Lecture 8
Heaps, Heapsort, Optimality of Heapsort/Mergesort (revisited)

CS 161 Design and Analysis of Algorithms

Ioannis Panageas

Heapsort

Consider the following version of Selection Sort (sometimes called Max sort)

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def maxSort(A,n):
    for k = n-1 downto 1
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But we can speed this up by using a binary heap.

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  - Most urgent priority may correspond to the lowest key value or to the highest key value, depending on the application.

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- In our examples, items are integers, key is the integer value

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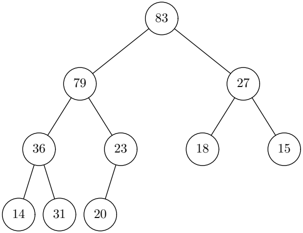

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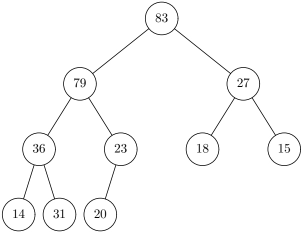

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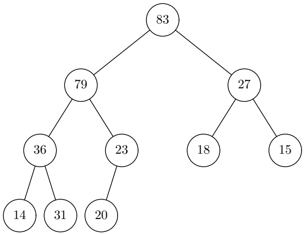

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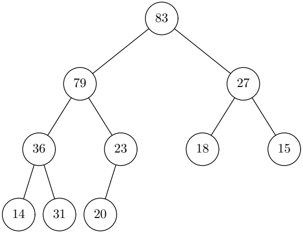

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- Parent of $H[i]$ is $H[\lfloor (i-1)/2 \rfloor]$ (provided $i > 0$)

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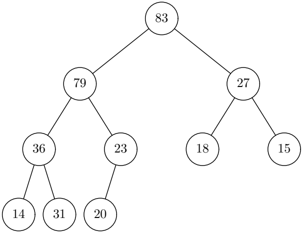

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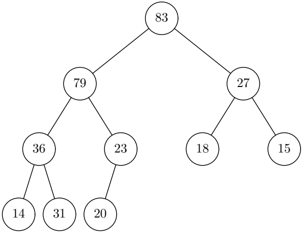

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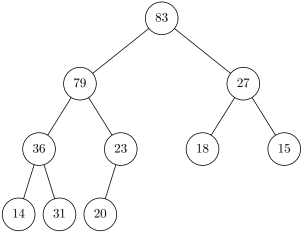

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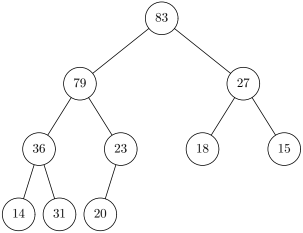

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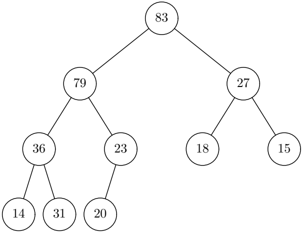

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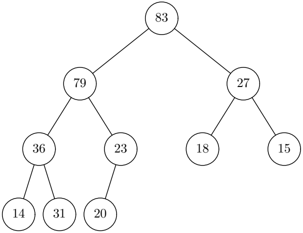

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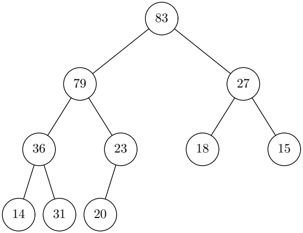

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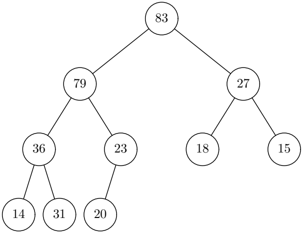

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def FindMax(H):
    return H[0]
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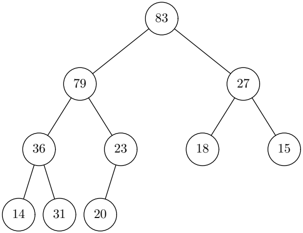

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[GT] calls these "up-heap bubbling" and "down-heap bubbling"

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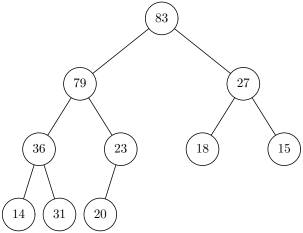

