Laplace Transform Lab: Solving ODEs using Laplace Transform in MATLAB

This lab will teach you to solve ODEs using a built in MATLAB Laplace transform function `laplace`.

There are five (5) exercises in this lab that are to be handed in. Write your solutions in a separate file, including appropriate descriptions in each step.

Include your name and student number in the submitted file.

Contents

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Using symbolic variables to define functions

In this exercise we will use symbolic variables and functions.

```
syms t s x y

f = cos(t)
h = exp(2*x)
```

```
f =

cos(t)


h =

exp(2*x)
```

Laplace transform and its inverse

```
% The routine |laplace| computes the Laplace transform of a function

F=laplace(f)
```

```
F =

s/(s^2 + 1)
```

By default it uses the variable `s` for the Laplace transform But we can specify which variable we want:

```
H=laplace(h)
laplace(h,y)

% Observe that the results are identical: one in the variable |s| and the
% other in the variable |y|
```

```
H =

1/(s - 2)


ans =

1/(y - 2)
```

We can also specify which variable to use to compute the Laplace transform:

```
j = exp(x*t)
laplace(j)
laplace(j,x,s)

% By default, MATLAB assumes that the Laplace transform is to be computed
% using the variable |t|, unless we specify that we should use the variable
% |x|
```

```
j =

exp(t*x)


ans =

1/(s - x)


ans =

1/(s - t)
```

We can also use inline functions with `laplace`. When using inline functions, we always have to specify the variable of the function.

```
l = @(t) t^2+t+1
laplace(l(t))
```

```
l =

  function_handle with value:

    @(t)t^2+t+1


ans =

(s + 1)/s^2 + 2/s^3
```

MATLAB also has the routine `ilaplace` to compute the inverse Laplace transform

```
ilaplace(F)
ilaplace(H)
ilaplace(laplace(f))
```

```
ans =

cos(t)


ans =

exp(2*t)


ans =

cos(t)
```

If `laplace` cannot compute the Laplace transform, it returns an unevaluated call.

```
g = 1/sqrt(t^2+1)
G = laplace(g)
```

```
g =

1/(t^2 + 1)^(1/2)


G =

laplace(1/(t^2 + 1)^(1/2), t, s)
```

But MATLAB "knows" that it is supposed to be a Laplace transform of a function. So if we compute the inverse Laplace transform, we obtain the original function

```
ilaplace(G)
```

```
ans =

1/(t^2 + 1)^(1/2)
```

The Laplace transform of a function is related to the Laplace transform of its derivative:

```
syms g(t)
laplace(diff(g,t),t,s)
```

```
ans =

s*laplace(g(t), t, s) - g(0)
```

Exercise 1

Objective: Compute the Laplace transform and use it to show that MATLAB 'knows' some of its properties.

Details:

(a) Define the function `f(t)=exp(2t)*t^3`, and compute its Laplace transform `F(s)`. (b) Find a function `f(t)` such that its Laplace transform is `(s - 1)*(s - 2))/(s*(s + 2)*(s - 3)` (c) Show that MATLAB 'knows' that if `F(s)` is the Laplace transform of `f(t)`, then the Laplace transform of `exp(at)f(t)` is `F(s-a)`

(in your answer, explain part (c) using comments).

Observe that MATLAB splits the rational function automatically when solving the inverse Laplace transform.

```
%a
syms f(t) f2(t)
f(t)=exp(2.*t).*t.^3
F=laplace(f(t))
%b
f2(t)=((s - 1)*(s - 2))/(s*(s + 2)*(s - 3))
f=ilaplace(f2)

%c
h = @(t) t^2
H1 = laplace(h(t))
H2 = laplace(exp(3*t) * h(t)) % is H1 with s shifted by 3 (s-3)
% it shows that MATLAB knows that the laplace transform of exp(at)f(t) is
% F(s-a) because it outputs the same value of s in H1 and replace it by
% (s-3) in H2 when the function is multiplied by exp(3t)
```

```
f(t) =

t^3*exp(2*t)


F =

6/(s - 2)^4


f2(t) =

((s - 1)*(s - 2))/(s*(s + 2)*(s - 3))


f(t) =

(6*exp(-2*t))/5 + (2*exp(3*t))/15 - 1/3


h =

  function_handle with value:

    @(t)t^2


H1 =

2/s^3


H2 =
```

```
2/(s - 3)^3
```

