$\boxed{Z1}$ $X \sim \text{Geom}(p)$ $\quad M_X(t) = \dfrac{pe^t}{1-(1-p)e^t}$

$$P(X=k) = (1-p)^{k-1} \cdot p$$

$$M_X(t) = E(e^{tX}) = \sum_{k=1}^{\infty} e^{tx_k} \cdot p_k = \sum_k e^{tx_k}(1-p)^{k-1} p =$$

$$= p \sum_k e^{tk}(1-p)^{k-1} = \frac{p}{1-p} \sum_k e^{tk}(1-p)^k =$$

$$= \frac{p}{1-p} \sum_k \left(e^t(1-p)\right)^k = \frac{p}{1-p} \cdot \frac{e^t(1-p)}{1-e^t(1-p)} = \frac{pe^t}{1-e^t(1-p)}$$

$$e^t(1-p) < 1$$

$$e^t < \frac{1}{1-p}$$

$$t < \ln \frac{1}{1-p}$$

$\boxed{\text{zadanie 2}}$

$$EX = M_X'(0) = \frac{p\left(e^t\left(1-e^t(1-p)\right) - \left((p-1)\cdot e^t\right)e^t\right)}{\left(1-e^t(1-p)\right)^2}(0) =$$

$$= p\,\frac{e^t - e^{2t}(1-p) - (p-1)e^{2t}}{\left(e^t(1-p)-1\right)^2}(0) = \frac{pe^t}{\left(e^t(1-p)-1\right)^2}(0) =$$

$$= \frac{p\cdot 1}{\left(1\cdot(1-p)-1\right)^2} = \frac{p}{p^2} = \frac{1}{p}$$

liczymy 2-gą pochodną:

$$M''_X(t) = \left(\frac{pe^t}{(e^t(1-p)-1)^2}\right)' = p\,\frac{e^t(e^t(1-p)-1)^2 - e^t\left(2e^{2t}(1-p)^2 - 2e^t(1-p)\right)}{(e^t(1-p)-1)^4} =$$

$$= p\,\frac{e^t(e^t(1-p)-1)^2 - e^t\cdot 2e^t(1-p)\cdot(e^t(1-p)-1)}{(e^t(1-p)-1)^4} = p\,\frac{e^t(e^t(1-p)-1) - 2e^{t\cdot 2}(1-p)}{(e^t(1-p)-1)^3} =$$

$$= \frac{pe^t}{(e^t(1-p)-1)^2} - \frac{2pe^{2t}(1-p)}{(e^t(1-p)-1)^3}$$ czyli

$$M''_X(0) = \frac{p}{p^2} + \frac{2p(1-p)}{p^3} =$$

$$= \frac{2-p}{p^2}$$ a więc $V(X) = E(X^2) - E(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$

$\boxed{Z3}$ $X \sim N(\mu, \sigma^2)$ , $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$ ; $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$$M_X(t) = \int_{\mathbb{R}} e^{tx} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)} dx = \left| \begin{array}{l} y = \frac{x-\mu}{\sigma} \\ dx = \sigma\, dy \end{array} \right| =$$

$\boxed{\text{niech } z = t\sigma}$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{t(\sigma y + \mu)} e^{-\frac{1}{2}y^2} dy = \frac{1}{\sqrt{2\pi}} e^{t\mu} \int_{\mathbb{R}} e^{t\sigma y} \cdot e^{-\frac{1}{2}y^2} dy =$$

$$= e^{t\mu} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2 + yz} dy = e^{t\mu} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-z)^2} \cdot e^{\frac{1}{2}z^2} dy =$$

$$= e^{t\mu} \cdot e^{\frac{1}{2}z^2} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \cdot e^{\left(-\frac{(y-z)^2}{2}\right)} dy = e^{t\mu + \frac{1}{2}t^2\sigma^2} \cdot \int_{\mathbb{R}} N(z, 1)\, dy =$$

$$= e^{t\mu + \frac{1}{2}t^2\sigma^2}$$

# Zadanie 4

$X_1 \dots X_n$ - n zal. $X_k \sim N(\mu, \sigma^2)$

znajdźmy f. tw. mom. zmiennej $X_s$ – suma $X_1 \dots X_n$ !