SiftUp: Sift an item up to its correct position

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def SiftUp(H,i):
    parent = (i-1)/2;
    if (i > 0) and (H[parent].key < H[i].key):
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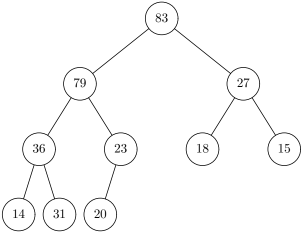

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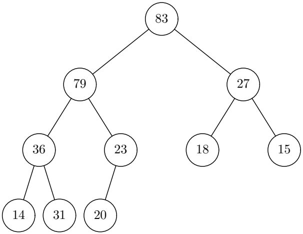
Analysis: at most 1 comparison at each level, so total time is $O(\log n)$



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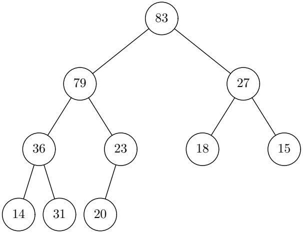

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def SiftDown(H,i):
    n = H.size // number of item in heap
    left = 2i+1; right = 2i+2
    if (right < n) and (H[right].key > H[left].key)
        largerChild = right
    else largerChild = left
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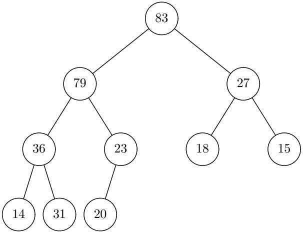

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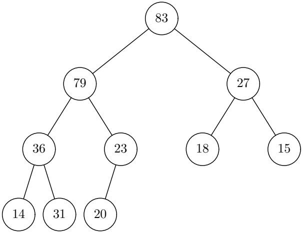
Analysis: at most 2 comparisons at each level, so total time is $O(\log n)$



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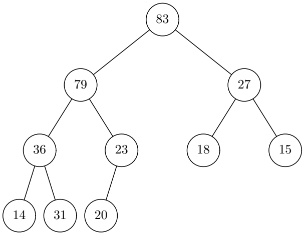

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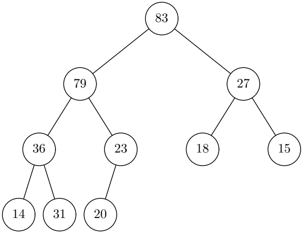

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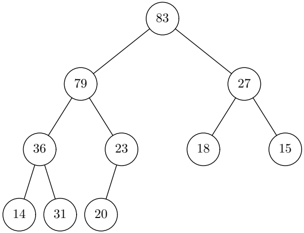

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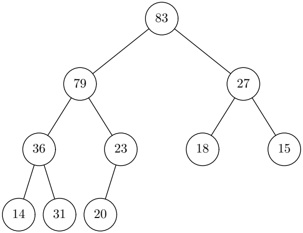

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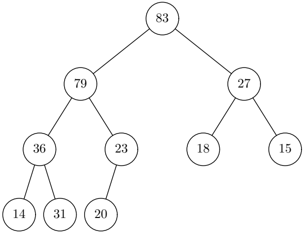

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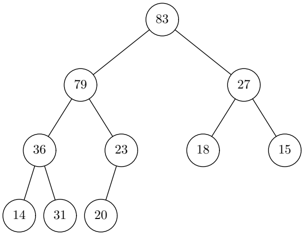

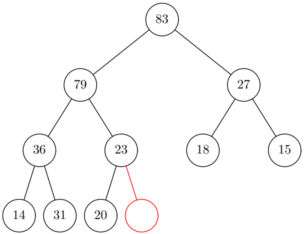

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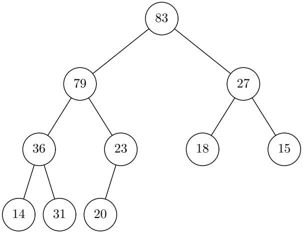

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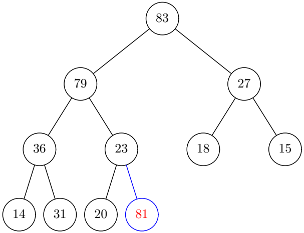

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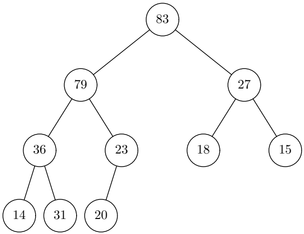

