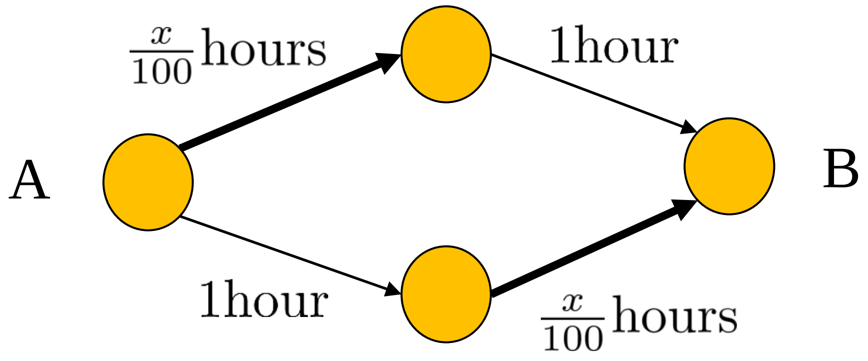
L07 Price of Anarchy

CS 295 Introduction to Algorithmic Game Theory

Ioannis Panageas

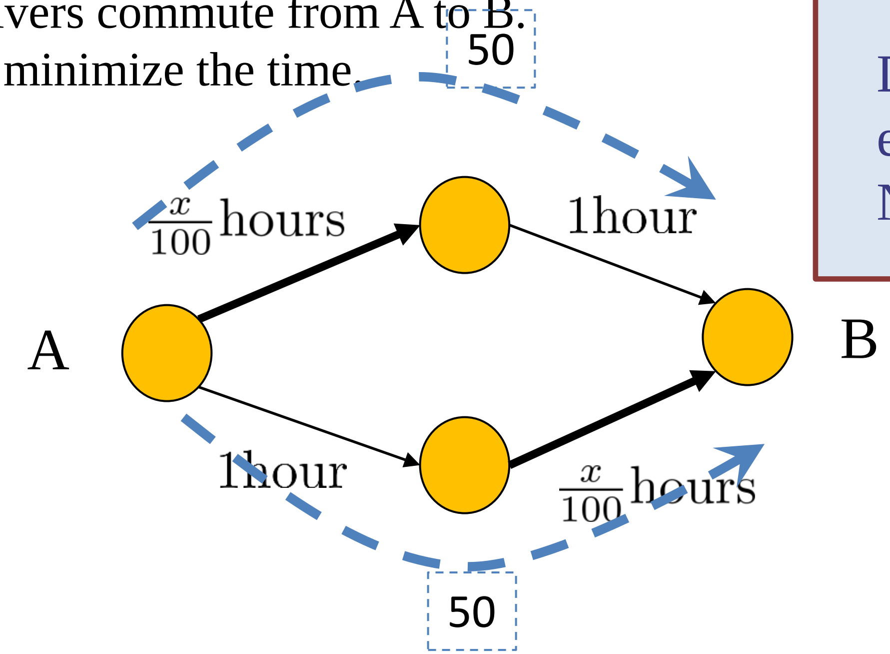
Price of Anarchy

Suppose 100 drivers commute from A to B.
Drivers want to minimize the time.



Price of Anarchy

Suppose 100 drivers commute from A to B.
Drivers want to minimize the time.

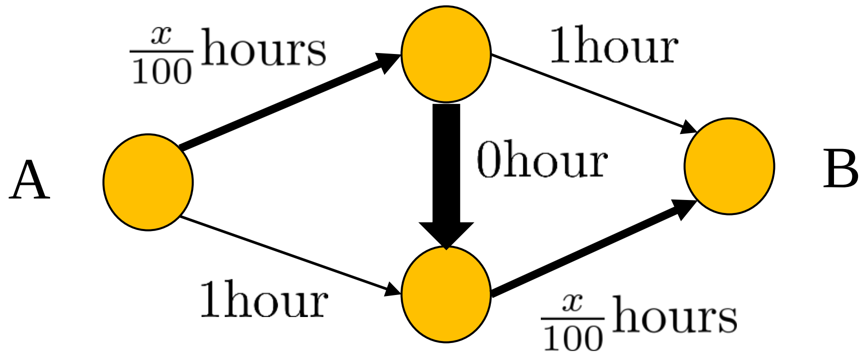


Delay is 1.5 hours for everybody at the unique Nash equilibrium.

Price of Anarchy

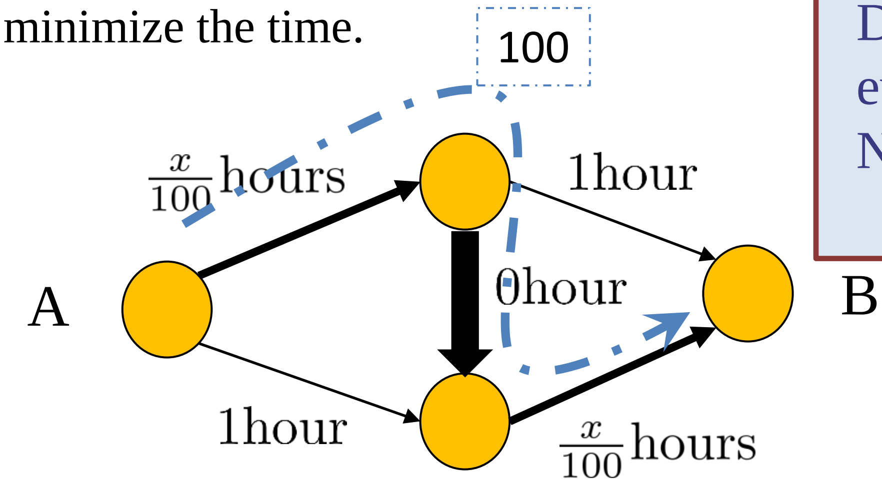
Suppose 100 drivers commute from A to B.
Drivers want to minimize the time.

Question: What if we add a new link?



Price of Anarchy

Suppose 100 drivers commute from A to B.
Drivers want to minimize the time.



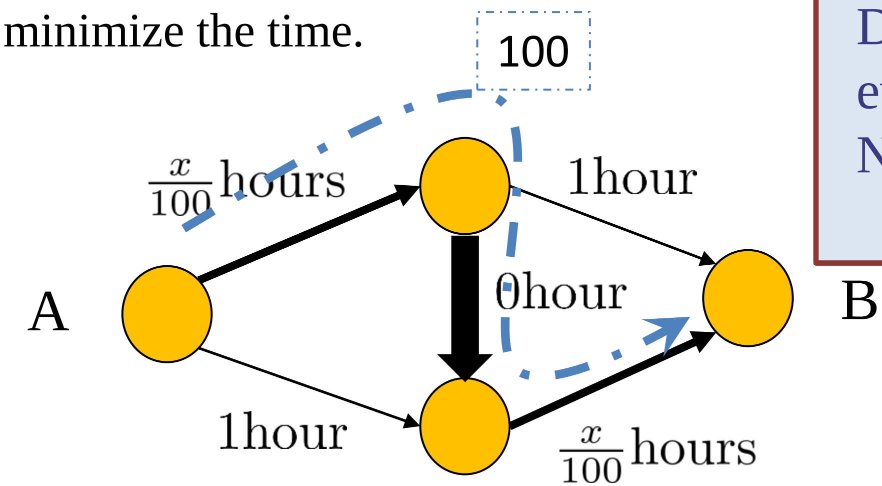
Delay is now 2 hours for everybody at the unique Nash equilibrium.

