Lecture Notes on Mathematics for Engineers

MA5101-Differential Equations

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This is the course material for the MA5101 course - Mathematics for engineers. It contains the differential equations part alone.

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1. Introduction

1.1 Mathematical Modeling

Let us revisit our school physics first. Do you know Newton's second law of motion? What does it say? Is it not the relationship between the force action on a body of mass and the acceleration caused by this force? If you remember your school physics, you can also remember that force is directly proportional to the acceleration. That is,

$$F = ma \tag{1.1}$$

Now, what is the meaning of acceleration? It is nothing but the name given to the velocity changing process. Isn't it? Mathematically, it is the rate of change of the velocity. What does it mean? Let us consider that we start from home or hostel at 8'O clock in the morning and reach the college which is located 10km away from your home or hostel. Have you ever noticed that, the driver of the vehicle changes the speed of the vehicle depending on the region which he travels, for example, at the school zone, driver slows the vehicle whereas on the highway he speeds up the vehicle. That means, the driver rides the vehicle at different speed at different location. How much changes was made between two different times? Let t_1 and t_2 be two different time where we would like to observe the velocity changes done by the driver and let us assume that v_1 and v_2 are the respective velocities at time t_1 and t_2. Then,

$$\Delta t = t_2 - t_1 \tag{1.2}$$

and

$$\Delta v = v_2 - v_1 \tag{1.3}$$

are the change in time and velocity respectively. Therefore, the average acceleration over a period of time is the ratio of change in velocity to the change in time. Mathematically,

$$a = \frac{\Delta v}{\Delta t} \tag{1.4}$$

If we need to find the instantaneous acceleration, then we should observe the change in velocity over an infinitesimal interval of time. In Calculus, this is achieved by taking the limit as $\Delta t \to 0$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \tag{1.5}$$

Newton's second law (1.1) can be rewritten with the help of (1.5) as follows:

$$F = m\frac{dv}{dt} \implies \frac{dv}{dt} = \frac{F}{m} \tag{1.6}$$

So, we ended up formulating a simple physical real world problems into a mathematical equation. In general, engineering or real world problem which are formulated using mathematical expression in terms of variables, functions and equations are known as mathematical modeling.

When you have attended the course Mechanical vibrations, you might have encountered the undamped free vibrations

$$m\ddot{x} + kx = 0 \tag{1.7}$$

or damped free vibrations

$$m\ddot{x} + \gamma\dot{x} + kx = 0 \tag{1.8}$$

or in general the mass-spring system as

$$m\ddot{x} + \gamma\dot{x} + kx = F(t) \tag{1.9}$$

This is another mathematical modeling in the mechanical vibrations for a mass-spring system. In fact, in a multi-degree freedom of system, this will become a system of ODEs which we will discuss in the future.

1.2 Differential Equation

What is an equation? A statement that the values of two mathematical expression are equal. $=$ sign indicates equal sign. When the equation contains derivatives of an unknown function is called differential equation. Since the function is unknown in the differential equation, our next point of interest will be searching whether any known function will satisfy the differential equation. The process of finding such a function is called method of solution finding. In this course, we will learn different ways to find a solution, its properties such as existence and uniqueness, geometrical interpretation and so on. Through out this lecture notes, we will call the unknown function as y. Since y is a function, as you aware, we need at least an independent variable and dependent variable to represent a function. In Newton's second law model, which we have discussed earlier, we have v as an unknown function which is a dependent variable depending on the independent variable t. An **ordinary differential equation (ODE)** is an equation that has one or several derivatives of an unknown function with or without the unknown function itself, which we usually call $y(x)$ if the independent variable is x. In general ODEs can be written as

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{1.10}$$

Why do we name it as an ordinary? Does there exist any special then? When the unknown function depends on more than one variable, then we used to obtain the derivatives with respect to one of the variable by treating the remaining variable as constant. These type of derivatives are called partial derivatives. Equation which contains partial derivatives with or without the unknown function itself is called **partial differential equation**. For instance,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1.11}$$

Here, u is an unknown function of two independent variables x and y.

An ODE (1.10) is said to be of order n if the highest derivative involved in the ODE is $y^{(n)}$. When $n = 1$, it is the first-order ODE. When $n = 2$, it is the second-order ODE. In this course, we will deal only first and second-order ODEs. In the next chapter, we will deal with first-order ODEs.

1.3 A few more examples on Modeling

■ **Example 1.1** There was a murder in a hotel at room number 315 at 4:30PM. Police arrested Arjun, who was in the next room at 5:00PM. But, Arjun claims that he was not in his room for at least half an hour. The police check the water temperature of his tea kettle in his room at the instant of arrest and again 30 minutes later, obtaining the values $87°C$ and $43°C$, respectively. Can you investigate the case as an inspector? Is it possible to claim that Arjun is the murderer?

