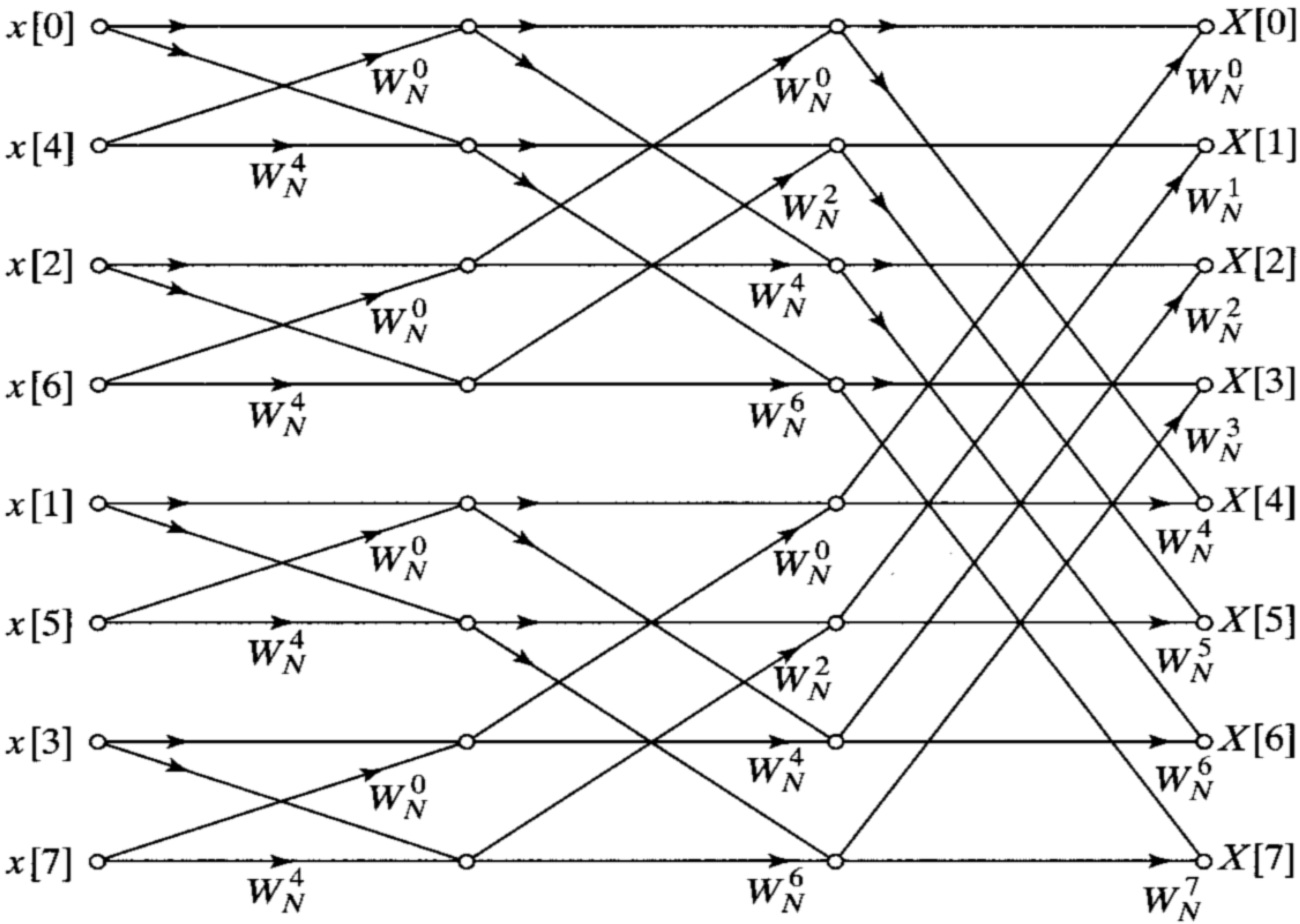
$$X[0] = x[0] + x[1]$$
$$X[1] = x[0] + e^{-j2\pi 1/2} x[1] = x[0] - x[1]$$

which in signal flowgraph notation looks like ...

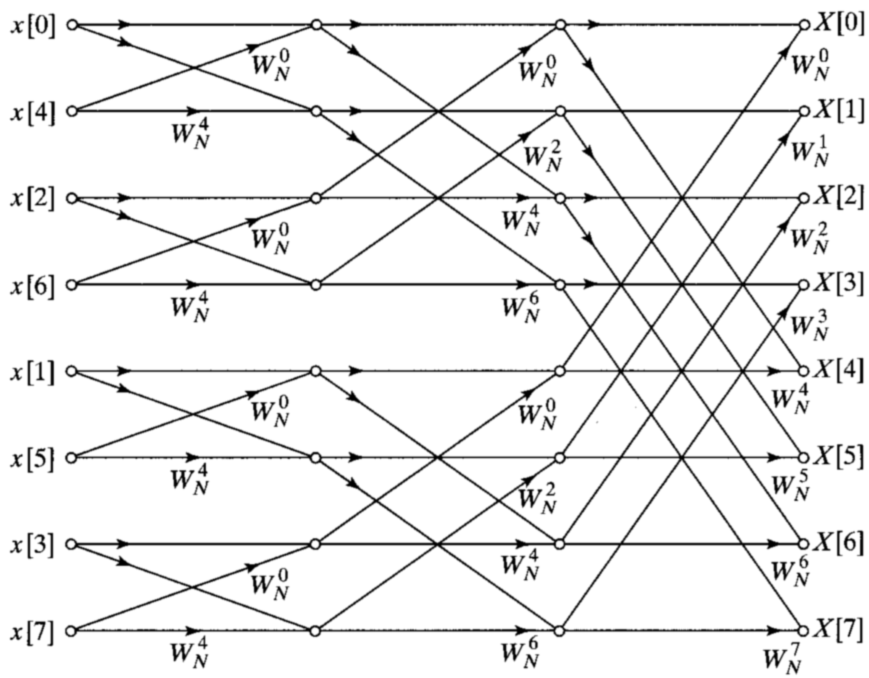


This topology is called the basic "butterfly"

