Summary Sheet for Power Series and Fourier Transform

$$y'' + p(x)y' + qy = 0 - - - - - (1)$$

If $p(x)$ and $q(x)$ of the equation (1) are analytic at x_0 (that is there exists a power series in powers of $x - x_0$), then the solution to (1) has the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Legendre Equation:

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

Any solution of Legendre equation is called a Legendre function

Recurrence Relation:

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s$$

$$y_1 = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots$$

$$y_2 = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots$$

Legendre Polynomial:

$$P_n(x) = \sum_{m=0}^{M} \frac{(-1)^m (2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!} x^{n-2m}, M = \frac{n}{2} or \frac{n-2}{2}$$

Rodrigues Formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Properties of Legendre Polynomial:

$P_n(-x) = (-1)^n P_n(x)$ $\quad$ $P_n(1) = 0$ $\quad$ $P_n(-1) = (-1)^n$ $\quad$ $P_{2n-1}(0) = 0$

$P'_{2n}(0) = 0$ $\quad$ $(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1}$ $\quad$ $P'_{n+1} - P'_{n-1} = (2n+1)P_n$

$$\int_{-1}^{1} P_n P_m dx = \begin{cases} 0 & if\ n \neq m \\ \dfrac{2}{2n+1} & if\ n = m \end{cases}$$

Frobenius Method

Consider the following PDE

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0 - - - - - (2)$$

If $b(x)$ and $c(x)$ in equation (2) are analytic at $x = 0$. Then Eq.(2) has at least one solution that can be represented in the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n --- (3)$$

where the exponent r may be any number (real or complex). The ODE (2) has a second solution that may be similar to (3) but with a different r and different coefficients or may contain a logarithmic term. Since $b(x)$ and $c(x)$ are analytic, it can also be written as

$$b(x) = \sum_{n=0}^{\infty} b_n x^n \, , c(x) = \sum_{n=0}^{\infty} c_n x^n$$

Then the indicial equation in r is $r(r-1) + b_0 r + c_0 = 0$. Let r_1, r_2 be two roots of the indicial equation

Case 1: Distinct Roots not differing by an integer($r_1 \neq r_2, r_1 - r_2$ is not divisible by an integer)

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \, , y_2(x) = x^{r_1} \sum_{n=0}^{\infty} A_n x^n$$

Case 2: Distinct Roots but differing by an integer($r_1 \neq r_2, r_1 - r_2$ is divisible by an integer)

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \, , y_2(x) = k y_1(x) \ln x + x^{r_2} \sum_{n=0}^{\infty} A_n x^n$$

Case 1: Double Roots ($r_1 = r_2 = r$)

$$y_1(x) = x^r \sum_{n=0}^{\infty} a_n x^n \, , y_2(x) = y_1(x) \ln x + x^r \sum_{n=0}^{\infty} A_n x^n$$

Bessel's Equation:

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

Bessel Functions of the First Kind

$r = \nu$:

$$a_{2n} = -\frac{1}{2^2 n(n+\nu)} a_{2n-2} = \frac{(-1)^n}{2^{2n+\nu} n! \, \Gamma(n+\nu+1)} a_0$$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, \Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$r = -\nu$:

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, \Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu}$$

Bessel Functions of the Second Kind

$$Y_\nu(x) = \frac{cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{sin\nu\pi}$$

Properties of Bessel Functions: n is an integer

$$[x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x) \qquad [x^{-\nu} J_\nu(x)]' = -x^\nu J_{\nu+1}(x)$$

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \tfrac{2\nu}{x} J_\nu(x) \qquad J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x)$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \qquad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$J_{-n}(x) = (-1)^n J_n(x) \qquad J_n(-x) = (-1)^n J_n(x) \qquad \lim_{x\to 0} Y_n(x) = -\infty$$

$$J_n(0) = \begin{cases} 0 & if\ n > 0 \\ 1 & if\ n = 0 \end{cases} \qquad \int_0^1 x J_n(\lambda x) J_n(\mu x)\, dx = \begin{cases} 0 & if\ \lambda \neq \mu > 0 \\ \frac{1}{2} J_{n+1}^2(\lambda) & if\ \lambda = \mu \end{cases}$$

$$\cos x = J_0(x) + 2\textstyle\sum_{n=1}^{\infty} (-1)^n J_{2n}(x) \qquad \sin x = 2\textstyle\sum_{n=1}^{\infty} (-1)^n J_{2n+1}(x)$$

Fourier Series

Suppose $f(x)$ is a given function of period 2π

$$f(x) = a_0 + \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx\,, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx\,, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Representation by a Fourier Series

Let $f(x)$ be periodic with period 2π and piecewise continuous in the interval $[-\pi, \pi]$. Furthermore, let $f(x)$ have a left-hand derivative and a righthand derivative at each point of that interval. Then the Fourier series of $f(x)$ converges. Its sum is $f(x)$, except at points $x_0$where $f(x)$ is discontinuous. There sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0.

Fourier Series (period $2L$)

$$f(x) = a_0 + \sum_{n=0}^{\infty} (a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx\,, a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos n\frac{n\pi}{L}x\, dx\,, b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin n\frac{n\pi}{L}x\, dx$$

If f is even and $L = \pi$

$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos nx, \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx\,, a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

If f is odd and $L = \pi$

$$f(x) = \sum_{n=0}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

Sturm-Liouville Problems:

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0, \;\; x \in [a, b]$$

$$k_1 y(a) + k_2 y'(a) = 0, \qquad l_1 y(b) + l_2 y'(b) = 0$$

Here λ is a parameter, k_i, l_i are real constants.

Orthogonality of Eigenfunction:

Suppose that the functions p, q, r, p' in the Sturm-Liouville problem are real valued and continuous and $r(x) > 0$ on $[a, b]$. Let y_m, y_n be eigenfunctions of the Sturm-Liouville problem, that correspond to different eigenvalues λ_m, λ_n respectively. Then y_m, y_n are orthogonal on that interval w.r.to the weight function r,

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x) dx = 0\,, m \neq n$$

Generalized Fourier Series

Let $y_0, y_1, y_2, \ldots$ be orthogonal w.r.to a weight function $r(x)$ on $[a, b]$. Let $f(x)$ be a function that can be represented by a convergent series

$$f(x) = \sum_{n=0}^{\infty} a_n y_n(x)$$

This is called orthogonal series, orthogonal expansion or generalized Fourier Series. If y_n's are eigenfunctions of the Sturm-Liouville problem, then it is called as an eigenfunction expansion.

Fourier Integral

$$f(x) = \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx] dw$$

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv\,, B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv$$

Fourier Cosine Integral

$$f(x) = \int_0^{\infty} A(w) \cos wx \, dw\,, A(w) = \frac{2}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv$$

Fourier Cosine Integral

$$f(x) = \int_0^{\infty} A(w) \sin wx \, dw\,, A(w) = \frac{2}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv$$