Tight polyhedral models of isoparametric families, and PL-taut submanifolds

THOMAS F. BANCHOFF AND WOLFGANG KÜHNEL

ABSTRACT: We present polyhedral models for isoparametric families in the sphere with at most three principal curvatures. Each member of the family (including the analogues of the focal sets) is tight in the boundary complex of an ambient convex polytope. In particular, the tube around the real (or complex) Veronese surface is represented as a tight polyhedron in 5-space (or 8-space). The examples are based on a certain Bier sphere triangulation of S^4 or S^7, respectively. In the 4-dimensional case there are simplicial branched coverings of these triangulations in the complex projective plane and in $S^2 \times S^2$ which are branched precisely along the polyhedral analogues of the Veronese surface. Moreover, we introduce a notion of PL-tautness and discuss its relationship with tightness of polyhedra. In particular, each member of our polyhedral isoparametric family is PL-taut. For an extended abstract see [17].

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1. Introduction and Result

By a theorem of E.Cartan [11] all isoparametric families of hypersurfaces in the sphere with at most three principal curvatures are given by the following list:

1. tubes around a point in S^n
2. tubes around a great sphere $S^k \subset S^n$ where $1 \le k \le n-2$
3. tubes around any of the Veronese-type standard embeddings of the projective planes $\mathbb{R}P^2 \to S^4$, $\mathbb{C}P^2 \to S^7$, $\mathbb{H}P^2 \to S^{13}$, or $\mathbb{O}P^2 \to S^{25}$.

In these three cases we have 1, 2 or 3 constant principal curvatures, respectively. Topologically, the hypersurfaces in Case 3 are total spaces of S^k-bundles over the projective plane over $\mathbb{F}(k)$ where the real dimension of $\mathbb{F}$ is k. In particular, the dimension of the total space is $3k$ in each of the cases. In addition isoparametric hypersurfaces have the geometric property of tightness and tautness, see [13]. Recall that in homogeneous coordinates the standard embeddings of $\mathbb{R}P^2, \mathbb{C}P^2, \mathbb{H}P^2$ are given by $[z_0, z_1, z_2] \mapsto (z_0\overline{z_0}, z_1\overline{z_1}, z_2\overline{z_2}, \sqrt{2}z_0\overline{z_1}, \sqrt{2}z_1\overline{z_2}, \sqrt{2}z_2\overline{z_0})$ with the normalization $z_0\overline{z_0} + z_1\overline{z_1} + z_2\overline{z_2} = 1$.

Definition An embedding $M \to \mathbb{E}^N$ of a compact manifold is called tight, if for any open half space $E_+ \subset \mathbb{E}^N$ the induced homomorphism

$$H_*(M \cap E_+) \longrightarrow H_*(M)$$

is injective where H_* denotes an appropriate homology theory with coefficients in a certain field. The notion of k-tightness refers to the injectivity in the low dimensions $H_i(M \cap E_+) \to H_i(M)$, $i = 0, \dots, k$. The notion of tightness is projectively invariant. Tightness of a subset means that it is embedded *as convexly as possible*. In the smooth case (and, with certain modifications, also in the polyhedral case) this is equivalent to the condition that almost all height functions on M are perfect functions, i.e., have the minimum number of critical points, see [6]. The similar notion of tautness refers to the condition that almost all distance functions are perfect functions, see [13]. A polyhedral analogue of tautness will be discussed in Section 4 below.

It is well known that the ε-tube around any taut submanifold or around any embedded tight submanifold is again taut or tight, respectively. The reason is that the cohomology ring of the total space of the unit normal bundle is isomorphic to the tensor product of the cohomology rings of the base and the fibre, compare [10].

Even though it is not obvious what constant principal curvatures would mean for polyhedral hypersurfaces in general, the tightness condition can be carried over easily. Tightness for polyhedra was first introduced by the first author in [3], for the further development see [23], [16], [6]. A polyhedral model for Cartan's isoparametric hypersurface in the 4-sphere was given in [7], combinatorially as an 8-fold quotient of the 600-cell, but that was not tight. In this paper we present polyhedral versions of the whole family in all of the cases mentioned above (and such that each member of the polyhedral family has the tightness property) except for the last one which is still open. In fact, a tight polyhedral octonion projective plane itself is still missing. In the quaternionic case a polyhedral model is available but it seems to be beyond reach to give a complete proof. Our main result is the following:

Theorem 1 *In each of the cases of an isoparametric hypersurface of S^n mentioned above (except possibly for the octonion case) there is a simplicial n-sphere in Euclidean space such that the following conditions are satisfied:*

1. *It contains two disjoint simplicial subcomplexes triangulating the two focal sets of the isoparametric family as a kind of "top" and "bottom" of the simplicial n-sphere (for the case of $\mathbb{H}P^2$ see below),*

2. *each member of the isoparametric family corresponds to a slice through this n-sphere between top and bottom,*

3. *each member of the family (including the focal sets) is a tight polyhedral submanifold in the boundary complex of a certain convex $(n+1)$-polytope. So in particular the real Cartan hypersurface is tight in the boundary complex of a 5-polytope, the complex Cartan hypersurface is tight in the boundary complex of an 8-polytope.*

In the case of the quaternionic Cartan hypersurface these polyhedral models exist, but a complete proof of their topological properties is not available. In the case of the octonion Cartan hypersurface an appropriate triangulation of the focal set is still missing. If there exists a tight 27-vertex triangulation of $\mathbb{O}P^2$ then the theorem includes this case as well.

One reason for the missing proof in the quaternionic case is that a combinatorial calculation of Pontrjagin classes seems to be extremely difficult. A combinatorial analogue of the focal set is available [9] which leads to the whole isoparametric family. However, so far the topology of the polyhedral focal set cannot be determined. In the octonion case the problem is that such a triangulation of the focal set $\mathbb{O}P^2$ would have a huge number of simplices. In particular, it would have 100386 top-dimensional simplices, see [16, Sect.4C].

