Lecture Notes on Mathematics for Engineers

MA5101-Differential Equations

Panchatcharam Mariappan

Illustration

Contents

1. Introduction

1.1 Mathematical Modeling

Let us revisit our school physics first. Do you know Newton's second law of motion? What does it say? Is it not the relationship between the force action on a body of mass and the acceleration caused by this force? If you remember your school physics, you can also remember that force is directly proportional to the acceleration. That is,

$$F = ma \tag{1.1}$$

Now, what is the meaning of acceleration? It is nothing but the name given to the velocity changing process. Isn't it? Mathematically, it is the rate of change of the velocity. What does it mean? Let us consider that we start from home or hostel at 8'O clock in the morning and reach the college which is located 10km away from your home or hostel. Have you ever noticed that, the driver of the vehicle changes the speed of the vehicle depending on the region which he travels, for example, at the school zone, driver slows the vehicle whereas on the highway he speeds up the vehicle. That means, the driver rides the vehicle at different speed at different location. How much changes was made between two different times? Let t_1 and t_2 be two different time where we would like to observe the velocity changes done by the driver and let us assume that v_1 and v_2 are the respective velocities at time t_1 and t_2. Then,

$$\Delta t = t_2 - t_1 \tag{1.2}$$

and

$$\Delta v = v_2 - v_1 \tag{1.3}$$

are the change in time and velocity respectively. Therefore, the average acceleration over a period of time is the ratio of change in velocity to the change in time. Mathematically,

$$a = \frac{\Delta v}{\Delta t} \tag{1.4}$$

If we need to find the instantaneous acceleration, then we should observe the change in velocity over an infinitesimal interval of time. In Calculus, this is achieved by taking the limit as $\Delta t \to 0$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \tag{1.5}$$

Newton's second law (1.1) can be rewritten with the help of (1.5) as follows:

$$F = m\frac{dv}{dt} \implies \frac{dv}{dt} = \frac{F}{m} \tag{1.6}$$

So, we ended up formulating a simple physical real world problems into a mathematical equation. In general, engineering or real world problem which are formulated using mathematical expression in terms of variables, functions and equations are known as mathematical modeling.

When you have attended the course Mechanical vibrations, you might have encountered the undamped free vibrations

$$m\ddot{x} + kx = 0 \tag{1.7}$$

or damped free vibrations

$$m\ddot{x} + \gamma\dot{x} + kx = 0 \tag{1.8}$$

or in general the mass-spring system as

$$m\ddot{x} + \gamma\dot{x} + kx = F(t) \tag{1.9}$$

This is another mathematical modeling in the mechanical vibrations for a mass-spring system. In fact, in a multi-degree freedom of system, this will become a system of ODEs which we will discuss in the future.

1.2 Differential Equation

What is an equation? A statement that the values of two mathematical expression are equal. $=$ sign indicates equal sign. When the equation contains derivatives of an unknown function is called differential equation. Since the function is unknown in the differential equation, our next point of interest will be searching whether any known function will satisfy the differential equation. The process of finding such a function is called method of solution finding. In this course, we will learn different ways to find a solution, its properties such as existence and uniqueness, geometrical interpretation and so on. Through out this lecture notes, we will call the unknown function as y. Since y is a function, as you aware, we need at least an independent variable and dependent variable to represent a function. In Newton's second law model, which we have discussed earlier, we have v as an unknown function which is a dependent variable depending on the independent variable t. An **ordinary differential equation (ODE)** is an equation that has one or several derivatives of an unknown function with or without the unknown function itself, which we usually call $y(x)$ if the independent variable is x. In general ODEs can be written as

$$F(x, y, y', y'', ..., y^{(n)}) = 0 \tag{1.10}$$

Why do we name it as an ordinary? Does there exist any special then? When the unknown function depends on more than one variable, then we used to obtain the derivatives with respect to one of the variable by treating the remaining variable as constant. These type of derivatives are called partial derivatives. Equation which contains partial derivatives with or without the unknown function itself is called **partial differential equation**. For instance,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1.11}$$

Here, u is an unknown function of two independent variables x and y.

An ODE (1.10) is said to be of order n if the highest derivative involved in the ODE is $y^{(n)}$. When $n = 1$, it is the first-order ODE. When $n = 2$, it is the second-order ODE. In this course, we will deal only first and second-order ODEs. In the next chapter, we will deal with first-order ODEs.

1.3 A few more examples on Modeling

■ **Example 1.1** There was a murder in a hotel at room number 315 at 4:30PM. Police arrested Arjun, who was in the next room at 5:00PM. But, Arjun claims that he was not in his room for at least half an hour. The police check the water temperature of his tea kettle in his room at the instant of arrest and again 30 minutes later, obtaining the values $87°C$ and $43°C$, respectively. Can you investigate the case as an inspector? Is it possible to claim that Arjun is the murderer?

Mathematical Model: Let $T(t)$ be the temperature of the the tea kettle at any given time t. Assume that the room temperature is T_A. From Physics experiment, it was proved that the time rate of change of the temperature T of a body conducts heat well is proportional to the difference between T and temperature of the surrounding medium, which is called as Newton's law of cooling. Therefore, we obtain

$$\frac{dT}{dt} \propto (T - T_A) \tag{1.12}$$

$$\frac{dT}{dt} = k(T - T_A) \tag{1.13}$$

where k is a proportionality constant. Let us assume that the murder time as 0 hrs, Arjun arrested or temperature measured time as 0.5 hrs and temperature measure 30 minutes later as 1 hrs. Then,

$$\frac{dT}{dt} = k(T - T_A)$$
$$T(0.5) = 87, T(1) = 43$$

We need to find $T(0)$. Other way to formulate is assume the temperature measured time as 0, 30 minutes later as 0.5 and murder time as -0.5, then,

$$\frac{dT}{dt} = k(T - T_A)$$
$$T(0) = 87, T(0.5) = 43$$

We need to find $T(-0.5)$. For solution see A.

■ **Example 1.2** If the temperature of a cake is $150°C$ when it leaves the oven and is $100°C$ ten minutes later, when will it reach the room temperature $20°C$?

Mathematical Model for this problem is similar to 1.1. Then,

$$\frac{dT}{dt} = k(T - T_A)$$
$$T(0) = 150, T(0.1) = 100$$

Hint: It will take a long time, mathematically, $t \to \infty$

■ **Example 1.3** In a hostel, there is a cylindrical water tank of diameter 2m and height 2.25m. On a fine day, when Ragu was the first person to take the shower at 7AM in the hostel, the tank was empty. After an inspection by the hostel warden, it was found that there is a circular hole in the water tank at the bottom. When the hostel watchman switched off the power button of the water tank at 1AM on the previous day, it was completely filled. Without manually measuring the diameter of the hole, could you calculate the diameter of the hole?

