Outline of these notes

- Review of basic data structures
- Searching in a sorted array/binary search: the algorithm, analysis, proof of optimality
- Sorting, part 1: insertion sort, selection sort

Basic Data structures

Prerequisite material. Review [GT Chapters 2–4, 6] as necessary)

- Arrays, dynamic arrays
- Linked lists
- Stacks, queues
- Dictionaries, hash tables
- Binary trees

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Disadvantages on next slide.

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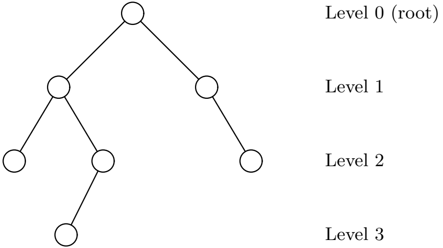
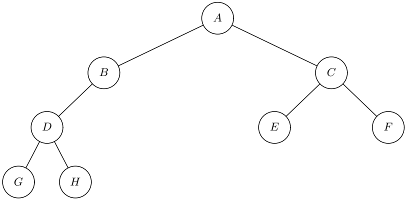


The depth of a binary tree is the maximum of the levels of all its leaves.

Traversing binary trees



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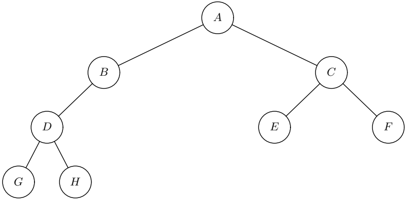

- Preorder: root, left subtree (in preorder), right subtree (in preorder): *ABDGHCEF*

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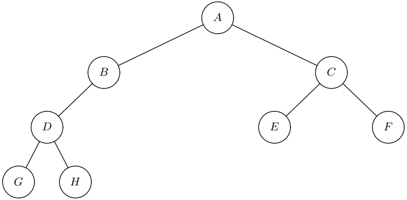

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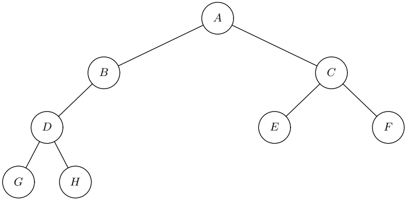

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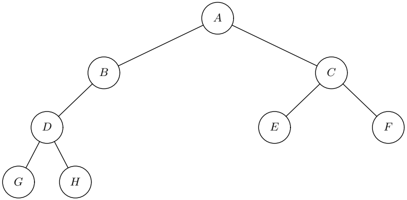

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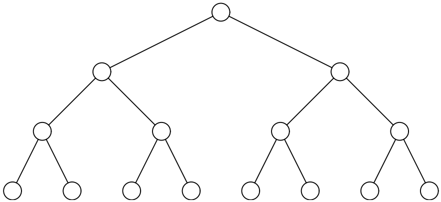
Facts about binary trees



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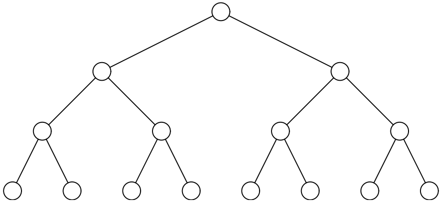

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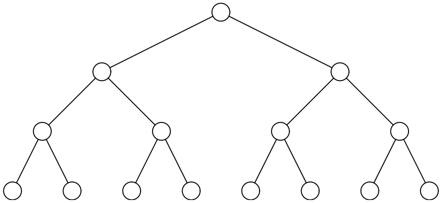

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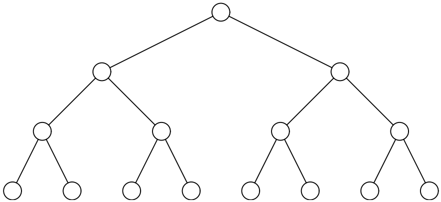

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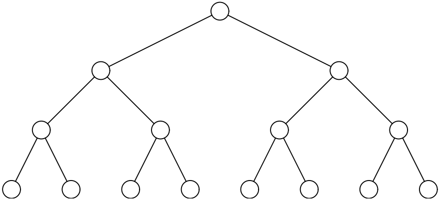

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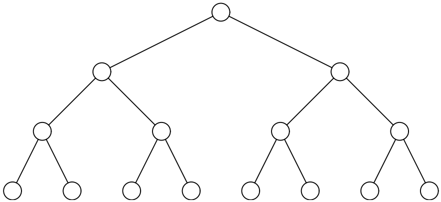