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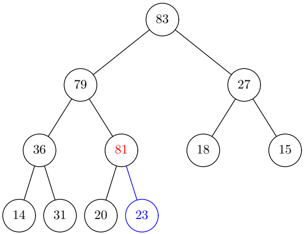

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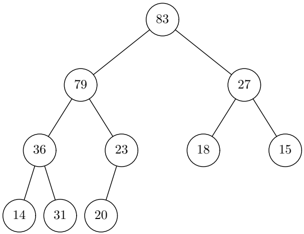

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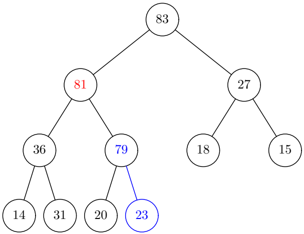

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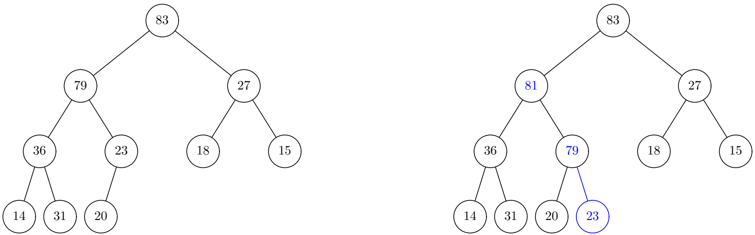

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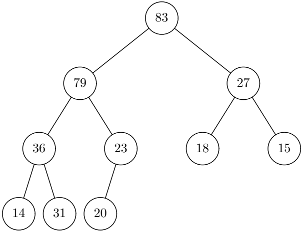

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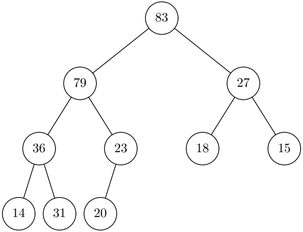

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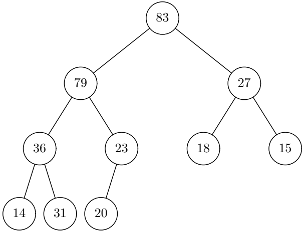

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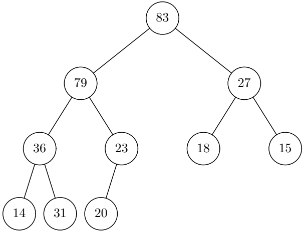

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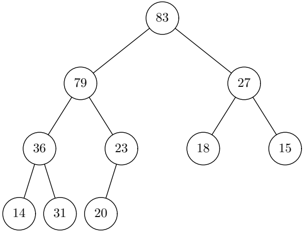

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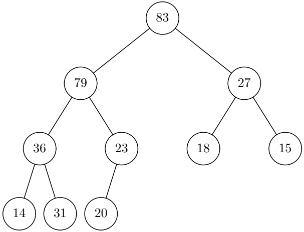

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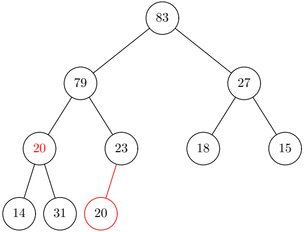

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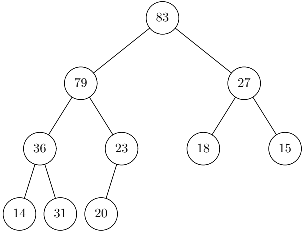

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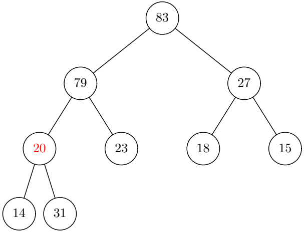

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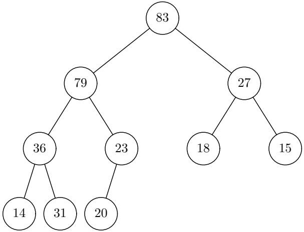

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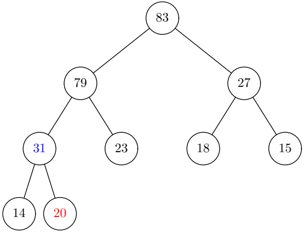