Mathematical Model: Under the influence of gravity the out-flowing water from the hole at the bottom has velocity

$$v(t) = 0.6\sqrt{2gh(t)}$$

Here 0.6 was introduced by J. C. Borda after experimental data,[1]. Also, $h(t)$ denotes the water level to the outflow. Let us denote the volume of the water outflow during the Δt time as ΔV. If A denote the area of the hole, Then $\Delta V = Av\Delta t$. If B denotes the cross-sectional area of the tank, then the change of the volume of the water in the tank can also be given by $\Delta V* = -B\Delta h$, $\Delta h > 0$ denotes the decrease of the height. The minus sign denotes the decreasing water level in the tank.

$$\Delta V = \Delta V* \implies Av\Delta t = -B\Delta h$$
$$\implies \frac{\Delta h}{\Delta t} = -\frac{A}{B}v = -\frac{A}{B}0.6\sqrt{2gh}$$
$$\Delta t \to 0 \implies \frac{dh}{dt} = -\frac{A}{B}2.656\sqrt{h}$$

The mathematical model is

$$\frac{dh}{dt} = -\frac{A}{B}2.656\sqrt{h}$$
$$h(0) = 2.25, h(21600) = 0$$
$$B = \pi * r^2 = \pi * 1 * 1 = \pi$$

We have to find the value of A. Note that, the dimension should be properly used. Here we have used the fact that $g = 9.8m/s^2$. Therefore, all units should be in m and s related to length, height, area, and time. If not, there will be a mismatch in calculation. Therefore, 6 hour duration is converted to seconds. However, in previous examples, it won't make any change in final outcome, although constants of general solutions will be different.

■ **Example 1.4** The outflow of water from a cylindrical tank with a hole at the bottom. You are asked to find the height of the water in the tank at any time if the tank has diameter 2 m, the hole has diameter 1 cm, and the initial height of the water when the hole is opened is 2.25 m. When will the tank be empty?

Mathematical model for this problem is similar to 1.3. The IVP is given as follows

$$\frac{dh}{dt} = -\frac{A}{B}2.656\sqrt{h}, h(0) = 2.25$$
$$A = \pi * 0.005 * 0.005, B = \pi * 1 * 1$$

We have to find t when $h = 0$.

■ **Example 1.5** One hour before a surgery, certain drug at a constant amount was injected to the patient's blood stream. Certain amount of drug is removed simultaneously to avoid over dosage of drugs which is proportional to the amount of the drug present at time t.

Mathematical Model: Let us assume that the constant amount injected to the patient's blood stream as A. It is given that the drug remove from the patient's body is proportional to the amount of drug present at time t. That is, $\frac{dy}{dt} \propto y$. Since the amount is removed, so the proportionality constant say k is negative.

$$\frac{dy}{dt} = Injected - Removed$$
$$\frac{dy}{dt} = A - ky$$

■ **Example 1.6** It was found that hormone level of a patient varies w.r.to time. The rate of change of the hormone w.r.to time is the difference between the sinusoidal input of a 24-hour

period from thyroid gland and a continuous removal rate proportional to the level. Find and solve the hormone level model?

Mathematical Model: Let $y(t)$ denote the hormone level at time t. The removal rate is Ky. It is given that input rate is sinusoidal input rate. That is $A + B\sin\omega t$, where $\omega = \frac{2\pi}{24} = \frac{\pi}{12}$. $A \geq B$ so that input rate is non-negative, A is the average input rate. The constants, A, B, K should be obtained from measurements.

$$\frac{dy}{dt} = In - Out$$
$$\frac{dy}{dt} = A + B\sin\omega t - ky$$
$$y' + Ky = A + B\sin\omega t$$

■ **Example 1.7** In a room containing 20 cubic m^3 of air, 600 m^3 of fresh air flows in per minute, and the mixture (made practically uniform by circulating fans) is exhausted at a rate of 600 cubic metre per minute. What is the amount of fresh air at any time if there are no initial fresh air? After what time will 90% of the air be fresh?

Mathematical Model: The model is similar to the earlier two models. The rate of change of fresh air is the difference between the fresh air in and the fresh air out per minute. If $y(t)$ denotes the fresh air in the room at time t, then

$$y'(t) = 600 - \frac{600y}{20000}, y(0) = 0$$

Solving this equation and find time t when $y = 18000$.

■ **Example 1.8** A hybrid fuel tank in a rocket works on the principle of mixing two different fuel substances for combustion which in turn produces fuel supply for the throttle. The first tank contains 2 million litres of fuel in which 0.18 million kg of solid fuel substance is dissolved. Each 50 litre of the fuel fed into the throttle after mixing contains $(1 + cost)$ kg of the dissolved solid fuel substance. The mixture is uniform and runs to the throttle at the same rate. What is the amount of solid fuel substance at any time t?

Mathematical Model: The model is similar to the earlier model.

$$y'(t) = 50(1 + cost) - 0.000025y, y(0) = 0$$

■ **Example 1.9** Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank contains 1000 litres of water in which initially 100 kg of salt is dissolved. Brine runs in at a rate of 10 litre min, and each litre contains 5 kg of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 litre per min. Find the amount of salt in the tank at any time t.

Mathematical Model:

$$y'(t) = 50 - 0.01y, y(0) = 100$$

■ **Example 1.10** A model for the spread of contagious diseases is obtained by assuming that the rate of spread is proportional to the number of contacts between infected and non-infected persons, who are assumed to move freely among each other.

Mathematical Model: It is given that the rate of spread is proportional to both infected and non-infected persons. If we assume that y is the percentage of people who got infected at any given time, then $1 - y$ represent the non-infected persons. Therefore,
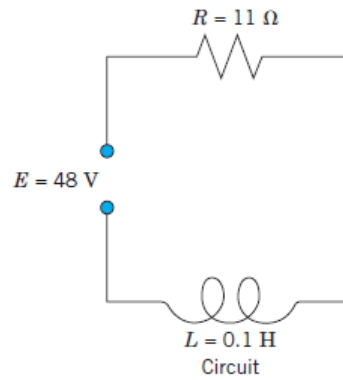
$$y' = k(1 - y)y$$

Figure 1.1: RL Circuit, Source: [2]

■ **Example 1.11** Model the following RL-Circuit for the current under the assumption that the initial current is zero.

Mathematical Model: According to Ohm's law, a current I in the circuit causes a voltage drop RI across the resistor. Also, a current in the circuit causes a voltage drop across the conductor, LI'. Kirchoff's Voltage law states that sum of these two voltage drops equals the EMF. The standard form of this model is