The proof will make use of the following three ingredients:

1. Higher-dimensional octahedra,
2. Tight triangulations of the projective planes over $\mathbb{R}$ and $\mathbb{C}$,
3. Sarkaria's deleted join of a simplicial complex with itself, and the Bier sphere.

2. Proof: Subcomplexes and slices of cross polytopes

In order to give a unified description for all cases, we will use the $(n+1)$-dimensional cross polytope β_{n+1} (also called $(n+1)$-dimensional octahedron) which is defined as the convex hull of the points

$$(0,\dots,0,\underbrace{\pm 1}_{i},0,\dots,0), \quad i=0,1,\dots,n$$

in $(n+1)$-space.

Case 1: In the first case we pick two antipodal vertices, say, $(\pm 1,0,\dots,0)$. Then the polyhedral model of the isoparametric family with one principal curvature is just given by all the slices through $\partial\beta_{n+1}$ by hyperplanes orthogonal to $(1,0,\dots,0)$. Combinatorially each member of the family is a $\partial\beta_n$, except for the two degenerate cases which are just two points. Each member of the family is a convex polyhedron in n-space and is therefore tight.

Case 2: In the case of two principal curvatures we start with a β_{k+1} as the subcomplex of β_{n+1} given by all vertices above where $0 \le i \le k$ and a complementary β_{n-k} given by all vertices with $k+1 \le i \le n$. As a matter of fact, the boundary $\partial\beta_{n+1}$ is just the join complex $\partial\beta_{k+1} * \partial\beta_{n-k}$ where, as usual, the join $\triangle^k * \triangle^{n-k-1}$ of two simplices $\triangle^k = \langle v_0, v_1, \ldots, v_k\rangle$ and $\triangle^{n-k-1} = \langle v_{k+1}, \ldots, v_n\rangle$ is defined as the simplex $\triangle^n = \langle v_0, v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\rangle$. Since each of the vertices of β_{n+1} is either in β_{k+1} or in the complementary β_{n-k}, we can define a simplexwise linear function f on the boundary complex of β_{n+1} which is 0 on $\partial\beta_{k+1}$ and 1 on $\partial\beta_{n-k}$. More precisely, f is assumed to be affine linear in the barycentric coordinates on each simplex, i.e., $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$ where $\sum_i \lambda_i = 1$.

Now the polyhedral analogue of the isoparametric family is given by the levels of the function f. Clearly each level set $f^{-1}(t)$ defines a polyhedral manifold, for $0 < t < 1$ the level set is a polyhedral decomposition of $S^k \times S^{n-k-1}$ as a subset of $\partial\beta_{n+1} \cong S^n$. Similarly, $f^{-1}([0,t])$ and $f^{-1}([t,1])$ are polyhedral "solid tori" as tubes around the k-sphere or $(n-k-1)$-sphere, respectively. Note that this family is also given by the level sets of the intrinsic distance function in $\partial\beta_{n+1}$ from each of the two "focal sets". This function is nothing but f or $1-f$.

We show that each of these level sets $f^{-1}(t)$ is tightly embedded into $(n+1)$-space. Recall that the original isoparametric family is given by all points $(\sin tx, \cos ty) \in S^n$ where $0 < t < 1,\ x \in S^k,\ y \in S^{n-k-1}$. Each t-level set is a cartesian product of a k-sphere and an $(n-k-1)$-sphere, like the Clifford torus. Similarly, the polyhedral version is given by all points $(tx, (1-t)y)$ where $0 < t < 1,\ x \in \partial\beta_{k+1},\ y \in \partial\beta_{n-k}$. Again this is a cartesian product of two tight polyhedral spheres and is therefore tight. Case 1 may be considered as the special case $k = 0$ in Case 2.

Case 3: In the third case of three principal curvature we have to consider the tubes around two antipodal real or complex Veronese-type embeddings $\mathbb{R}P^2 \to S^4$ or $\mathbb{C}P^2 \to S^7$, respectively. The quaternionic case will be discussed at the end.

First of all, there are tight polyhedral analogues of these Veronese-type embeddings themselves. These are the canonical embeddings of the unique 6-vertex triangulation $\mathbb{R}P^2_6$ of $\mathbb{R}P^2$ into the 5-simplex and of the unique 9-vertex triangulation $\mathbb{C}P^2_9$ of $\mathbb{C}P^2$ into the 8-simplex, see [18],[19].
For our purpose we have to find an appropriate triangulation of the 4-sphere or 7-sphere, respectively, which contains two antipodal copies of them, linking one another as required by the Cartan decomposition.

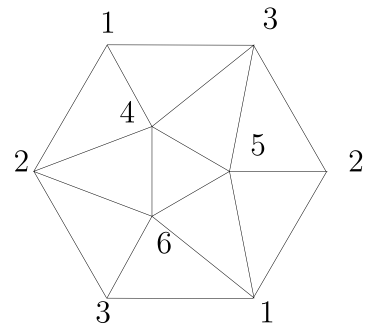

Figure 1: The 6-vertex triangulation of $\mathbb{R}P^2$

Definition

(1) The deleted join $K *_\Delta K$ of a simplicial complex K with itself is a part of the ordinary join of two disjoint copies K_1 and K_2 of K where we take the join of only those two simplices in K_1 and K_2, respectively, which are disjoint in K. So in particular, each vertex of K leads to a missing edge (a *diagonal*) in $K *_\Delta K$.

(2) Similarly we have the deleted join $K *_\Delta K^*$ of an n-vertex simplicial complex K with its combinatorial Alexander dual K^* where the combinatorial Alexander dual is defined as the set of the complements of the non-faces of K. Here we think of a face as a subset of $\{1, 2, \dots, n\}$ and its complement as the set-theoretic complement. Accordingly, a non-face is a subset that does not correspond to a face in the complex. The vertex set of the deleted join will be denoted by $\{1, 2, \dots, n, \overline{1}, \overline{2}, \dots, \overline{n}\}$ with diagonals $1\overline{1}, 2\overline{2}, \dots, n\overline{n}$.

