Queen Mary
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EBU6018
Advanced Transform Methods

Week 4.3 – Wigner-Ville Distribution (WVD)

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Lecture Outline

- **Wigner-Ville Distribution**
  - Definition
  - Pros and cons
  - Connection to other transform methods

Wigner-Ville Distribution - Background

- So far we have looked at transforms that compute the correlation between a signal and basis functions that are functions of time and frequency (or of scale and translation). The time-frequency resolution is determined by the basis functions
  - E.g., STFT, Wavelet Transform

- An alternative approach is to compute directly:
  - ❑ Time-frequency energy density- signal's energy density in both time and frequency
    - In contrast to power spectrum: energy in frequency only
  - ❑ An example of this is the Wigner-Ville distribution

Wigner-Ville Distribution - Background

- Comparison of STFT and CWT
  - Similarities:
    - They are both Windowed Transforms
    - signal is multiplied by a function, and the transform is computed separately for different segments of signals.
    - can be written in inner product form

$$STFT(b,\omega) = \left\langle s(t), \gamma(t-b)e^{j\omega t} \right\rangle$$

$$CWT(b,a) = \left\langle s(t), \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) \right\rangle$$

  - Difference:
    - Fixed time duration and frequency bandwidths of $\gamma(t)$
    - Variable time duration and bandwidth of $\psi(t)$

Wigner-Ville Distribution – Energy Distribution

- The purpose of energy distributions is to distribute the energy of the signal over time and frequency.
- The energy of a signal s(t) can be found from the squared modulus of either the signal or its Fourier Transform:
- $E_s = \int |s(t)|^2 \mathrm{dt} = \int |s(\omega)|^2 \mathrm{d}\omega$
- $|s(t)|^2$ and $|s(\omega\ )|^2$ can be interpreted as energy densities in time and frequency respectively.

Wigner-Ville Distribution – Wiener-Khinchin Theorem

- The energy density and autocorrelation function of a signal are related by the Wiener-Khinchin Theorem.

- According to the Wiener-Khinchin theorem, the power spectrum is the Fourier Transform of the Autocorrelation Function.

$$P(\omega) = |s(\omega)|^2 = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega t} \mathrm{d}\tau$$

(For an example of the derivation look at:

https://mathworld.wolfram.com/Wiener-Khinchin Theorem.html)

Wigner-Ville Distribution – Autocorrelation

$$\boxed{R(\tau) = \int s(t)s(t+\tau)dt}$$

τ is the shift of the signal with respect to itself.

- In the standard autocorrelation function, time is integrated out of the result, and $R(\tau)$ is a function of only the time lag τ.
- There is a class of distribution called Cohen Class (of which the Wigner-Ville Distribution is a member) that uses a variation of the autocorrelation function where time remains in the result, this is the Instantaneous Autocorrelation Function:

$R(t, \tau)$ = s(t+ τ/2)s^*(t- τ/2) (If the signal is real, then s^* = s)

  - where τ is the time lag and * represents the complex conjugate of signal s.

Wigner-Ville Distribution – Instantaneous power spectrum

Recall the power spectrum:

$$P(\omega) = |S(\omega)|^2 = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau$$

Power spectrum of a signal is the Fourier Transform of its autocorrelation function

where $R(\tau)$ is the autocorrelation function (acf)

$$R(\tau) = \int_{-\infty}^{\infty} s(t)s^*(t-\tau)dt = \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)dt$$

What happens if we use *instantaneous* autocorrelation:

$$R(t,\tau) = s(t+\tau/2)s^*(t-\tau/2)$$

instead of $R(\tau) = \int_{-\infty}^{\infty} R(t,\tau)dt$? We get:

The WVD is the Fourier Transform of the instantaneous autocorrelation function

$$WVD_s(t,\omega) = \int_{-\infty}^{\infty} s(t+\tau/2)s^*(t-\tau/2)e^{-j\omega\tau}d\tau$$

which is the *Wigner-Ville Distribution* (WVD).

Wigner-Ville Distribution

- The Wigner-Ville Distribution is not a Windowed Transform in a similar way to the STFT

- However, the shifted version of the same signal in the autocorrelation function could be considered as a window.

- The WVD compares the information in the signal with its own information at other times and frequencies.

- The WVD has some interesting and useful properties.

Wigner-Ville Distribution

➢ Cross-WVD vs auto-WVD

Since we can define a cross-correlation,

we can also define a cross-Wigner-Ville distribution:

$$WVD_{s,g}(t,\omega) = \int_{-\infty}^{\infty} s(t+\tau/2)g*(t-\tau/2)e^{-j\omega\tau}d\tau$$

Taking complex conjugates we find

$$WVD_{s,g}(t,\omega) = WVD*_{g,s}(t,\omega)$$

So for the usual WVD (auto-WVD) we have

$$WVD_s(t,\omega) = WVD_{s,s}(t,\omega) = WVD*_s(t,\omega)$$

so the auto-WVD is always real.