$$I' + \frac{R}{L}I = \frac{E(t)}{L}, I(0) = 0$$

From figure, $E = 48, L = 0.1, R = 11$. Therefore,

$$I' + 110I = 480, I(0) = 0$$
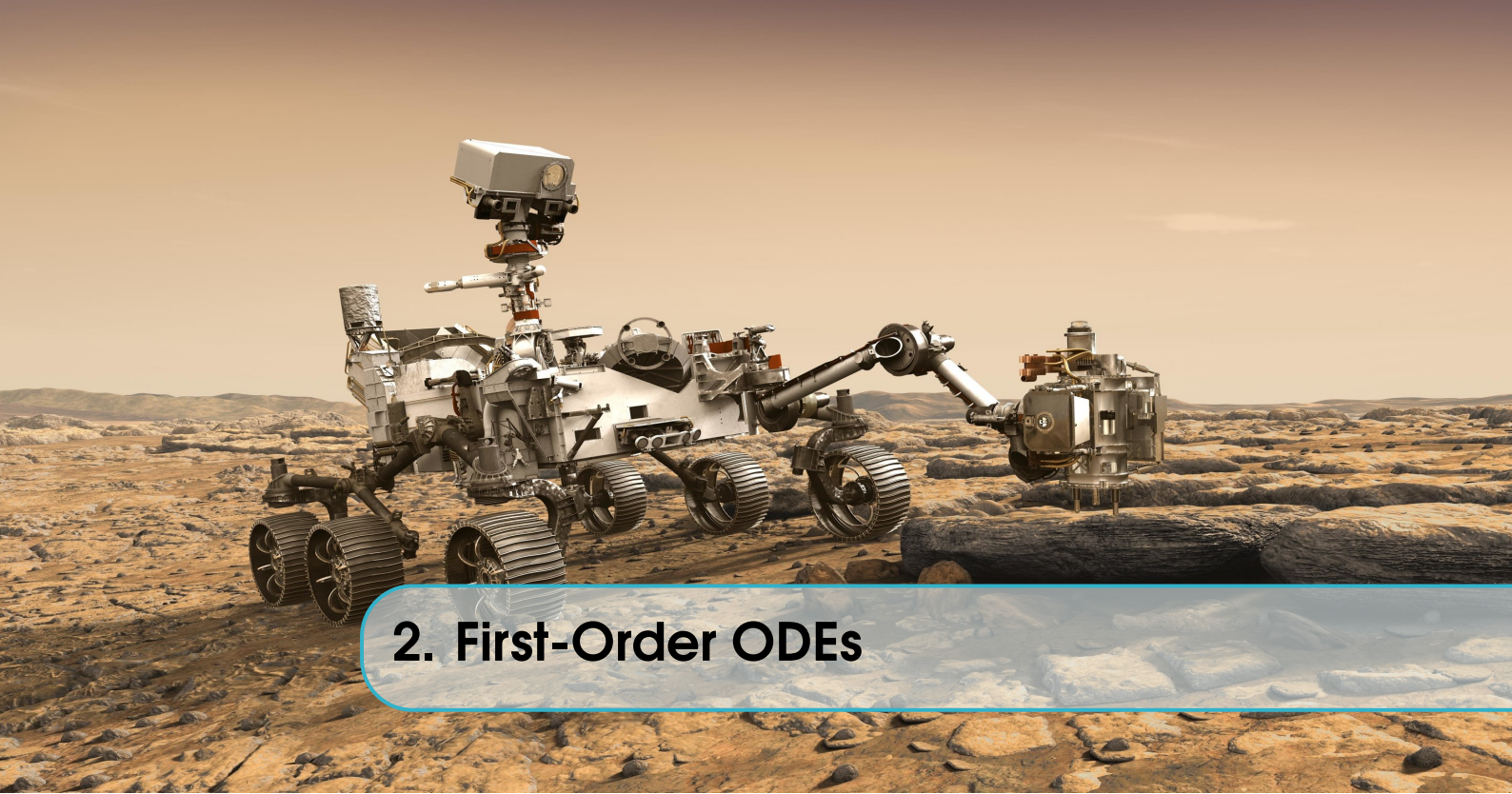
2. First-Order ODEs

2.1 First-Order ODE

Equation that contains only first derivative y' with or without y. A first-order ODE can be written in two forms: Implicit form (2.1) and Explicit form (2.2).

$$F(x, y, y') = 0 \tag{2.1}$$

Here, x is an independent variable, $y(x)$ is an unknown function to be determined and $y' = \frac{dy}{dx}$

$$y' = f(x, y) \tag{2.2}$$

Geometrically, $y'(x)$ is the slope of the curve $y(x)$. Therefore, the solution of (2.2) that passes through a point (x_0, y_0) must satisfy

$$y'(x_0) = f(x_0, y_0) \tag{2.3}$$

2.2 Initial Value Problem

Let us consider the following problem. In the morning, a cook prepared tea and poured the tea into a cup and served it to you at 7'O clock. The brewing temperature of the tea is $85°C$. If you had a phone call and forgot to drink the tea until 7.30AM, what will happen? Will the temperature of the tea be same? Assume that, the room temperature is $20°C$. Newton's law of cooling states that the rate of heat loss of an object is directly proportional to the difference between the temperature of the object and its surrounding temperature. Mathematically, it is written as

$$\frac{dT}{dt} = k(T - T_A) \tag{2.4}$$

where T and T_A represents the temperature of the tea and room respectively, k is the proportionality constant. The tea was served at 7'O clock, that is considered as the initial time for the tea. The tea was at $85°C$ at 7'O clock is treated as the initial condition. Mathematically, it is written as

$$T(0) = 85°C \tag{2.5}$$

Equations (2.4) and (2.5) constitute initial value problem. In general, an ODE with initial condition is called an initial value problem (IVP) and is given as follows, in case of explicit ODE.

$$y' = f(x,y), \quad y(x_0) = y_0 \tag{2.6}$$

2.2.1 What is a solution?

A value or values or a function which makes the equation valid. A function $h : (a,b) \to \mathbb{R}$ is said to be solution of (2.2) if h is differentiable in (a,b) and satisfies (2.2) while replacing y and y' by respectively h and h'. The graph or the curve of h is called a solution curve.

Let us consider the simple problem $y' = f(x)$. Then, it becomes a problem in integral calculus. We can obtain the solution by integration, that is

$$y = \int_{x_0}^{x} f(x)dx + c \tag{2.7}$$

where c is uniquely determined if $y(x_0) = y_0$ is specified.

2.3 Linear ODE

What is linear? Have you ever seen this term earlier? A transformation $T : \mathbb{V} \to \mathbb{W}$ is linear for any two scalars, $\alpha, \beta \in \mathbb{R}$, and any two vectors $u, v \in \mathbb{V}$ if T satisfies

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \tag{2.8}$$

Let us consider the following function

$$L(y, y') = F(x, y, y') \tag{2.9}$$

If L linear w.r.to y and y', then the ODE is said to be first-order linear ODE, otherwise, it is a first-order non-linear ODE. That is, for any $\alpha, \beta \in \mathbb{R}$ and y_1, y_2, y'_1, y'_2.

$$L(\alpha y_1 + \beta y_2, y') = \alpha L(y_1, y) + \beta L(y_2, y)$$

and

$$L(y, \alpha y'_1 + \beta y'_2) = \alpha L(y, y'_1) + \beta L(y, y'_2)$$

Note: We don't say it is linear in x, we insist that it is linear w.r.to y and y' only. The general form of the first-order linear ODE is

$$A_1(x)\frac{dy}{dx} + A_2(x)y = B(x) \tag{2.10}$$

provided, $A_1(x) \neq 0$. When the right hand side term $B(x) = 0$, then the ODE is said to be and ODE with null term or homogeneous ODE, on the other hand, if $B(x) \neq 0$, it is called non-homogeneous ODE. Since $A_1(x) \neq 0$, (2.10) can be re-written as

$$\frac{dy}{dx} + \frac{A_2(x)}{A_1(x)}y = \frac{B(x)}{A_1(x)}$$

or

$$\frac{dy}{dx} + p(x)y = r(x) \tag{2.11}$$

■ **Example 2.1** Let us consider the following problem,

$$x\frac{dy}{dx} - 3y = 0 \tag{2.12}$$

Solution: Assume that, $x = 0$ and we get

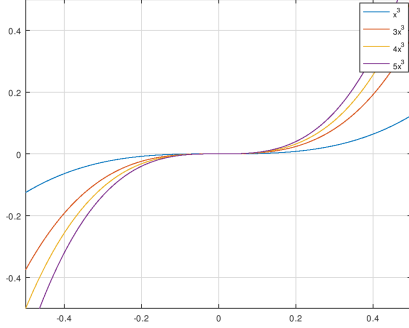$$\frac{dy}{dx} = 3\frac{y}{x} \implies \frac{dy}{y} = 3\frac{dx}{x}$$