The notion of the deleted join is due to K.Sarkaria [31]. Sarkaria also pointed out to the second author that the deleted join of $\mathbb{R}P^2_6$ with itself is a triangulated 4-sphere. In fact, from Figure 1 it is easily seen that $(\mathbb{R}P^2_6)^* = \mathbb{R}P^2_6$. Similarly, we have $(\mathbb{C}P^2_9)^* = \mathbb{C}P^2_9$. Together with this observation our main result is based on the following very general theorem:

Theorem 2 [30, p.112],[24] *For any given simplicial complex K with n vertices the deleted join of K with its combinatorial Alexander dual K^* is a triangulated $(n-2)$-sphere with at most $2n$ vertices. It is called the* Bier sphere *$Bier_n(K)$ because it was discovered by Thomas Bier in* 1992. *After subdivision, the Bier sphere coincides with the first barycentric subdivision of an $(n-1)$-simplex* [24].

In the case of $K = \mathbb{R}P^2_6 = (\mathbb{R}P^2_6)^*$ a direct verification of this theorem is not hard to obtain: Since the combinatorial automorphism group acts transitively on vertices of $\mathbb{R}P^2_6$, it also acts transitively on vertices of $\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$. So if one can show that the link of each vertex is a 3-sphere, then this triangulation is a combinatorial 4-manifold. Moreover, since $\langle 456\rangle$ is a triangle of $\mathbb{R}P^2_6$ (see Figure 1) then by construction the span of $\{1,2,3,\bar{4},\bar{5},\bar{6}\}$ in $\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$ is nothing but the join $\partial\langle 123\rangle * \langle\bar{4}\bar{5}\bar{6}\rangle$ which is a polyhedral ball. By symmetry the span of $\{\bar{1},\bar{2},\bar{3},4,5,6\}$ is a polyhedral ball as well. Then it follows that $\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$ is a 4-sphere, since it is represented as the union of two polyhedral 4-balls glued together along their boundaries. In fact $\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$ is the only triangulation of the 4-sphere with 12 vertices admitting a vertex transitive automorphism group, see [25, p.52].

Let us recall the following facts about triangulations of projective planes with the minimum number of vertices.

Theorem 3

1. [8] *Any simplicial n-vertex triangulation of a combinatorial $2k$-manifold M satisfies $n \geq 3k+3$ unless M is a sphere. In case of equality $n = 3k+3$ we have necessarily $k = 0,1,2,4,8$, and for $k \geq 1$ M has the same cohomology ring as the projective plane over $\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$, respectively. Moreover, for $k = 1,2$ the triangulation is combinatorially isomorphic with $\mathbb{R}P^2_6$ or $\mathbb{C}P^2_9$, respectively.*

2. [2] *Any combinatorial $2k$-manifold with $n = 3k+3$ vertices (which is not a sphere) satisfies the following* combinatorial complementarity condition*:*

   - Any subset of vertices spans a simplex in the triangulation if and only if the complementary subset does not.

   In particular, if K denotes the simplicial complex triangulating the manifold, then we have $K^* = K$, and K is $(k+1)$-neighborly which means that any $(k+1)$-tuple of vertices spans a simplex in K.

3. [9] *In the cases $k = 0,1,2,4$ there exists such a combinatorial manifold with $3,6,9,15$ vertices, respectively. It is unique for $k = 0,1,2$ and not unique for $k = 4$. For $k = 8$ the existence is still open.*

Corollary 4 *If K denotes any simplicial complex triangulating a combinatorial $2k$-manifold with $n = 3k+3$ vertices which is not a sphere, then the deleted join $Bier_n(K) = K *_\Delta K$ is a combinatorial sphere of dimension $n-2$ with $2n$ vertices. It can be regarded as a subcomplex of the cross polytope β_n.*

In particular, this applies to the triangulations $K = \mathbb{R}P^2_6$ and $K = \mathbb{C}P^2_9$. In these cases the deleted join coincides with the Bier sphere $Bier_6(\mathbb{R}P^2_6)$ or $Bier_9(\mathbb{C}P^2_9)$, respectively.

Recall that the Veronese-type standard embedding maps each of the four projectives planes into the $(n-2)$-sphere where $n = 3k+3$. Compare [29] for the topological significance of these embeddings and their normal bundles or tubes around them.

Exercise: It might be instructive to study the trivial case $k = 0$ first. In this case the Bier sphere is the deleted join of a discrete 3-point space with itself. In fact, a 3-point space satisfies the combinatorial complementarity condition. If we realize the two sets of points as the vertex sets of the top and bottom triangles in an ordinary octahedron in 3-space, then the Bier sphere is a skew hexagon going up and down in the edge graph of the octahedron. Then the "isoparametric family" is the set of all levels in between. Each member of the family consists of six points as the intersection of the skew hexagon with a plane parallel to top and bottom of the octahedron. These fix points degenerate to three points in two different ways, corresponding to the two "focal sets". In the sequel we are going to generalize this construction to the cases $k = 1$ and $k = 2$.

Proposition 5 *$\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$ is a triangulated 4-sphere and $\mathbb{C}P^2_9 *_\Delta \mathbb{C}P^2_9$ is a triangulated 7-sphere. By construction we can regard the former one as a subcomplex of β_6 and the latter one as a subcomplex of β_9, respectively. If we triangulate the two antipodal Veronese-type embeddings $\mathbb{R}P^2$ into S^4 and of $\mathbb{C}P^2$ into S^7 by two copies of $\mathbb{R}P^2_6$ or $\mathbb{C}P^2_9$, respectively, then this extends to a PL homeomorphism between the standard PL structure on the ordinary sphere and the Bier sphere triangulation. Moreover, the two antipodal Veronese-type embeddings of $\mathbb{R}P^2$ into S^4 and of $\mathbb{C}P^2$ into S^7 are isotopic to the two canonical embeddings of $\mathbb{R}P^2_6$ into $\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$ and of $\mathbb{C}P^2_9$ into $\mathbb{C}P^2_9 *_\Delta \mathbb{C}P^2_9$, respectively.*

