L12 Monotone Allocations and Myerson's Lemma

CS 280 Algorithmic Game Theory

Ioannis Panageas

Inspired and some figures by Tim Roughgarden notes

Recap

Three desirable guarantees

1. DSIC: Truthful bidding is a dominant strategy.

   Easy to play for bidders, Predict outcome.

2. Social surplus maximization:

   $$\sum_{i=1}^{n} x_i v_i$$

   where x_i is the amount allocated to i.

3. The auction can be implemented in polynomial time.

An Example: Sponsored Search Auctions

Every time you type a query into a search engine, an auction is run to decide which advertisers' links are shown, the order of the links, and how advertisers are charged.

An Example: Sponsored Search Auctions

Every time you type a query into a search engine, an auction is run to decide which advertisers' links are shown, the order of the links, and how advertisers are charged.

Probability to get a click

- Items for sale are k "slots"
- Bidders are the advertisers.
- Each slot j has CTR (click-through-rate) a_j.
- Each bidder i has private valuation v_i and gets value $a_j \cdot v_i$. Note $a_1 \geq \ \dots \geq \ a_k$

Definitions

Definition (Single parameter environments). *A single parameter environment is defined:*

- *n bidders with private v_i,*
- *Feasible set $\mathcal{X}$, each element of which is a n-dimensional vector $(x_1, \ldots, x_n)$ in which x_i is the amount of "stuff" given to i.*

Examples:

1. Single-item auctions: $\mathcal{X}$ is 0-1 vectors with at most one 1, i.e., $\sum x_i \leq 1$.
2. k identical goods, each bidder gets at most one: $\mathcal{X}$ is 0-1 vectors with $\sum x_i \leq k$.
3. In sponsored search, $\mathcal{X}$ is the set of n-vectors with x_i being a_j if slot j is assigned to bidder i.

More Definitions

Examples:

1. Single-item auctions: $\mathcal{X}$ is 0-1 vectors with at most one 1, i.e., $\sum x_i \leq 1$.
2. k identical goods, each bidder gets at most one: $\mathcal{X}$ is 0-1 vectors with $\sum x_i \leq k$.
3. In sponsored search, $\mathcal{X}$ is the set of n-vectors with x_i being a_j if slot j is assigned to bidder i.

Definition (Allocation and Payments). *A sealed-bid auction is defined:*

1. *Bidders report bids* $b = (b_1, .., b_n)$,
2. *Auctioneer chooses feasible allocation* $x(b) \in \mathcal{X}$.
3. *Auctioneer chooses payments* $p(b) \in \mathbb{R}^n$.
4. *Bidder i gets utility* $u_i = v_i \cdot x_i(b) - p_i(b)$.

Monotone Allocations and Myerson's Lemma

Definition (Monotone Allocations). *An allocation rule x for a single-parameter environment is monotone if for every bidder i and bids b_{-i} by rest of bidders, the allocation*

$$x_i(z, b_{-i}) \text{ is nondecreasing in } z.$$

Monotone Allocations and Myerson's Lemma

Definition (Monotone Allocations). *An allocation rule x for a single-parameter environment is monotone if for every bidder i and bids b_{-i} by rest of bidders, the allocation*

$$x_i(z, b_{-i}) \text{ is nondecreasing in } z.$$

Theorem (Myerson's Lemma). *Let (x, p) be a mechanism. We assume that $p_i(b) = 0$ whenever $b_i = 0$, for all bidders i.*

1. *It holds that if (x, p) is DSIC mechanism then x is monotone.*
2. *If x is a monotone allocation, then there is a unique payment rule such that (x, p) is DSIC.*

Myerson's Lemma: Monotone

Proof. Suppose (x, p) is a DSIC and let $0 \leq y \leq z$.

If bidder i has **private valuation** z, to avoid reporting y, DSIC demands

$$z \cdot x_i(z) - p_i(z) \geq z \cdot x_i(y) - p_i(y) \text{ for all } i.$$

Myerson's Lemma: Monotone

Proof. Suppose (x, p) is a DSIC and let $0 \le y \le z$.