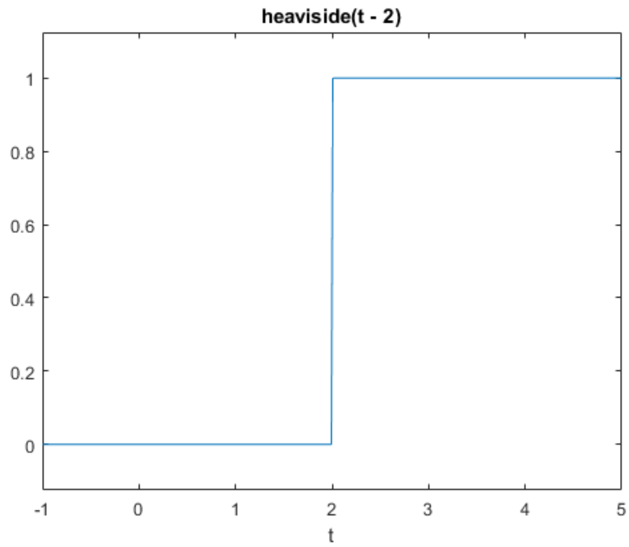
Heaviside and Dirac functions

These two functions are builtin to MATLAB: `heaviside` is the Heaviside function `u_0(t)` at `0`

To define `u_2(t)`, we need to write

```
f=heaviside(t-2)
ezplot(f,[-1,5])

% The Dirac delta function (at |0|) is also defined with the routine |dirac|

g = dirac(t-3)

% MATLAB "knows" how to compute the Laplace transform of these functions

laplace(f)
laplace(g)
```

```
f =

heaviside(t - 2)


g =

dirac(t - 3)


ans =

exp(-2*s)/s


ans =

exp(-3*s)
```



Exercise 2

Objective: Find a formula comparing the Laplace transform of a translation of `f(t)` by `t-a` with the Laplace transform of `f(t)`

Details:

- Give a value to `a`
- Let `G(s)` be the Laplace transform of `g(t)=u_a(t)f(t-a)` and `F(s)` is the Laplace transform of `f(t)`, then find a formula relating `G(s)` and `F(s)`

In your answer, explain the 'proof' using comments.

```
syms g(t) f(t)
a=5
u_a(t)=heaviside(t)
g(t)=u_a(t-a).*f(t-a)
G=laplace(g)
F=laplace(f)
%G(s) = exp(-as)*F
```

```
%when shifted by t-a (shift to the right by a), the laplace transform
%will also shift into the original laplace transform multiplied by exp(-as)
%this agrees with the property of laplace transform as proven in tutorial
```

```
a =

     5


u_a(t) =

heaviside(t)


g(t) =

f(t - 5)*heaviside(t - 5)


G =

exp(-5*s)*laplace(f(t), t, s)


F =

laplace(f(t), t, s)
```