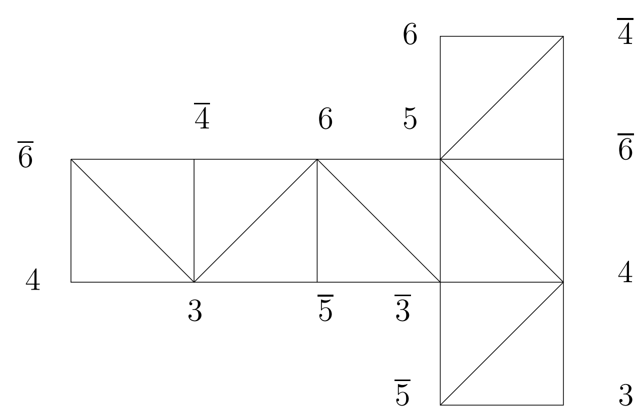

Figure 2: The link of the edge $\langle 1\overline{2}\rangle$ in $\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$

The first claim of Proposition 5 is contained in Theorem 2. See Figure 2 for a picture of the link of a typical edge. One easily recognizes a subdivided cube. Observe the two disjoint zigzags given by the diagonals in the cube. The second claim is obvious from the construction. In order to prove the last two claims we can use the fact that there are some smoothings of the polyhedral models within the same isotopy class. On the other hand the normal bundle of any embedding of $\mathbb{R}P^2$ into S^4 and of $\mathbb{C}P^2$ into S^7 is uniquely determined, see [29]. Furthermore we know that the complement is a neighborhood of the antipodal (smooth or polyhedral) $\mathbb{R}P^2$ in S^4 or $\mathbb{C}P^2$ in S^7, respectively. This implies that there is an isotopy respecting the antipodal pairs. In other words, we can find a deformation of the polyhedral Bier sphere into the standard smooth sphere respecting the antipodal pairs of focal sets. In the complex case it is even true that a smooth embedding of $\mathbb{C}P^2$ into S^7 is essentially unique (up to mirror image), see [15].

In more concrete and geometric terms one can argue as follows: If we think of the 10 triangles of $\mathbb{R}P^2_6$ as a pattern on the Veronese surface then the process of pulling each of them straight into a planar triangle by a kind of convex combination can be realized by a homeomorphism of the ambient 4-sphere. The linking behavior of the two antipodal copies is the same for the Veronese-type models and for the polyhedral models: The complement of each of the focal sets retracts (or collapses) onto the other, and the complement itself is an open ball bundle over the opposite focal set. Therefore, the same procedure can be carried out for the pair of two disjoint antipodal Veronese surfaces and the 20 triangles of them. It remains to find the simplices of the Bier sphere in the round 4-sphere. For carrying this out we can think of a distorted 4-sphere as the boundary of the convex hull of the two antipodal Veronese surfaces. In the boundary of this convex hull we find certain flat parts which correspond to the union of the interiors of the simplices of the Bier sphere. Then it remains to deform the boundary of that convex hull into the round sphere by central projection. The same procedure can be carried out in the complex case. $\square$

A particular consequence of Proposition 5 is that the tubes around the classical Veronese models and around the embeddings $\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6 \to \beta_6$ and $\mathbb{C}P^2_9 *_\Delta \mathbb{C}P^2_9 \to \beta_9$ within the Bier sphere are topologically equivalent. In our case the two polyhedral focal sets are represented as the canonical embedding of $\mathbb{R}P^2_6$ into one 5-dimensional facet of β_6 spanned by vertices 1,2,3,4,5,6 and into its antipodal facet spanned by vertices $\bar{1}, \ldots, \bar{6}$. Moreover, the polyhedral analogue of the isoparametric family can be defined by the levels of the simplexwise linear function f attaining the value 0 at the vertices $1, \ldots, 6$ (the bottom of the cross polytope) and the value 1 at the vertices $\bar{1}, \ldots, \bar{6}$ (the top). Each level consists of 60 prisms which are slices through 4-simplices, for a typical case see Figure 3. Similarly, in the complex case we have the two antipodal copies of

$\mathbb{C}P^2_9$ in 8-dimensional facets spanned by vertices $1, 2, \ldots, 9$ and $\bar{1}, \ldots, \bar{9}$, respectively, and the analogues levels of the function f. Observe that the link of a typical vertex in the slice is combinatorially equivalent to the link of the corresponding edge in the Bier sphere itself, see Figure 4 and Figure 2.

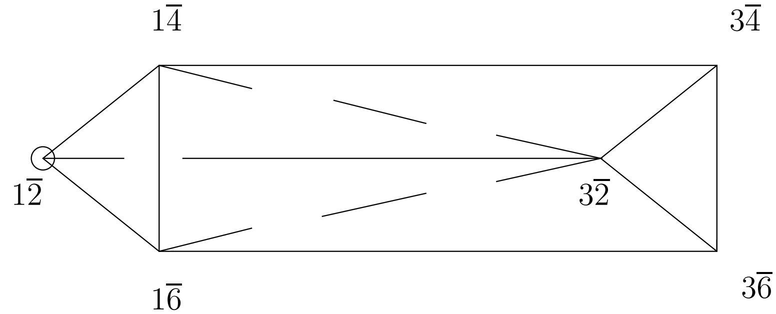

Figure 3: slice through the simplex $\langle 13\bar{2}\bar{4}\bar{6}\rangle$ together with the link of $1\overline{2}$

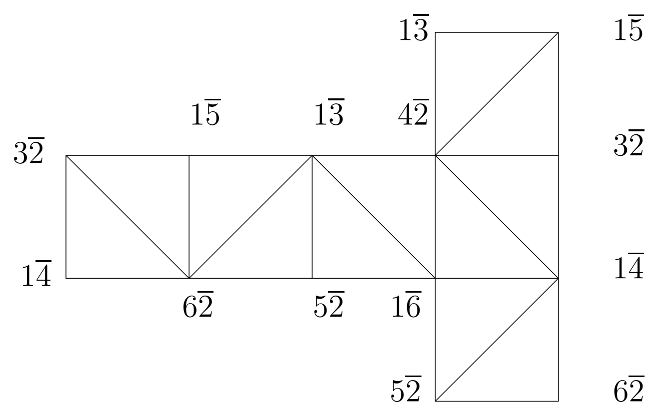

Figure 4: The link of the vertex $1\overline{2}$ in the slice through $\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$

The proof of our main Theorem 1 will be completed by the following lemma:

Lemma 6 *For any $0 \leq t \leq 1$ the two embeddings*

$$f^{-1}(t) \cap \mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6 \longrightarrow f^{-1}(t) \cap \partial\beta_6 \longrightarrow \mathbb{E}^5$$

$$f^{-1}(t) \cap \mathbb{C}P^2_9 *_\Delta \mathbb{C}P^2_9 \longrightarrow f^{-1}(t) \cap \partial\beta_9 \longrightarrow \mathbb{E}^8$$

are tight with respect to $\mathbb{Z}_2$. These are polyhedral analogues of the family of isoparametric tubes around the real or complex Veronese-type surface, respectively.

