Zadanie 1

$D: (X,Y);\ f(x,y)$

P. DYSKRETNY:

$$E(X+Y) = \sum_i \sum_j (x_i + y_j) \cdot p_{ij} = \sum_i \sum_j x_i p_{ij} + y_j p_{ij} =$$

$$= \sum_i x_i \sum_j p_{ij} + \sum_j y_j \sum_i p_{ij} =$$

$$= \sum_i x_i p_{i\bullet} + \sum_j y_j p_{\bullet j} = EX + EY \blacksquare$$

P. CIĄGŁY:

$$E(X+Y) = \iint_{\mathbb{R}} (x+y) \cdot f(x,y)\, dy\, dx = \iint_{\mathbb{R}} x \cdot f(x,y)\, dy\, dx + \iint_{\mathbb{R}} y \cdot f(x,y)\, dy\, dx$$

$$= \int_{\mathbb{R}} x \cdot f_1(x)\, dx + \int_{\mathbb{R}} y \cdot f_2(y)\, dy = E(X) + E(Y)$$

Zadanie 2 X – dyskretna

teza: $Y = aX + b \Rightarrow V(Y) = a^2 \cdot V(X)$, $a, b \in \mathbb{R}$

$$V(\cancel{X}\, aX + b) = E\left((aX+b)^2\right) - \left(E(aX+b)\right)^2 =$$

$$= E\left(a^2X^2 + 2abX + b^2\right) - \left(aE(X) + b\right)^2 =$$

$$= a^2E(X^2) + 2abE(X) + b^2 - \left(a^2(E(X))^2 + 2abE(X) + b^2\right) =$$

$$= a^2\left(E(X^2) - (E(X))^2\right) = a^2 \cdot V(X)$$

zadanie 3

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\left(\frac{x^2}{2}\right)} \Leftrightarrow X \sim N(0\ 1) \quad , x \in \mathbb{R} \qquad Y \stackrel{o}{=} X^2$$

$$F_Y(t) = P(Y < t) = P(X^2 < t) = P(-\sqrt{t} < X < \sqrt{t}) = P(X < \sqrt{t}) - P(X < -\sqrt{t}) =$$

$$= F_X(\sqrt{t}) - F_X(-\sqrt{t})$$

$$f_Y(t) = F_Y'(t) = F_X'(\sqrt{t}) - F_X'(-\sqrt{t}) = f_X(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} + f_X(-\sqrt{t}) \frac{1}{2\sqrt{t}} =$$

$$= \frac{1}{2\sqrt{t}} \left(\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{t}{2}} \right) = \frac{2e^{-\frac{t}{2}}}{2\sqrt{t} \cdot \sqrt{2\pi}} =$$

$$= \frac{e^{-\frac{t}{2}}}{\sqrt{t}} \cdot \frac{1}{\sqrt{2\pi}}$$

zadanie 4 $\Gamma(1/2) \overset{?}{=} \sqrt{\pi}$, $\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt$, $p > 0$

$$\int_0^\infty t^{-\frac{1}{2}} \cdot e^{-t}\, dt = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\, dt = \left| \begin{array}{l} t = x^2/2 \\ dt = \frac{1}{2} \cdot 2x\, dx \\ \quad dt = x \cdot dx \end{array} \right| = \int_0^\infty \frac{e^{-\frac{x^2}{2}}}{x \cdot \frac{\sqrt{2}}{2}}\, x\, dx =$$

$$= \int_0^\infty \sqrt{2} \cdot e^{-\frac{x^2}{2}}\, dx = \sqrt{2} \int_0^\infty e^{-\frac{x^2}{2}}\, dx = \frac{\sqrt{2}}{2} \int_{-\infty}^\infty e^{-\frac{x^2}{2}}\, dx$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-\left(\frac{x^2}{2} + \frac{y^2}{2}\right)}\, dy\, dx = \frac{1}{2} \cdot 2\pi = \pi$$

czyli $\Gamma(1/2) = \sqrt{\pi}$

Zadanie 5

$X \sim \mathrm{Gamma}(b, p)$

$$f(x) = \frac{b^p}{\Gamma(p)} x^{p-1} e^{-bx}, \quad x \in (0, \infty)$$

$$f_Y(t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{t}{2}}}{\sqrt{t}}$$, czyli jest szansa

weźmy $p = \frac{1}{2}$, $b = \frac{1}{2}$:

$$f(x) = \frac{\sqrt{b}}{\Gamma(\frac{1}{2})} \cdot \frac{e^{-bx}}{\sqrt{x}} = \frac{\sqrt{b}}{\sqrt{\pi}} \cdot \frac{e^{-bx}}{\sqrt{x}} = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{x}{2}}}{\sqrt{x}} = f_Y(x)$$