$$M_{X_s}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \dots \cdot M_{X_n}(t) =$$

$$= e^{t\mu_1 + \frac{1}{2}t^2\sigma_1^2} \cdot \dots \cdot e^{t\mu_n + \frac{1}{2}t^2\sigma_2^2} =$$

$$= \exp\left\{\sum_{k=1}^{n} t\mu_k\right\} \exp\left\{\sum_{k=1}^{n} \frac{1}{2}t^2\sigma_k^2\right\}$$

$$M_{\overline{X}}(t) = M_{\frac{1}{n} \cdot X_s}(t) = M_{X_s}\left(\frac{1}{n} \cdot t\right) = \exp\left\{\frac{1}{n} t \sum_{k=1}^{n} \mu_k\right\} \exp\left\{\frac{t^2}{2n^2} \sum_{k=1}^{n} \sigma_k^2\right\}$$

$\boxed{\text{zadanie 5}}$ $X \sim N(\mu, \sigma^2)$ , $Y = \left(\frac{X-\mu}{\sigma}\right)^2$

$$F_Y(t) = P(Y<t) = P\left(\left(\frac{X-\mu}{\sigma}\right)^2 < t\right) = P\left((X-\mu)^2 < t\sigma^2\right)$$

$$= P\left(-\sqrt{t} < \frac{X-\mu}{\sigma} < t\right) = P\left(\frac{X-\mu}{\sigma} < \sqrt{t}\right) - P\left(\frac{X-\mu}{\sigma} < -\sqrt{t}\right) =$$

$$= P(X < \sigma\sqrt{t}+\mu) - P(X < -\sigma\sqrt{t}+\mu) =$$

$$= F_X(\sigma\sqrt{t}+\mu) - F_X(-\sigma\sqrt{t}+\mu)$$

$$f_Y(t) = f_X(\sigma\sqrt{t}+\mu)\cdot\frac{\sigma}{2\sqrt{t}} + f_X(-\sigma\sqrt{t}+\mu)\frac{\sigma}{2\sqrt{t}} =$$

$$= \frac{\sigma}{2\sqrt{t}}\left(\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(\sigma\sqrt{t})^2}{2\sigma^2}\right\} + \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(-\sigma\sqrt{t})^2}{2\sigma^2}\right\}\right) =$$

$$= \frac{\sigma}{2\sqrt{t}}\cdot\frac{1}{\sqrt{2\pi}\,\sigma}\left(e^{-\frac{t}{2}} + e^{-\frac{t}{2}}\right) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{t}{2}}$$

## zadanie 7

niezależne

$X_1 \dots X_n$, $X_k \sim Gamma(b, p_k) \iff f(x) = \frac{b^{p_k}}{\Gamma(p_k)} x^{p_k - 1} e^{-bx}, x \in (0, \infty)$

$$M_Z(t) = M_{X_1 + \dots + X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \dots \cdot M_{X_n}(t)$$

policzmy $M_X(t)$, gdzie $X \sim Gamma(b, p)$:

$$M_X(t) = \frac{b^p}{\Gamma(p)} \int_0^\infty e^{tx} x^{p-1} e^{-bx}\, dx = \left| \begin{array}{l} u = (b-t)x \\ du = (b-t)\,dx \end{array} \right| =$$

$$= \frac{b^p}{\Gamma(p)} \int_0^\infty e^{-u} \left(\frac{u}{b-t}\right)^{p-1} \frac{du}{b-t} = \left(\frac{b}{b-t}\right)^p \cdot \frac{1}{\Gamma(p)} \underbrace{\int_0^\infty u^{p-1} e^{-u}\, du}_{= \Gamma(p)} =$$

$$= \left(\frac{b}{b-t}\right)^p$$

czyli $M_Z(t) = \left(\frac{b}{b-t}\right)^{p_1} \cdot \dots \cdot \left(\frac{b}{b-t}\right)^{p_n} = \left(\frac{b}{b-t}\right)^{\sum_i^n p_i}$

