



## Lecture 13

More problems on Maxflow, Maximum Matching, Baseball elimination, Vertex cover

CS 161 Design and Analysis of Algorithms

Ioannis Panageas

# Flow of a Network (Recap)

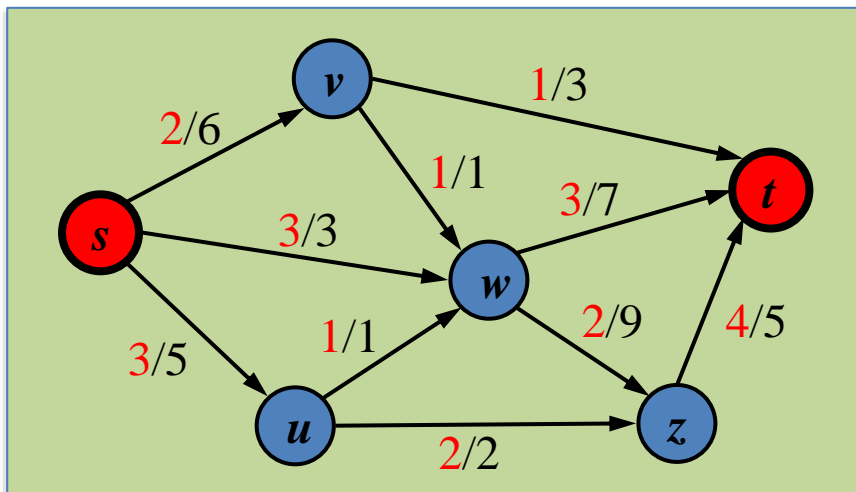
**Definition:** Function  $f : E \rightarrow \mathbb{N}$  from edges to non-negative integers so that for each edge  $e$  it holds

$$0 \leq f(e) \leq c(e) \quad \text{Capacity constraint}$$

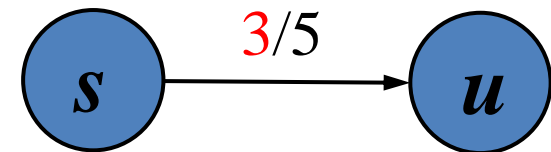
$$\sum_{e \in \text{outgoing}(u)} f(e) = \sum_{e \in \text{incoming}(u)} f(e) \quad \text{Conservation rule}$$

for all  $u \neq s, t$

**Example:**



**Capacity constraint:**



$$0 \leq 3 \leq 5$$

# Flow of a Network (Recap)

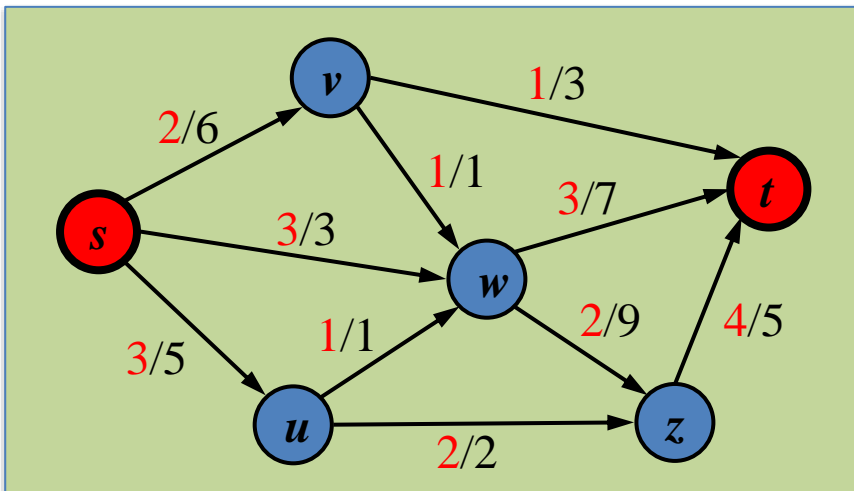
**Definition:** Function  $f : E \rightarrow \mathbb{N}$  from edges to non-negative integers so that for each edge  $e$  it holds

$$0 \leq f(e) \leq c(e) \quad \text{Capacity constraint}$$

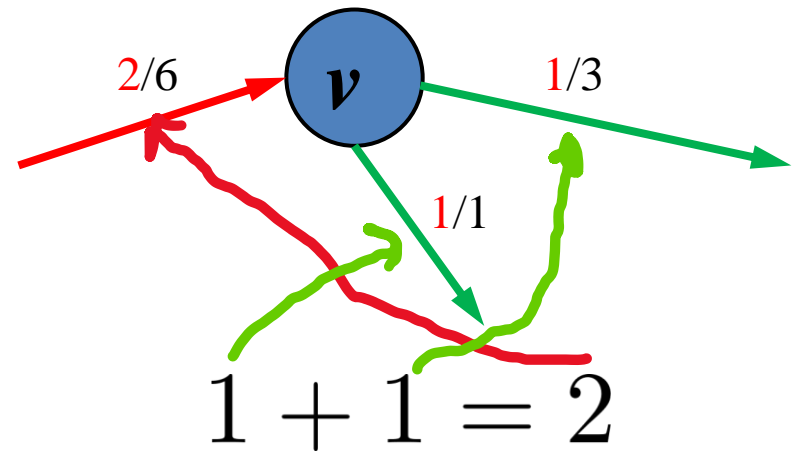
$$\sum_{e \in \text{outgoing}(u)} f(e) = \sum_{e \in \text{incoming}(u)} f(e) \quad \text{Conservation rule}$$

for all  $u \neq s, t$

**Example:**



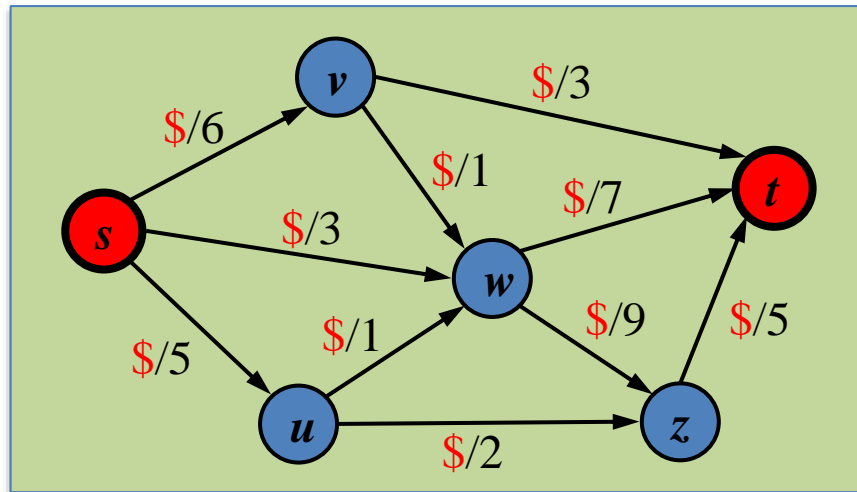
**Conservation rule:**



# Maxflow Problem (Recap)

**Problem:** Given a network  $G$ , a source  $s$  and a sink  $t$ , and capacities on the edges, compute the **maximum** possible **flow value**  $|f^*|$ .