Mathematical Model: Let $T(t)$ be the temperature of the the tea kettle at any given time t. Assume that the room temperature is T_A. From Physics experiment, it was proved that the time rate of change of the temperature T of a body conducts heat well is proportional to the difference between T and temperature of the surrounding medium, which is called as Newton's law of cooling. Therefore, we obtain

$$\frac{dT}{dt} \propto (T - T_A) \tag{1.12}$$

$$\frac{dT}{dt} = k(T - T_A) \tag{1.13}$$

where k is a proportionality constant. Let us assume that the murder time as 0 hrs, Arjun arrested or temperature measured time as 0.5 hrs and temperature measure 30 minutes later as 1 hrs. Then,

$$\frac{dT}{dt} = k(T - T_A)$$
$$T(0.5) = 87, T(1) = 43$$

We need to find $T(0)$. Other way to formulate is assume the temperature measured time as 0, 30 minutes later as 0.5 and murder time as -0.5, then,

$$\frac{dT}{dt} = k(T - T_A)$$
$$T(0) = 87, T(0.5) = 43$$

We need to find $T(-0.5)$. For solution see A.

■ **Example 1.2** If the temperature of a cake is $150°C$ when it leaves the oven and is $100°C$ ten minutes later, when will it reach the room temperature $20°C$?

Mathematical Model for this problem is similar to 1.1. Then,

$$\frac{dT}{dt} = k(T - T_A)$$
$$T(0) = 150, T(0.1) = 100$$

Hint: It will take a long time, mathematically, $t \to \infty$

■ **Example 1.3** In a hostel, there is a cylindrical water tank of diameter 2m and height 2.25m. On a fine day, when Ragu was the first person to take the shower at 7AM in the hostel, the tank was empty. After an inspection by the hostel warden, it was found that there is a circular hole in the water tank at the bottom. When the hostel watchman switched off the power button of the water tank at 1AM on the previous day, it was completely filled. Without manually measuring the diameter of the hole, could you calculate the diameter of the hole?

Mathematical Model: Under the influence of gravity the out-flowing water from the hole at the bottom has velocity

$$v(t) = 0.6\sqrt{2gh(t)}$$

Here 0.6 was introduced by J. C. Borda after experimental data,[1]. Also, $h(t)$ denotes the water level to the outflow. Let us denote the volume of the water outflow during the Δt time as ΔV. If A denote the area of the hole, Then $\Delta V = Av\Delta t$. If B denotes the cross-sectional area of the tank, then the change of the volume of the water in the tank can also be given by $\Delta V* = -B\Delta h$, $\Delta h > 0$ denotes the decrease of the height. The minus sign denotes the decreasing water level in the tank.

$$\begin{aligned} \Delta V = \Delta V* &\implies Av\Delta t = -B\Delta h \\ \implies \frac{\Delta h}{\Delta t} &= -\frac{A}{B}v = -\frac{A}{B}0.6\sqrt{2gh} \\ \Delta t \to 0 \implies \frac{dh}{dt} &= -\frac{A}{B}2.656\sqrt{h} \end{aligned}$$

The mathematical model is

$$\begin{aligned} \frac{dh}{dt} &= -\frac{A}{B}2.656\sqrt{h} \\ h(0) &= 2.25, h(21600) = 0 \\ B &= \pi * r^2 = \pi * 1 * 1 = \pi \end{aligned}$$

We have to find the value of A. Note that, the dimension should be properly used. Here we have used the fact that $g = 9.8m/s^2$. Therefore, all units should be in m and s related to length, height, area, and time. If not, there will be a mismatch in calculation. Therefore, 6 hour duration is converted to seconds. However, in previous examples, it won't make any change in final outcome, although constants of general solutions will be different.

■ **Example 1.4** The outflow of water from a cylindrical tank with a hole at the bottom. You are asked to find the height of the water in the tank at any time if the tank has diameter 2 m, the hole has diameter 1 cm, and the initial height of the water when the hole is opened is 2.25 m. When will the tank be empty?

Mathematical model for this problem is similar to 1.3. The IVP is given as follows

$$\begin{aligned} \frac{dh}{dt} &= -\frac{A}{B}2.656\sqrt{h}, h(0) = 2.25 \\ A &= \pi * 0.005 * 0.005, B = \pi * 1 * 1 \end{aligned}$$

We have to find t when $h = 0$.

■ **Example 1.5** One hour before a surgery, certain drug at a constant amount was injected to the patient's blood stream. Certain amount of drug is removed simultaneously to avoid over dosage of drugs which is proportional to the amount of the drug present at time t.

