zadanie 1

$$f(x,y) = c\cdot(x+y)e^{-(x+y)}, \quad x,y>0$$

1. $f(x,y) \geq 0$ czyli wystarczy $c \geq 0$

2. $c\cdot\int_0^\infty\int_0^\infty (x+y)e^{-(x+y)}\,dy\,dx = 1$

$$\int_0^\infty (x+y)e^{-(x+y)}\,dy =$$

$$= e^{-x}\int_0^\infty (x+y)e^{-y}\,dy =$$

$$= e^{-x}\left(-e^{-y}(x+y)\Big|_0^\infty + \int_0^\infty e^{-y}\,dy\right) =$$

$$= e^{-x}(x+1)$$

$$\int_0^\infty e^{-x}(x+1)\,dx = -e^{-x}(x+1)\Big|_0^\infty + \int_0^\infty e^{-x}\,dx =$$

$= 1 + 1 = 2$, czyli $\boxed{c = \frac{1}{2}}$

GĘSTOŚCI BRZEGOWE:

$$f_1(x) = \int_0^\infty \frac{1}{2}(x+y)e^{-(x+y)}\,dy = \frac{1}{2}e^{-x}(x+1)$$

$$f_2(y) = \int_0^\infty \frac{1}{2}(x+y)e^{-(x+y)}\,dx = \frac{1}{2}e^{-y}(y+1)$$

$$f_1(x)\cdot f_2(y) = \frac{1}{4}e^{-(x+y)}(x+1)(y+1) \neq f(x,y)$$

czyli zmienne są zależne

czyli zmienne są zależne

MOMENTY:

$$m_{10} = \int_0^\infty\int_0^\infty x f(x,y)\,dy\,dx = \int_0^\infty\int_0^\infty x\cdot\frac{1}{2}(x+y)e^{-(x+y)}\,dy\,dx =$$

$$= \frac{1}{2}\int_0^\infty \cancel{\tfrac{1}{2}}\, x\cdot e^{-x}(x+1)\,dx = \frac{1}{2}\left(\underbrace{-e^{-x}(x^2+x)\Big|_0^\infty}_{=0} - \int_0^\infty -e^{-x}(2x+1)\,dx\right) =$$

$$= \frac{1}{2}\int_0^\infty e^{-x}(2x+1)\,dx = \frac{1}{2}\left(-e^{-x}(2x+1)\Big|_0^\infty + 2\int_0^\infty e^{-x}\right) = \frac{1}{2}(1+2) =$$

symetrycznie $m_{01} = \frac{3}{2}$

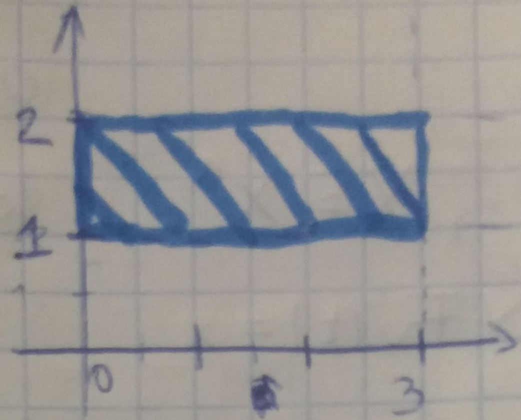
zadanie 2

$$f(x,y) = Cxy + x + y$$

aby $f(x,y) \geq 0$: $C \geq 0$ (1)

chcemy 1

$$\int_0^3 \int_1^2 Cxy + x + y \, dy \, dx$$



$$Cx \int_1^2 y \, dy = Cx \frac{y^2}{2}\Big|_1^2 = Cx\left(2 - \frac{1}{2}\right) = \frac{3}{2}Cx$$

$$\frac{x^2}{2}\Big|_0^3 = \frac{9}{2}$$

$$xC \int_1^2 y \, dy + \int_1^2 x \, dy + \int_1^2 y \, dy = Cx \cdot \frac{3}{2} + x + \frac{3}{2}$$

$$\int_0^3 Cx\frac{3}{2} + x + \frac{3}{2} \, dx = \frac{3}{2}C\int_0^3 x \, dx + \int_0^3 x \, dx + \frac{3}{2}\int_0^3 1 \, dx =$$

$$= \frac{3}{2}C \cdot \frac{9}{2} + \frac{9}{2} + \frac{9}{2} = \frac{9}{2}\left(\frac{3}{2}C + 2\right) = \frac{27}{4}C + 9 = 1$$

czyli $C = \frac{4 \cdot (-8)}{27} = -\frac{32}{27} < 0$

co sprzeczne z (1)

