$\boxed{\text{zadanie 1}}$ rozkład Cauchy'ego: $f(x) = \frac{1}{\pi(x^2+1)}$

$$E(X) = \int_{-\infty}^{\infty} \frac{x}{\pi(x^2+1)}\, dx$$

. Policzmy całkę nieoznaczoną:

$$\frac{1}{\pi} \int \frac{x}{x^2+1}\, dx = \left| \begin{matrix} u = x^2+1 \\ du = 2x\, dx \end{matrix} \right| = \frac{1}{\pi} \int \frac{x}{u} \cdot \frac{du}{2x} = \frac{1}{2\pi} \int \frac{1}{u}\, du =$$

$$= \frac{1}{2\pi} \cdot \ln(x^2+1) + C$$

$$E(X) = \int_{-\infty}^{0} \frac{x}{\pi(x^2+1)}\, dx + \int_{0}^{\infty} \frac{x}{\pi(x^2+1)}\, dx = \frac{1}{2\pi} \left(\ln(x^2+1) \Big|_{-\infty}^{0} + \ln(x^2+1) \Big|_{0}^{\infty} \right) =$$

$$= \frac{1}{2\pi} \left(-\infty + \infty \right) = \text{undefined}$$

zadanie 4

$$F_X(x) = 1 - \frac{9}{x^2} \quad ; x \in [3, \infty) \qquad EX, VX = ?$$

$$f(x) = F_X'(x) = 0 - 9 \cdot (-2) \left(\frac{1}{x^2}\right)' = 18/x^3 \quad , x \in [3, \infty)$$

$$EX = \int_3^\infty \frac{18}{x^2}\, dx = 18 \int_3^\infty x^{-2}\, dx = 18 \cdot \left(-\frac{1}{x}\right)\Big|_3^\infty = 6$$

$$VX = \int_3^\infty (x-6)^2 \cdot \frac{18}{x^3}\, dx = 18 \left(\int_3^\infty \frac{1}{x}\, dx - 12 \int_3^\infty \frac{1}{x^2}\, dx + 36 \int_3^\infty \frac{1}{x^3}\, dx \right) = \text{undefined}$$

czyli:

$\boxed{\text{zadanie 6}}$ $f(x_1, x_2) = \frac{1}{\pi}$ $0 < x_1^2 + x_2^2 < 1$ $-\sqrt{1-x_2^2} < x_1 < \sqrt{1-x_2^2}$

$$f_1(x_1) = \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} 1/\pi \, dx_2 = \frac{1}{\pi} \cdot x_2 \Big|_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} = \frac{1}{\pi} \cdot 2\sqrt{1-x_1^2} =$$

$$= \frac{2\sqrt{1-x_1^2}}{\pi}$$

analogicznie $f_2(x_2) = \dfrac{2\sqrt{1-x_2^2}}{\pi}$

zadanie 7 $\operatorname{cor}(x,y) = \dfrac{\operatorname{cov}(x,y)}{\sqrt{VX \cdot VY}} = \dfrac{E[(X-EX)(Y-EY)]}{\sqrt{VX \cdot VY}}$

$$EX = \int_{-1}^{1} \frac{2\sqrt{1-x^2}}{\pi} \cdot x \, dx = \frac{2}{\pi}\int_{-1}^{1} x\sqrt{1-x^2} = \frac{-2}{3\pi}\left(1-x^2\right)^{\frac{3}{2}}\Big|_{-1}^{1} = 0$$

$$\int x\sqrt{1-x^2}\,dx = \left| \begin{matrix} t = 1-x^2 \\ dt = -2x\,dx \end{matrix} \right| = \int \frac{x\cdot\sqrt{t}}{-2x}\,dt = -\frac{1}{2}\int\sqrt{t} =$$

$$= -\frac{1}{2}\cdot\frac{2}{3}t^{\frac{3}{2}} = -\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}$$

analogicznie $EY = \displaystyle\int_{-1}^{1} \frac{2\sqrt{1-y^2}}{\pi}\cdot y\,dy = 0$

licznik $\operatorname{cor}(x,y) = E[(X-0)(Y-0)] = E(XY) = m_{11}$

$$m_{11} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{xy}{\pi}\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{xy}{\pi}\,dy\,dx \overset{*}{=} \int_{-1}^{1} 0\,dx = $$

$$\boxed{\overset{*}{\ } \frac{x}{\pi}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \cancel{x}y\,dy = \frac{x}{\pi}\,\frac{y^2}{2}\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = 0}$$

$= 0$ ∎

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wystarczy pokazać $f(x,y) \neq f_1(x)\cdot f_2(y)$

dla $x = y = 1/2$:

$f(x,y) = 1/\pi$, $\quad f_1(x) = f_2(y) = \dfrac{2\,\frac{\sqrt{3}}{2}}{\pi} = \dfrac{\sqrt{3}}{\pi}$

zadanie 8

$0 < Y_1 < 1, \; 0 \leq Y_2 \leq 2\pi$

$$\begin{cases} X_1 = Y_1 \cos Y_2 \\ X_2 = Y_1 \sin Y_2 \end{cases}$$

$$J = \begin{vmatrix} \frac{\delta x_1}{\delta y_1} & \frac{\delta x_1}{\delta y_2} \\ \frac{\delta x_2}{\delta y_1} & \frac{\delta x_2}{\delta y_2} \end{vmatrix} = \begin{vmatrix} \cos Y_2 & -Y_1 \sin Y_2 \\ \sin Y_2 & Y_1 \cos Y_2 \end{vmatrix} =$$

$$= \left| Y_1 \cos^2 Y_2 + Y_1 \sin^2 Y_2 \right| = Y_1$$

$$g(y_1, y_2) = f(x_1(y_1, y_2), x_2(y_1, y_2)) \cdot |J| = \frac{Y_1}{\pi}$$

$$g_1(y_1) = \int_0^{2\pi} \frac{y_1}{\pi} \, dy_2 = \frac{y_1}{\pi} \cdot y_2 \Big|_0^{2\pi} = 2y_1$$

$$g_2(y_2) = \int_0^1 \frac{y_1}{\pi} \, dy_1 = \frac{1}{\pi} \cdot \frac{y_1^2}{2} \Big|_0^1 = \frac{1}{2\pi}$$

zmienne Y_1, Y_2 są niezależne: $g(y_1, y_2) = \frac{y_1}{\pi} = \frac{2y_1}{2\pi}$

$\boxed{\text{zadanie 9}}$ mamy $X = (x_1 \dots x_n)^T$

Y: $y_1 = \bar{x}$, $y_k = x_k - \bar{x}$:

$$y_1 = \bar{x}$$
$$y_2 = x_2 - \bar{x}$$
$$y_3 = x_3 - \bar{x}$$
$$\vdots$$
$$y_n = x_n - \bar{x}$$

czyli

$$\boxed{\begin{aligned} x_2 &= y_2 + \bar{x} \\ x_3 &= y_3 + \bar{x} \\ &\vdots \\ x_n &= y_n + \bar{x} \end{aligned}}$$

trzeba jeszcze wyznaczyć x_1:

$$x_1 = n\bar{x} - x_2 - x_3 - \dots - x_n = \bar{x} - (x_2 - \bar{x}) - (x_3 - \bar{x}) - \dots - (x_n - \bar{x}) =$$

$$= y_1 - y_2 - \dots - y_{n-1} - y_n$$

1.-szy wiersz Jacobianu: $\frac{\delta x_1}{\delta y_1}, \frac{\delta x_1}{\delta y_2}, \dots, \frac{\delta x_1}{\delta y_n}$

(ponieważ $x_1 = y_1 - y_2 - y_3 - \dots - y_n$)

$$\frac{\delta x_1}{\delta y_1} = 1, \quad \frac{\delta x_1}{\delta y_2} = -1, \quad \dots, \quad \frac{\delta x_1}{\delta y_n} = -1$$

każdy kolejny wiersz: $\frac{\delta x_k}{\delta y_1}, \frac{\delta x_k}{\delta y_2}, \dots, \frac{\delta x_k}{\delta y_k}, \dots, \frac{\delta x_k}{\delta y_n}$

(ponieważ $x_k = y_k + y_1$)

$$\frac{\delta x_k}{\delta y_1} = 1, \quad \frac{\delta x_k}{\delta y_2} = 0, \quad \dots, \quad \frac{\delta x_k}{\delta y_k} = 1, \quad \dots, \quad \frac{\delta x_k}{\delta y_n} = 0$$

$$J = \begin{vmatrix} 1 & -1 & -1 & \dots & -1 \\ 1 & 1 & & 0 & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & 0 & & & 1 \end{vmatrix} = n \quad \text{(Lista 0, zad. 5)}$$

zadanie 10 $\quad X_1 \dots X_n$ - nzal

$$\sum_{k=1}^{n}(X_k-\mu)^2=\sum_{k=1}^{n}(X_k-\bar{X})^2+n(\bar{X}-\mu)^2$$

$$\sum_{k=1}^{n}(X_k-\mu)^2=\sum_{k=1}^{n}(X_k-\bar{X}+\bar{X}-\mu)^2=\sum_{k=1}^{n}(X_k-\bar{X})^2+\sum_{k=1}^{n}(\bar{X}-\mu)^2+\sum_{k=1}^{n}2(X_k-\bar{X})(\bar{X}-\mu)=$$

$$=\sum_{k=1}^{n}(X_k-\bar{X})^2+n(\bar{X}-\mu)^2+\sum_{k=1}^{n}2(X_k-\bar{X})(\bar{X}-\mu) \leftarrow$$ wystarczy pokazać, że to jest równe 0

$$\sum_{k=1}^{n}2(X_k-\bar{X})(\bar{X}-\mu)=2(\bar{X}-\mu)\sum_{k=1}^{n}(X_k-\bar{X})=2(\bar{X}-\mu)\left(\sum_{k=1}^{n}X_k-n\bar{X}\right)=$$