Wigner-Ville Distribution

- Example: Gaussian function

Signal $s(t) = \sqrt[4]{\frac{\alpha}{\pi}} e^{-\alpha t^2/2}$ (normalized to unit energy)

For WVD, we get

$$WVDs(t,\omega) = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{\alpha}{2}\left[\left(t+\frac{\tau}{2}\right)^2 + \left(t-\frac{\tau}{2}\right)^2\right]\right\} e^{-j\omega\tau} d\tau$$

$$= e^{-\alpha t^2} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{\alpha}{4}\tau^2\right\} e^{-j\omega\tau} d\tau$$

Gaussian in time,
Gaussian in t-f

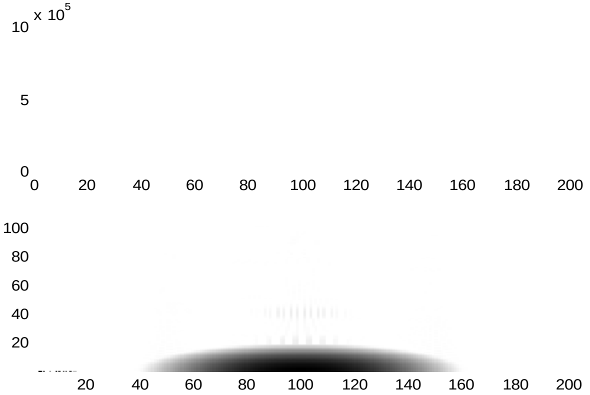
$$= 2\exp\left\{-\left[\alpha t^2 + \frac{1}{\alpha}\omega^2\right]\right\}$$

i.e. concentrated around (0,0).

α controls spread:

"time−width": $|t| < \sqrt{\frac{1}{\alpha}}$

"freq−width": $|\omega| < \sqrt{\alpha}$ Similar to Δt.Δω = k from the UP



Wigner-Ville Distribution

- Example: Gaussian chirplet

Signal: $s(t) = \sqrt[4]{\frac{\alpha}{\pi}} \exp\left\{-\frac{\alpha}{2}t^2 + j\frac{\beta}{2}t^2\right\}$

Power spectrum: $|S(\omega)|^2 = \sqrt{\frac{4\pi(\alpha^2+\beta^2)}{\alpha}} \exp\left\{-\frac{\alpha}{\alpha^2+\beta^2}\omega^2\right\}$

tells us which freqs $s(t)$ contains, not when. Compare:

$$WVDs(t,\omega) = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}\left[\left(t+\frac{\tau}{2}\right)^2+\left(t-\frac{\tau}{2}\right)^2\right]+\frac{j\beta}{2}\left[\left(t+\frac{\tau}{2}\right)^2-\left(t-\frac{\tau}{2}\right)^2\right]} e^{-j\omega\tau} d\tau$$

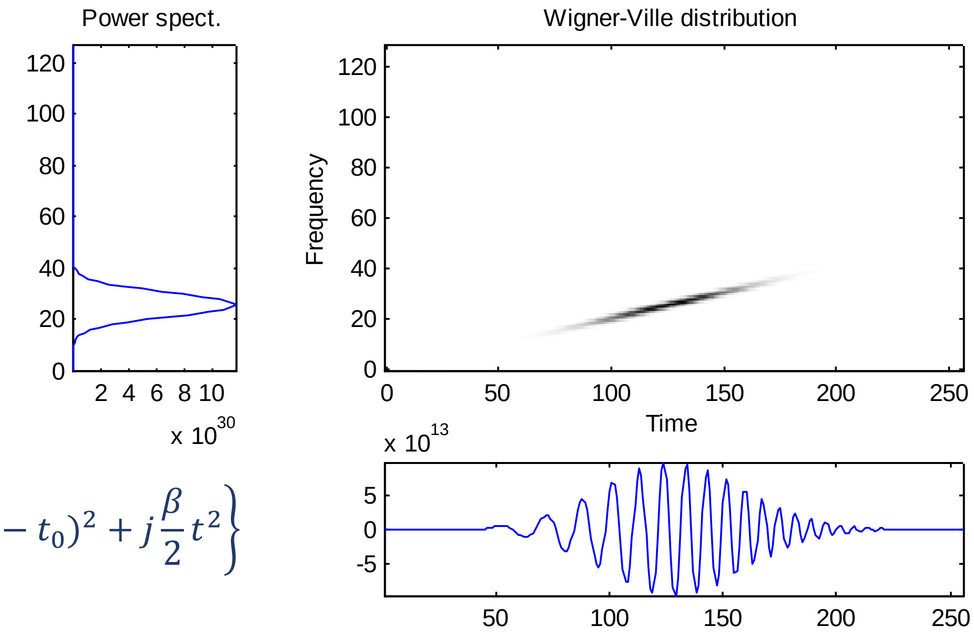
$$= e^{-\alpha t^2} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{4}\tau^2} e^{-j(\omega-\beta t)\tau} d\tau$$

$$= 2e^{-\left[\alpha t^2+\frac{1}{\alpha}(\omega-\beta t)^2\right]}$$

i.e. energy concentrated at $\omega = \beta t$,changing with time.