Mathematical Model: Let us assume that the constant amount injected to the patient's blood stream as A. It is given that the drug remove from the patient's body is proportional to the amount of drug present at time t. That is, $\frac{dy}{dt} \propto y$. Since the amount is removed, so the proportionality constant should be negative, that is $-k, k > 0$.

$$\begin{aligned} \frac{dy}{dt} &= Injected - Removed \\ \frac{dy}{dt} &= A - ky \end{aligned}$$

■ **Example 1.6** It was found that hormone level of a patient varies w.r.to time. The rate of change of the hormone w.r.to time is the difference between the sinusoidal input of a 24-hour period from thyroid gland and a continuous removal rate proportional to the level. Find and solve the hormone level model?

Mathematical Model: Let $y(t)$ denote the hormone level at time t. The removal rate is Ky. It is given that input rate is sinusoidal input rate. That is $A + Bsin\omega t$, where $\omega = \frac{2\pi}{24} = \frac{\pi}{12}$. $A \geq B$ so that input rate is non-negative, A is the average input rate. The constants, A, B, K should be obtained from measurements.

$$\frac{dy}{dt} = In - Out$$
$$\frac{dy}{dt} = A + Bsin\omega t - ky$$
$$y' + Ky = A + Bsin\omega t$$

■ **Example 1.7** In a room containing 20 cubic m^3 of air, 600 m^3 of fresh air flows in per minute, and the mixture (made practically uniform by circulating fans) is exhausted at a rate of 600 cubic metre per minute. What is the amount of fresh air at any time if there are no initial fresh air? After what time will 90% of the air be fresh?

Mathematical Model: The model is similar to the earlier two models. The rate of change of fresh air is the difference between the fresh air in and the fresh air out per minute. If $y(t)$ denotes the fresh air in the room at time t, then

$$y'(t) = 600 - \frac{600y}{20000}, y(0) = 0$$

Solving this equation and find time t when $y = 18000$.

■ **Example 1.8** A hybrid fuel tank in a rocket works on the principle of mixing two different fuel substances for combustion which in turn produces fuel supply for the throttle. The first tank contains 2 million litres of fuel in which 0.18 million kg of solid fuel substance is dissolved. Each 50 litre of the fuel fed into the throttle after mixing contains $(1 + cost)$ kg of the dissolved solid fuel substance. The mixture is uniform and runs to the throttle at the same rate. What is the amount of solid fuel substance at any time t?

Mathematical Model: The model is similar to the earlier model.

$$y'(t) = 50(1 + cost) - 0.000025y, y(0) = 0$$

■ **Example 1.9** Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank contains 1000 litres of water in which initially 100 kg of salt is dissolved. Brine runs in at a rate of 10 litre min, and each litre contains 5 kg of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 litre per min. Find the amount of salt in the tank at any time t.

Mathematical Model:

$$y'(t) = 50 - 0.01y, y(0) = 100$$

■ **Example 1.10** A model for the spread of contagious diseases is obtained by assuming that the rate of spread is proportional to the number of contacts between infected and non-infected persons, who are assumed to move freely among each other.

Mathematical Model: It is given that the rate of spread is proportional to both infected and non-infected persons. If we assume that y is the percentage of people who got infected at any given time, then $1 - y$ represent the non-infected persons. Therefore,

$$y' = k(1 - y)y$$

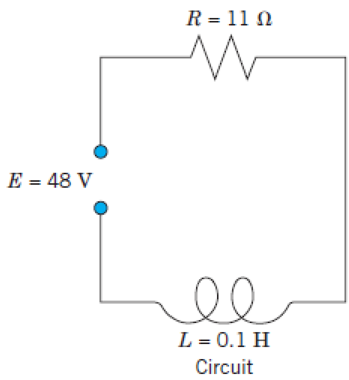


Figure 1.1: RL Circuit, Source: [2]

