Lecture 4
Binary search (cont.), insertion/selection sort, analysis of quick sort

CS 161 Design and Analysis of Algorithms

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Binary Search: Searching in a sorted array

Input:
- A: Sorted array with n entries $[0..n-1]$
- x: Item we are seeking

Output: Location of x, if x found
-1, if x not found

```
def binarySearch(A,x,first,last)
if first > last:
  return (-1)
else:
  mid = ⌊(first+last)/2⌋
  if x == A[mid]:
    return mid
  else if x < A[mid]:
    return binarySearch(A,x,first,mid-1)
  else:
    return binarySearch(A,x,mid+1,last)

binarySearch(A,x,0,n-1)
```

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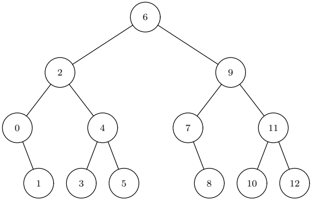

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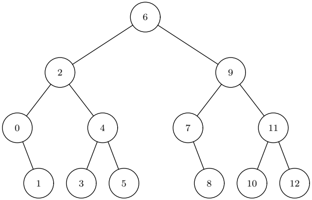

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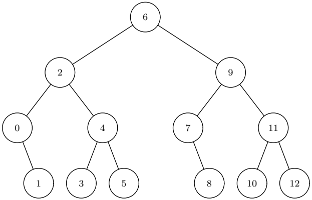

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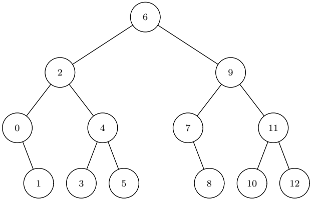

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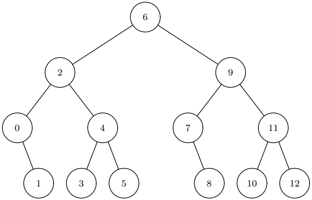

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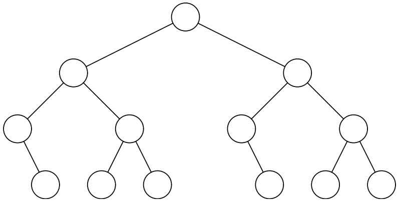

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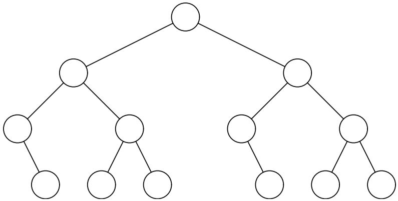

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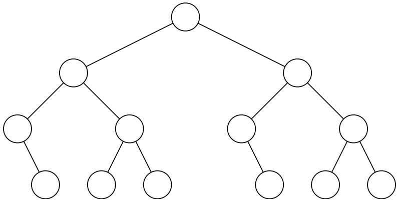

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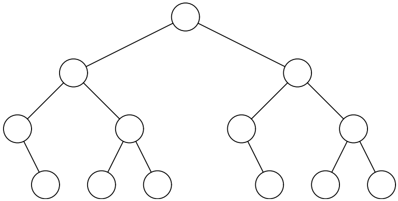

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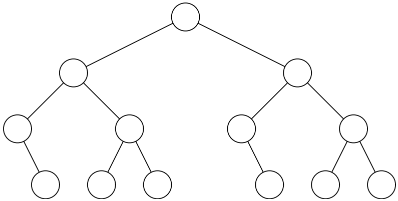

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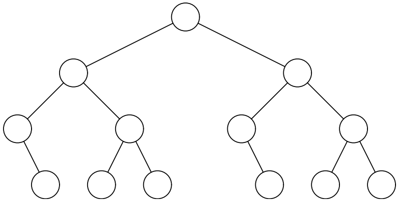

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So binary search is optimal with respect to worst-case performance.

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We will discuss in the class

- Comparison-based sorting algorithms (Insertion sort, Selection Sort, Quicksort, Mergesort, Heapsort)
- Bucket-based sorting methods