Integrating on both sides, we get

$$\int \frac{dy}{y} = 3 \int \frac{dx}{x}$$
$$\implies log\, y = 3log\, x + c_1$$
$$\implies log\, y = 3log\, x + log\, c$$
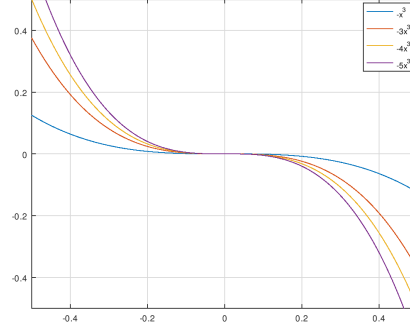$$\implies log\, y = log\, (cx^3)$$
$$\implies y = cx^3$$

Therefore, $y = cx^3$ is the solution to (2.12). For,

$$\frac{dy}{dx} = 3cx^2 \implies x\frac{dy}{dx} = 3cx^3 \implies \frac{dy}{dx} = 3y \implies (2.12)$$

Case 1: When we consider $y(0) = 0$, then $y = cx^3$ satisfies irrespective of the values of c. It

(a) $c > 0$ (b) $c < 0$

Figure 2.1: The graph of $y = cx^3$

means that, there are infinitely many solutions.
Case 2: When $y(0) = 1$, then there exists no c satisfying $y(0) = 1$. Therefore, there exists no solution.
case 3: When $x_0 \neq 0$ and $y(x_0) = y_0$, then $c = \frac{y_0}{x_0^3}$ and there exists a unique solution, $y = \frac{y_0}{x_0^3}x^3$.

2.3.1 Simple Case

Let us consider the simple case where $p(x) = 0$. Then (2.11) becomes

$$\frac{dy}{dx} = r(x) \tag{2.13}$$

This can be solved using the fundamental theorem of calculus.

Theorem 2.3.1 **Fundamental Theorem of Calculus:** If f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x) \tag{2.14}$$

If F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a) \tag{2.15}$$

Existence: Now, for a given $r(x)$ in the simple case (2.13), with the assumption that $r(x)$ is continuous on an interval $[x_0, x]$, by using fundamental theorem of calculus, one can find $R(x) = \int_{x_0}^x r(x)dx$ such that $R'(x) = r(x)$. That is, $y = R(x)$ solves the equation (2.13). **Note:** $R(x)$ is denotes as an antiderivative of $r(x)$

Question: Does there exist any other antiderivative that satisfies (2.13)?
Yes. $y(x) = R_c(x) = R(x) + c$, where c is any constant, also satisfies (2.13).

Remark: If S is any other antidertivative that solves (2.13), then S takes the form, $S(x) = R(x) + c$.

Uniqueness: If $y(x_0) = y_0$ is the initial condition for (2.13), then it is an IVP. $y(x_0) = R(x_0) + c \implies c = y(x_0) - R(x_0)$. From the definition of integral $R(x_0) = \int_{x_0}^{x_0} r(x)dx = 0$. Hence, $c = y_0$ and there exists a unique solution $y(x) = y_0 + R(x)$ to the given IVP.

2.4 Homogeneous ODE

Let us now consider (2.11) with the assumption that $r(x) = 0$. Then we get

$$\frac{dy}{dx} + p(x)y = 0 \tag{2.16}$$

In order to solve (2.16), we need to revise our knowledge on the separable ODEs.

2.4.1 Revisit: Separable ODE

A class of ODEs can be reduced to the seperable equation form after algebraic manipulations,

$$g(y)y' = f(x) \tag{2.17}$$

Here, we assume that f and g are continuous. Integrating on both sides w.r.to x, we obtain

$$\int g(y)y'dx = \int f(x)dx + c \implies \int g(y)dy = \int f(x)dx + c$$

Since f and g are continuous, both integrals exist. By evaluating the integrals we obtain the general solution of (2.17). This method of solving ODEs is called the method of separating variables.

■ **Example 2.2** Solve the following IVP

$$y' = (x+y-2)^2, y(0) = 2$$

Solution: Let $v = x + y - 2$, then

$$v' = 1 + y'$$
$$\implies y' = v' - 1$$
$$and\, y(0) = 2 \implies v(0) = 0$$

Replacing y' and $(x+y-2)$ by $1-v'$ and v in the given equation, we obtain

$$v'-1=v^2$$
$$\implies v'=1+v^2$$
$$\implies \frac{dv}{dx}=1+v^2$$
$$\implies \frac{dv}{1+v^2}=dx$$

Integrating on both sides

$$\int \frac{1}{1+v^2}dv = \int dx + c$$
$$\implies tan^{-1}(v)=x+c$$
$$\implies v=tan(x+c)$$
$$v(0)=0 \implies tan(c)=0$$
$$\implies c=n\pi$$
$$\implies x+y-2=tan(x+n\pi)$$
$$\implies y=tan(x+n\pi)-x+2$$

■ **Example 2.3** Solve the Newtons law of cooling problem for the tea cub which is explained at section 2.2.
Solution: Given,

$$\frac{dT}{dt}=k(T-T_A), T(0)=85°C, T_A=20°C$$

After rearrangement, we obtain

$$\frac{dT}{T-20}=kdt$$
$$\implies \int \frac{dT}{T-20}=k\int dt$$
$$\implies ln|T-20|=kt+c_1$$
$$\implies |T-20|=e^{kt+c_1}$$
$$\implies |T-20|=ce^{kt}, c=e^{c_1}$$
$$\implies T-20=ce^{kt} \text{(Why?!)}$$
$$\implies T=20+ce^{kt}$$
$$T(0)=85 \implies 85=20+ce^0$$
$$\implies c=65 \implies T=20+65e^{kt}$$

To find k, we need an additional condition, let us assume that $T(1)=40°C$. That is, the temperature of the tea in the tea cup at 8AM is $40°C$. Then,

$$T(1)=40=20+65e^k \implies e^k=\frac{20}{65}$$
$$\implies k=ln(4/13)\approx -1.178$$
$$\implies T=20+65e^{-1.178t}$$

When the time is 7.30AM, then $t=0.5$, and we obtain the temperature of the tea cup at time as $T(0.5)=20+65e^{-1.178/2}\approx 56°C$

2.4.2　Solution of Homogeneous ODE

We want to solve (2.16) in some interval. Now, (2.16) can be rewritten as follows

$$\frac{dy}{y} = -p(x)dx \implies \int \frac{dy}{y} = -\int p(x)dx$$

Upon integration, we obtain

$$ln|y| = -\int p(x)dx + c_1$$

Taking exponents on both sides, we obtain

$$y = ce^{-\int p(x)dx}, c = \pm e^{c_1} \text{ depending on } y > 0 \text{ or } y < 0$$

If y satisfies the initial condition $y(x_0) = y_0$, then

$$y_0 = ce^{-\int_{x_0}^{x_0} p(x)dx} \implies c = y_0 \implies y = y_0 e^{-\int p(x)dx}$$

2.4.3　Exact ODE

If a function $u(x, y)$ has continuous partial derivatives, its total differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{2.18}$$