Braess's paradox

Adding a fast link is not always a good idea!

Price of Anarchy

Suppose 100 drivers commute from A to B.
Drivers want to minimize the time.



Delay is now 2 hours for everybody at the unique Nash equilibrium.
Braess's paradox

Adding a fast link is not always a good idea!

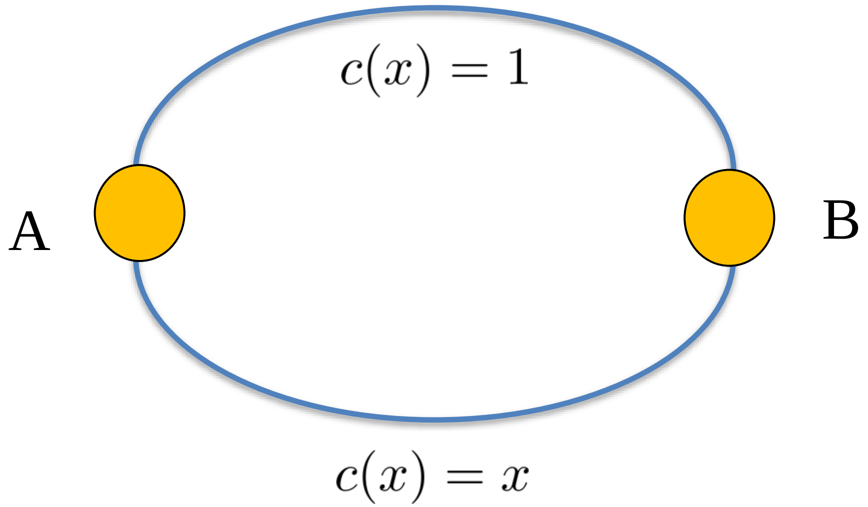
$$\text{PoA} = \frac{\text{performance of worst case NE}}{\text{optimal performance if agents do not decide on their own}}$$

Price of Anarchy (Koutsoupias, Papadimitriou 99').

4/3!!

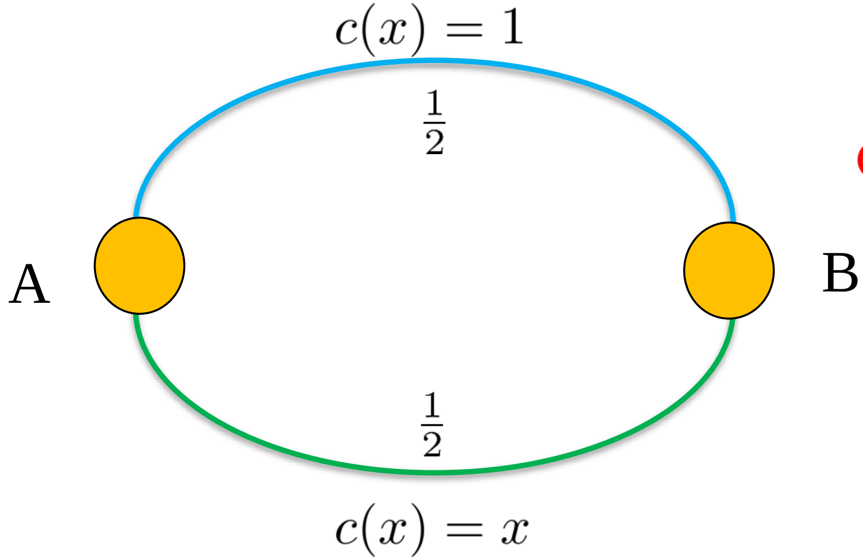
Non-atomic selfish routing

Example: Simpler example. **Pigou network.**



Non-atomic selfish routing

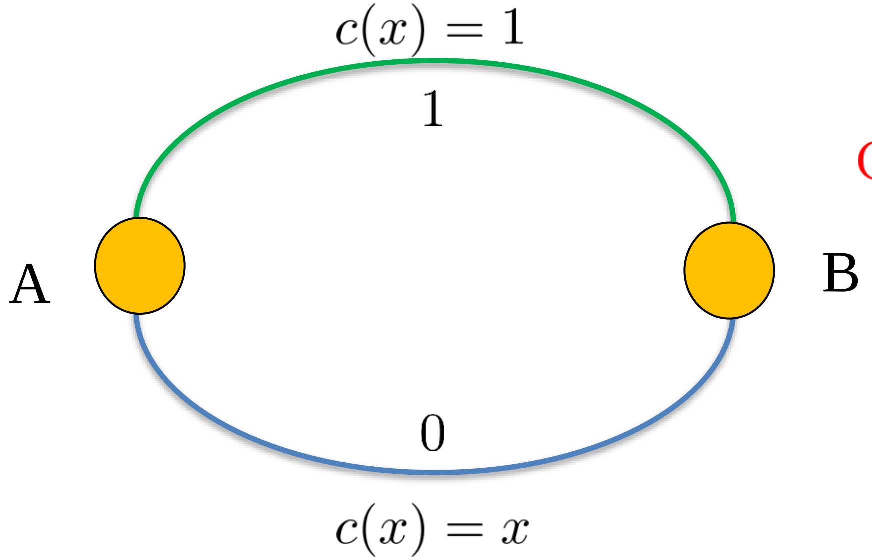
Example: Simpler example. **Pigou network.**