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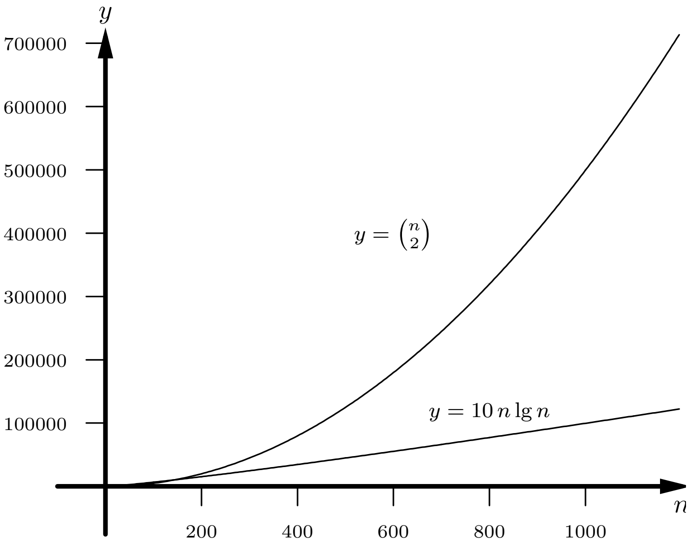
  - Misleading if other operations dominate (e.g., if we sort by moving items around without comparing them)
- Comparison-based sorting has lower bound of $\Omega(n \log n)$ comparisons. (We will prove this.)

$\Theta(n \log n)$ work vs. quadratic ($\Theta(n^2)$) work



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Example: The list

$$18 \quad 29 \quad 12 \quad 15 \quad 32 \quad 10$$

has 9 inversions:

$$\{(18,12), (18,15), (18,10), (29,12), (29,15), \\ (29,10), (12,10), (15,10), (32,10)\}$$

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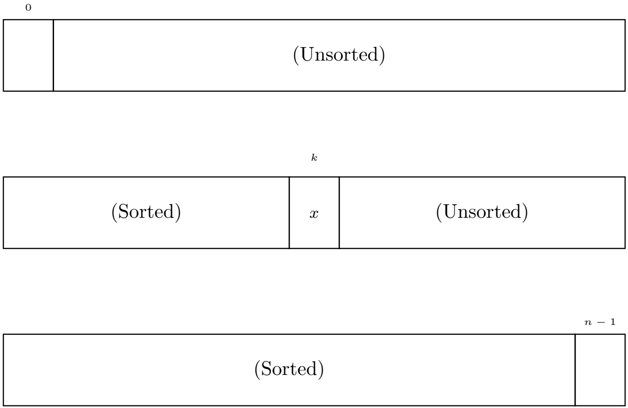

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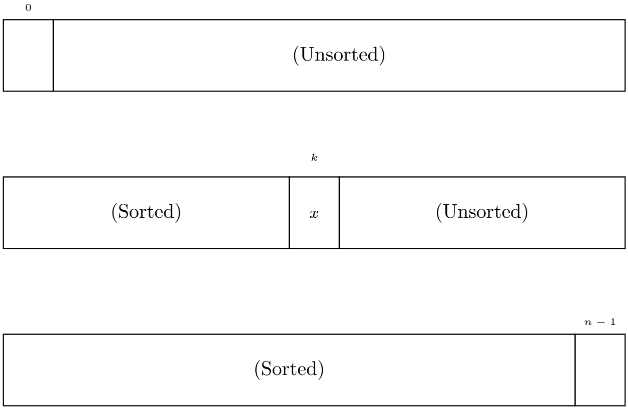
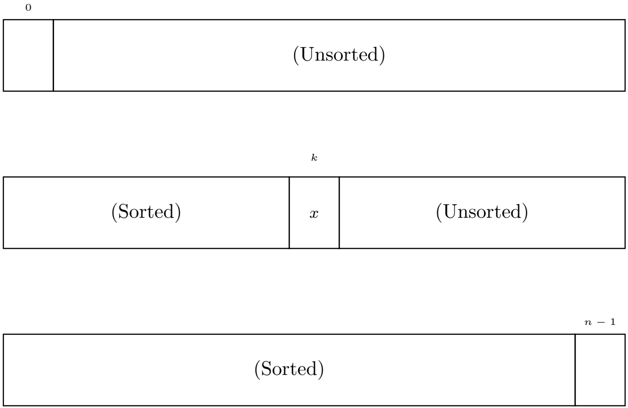
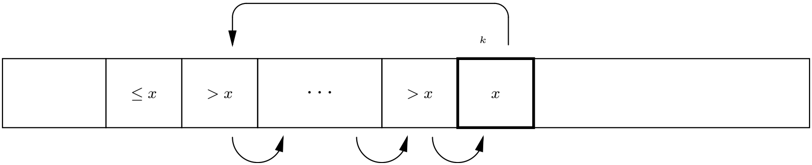


Insertion sort

- Work from left to right across array
- Insert each item in correct position with respect to (sorted) elements to its left



