zadanie 18

$x_1 \dots x_n$ - niezależne obserwacje

$$X \sim Geo(p) \iff f(x) = (1-p)^{x-1} p$$

$$L(p \mid x_1 \dots x_n) = f(x_1) \cdot \dots \cdot f(x_n) = (1-p)^{(\sum_i x_i) - n} \cdot p^n$$

$$\ln L(p) = \ln (1-p)^{\sum x_i - n} + \ln p^n = \left(\sum_i x_i - n\right) \ln(1-p) + n \ln p$$

$$\frac{\delta \ln L(p)}{\delta p} =$$

$$= \frac{n}{p} + \frac{n - \sum_i x_i}{1-p} = 0$$

$$n(1-p) + (n - n\bar{x})p = 0$$

$$1 - p = \bar{x}p - p$$

$$\boxed{p = \frac{1}{\bar{x}}}$$

zadanie 2 $x_1 \ldots x_n$ - niezależne obs.

$X \sim \text{Pareto}(k, a) \iff f(x) = \dfrac{ka^k}{x^{k+1}}, \quad x \in \langle a, \infty) \quad k$ - znane

$$L(a | x_1 \ldots x_n) = f(x_1) \cdot \ldots \cdot f(x_n) = \frac{k^n a^{kn}}{\prod_i x_i^{k+1}}$$

$$\ln L(a,k | x_1 \ldots x_n) = \ln k^n + \ln a^{kn} - \ln \prod_i x_i^{k+1} =$$
$$= n \ln k + kn \ln a - (k+1) \sum_i \ln x_i$$

$$\frac{\delta \ln L(a,k)}{\delta a} = \frac{kn}{a} = 0 \quad ???$$

ale mamy $a \leq x_i$

a jedynym miejscem, gdzie

~~skoro a nie zależy~~ występuje a jest $\ldots + kn \ln a + \ldots$

czyli a chcemy jak największe, aby zmaksymalizować logarytm $\Rightarrow$ $\boxed{a = \min_i (x_i)}$

zadanie 3

z poprzedniego zadania:

$$\ln L(a,k) = n\ln k + kn\ln a - (k+1)\sum_i \ln x_i$$

$$\frac{\partial \ln L(a,k)}{\partial k} = \frac{n}{k} + n\ln a - \sum_i \ln x_i = 0$$

$$\frac{n}{k} = \sum_i \ln x_i - n\ln a$$

$$\boxed{k = \frac{n}{\sum_i \ln x_i - n\ln a}}$$

zadanie 4

$$X \sim \exp(\lambda) \Leftrightarrow f(x) = \lambda e^{-\lambda x}, \quad x \in (0, \infty)$$

$$L(\lambda) = f(x_1) \cdot \ldots \cdot f(x_n) = \lambda^n e^{-\lambda(x_1 + x_2 + \ldots + x_n)}$$

$$\ln L(\lambda) = \ln \lambda^n - \lambda(x_1 + \ldots + x_n) \cdot 1$$

$$\frac{\delta \ln L(\lambda)}{\delta \lambda} = \frac{n}{\lambda} - \sum_i x_i = 0$$

$$\lambda = \frac{n}{\sum x_i} = \frac{n}{n\overline{x}} = \frac{1}{\overline{x}}$$

$\boxed{\text{zadanie 6}}$ $(x_1, y_1) \dots (x_n, y_n)$ $y = a + bx + cx^2$

szukamy minimum

$$f(a,b,c) = \sum_{i=1}^{n} (a + bx_i + cx_i^2 - y_i)^2$$

$$\frac{\delta f}{\delta a} = \sum_{i=1}^{n} 2(a + bx_i + cx_i^2 - y_i) = 0 \iff \underline{na + bn\bar{x} + c\sum x_i^2 = n\bar{y}}$$

$$\frac{\delta f}{\delta b} : 2\sum_{i=1}^{n} x_i(a + bx_i + cx_i^2 - y_i) = 0 \iff \underline{na\bar{x} + b\sum x_i^2 + c\sum x_i^3 = \sum x_i y_i}$$

$$\frac{\delta f}{\delta c} : \underline{a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4 = \sum x_i^2 y_i}$$

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}$$

zadanie 13

$$M_x(t) = \frac{e^t + e^{-t} + 4}{6} = \frac{1}{6}e^t + \frac{1}{6}e^{-t} + \frac{2}{3}$$

$$M_x'(t) = \frac{1}{6}e^t - \frac{1}{6}e^{-t}$$

$$M_x''(t) = \frac{1}{6}e^t + \frac{1}{6}e^{-t}$$

$$M_x'''(t) = \frac{1}{6}e^t - \frac{1}{6}e^{-t}$$

Łatwo zauważyć, że:

$$M_x^{(n)}(t) = \begin{cases} \frac{e^t + e^{-t}}{6}, & n \text{ parzyste} \\ \frac{e^t - e^{-t}}{6}, & n \text{ nieparzyste} \end{cases}$$

czyli $m_k = \begin{cases} 1/3, & k \text{ parzyste} \\ 0, & k \text{ nieparzyste} \end{cases}$