**Example:**



Find the \$ to get maxflow  $|f^*|$

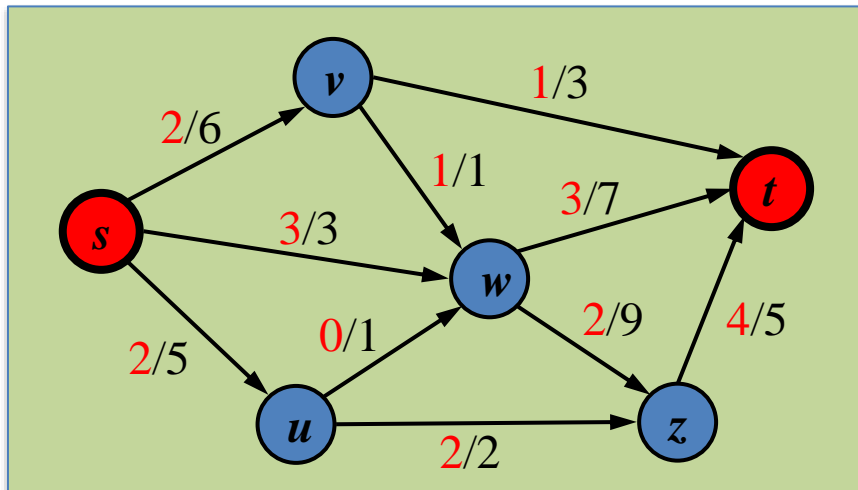
# Augmenting paths (Recap)

We are given a network  $G$  with edge capacities  $c$  and a flow  $f$ .  
Let  $(u, v)$  be an edge from  $u$  to  $v$ .

**Residual capacity** from  $u$  to  $v$  is  $\Delta_f(u, v) = c(u, v) - f(u, v)$ .

**Residual capacity** from  $v$  to  $u$  is  $\Delta_f(v, u) = f(u, v)$ .

Imply the **residual graph**.



**Augmenting path:** Path from  $s$  to  $t$  with **positive residual** capacities.

$s \rightarrow v \rightarrow t$  augmenting path

$s \rightarrow u \rightarrow w \rightarrow v \rightarrow t$  augmenting path

$s \rightarrow u \rightarrow z \rightarrow t$  is **not**

# The Ford-Fulkerson Algorithm

**Main idea:** Repeatedly search for an augmenting path  $\pi$ :

- If there is **an augmenting path**, augment flow with  $\Delta_f(\pi)$  (minimum residual capacity among the edges of  $\pi$ ) along the edges of  $\pi$ .
- If there is **no augmenting path**, **terminate**.

**Remark:** You can use **DFS** (or **BFS**) to search for an augmenting path.

**Running time:**

Time to search for an augmenting path  $\times$  number of updates.

$$\Theta(|V| + |E|) \cdot |f^*|$$

Running time of DFS or BFS

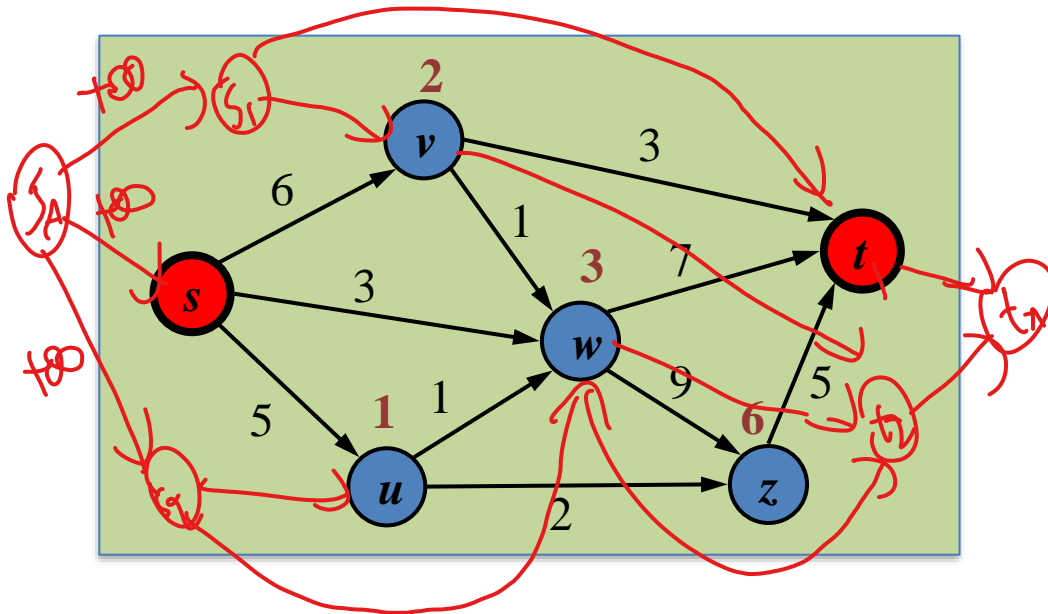
Updates increase flow by 1 unit only

# Case study I: Capacities on the vertices

**Problem:** Given a network  $G$ , a source  $s$  and a sink  $t$ , and capacities on the edges **and vertices**, compute the **maximum** possible **flow** value  $|f^*|$ .

$$\sum_{e \in \text{outgoing}(u)} f(e) = \sum_{e \in \text{incoming}(u)} f(e) \leq c(u)$$

