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EBU6018
Advanced Transform Methods

Tutorial - KLT

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Tutorial Example 1

- **Question:** Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

Guidance:

Step 1: Compute the covariance matrix between variable x and y

Step 2: Calculate the eigenvalues of the covariance matrix

Step 3: Calculate the eigenvectors

Step 4: Normalise the eigenvectors

x	y
-3.01	-2.67
0.18	-2.12
-6.56	-4.24
1.60	1.92
2.78	-1.20
2.03	2.14

Assuming this is a sample of a larger population

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Tutorial Example 1 - Solutions

- **Question**: Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

- **Step 1**: Compute the covariance matrix R_{xy}
  - To compute the covariance matrix, we need the following quantities
    - $x_{ave} =$
    - $y_{ave} =$
    - $Var_x = \frac{\sum_1^N (x_i - x_{ave})^2}{N-1} =$
    - $Var_y = \frac{\sum_1^N (y_i - y_{ave})^2}{N-1} =$
    - $Cov_{x,y} = \frac{\sum_1^N (x_i - x_{ave})(y_i - y_{ave})}{N-1} =$
    - $Cov_{y,x} = \frac{\sum_1^N (x_i - x_{ave})(y_i - y_{ave})}{N-1} =$

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Tutorial Example 1 - Solutions

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- **Step 1**: Compute the covariance matrix R_{xy}

  - The covariance matrix = $\begin{bmatrix} Var_x & Covar_{x,y} \\ Covar_{y,x} & Var_y \end{bmatrix}$

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Tutorial Example 1 - Solutions

- **Question**: Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

- **Step 2**: Compute the **eigenvalues** of the covariance matrix R_{xy}

$$\left|R_{xy} - \lambda I\right| =$$

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- **Question**: Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

- **Step 3**: Find the normalized **eigenvectors**

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Tutorial Example 2

- **Question**: Find the normalized eigenvectors of the following matrix:

$$R_{xy} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

Hint:

The Eigenvalues, λ, of a square matrix R_{xy} are the solutions of:

$$\left|R_{xy} - \lambda I\right| = 0$$

The Eigenvectors, v, are the solutions of:

$$\left(R_{xy} - \lambda I\right)\text{v} = 0$$

Tutorial Example 2

- **Question**: Find the normalized eigenvectors of the following matrix $R_{xy} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

- **Solution:** Compute the **eigenvalues**

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- **Solution:** Find the normalized **eigenvectors**

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Tutorial Example 2

- **Question**: Find the normalized eigenvectors of the following matrix:

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

Solutions:

Tutorial Example 3

- **Question**: Find the eigenvalues of the following 3x3 matrix:

$$A = \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix}$$

Hint:

The Eigenvalues, λ, of a square matrix A are the solutions of:

$$|A - \lambda I| = 0$$

Summary

- Covariance matrix
  - Refers to the measure of the directional relationship between two random variables.
  - Always symmetrical, so the eigenvalues will be real and the eigenvectors will be orthogonal.
- Eigenvalues/Eigenvectors
  - Frequently used in matrix decomposition for dimension reduction
  - Equation of eigenvalue: $|A - \lambda I| = 0$
  - Equation of eigenvector: $(A - \lambda I)v = 0$
- Calculating the Eigenvector Matrix is a relatively computation-intensive process.
  - This is a disadvantage of the Karhunen-Loeve Transform, which is based on multivariable statistics.

Karhunen Loève Transform (KLT) – Procedures

$$\boxed{\mathbf{X} = [\vec{x}_0, \vec{x}_1, \ldots \vec{x}_{N-1}]}$$

1. Find mean vector for input data $E(\mathbf{X}) = \frac{1}{N}\sum_{i=0}^{N-1} \vec{x}_i$

2. Find covariance matrix $\mathbf{R_{XX}} = \frac{1}{N-1}\sum_{i=0}^{N-1} (\vec{x}_i - E(\vec{x}))(\vec{x}_i - E(\vec{x}))^T$

3. Find eigenvalues of the covariance matrix $|\mathbf{R_{XX}} - \lambda \mathbf{I}| = 0$

4. Find eigenvectors of the covariance matrix $(\mathbf{R_{XX}} - \lambda_i \mathbf{I})\vec{\varphi}_i = 0$

5. Normalise the eigenvectors $\vec{\varphi}^*{}_i = \frac{\vec{\varphi}_i}{|\vec{\varphi}_i|}$ so that $\langle \vec{\varphi}_i, \vec{\varphi}_i \rangle = 1$

6. Transform the input $\mathbf{Y} = \varphi^T \mathbf{X}$, where $\varphi^T = [\vec{\varphi}^*{}_1, \vec{\varphi}^*{}_2, \ldots]$

Karhunen Loève Transform (KLT) – Tutorial Question 1

- Find the KLT of the given dataset (sampled from a population):

a	b
-1	0
-2	-1
0	2
0	-1
2	4

Karhunen Loève Transform (KLT) – Tutorial Question 2

- Find the covariance matrix of the given dataset (sampled from a population):

a	b	c
-1	0	2
-2	-1	4
0	2	0
0	-1	-2
2	4	0

Pop quiz

What is the dimension of this covariance matrix?

a. 2 x 2
b. 2 x 3
c. 3 x 3
d. 3 x 2
e. 5 x 5

* Same data a and b as in Question 1.
* You can directly use the solutions of Question 1 to save some work.

$$R_{ab} = \begin{bmatrix} 2.2 & 2.7 \\ 2.7 & 4.7 \end{bmatrix}$$

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