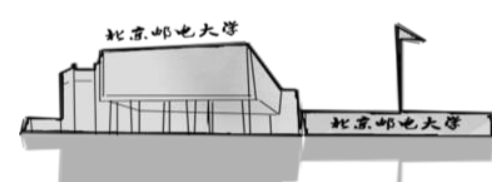
Chapter 6

Bandpass Transmission of Digital Signals

School of Information and Communication Engineering

Beijing University of Posts and Telecommunications

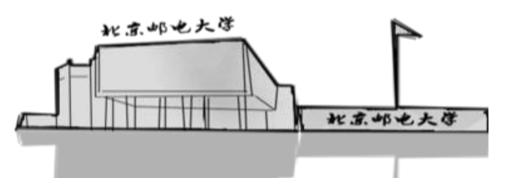


Bandpass Transmission of Digital Signals

- Introduction
- Sinusoidal carrier modulation of the binary digital signal
- Quadrature phase shift keying
- M-ary digital modulation

M-ary Digital Modulation

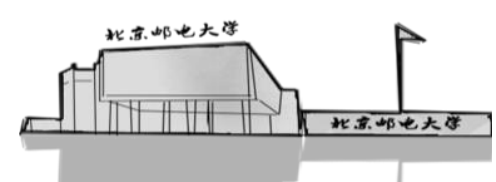
- Introduction
- Vector Representation of Digital Modulation Signals
- Statistical Decision Theory
- Optimal reception of M-ary digital modulation signals with AWGN
- MASK
- MPSK
- MQAM
- MFSK



M-ary digital modulation

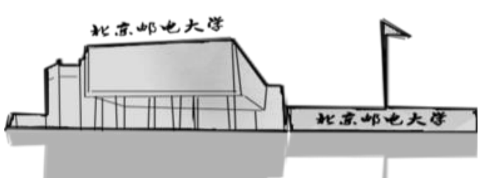
- **M-ary digital modulation (M > 2)**
  - **Improve spectrum efficiency**
  - **Ensure the reliability of transmission through increasing the average power of signal.**

- **General M-ary modulations**
  - **2PSK, QPSK, 8PSK, etc.**
  - **When M > 8, QAM has better performance, e.g. , 16QAM, 32QAM, 64QAM, etc.**



Vector Representation of Digital Modulation Signals

- **Using multi-dimension vectors to represent *M*-ary signals**
  - **can simplify the generation and demodulation of the signal.**
  - **can also make it easier to calculate the bit error rate.**
- **It is based on**
  - **orthogonal vector space theory**
  - **orthogonal signal space theory**



Orthogonal Vector Space

- **The geometric representation of signals involves expressing any signal *V* as a linear combination of N orthogonal basis functions.**

$$\mathrm{V} = \sum_{i=1}^{N} v_i \mathrm{e_i}$$

where, e_i **is a unit vector, and** $\mathrm{e_i} \cdot \mathrm{e_j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$v_i = \mathrm{V} \cdot \mathrm{e_i}$ ~ **projection of *V* to** e_i

$$\Longrightarrow \quad \mathrm{V} = [v_1, v_2, \ldots, v_N]$$

$(\mathrm{e_1}, \mathrm{e_2}, \ldots, \mathrm{e_N})$ **Orthogonal unit vectors determine the orthogonal vector space.**

Orthogonal Vector Space

- **Suppose $s(t)$ is a known real signal, with energy**

$$E_s = \int_{-\infty}^{\infty} s^2(t)dt$$

- **And we have N normalized orthogonal basis functions**

$\{f_n(t), n = 1,2,\dots,N\}$: $$\int_{-\infty}^{\infty} f_n(t) f_m(t)dt = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

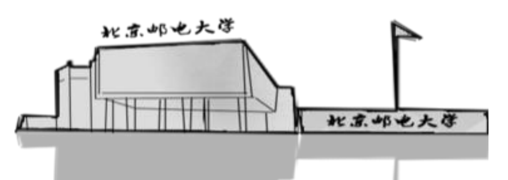
We can use the linear combination of $\{f_n(t)\}$ to approximatively denote the signal $s(t)$ as: $\hat{s}(t) = \sum_{n=1}^{N} s_n \cdot f_n(t)$, **$s_n$ ~ coefficient**

$$e(t) = s(t) - \hat{s}(t)$$

$$E_e = \int_{-\infty}^{\infty} e^2(t)dt = \int_{-\infty}^{\infty} [s(t) - \hat{s}(t)]^2 dt = \int_{-\infty}^{\infty} \left[s(t) - \sum_{k=1}^{N} s_k f_k(t) \right]^2 dt$$

$$\frac{\partial E_e}{\partial s_n} = 0 \quad \Longrightarrow \quad \int_{-\infty}^{\infty} \left[s(t) - \sum_{k=1}^{N} s_k f_k(t) \right] \cdot f_n(t)dt = 0, \quad n = 1, 2, \dots, N$$

$$s_n = \int_{-\infty}^{\infty} s(t) \cdot f_n(t)dt, \quad n = 1, 2, \dots, N$$



Orthogonal Vector Space

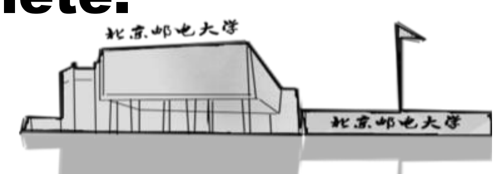
$$\therefore \quad E_{e\min} = \int_{-\infty}^{\infty} s^2(t)dt - 2\int_{-\infty}^{\infty}\left[\sum_{k=1}^{N} s_k f_k(t)\right]s(t)dt + \int_{-\infty}^{\infty}\left[\sum_{k=1}^{N} s_k f_k(t)\right]^2 dt$$

$$= E_s - 2\sum_{k=1}^{N} s_k \cdot s_k + \sum_{k=1}^{N} s_k^2 = E_s - \sum_{k=1}^{N} s_k^2$$

When E_{emin}= 0, $E_s = \sum_{k=1}^{N} s_k^2 = \int_{-\infty}^{\infty} s^2(t)dt$

then $s(t) = \sum_{k=1}^{N} s_k f_k(t)$

For any energy-limited signal, if its orthogonal expansion with an orthogonal set of basis functions $\{f_n(t)\}$ satisfies E_{emin}= 0, then $\{f_n(t)\}$ will be complete.