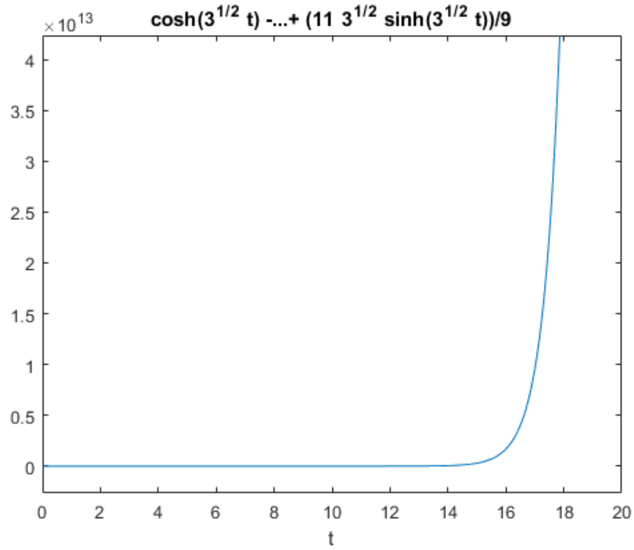
Solving IVPs using Laplace transforms

Consider the following IVP, `y''-3y = 5t` with the initial conditions `y(0)=1` and `y'(0)=2`. We can use MATLAB to solve this problem using Laplace transforms:

```
% First we define the unknown function and its variable and the Laplace
% transform of the unknown

syms y(t) t Y s

% Then we define the ODE

ODE=diff(y(t),t,2)-3*y(t)-5*t == 0

% Now we compute the Laplace transform of the ODE.

L_ODE = laplace(ODE)

% Use the initial conditions

L_ODE=subs(L_ODE,y(0),1)
L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),2)

% We then need to factor out the Laplace transform of |y(t)|

L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
Y=solve(L_ODE,Y)

% We now need to use the inverse Laplace transform to obtain the solution
% to the original IVP

y = ilaplace(Y)

% We can plot the solution

ezplot(y,[0,20])

% We can check that this is indeed the solution

diff(y,t,2)-3*y
```

```
ODE =

diff(y(t), t, t) - 3*y(t) - 5*t == 0


L_ODE =

s^2*laplace(y(t), t, s) - subs(diff(y(t), t), t, 0) - s*y(0) - 5/s^2 - 3*laplace(y(t), t, s) == 0


L_ODE =

s^2*laplace(y(t), t, s) - subs(diff(y(t), t), t, 0) - s - 5/s^2 - 3*laplace(y(t), t, s) == 0


L_ODE =

s^2*laplace(y(t), t, s) - s - 5/s^2 - 3*laplace(y(t), t, s) - 2 == 0
```

```
L_ODE =

Y*s^2 - s - 3*Y - 5/s^2 - 2 == 0


Y =

(s + 5/s^2 + 2)/(s^2 - 3)


y =

cosh(3^(1/2)*t) - (5*t)/3 + (11*3^(1/2)*sinh(3^(1/2)*t))/9


ans =

5*t
```



Exercise 3

Objective: Solve an IVP using the Laplace transform

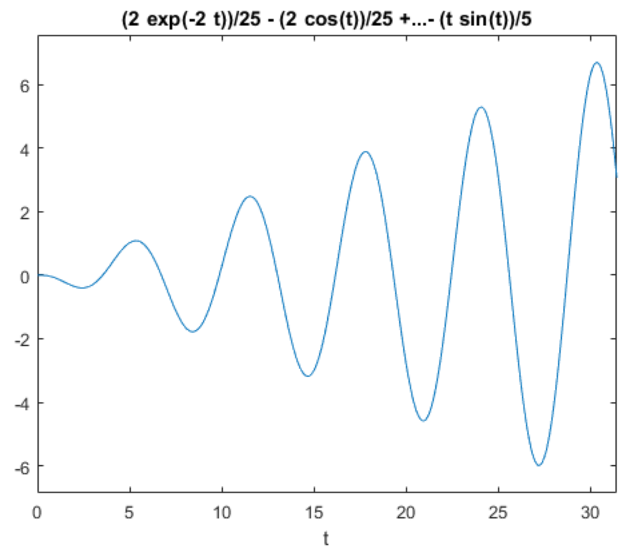
Details: Explain your steps using comments