$\boxed{\text{zadanie 8}}$ $X_1 \dots X_n$ – niezależne

$X_k \sim B(m_k, p)$

$M_Z(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \ldots \cdot M_{X_n}(t)$

policzmy $M_X(t)$, gdzie $X \sim B(m, p)$:

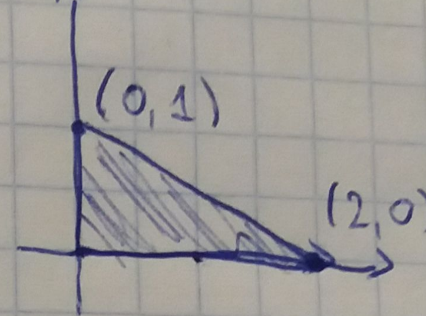
$$M_X(t) = \sum_k^{m} e^{tk} p_k = \sum_k^{m} e^{tk} \binom{m}{k} p^k (1-p)^{m-k} =$$

$$= \sum_k^{m} \binom{m}{k} (pe^t)^k (1-p)^{m-k} = (pe^t + 1 - p)^m$$

czyli $M_{X_i}(t) = (pe^t + 1 - p)^{m_i}$

$$M_Z(t) = (pe^t + 1 - p)^{\sum_i^n m_i}$$

**zadanie 9** $f(x,y) = \frac{15}{2}x^2y$ , $T = \frac{X}{Y}$




$$\begin{cases} T = \frac{X}{Y} \\ s = Y \end{cases} \qquad \begin{cases} X = T \cdot S \\ Y = S \end{cases}$$

liczymy Jakobian: $J = \begin{vmatrix} \frac{\delta x}{\delta t} & \frac{\delta x}{\delta s} \\ \frac{\delta y}{\delta t} & \frac{\delta y}{\delta s} \end{vmatrix} = \begin{vmatrix} s & t \\ 0 & 1 \end{vmatrix} = s$

mamy więc $g(t,s) = f(x(t,s), y(t,s)) \cdot |J| = \frac{15}{2} t^2 \cdot s^2 \cdot s \cdot s = \frac{15t^2s^4}{2}$

Chcemy wyznaczyć gęst. brzegową $g_1(t)$. Określmy przedz. całkowania:

$$\begin{cases} 0 < x < 2 \\ 0 < y < -\frac{1}{2}x + 1 \end{cases} \qquad \begin{cases} 0 < ts < 2 \\ 0 < s < -\frac{1}{2}ts + 1 \end{cases} \qquad \begin{cases} 0 < s < \frac{2}{t} \\ 0 < s < \frac{2}{t+2} \end{cases}$$

czyli $g_1(t) = \int_0^{\frac{2}{t+2}} \frac{15}{2} t^2 s^4 \, ds = \frac{48t^2}{(t+2)^5}$

## zadanie 10

$$M_U(t) = \frac{2}{2-3t}$$

a) $M_U'(t) = \frac{6}{(2-3t)^2}$ ;

czyli $E(U) = M_U'(0) = \frac{6}{4}$

b) $M_U''(t) = \frac{-6(18t-12)}{(2-3t)^4}$ ; czyli $M_U''(0) = \frac{9}{2}$ , a więc

$$V(U) = \frac{9}{2} - \frac{36}{16} = \frac{9}{4}$$

c) $Y = 0{,}9 \cdot U$

$$M_Y(t) = M_{0{,}9U}(t) = M(0{,}9 \cdot t) = \frac{2}{2-2{,}7t}$$