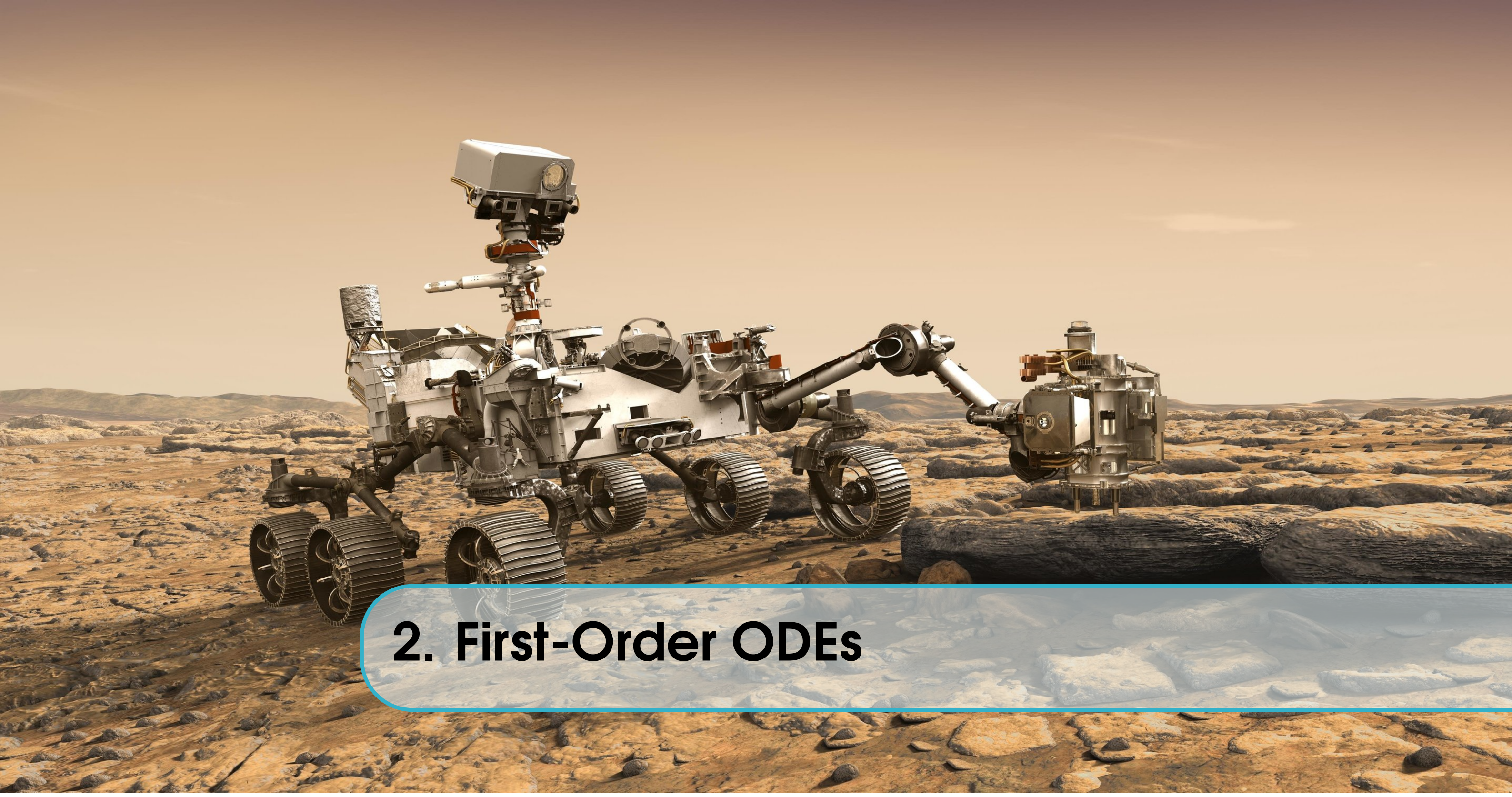
■ **Example 1.11** Model the following RL-Circuit for the current under the assumption that the initial current is zero.

Mathematical Model: According to Ohm's law, a current I in the circuit causes a voltage drop RI across the resistor. Also, a current in the circuit causes a voltage drop across the conductor, LI'. Kirchoff's Voltage law states that sum of these two voltage drops equals the EMF. The standard form of this model is

$$I' + \frac{R}{L}I = \frac{E(t)}{L}, I(0) = 0$$

From figure, $E = 48, L = 0.1, R = 11$. Therefore,

$$I' + 110I = 480, I(0) = 0$$



2. First-Order ODEs

2.1 First-Order ODE

Equation that contains only first derivative y' with or without y. A first-order ODE can be written in two forms: Implicit form (2.1) and Explicit form (2.2).

$$F(x,y,y') = 0 \tag{2.1}$$

Here, x is an independent variable, $y(x)$ is an unknown function to be determined and $y' = \frac{dy}{dx}$

$$y' = f(x,y) \tag{2.2}$$

Geometrically, $y'(x)$ is the slope of the curve $y(x)$. Therefore, the solution of (2.2) that passes through a point (x_0, y_0) must satisfy

$$y'(x_0) = f(x_0, y_0) \tag{2.3}$$

2.2 Initial Value Problem

Let us consider the following problem. In the morning, a cook prepared tea and poured the tea into a cup and served it to you at 7'O clock. The brewing temperature of the tea is $85°C$. If you had a phone call and forgot to drink the tea until 7.30AM, what will happen? Will the temperature of the tea be same? Assume that, the room temperature is $20°C$. Newton's law of cooling states that the rate of heat loss of an object is directly proportional to the difference between the temperature of the object and its surrounding temperature. Mathematically, it is written as

$$\frac{dT}{dt} = k(T - T_A) \tag{2.4}$$

where T and T_A represents the temperature of the tea and room respectively, k is the proportionality constant. The tea was served at 7'O clock, that is considered as the initial time for the tea. The tea was at $85°C$ at 7'O clock is treated as the initial condition. Mathematically, it is written as

$$T(0) = 85°C \tag{2.5}$$

Equations (2.4) and (2.5) constitute initial value problem. In general, an ODE with initial condition is called an initial value problem (IVP) and is given as follows, in case of explicit ODE.

$$y' = f(x,y), \quad y(x_0) = y_0 \tag{2.6}$$

2.2.1 What is a solution?

A value or values or a function which makes the equation valid. A function $h : (a,b) \to \mathbb{R}$ is said to be solution of (2.2) if h is differentiable in (a,b) and satisfies (2.2) while replacing y and y' by respectively h and h'. The graph or the curve of h is called a solution curve.

