L06
Online Optimization and Learning: Applications

CS 295 Optimization for Machine Learning

Ioannis Panageas

Multiplicative Weights Update (recap)

Algorithm (MWUA). *We define the following algorithm:*

1. Initialize $w_i^0 = 1$ for all $i \in [n]$.
2. **For** t=1 ... T **do**
3. **Choose** action i with probability proportional to w_i^{t-1}.
4. **For** each action i **do**
5. $w_i^t = (1-\epsilon)^{c_i^t} w_i^{t-1}$.
6. **End For**
7. **End For**

Remarks:

- $\epsilon := \sqrt{\frac{\log n}{T}}$
- We choose i with probability $p_i^t = \frac{w_i^{t-1}}{\sum_j w_j^{t-1}}$.
- c_i^t is the cost of action i at time t chosen by the adversary.

MWUA general setting

Theorem (MWUA). *Let* $OPT = \min_i \sum_{t=1}^{T} c_i^t$

$$\mathbb{E}[cost_{MWUA}] \leq OPT + \epsilon T + \frac{\log n}{\epsilon}.$$

Proof. Let's define the potential function $\phi_t = \sum_i w_i^t$.

Let best action in handsight be i^* then,
we have

$$\phi_T > w_{i^*}^T = (1-\epsilon)^{OPT}.$$

$$\text{Now } \phi_{t+1} = \sum w_i^{t+1} = \sum w_i^t (1-\epsilon)^{c_i^t}$$

MWUA general setting

Theorem (MWUA). *Let* $OPT = \min_i \sum_{t=1}^{T} c_i^t$

$$\mathbb{E}[cost_{MWUA}] \leq OPT + \epsilon T + \frac{\log n}{\epsilon}.$$

Proof. Let's define the potential function $\phi_t = \sum_i w_i^t$.

Let best action in handsight be i^* then,
we have

$$\phi_T > w_{i^*}^T = (1-\epsilon)^{OPT}.$$

$$\text{Now } \phi_{t+1} = \sum w_i^{t+1} = \sum w_i^t (1-\epsilon)^{c_i^t}$$

$$= \sum \phi_t p_i^{t+1} (1-\epsilon)^{c_i^t}$$

MWUA general setting

Theorem (MWUA). *Let* $OPT = \min_i \sum_{t=1}^{T} c_i^t$

$$\mathbb{E}[cost_{MWUA}] \leq OPT + \epsilon T + \frac{\log n}{\epsilon}.$$

Proof. Let's define the potential function $\phi_t = \sum_i w_i^t$.

Let best action in handsight be i^* then,
we have

$$\phi_T > w_{i^*}^T = (1-\epsilon)^{OPT}.$$

$$\text{Now } \phi_{t+1} = \sum w_i^{t+1} = \sum w_i^t (1-\epsilon)^{c_i^t}$$

$$= \sum \phi_t p_i^{t+1} (1-\epsilon)^{c_i^t}$$

$$= \phi_t \sum p_i^{t+1} (1-\epsilon)^{c_i^t}$$

MWUA general setting

Proof cont. Therefore

$$\phi_{t+1} = \phi_t \sum p_i^{t+1} (1-\epsilon)^{c_i^t}$$

MWUA general setting

Proof cont. Therefore

$$\phi_{t+1} = \phi_t \sum p_i^{t+1} (1-\epsilon)^{c_i^t}$$

$$\leq \phi_t \sum p_i^{t+1} (1 - \epsilon \cdot c_i^t)$$

Note $(1-\epsilon)^x \leq 1 - \epsilon x$ for $x \in [0,1], \epsilon \in [0, 1/2]$.

MWUA general setting

Proof cont. Therefore

$$\phi_{t+1} = \phi_t \sum p_i^{t+1} (1-\epsilon)^{c_i^t}$$

$$\leq \phi_t \sum p_i^{t+1} (1 - \epsilon \cdot c_i^t)$$

$$= \phi_t (1 - \epsilon \cdot \mathbb{E}[\mathrm{cost(t)}_{\mathrm{MWUA}}])$$

Note $(1-\epsilon)^x \leq 1 - \epsilon x$ for $x \in [0,1], \epsilon \in [0, 1/2]$.