Wigner-Ville Distribution

- Illustration: chirplet



$$s(t) = \sqrt[4]{\frac{\alpha}{\pi}} \exp\left\{-\frac{\alpha}{2}(t-t_0)^2 + j\frac{\beta}{2}t^2\right\}$$

Wigner-Ville Distribution

➢ Time-limited or band-limited signals

- If $s(t)$ is time$-$limited, i.e. $s(t) = 0$ outside some interval $[t_{0,}t_1]$, then the WVD is also time$-$limited, i.e.

$$WVD_s(t,\omega) = 0 \text{ for } t \notin [t_0, t_1]$$

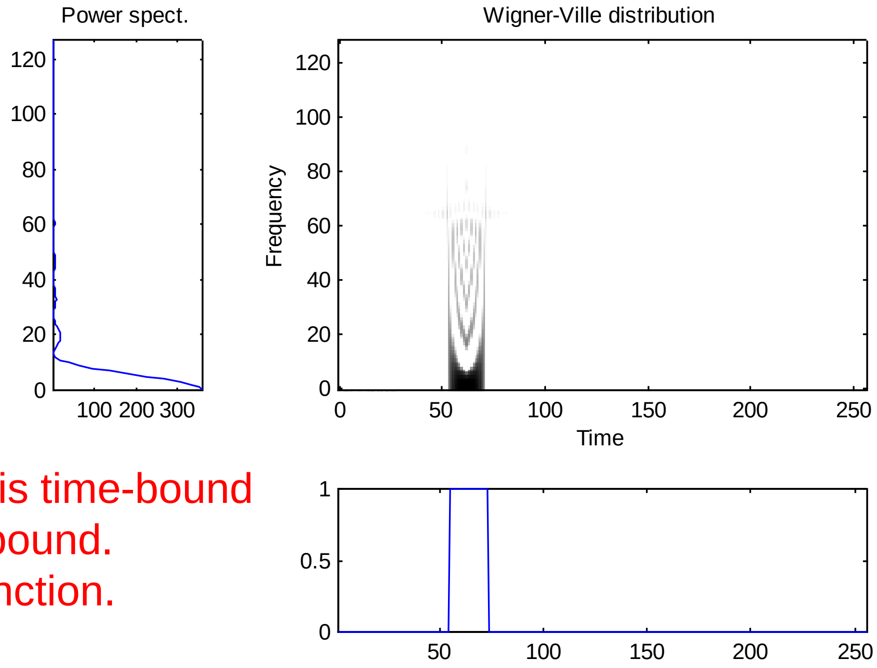
since no value for τ can make both $s(t + \tau/2)$ and $s(t - \tau/2)$ non$-$zero if t is outside this range.

(Either s(t+ τ/2) or s(t- τ/2) will be zero)

- A similar result is true for frequency band$-$limited signals.

Wigner-Ville Distribution

- Example: WVD of a Time-limited signal



The isolated pulse is time-bound but not frequency-bound.
The FT is a sinc function.

Wigner-Ville Distribution

- WVD Properties: Time Marginal Condition

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} WVD_s(t,\omega)d\omega = \int_{-\infty}^{\infty} s\left(t+\frac{\tau}{2}\right)s*\left(t-\frac{\tau}{2}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}d\omega\, d\tau$$

$$= \int_{-\infty}^{\infty} s\left(t+\frac{\tau}{2}\right)s*\left(t-\frac{\tau}{2}\right)\delta(\tau)d\tau$$

$$= |s(t)|^2$$

So integral over frequency of WVD is the signal power density at time t

Compare similar result for probability densities:

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)dy$$

Wigner-Ville Distribution

- WVD Properties: Frequency Marginal Condition

$$\int_{-\infty}^{\infty} WVD_s(t,\omega)dt = \int_{-\infty}^{\infty} s\left(t+\frac{\tau}{2}\right) s*\left(t-\frac{\tau}{2}\right) \int_{-\infty}^{\infty} e^{-j\omega\tau} dt\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j\omega\tau} \int_{-\infty}^{\infty} s(t)s*(t-\tau)dt\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j\omega\tau} R(\tau)d\tau$$

$$= |S(\omega)|^2$$

So integral over time of WVD is the power spectral density

We also have:

$$\frac{1}{2\pi}\int_{-\infty}^{-\infty}\int_{-\infty}^{\infty} WVD_s(t,\omega)dt\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{-\infty} |S(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |s(t)|^2 dt$$

i.e. the WVD is unitary: the energy in $WVD_s(t,\omega)$ is equal to energy in original signal $s(t)$.