Orthogonal Vector Space

- **Geometric Representaition of signals**

$$\mathbf{s} = [s_1, s_2, \ldots, s_N]$$ ~ **A vector in the *N*-dimension signal space**

where $$s_n = \int_{-\infty}^{\infty} s(t) \cdot f_n(t) dt$$ ~ **projection of *s*(*t*) to *f***

- **represent *M* energy-limited signals {$s_i(t)$, $i = 1,\ldots,M$} with {$f_n(t)$, $n=1,2,\ldots,N$}**

$$s_i(t) = \sum_{k=1}^{N} s_{in} f_n(t), \quad i = 1, 2, \ldots, M$$

where $$s_{in} = \int_{-\infty}^{\infty} s_i(t) \cdot f_n(t) dt, \quad i = 1, 2, \ldots, M; \quad n = 1, \ldots, N$$

then $$\mathbf{s}_i = [s_{i1}, s_{i2}, \ldots, s_{iN}], \quad i = 1, 2, \ldots, M$$

$$E_i = \int_{-\infty}^{\infty} [s_i(t)]^2 dt = \sum_{i=1}^{N} s_{in}^2 = \left| \mathbf{s}_i \right|^2$$

Orthogonal Vector Space

- **Two parameters related to BER**
  - **Cross-correlation coefficient**

$$\rho_{mk} = \frac{1}{\sqrt{E_m E_k}} \int_{-\infty}^{\infty} s_m(t) s_k(t) dt$$

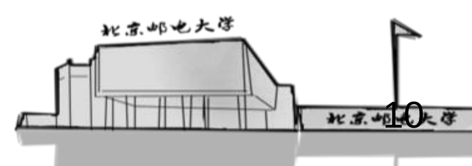
$$= \frac{\mathbf{s}_m \cdot \mathbf{s}_k}{\sqrt{E_m E_k}} = \frac{\mathbf{s}_m \cdot \mathbf{s}_k}{|\mathbf{s}_m| \cdot |\mathbf{s}_k|}$$

$E_k \sim$ energy of $s_k(t)$

$E_m \sim$ energy of $s_m(t)$

where $\quad \mathbf{s}_m \cdot \mathbf{s}_k = \sum_{n=1}^{N} s_{mn} s_{kn}$

$\rho \in$ **[-1, +1] characterizes the similarity between two signals.**



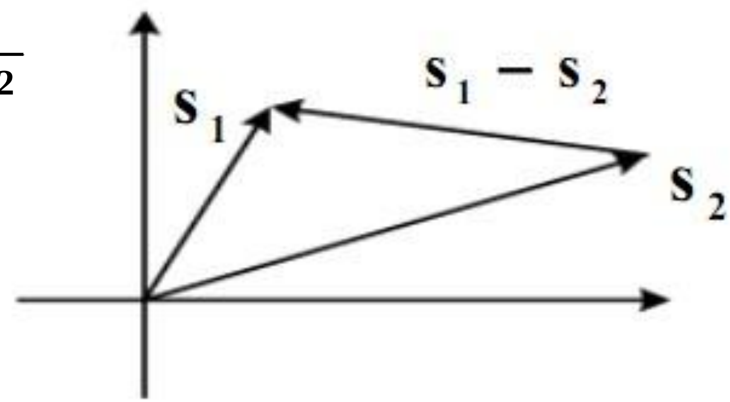
Orthogonal Vector Space

- **Euclidean distance**

$$d_{km} = \left\{ \int_{-\infty}^{\infty} \left[s_m(t) - s_k(t) \right]^2 dt \right\}^{1/2}$$

$$= \left(E_m + E_k - 2\sqrt{E_m E_k}\,\rho_{km} \right)^{1/2}$$

$$= \left| \mathbf{s}_m - \mathbf{s}_k \right| = \sqrt{\sum_n (s_{mn} - s_{kn})^2}$$



$$If \;\; E_k = E_m = E,$$

$$d_{km} = \sqrt{2E(1 - \rho_{km})}$$

~ is also used to evaluate the similarity between two signals.

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Orthogonal Vector Space

M **energy-limited signals** → **Mapping to** → ***M*** **vectors in** ***N*****-dimension signal space**

- **Signal constellation: Collection of *M* vectors in *N*-dimension signal space**
- **Square of vector length: Signal energy**
- **Distance between two vectors: Euclidean distance**
- **Square of Euclidean distance: the energy of the difference-signal between the two signals.**

Vector Representation of Digital Modulation Signals

- **OOK signal**

$$s(t)=\begin{cases} s_1(t)=\sqrt{\dfrac{2E_1}{T_b}}\cos\omega_c t \quad ,0\le t\le T_b \\ s_2(t)=0 \end{cases}$$

$$\boxed{E_1=\frac{A^2T_b}{2}\Rightarrow A=\sqrt{\frac{2E_1}{T_b}}}$$

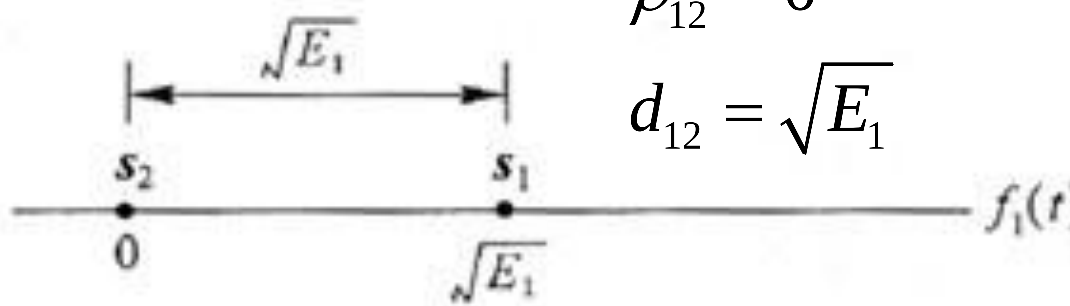
$$f_1(t)=\sqrt{\frac{2}{T_b}}\cos\omega_c t, 0\le t\le T_b \Rightarrow \begin{cases} s_1(t)=\sqrt{E_1}\,f_1(t) \\ s_2(t)=0 \end{cases}$$