If bidder i has **private valuation** z, to avoid reporting y, DSIC demands

$$z \cdot x_i(z) - p_i(z) \ge z \cdot x_i(y) - p_i(y) \text{ for all } i.$$

If bidder i has **private valuation** y, to avoid reporting z, DSIC demands

$$y \cdot x_i(y) - p_i(y) \ge y \cdot x_i(z) - p_i(z) \text{ for all } i.$$

Myerson’s Lemma: Monotone

Proof. Suppose (x, p) is a DSIC and let $0 \leq y \leq z$.

If bidder i has **private valuation** z, to avoid reporting y, DSIC demands

$$z \cdot x_i(z) - p_i(z) \geq z \cdot x_i(y) - p_i(y) \text{ for all } i.$$

If bidder i has **private valuation** y, to avoid reporting z, DSIC demands

$$y \cdot x_i(y) - p_i(y) \geq y \cdot x_i(z) - p_i(z) \text{ for all } i.$$

Combining the two inequalities:

$$\boxed{z \cdot (x_i(y) - x_i(z)) \leq p(y) - p(z) \leq y \cdot (x_i(y) - x_i(z))}$$

Myerson's Lemma: Monotone

Proof. Suppose (x, p) is a DSIC and let $0 \leq y \leq z$.

If bidder i has **private valuation** z, to avoid reporting y, DSIC demands

$$z \cdot x_i(z) - p_i(z) \geq z \cdot x_i(y) - p_i(y) \text{ for all } i.$$

If bidder i has **private valuation** y, to avoid reporting z, DSIC demands

$$y \cdot x_i(y) - p_i(y) \geq y \cdot x_i(z) - p_i(z) \text{ for all } i.$$

Combining the two inequalities:

$$\boxed{z \cdot (x_i(y) - x_i(z)) \leq p(y) - p(z) \leq y \cdot (x_i(y) - x_i(z))}$$

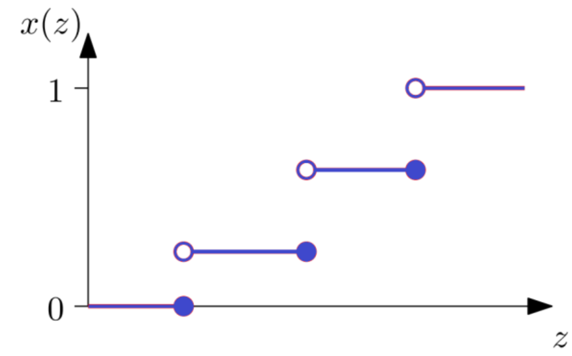
$$\boxed{x_i(y) \leq x_i(z)}$$

Myerson's Lemma: Payments

$$z \cdot (x_i(y) - x_i(z)) \leq p(y) - p(z) \leq y \cdot (x_i(y) - x_i(z))$$

Proof cont. Assume x is monotone for the rest of the proof and x is piecewise constant (simple function). if there is a jump at z (say of magnitude h) then as $y \to z$ from left we get

$$z \cdot h \leq p(y) - p(z) \leq y \cdot h.$$

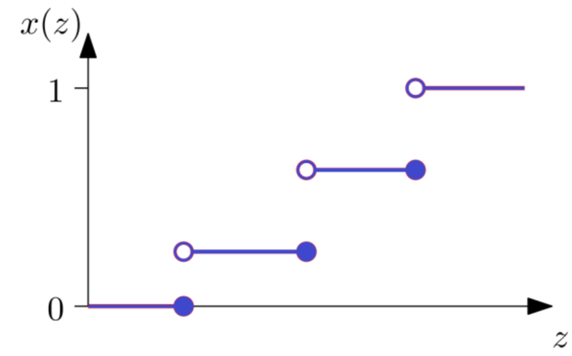


Myerson's Lemma: Payments

$$z \cdot (x_i(y) - x_i(z)) \le p(y) - p(z) \le y \cdot (x_i(y) - x_i(z))$$

Proof cont. Assume x is monotone for the rest of the proof and x is piecewise constant (simple function). if there is a jump at z (say of magnitude h) then as $y \to z$ from left we get