**Example:**



$$c(u) = 1$$

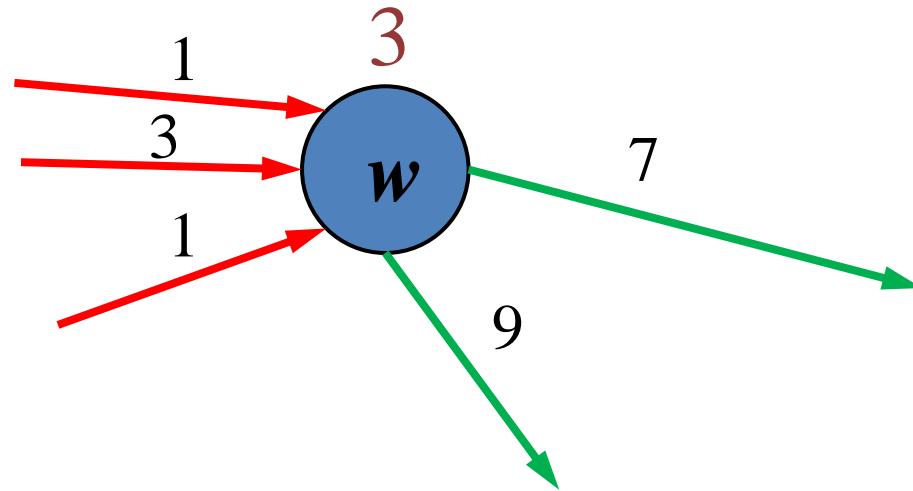
$$c(v) = 2$$

$$c(w) = 3$$

$$c(z) = 6$$

# Case study I: Capacities on the vertices

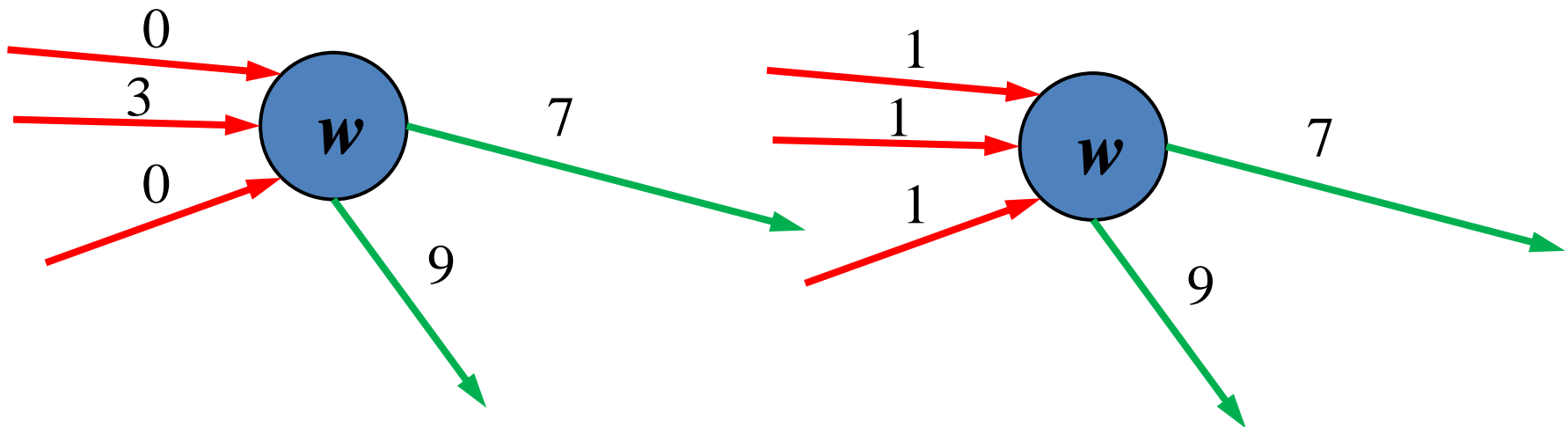
**Main idea:** **Reduce** it to classic Maxflow problem. To do that, we need to remove the capacity constraints on vertices. How?





# Case study I: Capacities on the vertices

**Main idea:** **Reduce** it to classic Maxflow problem. To do that, we need to remove the capacity constraints on vertices. How?