$$\mathbf{s_i}=[s_{i1}], s_{i1}=\int_{-\infty}^{\infty}\mathrm{s}_i(t)\,f_1(t)\mathrm{d}t, i=1,2$$

$$\begin{cases} \mathbf{s_1}=[\sqrt{E_1}\,] \\ \mathbf{s_2}=[0] \end{cases}$$

$$\rho_{12}=0$$

$$d_{12}=\sqrt{E_1}$$



Orthogonal 2FSK signal

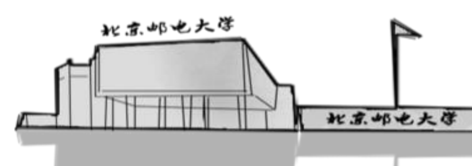
$$s(t)=\begin{cases} s_1(t)=\sqrt{2E_b/T_b}\cos\omega_1 t \\ s_2(t)=\sqrt{2E_b/T_b}\cos\omega_2 t \end{cases} ,0\le t\le T_b$$

$$f_1-f_2=k/2T_b \Rightarrow \rho_{12}=\frac{1}{E_b}\int_{-\infty}^{\infty}s_1(t)s_2(t)dt=0$$

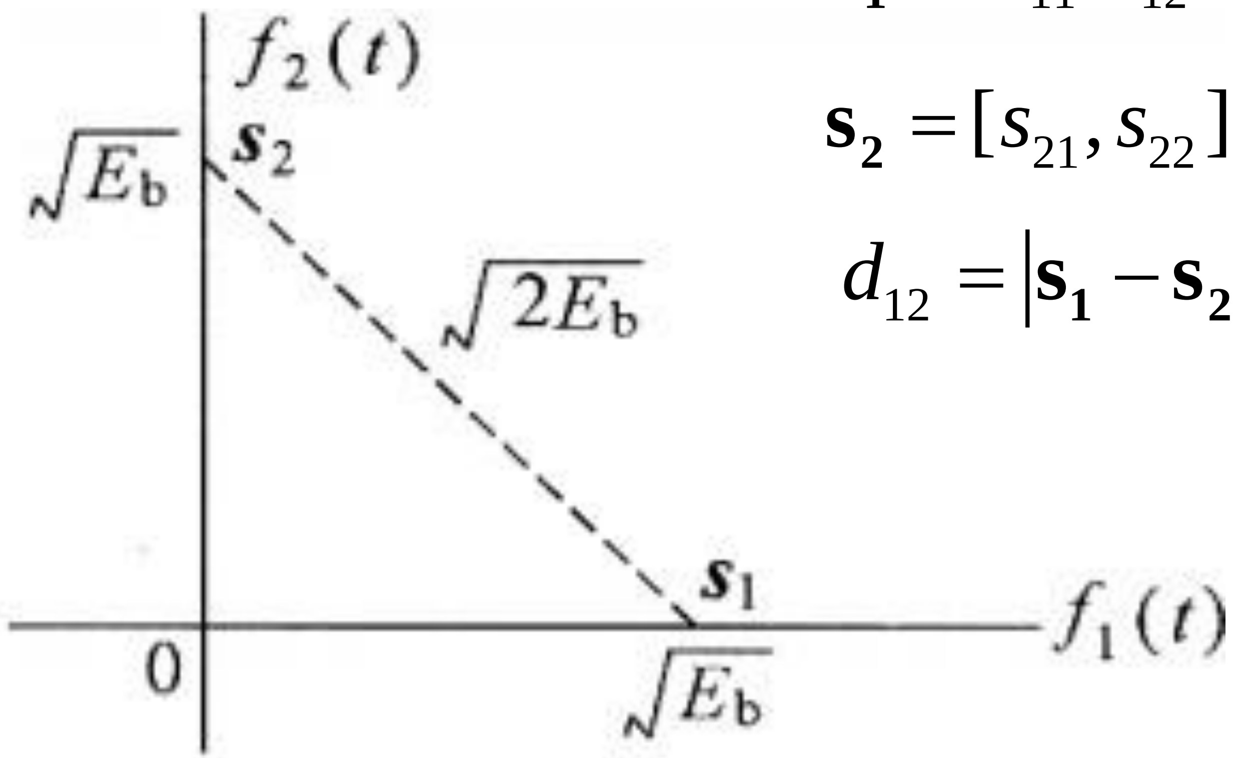
$$f_1(t)=\sqrt{2/T_b}\cos\omega_1 t,0\le t\le T_b$$

$$f_2(t)=\sqrt{2/T_b}\cos\omega_2 t,0\le t\le T_b$$

$$\Rightarrow \begin{cases} s_1(t)=\sqrt{E_b}\,f_1(t) \\ s_2(t)=\sqrt{E_b}\,f_2(t) \end{cases}$$