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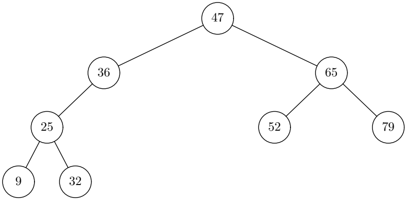

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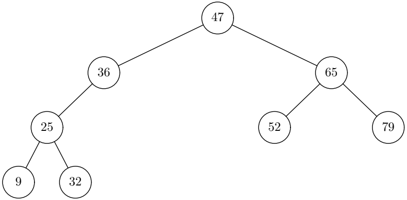

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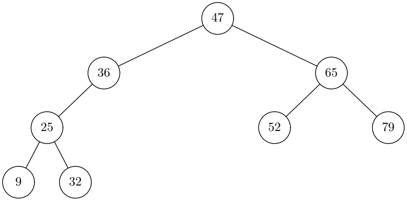

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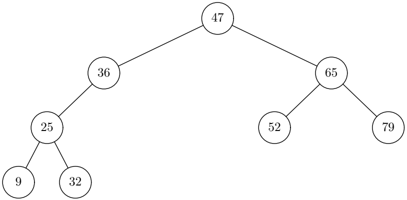

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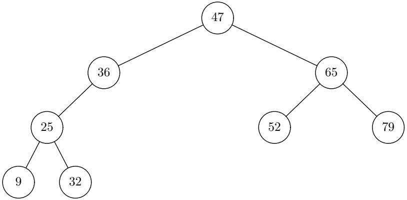

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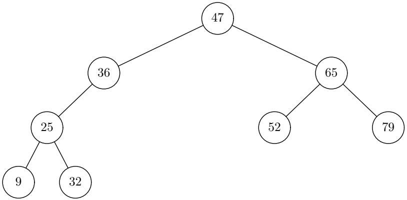

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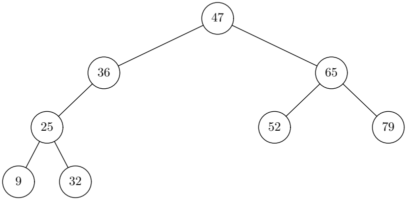

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- [GT] Chapters 3–4 for details

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  We will focus on the first variant.

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- Several variants of the problem, for example...
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  2. Find the largest i such that $A[i] \leq x$ (with a reasonable convention if $x < A[0]$).

  We will focus on the first variant.
- We will show that binary search is an optimal algorithm for solving this problem.

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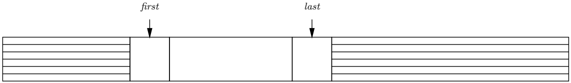
```
def binarySearch(A,x,first,last)
if first > last:
  return (-1)
else:
  mid = ⌊(first+last)/2⌋
  if x == A[mid]:
    return mid
  else if x < A[mid]:
    return binarySearch(A,x,first,mid-1)
  else:
    return binarySearch(A,x,mid+1,last)

binarySearch(A,x,0,n-1)
```

Correctness of Binary Search



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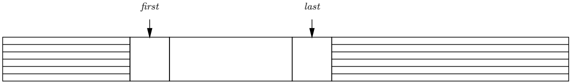

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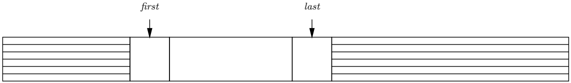

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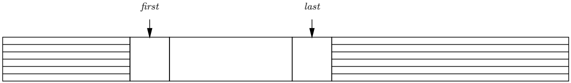

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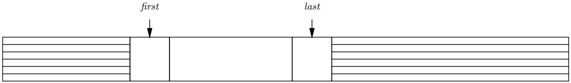
2. On each recursive call, the difference *last* − *first* gets strictly smaller.



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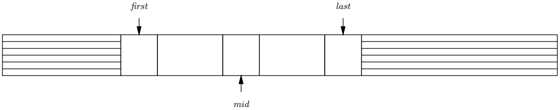

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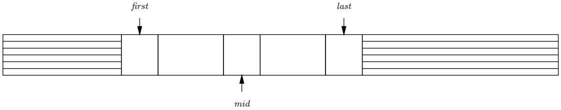

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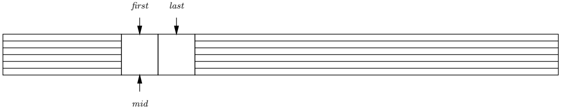

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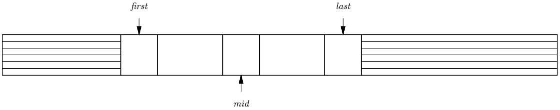

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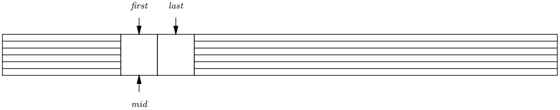

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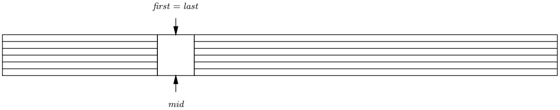

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  2. Gives us a way to compare binary search against other algorithms that solve the same problem: searching for an item in an array by comparing the item against array entries.

