L13 Myerson's Lemma cont (Bayesian).

CS 280 Algorithmic Game Theory

Ioannis Panageas

Inspired and some figures by Tim Roughgarden notes

Recap (Single parameter)

Three desirable guarantees

1. DSIC: Being truthful is a dominant strategy.
2. Social surplus maximization.
3. Implementation in polynomial time.

Recap (Single parameter)

Three desirable guarantees

1. DSIC: Being truthful is a dominant strategy.
2. Social surplus maximization.
3. Implementation in polynomial time.

Theorem (Myerson's Lemma). *Let (x, p) be a mechanism. We assume that $p_i(b) = 0$ whenever $b_i = 0$, for all bidders i.*

1. *It holds that if (x, p) is DSIC mechanism then x is monotone.*
2. *If x is a monotone allocation, then there is a unique payment rule such that (x, p) is DSIC.*

A (computationally) hard example: Knapsack auctions

- Each bidder i has a publicly known size w_i and a private valuation v_i.
- The seller has capacity W.
- Feasibility set X is all 0-1 n-vectors $(x_1, \ldots, x_n)$ so that $\sum x_i w_i \leq W$.

A (computationally) hard example: Knapsack auctions

- Each bidder i has a publicly known size w_i and a private valuation v_i.
- The seller has capacity W.
- Feasibility set X is all 0-1 n-vectors $(x_1, \ldots, x_n)$ so that $\sum x_i w_i \leq W$.

Remark:

- k-identical item auction is a special case (why)?

Knapsack auctions

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we design the allocation so that we can maximize surplus?
- Step 2: Given our answer to Step 1, how should we set the payments so that DSIC holds?

Knapsack auctions

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we design the allocation so that we can maximize surplus? Let $b_1, \dots, b_n$ the bids of the agents:

$$\begin{aligned} &\max_x \textstyle\sum_{i=1}^n x_i b_i \\ &\text{s.t } \textstyle\sum_{i=1}^n x_i w_i \leq W, \\ &\qquad x_i \in \{0,1\} \text{ for all } i. \end{aligned}$$

- Step 2: Given our answer to Step 1, how should we set the payments so that DSIC holds?

Knapsack auctions

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we design the allocation so that we can maximize surplus? Let $b_1, \ldots, b_n$ the bids of the agents:

$$\begin{aligned} &\max_x \textstyle\sum_{i=1}^n x_i b_i \\ &\text{s.t } \textstyle\sum_{i=1}^n x_i w_i \leq W, \\ &\qquad x_i \in \{0,1\} \text{ for all } i. \end{aligned}$$

This is not LP! It is IP (integer programming).

- Step 2: Given our answer to Step 1, how should we set the payments so that DSIC holds?

Knapsack auctions

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we design the allocation so that we can maximize surplus? Let $b_1, \dots, b_n$ the bids of the agents:

$$
\begin{aligned}
&\max_x \textstyle\sum_{i=1}^n x_i b_i \\
&\text{s.t } \textstyle\sum_{i=1}^n x_i w_i \le W, \\
&\qquad x_i \in \{0,1\} \text{ for all } i.
\end{aligned}
$$

This is not LP! It is IP (integer programming).
The above is called Knapsack, it is NP-complete!

- Step 2: Given our answer to Step 1, how should we set the payments so that DSIC holds?

Knapsack auctions

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we design the allocation so that we can maximize surplus? Let $b_1, \ldots, b_n$ the bids of the agents:

$$\begin{aligned} &\max_x \textstyle\sum_{i=1}^n x_i b_i \\ &\text{s.t } \textstyle\sum_{i=1}^n x_i w_i \leq W, \\ &\qquad x_i \in \{0,1\} \text{ for all } i. \end{aligned}$$

This is not LP! It is IP (integer programming).
The above is called Knapsack, it is NP-complete!

- Step 2: Given our answer to Step 1, how should we set the payments so that DSIC holds? Payment rule from Myerson's Lemma.

Remark: Theory people are not happy with the solution above.

Relaxing Knapsack auctions: Approximation

Approach:

- Step 1 was computationally **intractable**. Instead, how should we design the allocation so that we can **approximately** maximize surplus (**monotone allocation**)? Let $b_1, \dots, b_n$ the bids of the agents:

  First remove all i: $w_i > W$.