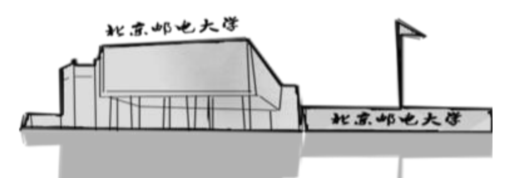
- **Orthogonal 2FSK signal**



$$\mathbf{s}_1 = [s_{11}, s_{12}] = [\sqrt{E_b}, 0]$$

$$\mathbf{s}_2 = [s_{21}, s_{22}] = [0, \sqrt{E_b}]$$

$$d_{12} = |\mathbf{s}_1 - \mathbf{s}_2| = \sqrt{2E_b}$$



Vector Representation of Digital Modulation Signals

- **BPSK signal**

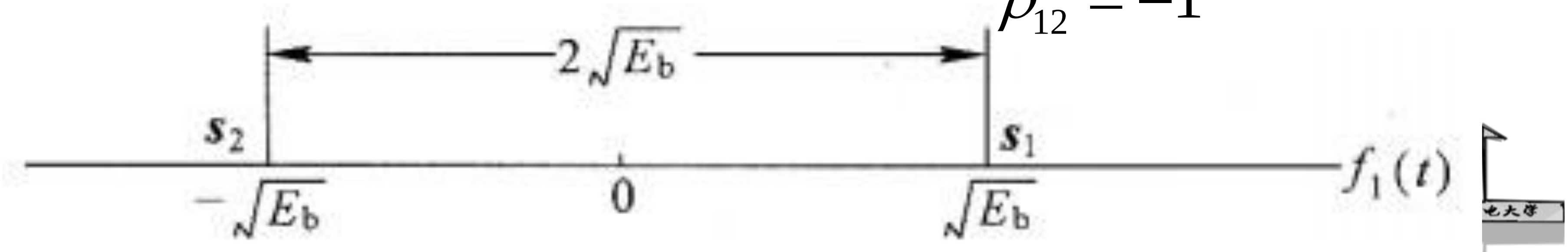
$$s(t)=\begin{cases} s_1(t)=\sqrt{2E_b/T_b}\cos\omega_c t \\ s_2(t)=-\sqrt{2E_b/T_b}\cos\omega_c t \end{cases} ,0\le t\le T_b$$

$$f_1(t)=\sqrt{2/T_b}\cos\omega_c t,0\le t\le T_b \Rightarrow \begin{cases} s_1(t)=\sqrt{E_b}\,f_1(t) \\ s_2(t)=-\sqrt{E_b}\,f_1(t) \end{cases}$$

$$\mathbf{s}_1=[\sqrt{E_b}],\quad \mathbf{s}_2=[-\sqrt{E_b}]$$

$$d_{12}=|\mathbf{s}_1-\mathbf{s}_2|=2\sqrt{E_b}$$

$$\rho_{12}=-1$$



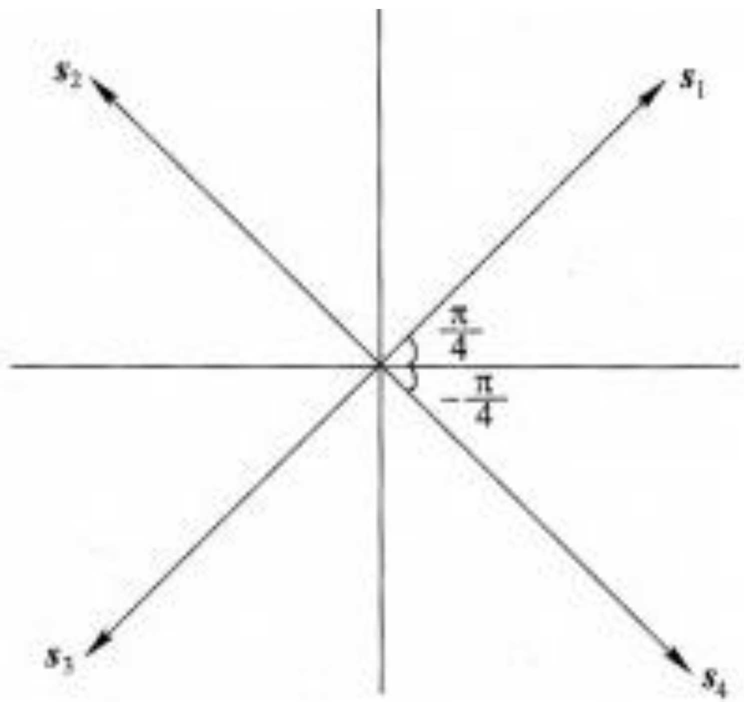
Vector Representation of Digital Modulation Signals

- **QPSK signal**

$$s_i(t) = A\cos\left(\omega_c t + (2i-1)\frac{\pi}{4}\right)$$

$$\boxed{E_s = \frac{A^2 T_s}{2}, A = \sqrt{\frac{2E_s}{T_s}}}$$

$$= \frac{A}{\sqrt{2}} I(t)\cos\omega_c t - \frac{A}{\sqrt{2}} Q(t)\sin\omega_c t, I(t), Q(t) \in \{\pm 1\}$$



$$f_1(t) = \sqrt{\frac{2}{T_s}}\cos 2\pi f_c t,\ f_2(t) = -\sqrt{\frac{2}{T_s}}\sin 2\pi f_c t,$$

$$\mathrm{s}_1 = [\sqrt{\frac{E_s}{2}}, \sqrt{\frac{E_s}{2}}], \mathrm{s}_2 = [-\sqrt{\frac{E_s}{2}}, \sqrt{\frac{E_s}{2}}],$$

$$\mathrm{s}_3 = [-\sqrt{\frac{E_s}{2}}, -\sqrt{\frac{E_s}{2}}], \mathrm{s}_4 = [\sqrt{\frac{E_s}{2}}, -\sqrt{\frac{E_s}{2}}]$$

$$d_{\min} = |\mathbf{s}_1 - \mathbf{s}_2| = \sqrt{2E_s}$$

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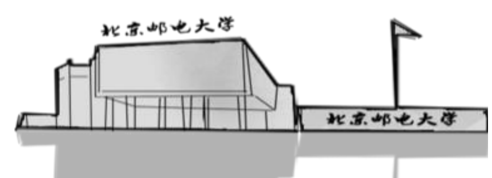


Statistical Decision Theory

- **Suppose $s_i(t)$ (i=1,2,…,M) was transmitted with a probability $P(s_i)$ (Priori Probability) and encountered AWGN $n_w(t)$. The received signal was denoted as**

$$r(t) = s_i(t) + n_w(t), i = 1, 2, \dots, M, 0 \le t \le T_s$$

- **When we make decision at the receiver, it is to decide which $s_i(t)$ has been transmitted.**

- **Statistical Decision Theory is of designing an optimal reception according to minimum average BER criterion.**

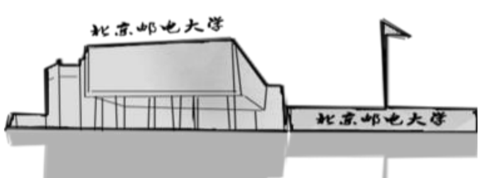
Statistical Decision Theory

- **Definitions of probabilities**
  - **A Priori probability** is the probability of transmitting each symbol, noted as $P(s_i)$.
  - **Transition probability of the channel** is a conditional probability of receiving r under the condition of transmitting s_i, noted as $P(r|s_i)$.
  - For continuous r, we also discuss the PDF $p(r|s_i)$, named **Likelihood Function.**
  - **A Posterior probability** is the probability of s_i transmitted, given that r is received, noted as $P(s_i|r)$,

    where r is named **Observation Vector**, noted as $\mathrm{r} = [r_1, \ldots, r_N]$, and r_i is the projection of the decision variable $r(T_s)$ to the ith basis function $f_i(t)$.