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- So binary search does $\lfloor \lg n \rfloor + 1$ 3-way comparisons on an array of size n, in the worst case.

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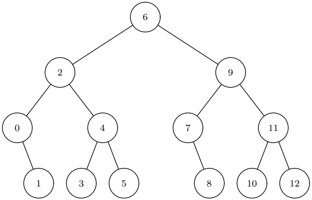
Example: Decision tree for binary search with $n = 13$:



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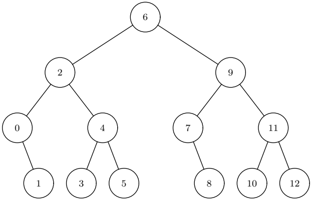

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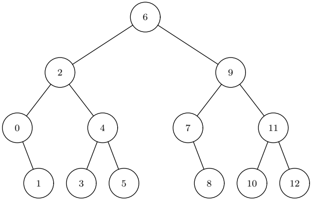

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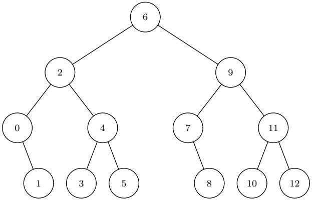

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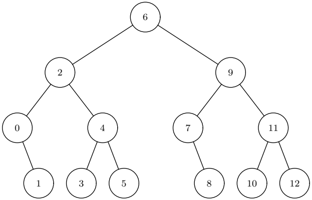

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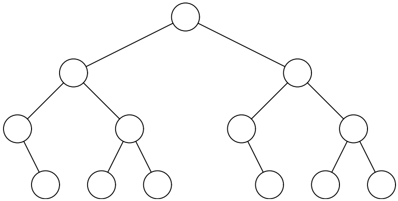

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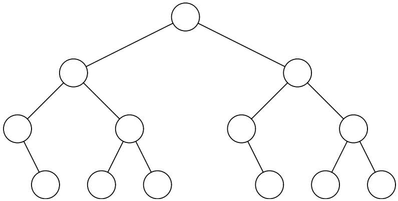

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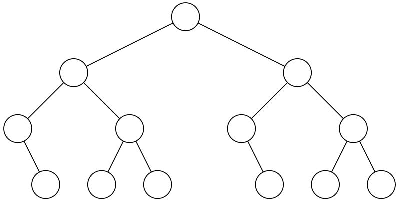

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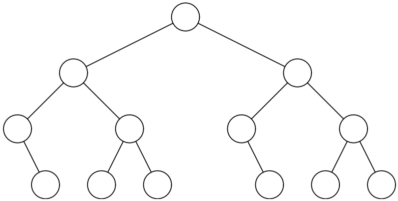

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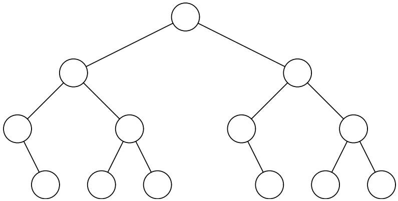

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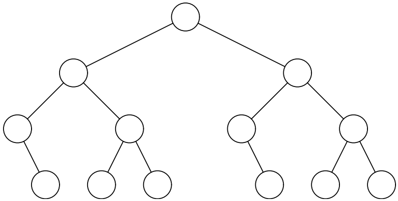

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So binary search is optimal with respect to worst-case performance.

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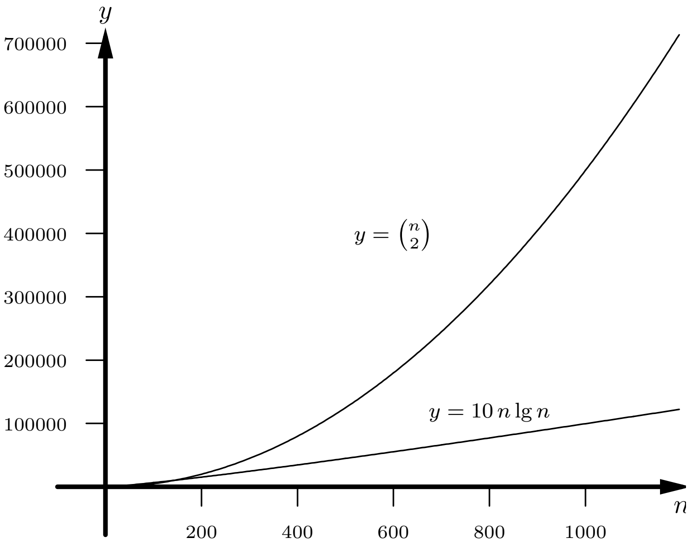
- Comparison-based sorting has lower bound of $\Omega(n \log n)$ comparisons. (We will prove this.)