Let us consider the simple problem $y' = f(x)$. Then, it becomes a problem in integral calculus. We can obtain the solution by integration, that is

$$y = \int_{x_0}^{x} f(x)dx + c \tag{2.7}$$

where c is uniquely determined if $y(x_0) = y_0$ is specified.

2.3 Linear ODE

What is linear? Have you ever seen this term earlier? A transformation $T : \mathbb{V} \to \mathbb{W}$ is linear for any two scalars, $\alpha, \beta \in \mathbb{R}$, and any two vectors $u, v \in \mathbb{V}$ if T satisfies

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \tag{2.8}$$

Let us consider the following function

$$L(y,y') = F(x,y,y') \tag{2.9}$$

If L is linear w.r.to y and y', then the ODE is said to be first-order linear ODE, otherwise, it is a first-order non-linear ODE. That is, for any $\alpha, \beta \in \mathbb{R}$ and y_1, y_2, y_1', y_2'.

$$L(\alpha y_1 + \beta y_2, y') = \alpha L(y_1, y) + \beta L(y_2, y)$$

and

$$L(y, \alpha y_1' + \beta y_2') = \alpha L(y, y_1') + \beta L(y, y_2')$$

Note: We don't say it is linear in x, we insist that it is linear w.r.to y and y' only. The general form of the first-order linear ODE is

$$A_1(x)\frac{dy}{dx} + A_2(x)y = B(x) \tag{2.10}$$

provided, $A_1(x) \neq 0$. When the right hand side term $B(x) = 0$, then the ODE is said to be and ODE with null term or homogeneous ODE, on the other hand, if $B(x) \neq 0$, it is called non-homogeneous ODE. Since $A_1(x) \neq 0$, (2.10) can be re-written as

$$\frac{dy}{dx} + \frac{A_2(x)}{A_1(x)}y = \frac{B(x)}{A_1(x)}$$

or

$$\frac{dy}{dx} + p(x)y = r(x) \tag{2.11}$$

■ **Example 2.1** Let us consider the following problem,

$$x\frac{dy}{dx} - 3y = 0 \tag{2.12}$$

Solution: Assume that, $x = 0$ and we get

$$\frac{dy}{dx} = 3\frac{y}{x} \implies \frac{dy}{y} = 3\frac{dx}{x}$$

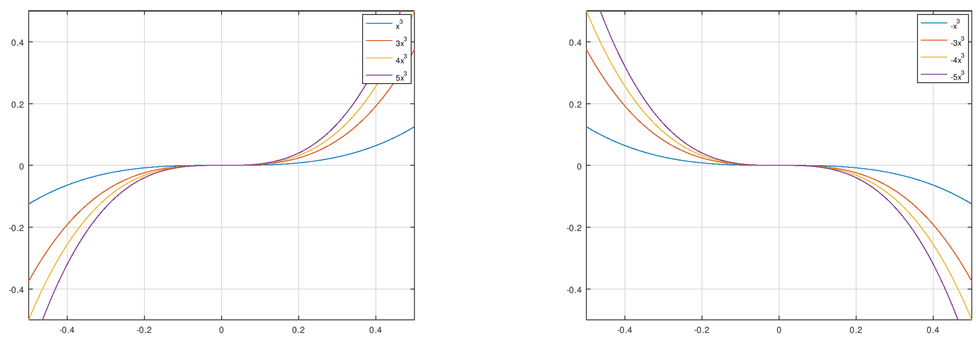
Integrating on both sides, we get

$$\int \frac{dy}{y} = 3\int \frac{dx}{x}$$
$$\implies log\ y = 3log\ x + c_1$$
$$\implies log\ y = 3log\ x + log\ c$$
$$\implies log\ y = log\ (cx^3)$$
$$\implies y = cx^3$$

Therefore, $y = cx^3$ is the solution to (2.12). For,

$$\frac{dy}{dx} = 3cx^2 \implies x\frac{dy}{dx} = 3cx^3 \implies \frac{dy}{dx} = 3y \implies (2.12)$$

Case 1: When we consider $y(0) = 0$, then $y = cx^3$ satisfies irrespective of the values of c. It



(a) $c > 0$ (b) $c < 0$

Figure 2.1: The graph of $y = cx^3$

means that, there are infinitely many solutions.
Case 2: When $y(0) = 1$, then there exists no c satisfying $y(0) = 1$. Therefore, there exists no solution.
case 3: When $x_0 \neq 0$ and $y(x_0) = y_0$, then $c = \frac{y_0}{x_0^3}$ and there exists a unique solution, $y = \frac{y_0}{x_0^3}x^3$.