$f(x,y)$ nie może być w tym wypadku f. gęstości

$-x(xy+1$

$\boxed{\text{zadanie 3}}$ $f_{xy}(x,y) = -xy + x$; $0 \le x \le 2$, $0 \le y \le 1$

$$f_1(x) = \int_0^1 -xy + x \, dy = -\int_0^1 xy \, dy + x \int_0^1 1 \, dy = -x \frac{y^2}{2}\Big|_0^1 + x = \frac{1}{2}x$$

$$f_2(y) = \int_0^2 -xy + x \, dx = -y \int_0^2 x \, dy + \int_0^2 x \, dx = -2y + 2 =$$

$$= -2y + 2$$

$f_1(x) \cdot f_2(y) = -yx + x = f(x,y)$ czyli są niezależne

zadanie 4

$$P(1 \leq X \leq 3,\ 0 \leq Y \leq 0{,}5) \quad \int_1^2 \int_0^{1/2} -xy + x \, dy\, dx = ?$$

$$= \int_0^{1/2} -xy + x \, dy = -x \int_0^{1/2} y \, dy + \int_0^{1/2} x \, dy = -\frac{1}{8}x + \frac{1}{2}x = \frac{3}{8}x$$

$$\int_1^2 \frac{3}{8} x \, dx = \frac{3}{8} \cdot \frac{x^2}{2} \Big|_1^2 = \frac{9}{16}$$

zadanie 5

$0 \leq y \leq 1$

$X \sim U[0,1]$, $Y = X^n$, teza: $f_Y(y) = \dfrac{y^{\frac{1}{n}-1}}{n}$

$f_X(x) = 1 \Rightarrow F_X(t) = x$

$$F_Y(t) = P(Y < t) = P(X^n < t) = P(X < \sqrt[n]{t}) = F_X(\sqrt[n]{t})$$

$$f_X(y) = (F_X(\sqrt[n]{y}))' = 1 \cdot (\sqrt[n]{y})' = \frac{y^{\frac{1}{n}-1}}{n} \quad \blacksquare$$

zadanie 6

$\boxed{\text{zadanie 6}}$

$X \sim U[-1, 1]$, $Y = |X|$

$f_X(x) = \frac{1}{2} \Rightarrow F_X(t) = \frac{1}{2}x$

$t \in [0, 1]$

$F_Y(t) = P(Y < t) = P(|X| < t) = P(-t < X < t) =$

$= P(X < t) - P(X < -t) = F_X(t) - F_X(-t) =$

$= \frac{1}{2}t - (-\frac{1}{2}t) = t = F_Y(t) \Rightarrow f_Y(y) = 1$

Zadanie 8

$Y = X^2$, X na $\mathbb{R}$ teza: $f_X(y) = \dfrac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}$ dla $y>0$

$F_Y(t) = P(Y \leq t) = P(X^2 \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t}) =$

$= P(X \leq \sqrt{t}) - P(X \leq -\sqrt{t}) = F_X(\sqrt{t}) - F_X(-\sqrt{t})$

$$f_Y(y) = F_Y'(y) = F_X'(\sqrt{y}) - F_X'(-\sqrt{y}) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) - \left(-\frac{1}{2\sqrt{y}} f_X(-\sqrt{y})\right) =$$

$$= \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \quad \blacksquare$$

zadanie 9

$$f_x(x) = xe^{-x} \quad , x \geq 0 \quad , \quad Y = X^2$$

$$F_y(t) = P(Y < t) = P(X^2 < t) = P(X < \sqrt{t}) = F_x(\sqrt{t})$$

$$F_x(t) = \int_0^t f_x(x)\,dx = \int_0^t xe^{-x}\,dx = -e^{-t}(t+1) + 1$$

$$F_x(\sqrt{t}) = -e^{-\sqrt{t}}(\sqrt{t}+1) + 1 = F_y(t)$$

$$f_y(t) = F_y'(t) = (-e^{-\sqrt{t}})'(\sqrt{t}+1) + (-e^{-\sqrt{t}})(\sqrt{t}+1)' =$$