Insertion sort pseudocode




```
def insertionSort(n, A):
    for k = 1 to n-1:
        x = A[k]
        j = k-1
        while (j >= 0) and (A[j] > x):
            A[j+1] = A[j]
            j = j-1
        A[j+1] = x
```

Insertion sort example

23	19	42	17	85	38

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- Storage: in place: $O(1)$ extra storage

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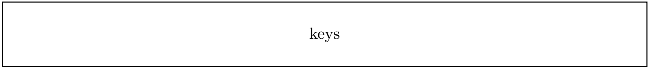

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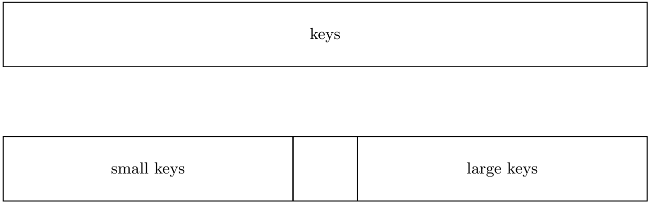

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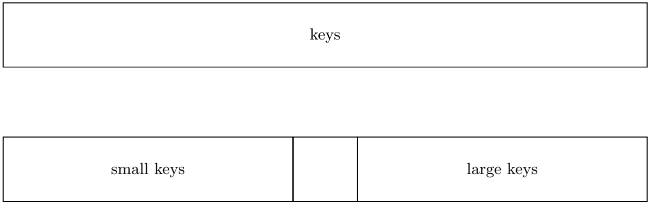
- Rearrange keys so small keys precede all large keys.
- Recursively sort small keys, recursively sort large keys.



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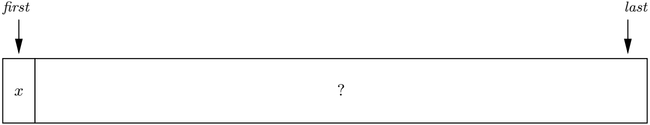

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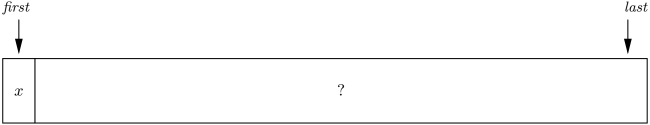
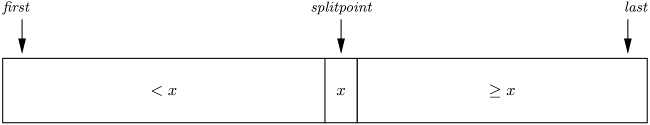
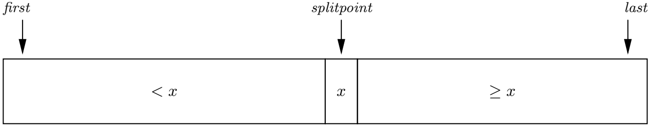
  - Small keys are the keys $< x$.
  - Large keys are the keys $\geq x$.





Pseudocode for Quicksort

```
def quickSort(A,first,last):
    if first < last:
        splitpoint = split(A,first,last)
        quickSort(A,first,splitpoint-1)
        quickSort(A,splitpoint+1,last)
```



The split step

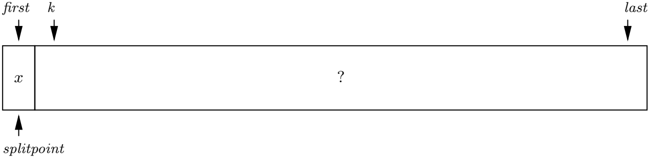
```
def split(A,first,last):
    splitpoint = first
    x = A[first]
    for k = first+1 to last do:
        if A[k] < x:
            A[splitpoint+1] ↔ A[k]
            splitpoint = splitpoint + 1
    A[first] ↔ A[splitpoint]
    return splitpoint
```

Loop invariants:

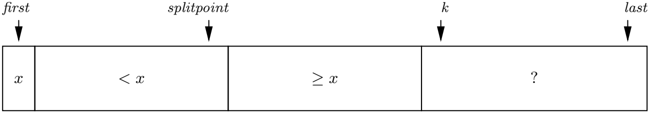
- `A[first+1..splitpoint]` contains keys $< x$.
- `A[splitpoint+1..k-1]` contains keys $\geq x$.
- `A[k..last]` contains unprocessed keys.

