Lecture 3
Recap of basic data structures, binary search, insertion/selection sort

CS 161 Design and Analysis of Algorithms

Ioannis Panageas

Outline of these notes

- Review of basic data structures
- Searching in a sorted array/binary search: the algorithm, analysis, proof of optimality
- Sorting, part 1: insertion sort, selection sort

Basic Data structures

Prerequisite material. Review [GT Chapters 2–4, 6] as necessary)

- Arrays, dynamic arrays
- Linked lists
- Stacks, queues
- Dictionaries, hash tables
- Binary trees

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  - Elements are inserted at the rear of the queue and are removed from the front

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Disadvantages on next slide.

Binary Trees: a quick review

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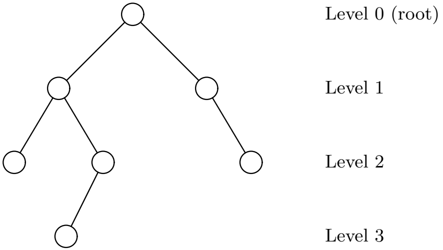
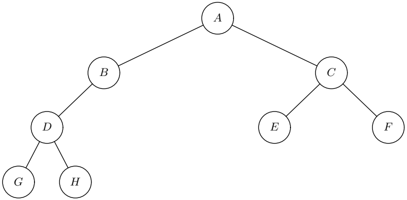


The depth of a binary tree is the maximum of the levels of all its leaves.

Traversing binary trees



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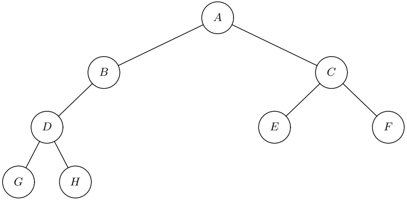

- Preorder: root, left subtree (in preorder), right subtree (in preorder): *ABDGHCEF*

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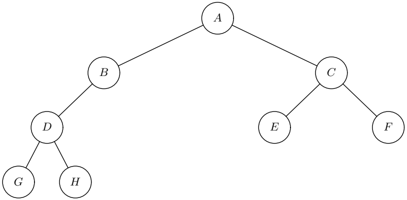

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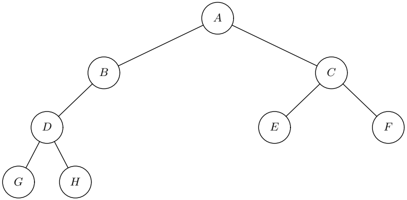

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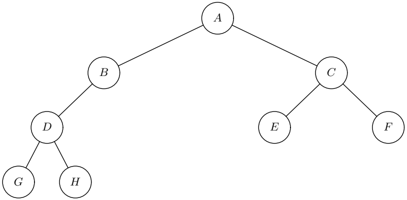

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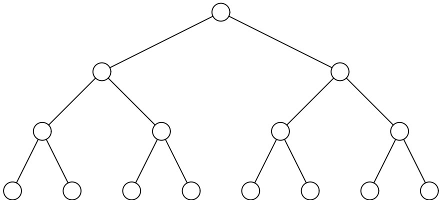
- Breadth-first order (level order): level 0 left-to-right, then level 1 left-to-right, . . . : *ABCDEFGH*

Facts about binary trees



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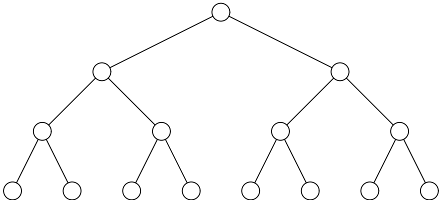

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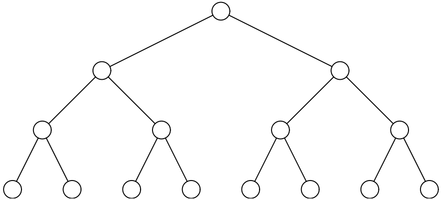

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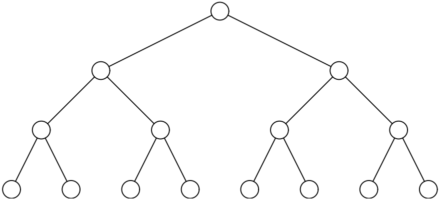

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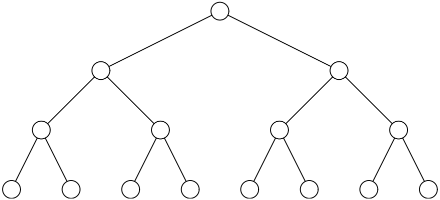