$\Theta(n \log n)$ work vs. quadratic ($\Theta(n^2)$) work



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Example: The list

18 29 12 15 32 10

has 9 inversions:

$$\{(18,12), (18,15), (18,10), (29,12), (29,15), (29,10), (12,10), (15,10), (32,10)\}$$

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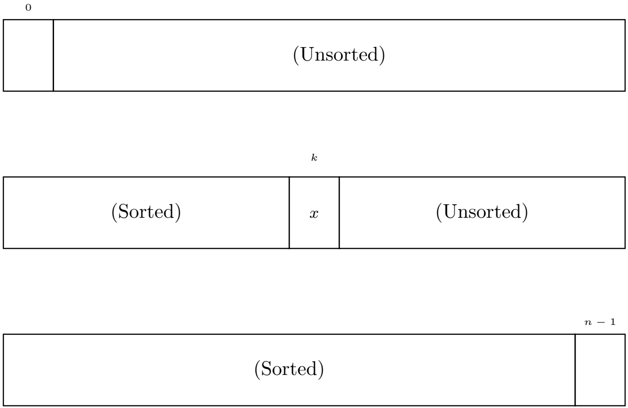

Insertion sort

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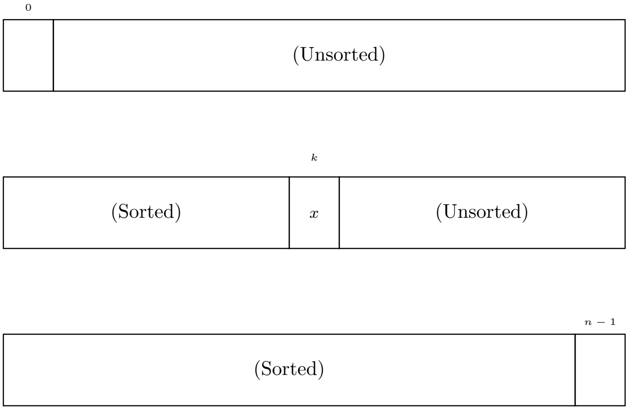
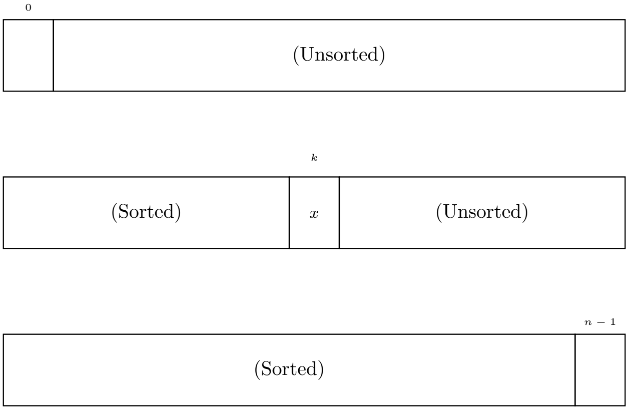
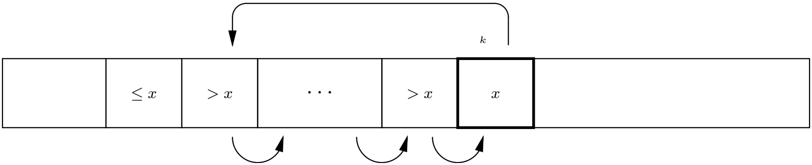


Insertion sort

- Work from left to right across array
- Insert each item in correct position with respect to (sorted) elements to its left



Insertion sort pseudocode




```
def insertionSort(n, A):
    for k = 1 to n-1:
        x = A[k]
        j = k-1
        while (j >= 0) and (A[j] > x):
            A[j+1] = A[j]
            j = j-1
        A[j+1] = x
```

Insertion sort example

23	19	42	17	85	38

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19	23	42	17	85	38

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17	19	23	42	85	38

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- Storage: in place: $O(1)$ extra storage

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