L04
Online Learning and a proof of Minimax Theorem

CS 295 Introduction to Algorithmic Game Theory

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Playing the experts game

Definition. *For each day $t = 1...T$, you have to choose between alternatives A, B (e.g., rain or not rain).*

- *Choose A or B according to some rule.*
- *One of the alternatives realizes.*
- *If you choose correctly you are not penalized otherwise you lose one point.*
- *Imagine that there are n experts who on each day t, recommend either A or B.*

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Perform close to best expert!

Playing the experts game

Algorithm (Weighted Majority). *We define the following algorithm:*

1. Initialize $w_i^0 = 1$ for all $i \in [n]$.
2. **For** t=1 … T **do**
3. **If** $\sum_{i \text{ choose } A} w_i^{t-1} \geq \sum_{i \text{ choose } B} w_i^{t-1}$
4. **Choose** A, **otherwise** B.
5. **End If**
6. **For** expert i that made a mistake **do**
7. $w_i^t = (1-\epsilon) w_i^{t-1}$.
8. **End For**
9. **For** expert i that did not make a mistake **do**
10. $w_i^t = w_i^{t-1}$.
11. **End For**
12. **End For**

Remarks:

- ϵ is the stepsize (to be chosen later).
- Performs almost as good as ``best” expert (fewest mistakes)

Playing the experts game

Theorem (Weighted Majority)**.** *Let M_T, M_T^B be the total number of mistakes the algorithm and best expert make until step T, respectively. It holds that*

$$M_T \leq 2(1+\epsilon)M_T^B + \frac{\log n}{\epsilon}.$$

Proof. Let's define the potential function $\phi_t = \sum_i w_i^t$.

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- $\phi_{t+1} \leq \phi_t$ (why?).

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- $\phi_0 = n$.
- $\phi_{t+1} \leq \phi_t$ (why?).

Observe that if we make a mistake at time t then the majority was wrong, that is at least $\frac{\phi_t}{2}$ will be multiplied by $(1-\epsilon)$.

Hence, if we make a mistake then $\phi_{t+1} \leq (1-\epsilon)\frac{\phi_t}{2} + \frac{\phi_t}{2} = (1-\frac{\epsilon}{2})\phi_t$

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Proof. Let's

- ϕ_0
- ϕ_t

That is $\phi_{t+1} \leq (1 - \frac{\epsilon}{2})\phi_t$ when we do a mistake, otherwise just $\phi_{t+1} \leq \phi_t$. Since we have M_T mistakes, then

$$\phi_T \leq \left(1 - \frac{\epsilon}{2}\right)^{M_T} \phi_1.$$

Observe that … that is at least $\frac{\phi_t}{2}$ will be multiplied by $(1-\epsilon)$.

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Playing the experts game

Proof cont. Moreover, assuming the best expert (say i^*) did M_T^B mistakes, we have

$$\phi_T > w_{i^*}^T = (1-\epsilon)^{M_T^B}.$$

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We conclude that

$$(1-\epsilon)^{M_T^B} < \left(1 - \frac{\epsilon}{2}\right)^{M_T} n.$$

By taking the log, $M_T^B \log(1-\epsilon) < \log(1-\epsilon/2)M_T + \log n$.

Playing the experts game

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By taking the log, $M_T^B \log(1-\epsilon) < \log(1-\epsilon/2) M_T + \log n$.

Since $-x - x^2 < \log(1-x) < -x$, $\boxed{M_T^B(-\epsilon - \epsilon^2) < -M_T\epsilon/2 + \log n.}$

The general setting

Definition. *At each time step $t = 1 \ldots T$.*

- *Player chooses $x_t \in \Delta_n$.*
- *Adversary chooses $u_t \in [-1, 1]^n$.*
- *Player gets payoff $x_t^\top u_t$ and observes u_t.*

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Player's goal is to minimize the (time average) Regret, that is:

$$\frac{1}{T}\left[\max_{x \in \Delta_n} \sum_{t=1}^{T} x^\top u_t - \sum_{t=1}^{T} x_t^\top u_t\right].$$

$$= \frac{1}{T}\left[\max_{i^* \in [n]} \sum_{t=1}^{T} x^\top u_{t,i} - \sum_{t=1}^{T} x_t^\top u_t\right].$$

If Regret → 0 as T → ∞, the algorithm is called no-regret.

Multiplicative Weights Update

Algorithm (MWU). *We define the following algorithm:*

1. Initialize $p_i^0 = \frac{1}{n}$ for all $i \in [n]$.
2. **For** t=1 ... T **do**
3. **For** each i that gives payoff $u_{t,i}$ **do**
4. $p_i^{t+1} = p_i^t \frac{1+\epsilon u_{t,i}}{Z^t}$.
5. **End For**
6. **End For**

Remarks:

- ϵ is the stepsize (to be chosen later).
- Performs almost as good as ``best" expert (fewest mistakes).
- The algorithm is also called Multiplicative Weights Update!
- $Z^t = \sum_i p_i^t (1 + \varepsilon u_{t,i})$ is renormalization constant.