Cost $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

Non-atomic selfish routing

Example: Simpler example. **Pigou network.**

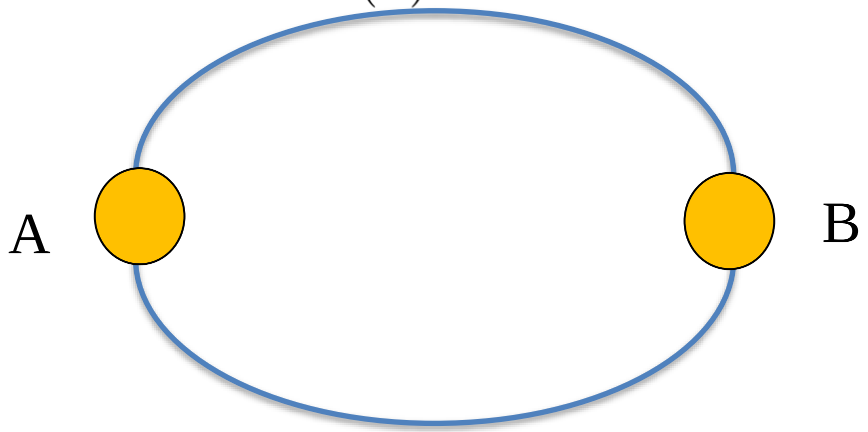


Cost 1 and $\mathbf{PoA} = \frac{4}{3}$

Non-atomic selfish routing

Example: Simpler example. **Pigou network.**

$c(x) = 1$



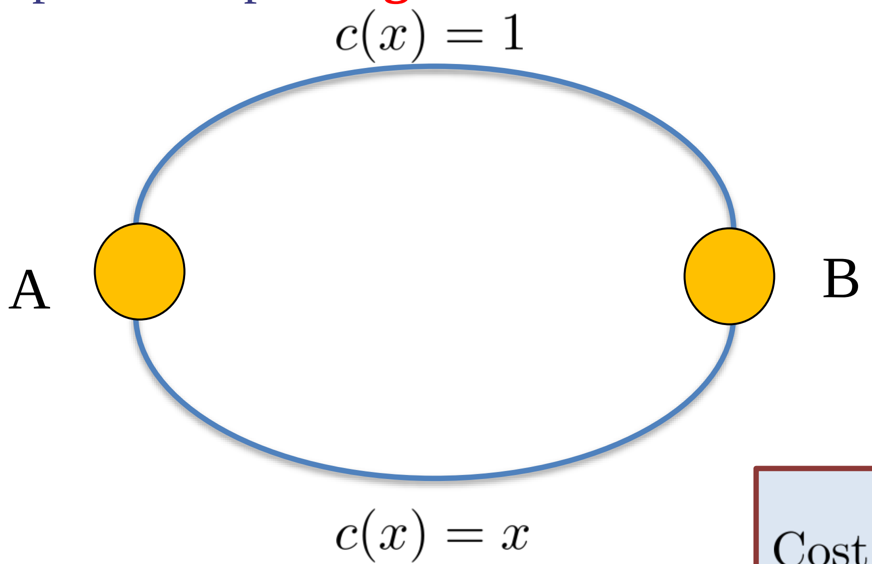
$c(x) = x$

A non-atomic selfish routing game is defined by:

- Graph $G(V, E)$.
- Source destination pairs $(s_1, t_1), ..., (s_k, t_k)$.
- r_i traffic from $s_i \rightarrow t_i$.
- $c_e(.) \geq 0$ cost function of edge e, continuous and non-decreasing.
- Flow is an equilibrium if all traffic is routed on cheapest paths.

Non-atomic selfish routing

Example: Simpler example. **Pigou network.**



A non-atomic selfish routing game is defined by:

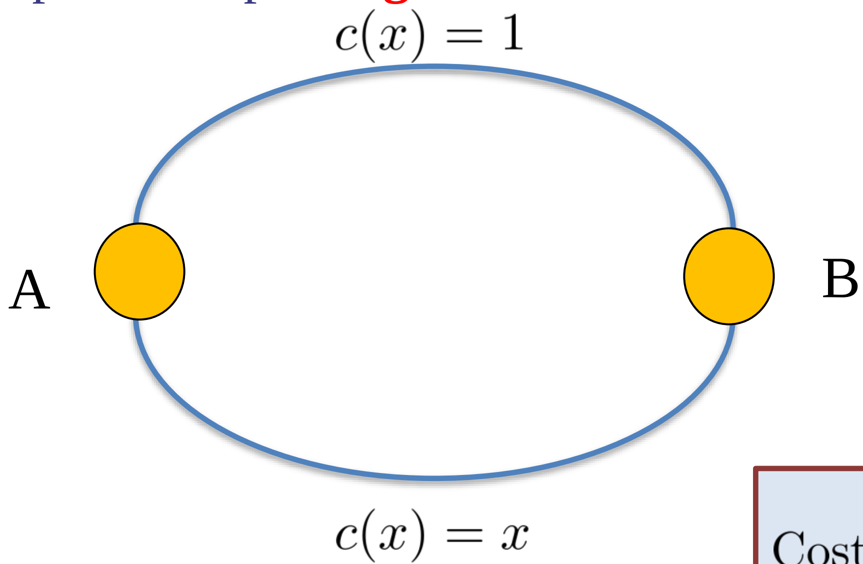
- Graph $G(V, E)$.
- Source destination pairs $(s_1, t_1), ..., (s_k, t_k)$.
- r_i traffic from $s_i \to t_i$.
- $c_e(.) \geq 0$ cost function of edge e, continuous and non-decreasing.
- Flow is an equilibrium if all traffic is routed on cheapest paths.

Cost of path: $c_p(f) = \sum_{e \in p} c_e(f)$

Social Cost $:= \sum_p f_p c_p(f)$

Non-atomic selfish routing

Example: Simpler example. **Pigou network.**



A non-atomic selfish routing game is defined by:

- Graph $G(V, E)$.
- Source destination pairs $(s_1, t_1), ..., (s_k, t_k)$.
- r_i traffic from $s_i \to t_i$.
- $c_e(.) \geq 0$ cost function of edge e, continuous and non-decreasing.
- Flow is an equilibrium if all traffic is routed on cheapest paths.

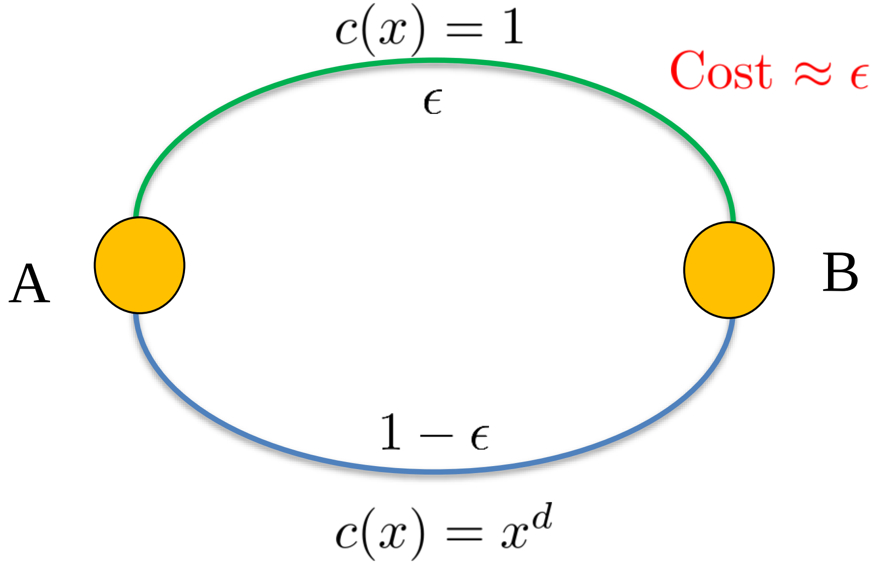
Cost of path: $c_p(f) = \sum_{e \in p} c_e(f)$

Social Cost $:= \sum_p f_p c_p(f)$

Remark: Equilibrium flow exists and is unique!