2.3.1 Simple Case

Let us consider the simple case where $p(x) = 0$. Then (2.11) becomes

$$\frac{dy}{dx} = r(x) \tag{2.13}$$

This can be solved using the fundamental theorem of calculus.

Theorem 2.3.1 Fundamental Theorem of Calculus: If f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x) \tag{2.14}$$

If F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a) \tag{2.15}$$

Existence: Now, for a given $r(x)$ in the simple case (2.13), with the assumption that $r(x)$ is continuous on an interval $[x_0, x]$, by using fundamental theorem of calculus, one can find $R(x) = \int_{x_0}^x r(x)dx$ such that $R'(x) = r(x)$. That is, $y = R(x)$ solves the equation (2.13). **Note:** $R(x)$ is denotes as an antiderivative of $r(x)$

Question: Does there exist any other antiderivative that satisfies (2.13)?
Yes. $y(x) = R_c(x) = R(x) + c$, where c is any constant, also satisfies (2.13).

Remark: If S is any other antidertivative that solves (2.13), then S takes the form, $S(x) = R(x) + c$.

Uniqueness: If $y(x_0) = y_0$ is the initial condition for (2.13), then it is an IVP. $y(x_0) = R(x_0) + c \implies c = y(x_0) - R(x_0)$. From the definition of integral $R(x_0) = \int_{x_0}^{x_0} r(x)dx = 0$. Hence, $c = y_0$ and there exists a unique solution $y(x) = y_0 + R(x)$ to the given IVP.

2.4 Homogeneous ODE

Let us now consider (2.11) with the assumption that $r(x) = 0$. Then we get

$$\frac{dy}{dx} + p(x)y = 0 \tag{2.16}$$

In order to solve (2.16), we need to revise our knowledge on the separable ODEs.

2.4.1 Revisit: Separable ODE

A class of ODEs can be reduced to the seperable equation form after algebraic manipulations,

$$g(y)y' = f(x) \tag{2.17}$$

Here, we assume that f and g are continuous. Integrating on both sides w.r.to x, we obtain

$$\int g(y)y'dx = \int f(x)dx + c \implies \int g(y)dy = \int f(x)dx + c$$

Since f and g are continuous, both integrals exist. By evaluating the integrals we obtain the general solution of (2.17). This method of solving ODEs is called the method of separating variables.

■ **Example 2.2** Solve the following IVP

$$y' = (x+y-2)^2, y(0) = 2$$

Solution: Let $v = x + y - 2$, then

$$\begin{aligned} v' &= 1 + y' \\ \implies y' &= v' - 1 \\ and y(0) = 2 \implies v(0) &= 0 \end{aligned}$$

Replacing y' and $(x+y-2)$ by $1-v'$ and v in the given equation, we obtain

$$\begin{aligned} v'-1 &= v^2 \\ \implies v' &= 1+v^2 \\ \implies \frac{dv}{dx} &= 1+v^2 \\ \implies \frac{dv}{1+v^2} &= dx \end{aligned}$$

Integrating on both sides

$$\begin{aligned} \int \frac{1}{1+v^2}dv &= \int dx + c \\ \implies tan^{-1}(v) &= x+c \\ \implies v &= tan(x+c) \\ v(0)=0 \implies tan(c) &= 0 \\ \implies c &= n\pi \\ \implies x+y-2 &= tan(x+n\pi) \\ \implies y &= tan(x+n\pi)-x+2 \end{aligned}$$

■ **Example 2.3** Solve the Newtons law of cooling problem for the tea cub which is explained at section 2.2.
Solution: Given,

$$\frac{dT}{dt} = k(T-T_A), T(0) = 85°C, T_A = 20°C$$

After rearrangement, we obtain

$$\begin{aligned} \frac{dT}{T-20} &= kdt \\ \implies \int \frac{dT}{T-20} &= k\int dt \\ \implies ln|T-20| &= kt+c_1 \\ \implies |T-20| &= e^{kt+c_1} \\ \implies |T-20| &= ce^{kt}, c = e^{c_1} \\ \implies T-20 &= ce^{kt} \text{(Why?!)} \\ \implies T &= 20+ce^{kt} \\ T(0)=85 \implies 85 &= 20+ce^0 \\ \implies c = 65 \implies T &= 20+65e^{kt} \end{aligned}$$

To find k, we need an additional condition, let us assume that $T(1) = 40°C$. That is, the temperature of the tea in the tea cup at 8AM is $40°C$. Then,

$$\begin{aligned} T(1) = 40 = 20+65e^k \implies e^k &= \frac{20}{65} \\ \implies k &= ln(4/13) \approx -1.178 \\ \implies T &= 20+65e^{-1.178t} \end{aligned}$$