If $u(x, y) = c$, then $du = 0$ A first-order ODE $M(x, y) + N(x, y)y' = 0$ written as

$$M(x, y)dx + N(x, y)dy = 0 \tag{2.19}$$

is called an exact differential equation if the differential $M(x, y)dx + N(x, y)dy$ is exact. Comparing (2.18) and (2.19), we get

$$\frac{\partial u}{\partial x} = M, \frac{\partial u}{\partial y} = N$$

Theorem 2.4.1　Suppose $M, N \in C^1(R), R = (a, b) \times (c, d)$, where R denotes the rectangle and C^1 denotes the set of all functions whose partial derivatives are continuous in R. Then there exists ϕ such that

$$\frac{\partial \phi}{\partial x} = M, \frac{\partial \phi}{\partial y} = N$$

if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof.　Suppose there exists ϕ such that

$$\frac{\partial \phi}{\partial x} = M, \frac{\partial \phi}{\partial y} = N$$

then

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial M}{\partial y}, \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

Conversely suppose

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Consider

$$M = \frac{\partial \phi(x,y)}{\partial x}$$

where y is treated as a parameter. Choose ϕ such that

$$\phi(x,y) = \int M(x,y)dx + h(y)$$

Note that as ϕ is a function of two variables and we are integrating over only x, the constant is replaced by an unknown function of y. Because, if take partial derivative w.r.to x, $h(y)$ will vanish. Now, differentiate ϕ w.r.to y.

$$\frac{\partial \phi}{\partial y} = \int \frac{\partial M}{\partial x}dx + h'(y) \implies h'(y) = \frac{\partial \phi}{\partial y} - \int \frac{\partial M}{\partial x}dx$$

We need

$$N = \frac{\partial \phi}{\partial y} \implies h'(y) = N - \int \frac{\partial M}{\partial x}dx$$

$$\implies h(y) = \int \left[N - \int \frac{\partial M}{\partial x}dx \right] dy$$

$$\implies \phi(x,y) = \int M(x,y)dx + \int Ndy - \int\int \frac{\partial M}{\partial x}dxdy$$

Hence ϕ satisfies the requirement and hence the proof.   ∎

> **Definition 2.4.1** Integrating Factor: Suppose $M(x,y)dx + N(x,y)dy$ is not exact and if there exists a function $\mu(x,y)$ such that
>
> $$\mu M dx + \mu N dy = 0$$
>
> is exact, then $\mu(x,y)$ is called an integrating factor of $M(x,y)dx + N(x,y)dy$.

By theorem 2.4.1,

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\implies \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

As a special case, suppose μ is a function of one variable say x, then $\mu_y = 0$ and $\mu_x = \mu'$. Therefore,

$$\mu M_y = \mu_x N + \mu N_x$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

Theorem 2.4.2 If $Mdx + Ndy = 0$ is not exact and if

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

depends only on x, then $Mdx + Ndy = 0$ has an integrating factor $\mu = \mu(x)$ which is given by

$$\mu(x) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}$$

2.4.4 Solution to Nonhomogeneous Linear ODE

If the right hand side of (2.11) is non-zero, then it is called nonhomogeneous linear ODE. Suppose there exists a function $\mu(x)$ such that

$$\mu y' + \mu p(x) y = d(\mu y)$$

That is, exactness is guaranteed, then multiplying μ on both sides of (2.11), we obtain

$$\mu y' + \mu p(x) y = \mu r(x) \tag{2.20}$$

Also,

$$\mu y' + \mu' y = d(\mu y) \tag{2.21}$$

Comparing left sides of (2.20) and (2.21), we obtain that

$$\mu' = \mu p(x) \implies \frac{d\mu}{\mu} = p(x)dx \implies \mu = e^{\int p(x)dx}$$

Once again, comparing the right sides of (2.20) and (2.21), we obtain that

$$d(\mu y) = \mu r(x)$$
$$\mu y = \int \mu r(x)dx + c$$
$$y = \frac{1}{\mu}\left(\int \mu r(x)dx + c\right)$$
$$y = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} r(x)dx + c\right)$$

The total output is the response to the input r and the response to the initial data. Therefore, the general solution to the problem (2.11)is given by

$$y = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} r(x)dx + c\right) \tag{2.22}$$

If a particular value is given for c, then the solution is called a particular solution. If initial condition $y(x_0) = y_0$ is prescribed we obtain once again that $c = y_0$ and hence

$$y = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} r(x)dx + y_0\right) \tag{2.23}$$

2.4.5 Bernoulli Equation

A few nonlinear ODEs can also be transformed to linear ODEs. For instance, the following Bernoulli equation (2.24) can be transformed to linear ODE format

$$y' + p(x)y = r(x)y^a, a \in \mathbb{R} \tag{2.24}$$

For $a = 0$ or $a = 1$, this will reduce to (2.11). Otherwise, it is nonlinear. To solve this type of problem, set

$$u(x) = y(x)^{1-a}$$
$$u' = (1-a)^{-a}y' = (1-a)(ry^a - py)$$
$$u' = (1-a)(r - py^{1-a}) = (1-a)(r - pu)$$
$$u' + (1-a)pu = (1-a)r$$

The last equation reduced to linear form and it can be solved using (2.22).

2.4.6 Autonomous ODE

An ODE $y' = f(x,y)$ in which x does not occur explicitly is of the form

$$y' = f(y) \tag{2.25}$$

and is called an autonomous ODE. If equation (2.25) has constant solutions, they are called equilibrium solution or equilibrium points. Equilibrium solutions are determined by the zeros of $f(y)$, as $f(y) = 0 \implies y' = 0 \implies y = c$. These zeros are called as critical points of (2.25). An equilibrium solution is called stable if solutions close to it for some x remain close to it for all further x. It is called unstable if solutions initially close to it do not remain close to it as t increases.

2.5 Existence and Uniqueness Theorem on IVP

> **Theorem 2.5.1** **Existence Theorem**: Consider the following initial value problem
>
> $$y' = f(x,y), y(x_0) = y_0$$
>
> Suppose, $f(x,y)$ is continuous for all points (x,y) in some rectangle
>
> $$R : |x - x_0| < a, |y - y_0| < b$$
>
> and bounded in R, that is there exists a number K such that
>
> $$|f(x,y)| \leq K, \text{ for all } (x,y) \in R$$
>
> Then the initial value problem has at least one solution $y(x)$. This solution exists at least for all x in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $a = min\{a, b/K\}$.

■ **Example 2.4** $|y'| + |y| = 0, y(0) = 1$ has no solution
$xy' = y - 1, y(0) = 1$ has infinitely many solutions

Definition 2.5.1 Lipschitz condition: A function $f : [a,b] \to \mathbb{R}$ is said to satisfy the Lipschitz condition, if there exists a constant M such that

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2|, \text{ for all } x_1, x_2 \in [a,b].$$

Theorem 2.5.2 Uniqueness Theorem: Consider the following initial value problem

$$y' = f(x,y), y(x_0) = y_0$$

Suppose, $f(x,y)$ is continuous for all points (x,y) in some rectangle

$$R : |x - x_0| < a, |y - y_0| < b$$

and bounded in R, that is there exists a number K such that

$$|f(x,y)| \leq K, \text{ for all } (x,y) \in R$$

Further, of partial derivative of f w.r.to y (that is $f_y = \frac{\partial f}{\partial y}$), is continuous for all $(x,y) \in R$ and bounded, say

$$|f_{(x,y)}| \leq M, \text{ for all } (x,y) \in R \tag{2.26}$$

for some M. Then the initial value problem has at most one solution $y(x)$. This solution exists at least for all x in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $a = min\{a, b/K\}$.