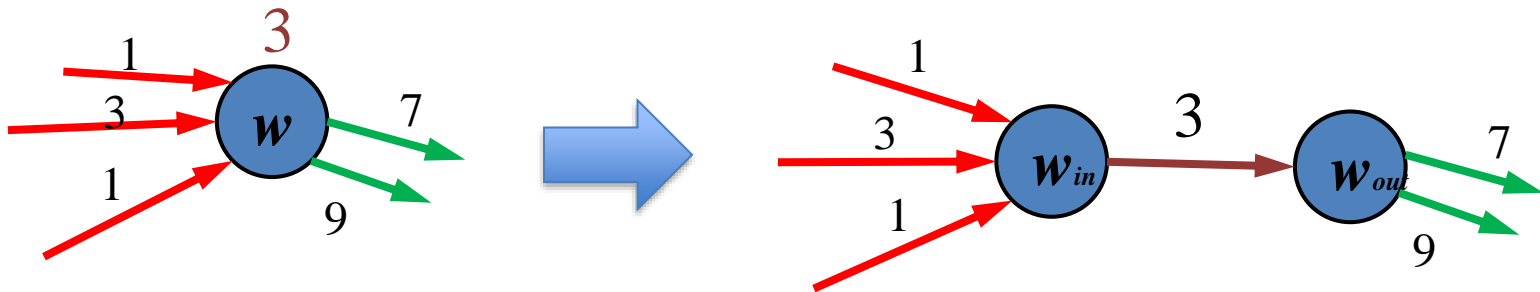


**Slow approach:** Consider all possible capacities on incoming edges. Run Maxflow in all possible flow networks.

**Exponentially many.**

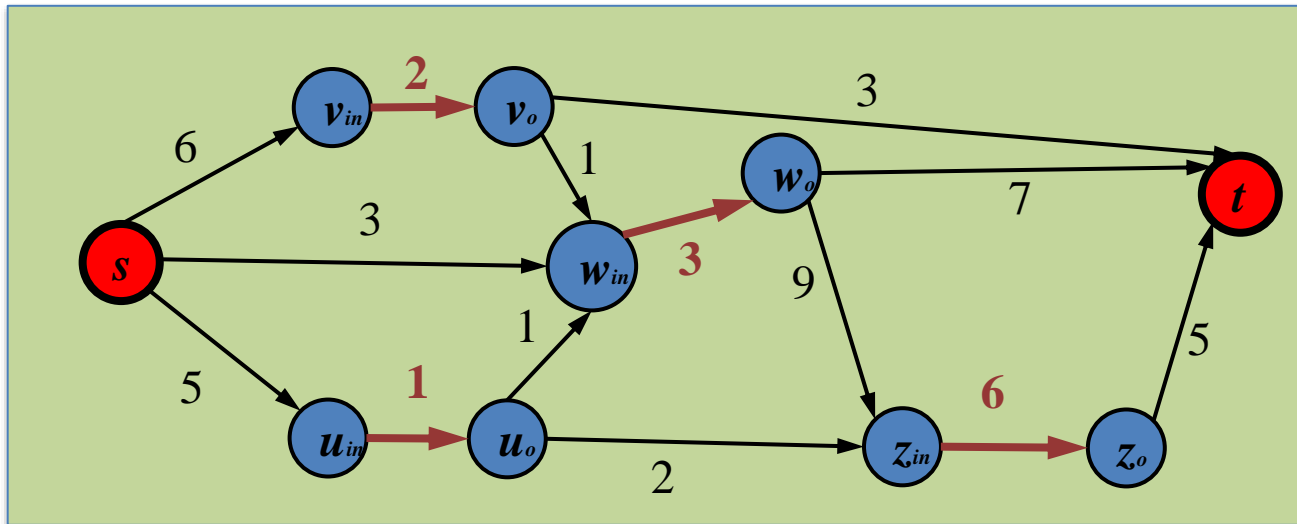
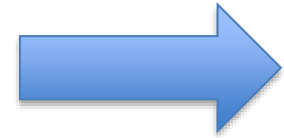
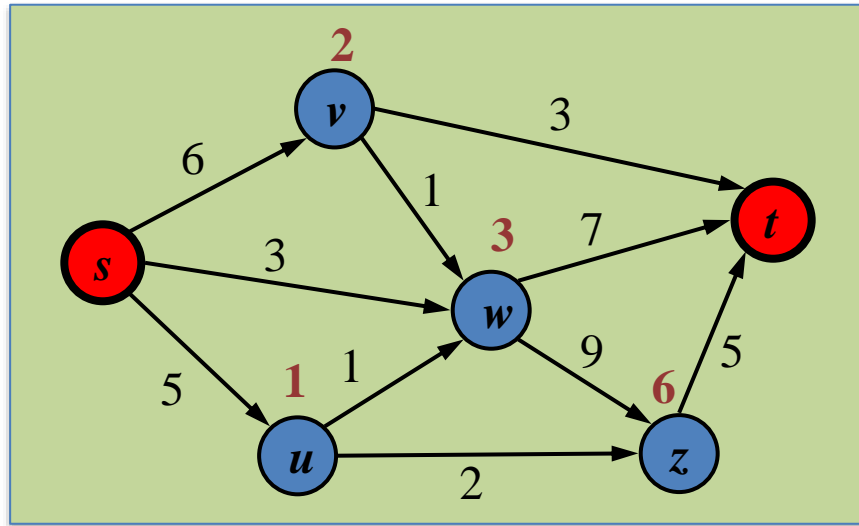
# Case study I: Capacities on the vertices

**Main idea:** **Reduce** it to classic Maxflow problem. To do that, we need to remove the capacity constraints on vertices. How?



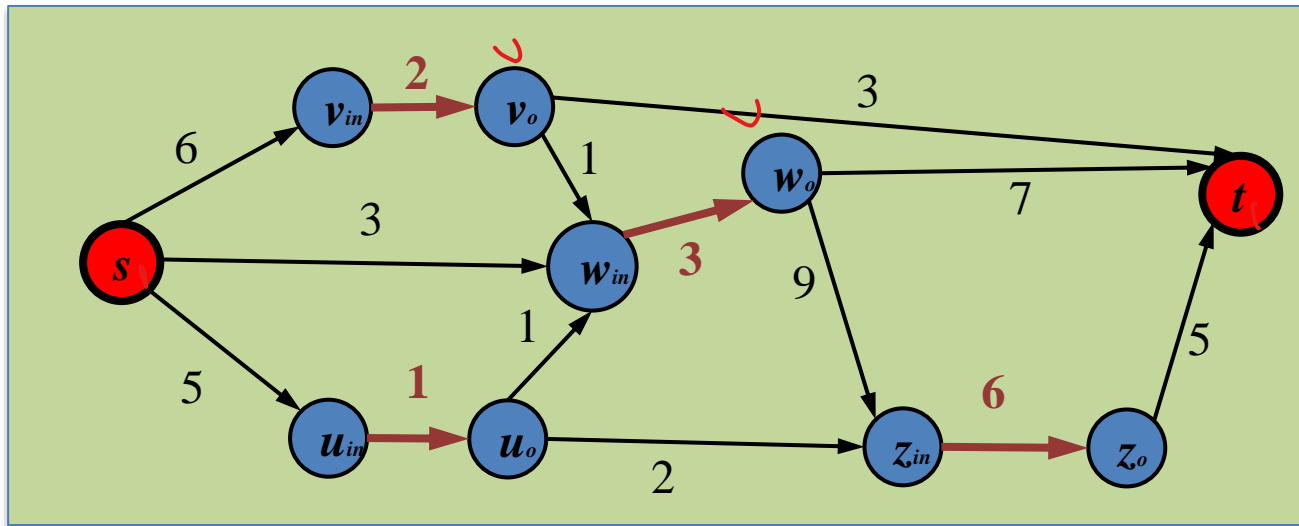
**Fast approach:** Transform the capacity on vertices to capacities on edges. For every vertex  $w$ , create two new vertices  $w_{in}$ ,  $w_{out}$  and connect all incoming edges of  $w$  to  $w_{in}$  and all outgoing edges to  $w_{out}$ .

# Case study I: Capacities on the vertices



# Case study I: Capacities on the vertices

**Running time:**  $\Theta(V + E)$  to create the **new graph**  
+ Ford-Fulkerson (or Edmonds-Karp).



**Another Question:** What if you have **multiple** sources and/or sinks?

# Case study II: Baseball Elimination

**Problem:** Let  $T$  be a set of teams in baseball league. At any point during the season, each team,  $i$  will have  $w_i$  wins, and will have  $g_i$  games left to play. The task is to determine whether it is possible for team  $i$  to finish the season in first place, given the games it has already won and the games it has left to play. Note that this depends on the games left for team  $i$  but also depends on the respective schedules of the other teams.