$$z \cdot h \le p(y) - p(z) \le y \cdot h.$$



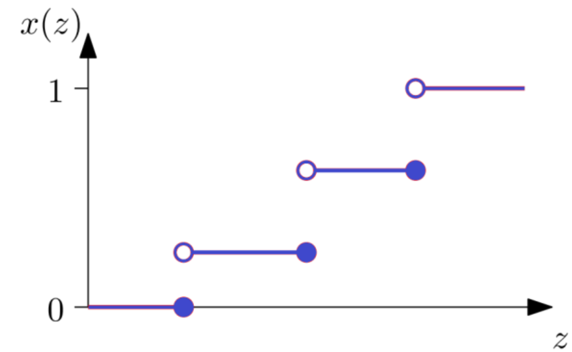
Hence there exists a jump in p so that

$$\text{jump in } p \text{ at } z = z \cdot \text{ jump in } x_i \text{ at } z$$

Myerson's Lemma: Payments

$$z \cdot (x_i(y) - x_i(z)) \leq p(y) - p(z) \leq y \cdot (x_i(y) - x_i(z))$$

Proof cont. Assume x is monotone for the rest of the proof and x is piecewise constant (simple function). if there is a jump at z (say of magnitude h) then as $y \to z$ from left we get



$$z \cdot h \leq p(y) - p(z) \leq y \cdot h.$$

Hence there exists a jump in p so that

$$\text{jump in } p \text{ at } z = z \cdot \text{ jump in } x_i \text{ at } z$$

We conclude that (given $p_i(0) = 0$)

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot \text{jump in } x_i(., b_{-i}) \text{ at } z_j,$$

where $z_1, ..., z_l$ are the breakpoints of $x_i(., b_{-i})$ in $[0, b_i]$.

Myerson's Lemma: Payments

$$z \cdot (x_i(y) - x_i(z)) \leq p(y) - p(z) \leq y \cdot (x_i(y) - x_i(z))$$

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot \text{jump in } x_i(., b_{-i}) \text{ at } z_j$$

Proof cont. Assume x is monotone and suppose that x is differentiable.

Myerson's Lemma: Payments

$$z \cdot (x_i(y) - x_i(z)) \leq p(y) - p(z) \leq y \cdot (x_i(y) - x_i(z))$$

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot \text{jump in } x_i(., b_{-i}) \text{ at } z_j$$

Proof cont. Assume x is monotone and suppose that x is differentiable.

If we devide both sides on the top inequality and let $y \to z$ we get

$$p_i'(z) \leq z \cdot x_i'(z)$$

Myerson's Lemma: Payments

$$z \cdot (x_i(y) - x_i(z)) \leq p(y) - p(z) \leq y \cdot (x_i(y) - x_i(z))$$

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot \text{jump in } x_i(., b_{-i}) \text{ at } z_j$$

Proof cont. Assume x is monotone and suppose that x is differentiable.