  Sort and re-index bidders: $\boxed{\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}.}$

Relaxing Knapsack auctions: Approximation

Approach:

- Step 1 was computationally **intractable**. Instead, how should we design the allocation so that we can **approximately** maximize surplus (**monotone allocation**)? Let $b_1, \ldots, b_n$ the bids of the agents:

  First remove all i: $w_i > W$.

  Sort and re-index bidders: $\boxed{\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}.}$

  Choose as many as possible (say S) so that $\sum_{i=1}^{S} w_i \leq W$ and $\sum_{i=1}^{S+1} w_i > W$.
  Allocate to highest feasible bidder or first S, whichever gives higher surplus.

Relaxing Knapsack auctions: Approximation

Approach:

- Step 1 was computationally **intractable**. Instead, how should we design the allocation so that we can **approximately** maximize surplus (**monotone allocation**)? Let $b_1, \ldots, b_n$ the bids of the agents:

  First remove all i: $w_i > W$.

  Sort and re-index bidders: $\boxed{\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}.}$

  Choose as many as possible (say S) so that $\sum_{i=1}^{S} w_i \leq W$ and $\sum_{i=1}^{S+1} w_i > W$.
  Allocate to highest feasible bidder or first S, whichever gives higher surplus.

- Step 2: Step 1 gives a monotone allocation (why)? We can use the payment rule from Myerson's Lemma.

Relaxing Knapsack auctions: Approximation

Approach:

- Step 1 was computationally **intractable**. Instead, how should we design the allocation so that we can **approximately** maximize surplus (**monotone allocation**)? Let $b_1, \dots, b_n$ the bids of the agents:

  First remove all i: $w_i > W$.

  Sort and re-index bidders: $\boxed{\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}.}$

  Choose as many as possible (say S) so that $\sum_{i=1}^{S} w_i \leq W$ and $\sum_{i=1}^{S+1} w_i > W$.
  Allocate to highest feasible bidder or first S, whichever gives higher surplus.

- Step 2: Step 1 gives a monotone allocation (why)? We can use the payment rule from Myerson's Lemma.

What guarantees the auctioneer has?

Knapsack auctions: 2-Approximation

Theorem (Approximation). *Assuming truthful bids, the surplus of the greedy allocation rule is at least 50% of the maximum-posible surplus.*

Knapsack auctions: 2-Approximation

Theorem (Approximation). *Assuming truthful bids, the surplus of the greedy allocation rule is at least 50% of the maximum-posible surplus.*

Proof. Let S be the number of agents chosen so that (if all agents fit, then this is optimal)

$$\sum_{i=1}^{S} w_i \leq W \text{ and } \sum_{i=1}^{S+1} w_i > W$$

It holds that

$$\sum_{i=1}^{S+1} v_i \geq \mathrm{OPT}.$$

Knapsack auctions: 2-Approximation

Theorem (Approximation). *Assuming truthful bids, the surplus of the greedy allocation rule is at least 50% of the maximum-posible surplus.*

Proof. Let S be the number of agents chosen so that (if all agents fit, then this is optimal)

$$\sum_{i=1}^{S} w_i \leq W \text{ and } \sum_{i=1}^{S+1} w_i > W$$

It holds that

$$\sum_{i=1}^{S+1} v_i \geq \text{OPT}.$$

Hence

$$\boxed{\max\left(\sum_{i=1}^{S} v_i, v_{S+1}\right) \geq \frac{1}{2}\text{OPT}.}$$

Knapsack auctions: 2-Approximation

Theorem (Approximation). *Assuming truthful bids, the surplus of the greedy allocation rule is at least 50% of the maximum-posible surplus.*

Proof. Let S be the number of agents chosen so that (if all agents fit, then this is optimal)

$$\sum_{i=1}^{S} w_i \leq W \text{ and } \sum_{i=1}^{S+1} w_i > W$$

It holds that

$$\sum_{i=1}^{S+1} v_i \geq \text{OPT}.$$

If $A + B \geq$ OPT then $\max(A, B) \geq \frac{OPT}{2}$

Hence

$$\boxed{\max\left(\sum_{i=1}^{S} v_i, v_{S+1}\right) \geq \frac{1}{2}\text{OPT}.}$$