Statistical Decision Theory

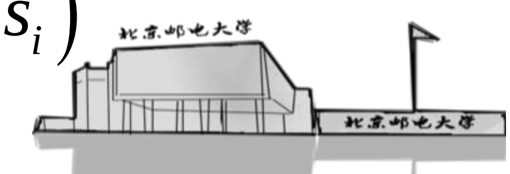
- **Decision criteria**
  - **Maximum a Posterior Probability (MAP) Criterion**
    - When selecting the transmitted signal based on the MAP criterion, we choose the symbol with the highest posterior probability from the M posterior probabilities.
    - This criterion is equivalent to using the minimum SER criterion with AWGN.
    - It is also equivalent to making a decision based on the decision area with AWGN.
  - **Maximum Likelihood (ML) Criterion**
    - According to the MAP criterion, we choose the symbol with the highest likelihood among the M likelihood probabilities as the transmitted signal.



Statistical Decision Theory

- **The optimal decision**
  - **Task 1: choose a proper decision criterion**
    - ✓ **MAP Criterion**
  - **Task 2: optimal division of decision area**
  - **Task 3: observe the observation vector *r*. If *r* is within the *i*th decision area, then choose the *i*th symbol as the transmitted signal.**
- **The optimal decision minimizes the average error decision probability P_e.**

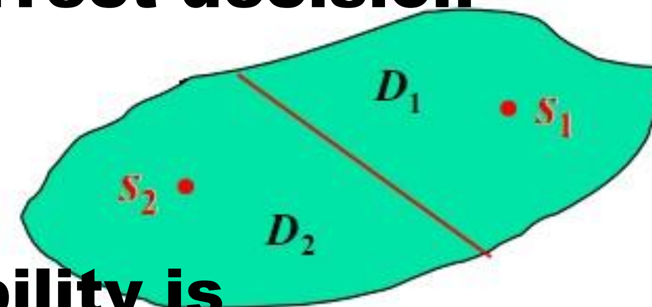
$$P_e = \sum_{i=1}^{M} P(s_i)\cdot P(\hat{s} \neq s_i | s_i) = \sum_{i=1}^{M} P(s_i)\cdot P(e | s_i)$$



Statistical Decision Theory

- **With $s_1(t)$ transmitted, the correct decision probability is**

$$P(\hat{s} = s_i \mid s_i) = \int_{D_i} p(\mathbf{r} \mid s_i)\mathrm{d}\mathbf{r}$$



- **And the error decision probability is**

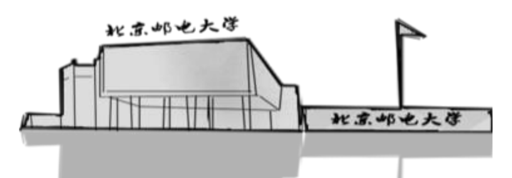
$$P(e \mid \mathbf{s_i}) = P(\hat{\mathbf{s}} \neq \mathbf{s_i} \mid \mathbf{s_i}) = 1 - \int_{D_i} p(\mathbf{r} \mid \mathbf{s_i})\mathrm{d}\mathbf{r}$$

- **By determining the bounds of decision areas, we can minimize the average error decision probability P_e and have the decision rule as**