Remark: The condition (2.26) may be replaced by the weak condition or Lipschitz condition,

$$|f(x,y_1) - f(x,y_2)| \leq M|y_1 - y_2|, \text{ for all } (x,y_1), (x,y_2) \in R.$$

■ **Example 2.5** Continuity of $f(x,y)$ is not enough to guarantee the uniqueness. For example, consider the initial value problem

$$y' = \sqrt{|y|}, y(0) = 0$$

$f(x,y) = \sqrt{|y|}$ is continuous for all y. However,

$$y_1 = 0 \text{ and } y_2 = \frac{x^2}{4}, x \geq 0, y_2 = -\frac{x^2}{4}, x < 0$$

are two solutions of the IVP.

3. Second-Order ODEs

3.1 Second-Order ODE

Similar to the first-order ODE, an equation that at most second derivatives is called second-order ODE. A second-order ODE can be written in the following form

$$F(x, y, y', y'') = 0 \tag{3.1}$$

These equations are important in engineering applications, especially in mechanical and electrical vibrations. Similar to the first-order ODE initial value problem, we have second-order ODE initial value problem which is given as follows:

$$F(x, y, y', y'') = 0, y(x_0) = y_0, y'(x_0) = y_1 \tag{3.2}$$

The initial value problem in first-order requires only one initial condition in order to eliminate the integration constant from its respective solution, whereas the solution of the second-order ODE contains two constants and hence we need at least two conditions to eliminate the constant. These constants can be initial conditions as given above. That is, initial values at x_0, the initial point on y and y' are provided. Similarly, if the second-order ODE is defined on an interval $[a, b]$, then two conditions one on each of the boundary point can be prescribed which is called as boundary value problems.

$$F(x, y, y', y'') = 0, y(a) = c, y(b) = d \tag{3.3}$$

The condition $y(a) = c, y(b) = d$ are called boundary conditions. Apart from this, a general boundary condition can also be prescribed.

$$F(x, y, y', y'') = 0$$
$$\alpha_1 y(a) + \beta_1 y'(a) = c$$
$$\alpha_2 y(b) + \beta_2 y'(b) = d$$

3.2 Second-order linear ODE

A second-order ODE is called linear if it can be written in the following form

$$y'' + p(x)y' + q(x)y = r(x) \tag{3.4}$$

and nonlinear if it cannot be written in this form. If the coefficient of y'' is not unity, then one can divide the equation by the coefficient and bring that to the standard form. If $r(x) \equiv 0$, then (3.4) is called homogeneous. If $r(x) \neq$ for some x, then (3.4) is called nonhomogeneous. It is also represented by $L(y, y', y'')$ where L is linear in both y, y' and y''. Compactly, the homogeneous and nonhomogeneous equation can be written as follows

$$L(y) = 0$$
$$L(y) = r(x)$$

■ **Example 3.1**

$$y'' + 25y' + 36y = x^2$$
$$y''y + y'^2 = 0$$

Similar to the solution of the first-order ODE, a function $y = h(x)$ is called a solution of a linear or nonlinear second-order ODE on some open interval (a, b) if h is defined, twice differentiable and replacing the h, h', h'' in the respective second-order ODE produces an identity.

Differential Operators: Instead of writing the equation as $y'' + p(x)y' + q(x)y = r(x)$, we can write it as $(D^2 + p(x)D + q(x)I)y = r(x)$. Here, D is called differential operator.

3.2.1 Superposition Principle or linearity principle

> **Theorem 3.2.1** Let y_1, y_2 be solutions of the homogeneous second-order ODE (3.4) on an interval $[a, b]$. Then the linear combination $y = c_1 y_1(x) + c_2 y_2(x)$ is also a solution of (3.4) on $[a, b]$. Here, c_1, c_2 are constants.

Proof.
$$L(y) = L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2) = c_1.0 + c_2.0 = 0$$

■

Remark: A constant multiple of $y = c_1 y_1(x)$ of a solution $y_1(x)$ of (3.4) is also a solution. Trivial solution is always a solution to (3.4).

> **Definition 3.2.1 — Linearly Dependent, Independent.** Two functions $y_1(x), y_2(x)$ is said to be linearly dependent on an interval $[a, b]$ if there exist constants c_1, c_2 not all zero such that
> $$c_1 y_1(x) + c_2 y_2(x) = 0$$
> for every x in the interval. If y_1, y_2 are not linearly dependent on the interval, it said to be linearly independent.

A solution $c_1 y_1 + c_2 y_2$ which contains arbitrary constants c_1, c_2 is called a general solution. If we choose a specific values for the constants, it is called particular solution.

> **Definition 3.2.2 — Basis.** If two independent solutions $y_1(x), y_2(x)$ of (3.4) are not proportional to each other, that is, if y_1, y_2 are independent, then y_1, y_2 are called basis or fundamental solution of (3.4).

> **Theorem 3.2.2** Let y_1, y_2 be any two solutions of (3.4), then $\alpha y_1 + \beta y_2$ is also a solution of (3.4) for any $\alpha, \beta \in \mathbb{R}$. Suppose y_1, y_2 are independent solution of (3.4), then any solution y can be written in the form $\alpha y_1 + \beta y_2$ for some $\alpha, \beta \in \mathbb{R}$. If S denotes the set of all solutions of (3.4), then S is a linear space and $dim\, S \leq 2$.

Proof. Let y be a solution and let y_1, y_2 be linearly independent solutions of (3.4). By superposition principle, $\alpha y_1 + \beta y_2$ is also a solution. Since, y, y_1, y_2 are solutions of (3.4), they are differentiable in $[a, b]$. Therefore, their derivative exists as some point $t \in [a, b]$. (We can take even the initial value conditions x_0). Suppose that $y(t) = y_0, y'(t) = y_1$. Then

$$\alpha y_1(t) + \beta y_2(t) = y_0$$
$$\alpha y'_1(t) + \beta y'_2(t) = y_1$$

In matrix form

$$\begin{bmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

The values of α and β can be uniquely determined if the determinant is non-zero. Define

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = y_1 y'_2 - y_2 y'_1$$

This $W(x)$ for any x is called as the Wronskian. If we prove that $W(t) \neq 0$, then α, β are uniquely determined, then it proves that y can be written in the form $y = \alpha y_1 + \beta y_2$ for some $\alpha, \beta \in \mathbb{R}$. It is enough to prove that $W(t) \neq 0$ if y_1, y_2 are independent.