Knapsack auctions: 2-Approximation

Theorem (Approximation)**.** *Assuming truthful bids, the surplus of the greedy allocation rule is at least 50% of the maximum-posible surplus.*

Proof cont. To show $\sum_{i=1}^{S+1} v_i \geq \mathrm{OPT}$, observe that the fractional version (relaxation of IP) has optimal solution $x_1 = \ldots = x_S = 1$ and $x_{S+1} = \frac{W - \sum_{i=1}^{S} w_i}{w_{S+1}}$

LP relaxation

$$\begin{aligned} &\max_x \textstyle\sum_{i=1}^{n} x_i v_i \\ &\text{s.t } \textstyle\sum_{i=1}^{n} x_i w_i \leq W, \\ &\qquad x_i \in [0,1] \text{ for all } i. \end{aligned}$$

Knapsack auctions: 2-Approximation

Theorem (Approximation). *Assuming truthful bids, the surplus of the greedy allocation rule is at least 50% of the maximum-posible surplus.*

Proof cont. To show $\sum_{i=1}^{S+1} v_i \geq \mathrm{OPT}$, observe that the fractional version (relaxation of IP) has optimal solution $x_1 = \ldots = x_S = 1$ and $x_{S+1} = \frac{W - \sum_{i=1}^{S} w_i}{w_{S+1}}$

LP relaxation

$$\begin{aligned} \max_x \textstyle\sum_{i=1}^n x_i v_i \\ \text{s.t } \textstyle\sum_{i=1}^n x_i w_i \leq W, \\ x_i \in [0,1] \text{ for all } i. \end{aligned}$$

Also we have $\boxed{\textbf{OPT of knapsack} \leq \textbf{OPT of LP relaxation}}$

Definitions: Bayesian Setting

Definition (Bayesian - Single parameter setting). *Bayesian setting single parameter environment is defined:*

- *n bidders with private v_i.*
- *Feasible set $\mathcal{X}$, each element of which is a n-dimensional vector $(x_1, ..., x_n)$ in which x_i is the amount of "stuff" given to i.*
- *The private valuation v_i of agent i is assumed to be drawn from a distribution F_i with density f_i and support $[0, v_{\max}]$.*
- *$F_1, ..., F_n$ are independent but not necessarily distributed and are known to the auctioneer.*

Definitions: Bayesian Setting

Definition (Bayesian - Single parameter setting). *Bayesian setting single parameter environment is defined:*

- *n bidders with private v_i.*
- *Feasible set $\mathcal{X}$, each element of which is a n-dimensional vector $(x_1, \ldots, x_n)$ in which x_i is the amount of "stuff" given to i.*
- *The private valuation v_i of agent i is assumed to be drawn from a distribution F_i with density f_i and support $[0, v_{\max}]$.*
- *$F_1, \ldots, F_n$ are independent but not necessarily distributed and are known to the auctioneer.*

Probability to have valuation higher than r.

Intuition:

- 1 item, 1 person. Suppose post price is r. Revenue is $r \cdot (1 - F(r))$

Definitions: Bayesian Setting

Probability to have valuation higher than r.

Intuition:

- 1 item, 1 person. Suppose post price is r. Revenue is

$$r \cdot (1 - F(r))$$

- Reserve price is r means that bidder needs to bid at least r.

Definitions: Bayesian Setting

Probability to have valuation higher than r.

Intuition:

- 1 item, 1 person. Suppose post price is r. Revenue is
$$r \cdot (1 - F(r))$$
- Reserve price is r means that bidder needs to bid at least r.

Question:

- 1 item, 1 person and F is uniform in [0,1]. Suppose post price is r. What r maximizes revenue?

Definitions: Bayesian Setting

Probability to have valuation higher than r.

Intuition:

- 1 item, 1 person. Suppose post price is r. Revenue is $r \cdot (1 - F(r))$
- Reserve price is r means that bidder needs to bid at least r.

