Lecture 14

Greedy Method: Fractional Knapsack, Interval scheduling

CS 161 Design and Analysis of Algorithms

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Greedy method

The greedy method is a general algorithm design technique, in which given:

- configurations: different choices we need to make
- objective function: a score assigned to all configurations, which we want to either maximize or minimize

We should make choices greedily: We can find a globally-optimal solution by a series of local improvements from a starting configuration.

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Example: Maxflow problem.

Configurations: All possible flow functions. Objective function: Maximize flow value.

Ford-Fulkerson makes choices greedily starting from flow $\boldsymbol{f = 0}$**.**

Greedy does not always work

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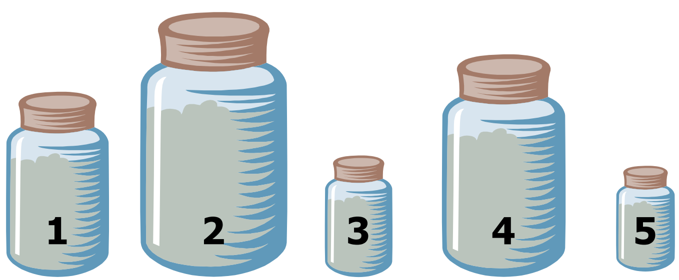
Fractional Knapsack

Problem: A set of n items, with each item i having positive weight w_i and positive value v_i. You are asked to choose items with maximum total value so that the total weight is at most W. We are allowed to take fractional amounts (some percentage of each item).

Example:

Items:



Weight:	4 ml	8 ml	2 ml	6 ml	1 ml
Value:	\$12	\$32	\$40	\$30	\$50
Value: (\$ per ml)	\$3	\$4	\$20	\$5	\$50

"knapsack" with 10ml

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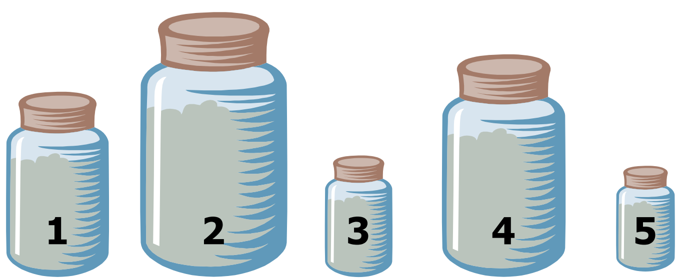

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“knapsack” with 10ml

Solution:

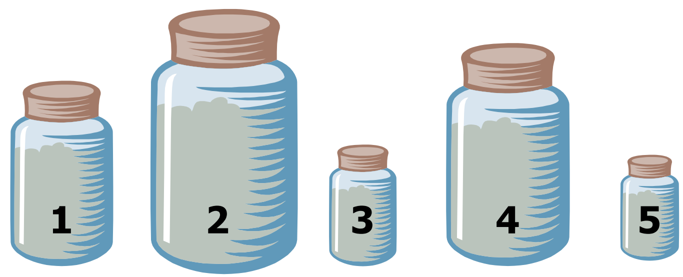
- 1 ml of 5
- 2 ml of 3
- 6 ml of 4
- 1 ml of 2

Total Value: \$124

Fractional Knapsack

Idea: Greedy approach. Keep taking item with highest value to weight ratio until knapsack is full or run out of items.

Items:



Weight:	4 ml	8 ml	2 ml	6 ml	1 ml
Value:	\$12	\$32	\$40	\$30	\$50
Value:	\$3	\$4	\$20	\$5	\$50

$W = 10$ ml

$value = \$0$

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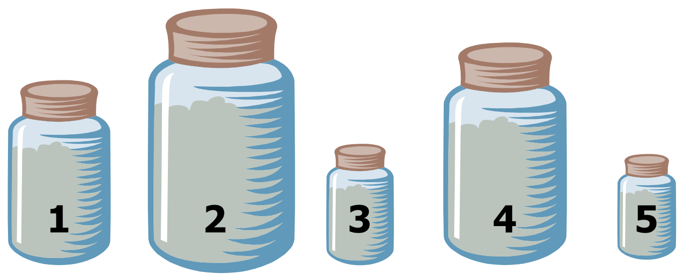
Items:



Weight:	4 ml	8 ml	2 ml	6 ml	**1 ml**
Value:	\$12	\$32	\$40	\$30	\$50
Value:	\$3	\$4	\$20	\$5	**\$50**

$W = 10$ ml

$value = \$0$

Fractional Knapsack

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Items:

Weight:	4 ml	8 ml	2 ml	6 ml
Value:	\$12	\$32	\$40	\$30
Value:	\$3	\$4	\$20	\$5

W = 9 ml

$value$ = \$50

Fractional Knapsack

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Items:

Weight:	4 ml	8 ml	**2 ml**	6 ml
Value:	\$12	\$32	\$40	\$30
Value:	\$3	\$4	**\$20**	\$5

$W = 9$ ml

$value = \$50$

Fractional Knapsack

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Items:

Weight:	4 ml	8 ml	6 ml
Value:	\$12	\$32	\$30
Value:	\$3	\$4	\$5

$W = 7$ ml

$value = \$90$

Fractional Knapsack

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Items:

Weight:	4 ml	8 ml	**6 ml**
Value:	\$12	\$32	\$30
Value:	\$3	\$4	**\$5**

$W = 7$ ml

$value = \$90$

Fractional Knapsack

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Items: 

Weight:	4 ml	8 ml
Value:	\$12	\$32
Value:	\$3	\$4

$W = 1$ ml

$value = \$120$

Fractional Knapsack

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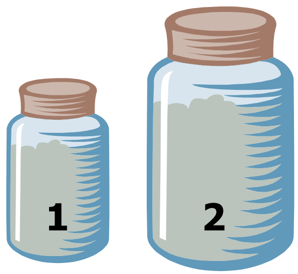
Items: 

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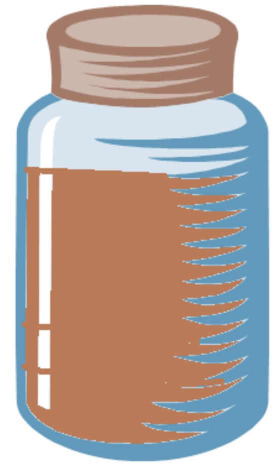

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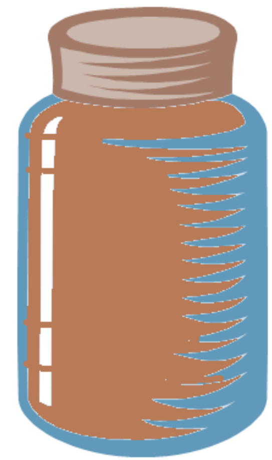

$W = 0$ ml
$value = \$124$

Running time: ?

Fractional Knapsack

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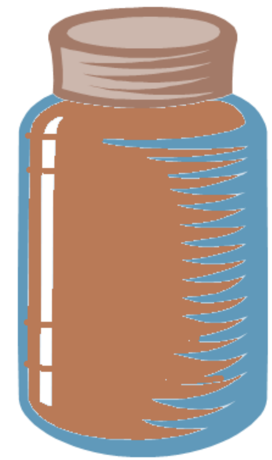

$W = 0$ ml
$value = \$124$

Running time: If we sort the items with respect to value to weight ratio then $\Theta(n \log n)$.

Fractional Knapsack

Pseudocode:

Items with $v[], w[]$, knapsack with W

For $i = 1$ to n **do**

$\quad \mathrm{r}[i] \leftarrow \frac{v[i]}{w[i]}$

$w \leftarrow 0$

$val \leftarrow 0$

While $w < W$ **do**

$\quad$ **Remove** item i with highest $r[i]$

$\quad$ **If** $w + w_i \leq W$ **then**

$\quad\quad w \leftarrow w + w_i$

$\quad\quad val \leftarrow val + v[i]$

$\quad$ **Else**

$\quad\quad w \leftarrow W,\ val \leftarrow val + (W - w) \cdot r[i]$

return val

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Compute the ratios

Initialization

While knapsack not full

If whole item fits

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return val

Compute the ratios

Initialization

While knapsack not full

If whole item fits

Percentage of item i that fits

Fractional Knapsack

Pseudocode:

Items with $v[], w[]$, knapsack with W

For $i = 1$ to n **do**

$\quad \mathrm{r}[i] \leftarrow \frac{v[i]}{w[i]}$

$w \leftarrow 0$

$val \leftarrow 0$

Sort $r[1], \ldots, r[n]$

While $w < W$ **do**

$\quad$ **Remove** item i with highest $r[i]$

$\quad$ **If** $w + w_i \leq W$ **then**

$\quad\quad w \leftarrow w + w_i$

$\quad\quad val \leftarrow val + v[i]$

$\quad$ **Else**

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return val

This is fast, in $O(1)$ time.

Fractional Knapsack

Why greedy works: General argument. Suppose there is a better solution. Assume items are order in decreasing order of value per weight, i.e., $r_1 \geq r_2 \ldots \geq r_n$.