Non-atomic selfish routing

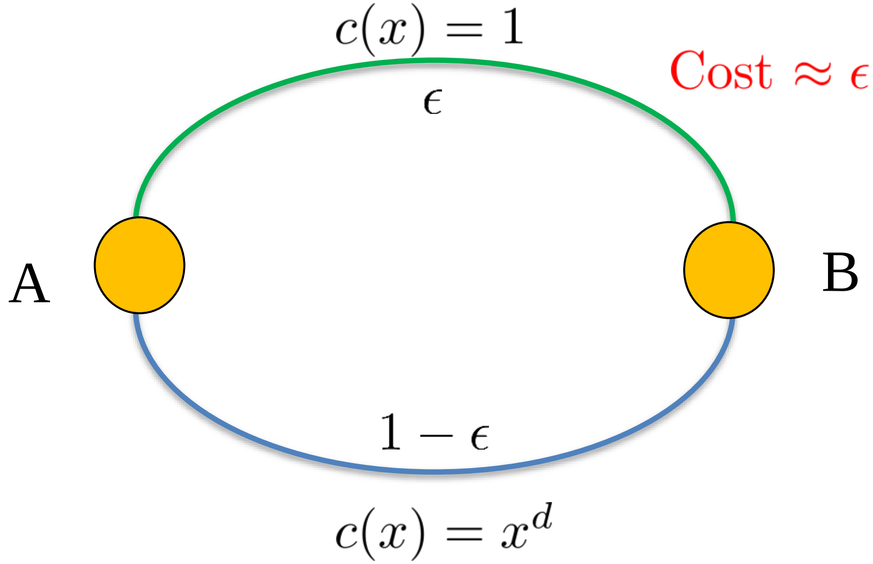
A bad Example. **Pigou network with large degree *d*.**



Cost $\approx \epsilon$ and $\mathbf{PoA} = \frac{1}{\epsilon}$ for d large.

Non-atomic selfish routing

A bad Example. **Pigou network with large degree *d*.**



Cost $\approx \epsilon$ and $\mathbf{PoA} = \frac{1}{\epsilon}$ for d large.

Questions:

1. When is PoA small (bounded)?
2. Can we find bounds on PoA for specific classes of cost functions?

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

Definition (Linear costs)**.** *Linear costs are of the form* $c_e(x) = a_e \cdot x + b_e$.

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

Definition (Linear costs)**.** *Linear costs are of the form* $c_e(x) = a_e \cdot x + b_e$.

Theorem (Roughgarden-Tardos 00′, PoA for linear costs)**.** *For every network with linear costs:*

$$\textit{cost of Nash flow} \leq \frac{4}{3} \cdot \textit{cost of optimal flow}.$$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

Definition (Linear costs). *Linear costs are of the form* $c_e(x) = a_e \cdot x + b_e$.

Theorem (Roughgarden-Tardos 00′, PoA for linear costs). *For every network with linear costs:*

$$\text{cost of Nash flow} \leq \frac{4}{3} \cdot \text{cost of optimal flow}.$$

Proof. Let f^* be a Nash flow and f another flow. We first show (Variational Inequality)

$$\boxed{\sum_e f_e^* c_e(f_e^*) \leq \sum_e f_e c_e(f_e^*).}$$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

Definition (Linear costs). *Linear costs are of the form* $c_e(x) = a_e \cdot x + b_e$.

Theorem (Roughgarden-Tardos 00', PoA for linear costs). *For every network with linear costs:*

$$\text{cost of Nash flow} \leq \frac{4}{3} \cdot \text{cost of optimal flow}.$$

Proof. Let f^* be a Nash flow and f another flow. We first show (Variational Inequality)

$$\boxed{\sum_e f_e^* c_e(f_e^*) \leq \sum_e f_e c_e(f_e^*).}$$

Observe that

f^* equilibrium flow $\Rightarrow$ if $f_p^* > 0$ then $c_p(f^*) \leq c_{p'}(f^*)$ for all paths p'.

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

Proof cont. Therefore all paths p so that $f_p^* > 0$ have same cost say L.

Hence $\sum_p f_p^* c_p(f^*) = L \cdot F$ where $F = \sum_p f_p^*$ is the total flow.

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

Proof cont. Therefore all paths p so that $f_p^* > 0$ have same cost say L.

Hence $\sum_p f_p^* c_p(f^*) = L \cdot F$ where $F = \sum_p f_p^*$ is the total flow.

Since $c_p(f^*) \geq L$ we conlude

$$\sum_p f_p c_p(f^*) \geq L \sum_p f_p = L \cdot F$$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

Proof cont. Therefore all paths p so that $f_p^* > 0$ have same cost say L.

Hence $\sum_p f_p^* c_p(f^*) = L \cdot F$ where $F = \sum_p f_p^*$ is the total flow.

Since $c_p(f^*) \geq L$ we conlude

$$\sum_p f_p c_p(f^*) \geq L \sum_p f_p = L \cdot F$$

Combining the above

$$\boxed{\sum_e f_e c_e(f^*) = \sum_p f_p c_p(f^*) \geq L \cdot F = \sum_p f_p^* c_p(f^*) = \sum_e f_e^* c_e(f^*)}$$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

$$\boxed{\sum_e f_e c_e(f^*) \geq \sum_e f_e^* c_e(f^*).}$$

Proof cont. We get that

$$\sum_e f_e^* c_e(f^*) \leq \sum_e f_e c_e(f) + \sum_e f_e(c_e(f^*) - c_e(f))$$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

$$\boxed{\sum_e f_e c_e(f^*) \geq \sum_e f_e^* c_e(f^*).}$$

Proof cont. We get that

$$\sum_e f_e^* c_e(f^*) \leq \sum_e f_e c_e(f) + \boxed{\sum_e f_e(c_e(f^*) - c_e(f))}$$

We also have that

$$\boxed{\sum_e f_e(c_e(f^*) - c_e(f))} \leq \frac{1}{4} \sum_e f_e^* c_e(f^*)$$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

$$\boxed{\sum_e f_e c_e(f^*) \geq \sum_e f_e^* c_e(f^*).}$$

Proof cont. We get that

$$\sum_e f_e^* c_e(f^*) \leq \sum_e f_e c_e(f) + \boxed{\sum_e f_e(c_e(f^*) - c_e(f))}$$

We also have that

$$\boxed{\sum_e f_e(c_e(f^*) - c_e(f))} \leq \frac{1}{4} \sum_e f_e^* c_e(f^*)$$

- Case $c_e(f^*) < c_e(f)$ trivially $f_e(c_e(f^*) - c_e(f)) \leq \frac{1}{4} f_e^* c_e(f^*)$.