Question:

- 1 item, 1 person and F is uniform in [0,1]. Suppose post price is r. What r maximizes revenue?

$$\boxed{\max_{r \in [0,1]} r - r^2 \Rightarrow r = \frac{1}{2}, \text{rev} = \frac{1}{4}}$$

More Definitions

Definition (Payments). *Assume bidders are truthful* ($b = v$). *Recall by Myerson's Lemma:*

$$p_i(v_i, v_{-i}) = \int_0^{v_i} z \cdot \frac{dx_i(z, v_{-i})}{dz} dz.$$

More Definitions

Definition (Payments). *Assume bidders are truthful ($b = v$). Recall by Myerson's Lemma:*

$$p_i(v_i, v_{-i}) = \int_0^{v_i} z \cdot \frac{dx_i(z, v_{-i})}{dz} dz.$$

Valuations are **random variables**, hence we care about the **expectation**:

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} p_i(v_i, v_{-i}) f(v_i) dv.$$

More Definitions

Definition (Payments). *Assume bidders are truthful* ($b = v$). *Recall by Myerson's Lemma:*

$$p_i(v_i, v_{-i}) = \int_0^{v_i} z \cdot \frac{dx_i(z, v_{-i})}{dz} dz.$$

Valuations are **random variables**, hence we care about the **expectation**:

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} p_i(v_i, v_{-i}) f(v_i) dv.$$

Plugging in the above:

$$\boxed{\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x_i'(z, v_{-i}) dz\right] f(v_i) dv_i.}$$

Payments reduce to welfare

Payments: $\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x_i'(z, v_{-i}) dz \right] f(v_i) dv_i.$

Reversing the integration we have

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \left[\int_z^{v_{\max}} f(v_i) dv_i \right] z \cdot x_i'(z, v_{-i}) dz.$$

Payments reduce to welfare

$$\text{Payments: } \mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x_i'(z, v_{-i}) dz \right] f(v_i) dv_i.$$

Reversing the integration we have

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \underbrace{\left[\int_z^{v_{\max}} f(v_i) dv_i \right]}_{1 - F_i(z)} z \cdot x_i'(z, v_{-i}) dz.$$

Payments reduce to welfare

Payments: $\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x_i'(z, v_{-i}) dz \right] f(v_i) dv_i.$

Reversing the integration we have

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \underbrace{\left[\int_z^{v_{\max}} f(v_i) dv_i \right]}_{1-F_i(z)} z \cdot x_i'(z, v_{-i}) dz.$$

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} (1 - F_i(z)) z \cdot x_i'(z, v_{-i}) dz.$$

Payments reduce to welfare

$$\text{Payments: } \mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x_i'(z, v_{-i}) dz \right] f(v_i) dv_i.$$

Reversing the integration we have

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \underbrace{\left[\int_z^{v_{\max}} f(v_i) dv_i \right]}_{1 - F_i(z)} z \cdot x_i'(z, v_{-i}) dz.$$

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} (1 - F_i(z)) z \cdot x_i'(z, v_{-i}) dz.$$

$$= \underbrace{(1 - F_i(z)) z \cdot x_i(z, v_{-i}) \Big|_0^{v_{\max}}}_{\text{this is zero}} - \int_0^{v_{\max}} x_i(z, v_{-i})(1 - F_i(z) - z f_i(z)) dz.$$

Payments reduce to welfare

$$\text{Payments: } \mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x_i'(z, v_{-i}) dz \right] f(v_i) dv_i.$$

Reversing the integration we have

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} \underbrace{\left[\int_z^{v_{\max}} f(v_i) dv_i \right]}_{1 - F_i(z)} z \cdot x_i'(z, v_{-i}) dz.$$

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} (1 - F_i(z)) z \cdot x_i'(z, v_{-i}) dz.$$

$$= \underbrace{(1 - F_i(z)) z \cdot x_i(z, v_{-i}) \Big|_0^{v_{\max}}}_{\text{this is zero}} - \int_0^{v_{\max}} x_i(z, v_{-i})(1 - F_i(z) - z f_i(z)) dz.$$

$$= - \int_0^{v_{\max}} x_i(z, v_{-i}) \frac{(1 - F_i(z) - z f_i(z))}{f_i(z)} f_i(z) dz.$$

Payments reduce to welfare

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = -\int_0^{v_{\max}} x_i(z, v_{-i}) \frac{(1 - F_i(z) - z f_i(z))}{f_i(z)} f_i(z) dz.$$

Set $\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$ (called **virtual** valuations) and we get