**Example:**

Team $i$	Wins $w_i$	Games Left $g_i$	Schedule ( $g_{i,j}$ )			
			LA	Oak	Sea	Tex
Los Angeles	81	8	-	1	6	1
Oakland	77	4	1	-	0	3
Seattle	76	7	6	0	-	1
Texas	74	5	1	3	1	-

# Case study II: Baseball Elimination

**Problem:** Let  $T$  be a set of teams in baseball league. At any point during the season, each team,  $i$  will have  $w_i$  wins, and will have  $g_i$  games left to play. The task is to determine whether it is possible for team  $i$  to finish the season in first place, given the games it has already won and the games it has left to play. Note that this depends on the games left for team  $i$  but also depends on the respective schedules of the other teams.

**Example:**

Team $i$	Wins $w_i$	Games Left $g_i$	Schedule ( $g_{i,j}$ )			
			LA	Oak	Sea	Tex
Los Angeles	81	8	-	1	6	1
Oakland	77	4	1	-	0	3
Seattle	76	7	6	0	-	1
Texas	74	5	1	3	1	-

**Question:** Can Texas finish first? What about Oakland?

# Case study II: Baseball Elimination

Example:

Team $i$	Wins $w_i$	Games Left $g_i$	Schedule ( $g_{i,j}$ )			
			LA	Oak	Sea	Tex
Los Angeles	81	8	-	1	6	1
Oakland	77	4	1	-	0	3
Seattle	76	7	6	0	-	1
Texas	74	5	1	3	1	-

Question: Can Texas finish first? What about Oakland?

Answer: For Texas the answer is NO. It can win at most 79 wins.  
For Oakland is also NO. Either LA or Seattle will definitely reach 82 wins.

# Case study II: Baseball Elimination

Example:

Team $i$	Wins $w_i$	Games Left $g_i$	Schedule ( $g_{i,j}$ )			
			LA	Oak	Sea	Tex
Los Angeles	81	8	-	1	6	1
Oakland	77	4	1	-	0	3
Seattle	76	7	6	0	-	1
Texas	74	5	1	3	1	-

Focus on team  $k$ .

Maximum possible number of wins is  $W = w_k + g_k$ .



# Case study II: Baseball Elimination

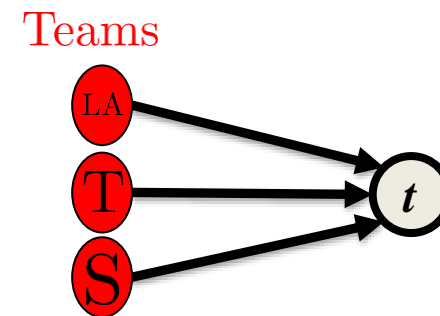
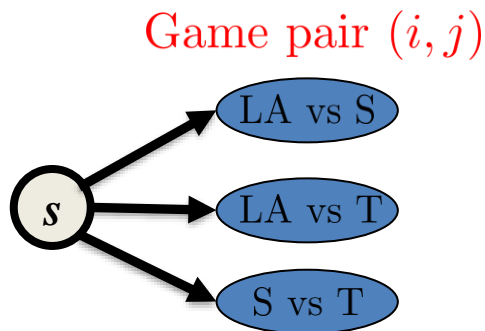
Example:

Team $i$	Wins $w_i$	Games Left $g_i$	Schedule ( $g_{i,j}$ )			
			LA	Oak	Sea	Tex
Los Angeles	81	8	-	1	6	1
Oakland	77	4	1	-	0	3
Seattle	76	7	6	0	-	1
Texas	74	5	1	3	1	-

Focus on team  $k$ .

Maximum possible number of wins is  $W = w_k + g_k$ .

If there exists a team  $i$  so that  $w_i > W$  then  $k$  is eliminated.



# Case study II: Baseball Elimination

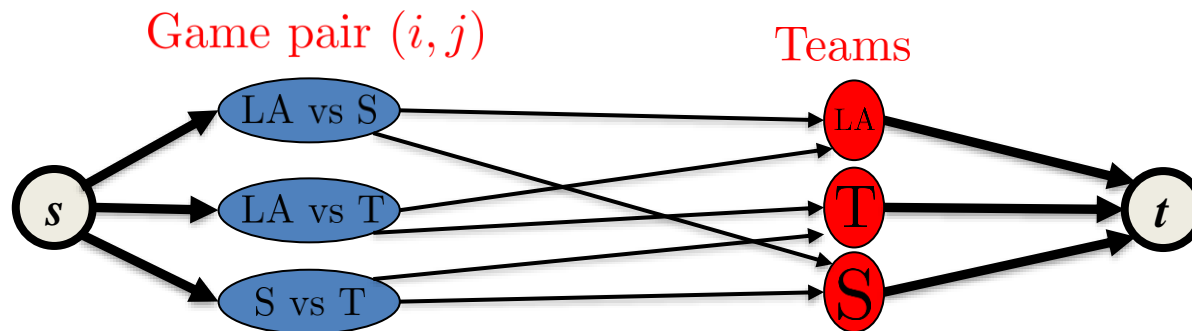
Example:

Team $i$	Wins $w_i$	Games Left $g_i$	Schedule ( $g_{i,j}$ )			
			LA	Oak	Sea	Tex
Los Angeles	81	8	-	1	6	1
Oakland	77	4	1	-	0	3
Seattle	76	7	6	0	-	1
Texas	74	5	1	3	1	-

Focus on team  $k$ .

Maximum possible number of wins is  $W = w_k + g_k$ .

If there exists a team  $i$  so that  $w_i > W$  then  $k$  is eliminated.



