# L16 – Week 8
# Introduction to Statistical Learning Theory: VC dimension and Learnability

CS 295 Optimization for Machine Learning

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# A motivating example

Recap:

- We saw that the hypothesis classes of finite cardinality are PAC learnable using Chernoff Bounds and Union Bound. What if the class is not finite?

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$$\mathcal{H} = \{h_a : a \in \mathbb{R}\}$$

*where* $h_a(x) = \mathbf{1}_{x<a}$. $\mathcal{H}$ *is PAC learnable using ERM algorithm (even if the class is infinite).*

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Remarks:

- Therefore it is not necessary that the hypothesis class is of finite cardinality.
- We will show the lemma above, i.e., $(\epsilon, \delta)$-learnable using $\frac{\log\frac{2}{\delta}}{\epsilon}$ samples.

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*Proof.* Let $D$ be the marginal distribution over the domain and fix $\epsilon, \delta$. We need to show that taking $S$ samples IID of size $\frac{\log(2/\delta)}{\epsilon}$ suffices so that with probability $1-\delta$ the generalization error is at most $\epsilon$.

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Let $a^*$ be a number such that $h_{a^*}$ has error zero (perfect fit).

Moreover, consider $a_0 < a^* < a_1$ such that

$$\Pr_{x \sim D}[x \in (a_0, a^*)] = \Pr_{x \sim D}[x \in (a^*, a_1)] = \epsilon.$$

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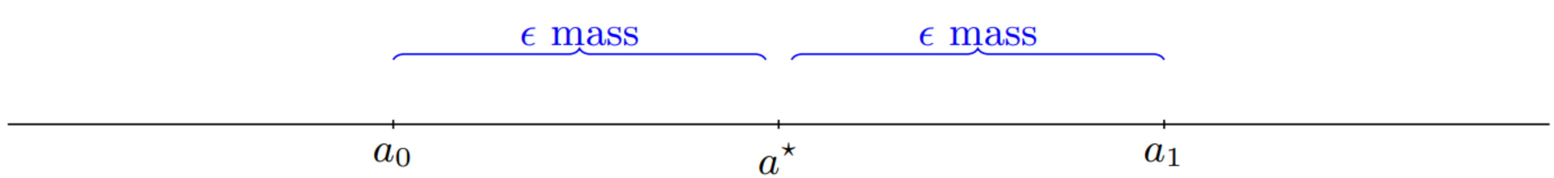


Observe that we might have to choose $a_0 = -\infty$ or $a_1 = +\infty$.

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*Proof cont.* Let $S$ be a set of IID samples and assume that the ERM algorithm returns a function $h_S$ with threshold $b_S$.

If $b_0$ is the maximum $x$ with label 1 and $b_1$ the minimum $x$ with label 0 it holds that

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By union bound we have

$$\Pr_{S\sim D^m}[(b_0 < a_0) \cup (b_1 > a_1)] \leq \Pr_{S\sim D^m}[(b_0 < a_0)] + \Pr_{S\sim D^m}[(b_1 > a_1)].$$

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*Proof cont.*

$$\Pr_{S\sim D^m}[(b_0 < a_0)] \leq \Pr_{S}[\forall x \in S, x \notin (a_0, a^*)] = (1-\epsilon)^m \leq e^{-\epsilon m}$$

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$$\boxed{m = \frac{\log(2/\delta)}{\epsilon}.}$$

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**All hypothesis classes are learnable then? Not really**

# VC dimension

**Definition** (Restriction). *Let $\mathcal{H}$ be a class of functions from $\mathcal{X}$ to $\{0,1\}$ and let $C = \{c_1, ..., c_m\}$. The restriction of $\mathcal{H}$ to $C$ is the set of functions from $C$ to $\{0,1\}$ that can be derived from $\mathcal{H}$. That is*

$$\mathcal{H}_C = \{h(c_1), ..., h(c_m)) : h \in \mathcal{H}\},$$

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**Definition** (VC dimension). *The VC-dimension hypothesis class $\mathcal{H}$, denoted $VCdim(\mathcal{H})$, is the maximal size of a set $C$ that can be shattered by $\mathcal{H}$. If $\mathcal{H}$ can shatter sets of arbitrarily large size we say that $\mathcal{H}$ has infinite VC-dimension.*