Proof. First of all, the two polyhedral focal sets are given by the 0-level and the 1-level, i.e., by the tight triangulations of $\mathbb{R}P^2_6$ or $\mathbb{C}P^2_9$, respectively. Since the tight polyhedral focal sets are 2-neighborly or 3-neighborly, respectively, it follows that each t-level between them contains the complete 1-skeleton or 2-skeleton of the boundary of its convex hull, respectively. Therefore, the embedding is 0-tight in the real case and 1-tight in the complex case. By duality, for the tightness of a 3-manifold or 6-manifold it is sufficient to prove in addition the 1-tightness or 2-tightness, respectively.

Here we need an explicit description of the $\mathbb{Z}_2$-homology of the spaces. It is well known that the cohomology ring of the tube around a submanifold is the tensor product of the cohomology rings of the base and the fibre. The cohomology ring of a projective plane is a truncated polynomial ring in one variable, the fibres are spheres. Consequently, the $1st$ homology of the real isoparametric hypersurface is generated by two generators c and $\overline{c}$ where c corresponds to the first homology of $\mathbb{R}P^2$ and $\overline{c}$ to the unit normal 1-sphere around the Veronese-type embedding. Similarly, the 2^{nd} homology of the complex isoparametric hypersurface is generated by two generators c and $\overline{c}$ where c corresponds to the second homology of $\mathbb{C}P^2$ and $\overline{c}$ to the unit normal 2-sphere around it. However, if we interchange the two focal sets then the roles of c and $\overline{c}$ are also interchanged. So we can regard $\overline{c}$ as being homologous to a generator of the middle homology in the antipodal $\mathbb{R}P^2$ or $\mathbb{C}P^2$, respectively. In [14] the following interpretation is given: One of the two focal sets is the point space of a projective plane over $\mathbb{R}$ or $\mathbb{C}$, respectively, and the other one is the line space of the same projective plane, i.e., the space of all projective lines. The tubular sphere around a point in the former then corresponds to the set of all projective lines through that point in the latter, and, conversely, a tubular sphere around a point in the latter (which is a projective line) corresponds to the set of all points on that line. Their intersection behavior in the Cartan hypersurface reflects the incidence structure of the projective plane. In our terminology these tubular spheres represent the generators c and $\overline{c}$ of the homology H_1 or H_2, respectively. Topologically, this means that c shrinks onto a projective line on one side and onto a tubular S^1 or S^2 around the other side, similarly for $\overline{c}$ where the roles are interchanged. All these considerations are analogous for the Hopf decomposition of the 3-sphere into tori. Here the two focal sets correspond to two opposite circles.

In order to prove the 1-tightness or 2-tightness of our polyhedral analogues of the Cartan hypersurfaces, let us denote the vertex set by $V \cup \overline{V}$ where each vertex occurs in a version $v \in V$ and in its antipodal version $\overline{v} \in \overline{V}$. Assume that V spans one focal set $\mathbb{R}P^2_6$ or $\mathbb{C}P^2_9$ and $\overline{V}$ spans the antipodal focal set $\overline{\mathbb{R}P}^2_6$ or $\overline{\mathbb{C}P}^2_9$. If the intersection with a halfspace H_+ contains two antipodal vertices v and $\overline{v}$ then it contains also the center point of the ambient cross polytope. Therefore in this case it contains at least one vertex of any antipodal pair, so at least 7 vertices of the original $\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$ or

10 of the original $\mathbb{C}P^2_9 *_\Delta \mathbb{C}P^2_9$, respectively. By duality, it is sufficient to consider the case where we have at most 6 or 9 vertices, respectively, not all belonging to V or $\overline{V}$. In this case the original vertices in H_+ split into $A \cup \overline{B}$ where $A \subset V, \overline{B} \subset \overline{V}$, $A \cap B = \emptyset$ and $A, B \neq \emptyset$. The question is: What is the $\mathbb{Z}_2$-homology H_1 or H_2, respectively, of the intersection of H_+ with the t-level of the function f?

Obviously, this will depend on the homology of the span of A in the focal set and the span of $\overline{B}$ in the antipodal focal set. If both are contractible then the intersection of H_+ with the t-level is also contractible. The 2-dimensional homology H_1 or H_2, respectively, is generated by the generator c and $\overline{c}$ of H_1 or H_2 in each of the antipodal focal sets, respectively. These correspond to pointwise tubular sphere "running around the focal set" in two different ways. Now, if the span of A is homologous to c and the span of B is homologous to zero, then the intersection of H_+ with the t-level is homologous to c also. Similarly, if the span of A is homologous to zero and the span of B is homologous to $\overline{c}$, then the intersection of H_+ with the t-level is homologous to $\overline{c}$. In any case, the intersection with H_+ injects at the homology level H_1 or H_2, respectively. The tightness follows.