$$=2(\bar{X}-\mu)\cdot 0=0 \quad \blacksquare$$

zadanie 11

$X_k \sim N(\mu, \sigma^2) \iff M_{X_k}(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$$M = \frac{n}{\sigma^2} \cdot (\overline{X} - \mu)^2 = \left(\sqrt{n}\, \frac{\overline{X} - \mu}{\sigma}\right)^2 = \left(\frac{\sqrt{n}}{\sigma}\overline{X} - \frac{\sqrt{n} \cdot \mu}{\sigma}\right)^2$$

$$M_{n\overline{X}}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \exp\left(n\mu t + \frac{n\sigma^2 t^2}{2}\right)$$

$$M_{\overline{X}}(t) = M_{n\overline{X}}\left(\frac{1}{n} \cdot t\right) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2n}\right)$$

$$M_{\frac{\sqrt{n}}{\sigma}\overline{X}}(t) = M_{\overline{X}}\left(\frac{\sqrt{n}}{\sigma} t\right) = \exp\left(\frac{\sqrt{n} \cdot \mu t}{\sigma} + \frac{\sigma^2 n t^2}{2\sigma^2 n}\right) = \exp\left(\frac{\sqrt{n}\mu t}{\sigma} + \frac{t^2}{2}\right)$$

$$M_{\frac{\sqrt{n}}{\sigma}\overline{X} - \frac{\mu\sqrt{n}}{\sigma}}(t) = \exp\left(-\frac{t\mu\sqrt{n}}{\sigma}\right) \cdot \exp\left(\frac{\sqrt{n} \cdot \mu t}{\sigma} + \frac{t^2}{2}\right) =$$

$$= \exp\left(\frac{t^2}{2}\right)$$ czyli $\frac{\sqrt{n}}{\sigma}\overline{X} - \frac{\mu\sqrt{n}}{\sigma} \sim N(0,1)$

więc $M \sim \chi^2(1)$ (ponieważ $\chi^2(1) = X^2$, gdzie $X \sim N(0,1)$)

zadanie 13

$X_1, X_2 \sim U[1,2]$

$$\begin{cases} Y_1 = 2X_1 + 2X_2 \\ Y_2 = X_1 X_2 \end{cases}$$

$$E(Y_1) = E(2X_1 + 2X_2) = 2E(X_1) + 2E(X_2) = 2\left(\int_1^2 x\,dx + \int_1^2 x\,dx\right) = \underline{6}$$

$$E(Y_2) = E(X_1 \cdot X_2) \overset{\text{niezależność}}{=} E(X_1) \cdot E(X_2) = \int_1^2 x\,dx \cdot \int_1^2 x\,dx = \left(\frac{3}{2}\right)^2 = \underline{\frac{9}{4}}$$

$$E(X_1^2) = E(X_2^2) = \int_1^2 x^2\,dx = \frac{7}{3}$$

$$E(Y_1^2) = E\left((2X_1 + 2X_2)^2\right) = E\left(4X_1^2 + 8X_1X_2 + 4X_2^2\right) = 4E(X_1^2) + 8E(X_1X_2) + 4E(X_2^2) =$$

$$= 4 \cdot 7/3 + 8 \cdot 9/4 + 4 \cdot 7/3 = 18 + 18\,2/3 = 36\,2/3$$

$$V(Y_1) = E(Y_1^2) - E(Y_1)^2 = 36\,2/3 - 6^2 = \underline{2/3}$$

$$E(Y_2^2) = E(X_1^2 \cdot X_2^2) \overset{\text{niezal.}}{=} E(X_1^2) \cdot E(X_2^2) = \frac{49}{9}$$

$$V(Y_2) = E(Y_2^2) - E(Y_2)^2 = \frac{49}{9} - \left(\frac{9}{4}\right)^2 = \frac{49 \cdot 16 - 81 \cdot 9}{144} = \underline{\frac{55}{144}}$$

zadanie 14

$$p = \frac{E((Y_1 - EY_1)(Y_2 - EY_2))}{\sqrt{VY_1 \cdot VY_2}} = \frac{E\left(Y_1 Y_2 - \frac{9}{4}Y_1 - 6Y_2 + 13\frac{1}{2}\right)}{\sqrt{55/216}}$$

niezależność X_1, X_2

$$E(Y_1 Y_2) = E((2X_1 + 2X_2)X_1 X_2) = 2E(X_1^2 X_2) + 2E(X_1 X_2^2) = 2 \cdot 7/3 \cdot 3/2 + 2 \cdot 3/2 \cdot 7/3 =$$

$$= 4 \cdot 7/3 \cdot 3/2 = 14$$

$$p = \frac{14 - 9/4 \cdot 6 - 6 \cdot 9/4 + 13\frac{1}{2}}{\sqrt{\frac{55}{216}}} = \frac{27\frac{1}{2} - 27}{\sqrt{\frac{55}{216}}} = \frac{1}{2\sqrt{\frac{55}{216}}} =$$

$$= \frac{1}{2} \cdot \sqrt{\frac{216}{55}} = \frac{1}{2} \cdot \frac{\sqrt{55 \cdot 216}}{55} = \frac{1}{2} \cdot \frac{\sqrt{36 \cdot 330}}{55} = \frac{3\sqrt{330}}{55} \blacksquare$$