Wigner-Ville Distribution

- Time-shift & Freq-moduln. invariant

  - If the WVD of $s(t)$ is $WVD_s(t,\omega)$,
    then the WVD of time−shifted signal $s_0(t) = s(t - t_0)$
    is a time−shifted WVD: $WVD_{s_0}(t,\omega) = WVD_s(t - t_0, \omega)$

  - Further, the WVD of frequency−modulated signal
    $s_1(t) = s(t)e^{j\omega_1 t}$ is a frequency−shifted WVD:
    $WVD_{s_1}(t,\omega) = WVD_s(t, \omega - \omega_1)$

(Both follow immediately from the formulas for WVD)

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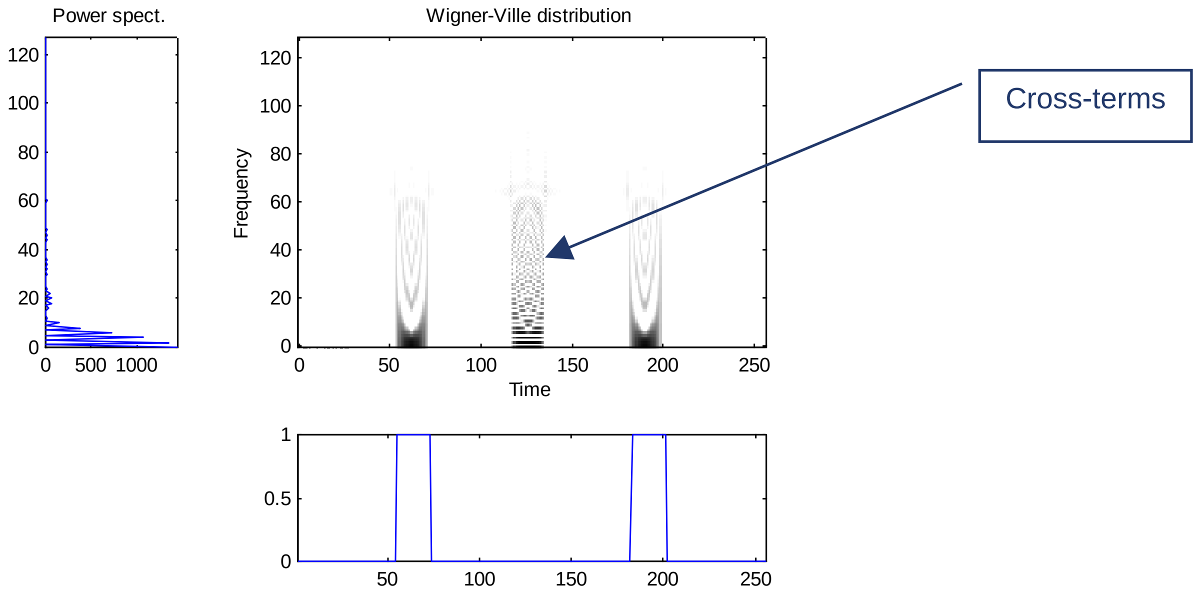
Wigner-Ville Distribution

➢ WVD of multiple signals: Cross-terms

- Wigner-Ville Distribution has many useful properties, and better resolution than STFT spectrogram. <u>BUT</u>
- Applications are limited due to *cross-term interference.*
- Consider composite signal $s(t) = s_1(t) + s_2(t)$. Then
  $WVD_s(t,\omega) = WVD_{s_1}(t,\omega) + WVD_{s_2}(t,\omega) + 2\,\mathrm{Re}\{WVD_{s_1,s_2}(t,\omega)\}$
  i.e. not only the sum of WVDs, but also the cross−term $WVD_{s_1,s_2}(t,\omega)$
- Also, cross−term is included at double the magnitude of the auto−terms, so often obscures useful patterns.

