1.2 Supposed $\boldsymbol{a} = xz^3\boldsymbol{e}_x - 2x^2yz\boldsymbol{e}_y + 2yz^4\boldsymbol{e}_z$, calculate rotation on point $M(1,-1,-1)$.

1.3 Supposed $\varphi(x,y,z) = 3x^2y - y^3z^2$, calculate $\nabla\varphi$ on point $M(1,-2,1)$.

1.4 calculate $\nabla\left(\frac{1}{r}\right)$.

1.2

设 $\boldsymbol{a} = xz^3\boldsymbol{e}_x - 2x^2yz\boldsymbol{e}_y + 2yz^4\boldsymbol{e}_z$， 求 M(1,-1,-1)点的旋度。

解 $\nabla\times\boldsymbol{a} = \left(\boldsymbol{e}_x\frac{\partial}{\partial x} + \boldsymbol{e}_y\frac{\partial}{\partial y} + \boldsymbol{e}_z\frac{\partial}{\partial z}\right)\times\left(xz^3\boldsymbol{e}_x - 2x^2yz\boldsymbol{e}_y + 2yz^4\boldsymbol{e}_z\right)$

$$= \left(\frac{\partial(2yz^4)}{\partial y} - \frac{\partial(-2x^2yz)}{\partial z}\right)\boldsymbol{e}_x + \left(\frac{\partial(xz^3)}{\partial z} - \frac{\partial(2yz^4)}{\partial x}\right)\boldsymbol{e}_y + \left(\frac{\partial(-2x^2yz)}{\partial x} - \frac{\partial(xz^3)}{\partial y}\right)\boldsymbol{e}_z$$

$$= (2z^4 + 2x^2y)\boldsymbol{e}_x + (3xz^2)\boldsymbol{e}_y + (-4xyz)\boldsymbol{e}_z$$

所以在 M(1,-1,-1)点的旋度为

$$\nabla\times\boldsymbol{a} = 3\boldsymbol{e}_y - 4\boldsymbol{e}_z$$

1.3 解：$\nabla\varphi = \frac{\partial\varphi}{\partial x}\boldsymbol{e}_x + \frac{\partial\varphi}{\partial y}\boldsymbol{e}_y + \frac{\partial\varphi}{\partial z}\boldsymbol{e}_z$

$$= \frac{\partial(3x^2y - y^3z^2)}{\partial x}\boldsymbol{e}_x + \frac{\partial(3x^2y - y^3z^2)}{\partial y}\boldsymbol{e}_y + \frac{\partial(3x^2y - y^3z^2)}{\partial z}\boldsymbol{e}_z$$

$$= 6xy\boldsymbol{e}_x + (3x^2 - 3y^2z^2)\boldsymbol{e}_y - 2y^3z\boldsymbol{e}_z$$

$$= -12\boldsymbol{e}_x - 9\boldsymbol{e}_y + 16\boldsymbol{e}_z$$

1.4 解：（1）在直角坐标系中，

$$\nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \nabla\left[(x^2+y^2+z^2)^{-1/2}\right]$$

$$= \frac{\partial(x^2+y^2+z^2)^{-1/2}}{\partial x}\boldsymbol{e}_x + \frac{\partial(x^2+y^2+z^2)^{-1/2}}{\partial y}\boldsymbol{e}_y + \frac{\partial(x^2+y^2+z^2)^{-1/2}}{\partial z}\boldsymbol{e}_z$$

$$= -x(x^2+y^2+z^2)^{-3/2}\boldsymbol{e}_x - y(x^2+y^2+z^2)^{-3/2}\boldsymbol{e}_y - z(x^2+y^2+z^2)^{-3/2}\boldsymbol{e}_z$$

$$= -\frac{x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z}{(x^2+y^2+z^2)^{3/2}} = -\frac{\boldsymbol{r}}{r^3}$$

（2）在球坐标系和圆柱坐标系中，

$$\nabla\left(\frac{1}{r}\right) = \boldsymbol{e}_r\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\boldsymbol{e}_r\frac{1}{r^2} = = -\frac{\boldsymbol{r}}{r^3}$$

1.9 There are vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, they satisfy

$$\boldsymbol{A} = \boldsymbol{e}_r z^2 \sin\phi + \boldsymbol{e}_\phi z^2 \cos\phi + \boldsymbol{e}_z 2rz\sin\phi$$
$$\boldsymbol{B} = \boldsymbol{e}_x(3y^2 - 2x) + \boldsymbol{e}_y x^2 + \boldsymbol{e}_z 2z$$

(1) which vector can be denoted as gradient of scalar function? And which vector can be denot as rotation of vector function?

(2) calculate the distribution of vector's source.

1.9 分析：一个无旋矢量场可用一标量函数的梯度来表示，一个无散矢量场可用一矢量函数的旋度来表示。若矢量的散度或旋度不为零，则分别表示了该矢量的源分布。

解：（1）在柱坐标系中 ***A*** 的旋度为

$$\nabla\times\boldsymbol{A} = \begin{vmatrix} \frac{\boldsymbol{e}_r}{r} & \boldsymbol{e}_\phi & \frac{\boldsymbol{e}_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ B_r & rB_\phi & B_z \end{vmatrix} = \begin{vmatrix} \frac{\boldsymbol{e}_r}{r} & \boldsymbol{e}_\phi & \frac{e_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ z^2\sin\phi & rz^2\cos\phi & 2rz\sin\phi \end{vmatrix}$$

$$= \boldsymbol{e}_r(2z\cos\phi - 2z\cos\phi) + \boldsymbol{e}_\phi(2z\sin\phi - 2z\sin\phi) + \boldsymbol{e}_z(\frac{z^2\cos\phi}{r} - \frac{z^2\cos\phi}{r}) = 0$$

在柱坐标中 ***A*** 的散度为

$$\nabla\cdot\boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rz^2\sin\phi) + \frac{1}{r}\frac{\partial}{\partial\phi}(z^2\cos\phi) + \frac{\partial}{\partial z}(2rz\sin\phi)$$
$$= \frac{z^2\sin\phi}{r} - \frac{z^2\sin\phi}{r} + 2r\sin\phi = 2r\sin\phi$$

可见，矢量 ***A*** 为一个有散无旋场，可以用一个标量函数的梯度表示，

其源分布为 $\nabla\cdot\boldsymbol{A} = 2r\sin\phi$

（2）在直角坐标系中 ***B*** 的散度和旋度分别为

$$\nabla\times\boldsymbol{B} = \boldsymbol{e}_x(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}) + \boldsymbol{e}_y(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}) + \boldsymbol{e}_z(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y})$$
$$= \boldsymbol{e}_x[\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z}(x^2)] + \boldsymbol{e}_y[\frac{\partial}{\partial z}(3y^2 - 2x) - \frac{\partial}{\partial x}(2z)] + \boldsymbol{e}_z[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(3y^2 - 2x)]$$
$$= 0 + 0 + \boldsymbol{e}_z(2x - 6y) = \boldsymbol{e}_z(2x - 6y)$$

$$\nabla\cdot\boldsymbol{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \frac{\partial}{\partial x}(3y^2 - 2x) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(2z) = -2 + 0 + 2 = 0$$

可见，矢量 $\boldsymbol{B}$ 为一个无散有旋矢量，可以用一个矢量的旋度表示，其源分布为

$$\nabla \times \boldsymbol{B} = \boldsymbol{e}_z (2x - 6y)$$