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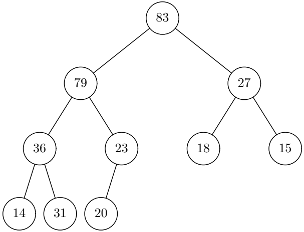

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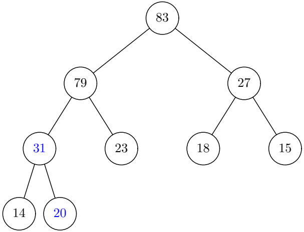

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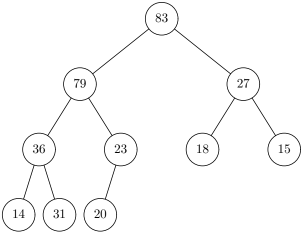

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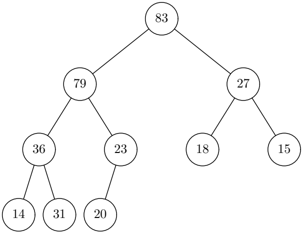

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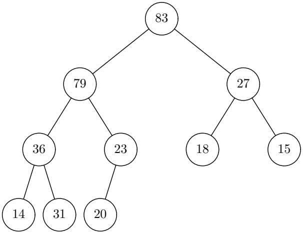

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There is a better way that only requires $O(n)$ time...

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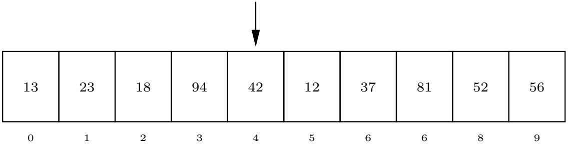
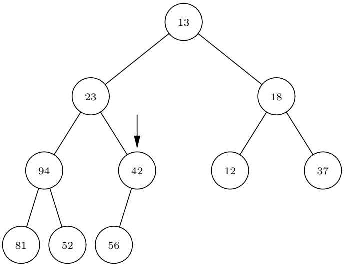
The code given above can be improved: We can start at $\mathrm{i} = \lfloor (\mathrm{n}\text{-}2)/2 \rfloor$ (or equivalently, $\mathrm{i} = \lfloor \mathrm{n}/2 \rfloor - 1$), rather than $i = n - 1$.

Heapify example

```
13 23 18 94 42 12 37 81 52 56
```





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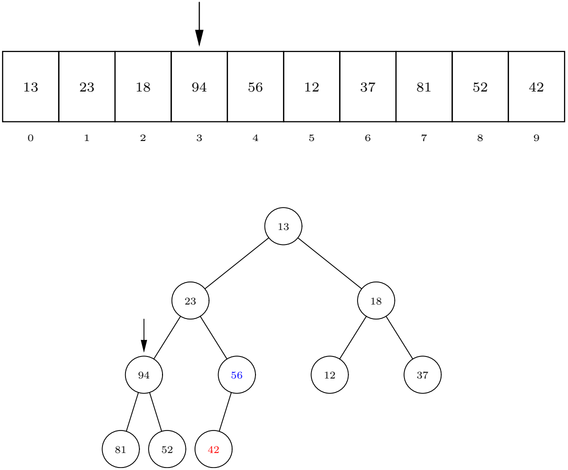

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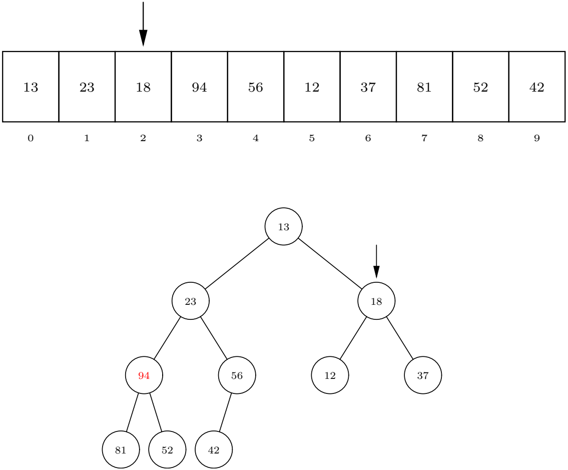

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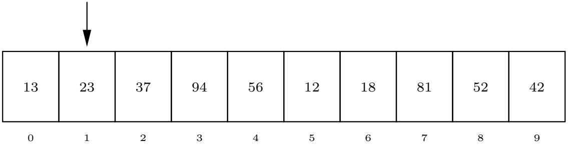