As an example, we consider the case $A = \{1,2,3\}, \overline{B} = \{\overline{4}, \overline{5}, \overline{6}\}$. The span of 1,2,3 is homologous to c, the other part is contractible, even collapsible. Therefore, the span of $A \cup \overline{B}$ in the t-level collapses onto the 1-cycle $1\overline{4}\ 2\overline{4}\ 3\overline{4}$ which injects at the homology level to one of the generators there. Similarly, the span of $\{1,2,3,\overline{1},\overline{4},\overline{5},\overline{6}\}$ collapses onto the figure eight spanned by the two 1-cycles $1\overline{4}\ 2\overline{4}\ 3\overline{4}$ and $2\overline{4}\ 2\overline{1}\ 2\overline{6}$ which injects also because it represents the sum of the two generators. □

Remarks: 1. In the quaternionic case there is a perfect candidate for a tight 15-vertex triangulation of $\mathbb{H}P^2$, see [9], compare also [25, p.65]. There is, however, no formal proof that it is really a triangulation of $\mathbb{H}P^2$ although it shares many properties with it. Its deleted join is the Bier sphere triangulation of the 13-sphere with 30 vertices. By construction it is a subcomplex of $\partial\beta_{15}$. By the same argument as above, the t-level sets in between the two focal sets define a family of tight polyhedral analogues of the quaternionic isoparametric family in some 14-space. Moreover, the results of [29] on the normal bundles of the embedded projective planes can be applied to the quaternionic and the octonion case also.

2. In the octonion case it is still open whether or not there is an appropriate tight 27-vertex triangulation. If yes, its deleted join would yield a Bier triangulation of the 25-sphere as a subcomplex of $\partial\beta_{27}$, and the level sets would be tight polyhedral versions of the corresponding isoparametric family.

3. One can ask whether there are similar polyhedral analogues of isoparametric families with 4 or 6 principal curvatures. It seems these are not yet available. There is a tight triangulation for one focal set in one of the standard examples with 4 principal

curvatures. The focal set is the "complexified sphere". There is a $(2n+3)$-vertex triangulation of the product $S^1 \times S^{n-1}$ for even n and of the twisted product $S^1 \underline{\times} S^{n-1}$ for odd n, see [16, 5B]. However, it is not clear how this can be embedded into an appropriate triangulated sphere which contains the other focal set as well. The Bier sphere is not suitable here because the combinatorial Alexander dual of the triangulation above is not a manifold.

3. Branched simplicial coverings related to $\mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$

It was pointed out by Massey in [28] that incidentally a number of interesting 4-manifolds (among them the complex projective plane) are (branched or non-branched) quotients of $S^2 \times S^2$. In particular, $\mathbb{C}P^2$ is the quotient of $S^2 \times S^2$ by the involution $\tau(x, y) = (y, x)$, and the 4-sphere is the quotient of $\mathbb{C}P^2$ modulo complex conjugation σ where $\sigma[z_0, z_1, z_2] = [\overline{z_0}, \overline{z_1}, \overline{z_2}]$. In the latter case the branch locus consists precisely of the real part which is a real projective plane. Opposite to it we find the complex quadric $z_0^2 + z_1^2 + z_2^2 = 0$ on which the involution σ acts freely. These two submanifolds happen to be the top and bottom critical level of the function $f\colon \mathbb{C}P^2 \to [0, 1]$ defined by

$$f(z_0, z_1, z_2) = |z_0^2 + z_1^2 + z_2^2|^2$$

with the normalization $|z_0|^2 + |z_1|^2 + |z_2|^2 = 1$. The levels in between can be interpreted as tubes around the real projective plane as well as the opposite 2-sphere.

An elementary polyhedral proof for $\mathbb{C}P^2/\sigma \cong S^4$ was given in [5] but the branched simplicial covering required a certain subdivision of a triangulation of $\mathbb{C}P^2$. The Sarkaria-Bier sphere $Bier_6(\mathbb{R}P^2_6) = \mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$ gives a possibility to realize this branched covering in a canonical and fairly symmetrical way as a simplicial map from a triangulated $\mathbb{C}P^2$ with 18 vertices onto a triangulated S^4 with 12 vertices, branched along a 6-vertex $\mathbb{R}P^2$. In this case all the data are contained in the ordinary icosahedron, a Platonic solid. Recall that branched simplicial coverings between 2-dimensional surfaces with the minimum number of vertices were investigated in [27] and that a minimal polyhedral model of the Hopf mapping $S^3 \to S^2$ was found in [26]. By a branched simplicial k-sheeted covering between two d-manifolds we mean a simplicial mapping which is simultaneously a branched k-sheeted covering. In particular, it is required that the preimage of any (open) d-simplex consists of k disjoint (open) d-simplices and that there is no collapsing of lower-dimensional simplices. Then the branch locus is a simplicial subcomplex of each of the two triangulated d-manifolds.

Proposition 7 *There is a branched simplicial 2-sheeted covering from a triangulated $\mathbb{C}P^2$ onto a triangulated 4-sphere which is branched along a subcomplex isomor-*

phic to $\mathbb{R}P^2_6$. We can denote it – by slight abuse of notation – as follows:

$$\mathbb{C}P^2_{18} := S^2_{12} *_\Delta \mathbb{R}P^2_6 \longrightarrow \mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6.$$

Here S^2_{12} denotes the icosahedral triangulation of the 2-sphere with its 2-fold simplicial covering $S^2_{12} \longrightarrow \mathbb{R}P^2_6$. The complex $S^2_{12} *_\Delta \mathbb{R}P^2_6$ does not literally denote the deleted join but the join where each simplex is deleted which involves one vertex of $\mathbb{R}P^2_6$ and any of the two corresponding antipodal vertices of the icosahedron S^2_{12}.

This branched simplicial covering has the minimum number of vertices which is possible for this mapping. In addition, the triangulation is highly symmetric since the group $A_5 \times C_2$ acts on the Bier sphere, transitively on vertices and on 4-dimensional simplices.