- Solve the IVP
- `y'''+2y''+y'+2*y=-cos(t)`
- `y(0)=0`, `y'(0)=0`, and `y''(0)=0`
- for `t` in `[0,10*pi]`
- Is there an initial condition for which `y` remains bounded as `t` goes to infinity? If so, find it.

```
syms y(t) t Y s
% Then we define the ODE

ODE=diff(y(t),t,3)+2*diff(y(t),t,2)+diff(y(t),t,1)+2*y(t)+cos(t) == 0
%y'''+2y''+y'+2*y=-cos(t)

% Now we compute the Laplace transform of the ODE.

L_ODE = laplace(ODE)

% Use the initial conditions

L_ODE=subs(L_ODE,y(0),0)
L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),0)
L_ODE=subs(L_ODE,subs(diff(y(t), t, 2), t, 0),0)

% We then need to factor out the Laplace transform of |y(t)|

L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
Y=solve(L_ODE,Y)

% We now need to use the inverse Laplace transform to obtain the solution
% to the original IVP

y = ilaplace(Y)

% We can plot the solution

ezplot(y,[0,10*pi])
```

```
% Using arbitrary constants a, b, and c we can find the general
% solution of the ODE
% a*exp(-2t) +bcos(t)+c*sin(t)-t*sin(t)/5+t*cos(t)/10
% There's no initial condition for which y remains bounded as t goes to
% infinity since when t approaches infinity y will continue growing while oscillating
% -a*exp(-2t) will approach 0
% bcos(t) will oscillate will oscillate between b and -b
% c*sin(t) will oscillate between c and -c
% which means the final solution will oscillate
% - t*sin(t)/5 + t*cos(t)/10 will diverge, this does not depend on
% constants a, b, and c or the initial conditions
% thus the final solution will oscillate while diverge, or in other words
% growing while oscillating no matter the initial conditions
```

```
ODE =

cos(t) + 2*y(t) + diff(y(t), t) + 2*diff(y(t), t, t) + diff(y(t), t, t, t) == 0


L_ODE =

s*laplace(y(t), t, s) - y(0) - s*subs(diff(y(t), t), t, 0) - 2*s*y(0) - 2*subs(diff(y(t), t), t, 0) + s/(s^2 + 1) + 2*s^2*laplace(y(t), t, s) + s^3*laplace(y(t


L_ODE =

s*laplace(y(t), t, s) - s*subs(diff(y(t), t), t, 0) - 2*subs(diff(y(t), t), t, 0) + s/(s^2 + 1) + 2*s^2*laplace(y(t), t, s) + s^3*laplace(y(t), t, s) - subs(di


L_ODE =

s*laplace(y(t), t, s) + s/(s^2 + 1) + 2*s^2*laplace(y(t), t, s) + s^3*laplace(y(t), t, s) - subs(diff(y(t), t, t), t, 0) + 2*laplace(y(t), t, s) == 0


L_ODE =

s*laplace(y(t), t, s) + s/(s^2 + 1) + 2*s^2*laplace(y(t), t, s) + s^3*laplace(y(t), t, s) + 2*laplace(y(t), t, s) == 0


L_ODE =

2*Y + Y*s + s/(s^2 + 1) + 2*Y*s^2 + Y*s^3 == 0


Y =

-s/((s^2 + 1)*(s^3 + 2*s^2 + s + 2))


y =

(2*exp(-2*t))/25 - (2*cos(t))/25 + (3*sin(t))/50 + (t*cos(t))/10 - (t*sin(t))/5
```



Exercise 4

Objective: Solve an IVP using the Laplace transform

Details:

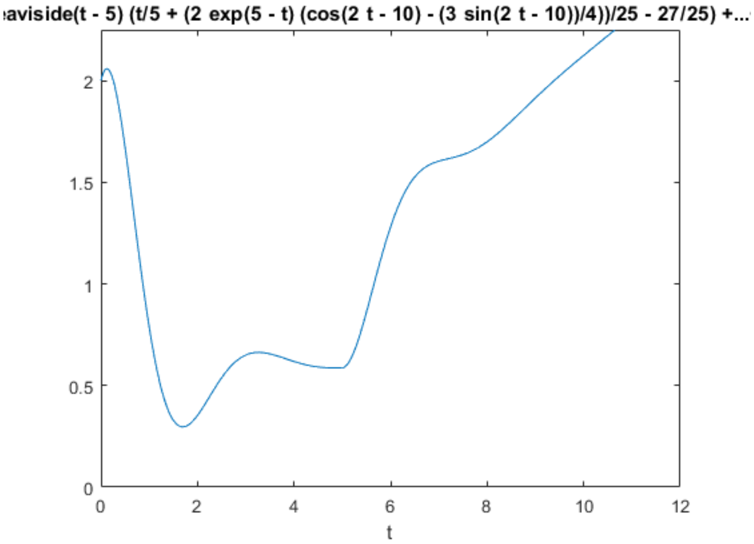
- Define
- `g(t) = 3 if 0 < t < 2`

- `g(t) = t+1 if 2 < t < 5`
- `g(t) = 5 if t > 5`
- Solve the IVP
- `y''+2y'+5y=g(t)`
- `y(0)=2 and y'(0)=1`
- Plot the solution for `t` in `[0,12]` and `y` in `[0,2.25]`.