MWUA general setting

Proof cont. Therefore

$$\begin{aligned}\phi_{t+1} &= \phi_t \sum p_i^{t+1}(1-\epsilon)^{c_i^t} \\ &\leq \phi_t \sum p_i^{t+1}(1-\epsilon \cdot c_i^t) \\ &= \phi_t(1-\epsilon \cdot \mathbb{E}[\mathrm{cost(t)}_{\mathrm{MWUA}}]) \\ &\leq \phi_t e^{-\epsilon \mathbb{E}[\mathrm{cost(t)}_{\mathrm{MWUA}}]}\end{aligned}$$

Note $(1-\epsilon)^x \leq 1-\epsilon x$ for $x \in [0,1], \epsilon \in [0,1/2]$.

MWUA general setting

Proof cont. Therefore

$$\phi_{t+1} = \phi_t \sum p_i^{t+1}(1-\epsilon)^{c_i^t}$$

$$\leq \phi_t \sum p_i^{t+1}(1-\epsilon \cdot c_i^t)$$

$$= \phi_t(1-\epsilon \cdot \mathbb{E}[\mathrm{cost(t)}_{\mathrm{MWUA}}])$$

$$\leq \phi_t e^{-\epsilon \mathbb{E}[\mathrm{cost(t)}_{\mathrm{MWUA}}]}$$

Telescopic product gives

$$\phi_T \leq \phi_1 e^{-\epsilon \mathbb{E}[\mathrm{cost}_{\mathrm{MWUA}}]}.$$

Therefore $(1-\epsilon)^{OPT} \leq e^{-\epsilon \mathbb{E}[\mathrm{cost}_{\mathrm{MWUA}}]} n$, or $OPT(-\epsilon-\epsilon^2) \leq \log n - \epsilon \mathbb{E}[\mathrm{cost}_{\mathrm{MWUA}}]$.

MWUA general setting

Proof cont. Therefore

Plugging in $\epsilon = \sqrt{\frac{\log n}{T}}$, gives $\frac{1}{T}(\mathbb{E}[\text{cost}_{\text{MWUA}}] - OPT) \leq 2\sqrt{\frac{\log n}{T}}$!

$$\leq \phi_t e^{-\epsilon \mathbb{E}[\text{cost(t)}_{\text{MWUA}}]}$$

Telescopic product gives

$$\phi_T \leq \phi_1 e^{-\epsilon \mathbb{E}[\text{cost}_{\text{MWUA}}]}.$$

Therefore $(1-\epsilon)^{OPT} \leq e^{-\epsilon \mathbb{E}[\text{cost}_{\text{MWUA}}]} n$, or $OPT(-\epsilon - \epsilon^2) \leq \log n - \epsilon \mathbb{E}[\text{cost}_{\text{MWUA}}]$.

Solving Linear Programs

Problem (Linear Program). *Suppose we are given a linear program in the standard form*

$$Ax \geq b$$
$$s.t\ x \geq 0.$$

Goal (Check feasibility). *Compute a vector $x^* \geq 0$ such that for some $\epsilon > 0$ we get*

$$\alpha_i^\top x^* \geq b_i - \epsilon, \textit{ for all } i.$$

Oracle access: Given a vector c and scalar d, does there exist a $x \geq 0$ such that $c^T x \geq d$.

Remark: Using the above and binary search, you can solve any linear program!

Solving Linear Programs

Problem (Linear Program). *Suppose we are given a linear program in the standard form*

$$Ax \geq b$$
$$s.t \ x \geq 0.$$

Goal (Check feasibility). *Compute a vector $x^* \geq 0$ such that for some $\epsilon > 0$ we get*

$$\alpha_i^\top x^* \geq b_i - \epsilon, \textit{ for all } i.$$

Oracle access: Given a vector c and scalar d, does there exist a $x \geq 0$ such that $c^T x \geq d$.