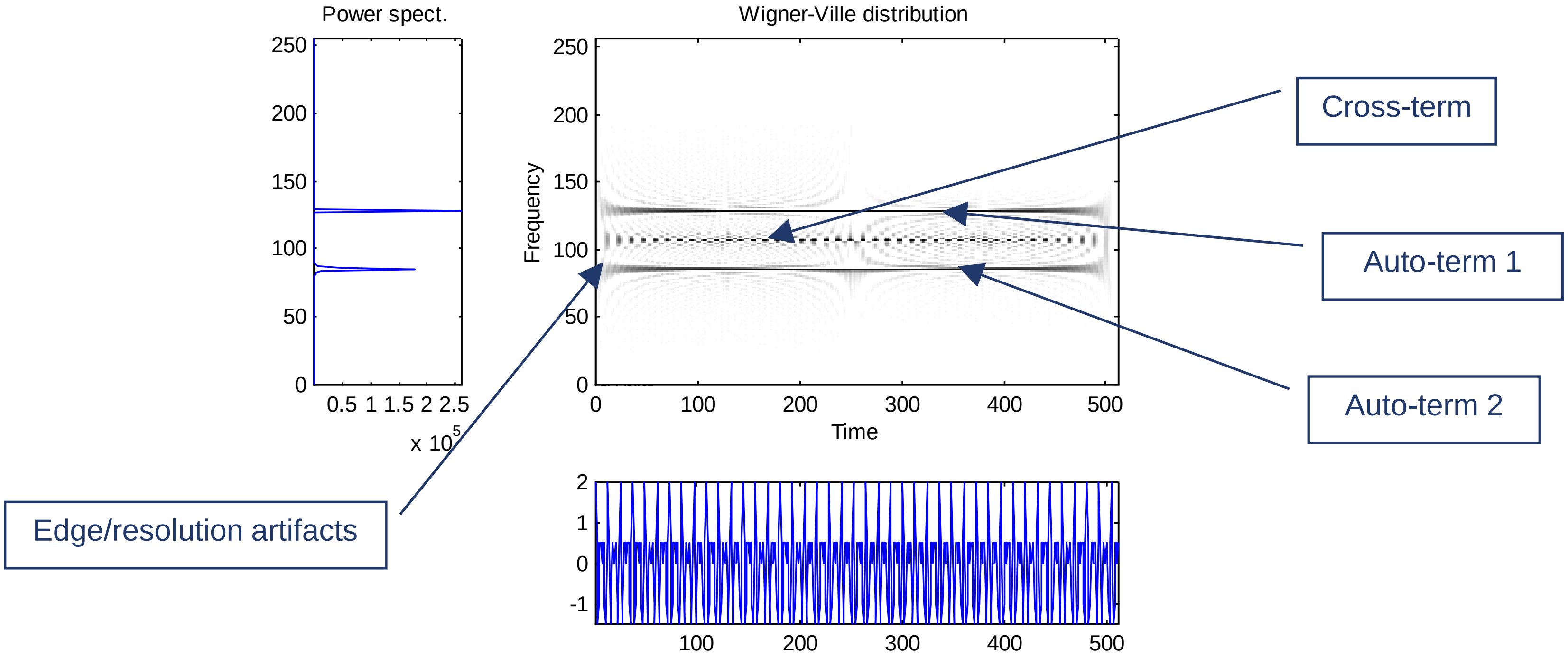
Wigner-Ville Distribution

- Example of cross-terms
  - WVD gives cross-terms for all but simplest signals, e.g.:



Wigner-Ville Distribution

- Example: sum of two sinusoids



Wigner-Ville Distribution

- Example: sum of two sinusoids

  - If $s(t) = \exp(j\omega_0 t)$ then
  $$WVD_s(t,\omega) = \int_{-\infty}^{\infty} \exp\left\{j\omega_0\left(t + \frac{t}{2} - t + \frac{t}{2}\right)\right\} e^{-j\omega\tau} d\tau = 2\pi\delta(\omega - \omega_0)$$
  i.e. the WVD is a "ridge" along frequency ω_0.

  - Now let $s(t) = \exp(j\omega_1 t) + \exp(j\omega_2 t)$. The power spectrum is
  $$|S(\omega)|^2 = 2\pi\delta(\omega_1) + 2\pi\delta(\omega_2)$$
  while the WVD is
  $$WVD_s(t,\omega) = 2\pi\delta(\omega - \omega_1) + 2\pi\delta(\omega - \omega_2) + 4\pi\delta(\omega - \omega_\mu)\cos(\omega_d t)$$
  where $\omega_\mu = \dfrac{\omega_1 + \omega_2}{2}$ and $\omega_d = \omega_1 - \omega_2$.