   <u>Claim</u>: Either $W \equiv 0$ or W is never zero.

$$W(x) = y_1 y'_2 - y_2 y'_1$$
$$W'(x) = y'_1 y'_2 + y_1 y''_2 - y'_2 y'_1 - y_2 y''_1$$
$$= y_1 y''_2 - y_2 y''_1$$
$$= y_1(-py'_2 - qy_2) - y_2(-py'_1 - qy_1)$$
$$= -py'_2 y_1 - qy_2 y_1 + py_2 y'_1 + qy_2 y_1$$
$$= -p(y_1 y'_2 - y_2 y'_1)$$
$$= -pW$$
$$\implies W' = -pW$$
$$\implies W = Ce^{-\int p(x)dx}$$

From this it is clear that if $C = 0$, $W \equiv 0$, and if $C \neq 0, W \neq 0$ as exponential will never be zero. <u>Claim</u>: $W \equiv 0$ if and only if y_1, y_2 are dependent Suppose y_1, y_2 are dependent, then either $y_1 = ky_2$ or $y_2 = ly_1$ for some k, l. Therefore, $y'_1 = ky'_2$

$$W(x) = y_1 y'_2 - y_2 y'_1 = k(y_2 y'_2 - y_2 y'_2) = 0 \implies W \equiv 0$$

Conversely assume that $W \equiv 0$. Therefore, $W(x) = 0$ for all x. If $y_1 \equiv 0$ or $y_2 \equiv 0$, nothing to prove. If $y_1 \neq 0, y_2 \neq 0$. Then there exists a point x_i such that $y(x_i) \neq 0$ for some x_i. Therefore, there exists an interval $[c, d] \subset [a, b]$ such that for all $x_i \in [c, d], y_1(x_i) \neq 0$.

$$0 = \frac{W(x)}{y_1^2} = \frac{y_1 y'_2 - y_2 y'_1}{y_1^2} = d\left(\frac{y_2}{y_2}\right) \implies \frac{y_2}{y_1} = k \implies y_2 = ky_1$$

This is true for all $x \in [c, d]$. By uniqueness property it is true everywhere in $[a, b]$. Hence, if y_1, y_2 are independent, then $W(x) \neq 0$. ∎

Theorem 3.2.3 dim S = 2

Proof. Let y_1, y_2 be two solutions of IVP such that

$$L(y_1, y_1', y_1'') = 0, y_1(t_0) = 1, y_1'(t_0) = 0$$
$$L(y_2, y_2', y_2'') = 0, y_2(t_0) = 0, y_2'(t_0) = 1$$

Then $W(x) \neq 0$ since $W(t_0) \neq 0$. Therefore, y_1, y_2 are independent. Hence *dim S = 2*. ∎

Definition 3.2.3 — Wronskian. Suppose y_1, y_2 are twice differentiable, then

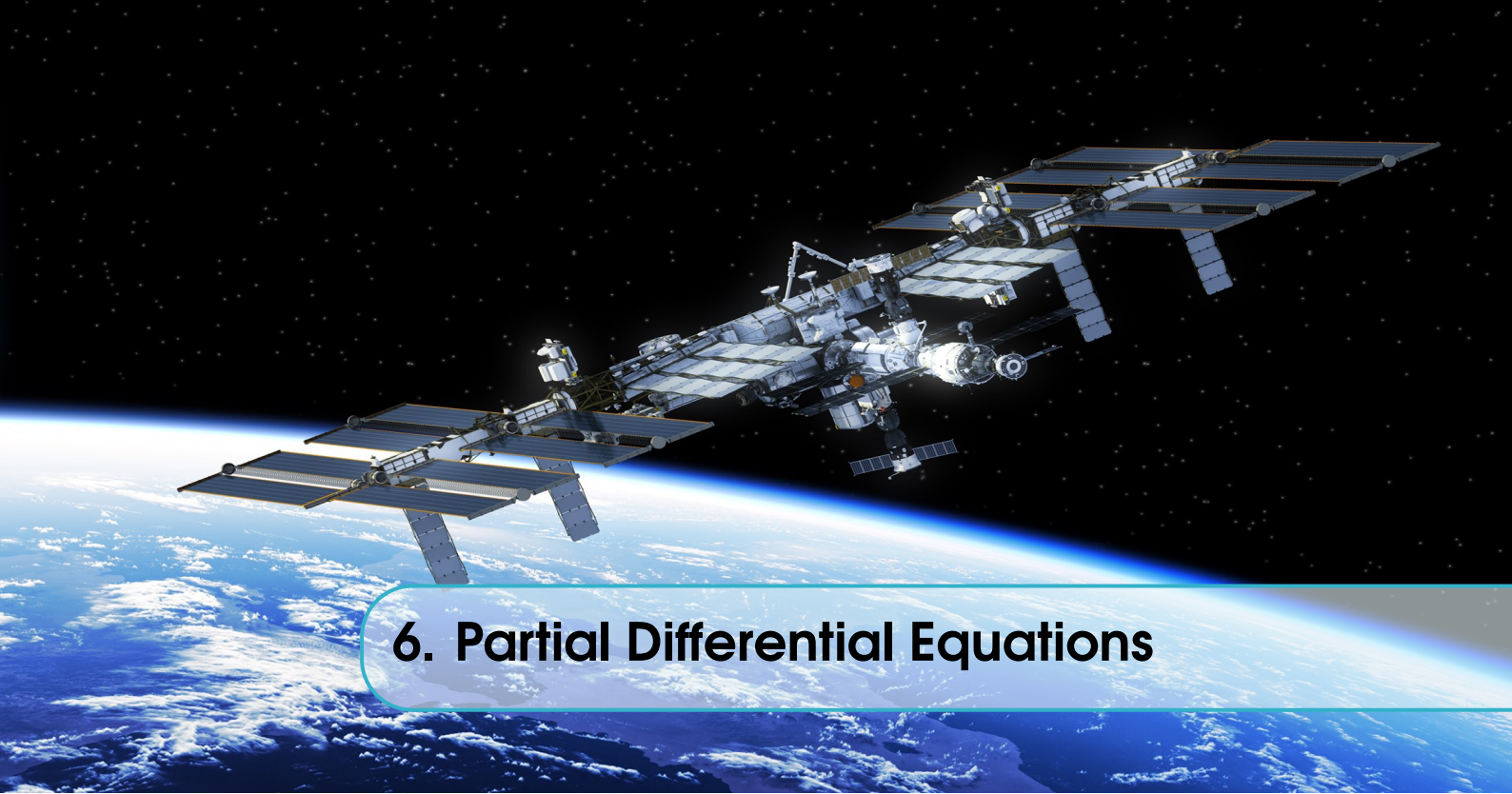$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

is called the Wronskian of y_1, y_2.

4. Power-Series, Legendre, Bessel

5. Fourier Series and Integral

6. Partial Differential Equations

Appendices

■ **Solution A.1** Solution to 1.1

$$\frac{dT}{dt} = k(T - T_A)$$

$$T(0.5) = 87, T(1) = 43$$

$$\text{or } T(0) = 87, T(0.5) = 43$$

By separation of variables and integrating, we obtain

$$\frac{dT}{T - T_A} = kdt$$

$$\int \frac{dT}{T - T_A} = k \int dt$$

$$ln|T - T_A| = kt + c_1$$

$$T(t) = T_A + ce^{kt}, (c = e^{c_1})$$

We have two unknowns k and c (Of course T_A is also an unknown, but this can be estimated). The room temperature can be between $15°C$ and $25°C$, let us take the average of them, that is $T_A = 20°C$

Case 1:

$$T(1) = 43 \implies 43 = 20 + ce^k \implies ce^k = 23 \implies c = e^{-k}(23)$$

$$T(0.5) = 87 \implies 87 = 20 + ce^{0.5k} \implies 67 = e^{-k}(23)e^{0.5k}$$

$$\implies \frac{67}{23} = e^{-0.5k} \implies e^{k/2} \approx 0.3433 \implies \frac{k}{2} \approx -1.0692 \implies k \approx -2.138$$

$$c = 195.17 \implies T(t) = 20 + 195.17e^{-2.138t}$$

When $t = 0$, we have

$$T(0) = T_A + ce^0 = T_A + c = 215.17$$

This is impossible as the water boils at $100°C$. Therefore Arjun is not a criminal and he might have been in his room just 10 minutes before, as $T(0.4) = 102°C$.
Case 2: If we assume that $T(0) = 87, T(0.5) = 43$, then

$$T(0) = 87 \implies 67 = c \implies T(0.5) = 43 \implies 23 = 67e^{0.5k} \implies \frac{67}{23} = e^{-0.5k}$$

$$\implies e^{k/2} \approx 0.3433 \implies \frac{k}{2} \approx -1.0692 \implies k \approx -2.138$$

$$\implies T(t) = 20 + 67 * e^{-2.138t} \implies T(-0.5) = 20 + 67 * e^{2.135/2} \approx 215.13$$

For both cases, we have obtained that an impossible temperature for the water in the kettle. If we assume that, the hotel is in Russia or the place where the temperature is around $-40°C$, then also, the temperature will be around $150°C$. But, a kettle switches off after the water reaches the boiling temperature. You can vary T_A and find the values. The ranges of T at 4.30 PM for the room temperature $[-40, 40]$ is $[154, 776]$, which is not our desired range. Also, you can vary t and find Arjun's arrival time in his room.