$$\hat{s} = \arg\max_{s_i} P(s_i)\, p(\mathbf{r} \mid s_i)$$

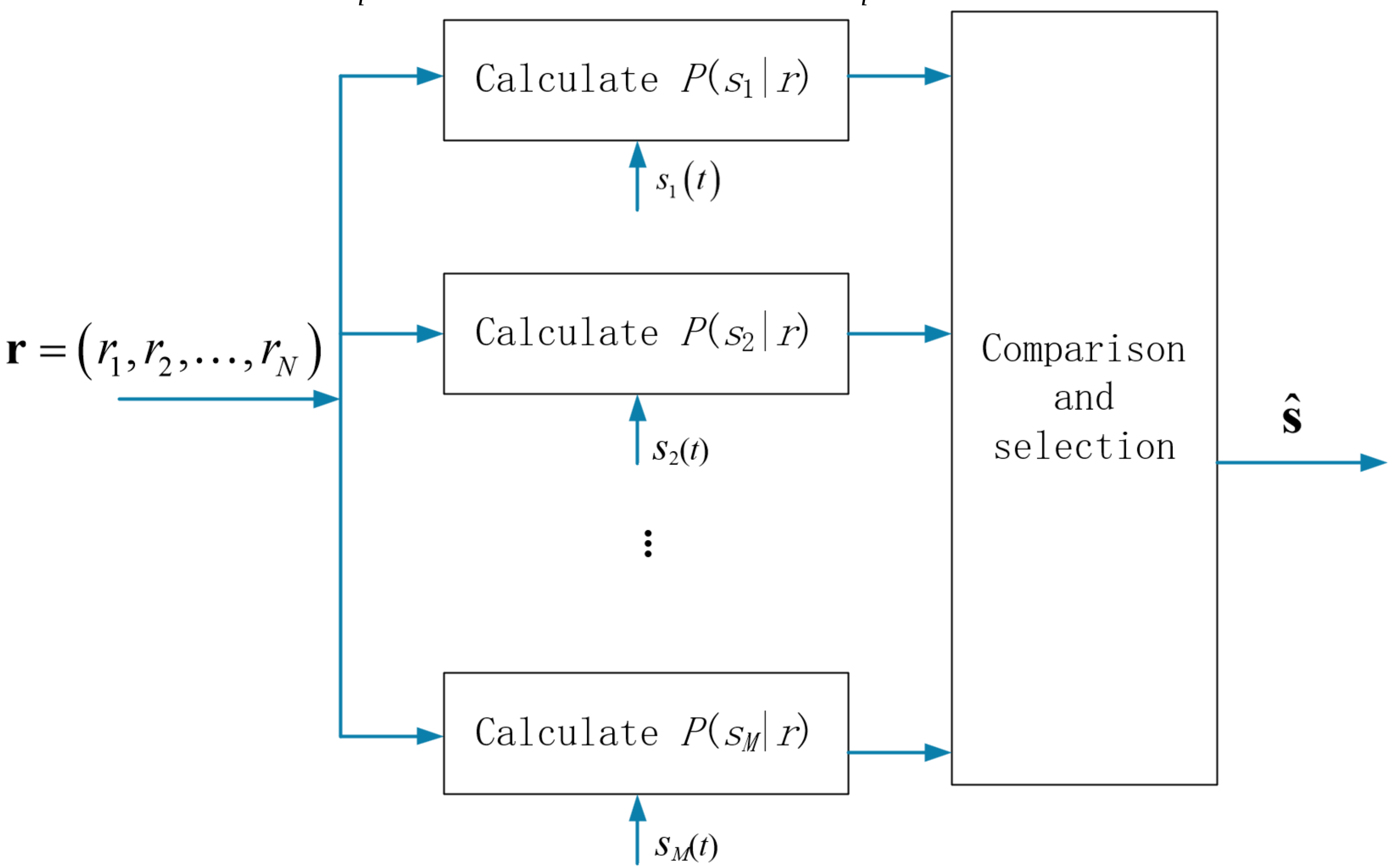
- **A posterior probability is as**

$$p(s_i \mid \mathbf{r}) = \frac{P(s_i)\, p(\mathbf{r} \mid s_i)}{p(\mathbf{r})}$$



Formulation of the MAP criterion

$$\hat{s} = \arg\max_{s_i} P(s_i \mid \mathbf{r}) = \arg\max_{s_i} P(s_i)\, p(\mathbf{r} \mid s_i)$$



Statistical Decision Theory

Formulation of the ML criterion

$$\hat{s} = \arg\max_{s_i} p(\mathbf{r} \mid s_i)$$

- **According to Bayes function**

$$p(s_i \mid \mathbf{r}) = \frac{p(\mathbf{r} \mid s_i) P(s_i)}{p(\mathbf{r})}$$

- **With equal a priori probability, the ML criterion is equivalent to the MAP criterion.**

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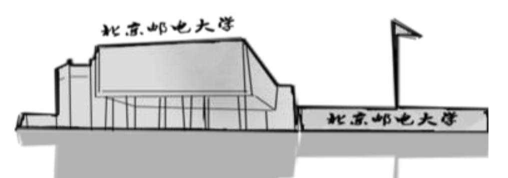


Optimal Reception of M-ary Digital Modulation Signals with AWGN

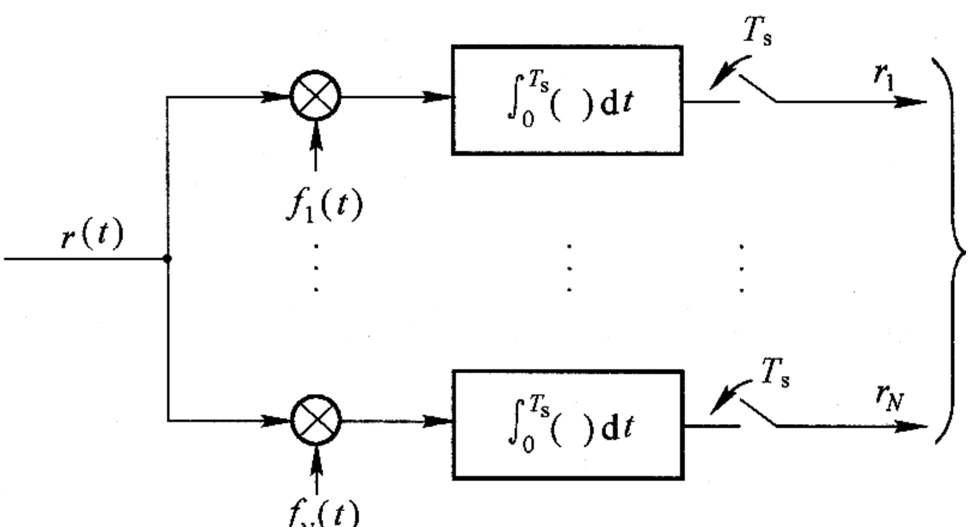
- **Vector representation of $r(t)$**

$$r(t)=s_i(t)+n_W(t),\quad i=1,\dots,M,\quad 0\le t\le Ts$$

$$\mathbf{r}=\left[r_1,r_2,\dots,r_N\right]$$

Where $r_k=s_{ik}+n_k,$

$$s_{ik}=\int_0^{T_s}s_i(t)\cdot f_k(t)dt,\quad n_k=\int_0^{T_s}n_w(t)\cdot f_k(t)dt$$



to detector

r_k is statistically independent from each other. The observation vector r is a sufficient statistic, i.e., r contains all information for decision-making.

Distribution of *r*

$$n_k = \int_0^T n_w(t) f_k(t)\mathrm{d}t \quad \sim\textbf{Gaussian}$$

$$E[n_k] = \int_0^T E[n_w(t)] f_k(t)\mathrm{d}t = 0, \quad E[r_k] = E\left\{s_{ik} + n_k\right\} = s_{ik}$$

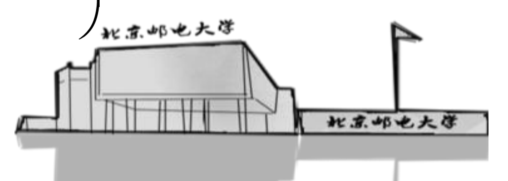
$$E[n_k n_m] = \int_0^T E[n_w(t) n_w(\tau)] f_k(t) f_m(\tau)\mathrm{d}t\mathrm{d}\tau = \int_0^T\int_0^T \frac{N_0}{2}\delta(t-\tau) f_k(t) f_m(\tau)\mathrm{d}t\mathrm{d}\tau$$

$$= \frac{N_0}{2}\int_0^T f_k(t) f_m(t)\mathrm{d}t = \frac{N_0}{2}\delta_{mk} = \begin{cases} \dfrac{N_0}{2}, & m = k \\ 0, & m \neq k \end{cases}$$

$$\boxed{\operatorname{cov}\left\{r_k, r_m\right\} = \begin{cases} 0, m \neq k \\ \dfrac{N_0}{2}, m = k \end{cases}}$$

$$\therefore\ p(r_k | s_{ik}) = \frac{1}{\sqrt{2\pi}\sigma_n}\exp\left(-\frac{(r_k - s_{ik})^2}{2\sigma_n^2}\right) = \frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{(r_k - s_{ik})^2}{N_0}\right)$$

$$p(\mathrm{r}|s_i) = p(r_1 \cdots r_N | s_i) = \prod_{k=1}^{N} p(r_k | s_i) = \frac{1}{(\pi N_0)^{N/2}}\exp\left(-\sum_{k=1}^{N}\frac{(r_k - s_{ik})^2}{N_0}\right)$$