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

$$\boxed{\sum_e f_e c_e(f^*) \geq \sum_e f_e^* c_e(f^*).}$$

Proof cont. We get that

$$\sum_e f_e^* c_e(f^*) \leq \sum_e f_e c_e(f) + \boxed{\sum_e f_e(c_e(f^*) - c_e(f))}$$

We also have that

$$\boxed{\sum_e f_e(c_e(f^*) - c_e(f))} \leq \frac{1}{4} \sum_e f_e^* c_e(f^*)$$

- Case $c_e(f^*) < c_e(f)$ trivially $f_e(c_e(f^*) - c_e(f)) \leq \frac{1}{4} f_e^* c_e(f^*)$.
- Case $c_e(f^*) \geq c_e(f) \Rightarrow f_e^* \geq f_e$. Linear costs $\Rightarrow$ LHS $= a_e f_e(f_e^* - f_e)$ and RHS $\geq \frac{1}{4} a_e {f_e^*}^2$.

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

$$\sum_e f_e c_e(f^*) \geq \sum_e f_e^* c_e(f^*).$$

Proof cont. We get that

$$\sum_e f_e^* c_e(f^*) \leq \sum_e f_e c_e(f) + \sum_e f_e(c_e(f^*) - c_e(f))$$

We also have that

$$\sum_e f_e(c_e(f^*) - c_e(f)) \leq \frac{1}{4} \sum_e f_e^* c_e(f^*)$$

- Case $c_e(f^*) < c_e$ … $e(f^*)$.

Since $xy - y^2 \leq \frac{x^2}{4} \Rightarrow$ LHS $\leq$ RHS.

- Case $c_e(f^*) \geq c_e(f) \Rightarrow f_e^* \geq f_e$. Linear costs $\Rightarrow$ LHS $= a_e f_e(f_e^* - f_e)$ and RHS $\geq \frac{1}{4} a_e {f_e^*}^2$.

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

$$\boxed{\sum_e f_e c_e(f^*) \geq \sum_e f_e^* c_e(f^*).}$$

Proof cont. We conclude that

$$\sum_e f_e^* c_e(f^*) \leq \sum_e f_e c_e(f) + \frac{1}{4} \sum_e f_e^* c_e(f^*)$$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

$$\boxed{\sum_e f_e c_e(f^*) \geq \sum_e f_e^* c_e(f^*).}$$

Proof cont. We conclude that

$$\sum_e f_e^* c_e(f^*) \leq \sum_e f_e c_e(f) + \frac{1}{4} \sum_e f_e^* c_e(f^*).$$

Or equivalently

$$\sum_e f_e^* c_e(f^*) \leq \frac{4}{3} \sum_e f_e c_e(f).$$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

$$\sum_e f_e c_e(f^*) \geq \sum_e f_e^* c_e(f^*).$$

Proof cont. V

Pigou example is tight!

).

Or equivalently

$$\sum_e f_e^* c_e(f^*) \leq \frac{4}{3} \sum_e f_e c_e(f).$$

Theorem (Roughgarden 02′, PoA for polynomial costs)**.** *For every network with polynomial costs with degree d:*

$$\textit{cost of Nash flow} \leq \Theta\left(\frac{d}{\log d}\right) \cdot \textit{cost of optimal flow}.$$

Price of Anarchy in Non-atomic selfish routing with *Linear* costs

$$\sum_e f_e c_e(f^*) \geq \sum_e f_e^* c_e(f^*).$$

Proof cont. V

Pigou example is tight!

Or equivalently

$$\sum_e f_e^* c_e(f^*) \leq \frac{4}{3} \sum_e f_e c_e(f).$$

Theorem (Roughgarden 02′, PoA for polynomial costs)**.** *For every network with polynomial costs with degree d:*

$$\textit{cost of Nash flow} \leq \Theta\left(\frac{d}{\log d}\right) \cdot \textit{cost of optimal fl}$$

HW2

Price of Anarchy in Congestion Games

Theorem (Christodoulou-Koutsoupias, PoA for linear costs)**.** *For every congestion game with linear costs:*

$$\textit{cost of worst Nash} \leq \frac{5}{2} \cdot \textit{cost of optimal welfare}.$$

Price of Anarchy in Congestion Games

Theorem (Christodoulou-Koutsoupias, PoA for linear costs). *For every congestion game with linear costs:*

$$\textit{cost of worst Nash} \leq \frac{5}{2} \cdot \textit{cost of optimal welfare}.$$

Proof. Let l^* be a Nash equilibrium in which i uses path P_i and asuume i deviates to path $\tilde{P}_i$. It holds (Variational Inequality)

$$\sum_{e \in P_i} c_e(l_e^*) \leq \sum_{e \in P_i \cap \tilde{P}_i} c_e(l_e^*) + \sum_{e \in \tilde{P}_i \setminus P_{-i}} c_e(l_e^* + 1)$$

Price of Anarchy in Congestion Games

Theorem (Christodoulou-Koutsoupias, PoA for linear costs)**.** *For every congestion game with linear costs:*

$$\textit{cost of worst Nash} \leq \frac{5}{2} \cdot \textit{cost of optimal welfare}.$$

Proof. Let l^* be a Nash equilibrium in which i uses path P_i and asuume i deviates to path $\tilde{P}_i$. It holds (Variational Inequality)

$$\sum_{e \in P_i} c_e(l_e^*) \leq \sum_{e \in P_i \cap \tilde{P}_i} c_e(l_e^*) + \sum_{e \in \tilde{P}_i \setminus P_{-i}} c_e(l_e^* + 1)$$

$$\leq \sum_{e \in P_i \cap \tilde{P}_i} c_e(l_e^* + 1) + \sum_{e \in \tilde{P}_i \setminus P_{-i}} c_e(l_e^* + 1)$$

Price of Anarchy in Congestion Games

Theorem (Christodoulou-Koutsoupias, PoA for linear costs). *For every congestion game with linear costs:*

$$\textit{cost of worst Nash} \leq \frac{5}{2} \cdot \textit{cost of optimal welfare}.$$

Proof. Let l^* be a Nash equilibrium in which i uses path P_i and asuume i deviates to path $\tilde{P}_i$. It holds (Variational Inequality)

$$\begin{aligned}\sum_{e \in P_i} c_e(l_e^*) &\leq \sum_{e \in P_i \cap \tilde{P}_i} c_e(l_e^*) + \sum_{e \in \tilde{P}_i \setminus P_{-i}} c_e(l_e^* + 1) \\ &\leq \sum_{e \in P_i \cap \tilde{P}_i} c_e(l_e^* + 1) + \sum_{e \in \tilde{P}_i \setminus P_{-i}} c_e(l_e^* + 1) \\ &= \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1).\end{aligned}$$