Note that, here we used the time in hours. If we change the time in seconds, values of k and c will change. However, the conclusion will not change. Therefore, one should know the dimension analysis to solve the problem.

■ **Solution A.2** Solution to 1.3

$$\frac{dh}{dt} = -\frac{A}{\pi}2.656\sqrt{h}$$
$$h(0) = 2.25, h(21600) = 0$$

Let us assume that hole is in circular form and its radius is r, then its area is $A = \pi * r^2$. Therefore $\frac{A}{\pi} = r^2$

$$\frac{dh}{\sqrt{h}} = -2.656r^2 dt \implies 2\sqrt{h} = -2.656r^2 t + c_1 \implies h = (-1.328r^2 t + c)^2$$

$$h(0) = 2.25 \implies c^2 = 2.25 \implies c = 1.5 \implies h(t) = (1.5 - 1.328r^2 t)^2$$

$$h(21600) = 0 \implies 1.5 - 1.328r^2 * 21600 = 0 \implies 28684.8r^2 = 1.5$$

$$\implies r^2 \approx 0.00723m \implies r \approx 0.723cm == 7.23mm$$

Diameter of the hole is approximately $14.46mm$

■ **Solution A.3** Solution to 1.5
$$\frac{dy}{dt} = A - ky$$

$$u = A - ky \implies \frac{du}{dt} = -k\frac{dy}{dt} \implies \frac{dy}{dt} = -\frac{1}{k}\frac{du}{dt} \implies \frac{du}{dt} = -ku$$

$$\frac{du}{u} = -kdt \implies ln|u| = -kt + c_1 \implies u = ce^{-kt}$$

$$A - ky = ce^{-kt} \implies ky = A - ce^{-kt} \implies y = \frac{1}{k}(A - ce^{-kt})$$

■ **Solution A.4** Solution to 1.6 It is a non-homogeneous ODE, therefore both general and particular solution should be found.

$$y' + Ky = A + B\sin\omega t$$

Comparing it with

$$y' + p(t)y = r(t) \implies p(t) = K, r(t) = A + B\sin\omega t$$

$$y' + p(t)y = r(t) \implies y = e^{\int -p(t)dt}\left(\int e^{\int p(t)dt} r(t)dt + c\right)$$

$$\int p\,dt = \int K\,dt = Kt \implies y = e^{-Kt}\left(\int e^{Kt}(A + B\sin\omega t)dt + c\right)$$

$$\implies y = e^{-Kt}\left(\int e^{Kt}A\,dt + \int e^{Kt}B\sin\omega t\,dt + c\right)$$

$$\implies y = e^{-Kt}\left(\frac{A}{K}e^{Kt} + \frac{B}{K^2 + \omega^2}e^{Kt}(K\sin\omega t - \omega\cos\omega t + c)\right)$$

$$\implies y = e^{-Kt}\left(\frac{A}{K}e^{Kt} + \frac{B}{K^2 + (\frac{\pi}{12})^2}e^{Kt}\left(K\sin\frac{\pi}{12}t - \frac{\pi}{12}\cos\frac{\pi}{12}t\right) + c\right)$$

$$\implies y = \frac{A}{K} + \frac{B}{K^2 + (\frac{\pi}{12})^2}\left(K\sin\frac{\pi}{12}t - \frac{\pi}{12}\cos\frac{\pi}{12}t\right) + ce^{-Kt}$$

When $t \to \infty$, the last term tends to zero (since K is positive) irrespective of the value of c. Except the last term, remaining terms are called the steady-state solution as they contain constant and periodic terms. y is a transient solution as it models from rest to the steady state.

■ **Solution A.5** Solution to 1.11

$$I' + \frac{R}{L}I = \frac{E}{L} \implies p = \frac{R}{L}, r = \frac{E}{L}$$

$$y' + p(t)y = r(t) \implies y = e^{\int -p(t)dt}\left(\int e^{\int p(t)dt} r(t)dt + c\right)$$

$$\implies I = e^{\int -\frac{R}{L}dt}\left(\int e^{\int \frac{R}{L}dt}\frac{E}{L}dt + c\right)$$

$$\implies I = e^{-\frac{R}{L}t}\left(\frac{E}{L}\frac{e^{\frac{R}{L}t}}{\frac{R}{L}} + c\right)$$

$$\implies I = \frac{E}{R} + ce^{-\frac{R}{L}t}$$

$$I(0) = 0 \implies c = -\frac{E}{R}$$

$$\implies I = \frac{E}{R}(1 - e^{-\frac{R}{L}t})$$

For the given problem, it is given that $E = 48, R = 11, \frac{R}{L} = 110$. Therefore,

$$I = \frac{48}{11}(1 - e^{-110t})$$

If $E = B\sin\omega t$, then

$$y = e^{-\frac{R}{L}t}\left(\int e^{\frac{R}{L}t}\frac{B}{L}\sin\omega t\,dt + c\right)$$

$$\implies I = e^{-\frac{R}{L}t}\left(\frac{\frac{B}{L}}{(\frac{R}{L})^2 + \omega^2}e^{\frac{R}{L}t}(B\sin\omega - \omega\cos\omega t) + c\right)$$

$$\implies I = \frac{BL}{R^2 + (\omega L)^2}(B\sin\omega t - \omega\cos\omega t) + ce^{-\frac{R}{L}t}$$

$$I(0) = 0 \implies I = \frac{BL}{R^2 + (\omega L)^2}(B\sin\omega t - \omega\cos\omega t + \omega e^{-\frac{R}{L}t})$$

■ **Solution A.6** Solution to 1.10

$$y' = k(1-y)y \implies y' = ky - ky^2$$

This is Bernoulli equation type with $a = 2$

$$u = y^{1-a} = y^{-1} \implies u' = -y^{-2}y' = -y^{-2}k(y-y^2) = -ky^{-1} + k = k - ku$$

$$u' + ku = k \implies u = ce^{-kt} + 1$$

$$y = \frac{1}{u} \implies y = \frac{1}{ce^{-kt} + 1}$$

Bibliography

[1] R. D. Driver, "Torricelli's law: An ideal example of an elementary ode", *The American Mathematical Monthly*, volume 105, number 5, pages 453–455, 1998 (cited on page 8).

[2] E. Kreyszig, H. Kreyszig, and E. J. Norminton, *Advanced Engineering Mathematics*, Tenth. Hoboken, NJ: Wiley, 2011 (cited on page 10).