Remark: Using the above and binary search, you can solve any linear program!

Use MWUA, what are the actions/costs?

Solving Linear Programs

Setting. *Consider every constraint* $a_i^\top x - b_i$ *as an action.*

- *Choose* $c_i(x) = \frac{a_i^\top x - b_i}{\rho}$ *with* ρ *chosen so that* $|c_i| \leq 1$.
- *Initiliazation* $w_i^0 = 1$ *(uniform distribution).*
- *For each* $t = 1, ..., T$, *ask oracle if there exists a point* $x \geq 0$ *such that* $c^\top x \geq d$ *where*
$$c = \sum p_i^t a_i, \; d = \sum p_i^t b_i.$$

Solving Linear Programs

Setting. *Consider every constraint* $a_i^\top x - b_i$ *as an action.*

- *Choose* $c_i(x) = \frac{a_i^\top x - b_i}{\rho}$ *with* ρ *chosen so that* $|c_i| \leq 1$.
- *Initiliazation* $w_i^0 = 1$ *(uniform distribution).*
- *For each* $t = 1, ..., T$, *ask oracle if there exists a point* $x \geq 0$ *such that* $c^\top x \geq d$ *where*

$$c = \sum p_i^t a_i, \; d = \sum p_i^t b_i.$$

If the answer is no, linear problem infeasible!

Solving Linear Programs

Setting. *Consider every constraint* $a_i^\top x - b_i$ *as an action.*

- *Choose* $c_i(x) = \frac{a_i^\top x - b_i}{\rho}$ *with* ρ *chosen so that* $|c_i| \leq 1$.
- *Initiliazation* $w_i^0 = 1$ *(uniform distribution).*
- *For each* $t = 1, ..., T$, *ask oracle if there exists a point* $x \geq 0$ *such that* $c^\top x \geq d$ *where*

$$c = \sum p_i^t a_i, \; d = \sum p_i^t b_i.$$

If the answer is no, linear problem infeasible!

If the answer is yes (returns a x^t), each action suffers cost $c_i^t = c_i(x^t)$.

Solving Linear Programs

From our theorem we get that

$$0 \leq \sum_t \sum_i p_i^t (a_i^\top x_i^t - b_i) \leq \sum_t \sum_i p_i^* (a_i^\top x_i^t - b_i) + 2\rho \sqrt{\frac{\log m}{T}}.$$

where p^* is the optimal handsight. But the RHS is at most (for all i)

$$\sum_t a_i^\top x_i^t - b_i + 2\rho \sqrt{\frac{\log m}{T}} = a_i^\top \sum_t x_i^t - T b_i + 2\rho \sqrt{\frac{\log m}{T}}.$$

Solving Linear Programs

From our theorem we get that

$$0 \leq \sum_t \sum_i p_i^t (a_i^\top x_i^t - b_i) \leq \sum_t \sum_i p_i^* (a_i^\top x_i^t - b_i) + 2\rho \sqrt{\frac{\log m}{T}}.$$

where p^* is the optimal handsight. But the RHS is at most (for all i)

$$\sum_t a_i^\top x_i^t - b_i + 2\rho \sqrt{\frac{\log m}{T}} = a_i^\top \sum_t x_i^t - T b_i + 2\rho \sqrt{\frac{\log m}{T}}.$$

Therefore, by choosing $T = \frac{4\rho^2 \log m}{\epsilon^2}$, $\tilde{x} = \frac{1}{T} \sum_t x^t$ we get that

$$a_i^\top \tilde{x} - b_i + \epsilon \geq 0 \text{ for all } i.$$

MWUA and Zero-sum games

Definition. *Consider a matrix A (called payoff). A_{ij} denotes the amount of money player x pays to player y. Example (Rock-Paper-Scissors):*