● Making decision of transmitted signal according to *r*

$$r(t) = s_i(t) + n_w(t)$$

$$= \sum_{k=1}^{N} s_{ik} f_k(t) + \sum_{k=1}^{N} n_k f_k(t) + n'(t) = \sum_{k=1}^{N} r_k f_k(t) + n'(t)$$

$$\Rightarrow \quad n'(t) = n_w(t) - \sum_{k=1}^{N} n_k f_k(t)$$

$$\boxed{n_k = \int_0^T n_w(t) f_k(t) dt}$$

$$E[n'(t) r_k] = E[n'(t) s_{ik}] + E[n'(t) n_k] = E[n'(t) n_k]$$

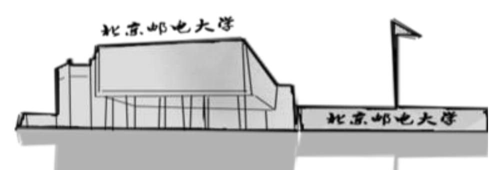
$$= E\left\{ \left[n_w(t) - \sum_{j=1}^{N} n_j f_j(t) \right] n_k \right\}$$

If *n'*(*t*) and *r*$_k$ are un-correlated, then *n'*(*t*) will not contribute to decision-making.

$$= \int_0^T E[n_w(t) n_w(\tau)] f_k(t) d\tau - \sum_{j=1}^{N} E(n_j n_k) f_j(t)$$

$$= \frac{N_0}{2} f_k(t) - \frac{N_0}{2} f_k(t) = 0$$

Proved.

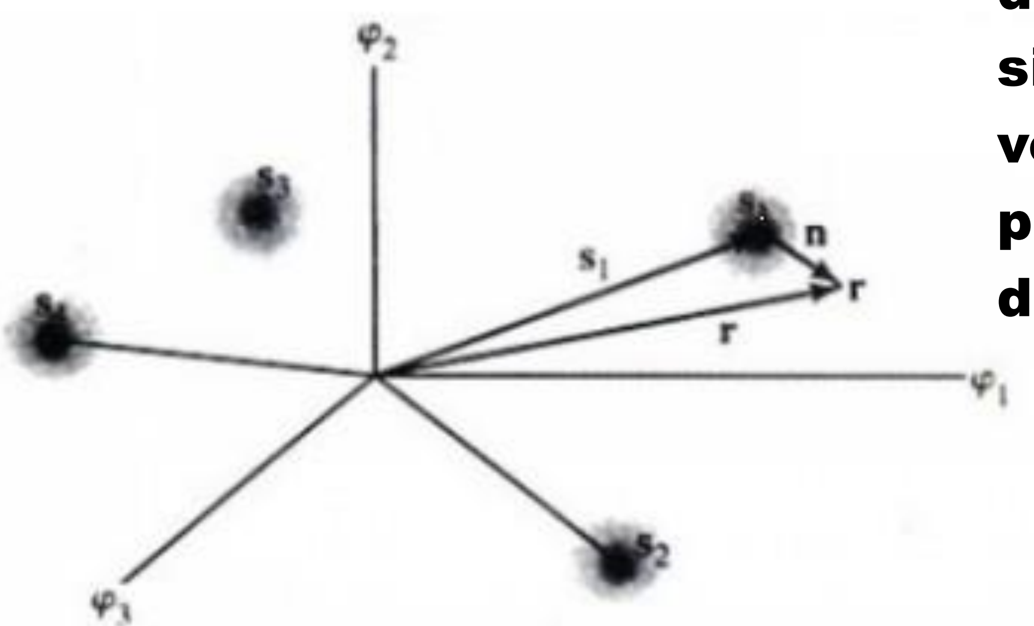


Optimal Reception of M-ary Digital Modulation Signals with AWGN

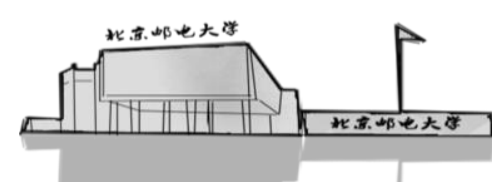
$$\mathbf{r} = \mathbf{s}_i + \mathbf{n}$$

Where: $\mathbf{r}=(r_1,r_2,\ldots r_N)$, $\mathbf{s_i} = (s_{i1}, s_{i2}, \ldots, s_{iN})$,

$$\mathbf{n} = (n_1, n_2, \ldots, n_N)$$



Signal detection aims to determine the transmitted signal based on the received vector, r, and maximize the probability of a correct decision.



Optimal Reception of M-ary Digital Modulation Signals with AWGN

$$r(t) \Rightarrow r = [r_1, r_2, \ldots, r_N]$$

- **Based on r's statistical characteristics, we use either MAP or ML criteria to decide on transmitted $s_i(t)$ and minimize P_e.**