Corollary 8 *The polyhedral Cartan hypersurface halfway between the two copies of $\mathbb{R}P^2_6$ in the Bier sphere lifts to a 2-fold covering halfway between S^2_{12} and $\mathbb{R}P^2_6$. This is a polyhedral decomposition of the lens space $L(4,1)$ which occurs as a tubular neighborhood of the real projective plane in the complex projective plane. Combinatorially, it consists of 120 triangular prisms.*

Proof: Since the intermediate levels do not intersect the branch locus, this defines a twofold non-branched covering of some 3-manifold onto the Cartan isoparametric hypersurface in S^4. Topologically the latter is the quaternion space S^3/Q where $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ denotes the quaternion group of order 8. Any twofold covering in between is a quotient of S^3 by a group of order 4 which is a subgroup of Q. This is possible only for the cyclic group C_4, e.g., for $\{\pm 1, \pm i\} \subset Q$. Consequently, the twofold covering of the Cartan hypersurface is a lens space $L(4,1)$ with fundamental group C_4. Its combinatorial automorphism group of order 240 acts transitively on the 120 prisms. $\square$

Proposition 9 *There is a branched simplicial 2-sheeted covering from a triangulated $S^2 \times S^2$ onto a triangulated $\mathbb{C}P^2$ which is branched along a subcomplex isomorphic to the icosahedral triangulation of S^2. We can denote it – with the same remark as in Proposition 7 above – as follows:*

$$(S^2 \times S^2)_{24} := S^2_{12} *_\Delta S^2_{12} \longrightarrow S^2_{12} *_\Delta \mathbb{R}P^2_6$$

where $S^2_{12} \longrightarrow \mathbb{R}P^2_6$ denotes the same 2-fold simplicial covering as above.

Corollary 10 *The polyhedral Cartan hypersurface halfway between the two copies of $\mathbb{R}P^2_6$ in the Bier sphere lifts to a 4-fold covering halfway between the two copies of S^2_{12} in the triangulated $S^2 \times S^2$. This is a polyhedral decomposition of the $\mathbb{R}P^3$*

which occurs as a tubular neighborhood of the diagonal in $S^2 \times S^2$. It consists of 240 *triangular prisms.*

Proof: As in Corollary 8 the intermediate hypersurface does not intersect the branch locus. Therefore, the branched simplicial 4-sheeted covering $(S^2 \times S^2)_{24} \longrightarrow \mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6$ induces a non-branched we obtain a 4-sheeted covering of the Cartan hypersurface S^3/Q. Hence we obtain a 2-fold quotient of S^3 which must be $\mathbb{R}P^3$. In fact, since the diagonal in $S^2 \times S^2$ is given by $\{(x,y) \mid x = y\}$, for small ε the tube is given by $\{(x,y) \mid dist(x,y) = \varepsilon\}$ which is homeomorphic with the total space of the unit tangent bundle of S^2 which in turn is known to be homeomorphic with $\mathbb{R}P^3$. The combinatorial automorphism group acts transitively on the 240 prisms. Its universal covering gives a decomposition of the 3-sphere into 480 triangular prisms. □

4. Polyhedral Tautness

Definition An embedding $M \to \mathbb{E}^N$ of a compact manifold is called taut, if for any open ball (or ball complement) $B \subset \mathbb{E}^N$ the induced homomorphism

$$H_*(M \cap B) \longrightarrow H_*(M)$$

is injective where H_* denotes an appropriate homology theory with coefficients in a certain field. The notion of k-tautness refers to the injectivity in the low dimensions $H_i(M \cap B) \to H_i(M)$, $i = 0, \ldots, k$. Tautness is conformally invariant. An equivalent formulation is that all nondegenerate distance functions are perfect functions. Any taut embedding is also tight, and a tight spherical embedding is also taut, compare [13]. Moreover, any taut smooth submanifold of $\mathbb{E}^N$ can be lifted to a spherical taut submanifold of $S^N \subset \mathbb{E}^{N+1}$ by inverse stereographic projection.

Examples of taut embeddings are the Veronese-type embeddings discussed above as well as the tubes in spheres around them. More generally, any isoparametric hypersurface in a sphere is taut [12].

This definition does not apply to polyhedral submanifolds since the distance spheres do not fit the piecewise linear structure. Even if one replaces the euclidean distance spheres by the distance spheres in the maximum norm (which then are euclidean cubes), this does not seem to give an appropriate analogue. Instead, we suggest the following definition of PL-tautness:

Definition A polyhedral complex $M \subset \mathbb{E}^N$ with convex faces is called PL-taut, if for any open ball (or ball complement) $B \subset \mathbb{E}^N$ the induced homomorphism

$$H_*(M \cap \langle B_0 \rangle) \longrightarrow H_*(M)$$

is injective where B_0 denotes the set of vertices in $M \cap B$, and $\langle B_0 \rangle$ refers to the subcomplex in M spanned by those vertices. We call an embedding of a manifold (or a submanifold) PL-taut if there is a polyhedral decomposition such that the image (or the submanifold itself) can be decomposed into a PL-taut polyhedral complex. Note that PL-tautness is not invariant under subdivision.

Obviously, any PL-taut embedding is also tight (consider very large balls), and a tight PL-embedding is PL-taut provided that it is PL-spherical in the following sense:

Definition and Proposition 11 A polyhedral complex with convex faces is called PL-spherical if all its vertices are contained in a certain euclidean sphere. *Then any tight and PL-spherical embedding is also PL-taut.*

Proof: Consider a ball (or ball complement) B and the set B_0 of vertices in $B \cap M$. Since all the vertices of M are contained in a sphere, it is possible to find a halfspace E_+ such that $M \cap E_+$ and $M \cap B$ contain precisely the same vertices. Consequently, the span of B_0 has the same homology as $M \cap E_+$, and the latter injects at the homology level by the assumption of tightness.

Corollary 12 *Any tight subcomplex of a higher-dimensional regular simplex or cube is PL-taut. This includes the class of tight triangulations of manifolds as well as the class of power complexes 2^K as subcomplexes of the n-cube where K denotes an arbitrary simplicial complex with n vertices, see* [16, Ch.3].