Wigner-Ville Distribution

- Example: cross-term (cont)

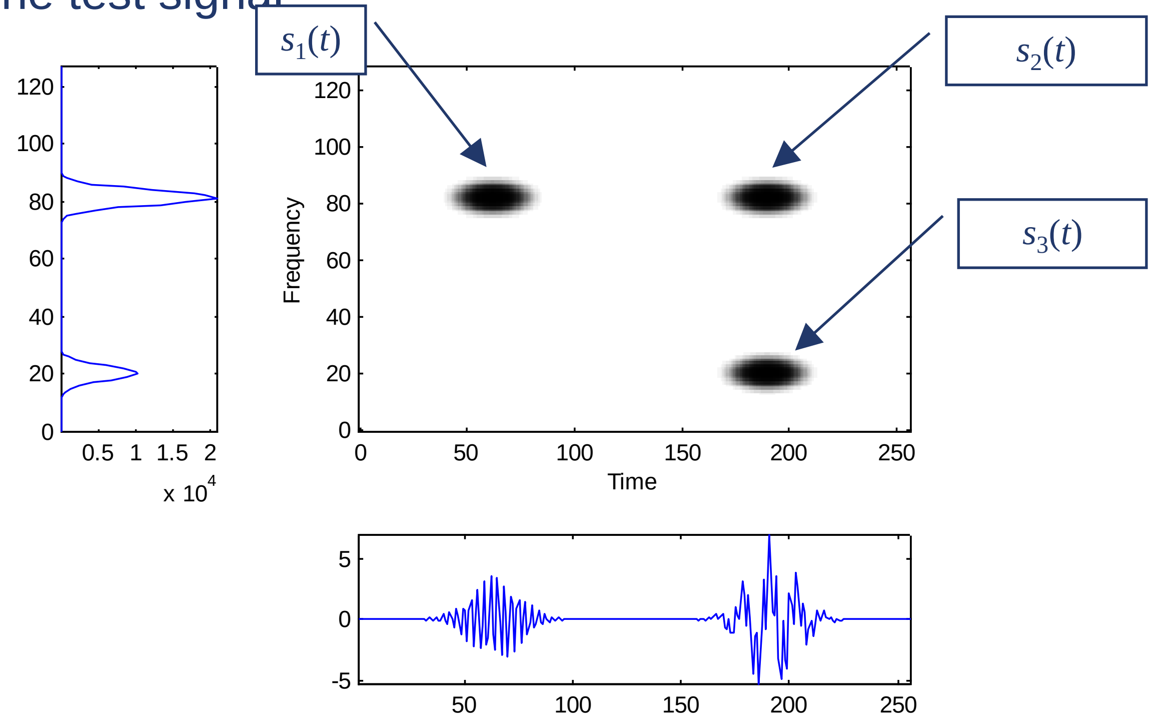
Get a large cross−term $4\pi\delta(\omega - \omega_\mu)\cos(\omega_d t)$ which varies as $\cos((\omega_1 - \omega_2)t)$ at ω_d, mid−way between the auto−terms.
While the auto−terms are $+$ve, this oscillates $+$ve & $\rightleftarrows$ $-$ve
Average of the cross−term is zero:

$$\int_{-\infty}^{\infty} 4\pi\delta(\omega - \omega_\mu)\cos(\omega_d t)dt = 0 \qquad \omega_d \neq 0$$

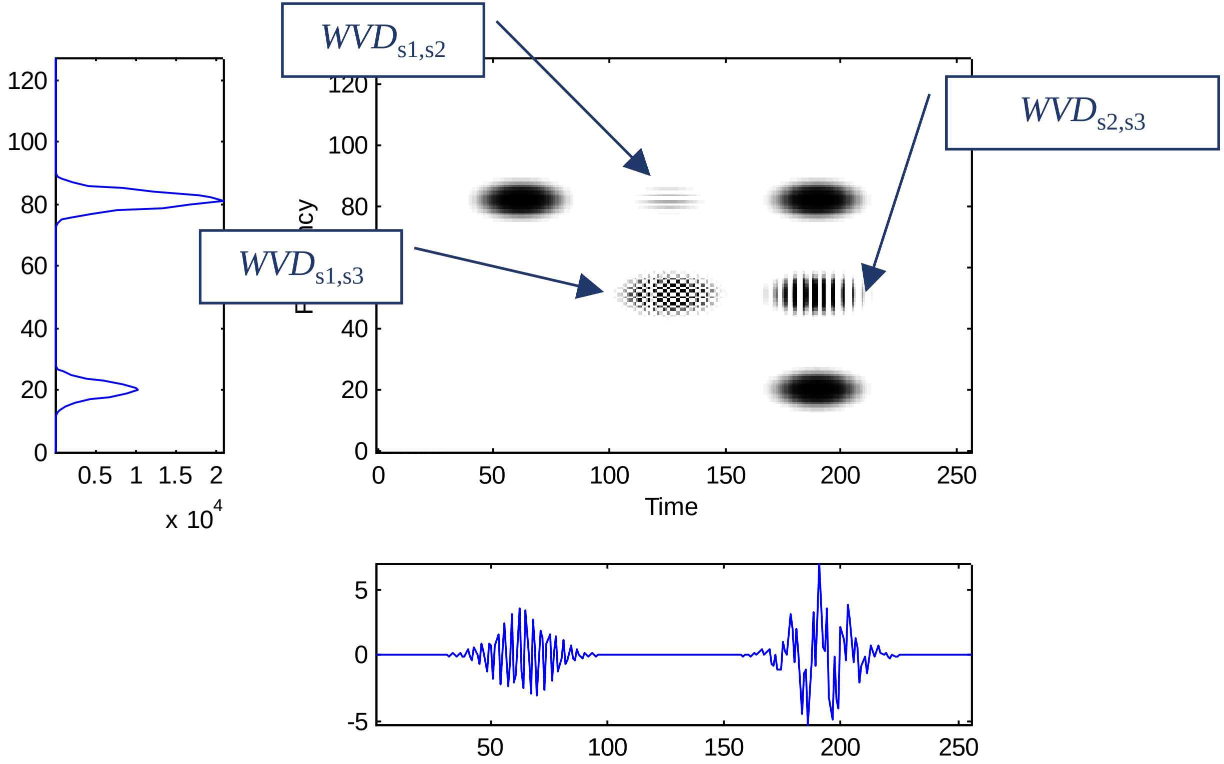
This suggests we may be able to remove these by smoothing (see later.)