The split step

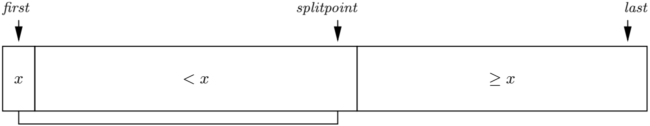
At start:



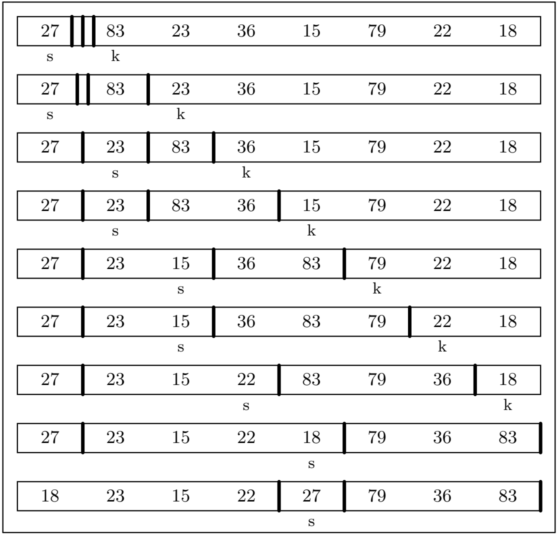
In middle:



At end:



Example of split step



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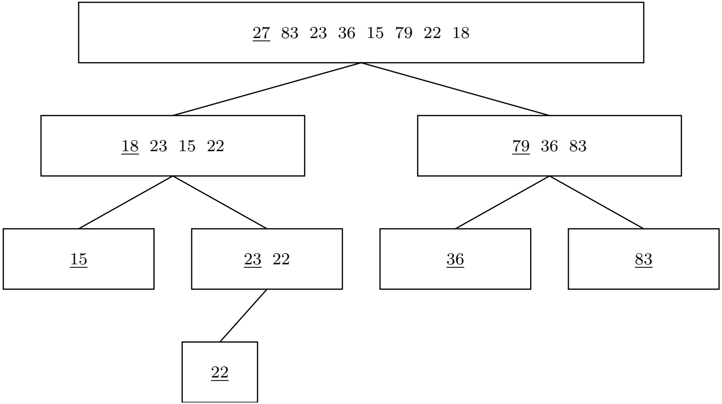

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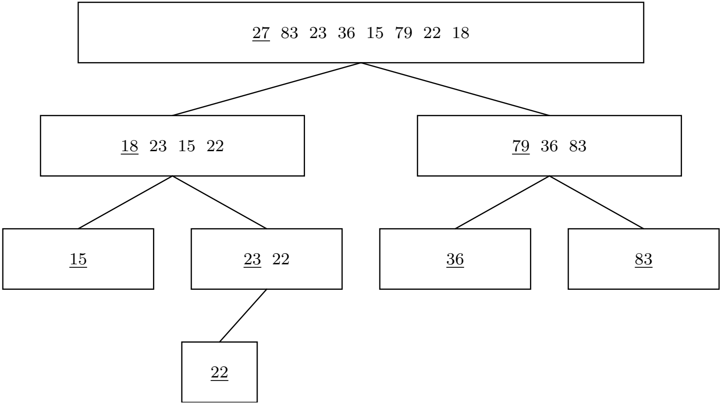

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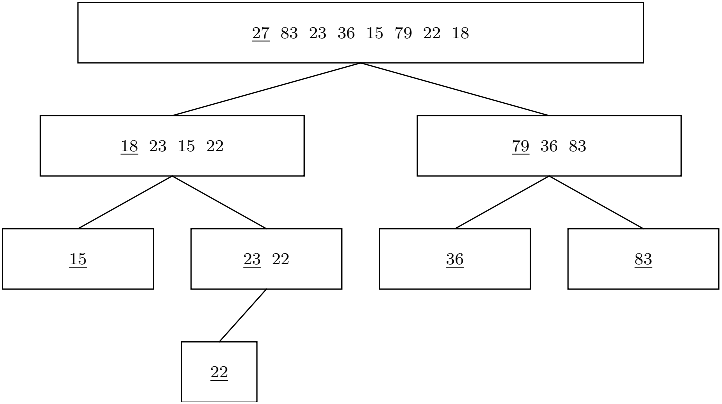
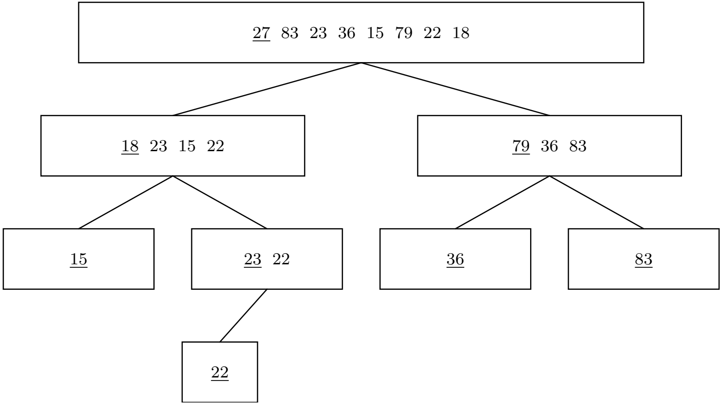


Analysis of Quicksort

We can visualize the lists sorted by quicksort as a binary tree.