# Case study II: Baseball Elimination

Example:

Team $i$	Wins $w_i$	Games Left $g_i$	Schedule ( $g_{i,j}$ )			
			LA	Oak	Sea	Tex
Los Angeles	81	8	-	1	6	1
Oakland	77	4	1	-	0	3
Seattle	76	7	6	0	-	1
Texas	74	5	1	3	1	-

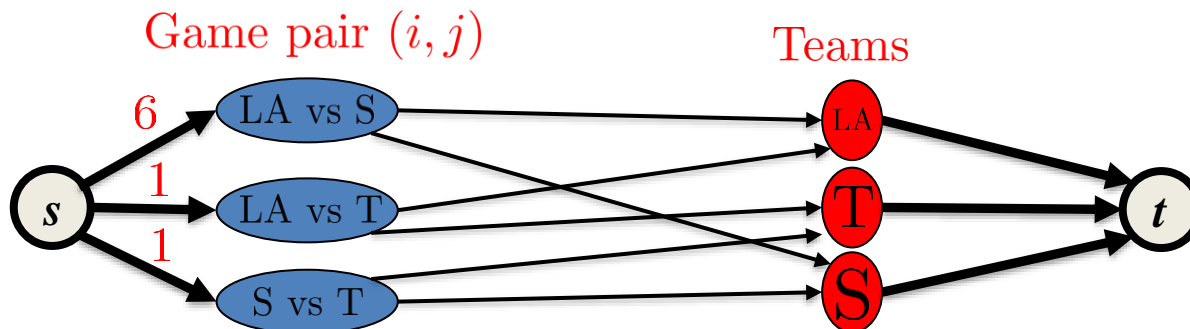
Focus on team  $k$ .

Maximum possible number of wins is  $W = w_k + g_k$ .

If there exists a team  $i$  so that  $w_i > W$  then  $k$  is eliminated.

$k = \text{Oakland}$ .

$W = 81$ .



# Case study II: Baseball Elimination

Example:

Team $i$	Wins $w_i$	Games Left $g_i$	Schedule ( $g_{i,j}$ )			
			LA	Oak	Sea	Tex
Los Angeles	81	8	-	1	6	1
Oakland	77	4	1	-	0	3
Seattle	76	7	6	0	-	1
Texas	74	5	1	3	1	-

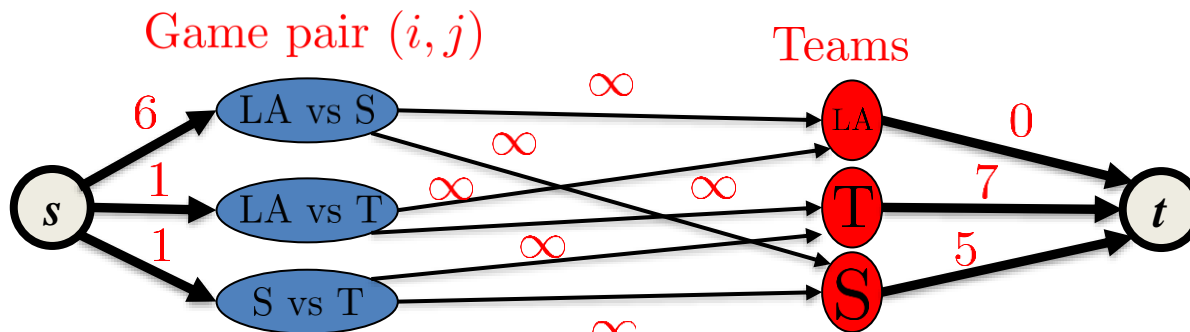
Focus on team  $k$ .

Maximum possible number of wins is  $W = w_k + g_k$ .

If there exists a team  $i$  so that  $w_i > W$  then  $k$  is eliminated.

$k = \text{Oakland}$ .

$W = 81$ .



# Case study II: Baseball Elimination

Example:

Team $i$	Wins $w_i$	Games Left $g_i$	Schedule ( $g_{i,j}$ )			
			LA	Oak	Sea	Tex
Los Angeles	81	8	-	1	6	1
Oakland	77	4	1	-	0	3
Seattle	76	7	6	0	-	1
Texas	74	5	1	3	1	-

Focus on team  $k$ .

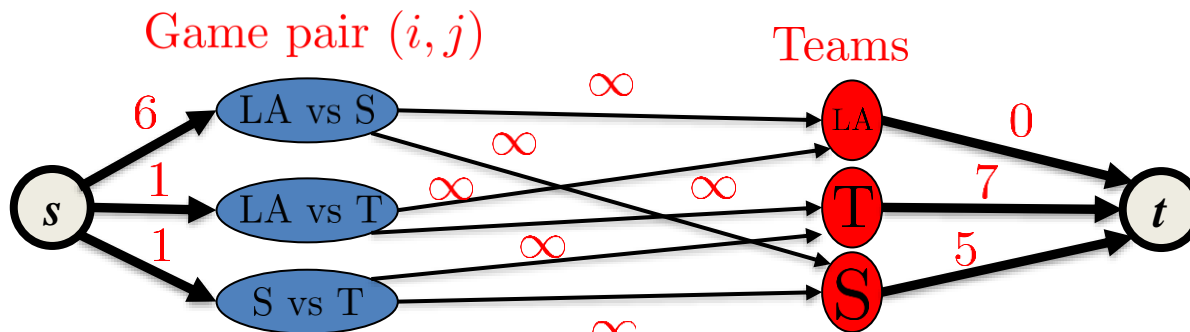
For team  $i$ , capacity is  $W - w_i$

Maximum possible number of wins is  $W = w_k + g_k$ .

If there exists a team  $i$  so that  $w_i > W$  then  $k$  is eliminated.

$k = \text{Oakland}$ .

$W = 81$ .



# Case study II: Baseball Elimination

**Intuition:** Suppose  $k$  wins all its games ( $W$  wins in total). All possible flows show how the rest of the games will proceed. Need to check if all games can be played **without** any other team **exceeding**  $W$  wins.

# Case study II: Baseball Elimination

**Intuition:** Suppose  $k$  wins all its games ( $W$  wins in total). All possible flows show how the rest of the games will proceed. Need to check if all games can be played **without** any other team **exceeding**  $W$  wins.

**Maxflow** in graph  
= # of remaining games (without  $k$ )  
 $k$  can finish first.

# Case study III: Vertex Cover on Bipartite Graphs

**Problem (Practice):** We use the term line of a matrix to mean either a row or a column. Given an  $n \times n$  matrix of **0**'s and **1**'s, your task is to find a **smallest number of lines** such that every **1** entry of the matrix is **contained in one of the selected lines**.