When the time is 7.30AM, then $t = 0.5$, and we obtain the temperature of the tea cup at time as $T(0.5) = 20+65e^{-1.178/2} \approx 56°C$

2.4.2 Solution of Homogeneous ODE

We want to solve (2.16) in some interval. Now, (2.16) can be rewritten as follows

$$\frac{dy}{y} = -p(x)dx \implies \int \frac{dy}{y} = -\int p(x)dx$$

Upon integration, we obtain

$$ln|y| = -\int p(x)dx + c_1$$

Taking exponents on both sides, we obtain

$$y = ce^{-\int p(x)dx}, c = \pm e^{c_1} \text{ depending on } y > 0 \text{ or } y < 0$$

If y satisfies the initial condition $y(x_0) = y_0$, then

$$y_0 = ce^{-\int_{x_0}^{x_0} p(x)dx} \implies c = y_0 \implies y = y_0 e^{-\int p(x)dx}$$

2.4.3 Exact ODE

If a function $u(x,y)$ has continuous partial derivatives, its total differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{2.18}$$

If $u(x,y) = c$, then $du = 0$ A first-order ODE $M(x,y) + N(x,y)y' = 0$ written as

$$M(x,y)dx + N(x,y)dy = 0 \tag{2.19}$$

is called an exact differential equation if the differential $M(x,y)dx + N(x,y)dy$ is exact. Comparing (2.18) and (2.19), we get

$$\frac{\partial u}{\partial x} = M, \frac{\partial u}{\partial y} = N$$

Theorem 2.4.1 Suppose $M,N \in C^1(R), R = (a,b) \times (c,d)$, where R denotes the rectangle and C^1 denotes the set of all functions whose partial derivatives are continuous in R. Then there exists ϕ such that

$$\frac{\partial \phi}{\partial x} = M, \frac{\partial \phi}{\partial y} = N$$

if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof. Suppose there exists ϕ such that

$$\frac{\partial \phi}{\partial x} = M, \frac{\partial \phi}{\partial y} = N$$

then

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial M}{\partial y}, \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

Conversely suppose

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Consider

$$M = \frac{\partial \phi(x,y)}{\partial x}$$

where y is treated as a parameter. Choose ϕ such that

$$\phi(x,y) = \int M(x,y)dx + h(y)$$

Note that as ϕ is a function of two variables and we are integrating over only x, the constant is replaced by an unknown function of y. Because, if take partial derivative w.r.to x, $h(y)$ will vanish. Now, differentiate ϕ w.r.to y.

$$\frac{\partial \phi}{\partial y} = \int \frac{\partial M}{\partial x} dx + h'(y) \implies h'(y) = \frac{\partial \phi}{\partial y} - \int \frac{\partial M}{\partial x} dx$$

We need

$$\begin{aligned} N = \frac{\partial \phi}{\partial y} &\implies h'(y) = N - \int \frac{\partial M}{\partial x} dx \\ &\implies h(y) = \int \left[N - \int \frac{\partial M}{\partial x} dx \right] dy \\ &\implies \phi(x,y) = \int M(x,y)dx + \int N dy - \int \int \frac{\partial M}{\partial x} dx dy \end{aligned}$$

Hence ϕ satisfies the requirement and hence the proof. ■

Definition 2.4.1 Integrating Factor: Suppose $M(x,y)dx + N(x,y)dy$ is not exact and if there exists a function $\mu(x,y)$ such that

$$\mu M dx + \mu N dy = 0$$

is exact, then $\mu(x,y)$ is called an integrating factor of $M(x,y)dx + N(x,y)dy$.

By theorem 2.4.1,

$$\begin{aligned} \frac{\partial}{\partial y}(\mu M) &= \frac{\partial}{\partial x}(\mu N) \\ \implies \mu_y M + \mu M_y &= \mu_x N + \mu N_x \end{aligned}$$

As a special case, suppose μ is a function of one variable say x, then $\mu_y = 0$ and $\mu_x = \mu'$. Therefore,

$$\begin{aligned} \mu M_y &= \mu_x N + \mu N_x \\ \frac{1}{\mu} \frac{d\mu}{dx} &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \end{aligned}$$