Examples: All the PL analogues of isoparametric hypersurfaces discussed in Theorem 1 are not only tight but also PL-taut because in each case all vertices lie on a common euclidean sphere. In particular Corollary 12 implies that the class of PL-taut submanifolds is much richer than the class of smooth taut submanifolds. Infinitely many surfaces admit tight triangulations. In higher dimensions there are a number of tight triangulations of manifolds including the K3 surface and the homogeneous space $SU(3)/SO(3)$ [20] as well as tight subcomplexes of cubes which are homeomorphic to connected sums of standard handles $S^k \times S^l$ [21]. By truncating the tight and PL-taut complex projective plane as a subcomplex $\mathbb{C}P^2_9$ of the regular 8-dimensional simplex and by inserting copies of the same triangulation one obtains an embedding of $\mathbb{C}P^2 \#(-\mathbb{C}P^2)^{\#9}$ as a PL-taut and PL-spherical submanifold of 8-space. The unique 7-vertex triangulation of the torus can be realized by a PL-taut submanifold in several ways, namely, as a PL-spherical and tight subcomplex of the cyclic 4-polytope with 7 vertices, or as a subcomplex of the 6-dimensional regular simplex, or as Császár's torus in 3-space, see [16]. The tight polyhedral Klein bottle in 5-space found in [4] is also PL-taut since all the vertices can be chosen in a sphere. Similarly, by truncating the

6-vertex triangulation of $\mathbb{R}P^2$ as a subcomplex of the regular 5-dimensional simplex one obtains a PL-taut and PL-spherical surface with $\chi = -5$ in 5-space by precisely the polyhedral decomposition which is depicted on the cover of [16]. The same surface admits 14 combinatorially distinct tight 10-vertex triangulations [1] and, therefore, as many non-congruent PL-taut realizations as a subcomplex of the 9-dimensional regular simplex.

Corollary 13 *Any PL-taut submanifold of $\mathbb{E}^N$ with convex faces can be lifted to a tight and PL-spherical submanifold of $\mathbb{E}^{N+1}$ by mapping the vertices by inverse stereographic projection and by replacing the convex faces of the original manifold by convex faces in the image.*

Example: The standard polyhedral square torus in 3-space based on a (4,4)-grid is PL-taut and can be lifted to a kind of a PL Clifford torus in 4-space where all the vertices are contained in a 3-sphere. By an appropriate choice of the stereographic projection it can be regarded as a realization by the power complex $2^{\{4\}} = \{4\} \times \{4\}$ which is a subcomplex of the 4-dimensional cube, compare [21].

Theorem 14 *Let $f\colon \Sigma^n \to \mathbb{E}^N$ be a PL-taut embedding of a certain homology sphere Σ^n, then $f(\Sigma^n)$ is the boundary of a convex polytope in $(n+1)$-space. If we regard only those vertices which are vertices of that polytope, then f is PL-spherical. In particular, Σ^n is the standard PL sphere S^n.*

Proof: Since f is, in particular, tight, the first claim follows from a well known statement about tightly embedded spheres, see [16, 3.6]. Now consider the PL-spherical lift into the $(n+1)$-sphere, according to Corollary 13 above. Since this is again tight, its image has to be part of an affine $(n+1)$-dimensional space. Because all vertices lie on a certain sphere, and since stereographic projection preserves spheres, all the vertices of the original $f(\Sigma^n)$ lie on a certain sphere as well. Therefore, f is PL-spherical. $\square$

Corollary 15 *(i) The image of any PL-taut embedding of an n-ball into N-space is a convex $(n+1)$-polytope such that all vertices lie on a common sphere.*

(ii) Any top-dimensional face of the image of a PL-taut embedding of any n-manifold is a PL-spherical convex n-polytope.

Definition A subset of a PL-taut submanifold M is called a top-set if it is the intersection of M with a supporting hyperplane.

Proposition 16 *Any top-set of any PL-taut submanifold is again PL-taut.*

Proof: *Let* $M \to \mathbb{E}^N$ be PL-taut, and let H be a supporting hyperplane. According to Corollary 13 we consider the lift of the embedding such that all vertices lie on the N-sphere. In this case a top-set $X = M \cap H$ will be mapped onto a certain topset X' with respect to a hyperplane H' which contains the stereographic preimage of H. It is well known that a top-set of a tight embedding is again tight. So now X' is PL-spherical and tight, hence PL-taut. Therefore X is also PL-taut as the image of X' under stereographic projection. □

Theorem 17 *The image of any PL-taut polyhedral embedding of* $\mathbb{C}P^2$ *into* 8-*space (not in a hyperplane and such that all vertices are in general position) is a subcomplex of an* 8-*dimensional simplex which is combinatorially equivalent to* $\mathbb{C}P^2_9$.

Proof: According to Corollary 13 we consider the lift of the embedding such that all vertices lie on the 8-sphere. If the image lies in an 8-dimensional affine subspace then the vertices lie in a 7-sphere, and the original embedding was PL-spherical. In this case the convex hull of the image of the original embedding is a simplicial 8-polytope, and the image itself is a subcomplex of it. In this case the assertion follows from [16, Thm.4.5] together with the uniqueness of the 9-vertex triangulation of $\mathbb{C}P^2$. Otherwise, if the image does not lie in any 8-dimensional affine subspace, we obtain a tight substantial embedding into some euclidean 9-space which is impossible by Theorem 14 in [22]. □

Conjecture: The assertion of Theorem 17 holds without the assumption about general position, and for any tight polyhedral embedding.

Final remark:
From the purely combinatorial point of view one might ask for discrete (i.e., finite) analogues of the examples in Theorem 1. We could start with two copies of the 7-point projective plane $PG(2,2)$ with automorphism group $GL(3,2) \cong PSL(2,7)$ of order 168. If these correspond to the two focal sets then the levels in between correspond to the set of the 21 flags in that plane with the same group acting. Therefore, the isotropy group has order 8 (as in the classical case of Cartan's hypersurface in the 4-sphere). However, it is not isomorphic to the quaternion group because it contains a non-cyclic abelian subgroup of order 4.

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Thomas F. Banchoff
Department of Mathematics
Brown University
Providence, RI 02912, USA E-mail: Thomas_Banchoff@brown.edu

Wolfgang Kühnel
Fachbereich Mathematik
Universität Stuttgart
70550 Stuttgart, Germany E-mail: kuehnel@mathematik.uni-stuttgart.de