**Example:**

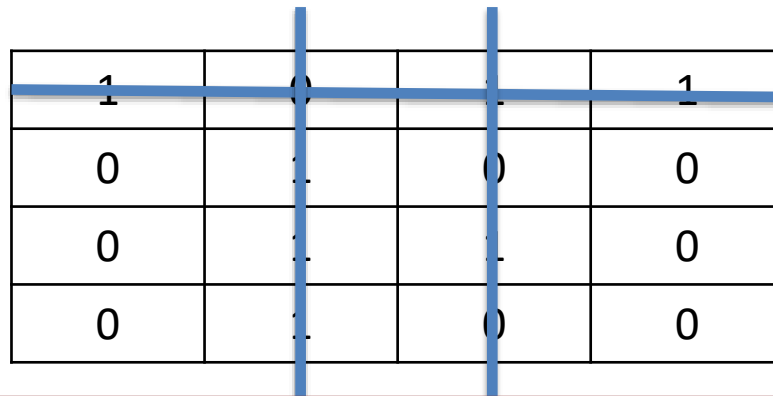
1	0	1	1
0	1	0	0
0	1	1	0
0	1	0	0



# Case study III: Vertex Cover on Bipartite Graphs

**Problem (Practice):** We use the term line of a matrix to mean either a row or a column. Given an  $n \times n$  matrix of **0**'s and **1**'s, your task is to find a **smallest number of lines** such that every **1** entry of the matrix is **contained in one of the selected lines**.

**Example:**



1	0	0	1
0	1	0	0
0	1	1	0
0	1	0	0

Three lines. Can you do it with two?