Payments reduce to welfare

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = -\int_0^{v_{\max}} x_i(z, v_{-i}) \frac{(1 - F_i(z) - z f_i(z))}{f_i(z)} f_i(z) dz.$$

Set $\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$ (called **virtual** valuations) and we get

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} x_i(z, v_{-i}) \phi(z) f_i(z) dz.$$

$$\Rightarrow \quad \boxed{\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \mathbb{E}_{v_i \sim F_i}[\phi(v_i) x_i(v_i, v_{-i})]}$$

Payments reduce to welfare

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = -\int_0^{v_{\max}} x_i(z, v_{-i}) \frac{(1 - F_i(z) - z f_i(z))}{f_i(z)} f_i(z) dz.$$

Set $\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$ (called **virtual** valuations) and we get

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} x_i(z, v_{-i}) \phi(z) f_i(z) dz.$$

$$\mathbb{E}_{v_i \sim F_i}[p_i(v_i, v_{-i})] = \mathbb{E}_{v_i \sim F_i}[\phi(v_i) x_i(v_i, v_{-i})]$$

$$\text{Rev} = \mathbb{E}_{v \sim F_1, \ldots, F_n}\left[\sum_i p_i(v)\right] = \mathbb{E}_{v \sim F_1, \ldots, F_n}\left[\sum_i x_i(v) \phi_i(v)\right]$$

Monotone Allocations for regular F

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we design the allocation so that we can maximize virtual social welfare, $\sum x_i(v)\phi_i(v)$?

- Step 2: Given our answer to Step 1, how should we set the payments so that DSIC holds? According to Myerson's Lemma

Monotone Allocations for regular F

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we design the allocation so that we can maximize virtual social welfare, $\sum x_i(v)\phi_i(v)$?
- Step 2: Given our answer to Step 1, how should we set the payments so that DSIC holds? According to Myerson's Lemma

Issue: Higher valuation v_i gives higher x_i (is the allocation monotone)? Depends on F_i.

Monotone Allocations for regular F

Approach:

- Step 1: Assume, without justification, that bidders bid truthfully. How should we design the allocation so that we can maximize virtual social welfare, $\sum x_i(v)\phi_i(v)$?
- Step 2: Given our answer to Step 1, how should we set the payments so that DSIC holds? According to Myerson's Lemma

Issue: Higher valuation v_i gives higher x_i (is the allocation monotone)? Depends on F_i.

Definition (Regular F). *A distribution F is regular if the corresponding virtual valuation function $v - \frac{1-F(v)}{f(v)}$ is strictly increasing.*

Monotone Allocations for regular F

Definition (Regular F)**.** *A distribution F is regular if the corresponding virtual valuation function $v - \frac{1-F(v)}{f(v)}$ is strictly increasing.*

It turns out that if F_i are regular, then in step 1,
x is monotone.

Monotone Allocations for regular F

Definition (Regular F). *A distribution F is regular if the corresponding virtual valuation function $v - \frac{1-F(v)}{f(v)}$ is strictly increasing.*

It turns out that if F_i are regular, then in step 1, x is monotone.

Example (Uniform is Regular): Let F be the uniform in $[0,1]$. The valuation is $2v - 1$ which is strictly increasing.

Example

Consider a single item and n bidders with same F being regular.

Question: What is the allocation rule and the payment?

Example

Consider a single item and n bidders with same F being regular.

Question: What is the allocation rule and the payment?

1) Give the item to the bidder with highest positive virtual valuation.

Example

Consider a single item and n bidders with same F being regular.

Question: What is the allocation rule and the payment?

1) Give the item to the bidder with highest positive virtual valuation.
2) Since virtual is strictly increasing, the winner is the highest bidder, thus the allocation is monotone!

Example

Consider a single item and n bidders with same F being regular.

Question: What is the allocation rule and the payment?

1) Give the item to the bidder with highest positive virtual valuation.
2) Since virtual is strictly increasing, the winner is the highest bidder, thus the allocation is monotone!
3) The winner i pays $\phi_i(v_i)$.

Observe that this is a Vickrey auction with reserve price $\phi^{-1}(0)$. If valuations come from $[0,1]$, to maximize welfare, set $r = \frac{1}{2}$.