Price of Anarchy in Congestion Games

$$\boxed{\sum_{e \in P_i} c_e(l_e^*) \leq \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1).}$$

Proof cont. Consider any configuration $\tilde{l}$, where each agent j uses path $\tilde{P}_j$. Summing for all agents i

$$\sum_{i \in [n]} \sum_{e \in P_i} c_e(l_e^*) \leq \sum_{i \in [n]} \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1).$$

Price of Anarchy in Congestion Games

$$\boxed{\sum_{e \in P_i} c_e(l_e^*) \le \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1).}$$

Proof cont. Consider any configuration $\tilde{l}$, where each agent j uses path $\tilde{P}_j$. Summing for all agents i

$$\sum_{i \in [n]} \sum_{e \in P_i} c_e(l_e^*) \le \sum_{i \in [n]} \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1).$$

$$= \sum_{e \in \tilde{P}_i} \tilde{l}_e c_e(l_e^* + 1).$$

Price of Anarchy in Congestion Games

$$\boxed{\sum_{e \in P_i} c_e(l_e^*) \leq \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1).}$$

Proof cont. Consider any configuration $\tilde{l}$, where each agent j uses path $\tilde{P}_j$. Summing for all agents i

$$\begin{aligned}
\sum_{i \in [n]} \sum_{e \in P_i} c_e(l_e^*) \leq \sum_{i \in [n]} \sum_{e \in \tilde{P}_i} c_e(l_e^* + 1). \\
= \sum_{e \in \tilde{P}_i} \tilde{l}_e c_e(l_e^* + 1). \\
= \sum_{e \in \tilde{P}_i} a_e \tilde{l}_e (l_e^* + 1) + b_e \tilde{l}_e.
\end{aligned}$$

Price of Anarchy in Congestion Games

$$\boxed{\sum_{e\in P_i} c_e(l_e^*) \le \sum_{e\in \tilde{P}_i} c_e(l_e^*+1).}$$

Proof cont. Consider any configuration $\tilde{l}$, where each agent j uses path $\tilde{P}_j$. Summing for all agents i

$$\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \le$$

Since $y(z+1) \le \frac{5}{3}y^2 + \frac{1}{3}z^2$ for naturals y, z

HW2

$$= \sum_{e\in\tilde{P}_i}$$

$$= \sum_{e\in\tilde{P}_i} a_e\tilde{l}_e(l_e^*+1) + b_e\tilde{l}_e.$$

$$\le \sum_{e\in\tilde{P}_i} a_e\left(\frac{5}{3}\tilde{l}_e^2 + \frac{1}{3}{l_e^*}^2\right) + b_e\tilde{l}_e.$$

Price of Anarchy in Congestion Games

$$\boxed{\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \le \sum_{e\in\tilde{P}_i} a_e\left(\frac{5}{3}\tilde{l}_e^2 + \frac{1}{3}{l_e^*}^2\right) + b_e\tilde{l}_e}$$

Proof cont. Observe that

$$\frac{5}{3}C(\tilde{l}) = \frac{5}{3}\sum_{i\in[n]}\sum_{e\in\tilde{P}_i} c_e(\tilde{l}_e) = \sum_{e\in\tilde{P}_i}\frac{5}{3}a_e\tilde{l}_e^2 + \frac{5}{3}b_e\tilde{l}_e \ge \sum_{e\in\tilde{P}_i}\frac{5}{3}a_e\tilde{l}_e^2 + b_e\tilde{l}_e$$

Price of Anarchy in Congestion Games

$$\boxed{\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \leq \sum_{e\in\tilde{P}_i} a_e\left(\frac{5}{3}\tilde{l}_e^2 + \frac{1}{3}{l_e^*}^2\right) + b_e\tilde{l}_e}$$

Proof cont. Observe that

$$\frac{5}{3}C(\tilde{l}) = \frac{5}{3}\sum_{i\in[n]}\sum_{e\in\tilde{P}_i} c_e(\tilde{l}_e) = \sum_{e\in\tilde{P}_i}\frac{5}{3}a_e\tilde{l}_e^2 + \frac{5}{3}b_e\tilde{l}_e \geq \sum_{e\in\tilde{P}_i}\frac{5}{3}a_e\tilde{l}_e^2 + b_e\tilde{l}_e$$

Therefore

$$C(l^*) \leq \frac{5}{3}C(\tilde{l}) + \frac{1}{3}\sum_{e\in\tilde{P}_i} a_e{l_e^*}^2$$

Price of Anarchy in Congestion Games

$$\boxed{\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \leq \sum_{e\in \tilde{P}_i} a_e\left(\frac{5}{3}\tilde{l}_e^2 + \frac{1}{3}{l_e^*}^2\right) + b_e\tilde{l}_e}$$

Proof cont. Observe that

$$\frac{5}{3}C(\tilde{l}) = \frac{5}{3}\sum_{i\in[n]}\sum_{e\in \tilde{P}_i} c_e(\tilde{l}_e) = \sum_{e\in \tilde{P}_i}\frac{5}{3}a_e\tilde{l}_e^2 + \frac{5}{3}b_e\tilde{l}_e \geq \sum_{e\in \tilde{P}_i}\frac{5}{3}a_e\tilde{l}_e^2 + b_e\tilde{l}_e$$

Therefore

$$C(l^*) \leq \frac{5}{3}C(\tilde{l}) + \frac{1}{3}\sum_{e\in \tilde{P}_i} a_e{l_e^*}^2$$

$$\leq \frac{5}{3}C(\tilde{l}) + \frac{1}{3}C(l^*)$$

Price of Anarchy in Congestion Games

$$\sum_{i\in[n]}\sum_{e\in P_i} c_e(l_e^*) \leq \sum_{e\in\tilde{P}_i} a_e\left(\frac{5}{3}\tilde{l}_e^2 + \frac{1}{3}{l_e^*}^2\right) + b_e\tilde{l}_e$$

Proof cont. Ob

$$C(l^*) \leq \tfrac{5}{2}C(\tilde{l}).$$

$$\frac{5}{3}C(\tilde{l}) = \quad \ldots \quad a_e\tilde{l}_e^2 + b_e\tilde{l}_e$$

Therefore

$$C(l^*) \leq \frac{5}{3}C(\tilde{l}) + \frac{1}{3}\sum_{e\in\tilde{P}_i} a_e {l_e^*}^2$$

$$\leq \frac{5}{3}C(\tilde{l}) + \frac{1}{3}C(l^*)$$

Remark:

1. The above bound is tight!
2. For polynomial cost functions the PoA is exponential in *d*.

Price of Anarchy and Balls & Bins

Definition (Balls and Bins)**.** *Consider*

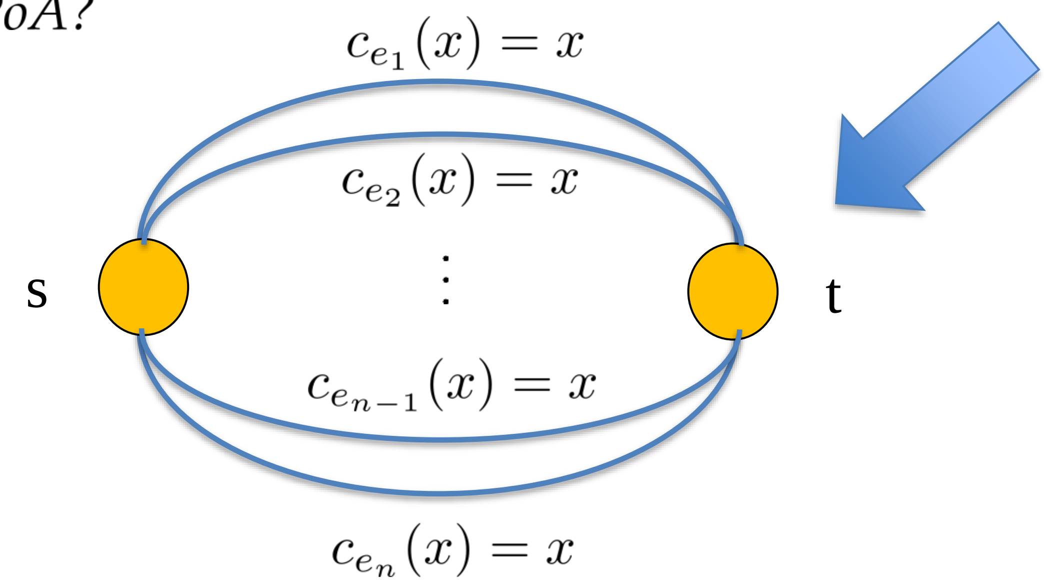
- *set of n balls and n bins $\{e_1, \ldots, e_n\}$.*
- *Each ball i chooses a bin j and pays the load of the bin j.*
- *Define social cost the* ***maximum load****.*
- *What is PoA? Is it $\frac{5}{2}$?*

Price of Anarchy and Balls & Bins

Definition (Balls and Bins). *Consider*

- *set of n balls and n bins $\{e_1, ..., e_n\}$.*
- *Each ball i chooses a bin j and pays the load of the bin j.*
- *Define social cost the **maximum load**.*
- *What is PoA?*

Congestion game!



Price of Anarchy and Balls & Bins

Theorem (Koutsoupias-Papadimitriou, PoA for balls & bins)**.** *The PoA is*

$$\Omega\left(\frac{\ln n}{\ln\ln n}\right).$$

Proof. We will use second moment method.

- Set every ball in a different bin. Hence optimal social cost is 1.
- Uniform $\left(\frac{1}{n}, ..., \frac{1}{n}\right)$ is a Nash Equilibrium (symmetry).

Price of Anarchy and Balls & Bins

Theorem (Koutsoupias-Papadimitriou, PoA for balls & bins)**.** *The PoA is*

$$\Omega\left(\frac{\ln n}{\ln \ln n}\right).$$

Proof. We will use second moment method.

- Set every ball in a different bin. Hence optimal social cost is 1.
- Uniform $\left(\frac{1}{n}, ..., \frac{1}{n}\right)$ is a Nash Equilibrium (symmetry).
- With high probability, we show that uniform gives max load $\Omega\left(\frac{\ln n}{\ln \ln}\right)$, which implies the expected max load is $\Omega\left(\frac{\ln n}{\ln \ln n}\right)$.

Price of Anarchy and Balls & Bins

Theorem (Koutsoupias-Papadimitriou, PoA for balls & bins)**.** *The PoA is*

$$\Omega\left(\frac{\ln n}{\ln \ln n}\right).$$

Proof. We will use second moment method.

- Set every ball in a different bin. Hence optimal social cost is 1.
- Uniform $\left(\frac{1}{n}, ..., \frac{1}{n}\right)$ is a Nash Equilibrium (symmetry).
- With high probability, we show that uniform gives max load $\Omega\left(\frac{\ln n}{\ln \ln}\right)$, which implies the expected max load is $\Omega\left(\frac{\ln n}{\ln \ln n}\right)$.

Claim 1: Bin i has at least $k \ll n$ balls with probability at least:

$$\binom{n}{k}\frac{1}{n^k}\left(1 - \frac{1}{n}\right)^{n-k} \geq \frac{1}{n^k}\left(\frac{n}{k}\right)^k \frac{1}{e} = \frac{1}{ek^k}.$$

Price of Anarchy and Balls & Bins

Theorem (Koutsoupias-Papadimitriou, PoA for balls & bins). *The PoA is*

$$\Omega\left(\frac{\ln n}{\ln\ln n}\right).$$

Proof. We will use second moment method.

- Set every ball in a different bin. Hence optimal social cost is 1.
- Uniform (
- With high … ad $\Omega\left(\frac{\ln n}{\ln\ln}\right)$,
  which imp

In general (HW2):

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k.$$

Claim 1: Bin i has at least $k \ll n$ balls with probability at least:

$$\binom{n}{k}\frac{1}{n^k}\left(1-\frac{1}{n}\right)^{n-k} \ge \frac{1}{n^k}\left(\frac{n}{k}\right)^k\frac{1}{e} = \frac{1}{ek^k}.$$

Price of Anarchy and Balls & Bins

Proof cont. Choosing $k = \frac{\ln n}{3 \ln \ln}$ we have that $k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln}} = n^{1/3}$.

.

Claim 1: Thus bin i has at least $\frac{\ln n}{3 \ln \ln}$ balls with probability at least $\frac{1}{en^{1/3}}$.

Price of Anarchy and Balls & Bins

Proof cont. Choosing $k = \frac{\ln n}{3 \ln \ln}$ we have that $k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln}} = n^{1/3}$.

.

Claim 1: Thus bin i has at least $\frac{\ln n}{3 \ln \ln}$ balls with probability at least $\frac{1}{en^{1/3}}$.

Let X_i be the indicator that bin i has at least $\frac{\ln n}{3 \ln \ln n}$ balls and X be the expected number of all bins with at least $\frac{\ln n}{3 \ln \ln n}$ balls.

Price of Anarchy and Balls & Bins

Proof cont. Choosing $k = \frac{\ln n}{3 \ln \ln}$ we have that $k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln}} = n^{1/3}$.

.