- The root is the top-level list (of all items to be sorted)
- The children of a node are the two sublists to be sorted.
- Identify each list with its split value.



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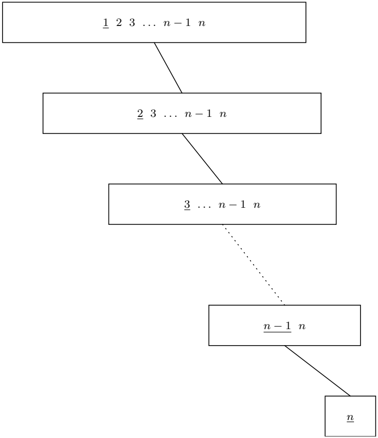
A bad case case for Quicksort: $1, 2, 3, \ldots, n-1, n$



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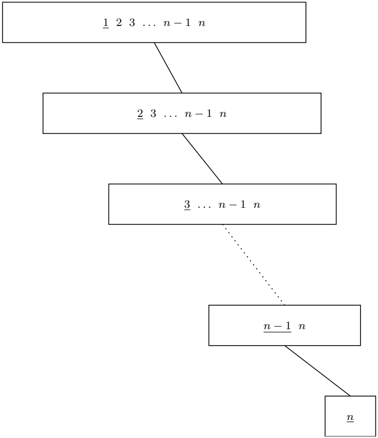

$\binom{n}{2}$ comparisons required. So the worst-case running time for Quicksort is $\Theta(n^2)$. But what about the average case ...?

Average-case analysis of Quicksort:

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4. Use this to compute the expected number of comparisons performed by Quicksort.

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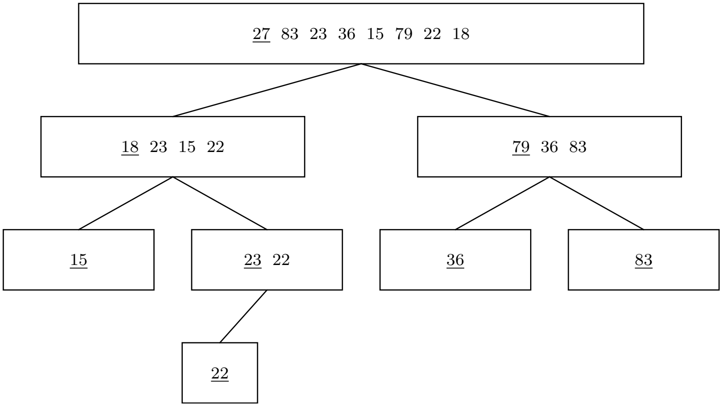

Sorted order: 15 18 22 23 27 36 79 83

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Examples:

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Examples:

- 23 and 22 (both statements true)

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Examples:

- 23 and 22 (both statements true)
- 36 and 83 (both statements false)

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$$
\begin{aligned}
P_{i,j} &= \text{The probability that the first key from } \\
&\quad \{S_i, S_{i+1}, \ldots, S_j\} \text{ to be chosen as a pivot value is} \\
&\quad \text{either } S_i \text{ or } S_j \\
&= \frac{2}{j-i+1}
\end{aligned}
$$

Average-case analysis of Quicksort

Define indicator random variables $\{X_{i,j} : 1 \leq i < j \leq n\}$

$$X_{i,j} = \begin{cases} 1 & \text{if keys } S_i \text{ and } S_j \text{ get compared} \\ 0 & \text{if keys } S_i \text{ and } S_j \text{ do \underline{not} get compared} \end{cases}$$

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3. The expected value of $X_{i,j}$ is:

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$$= \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k} \quad (k = j - i + 1)$$

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Hence the expected number of comparisons is

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i+1}^{n} E\left(X_{i,j}\right) &= \sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{2}{j-i+1} \\ &= \sum_{i=1}^{n}\sum_{k=2}^{n-i+1}\frac{2}{k} \quad (k=j-i+1) \\ &< \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{2}{k}\end{aligned}$$

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So the average time for Quicksort is $O(n \lg n)$.