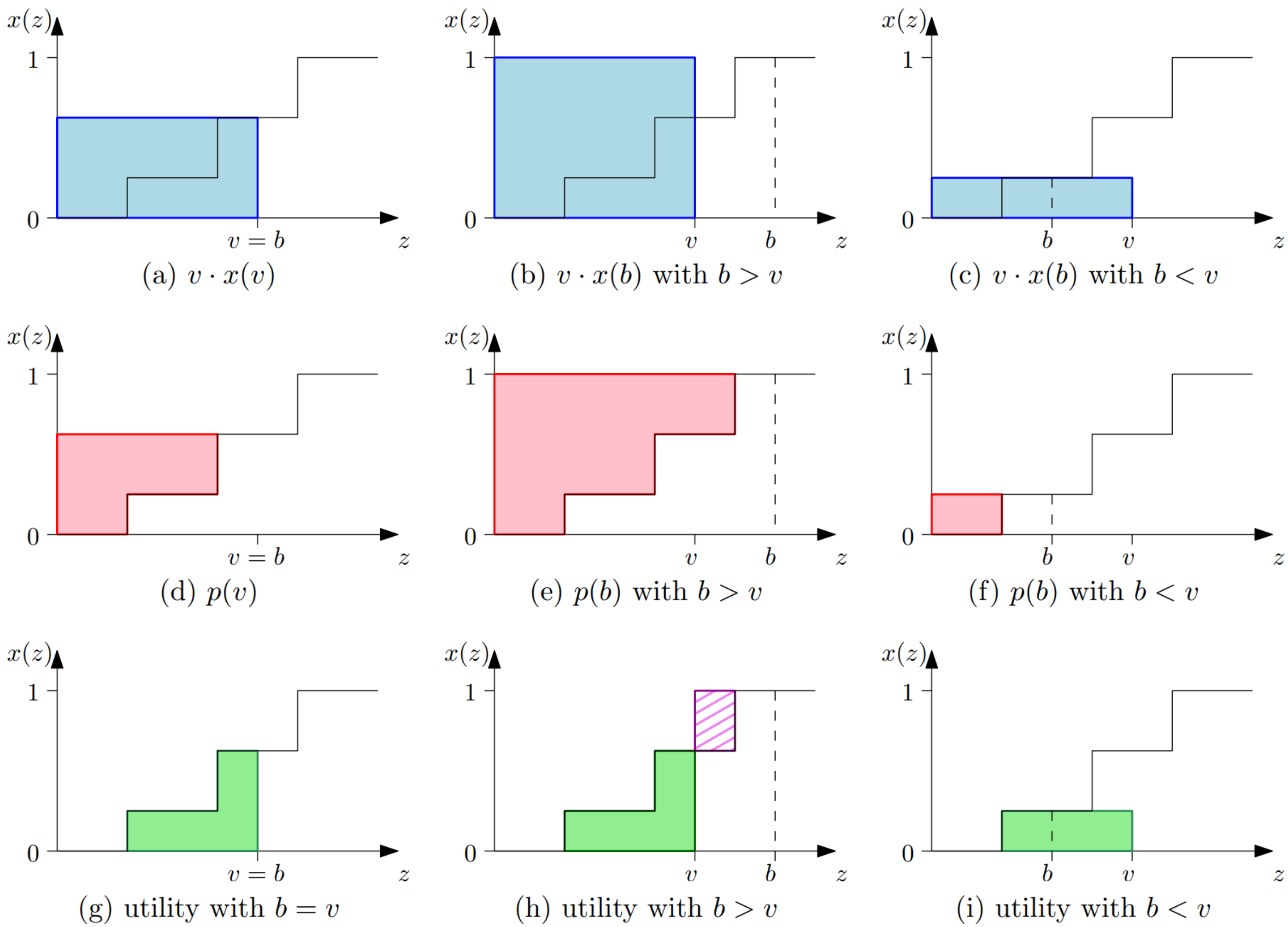
If we devide both sides on the top inequality and let $y \to z$ we get

$$p_i'(z) \leq z \cdot x_i'(z)$$

$$\Rightarrow \quad p_i(b_i, b_{-i}) = \int_0^{b_i} z \cdot \frac{dx_i(z, b_{-i})}{dz} dz.$$

Myerson's Lemma: DSIC

Proof cont. By picture.



Back to Sponsored Search Auctions

Remark: Myerson’s Lemma regenerates the Vickrey auction as a special case. (why?)

Back to Sponsored Search Auctions

Remark: Myerson's Lemma regenerates the Vickrey auction as a special case. (why?)
Answer: Fix i, b_{-i} and set $B = \max_{j \neq i} b_j$. Then $x_i(z, b_{-i})$ is 0 for $0 \leq z < B$ for and 1 for $z \geq B$. Moreover, $p_i(z, b_{-i}) = B$ for $z \geq B$ and 0 for $0 \leq z < B$.