The complete 8-point decimation-in-time FFT



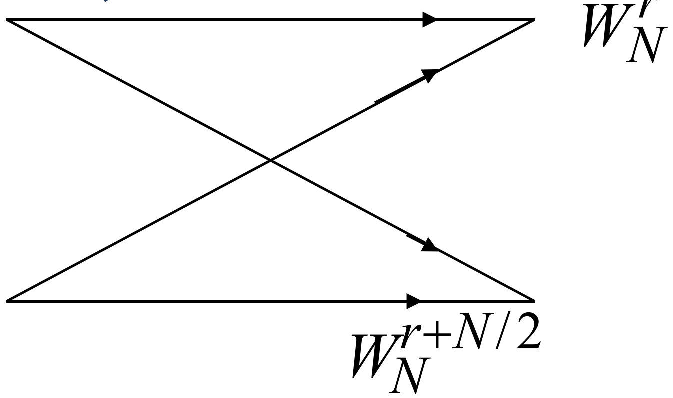
Number of multiplies for N-point FFT



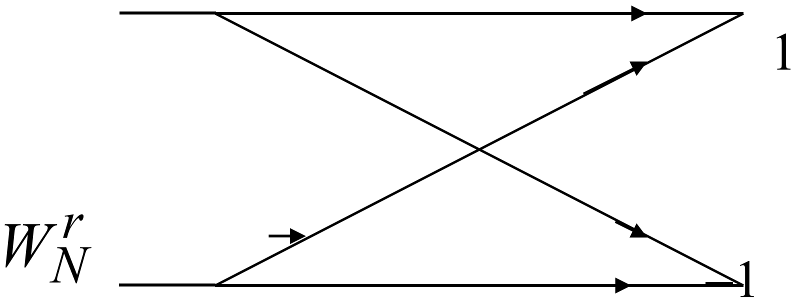
- Let $N = 2^{\nu}$ where $\nu = \log_2(N)$
- ($\log_2(N)$ columns)($N/2$ butterflys/column)(2 mults/butterfly) or approx $N\log_2(N)$ multiplications

Additional timesavers: reducing multiplications in the basic butterfly

- As we derived it, the basic butterfly is of the form

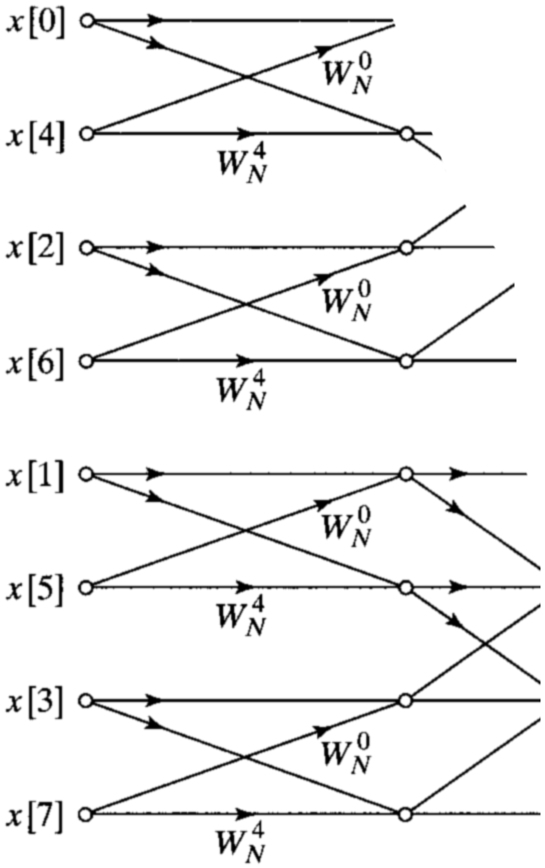


Since $W_N^{N/2} = -1$ we can reduce computation by 2 by premultiplying by W_N^r



Bit reversal of the input

- Recall the first stages of the 8-point FFT:



Consider the binary representation of the indices of the input:

0	000
4	100
2	010
6	110
1	001
5	101
3	011
7	111

If these binary indices are time reversed, we get the binary sequence representing
0,1,2,3,4,5,6,7

Hence the indices of the FFT inputs are said to be in bit-reversed order

Example with an 8-point input sequence

	POSITION	000	001	010	011	100	101	110	111
8-POINT	$x[n] =$	6	2	7	3	4	9	5	1

2 x 4-POINT: [6 7 4 5][2 3 9 1]

4 x 2-POINT: [6 4][7 5][2 9][3 1]

8 x 1 POINT: [6][4][7][5][2][9][3][1]

POSITION: 000 001 010 011 100 101 110 111

BIT REVERSAL OF ORIGINAL $x[n]$ GIVES $X[k]$

Some comments on bit reversal

- This implementation of FFT: input is bit reversed, output is in natural order
- Sometimes convenient to implement filtering applications by
  - Use FFTs with input in natural order, output in bit-reversed order
  - Multiply frequency coefficients together (in bit-reversed order)
  - Use inverse FFTs with input in bit-reversed order, output in natural order
- Computing in this fashion means we never have to compute bit reversal explicitly