Zadanie 1

$$\alpha + \beta = 1$$
$$\alpha, \beta \geq 0$$
$$\lambda, \mu > 0$$

$$P(X > t) = \alpha e^{-\lambda t} + \beta e^{-\mu t}, \quad t \geq 0$$

$$P(X < t) = 1 - \alpha e^{-\lambda t} - \beta e^{-\mu t} = F(t)$$

$$f_X(t) = (-\alpha)(-\lambda) e^{-\lambda t} - \beta \cdot (-\mu) e^{-\mu t} =$$
$$= \alpha \lambda e^{-\lambda t} + \beta \mu e^{-\mu t}$$

$$E(X) = \int_0^\infty x(\alpha \lambda e^{-\lambda x} + \beta \mu e^{-\mu x})\, dx = \alpha\lambda \int_0^\infty x e^{-\lambda x} dx + \beta\mu \int_0^\infty x e^{-\mu x} dx =$$
$$= \alpha \cdot \lambda \cdot \frac{1}{\lambda^2} + \beta \cdot \mu \cdot \frac{1}{\mu^2} = \boxed{\frac{\alpha}{\lambda} + \frac{\beta}{\mu}}$$

α^2

zadanie 2

$X_n: f_n(x) = \frac{c_n}{x^{n+1}}, \quad x \in [c_n, \infty) \quad n = 1, 2 \ldots$

Szukany c_n:

$$\int_{c_n}^{\infty} \frac{c_n}{x^{n+1}} dx = c_n \int_{c_n}^{\infty} \frac{1}{x^{n+1}} dx = c_n \cdot \left(-\frac{1}{nx^n}\right)_{c_n}^{\infty} = c_n \cdot \frac{1}{nc_n^n} =$$

$$= \frac{1}{nc_n^{n-1}} = \underline{1}$$

czyli $c_n = \left(\frac{1}{n}\right)^{\frac{1}{n-1}}$ dla $n > 1$ i $c_n = 1$ dla $n = 1$

$$E(X_n) = c_n \int_{c_n}^{\infty} \frac{1}{x^n} dx = \begin{cases} c_n \cdot (\ln x)_{c_n}^{\infty} = \text{undefined} & ; n = 1 \\ c_n \cdot \left(\frac{1}{(n-1)x^{n-1}}\right)\Big|_{c_n}^{\infty} & ; n > 1 \end{cases}$$

$$c_n \left(\frac{1}{(n-1)x^{n-1}}\right)_{c_n}^{\infty} = c_n \left(0 - \frac{1}{(n-1)c_n^{n-1}}\right) = -\frac{1}{(n-1)c_n^{n-2}} =$$

$$= -\frac{1}{n-1} \cdot \frac{1}{\left(\frac{1}{n}\right)^{\frac{n-2}{n-1}}} = -\frac{n^{\frac{n-2}{n-1}}}{n-1}$$

zadanie 3 $Z_n = \ln X_n$

$$F_{Z_n}(t) = P(Z_n < t) = P(\ln X_n < t) = P(X_n < e^t) = F_{X_n}(e^t)$$

$$f_{Z_n}(t) = (F_{X_n}(e^t))' = \frac{c_n}{e^{t(n+1)}} \cdot e^t = \frac{c_n}{e^{tn}}$$

Zadanie 6 $\{f_i\}_{i=1}^{n}$, $\{\alpha_i\}_{i=1}^{n}$: $\sum_{i}^{n} \alpha_i = 1$

$$f(x) = \sum_{i=1}^{n} \alpha_i f_i(x)$$

$$\int_{\mathbb{R}} f(x)\, dx = \int_{\mathbb{R}} \sum \alpha_i f_i(x) = \int_{\mathbb{R}} \alpha_1 f_1(x)\, dx + \ldots + \int_{\mathbb{R}} \alpha_n f_n(x)\, dx =$$

$$= \alpha_1 \int_{\mathbb{R}} f_1(x)\, dx + \ldots + \alpha_n \int_{\mathbb{R}} f_n(x)\, dx = \alpha_1 + \ldots + \alpha_n = 1$$

b) $Y_1, Y_2 \sim U[0,1]$ $\quad Z = \frac{Y_1 + Y_2}{2}$

$$\begin{cases} Y_1 = 2Z - Y_2 = 2Z - T \\ Y_2 = T \end{cases}$$

Przechodzimy do innych zmiennych, liczymy więc Jakobian:

$$J = \begin{vmatrix} \frac{\delta y_1}{\delta z} & \frac{\delta y_1}{\delta t} \\ \frac{\delta y_2}{\delta z} & \frac{\delta y_2}{\delta t} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

$$g(z,t) = f(z(y_1,y_2), t(y_1,y_2)) \cdot J = 2$$

Chcemy policzyć $f_1(z) = \int_{\mathbb{R}} g(z,t)\, dt$, czyli należy ~~[illegible]~~ wyznaczyć przedział całkowania:

$$\begin{cases} 0 < 2Z - T < 1 \\ 0 < T < 1 \end{cases}, \text{ czyli } \begin{cases} 2Z - 1 < T < 2Z \\ 0 < T < 1 \end{cases}$$

mamy więc $t \in [\max\{0, 2z-1\}, \min\{1, 2z\}]$

czyli dla ustalonego z mamy:

$$0 < t < 2z \quad \text{gdy } z \in [0, 1/2]$$

$$2z - 1 < t < 1 \quad \text{gdy } z \in [1/2, 1]$$

$$g_1(z) = \begin{cases} \int_0^{2z} 2\, dt = 4z & , z \in [0, 1/2] \\ \int_{2z-1}^{1} 2\, dt = 2(1 - 2z + 1) = -4z + 4 & , z \in [1/2, 1] \end{cases}$$

Zadanie 12

$X, Y \sim N(0,1)$ $\quad (X,Y) \longrightarrow (R, \Theta)$ gdzie:

$$\begin{cases} X = R\cos\Theta \\ Y = R\sin\Theta \end{cases}$$

$R > 0$

$0 < \Theta < 2\pi$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$$

$$R = \sqrt{x^2 + y^2}$$

$$\Theta = \tan^{-1}\frac{y}{x}$$

(przydatne w kolejnym zadaniu)

Przechodzimy do innych zmiennych, liczymy więc Jakobian:

$$J = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \Theta} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \Theta} \end{vmatrix} = \begin{vmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{vmatrix} =$$

$$= |R\cos^2\Theta + R\sin^2\Theta| = |R| = R$$

Funkcja gęstości 2-wymiarowej zmiennej (X,Y):

ponieważ X, Y niezależne: $f_{XY}(x,y) = f_X(x) \cdot f_Y(y) = \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right)$

$$g(r,\Theta) = f(x(r,\Theta), y(r,\Theta)) = \frac{1}{2\pi} \exp\left(-\frac{r^2\cos^2\Theta + r^2\sin^2\Theta}{2}\right) \cdot r =$$

$$= \frac{1}{2\pi} \cdot r \cdot \exp\left(-\frac{r^2}{2}\right)$$

zadanie 13 $X, Y \sim N(0,1)$ $(X,Y) \longrightarrow (D, \Theta)$

$D = R^2$

Θ jak poprzednio więc:

$$\begin{cases} X = \sqrt{D}\cos\Theta \\ Y = \sqrt{D}\sin\Theta \end{cases}$$

$$J = \begin{vmatrix} \frac{\delta x}{\delta d} & \frac{\delta x}{\delta \Theta} \\ \frac{\delta y}{\delta d} & \frac{\delta y}{\delta \Theta} \end{vmatrix} = \begin{vmatrix} \frac{\cos\Theta}{2\sqrt{D}} & -\sqrt{D}\sin\Theta \\ \frac{\sin\Theta}{2\sqrt{D}} & \sqrt{D}\cos\Theta \end{vmatrix} = \left| \frac{\cos^2\Theta}{2} + \frac{\sin^2\Theta}{2} \right| = \frac{1}{2}$$

tak samo jak w Z12: $f_{XY}(x,y) = f_X(x) \cdot f_Y(y) = \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)$

$$f(d,\Theta) = f_{XY}\big(x(d,\Theta), y(d,\Theta)\big) = \frac{1}{2\pi}\exp\left(-\frac{d\cos^2\Theta + d\sin^2\Theta}{2}\right) \cdot \frac{1}{2} =$$

$$= \underline{\frac{1}{4\pi}\exp\left(-\frac{d}{2}\right)}$$

Policzmy rozkłady d, Θ aby sprawdzić, czy zmienne są niezależne:

$$f_1(d) = \int_0^{2\pi} \frac{1}{4\pi} e^{-\frac{d}{2}}\, d\Theta = \frac{1}{4\pi} e^{-\frac{d}{2}} \cdot \Theta \Big|_0^{2\pi} = \underline{\frac{1}{2} e^{-\frac{d}{2}}}$$

$$f_2(\Theta) = \int_0^{\infty} \frac{1}{4\pi} e^{-\frac{d}{2}}\, dd = \frac{1}{4\pi}\int_0^{\infty} e^{-\frac{d}{2}}\, dd = \frac{1}{4\pi}\cdot\left(-2e^{-\frac{d}{2}}\right)\Big|_0^{\infty} = \frac{1}{4\pi}(0+2) = \frac{1}{2\pi}$$

$$f_1(d) \cdot f_2(\Theta) = \frac{1}{4\pi} e^{-\frac{d}{2}} = f(d,\Theta)$$, czyli

<u>zmienne D, Θ są NIEZALEŻNE</u>