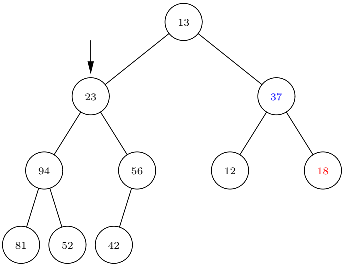

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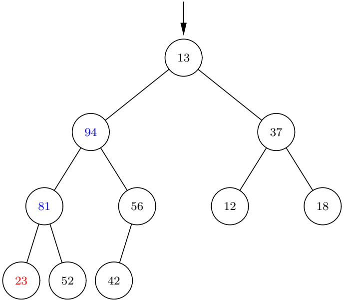

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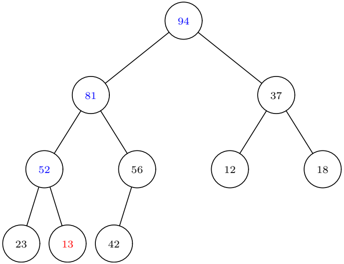

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- Running time: `Heapify` runs in $O(n)$ time. We will prove this on the next slide.

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So heap can be constructed using $O(n)$ comparisons.

Heapsort: version based on Max Sort

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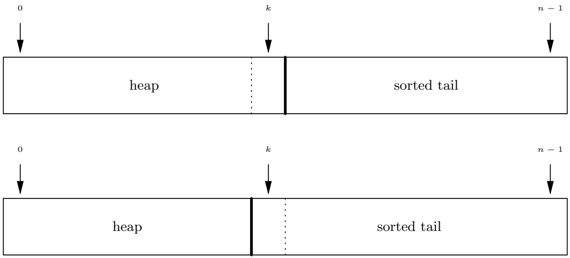
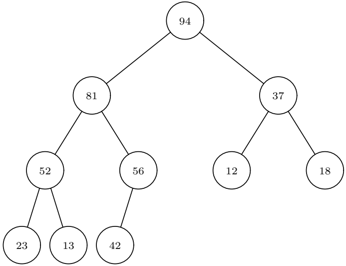


Heapsort example

Sort: 13 23 18 94 42 12 37 81 52 56

Heapify:



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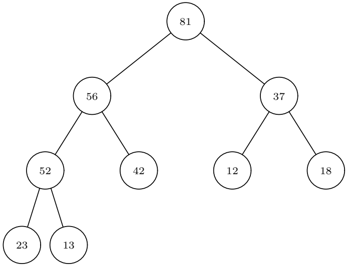
Heapsort example, continued



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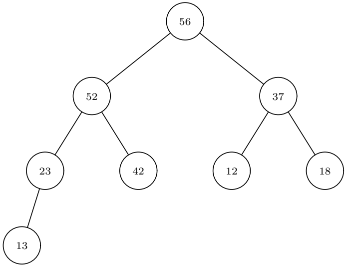

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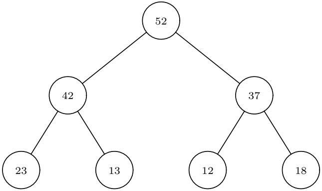

52	42	37	23	13	12	18	56	81	94
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Exercise: Finish this example.

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  - Hence total time is $O(n \log n)$.

Comparison-based sorts: Summary/Comparison

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Sort	Worst-case Time	Storage Requirement	Remarks
Insertion Sort	$O(n^2)$	In-place	Good if input is almost sorted.
QuickSort	$O(n^2)$	$O(\log n)$ extra for stack	$O(n \log n)$ expected time.
Mergesort	$O(n \log n)$	$O(n)$ extra for merge	
Heapsort	$O(n \log n)$	In-place	Can output k smallest in sorted order in $O(n + k \log n)$ time.