In your answer, explain your steps using comments.

```
% First we define the unknown function and its variable and the Laplace
% transform of the unknown

syms y(t) t Y s g(t)
%g(t) = piecewise(0 < t < 2, 3 , 2 < t < 5, t+1, t > 5, 5)
%g(t) = (3).*[heaviside(t)-heaviside(t-2)]+(t+1).*[heaviside(t-2)-heaviside(t-5)]+(5).*[heaviside(t-5)];
g(t) = (3).*(heaviside(t))+(t-2).*heaviside(t+3).*(heaviside(t-5));
%plot(t,y)
% Then we define the ODE

ODE=diff(y(t),t,2)+2.*diff(y(t),t) + 5*y(t) == g(t)

% Now we compute the Laplace transform of the ODE.

L_ODE = laplace(ODE)

% Use the initial conditions

L_ODE=subs(L_ODE,y(0),2)
L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),1)

% We then need to factor out the Laplace transform of |y(t)|

L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
Y=solve(L_ODE,Y)

% We now need to use the inverse Laplace transform to obtain the solution
% to the original IVP

y = ilaplace(Y)

% We can plot the solution

ezplot(y,[0, 12, 0, 2.25])
clear
```

```
ODE =

5*y(t) + 2*diff(y(t), t) + diff(y(t), t, t) == 3*heaviside(t) + heaviside(t + 3)*heaviside(t - 5)*(t - 2)


L_ODE =

2*s*laplace(y(t), t, s) - 2*y(0) - s*y(0) - subs(diff(y(t), t), t, 0) + s^2*laplace(y(t), t, s) + 5*laplace(y(t), t, s) == 3/s + (exp(-5*s)*(3*s + 1))/s^2


L_ODE =

2*s*laplace(y(t), t, s) - 2*s - subs(diff(y(t), t), t, 0) + s^2*laplace(y(t), t, s) + 5*laplace(y(t), t, s) - 4 == 3/s + (exp(-5*s)*(3*s + 1))/s^2


L_ODE =

2*s*laplace(y(t), t, s) - 2*s + s^2*laplace(y(t), t, s) + 5*laplace(y(t), t, s) - 5 == 3/s + (exp(-5*s)*(3*s + 1))/s^2


L_ODE =

5*Y - 2*s + 2*Y*s + Y*s^2 - 5 == 3/s + (exp(-5*s)*(3*s + 1))/s^2


Y =

(2*s + 3/s + (exp(-5*s)*(3*s + 1))/s^2 + 5)/(s^2 + 2*s + 5)


y =

heaviside(t - 5)*(t/5 + (2*exp(5 - t)*(cos(2*t - 10) - (3*sin(2*t - 10))/4))/25 - 27/25) + 2*exp(-t)*(cos(2*t) - sin(2*t)/2) - (3*exp(-t)*(cos(2*t) + sin(2*t)/
```



Exercise 5

Objective: Use the Laplace transform to solve an integral equation

Verify that MATLAB knowns about the convolution theorem by explaining why the following transform is computed correctly.

```
syms t tau y(tau) s
I=int(exp(-2*(t-tau))*y(tau),tau,0,t) %f(t-Tau) = exp(-2*(t-tau)) , g(tau)=y(tau)
L= laplace(I,t,s)
L= laplace(int(exp(-2*(tau))*y(t-tau),tau,0,t),t,s) %H

G=laplace(y(t), t, s)
F=laplace(exp(-2*(t)), t, s)
H=F*G
% as shown by the answer, the solution to the laplace of the integral of
% f(t-tau)g(tau) dtau is equivalent to the laplace of f(t) * laplace of g(t)
% which satisfies the convolution theorem that states that Laplace transform of a convolution integral is equal
% to the product of the individual Laplace transforms of the functions involved.
```

```
I =

int(exp(2*tau - 2*t)*y(tau), tau, 0, t)


L =

laplace(y(t), t, s)/(s + 2)


L =

laplace(y(t), t, s)/(s + 2)


G =

laplace(y(t), t, s)


F =

1/(s + 2)


H =

laplace(y(t), t, s)/(s + 2)
```

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