MAP

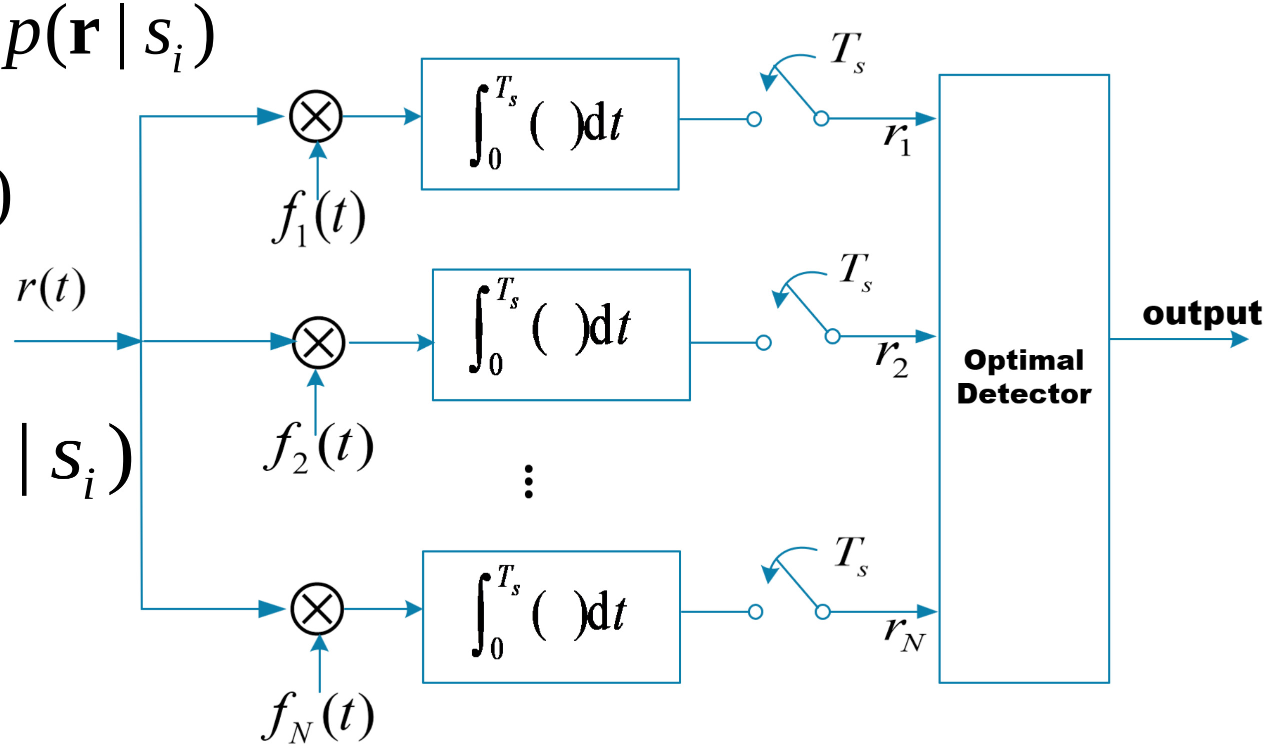
$$\hat{s} = \arg\max_{s_i} P(s_i)\, p(\mathbf{r} \mid s_i)$$

$$= \arg\max_{s_i} p(s_i \mid \mathbf{r})$$

or

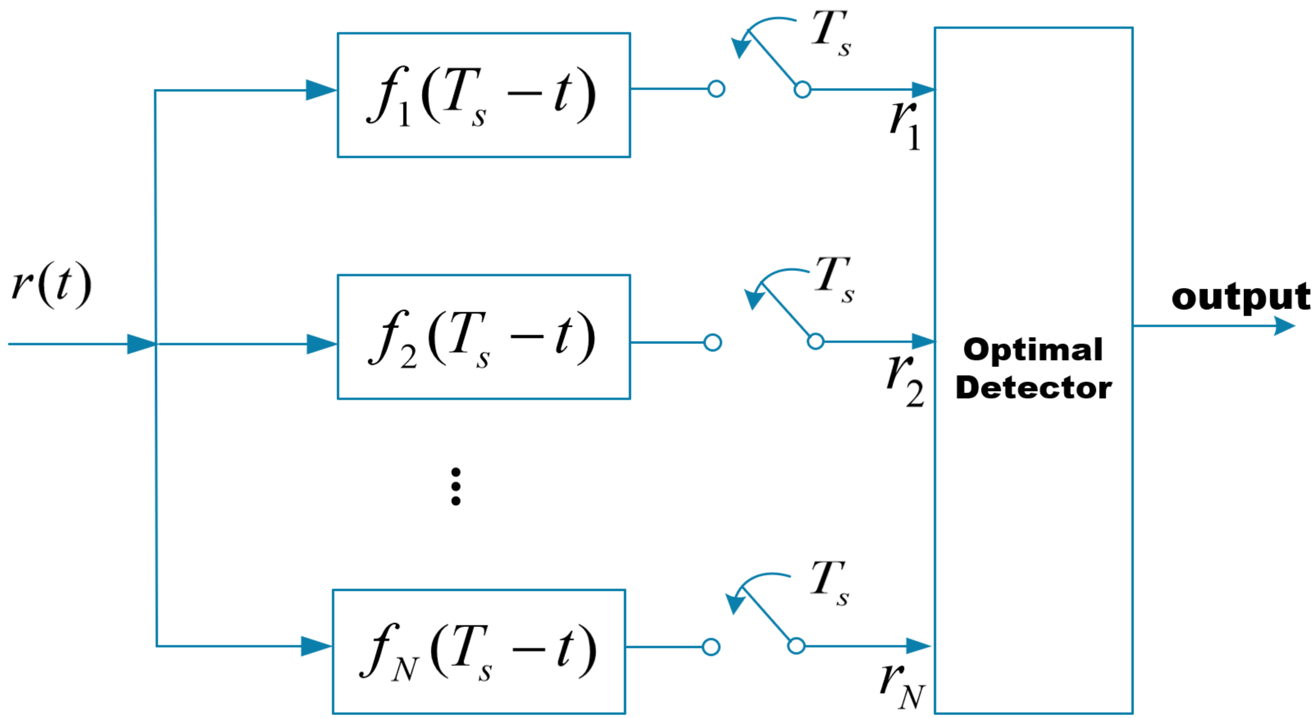
ML

$$\hat{s} = \arg\max_{s_i} p(\mathbf{r} \mid s_i)$$



Optimal Reception of M-ary Digital Modulation Signals with AWGN

● Equivalent optimal receiver



$$y_k(t) = \int_0^t r(\tau) h_k(t-\tau) d\tau = \int_0^t r(\tau) f_k(T_s - t + \tau) d\tau$$

$$\Rightarrow \quad y_k(T_s) = \int_0^{T_s} r(\tau) f_k(\tau) d\tau = r_k$$