$$= \frac{e^{-\sqrt{t}}}{2\sqrt{t}}(\sqrt{t}+1) + (-e^{-\sqrt{t}}) \cdot \frac{1}{2\sqrt{t}} =$$

$$= \frac{e^{-\sqrt{t}} \cdot \sqrt{t} + e^{-\sqrt{t}} - e^{-\sqrt{t}}}{2\sqrt{t}} = \frac{e^{-\sqrt{t}}}{2} \quad \blacksquare$$

zadanie 10

$X \sim U[a,b]$ czyli $f_X(x) = \frac{1}{b-a}$

policzmy $EX = \int_a^b x f(x) dx = \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_a^b = \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) =$

$= \frac{1}{2} \cdot \frac{1}{b-a} \cdot (b-a)(b+a) = \frac{b+a}{2}$

$VX = \int_a^b (x - EX)^2 f(x) dx = \frac{1}{b-a}\left(\int_a^b x^2 dx - 2\int_a^b EX \cdot x \, dx + \int_a^b EX^2 dx\right) =$

$= \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3} - (b+a)\left(\frac{b^2}{2} - \frac{a^2}{2}\right) + \frac{(b+a)^2}{4}(b-a)\right) =$

$= \cancel{\frac{1}{b-a}}\left(\frac{\cancel{(b-a)}(b^2+ab+a^2)}{3} - \frac{(b+a)^2\cancel{(b-a)}}{2} + \frac{(b+a)^2\cancel{(b-a)}}{4}\right) =$

$= \frac{a^2}{3} + \frac{ab}{3} + \frac{b^2}{3} - \frac{a^2}{2} - ab - \frac{b^2}{2} + \frac{a^2}{4} + \frac{ab}{2} + \frac{b^2}{4} =$

$= \frac{1}{12}a^2 - \frac{1}{6}ab + \frac{1}{12}b^2$

$$f_X(x) = \frac{1}{\pi(x^2+1)}, \quad x \in \mathbb{R} \qquad Y = \frac{1}{X}$$

$$F_Y(y) = P(Y<y) = P\left(\frac{1}{X}<y\right) = P\left(X > \frac{1}{y}\right) = 1 - P\left(X<\frac{1}{y}\right) =$$

$$= 1 - F_X\left(\frac{1}{y}\right)$$

$$F_X(t) = \frac{1}{\pi}\int_{-\infty}^{t} \frac{1}{x^2+1}\,dx = \frac{1}{\pi}\cdot \arctan x \Big|_{-\infty}^{t} = \frac{\arctan t}{\pi} + \frac{1}{2}$$

$$F_Y(y) = 1 - F_X(1/y) = 1 - \left(\frac{\arctan\frac{1}{y}}{\pi} + \frac{1}{2}\right) =$$

$$= \frac{1}{2} - \frac{\arctan \frac{1}{y}}{\pi}$$

$$f_Y(y) = F_Y'(y) = -\frac{1}{\pi}\left(\arctan\frac{1}{y}\right)' =$$

$$= -\frac{1}{\pi}\cdot \frac{1}{\frac{1}{y^2}+1}\cdot\frac{-1}{y^2} = \frac{1}{\pi(y^2+1)}$$

zadanie 11 dla $y=0$:

$$f_y(0) = \lim_{t \to 0} \left(1 - F_x\left(\frac{1}{t}\right)\right)' = \lim_{t \to 0} f_x\left(\frac{1}{t}\right) \cdot \frac{1}{t^2} =$$

$$= \lim_{t \to 0} \frac{1}{\pi\left(\frac{1}{t^2}+1\right)} \cdot \frac{1}{t^2} = \frac{1}{\pi} \lim_{t \to 0} \frac{1}{t^2+1} = \frac{1}{\pi}$$