Wigner-Ville Distribution

- Example: 3-tone test signal

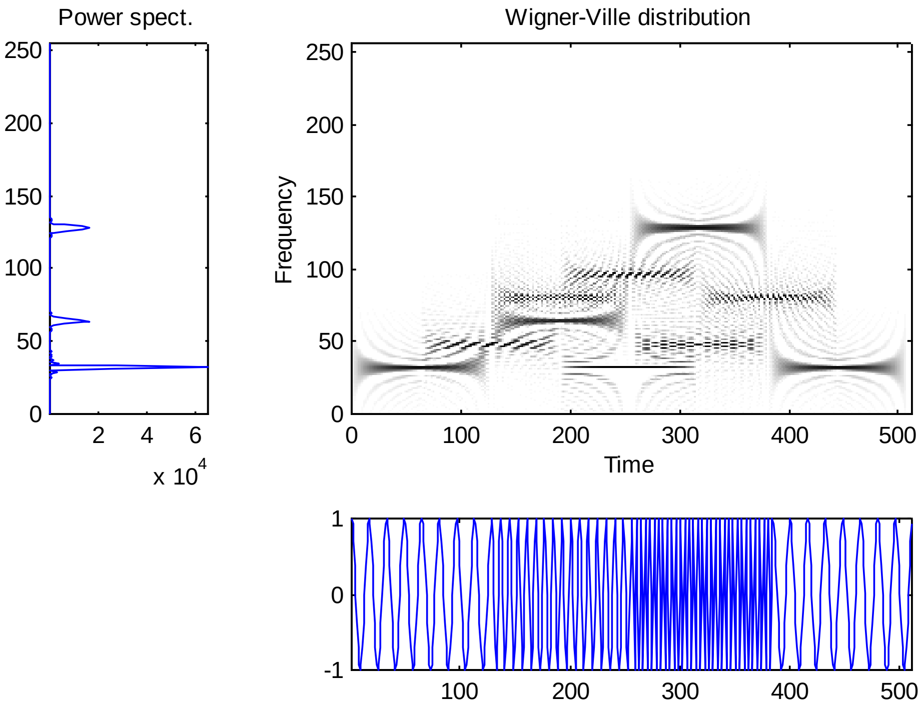


Wigner-Ville Distribution



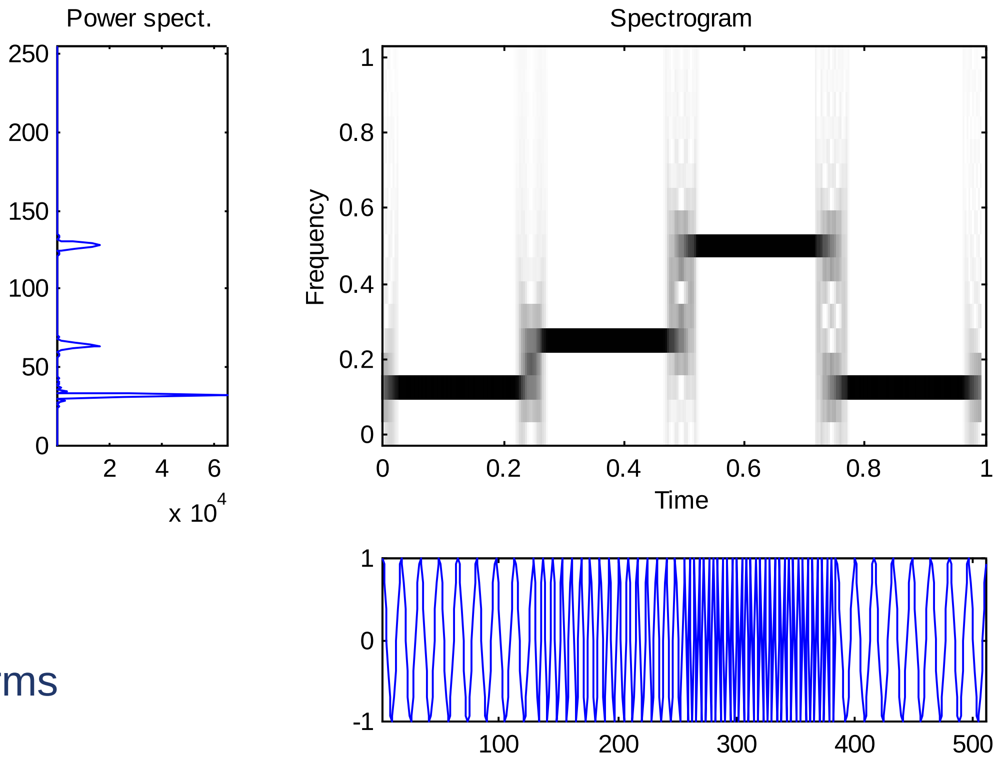
Wigner-Ville Distribution

- Another example: Frequency pulses



Wigner-Ville Distribution

- STFT spectrogram



Less resolution,
but no cross-terms

Wigner-Ville Distribution - Smoothing

- So, there is a trade-off between the beneficial properties of the WVD and the interference caused by the cross terms.

- To try to reduce the interference problem, we could use a windowed version of the WVD, called the Pseudo-WVD (PWVD):

$$PWVD_s(\mathrm{t},\, \omega) = \frac{1}{2\pi} \int s\left(t + \frac{\tau}{2}\right) s^*\left(t - \frac{\tau}{2}\right) h(\tau) e^{-j\omega\tau} \mathrm{d}\tau$$

$h(\tau)$ is the window function.

- This is equivalent to frequency smoothing (low-pass filtering) the WVD

Wigner-Ville Distribution

- Since the cross−terms WVD are usually strongly oscillating, try removing them by using 2D low−pass filtering, to give a "smoothed Wigner−Ville distribution" (SWVD):

$$SWVD_s(t,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x,y) WVD_s(t-x,\omega-y)\,dx\,dy$$

where $\varphi(x,y)$ is a 2D low−pass filter.

- Example: 2D Gaussian

$$\varphi(x,y) = e^{-\alpha t^2 - \beta\omega^2} \qquad \alpha,\beta > 0$$

- We have a trade−off:

more smoothing $\rightarrow$ less cross−terms, BUT

more smoothing $\rightarrow$ reduced resolution

i.e. not as good

Wigner-Ville Distribution

- STFT Spectrogram from WVD

  - The STFT spectrogram is a smoothed WVD,with the WVD of the analysis function $\gamma(t)$ doing the smoothing:

$$|STFT_s(t,\omega)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} WVD_\gamma(x,y)WVD_s(t-x,\omega-y)dx\,dy$$

Convolution of the WVD of s(t) and