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

MWUA and Zero-sum games

Definition. *Consider a matrix A (called payoff). A_{ij} denotes the amount of money player x pays to player y. Example (Rock-Paper-Scissors):*

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Definition (Nash Equilibrium). *A vector (x^*, y^*) is called a NE if*

$$x^{*\top} A y^* \geq x^{*\top} A \tilde{y} \text{ for all } \tilde{y} \in \Delta \text{ and } x^{*\top} A y^* \leq \tilde{x}^\top A y^* \text{ for all } \tilde{x} \in \Delta.$$

MWUA and Zero-sum games

Definition. *Consider a matrix A (called* payoff*). A_{ij} denotes the amount of money player x pays to player y. Example (Rock-Paper-Scissors):*

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Definition (Nash Equilibrium). *A vector (x^*, y^*) is called a NE if*

$$x^{*\top} A y^* \geq x^{*\top} A \tilde{y} \text{ for all } \tilde{y} \in \Delta \text{ and } x^{*\top} A y^* \leq \tilde{x}^\top A y^* \text{ for all } \tilde{x} \in \Delta.$$

How to compute NE? Let them run MWUA!

MWUA and Zero-sum games

Algorithm (MWUA). *We define the following algorithm for zero sum games:*

1. Initialize $p^0_{i,x} = 1/n$, $p^0_{i,y} = 1/n$ for all i (both players, uniform).
2. **For** t=1 … T **do**
3. Player x chooses i with probability $p^t_{i,x}$ and y with $p^t_{i,y}$ respectively.
4. **For** each action i **do**
5. $p^t_{i,x} = p^{t-1}_{i,x} \frac{(1-\epsilon)^{(Ap^{t-1}_y)_i}}{Z_x}$.
6. $p^t_{i,y} = p^{t-1}_{i,y} \frac{(1+\epsilon)^{(A^\top p^{t-1}_x)_i}}{Z_y}$.
7. **End For**
8. **End For**

Remarks:

- $\epsilon := \sqrt{\frac{\log n}{T}}$
- $c^t_i := \left(Ap^{t-1}_y\right)_i$ is the (expected cost) of action i at time t for player x.
- For player y is the expected utility…

MWUA and Zero-sum games

Theorem (MWUA). *Let* $\tilde{x} = \frac{1}{T}\sum_t p_x^t$ *and* $\tilde{y} = \frac{1}{T}\sum_t p_y^t$. *Assume that* A *has entries in* $[-1,1]$ *and* $T = \Theta\left(\frac{\log n}{\epsilon^2}\right)$. *It holds* $(\tilde{x},\tilde{y})$ *is an* ϵ*-approximate NE that is*

$$\tilde{x}^\top A\tilde{y} \leq x'^\top A\tilde{y} + \epsilon \text{ and } \tilde{x}^\top A\tilde{y} \geq x^\top Ay' - \epsilon.$$

MWUA and Zero-sum games

Theorem (MWUA). *Let* $\tilde{x} = \frac{1}{T}\sum_t p_x^t$ *and* $\tilde{y} = \frac{1}{T}\sum_t p_y^t$. *Assume that* A *has entries in* $[-1,1]$ *and* $T = \Theta\left(\frac{\log n}{\epsilon^2}\right)$. *It holds* $(\tilde{x},\tilde{y})$ *is an* ϵ*-approximate NE that is*

$$\tilde{x}^\top A\tilde{y} \leq x'^\top A\tilde{y} + \epsilon \text{ and } \tilde{x}^\top A\tilde{y} \geq x^\top Ay' - \epsilon.$$

Proof. **Exercise 6!**

MWUA and Zero-sum games

Theorem (MWUA). *Let $\tilde{x} = \frac{1}{T}\sum_t p_x^t$ and $\tilde{y} = \frac{1}{T}\sum_t p_y^t$. Assume that A has entries in $[-1,1]$ and $T = \Theta\left(\frac{\log n}{\epsilon^2}\right)$. It holds $(\tilde{x},\tilde{y})$ is an ϵ-approximate NE that is*