Multiplicative Weights Update

Theorem (MWU). *It holds that*

$$\frac{1}{T}\sum_t u_t^\top p^t \geq \max_x \sum_t x^\top u_t - \frac{\log n}{\epsilon T} - \epsilon.$$

Proof. Let's define the potential function $\phi_t = \sum_i w_i^t$ where $w_i^t = \prod_{s=0}^t (1 + \epsilon u_{s,i})$.

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Let the best strategy be i^*, we have

$$\phi_T > w_{i*}^T \geq e^{\epsilon \sum_{s=0}^T u_{s,i*} - \epsilon^2 (\sum_{s=0}^T u_{s,i*})^2}.$$

$$\text{Now } \phi_{t+1} = \sum w_i^{t+1} = \sum w_i^t (1+\epsilon u_{t,i})$$

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Proof cont. Therefore

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Telescopic product gives

$$\phi_T \leq \phi_1 e^{\epsilon \sum_t u_t^\top p^t} = n e^{\epsilon \sum_t u_t^\top p^t}.$$

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Therefore $e^{\epsilon \sum_{s=0}^{T} u_{s,i*} - \epsilon^2 (\sum_{s=0}^{T} u_{s,i*})^2} \leq n e^{\epsilon \sum_t u_t^\top p^t}$, or equivalently

$$\epsilon OPT - \epsilon^2 T \leq \epsilon OPT - \epsilon^2 OPT^2 \leq \log n + \epsilon \textstyle\sum_t u_t^\top p^t.$$

Multiplicative Weights Update

Proof cont. Therefore

Set $\varepsilon \rightarrow \sqrt{\frac{\ln n}{T}}$ and we get regret

$2\sqrt{\frac{\ln n}{T}}$ (No-regret!)

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Minimax Theorem

Theorem (Minimax by John von Neumann). *Let A a matrix of size $n \times m$.*

$$\min_{x \in \Delta_n} \max_{y \in \Delta_m} x^\top A y = \max_{y \in \Delta_m} \min_{x \in \Delta_n} x^\top A y$$

Remarks

- The above is the value of the game.
- Note that It is always true (min-max inequality):

$$\boxed{\inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x,y) \geq \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} f(x,y)}$$

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Define $g(z) \triangleq \inf_{w \in W} f(z, w)$.

$$\forall w, \forall z, g(z) \leq f(z, w)$$

$$\implies \forall w, \sup_z g(z) \leq \sup_z f(z, w)$$

$$\implies \sup_z g(z) \leq \inf_w \sup_z f(z, w)$$

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Proof. Let's use no-regret learning for both "players"!

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Let $x_1, ..., x_T$ and $y_1, ..., y_T$ be the iterates as advised by MWU and define $\hat{x} = \frac{1}{T}\sum_{i=1}^{T} x_i$ and $\hat{y} = \frac{1}{T}\sum_{i=1}^{T} y_i$ and $T = \Theta(\frac{1}{\eta^2})$.

Choose any x, then from the no-regret property for x we get that

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$$\frac{1}{T}\sum_t x_t^\top A y_t \leq \frac{1}{T}\sum_t x^\top A y_t + \eta$$

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$$\begin{aligned} \tfrac{1}{T}\textstyle\sum_t x_t^\top A y_t &\leq \tfrac{1}{T}\textstyle\sum_t x^\top A y_t + \eta \\ &= x^\top A \left(\frac{\sum_t y_t}{T}\right) + \eta. \end{aligned}$$

Minimax Theorem

Proof cont.

Choose any y, then from the no-regret property for y we get that

$$\begin{aligned}\tfrac{1}{T}\textstyle\sum_t x_t^\top A y_t &\geq \tfrac{1}{T}\textstyle\sum_t x_t^\top A y - \eta \\ &= \left(\frac{\sum x_t}{T}\right)^\top A y \;\; - \eta.\end{aligned}$$

Minimax Theorem

Proof cont.

Choose any y, then from the no-regret property for y we get that

$$\frac{1}{T}\sum_t x_t^\top A y_t \geq \frac{1}{T}\sum_t x_t^\top A y - \eta$$
$$= \left(\frac{\sum x_t}{T}\right)^\top A y \ - \eta.$$

We conclude that for all x, y we have

$$\boxed{\left(\frac{\sum x_t}{T}\right)^\top A y \ - \ 2\eta \leq \left(\frac{\sum_t y_t}{T}\right).}$$

Minimax Theorem

Proof cont.

Choose any y, then from the no-regret property for y we get that

$$\begin{aligned}\tfrac{1}{T}\textstyle\sum_t x_t^\top A y_t &\geq \tfrac{1}{T}\textstyle\sum_t x_t^\top A y - \eta \\ &= \left(\frac{\sum x_t}{T}\right)^\top Ay \;\; - \eta.\end{aligned}$$

We conclude that for all x, y we have

$$\boxed{\max_y \left(\frac{\sum x_t}{T}\right)^\top Ay \;\; - \;\; 2\eta \leq \min_x x^\top A \left(\frac{\sum_t y_t}{T}\right).}$$

Finally we get $\max_y \min_x x^\top Ay \geq \min_x x^\top A \left(\frac{\sum_y y_t}{T}\right)$

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Minimax Theorem

Proof cont.

Choose an

Set $\eta \to 0$ and we are done!

t

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