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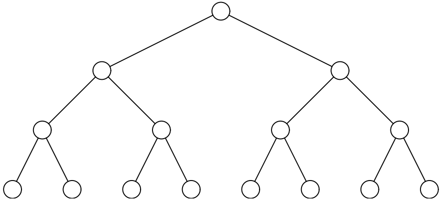

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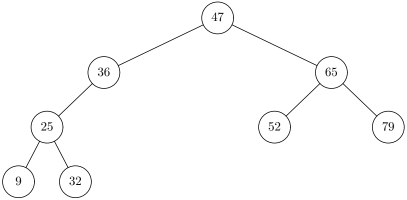

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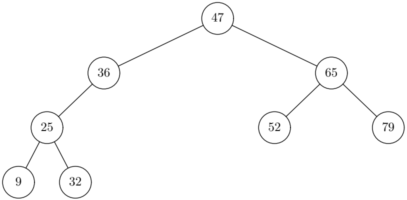

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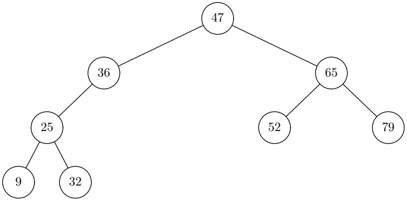

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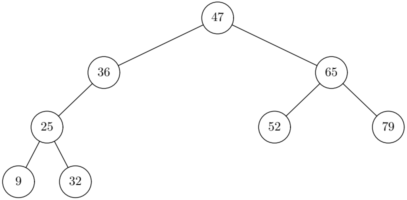

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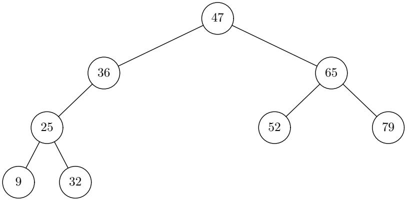

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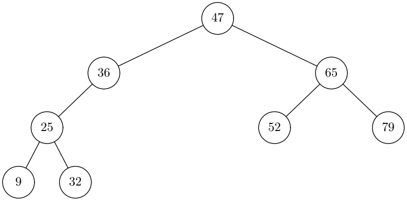

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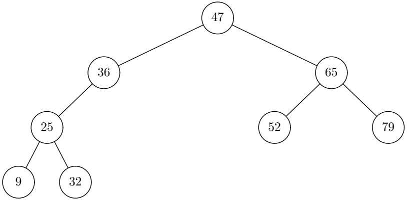

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- [GT] Chapters 3–4 for details

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  We will focus on the first variant.

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- Several variants of the problem, for example...
  1. Determine whether x is stored in the array
  2. Find the largest i such that $A[i] \leq x$ (with a reasonable convention if $x < A[0]$).

  We will focus on the first variant.
- We will show that binary search is an optimal algorithm for solving this problem.

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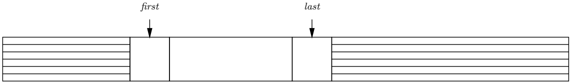
```
def binarySearch(A,x,first,last)
if first > last:
  return (-1)
else:
  mid = ⌊(first+last)/2⌋
  if x == A[mid]:
    return mid
  else if x < A[mid]:
    return binarySearch(A,x,first,mid-1)
  else:
    return binarySearch(A,x,mid+1,last)

binarySearch(A,x,0,n-1)
```

Correctness of Binary Search



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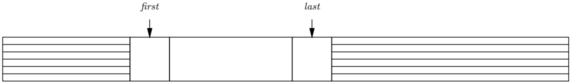

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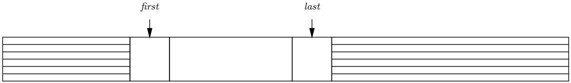

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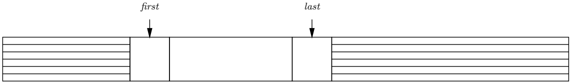

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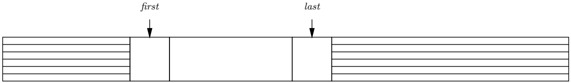
2. On each recursive call, the difference *last* $-$ *first* gets strictly smaller.