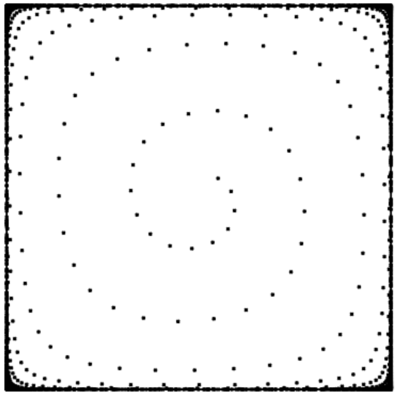
$$\tilde{x}^\top A\tilde{y} \leq x'^\top A\tilde{y} + \epsilon \text{ and } \tilde{x}^\top A\tilde{y} \geq x^\top Ay' - \epsilon.$$

Proof. **Exercise 6!**

Remark: The result above is not true for last iterate p_x^T, p_y^T.

Definition. *Matching Pennies:*

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow$$



General Family of no-regret Algorithms

Definition (Follow the Leader). *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be convex functions for all k, differentiable in some convex set $\mathcal{K}$. FTL is defined:*

Initialize at some x_0.
For t:=1 to T do

1. Choose $x_t = \text{argmin}_{x \in \mathcal{K}} \sum_{k=0}^{t-1} f_k(x)$.

Remark: The above can perform really poorly! Why?

General Family of no-regret Algorithms

Definition (Follow the Leader). *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be convex functions for all k, differentiable in some convex set $\mathcal{K}$. FTL is defined:*

Initialize at some x_0.
For t:=1 to T do

1. Choose $x_t = \text{argmin}_{x \in \mathcal{K}} \sum_{k=0}^{t-1} f_k(x)$.

Remark: The above can perform really poorly! Why?

Consider $n = 2$, $\mathcal{K} = \Delta_2$, $x_0 = (1/2, 1/2)$ and $f_k(x) = x^\top \ell_k$.

- $\ell_0 = (0, 1/2)$

General Family of no-regret Algorithms

Definition (Follow the Leader). *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be convex functions for all k, differentiable in some convex set $\mathcal{K}$. FTL is defined:*

Initialize at some x_0.
For t:=1 to T do

1. Choose $x_t = \operatorname{argmin}_{x \in \mathcal{K}} \sum_{k=0}^{t-1} f_k(x)$.

Remark: The above can perform really poorly! Why?

Consider $n = 2$, $\mathcal{K} = \Delta_2$, $x_0 = (1/2, 1/2)$ and $f_k(x) = x^\top \ell_k$.

- $\ell_0 = (0, 1/2)$
- Thus $x_1 = (1, 0)$

General Family of no-regret Algorithms

Definition (Follow the Leader). *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be convex functions for all k, differentiable in some convex set $\mathcal{K}$. FTL is defined:*

> Initialize at some x_0.
> For t:=1 to T do
>
> 1. Choose $x_t = \text{argmin}_{x \in \mathcal{K}} \sum_{k=0}^{t-1} f_k(x)$.

Remark: The above can perform really poorly! Why?

Consider $n = 2$, $\mathcal{K} = \Delta_2$, $x_0 = (1/2, 1/2)$ and $f_k(x) = x^\top \ell_k$.

- $\ell_0 = (0, 1/2)$
- $\ell_1 = (1, 0)$
- Thus $x_1 = (1, 0)$

General Family of no-regret Algorithms

Definition (Follow the Leader). *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be convex functions for all k, differentiable in some convex set $\mathcal{K}$. FTL is defined:*

Initialize at some x_0.
For t:=1 to T do

1. Choose $x_t = \operatorname{argmin}_{x \in \mathcal{K}} \sum_{k=0}^{t-1} f_k(x)$.

Remark: The above can perform really poorly! Why?