# Case study III: Vertex Cover on Bipartite Graphs

**Idea:** Create a bipartite graph.

1	0	1	1
0	1	0	0
0	1	1	0
0	1	0	0

Row 1 ●

Row 2 ●

Row 3 ●

Row 4 ●

● Col 1

● Col 2

● Col 3

● Col 4

# Case study III: Vertex Cover on Bipartite Graphs

**Idea:** Create a bipartite graph.

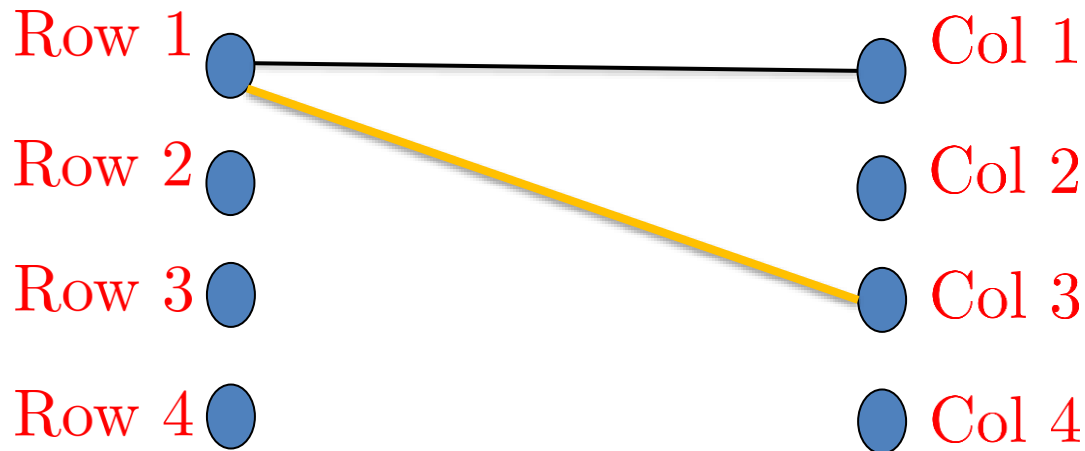
1	0	1	1
0	1	0	0
0	1	1	0
0	1	0	0



# Case study III: Vertex Cover on Bipartite Graphs

**Idea:** Create a bipartite graph.

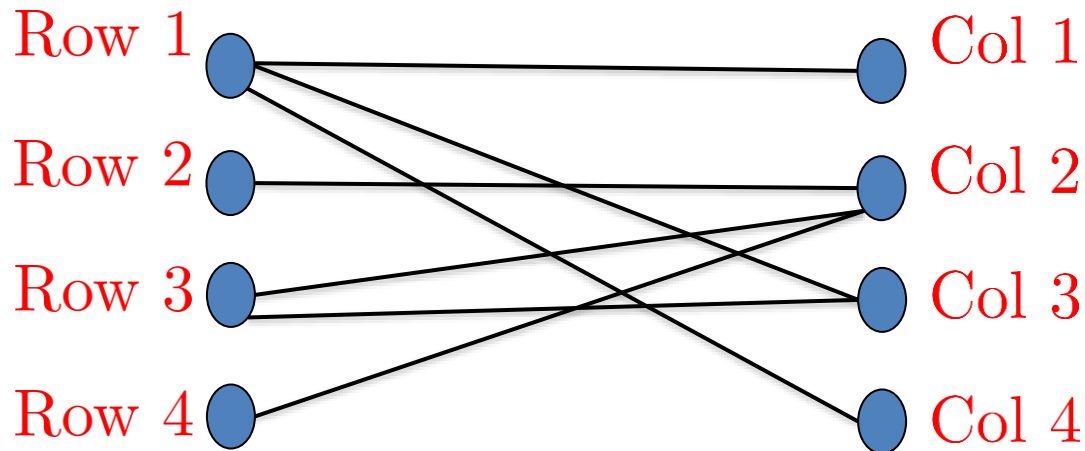
1	0	1	1
0	1	0	0
0	1	1	0
0	1	0	0



# Case study III: Vertex Cover on Bipartite Graphs

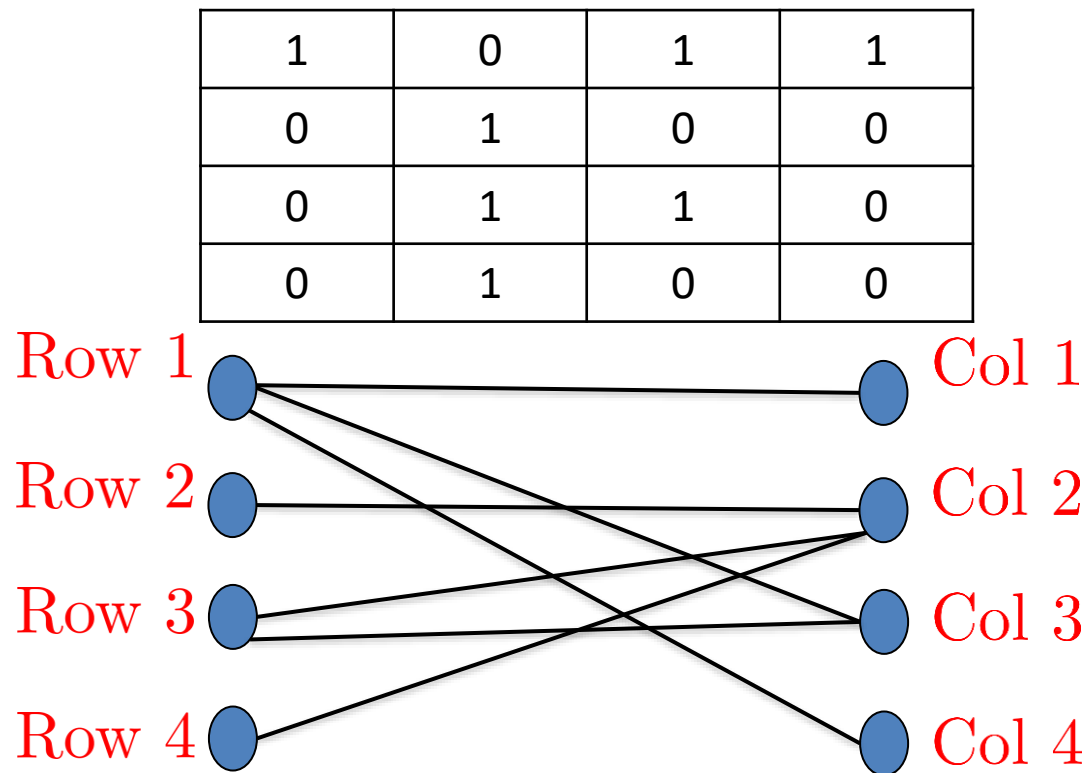
**Idea:** Create a bipartite graph.

1	0	1	1
0	1	0	0
0	1	1	0
0	1	0	0



# Case study III: Vertex Cover on Bipartite Graphs

Idea 2: Choose min number of vertices to cover all the edges  
(minimum vertex cover)

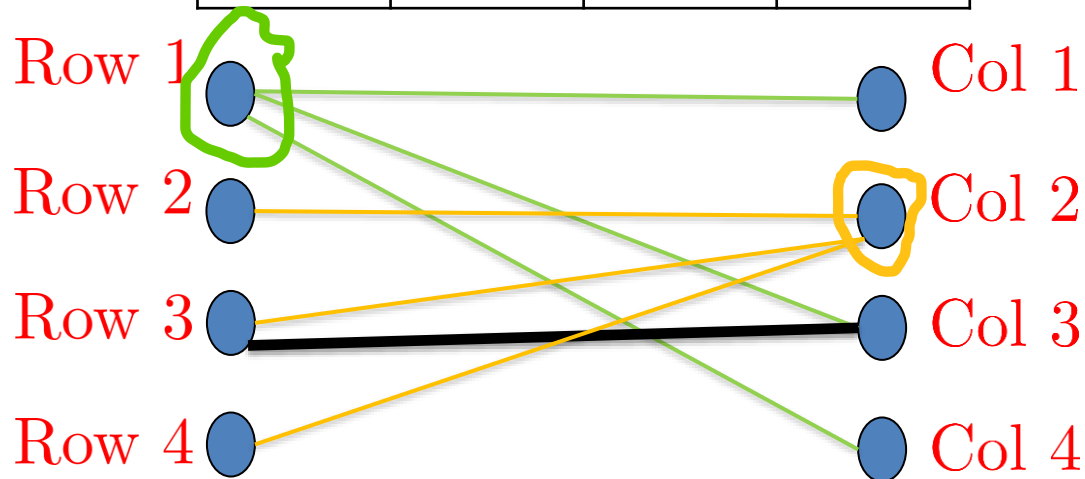


# Case study III: Vertex Cover on Bipartite Graphs

Idea 2: Choose min number of vertices to cover all the edges  
(minimum vertex cover)

Not Vertex Cover

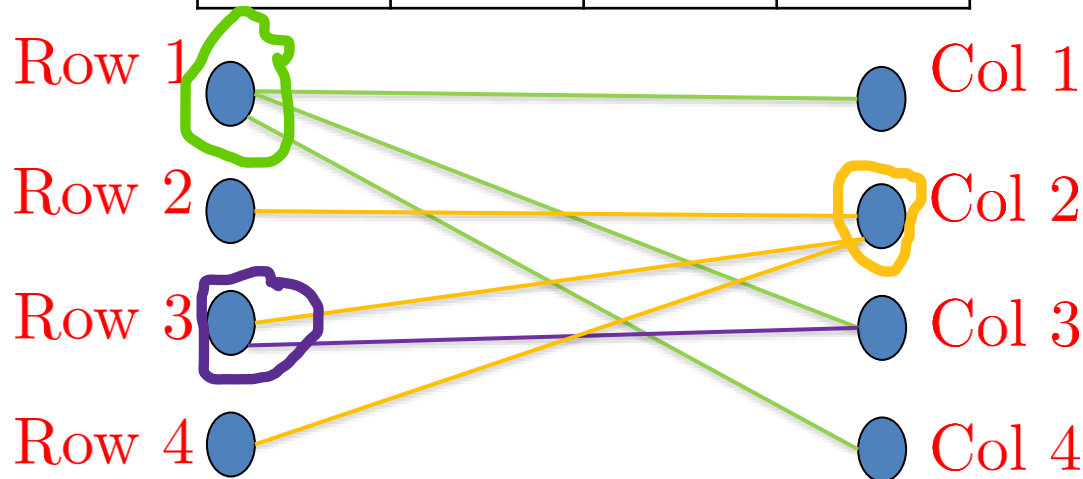
1	0	1	1
0	1	0	0
0	1	1	0
0	1	0	0



# Case study III: Vertex Cover on Bipartite Graphs

Idea 2: Choose min number of vertices to cover all the edges  
(minimum vertex cover)

1	0	1	1
0	1	0	0
0	1	1	0
0	1	0	0





# Case study III: Vertex Cover on Bipartite Graphs

**Idea 3:** Find a maximum Matching. Theorem (Konig): This is equal to min VC in bipartite graphs.

1	0	1	1
0	1	0	0
0	1	1	0
0	1	0	0