- Let $x_1, \ldots, x_n$ be the weight values of the items in the knapsack for the better solution.

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Part or all of item j is in the knapsack

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value per weight of item i is larger than j

Part or all of item j is in the knapsack

Not all of item i is in the knapsack

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- Since it is different from what greedy returns, there must be indices i, j so that $r_i > r_j$ and $x_j > 0$ and $x_i < w_i$.
- Exchange part of item i, with part of item j. How much?

Say the minimum of $w_i - x_i \; and \; x_j$.

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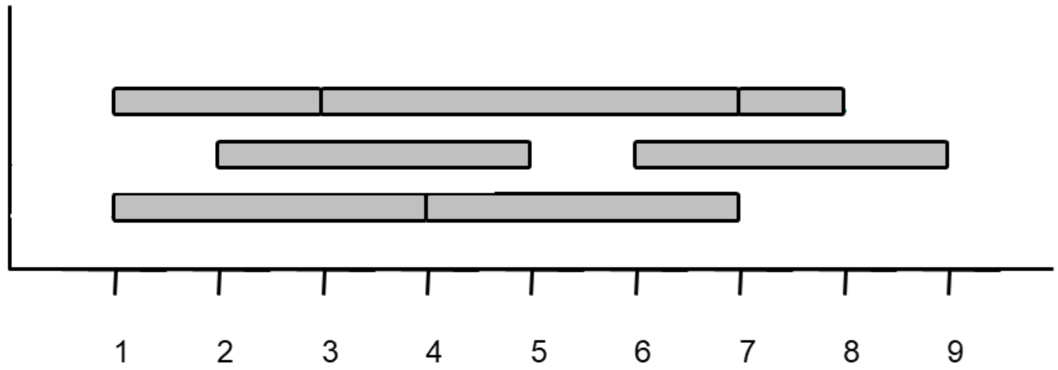
Say the minimum of $w_i - x_i \; and \; x_j$.

Total value will increase by $\left(r_i - r_j\right) \cdot \min(w_i - x_i, x_j)$

Task scheduling

Problem: Given: a set T of n tasks, each having a start time s_i and a finish time f_i (where $s_i < f_i$)
Goal: Perform all the tasks using a minimum number of machines.
A machine can serve one task at a given time.

Example: 7 Tasks, [1,4], [1,3], [2,5], [3,7], [4,7], [6,9], [7,8]



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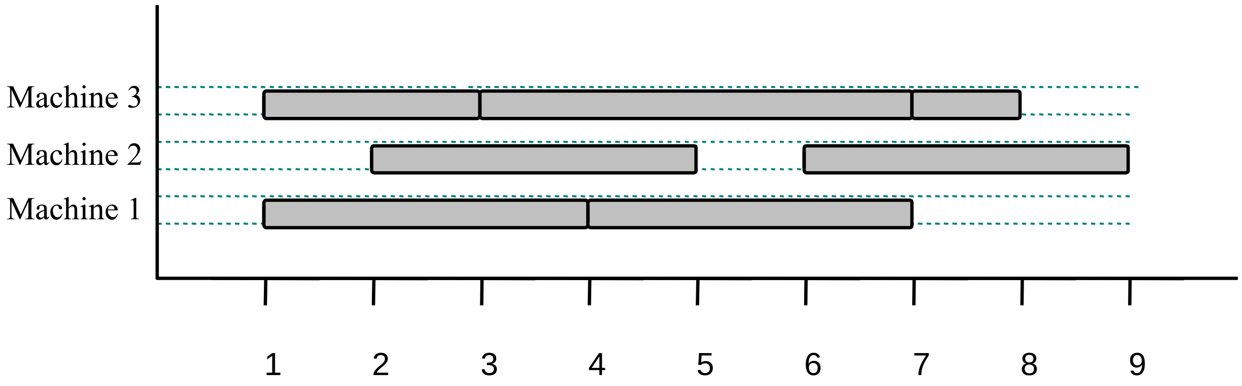

Task scheduling

Idea: Greedy approach. Consider tasks in increasing order of their start time. Assign first task to machine 1 and set $K = 1$.
When considering a new task, if all machines are busy, create a new machine, set $K = K + 1$ and assign the new task to the new machine otherwise assign the new task to an available machine.

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$K = 2$

Machine 1 [1,4]

Machine 2 [1,3]

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$K = 3$

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Machine 2 [1,3]

Machine 3 [2,5]

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- So we have k tasks that conflict with each other, we need k machines! Contradiction!

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- So we have k tasks that conflict with each other, we need k machines!