czyli $Y \sim \mathrm{Gamma}(\frac{1}{2}, \frac{1}{2})$

zadanie 6

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad Y = \delta X + \mu$$

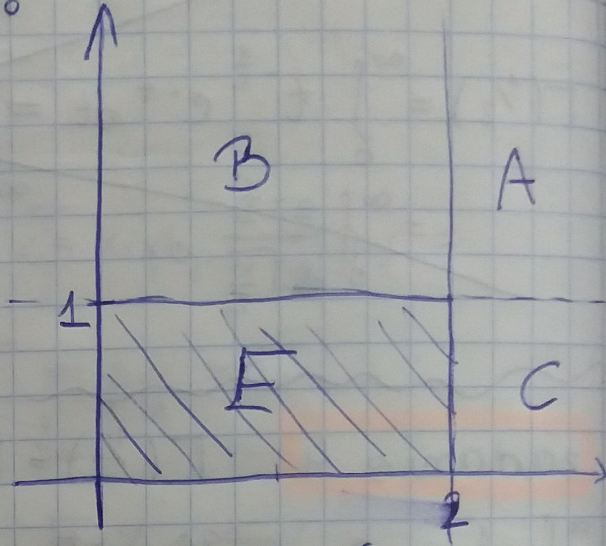
$$F_Y(t) = P(Y < t) = P(\delta X + \mu < t) = P\left(X < \frac{t-\mu}{\delta}\right) = F_X\left(\frac{t-\mu}{\delta}\right)$$

$$f_Y(t) = f_X\left(\frac{t-\mu}{\delta}\right) \cdot \frac{1}{\delta} = \frac{1}{\delta\sqrt{2\pi}} \cdot e^{-\frac{(t-\mu)^2}{2\delta^2}}$$

zadanie 7

$f(x,y) = xy$ na $[0,2]\times[0,1]$ $\qquad \int_0^2\int_0^1 xy \, dy\, dx = 1$ ✓

$$\int_{-\infty}^{s}\int_{-\infty}^{t} xy \, dy\, dx = ?$$



(D) $F_{X,Y}(x,y) = 0$

(A): $[2,\infty)\times[1,\infty)$

$F_{X,Y}(x,y) = 1$

D – II, III, IV ćwiartki

(E): $[0,2]\times[0,1]$

$$F_{X,Y}(x,y) = \int_0^s\int_0^t xy\, dy\, dx = \int_0^s x\cdot\frac{t^2}{2}\, dx = \frac{t^2}{2}\int_0^s x\, dx = \frac{t^2\cdot s^2}{2}$$

(B): $[0,2]\times[1,\infty)$:

$$F_{X,Y}(x,y) = \int_0^s\int_0^1 xy\, dy\, dx = \int_0^s x\cdot\frac{1}{2}\, dx = \frac{s^2}{4}$$

(C) $[2,\infty)\times[0,1]$

$$F_{X,Y}(x,y) = \int_0^2\int_0^t xy\, dy\, dx = \int_0^2 x\frac{t^2}{2}\, dx = \frac{t^2}{2}\cdot 2 = t^2$$

zadanie 8

$f(x,y) = xy$ na $[0,2]\times[0,1]$ $\qquad Z = X + Y$

$$\begin{cases} Z = X + Y \\ T = Y \end{cases} \qquad \begin{cases} X = Z - T \\ Y = T \end{cases} \qquad J = \begin{vmatrix} \frac{\delta x}{\delta z} & \frac{\delta x}{\delta t} \\ \frac{\delta y}{\delta z} & \frac{\delta y}{\delta t} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$g(z,t) = f\big(x(z,t), y(z,t)\big)\cdot |J| = (z-t)\,t = zt - t^2$$

Chcemy wyznaczyć gęst. brzegową funkcji g: $g_1(z)$

Określmy przedział całkowania.

$$\begin{cases} 0 \leq x \leq 2 \\ 0 < y < 1 \end{cases} \quad \begin{cases} 0 < z-t < 2 \\ 0 < t < 1 \end{cases} \quad \begin{cases} -z < -t < 2-z \\ 0 < t < 1 \end{cases} \quad \begin{cases} z-2 < t < z \\ 0 < t < 1 \end{cases}$$

czyli przedział całkowania to

$$[\max\{0, z-2\}, \min\{1, z\}]$$

dla $z \in [0,1]$: $[0, z]$

dla $z \in [1,2]$: $[0, 1]$

dla $z \in [2,3]$: $[z-2, 1]$

$$g_1(z) = \begin{cases} \int_0^z zt - t^2\, dt = z\cdot\frac{z^2}{2} - \frac{z^3}{3} = \frac{z^3}{6}, & z \in [0,1] \\ \int_0^1 zt - t^2\, dt = z\cdot\frac{1}{2} - \frac{1}{3} = \frac{z}{2} - \frac{1}{3}, & z \in [1,2] \\ \int_{z-2}^1 zt - t^2\, dt = \frac{zt^2}{2} - \frac{t^3}{3}\Big|_{z-2}^{1} = -\frac{z^3}{6} + \frac{5}{2}z - 3, & z \in [2,3] \end{cases}$$

$\boxed{\text{zadanie 9}}$

X, Y - nzal

$$Cov(X,Y) = E\Big((X-EX)\cdot(Y-EY)\Big) =$$

$$= E(XY - Y\cdot EX - X\cdot EY + EX\cdot EY) =$$

$$= E(XY) - E(X)\cdot E(Y) - E(Y)\cdot E(X) + E(X)\cdot E(Y) = E(X\cdot Y) - E(X)\cdot E(Y)$$

$$E(X\cdot Y) = \iint_{\mathbb{R}} xy\cdot f(x,y)\,dy\,dx \overset{nzal}{=} \iint_{\mathbb{R}} xy\cdot f_1(x)\cdot f_2(y)\,dy\,dx =$$

$$= \int_{\mathbb{R}} x f_1(x)\,dx \cdot \int_{\mathbb{R}} y\cdot f_1(y)\,dy = E(X)\cdot E(Y)$$

wobec tego $E(X\cdot Y) - E(X)\cdot E(Y) = cov(X,Y) = 0$ ■