the WVD of the STFT window function

Wigner-Ville Distribution

- Wavelet Scalogram from WVD
  - The scalogram (square of the wavelet transform) can be written in terms of the WVD:

$$SCAL_s(a,b) = |CWT_s(a,b)|^2$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} WVD_s(x,y)WVD_\psi\left(\frac{x-b}{a}, ay\right) dx\, dy$$

where $WVD_s(x,y)$ is the WVD of the signal $s(t)$ and $WVD_\psi\left(\frac{x-b}{a}, ay\right)$ is the WVD of the mother wavelet $\psi(t)$.

  - This operation is known as affine correlation.
    - Affine means "related to" and in mathematics can refer to a transformation that maps from one space to another while preserving shape.

Wigner-Ville Distribution - From WVD to …?

$$|s(t)|^2$$

Integrate over ω

$$|S(\omega)|^2$$

Integrate over t

$$WVD_s(t,\omega)$$

Affine correlation with WVD of wavelet function

$$|CWT_s(a,b)|^2$$

Convolution with WVD of STFT window function

$$|STFT_s(t,\omega)|^2$$

Both the STFT spectrogram and the WT scalogram are smoothed versions of the WVD, explaining why the WVD has the best time-frequency resolution.
It is still constrained by the Uncertainty Principle.

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Wigner-Ville Distribution - Summary

- **Kind of Decomposition**

  Time-Frequency

- **Analyzing Function**

  Uses the signal itself. Motivated by time-frequency energy density (c.f. a probability density).

- **Variable**

  Time and Frequency. Has high resolution in both time and frequency.

- **Suited for**

  Simple (i.e. not composite) signals (non-stationary), e.g. linear chirp, gaussian pulse

- **Notes**

  More “complex” (composite) signals lead to undesired “cross-terms”. Can be suppressed with smoothing, but lose high resolution in the process.

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