Claim 1: Thus bin i has at least $\frac{\ln n}{3 \ln \ln}$ balls with probability at least $\frac{1}{en^{1/3}}$.

Let X_i be the indicator that bin i has at least $\frac{\ln n}{3 \ln \ln n}$ balls and X be the expected number of all bins with at least $\frac{\ln n}{3 \ln \ln n}$ balls.

$$\boxed{X = \sum_i X_i \Rightarrow E[X] = \sum_i E[X_i].}$$

Price of Anarchy and Balls & Bins

Proof cont. Choosing $k = \frac{\ln n}{3 \ln \ln}$ we have that $k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln}} = n^{1/3}$.
.

Claim 1: Thus bin i has at least $\frac{\ln n}{3 \ln \ln}$ balls with probability at least $\frac{1}{en^{1/3}}$.

Let X_i be the indicator that bin i has at least $\frac{\ln n}{3 \ln \ln n}$ balls and X be the expected number of all bins with at least $\frac{\ln n}{3 \ln \ln n}$ balls.

$$X = \sum_i X_i \Rightarrow E[X] = \sum_i E[X_i].$$

Observe that $E[X] \geq \frac{n^{2/3}}{e} \gg 1$ but this does not imply $X \geq 1$ with high probability. We need to argue about the variance (second moment).

Price of Anarchy and Balls & Bins

Proof cont. Choosing $k = \frac{\ln n}{3 \ln \ln}$ we have that $k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln}} = n^{1/3}$.
.
Claim 1: Thus bin i has at least $\frac{\ln n}{3 \ln \ln}$ balls with probability at least $\frac{1}{en^{1/3}}$.

Let X_i be the indicator that bin i has at least $\frac{\ln n}{3 \ln \ln n}$ balls and X be the expected number of all bins with at least $\frac{\ln n}{3 \ln \ln n}$ balls.

$$X = \sum_i X_i \Rightarrow E[X] = \sum_i E[X_i].$$

Observe that $E[X] \geq \frac{n^{2/3}}{e} \gg 1$ but this does not imply $X \geq 1$ with high probability. We need to argue about the variance (second moment).

Chebyshev's inequality gives $Pr[|X - E[X]| \geq tE[X]] \leq \frac{Var[X]}{t^2 E^2[X]}$,

Price of Anarchy and Balls & Bins

Proof cont. Choosing $k = \frac{\ln n}{3 \ln \ln}$ we have that $k^k \leq (\ln n)^k = (\ln n)^{\frac{\ln n}{3 \ln \ln}} = n^{1/3}$.
.

Claim 1: Thus bin i has at least $\frac{\ln n}{3 \ln \ln}$ balls with probability at least $\frac{1}{en^{1/3}}$.

Let X_i be the indicator that bin i has at least $\frac{\ln n}{3 \ln \ln n}$ balls and X be the expected number of all bins with at least $\frac{\ln n}{3 \ln \ln n}$ balls.

$$X = \sum_i X_i \Rightarrow E[X] = \sum_i E[X_i].$$

Observe that $E[X] \geq \frac{n^{2/3}}{e} \gg 1$ but this does not imply $X \geq 1$ with high probability. We need to argue about the variance (second moment).

Chebyshev's inequality gives $Pr[|X - E[X]| \geq tE[X]] \leq \frac{Var[X]}{t^2 E^2[X]}$,

thus $Pr[X = 0] \leq Pr[|X - E[X]| \geq E[X]] \leq \frac{Var[X]}{E^2[X]}$.

Price of Anarchy and Balls & Bins

Proof cont. $\boxed{Pr[X = 0] \leq \frac{Var[X]}{E^2[X]}.}$

.

From *negative correlation* we have that $Var[X] \leq \sum_i Var[X_i]$.

.

Morever $Var[X_i] = E[X_i^2] - E^2[X_i] \leq E[X_i^2] = E[X_i] \leq 1$

.

Price of Anarchy and Balls & Bins

Proof cont. $\boxed{Pr[X=0] \leq \frac{Var[X]}{E^2[X]}.}$

.

From *negative correlation* we have that $Var[X] \leq \sum_i Var[X_i]$.

.

Morever $Var[X_i] = E[X_i^2] - E^2[X_i] \leq E[X_i^2] = E[X_i] \leq 1$

.

We conclude that

.

$$Pr[X=0] \leq \frac{n}{e^2 n^{4/3}} = \frac{n^{-1/3}}{e^2}.$$

.

Price of Anarchy and Balls & Bins

Proof cont. $\boxed{Pr[X=0] \le \frac{Var[X]}{E^2[X]}.}$

From *negative correlation* we have that $Var[X] \le \sum_i Var[X_i]$.

Morever $Var[X_i] = E[X_i^2] - E^2[X_i] \le E[X_i^2] = E[X_i] \le 1$

We conclude that

$$Pr[X=0] \le \frac{n}{e^2 n^{4/3}} = \frac{n^{-1/3}}{e^2}.$$

Therefore

$$\boxed{Pr[X \ge 1] = 1 - Pr[X=0] \ge 1 - \frac{n^{-1/3}}{e^2} \to 1.}$$

Congestion Games

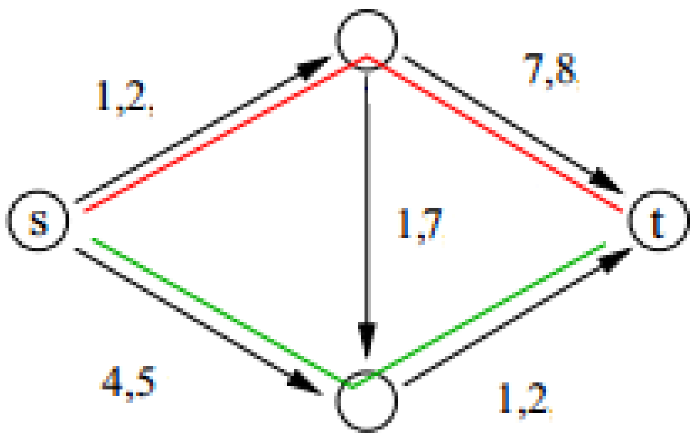
A congestion game is defined by:

- n set of players.
- E set of edges/facilities/ bins.
- $S_i \subset 2^E$ the set of strategies of player i.
- $c_e : \{1, ..., n\} \rightarrow \mathbb{R}^+$ cost function of edge e.

For any $s = (s_1, ..., s_n)$

- $l_e(s)$ number of players (load) that use edge e.
- $c_i(s) = \sum_{e \in s_i} c_e(l_e)$ the cost function of player i.

Congestion Games



For this game:

$n = \{1, 2\}$ (red, green)
E are the edges of the network.
S_i is all $s - t$ paths.
c_e on edges.

Remark: Defined by Rosenthal in 1973. Capture atomic routing **games!**