# Examples

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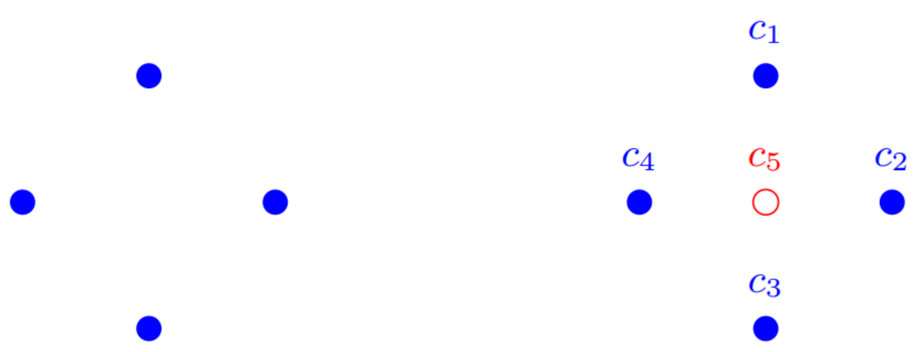


**Figure 6.1** Left: 4 points that are shattered by axis aligned rectangles. Right: Any axis aligned rectangle cannot label $c_5$ by 0 and the rest of the points by 1.

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**Figure 6.1** Left: 4 points that are shattered by axis aligned rectangles. Right: Any axis aligned rectangle cannot label $c_5$ by 0 and the rest of the points by 1.

- Any finite class H has VC dimension at most $\log|H|$. Why?

# VC dimension of halfspaces

**Theorem** (Halfspaces)**.** *The VC dimension of the class $\mathcal{H}$ of homogenous halfspaces in $\mathbb{R}^d$ is $d$. Note that $\mathcal{H} = \{h_w(x) : h_w(x) := sign(w^\top x)\}$.*

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This set is shattered by the class of homogenous halfspaces because for every binary vector $y_1, ..., y_d$, and for $w = (y_1, ..., y_d)$, we get that $h_w(e_i) = y_i$.

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We need now to show that VC dimension is less than $d+1$. Let $x_1, ..., x_{d+1}$ be a set of $d+1$ vectors in $\mathbb{R}^d$.

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*Proof cont.* Then, there must exist real numbers $a_1, ..., a_{d+1}$, not all of them are zero, such that

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$$\text{Let } I = \{i : a_i > 0\} \text{ and } J = \{j : a_j < 0\}.$$

If both $I, J$ are non-empty then

$$\sum_{i \in I} a_i x_i = \sum_{j \in J} |a_j| x_j.$$

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**Contradiction!**

# Example of infinite VC

**Theorem** (sin has infinite VC). *Consider the real line and let*

$$\mathcal{H} = \{x \to \lceil \sin(\theta x) \rceil : \theta \in \mathbb{R}\}.$$

*The VC dimension of the hypothesis class above is infinite.*

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Fix $d$ and consider $C = \{1/2, 1/4, ..., 1/2^d\}$ and moveover choose any binary vector of labels $(y_1, ..., y_d)$. Set $x = 0.y_1 ... y_d 1$ and use the above.

# Why do we care about VC?

**Theorem** (Fundamental Theorem of Learnability)**.** *The following are equivalent:*

- *$\mathcal{H}$ is PAC learnable.*
- *Any ERM rule is a successful PAC learner for $\mathcal{H}$.*
- *$\mathcal{H}$ has finite VC dimension.*

Remarks:

- The number of samples needed is $O\left(\frac{d\log\frac{1}{\epsilon}+\log\frac{1}{\delta}}{\epsilon}\right)$ where $d$ is the VC dimension of the hypothesis class.

# Conclusion

- Introduction to Statistical Learning.
  - VC dimension.
  - Examples.
  - Fundamental theorem of Learnability
- Last lecture we be about Stochastic Games and Multi-agent RL.