Consider $n = 2$, $\mathcal{K} = \Delta_2$, $x_0 = (1/2, 1/2)$ and $f_k(x) = x^\top \ell_k$.

- $\ell_0 = (0, 1/2)$
- $\ell_1 = (1, 0)$

- Thus $x_1 = (1, 0)$
- Thus $x_2 = (0, 1)$

General Family of no-regret Algorithms

Definition (Follow the Leader)**.** *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be convex functions for all k, differentiable in some convex set $\mathcal{K}$. FTL is defined:*

Initialize at some x_0.
For t:=1 to T do

1. Choose $x_t = \mathrm{argmin}_{x \in \mathcal{K}} \sum_{k=0}^{t-1} f_k(x)$.

Remark: The above can perform really poorly! Why?

Consider $n = 2$, $\mathcal{K} = \Delta_2$, $x_0 = (1/2, 1/2)$ and $f_k(x) = x^\top \ell_k$.

- $\ell_0 = (0, 1/2)$
- $\ell_1 = (1, 0)$

- Thus $x_1 = (1, 0)$
- Thus $x_2 = (0, 1)$

Regret T/2 hence average Regret not vanishing!

General Family of no-regret Algorithms

Definition (Follow the Regularized Leader). *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be convex for all k, differentiable in some convex set $\mathcal{K}$. Moreover, let R be a strongly convex function. FTRL is defined:*

> Initialize at some x_0.
> For t:=1 to T do
>
> 1. Choose $x_t = \text{argmin}_{x \in \mathcal{K}} \{\epsilon_{t-1} \cdot \sum_{k=0}^{t-1} f_k(x) + R(x)\}$.

What happens when $R(x) = \frac{1}{2}\|x\|^2$ and $f_k(x) = x^T c_k$ (linear in x)?

General Family of no-regret Algorithms

Definition (Follow the Regularized Leader). *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be convex for all k, differentiable in some convex set $\mathcal{K}$. Moreover, let R be a strongly convex function. FTRL is defined:*

> Initialize at some x_0.
> For t:=1 to T do
>
> 1. Choose $x_t = \operatorname{argmin}_{x \in \mathcal{K}} \{\epsilon_{t-1} \cdot \sum_{k=0}^{t-1} f_k(x) + R(x)\}$.

What happens when $R(x) = \frac{1}{2}\|x\|^2$ and $f_k(x)$

Online GD!

General Family of no-regret Algorithms

Definition (Follow the Regularized Leader). *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be convex for all k, differentiable in some convex set $\mathcal{K}$. Moreover, let R be a strongly convex function. FTRL is defined:*

Initialize at some x_0.
For t:=1 to T do

1. Choose $x_t = \text{argmin}_{x \in \mathcal{K}} \{\epsilon_{t-1} \cdot \sum_{k=0}^{t-1} f_k(x) + R(x)\}$.

What happens when $R(x) = \frac{1}{2}\|x\|^2$ and $f_k(x)$ **Online GD!**

What happens when $R(x) = -\sum x_i \log x_i$ (entropy) and $f_k(x) = x^T c_k$ (linear in x)?

General Family of no-regret Algorithms

Definition (Follow the Regularized Leader). *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be convex for all k, differentiable in some convex set $\mathcal{K}$. Moreover, let R be a strongly convex function. FTRL is defined:*

Initialize at some x_0.
For t:=1 to T do

1. Choose $x_t = \operatorname{argmin}_{x \in \mathcal{K}} \{\epsilon_{t-1} \cdot \sum_{k=0}^{t-1} f_k(x) + R(x)\}$.

What happens when $R(x) = \frac{1}{2}\|x\|^2$ and $f_k(x)$ **Online GD!**

What happens when $R(x) = -\sum x_i \log x_i$ (entropy) a **MWUA!**

Exercise 7! (MWUA)

Conclusion

- Introduction to Online Optimization and Learning.
  - Applications of MWUA.
  - Introduction to FTRL
- Next week we will talk about accelerated methods!