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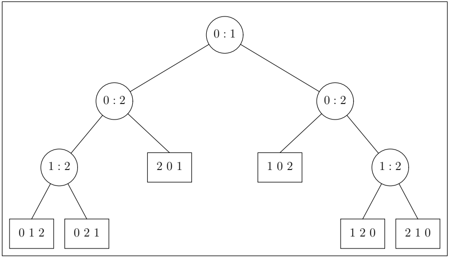

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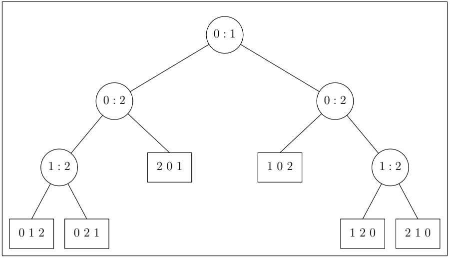

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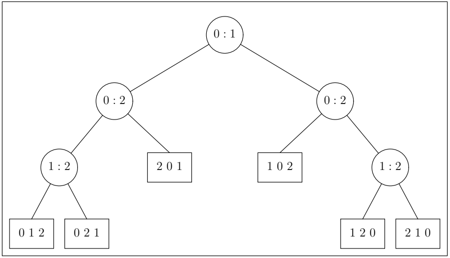

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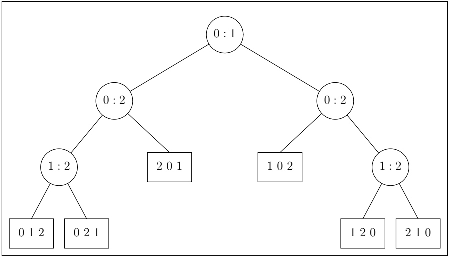

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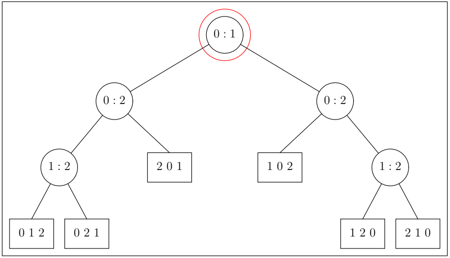

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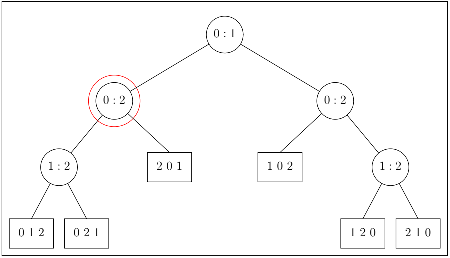

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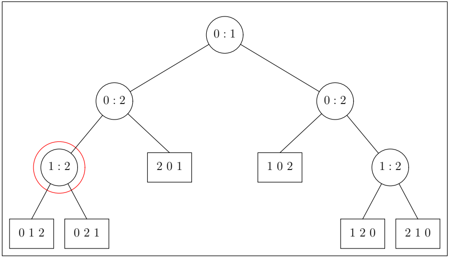

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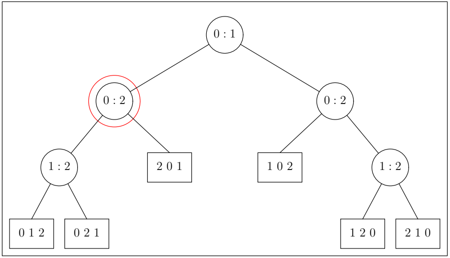

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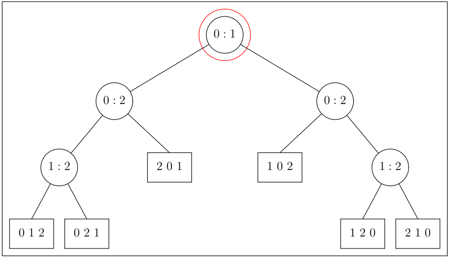

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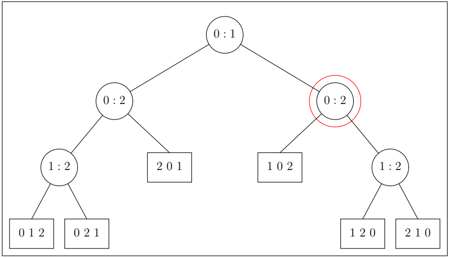

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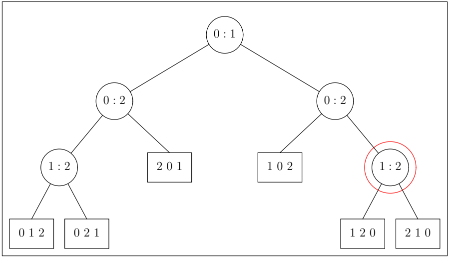

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Conclusions:

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