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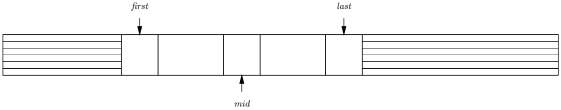

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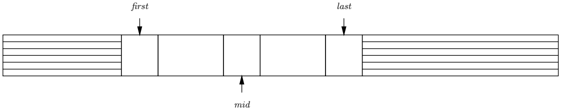

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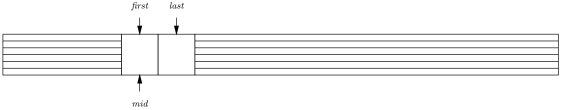

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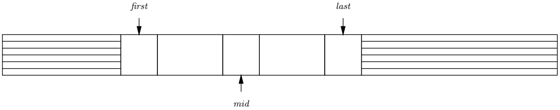

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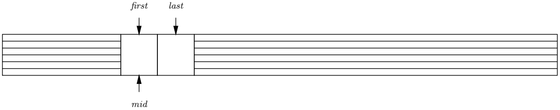

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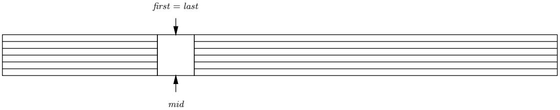

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  2. Gives us a way to compare binary search against other algorithms that solve the same problem: searching for an item in an array by comparing the item against array entries.

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- So binary search does $\lfloor \lg n \rfloor + 1$ 3-way comparisons on an array of size n, in the worst case.

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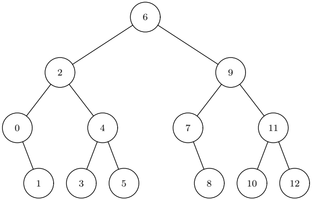

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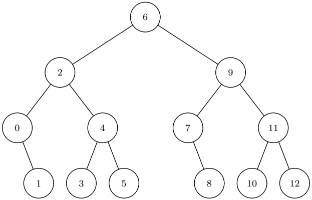

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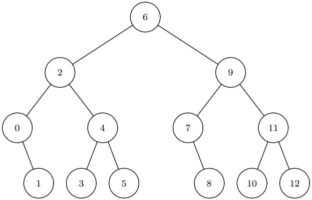

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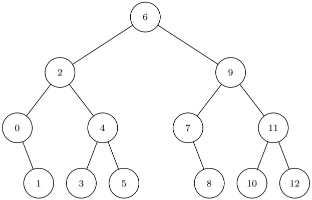

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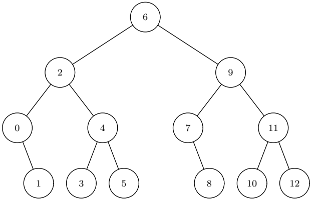

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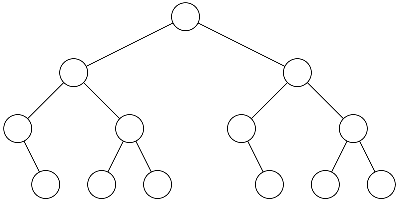

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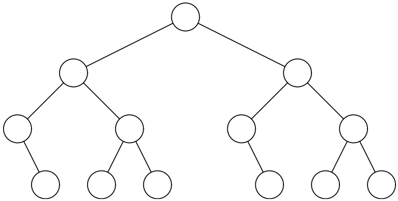

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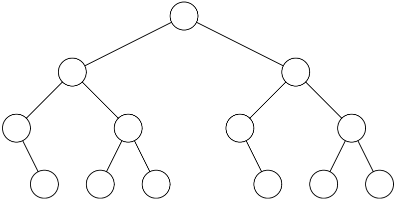

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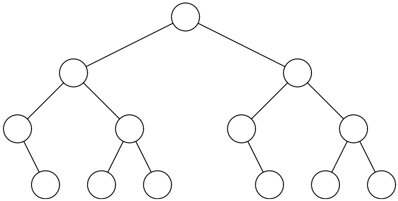

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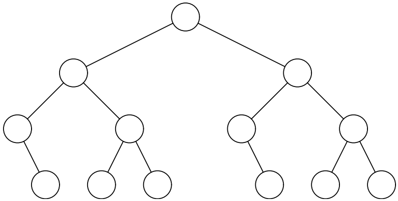

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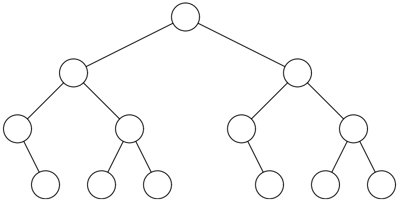

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So binary search is optimal with respect to worst-case performance.

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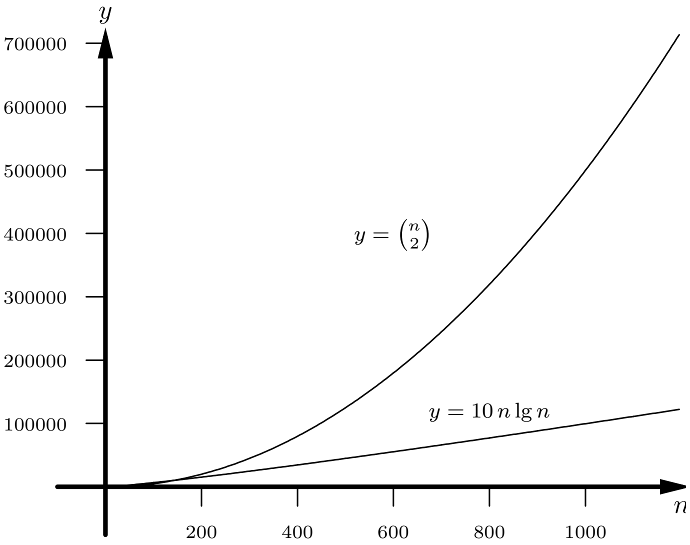
- Comparison-based sorting has lower bound of $\Omega(n \log n)$ comparisons. (We will prove this.)