Back to Sponsored Search Auctions

Remark: Myerson's Lemma regenerates the Vickrey auction as a special case. (why?)
Answer: Fix i, b_{-i} and set $B = \max_{j \neq i} b_j$. Then $x_i(z, b_{-i})$ is 0 for $0 \leq z < B$ for and 1 for $z \geq B$. Moreover, $p_i(z, b_{-i}) = B$ for $z \geq B$ and 0 for $0 \leq z < B$.

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we assign bidders to slots so that we can maximize surplus?
- Step 2: Given our answer to Step 1, how should we set selling prices so that DSIC holds?

Back to Sponsored Search Auctions

Remark: Myerson's Lemma regenerates the Vickrey auction as a special case. (why?) Answer: Fix i, b_{-i} and set $B = \max_{j \neq i} b_j$. Then $x_i(z, b_{-i})$ is 0 for $0 \leq z < B$ for and 1 for $z \geq B$. Moreover, $p_i(z, b_{-i}) = B$ for $z \geq B$ and 0 for $0 \leq z < B$.

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we assign bidders to slots so that we can maximize surplus?
- Step 2: Given our answer to Step 1, how should we set selling prices so that DSIC holds?

- Assign to the j-th highest bidder the j-th highest slot for $j = 1, \ldots, k$. Note that this can be done in polynomial time. Moreover, the allocation is monotone!

Back to Sponsored Search Auctions

Remark: Myerson's Lemma regenerates the Vickrey auction as a special case. (why?)
Answer: Fix i, b_{-i} and set $B = \max_{j \neq i} b_j$. Then $x_i(z, b_{-i})$ is 0 for $0 \leq z < B$ for and 1 for $z \geq B$. Moreover, $p_i(z, b_{-i}) = B$ for $z \geq B$ and 0 for $0 \leq z < B$.

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we assign bidders to slots so that we can maximize surplus?
- Step 2: Given our answer to Step 1, how should we set selling prices so that DSIC holds?

- Assign to the j-th highest bidder the j-th highest slot for $j = 1, \ldots, k$. Note that this can be done in polynomial time. Moreover, the allocation is monotone!

From Myerson's Lemma, there are payments that make the above DSIC.

Back to Sponsored Search Auctions

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we assign bidders to slots so that we can maximize surplus?
- Step 2: Given our answer to Step 1, how should we set selling prices so that DSIC holds?

- Assign to the j-th highest bidder the j-th highest slot for $j = 1, \dots, k$. Note that this can be done in polynomial time. Moreover, the allocation is monotone!

From Myerson's Lemma, there are payments that make the above DSIC.

Consider $b_1 \geq \ \dots \geq \ b_n$. Focus on first bidder (fix other bidders) and assume bid ranges from 0 to b_1. The allocation $x_1(z, b_{-1})$ ranges from 0 to a_1 with a jump at b_{j+1} of $a_j - a_{j+1}$ (when bidder 1 becomes j-th highest effectively).

Back to Sponsored Search Auctions

Approach:

- Assign to the j-th highest bidder the j-th highest slot for $j = 1, \dots, k$. Note that this can be done in polynomial time. Moreover, the allocation is monotone!

From Myerson's Lemma, there are payments that make the above DSIC.

Consider $b_1 \geq \dots \geq b_n$. Focus on first bidder (fix other bidders) and assume bid ranges from 0 to b_1. The allocation $x_1(z, b_{-1})$ ranges from 0 to a_1 with a jump at b_{j+1} of $a_j - a_{j+1}$ (when bidder 1 becomes j-th highest effectively). Hence for the i-th highest bidder we get the payment

$$\boxed{p_i(b) = \sum_{j=i}^{k} b_{j+1}(a_j - a_{j+1})}$$