$\Theta(n \log n)$ work vs. quadratic ($\Theta(n^2)$) work



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Example: The list

$$18 \quad 29 \quad 12 \quad 15 \quad 32 \quad 10$$

has 9 inversions:

$$\{(18,12), (18,15), (18,10), (29,12), (29,15), (29,10), (12,10), (15,10), (32,10)\}$$

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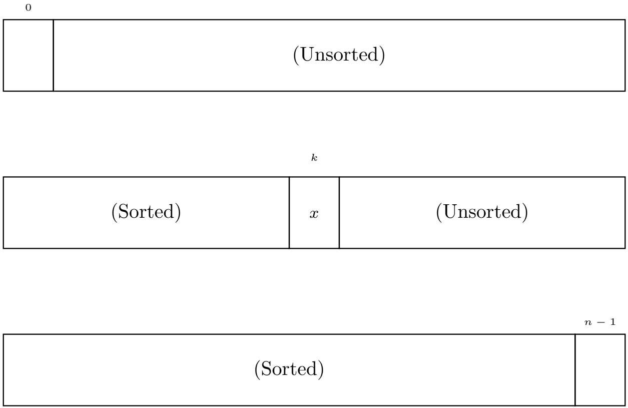
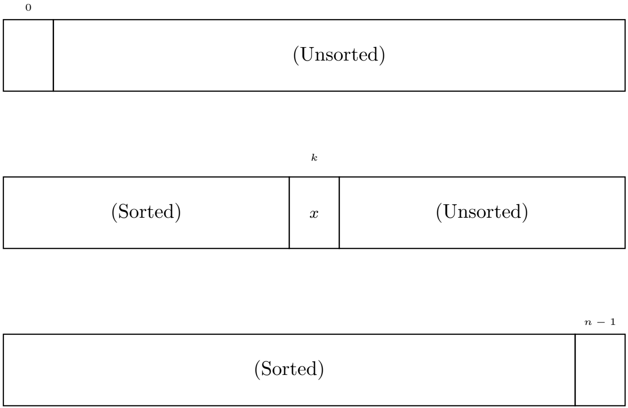


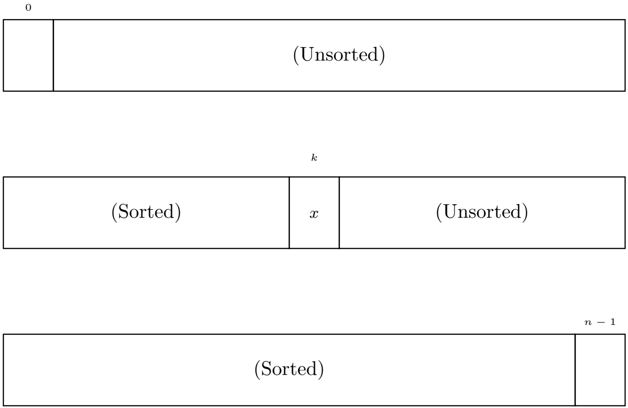
Insertion sort

- Work from left to right across array

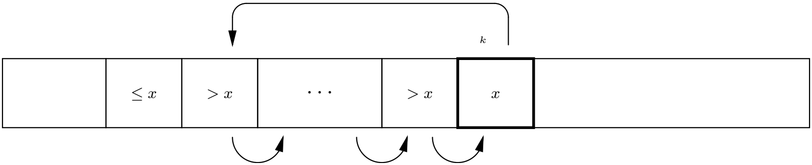


Insertion sort

- Work from left to right across array
- Insert each item in correct position with respect to (sorted) elements to its left



Insertion sort pseudocode




```
def insertionSort(n, A):
    for k = 1 to n-1:
        x = A[k]
        j = k-1
        while (j >= 0) and (A[j] > x):
            A[j+1] = A[j]
            j = j-1
        A[j+1] = x
```

Insertion sort example

23	19	42	17	85	38
23	19	42	17	85	38
19	23	42	17	85	38
19	23	42	17	85	38
17	19	23	42	85	38
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- Storage: in place: $O(1)$ extra storage

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