Theorem 2.4.2 If $Mdx+Ndy=0$ is not exact and if

$$\frac{1}{\mu}\frac{d\mu}{dx}=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$

depends only on x, then $Mdx+Ndy=0$ has an integrating factor $\mu=\mu(x)$ which is given by

$$\mu(x)=e^{\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx}$$

2.4.4 Solution to Nonhomogeneous Linear ODE

If the right hand side of (2.11) is non-zero, then it is called nonhomogeneous linear ODE. Suppose there exists a function $\mu(x)$ such that

$$\mu y'+\mu p(x)y=d(\mu y)$$

That is, exactness is guaranteed, then multiplying μ on both sides of (2.11), we obtain

$$\mu y'+\mu p(x)y=\mu r(x) \tag{2.20}$$

Also,

$$\mu y'+\mu' y=d(\mu y) \tag{2.21}$$

Comparing left sides of (2.20) and (2.21), we obtain that

$$\mu'=\mu p(x) \implies \frac{d\mu}{\mu}=p(x)dx \implies \mu=e^{\int p(x)dx}$$

Once again, comparing the right sides of (2.20) and (2.21), we obtain that

$$\begin{aligned} d(\mu y)&=\mu r(x)\\ \mu y&=\int \mu r(x)dx+c\\ y&=\frac{1}{\mu}\left(\int \mu r(x)dx+c\right)\\ y&=e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}r(x)dx+c\right)\end{aligned}$$

The total output is the response to the input r and the response to the initial data. Therefore, the general solution to the problem (2.11)is given by

$$y=e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}r(x)dx+c\right) \tag{2.22}$$

If a particular value is given for c, then the solution is called a particular solution. If initial condition $y(x_0)=y_0$ is prescribed we obtain once again that $c=y_0$ and hence

$$y=e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}r(x)dx+y_0\right) \tag{2.23}$$

2.4.5 Bernoulli Equation

A few nonlinear ODEs can also be transformed to linear ODEs. For instance, the following Bernoulli equation (2.24) can be transformed to linear ODE format

$$y' + p(x)y = r(x)y^a, a \in \mathbb{R} \tag{2.24}$$

For $a = 0$ or $a = 1$, this will reduce to (2.11). Otherwise, it is nonlinear. To solve this type of problem, set

$$\begin{aligned} u(x) &= y(x)^{1-a} \\ u' &= (1-a)^{-a}y' = (1-a)(ry^a - py) \\ u' &= (1-a)(r - py^{1-a}) = (1-a)(r - pu) \\ u' + (1-a)pu &= (1-a)r \end{aligned}$$

The last equation reduced to linear form and it can be solved using (2.22).

2.4.6 Autonomous ODE

An ODE $y' = f(x,y)$ in which x does not occur explicitly is of the form

$$y' = f(y) \tag{2.25}$$

and is called an autonomous ODE. If equation (2.25) has constant solutions, they are called equilibrium solution or equilibrium points. Equilibrium solutions are determined by the zeros of $f(y)$, as $f(y) = 0 \implies y' = 0 \implies y = c$. These zeros are called as critical points of (2.25). An equilibrium solution is called stable if solutions close to it for some x remain close to it for all further x. It is called unstable if solutions initially close to it do not remain close to it as t increases.

2.5 Existence and Uniqueness Theorem on IVP

Theorem 2.5.1 **Existence Theorem**: Consider the following initial value problem

$$y' = f(x,y), y(x_0) = y_0$$

Suppose, $f(x,y)$ is continuous for all points (x,y) in some rectangle

$$R : |x - x_0| < a, |y - y_0| < b$$

and bounded in R, that is there exists a number K such that

$$|f(x,y)| \leq K, \text{ for all } (x,y) \in R$$

Then the initial value problem has at least one solution $y(x)$. This solution exists at least for all x in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $a = min\{a, b/K\}$.

■ **Example 2.4** $|y'| + |y| = 0, y(0) = 1$ has no solution
$xy' = y - 1, y(0) = 1$ has infinitely many solutions

Definition 2.5.1 Lipschitz condition: A function $f:[a,b]\to\mathbb{R}$ is said to satisfy the Lipschitz condition, if there exists a constant M such that

$$|f(x_1)-f(x_2)|\le M|x_1-x_2|,\ \text{for all } x_1,x_2\in[a,b].$$

Theorem 2.5.2 Uniqueness Theorem: Consider the following initial value problem

$$y' = f(x,y), y(x_0) = y_0$$

Suppose, $f(x,y)$ is continuous for all points (x,y) in some rectangle

$$R : |x-x_0| < a, |y-y_0| < b$$

and bounded in R, that is there exists a number K such that

$$|f(x,y)|\le K,\ \text{for all } (x,y)\in R$$

Further, of partial derivative of f w.r.to y (that is $f_y = \frac{\partial f}{\partial y}$), is continuous for all $(x,y)\in R$ and bounded, say

$$|f_(x,y)|\le M,\ \text{for all } (x,y)\in R \tag{2.26}$$

for some M. Then the initial value problem has at most one solution $y(x)$. This solution exists at least for all x in the subinterval $|x-x_0| < \alpha$ of the interval $|x-x_0| < a$, here $a = min\{a,b/K\}$.

Remark: The condition (2.26) may be replaced by the weak condition or Lipschitz condition,

$$|f(x,y_1)-f(x,y_2)|\le M|y_1-y_2|,\ \text{for all } (x,y_1),(x,y_2)\in R.$$

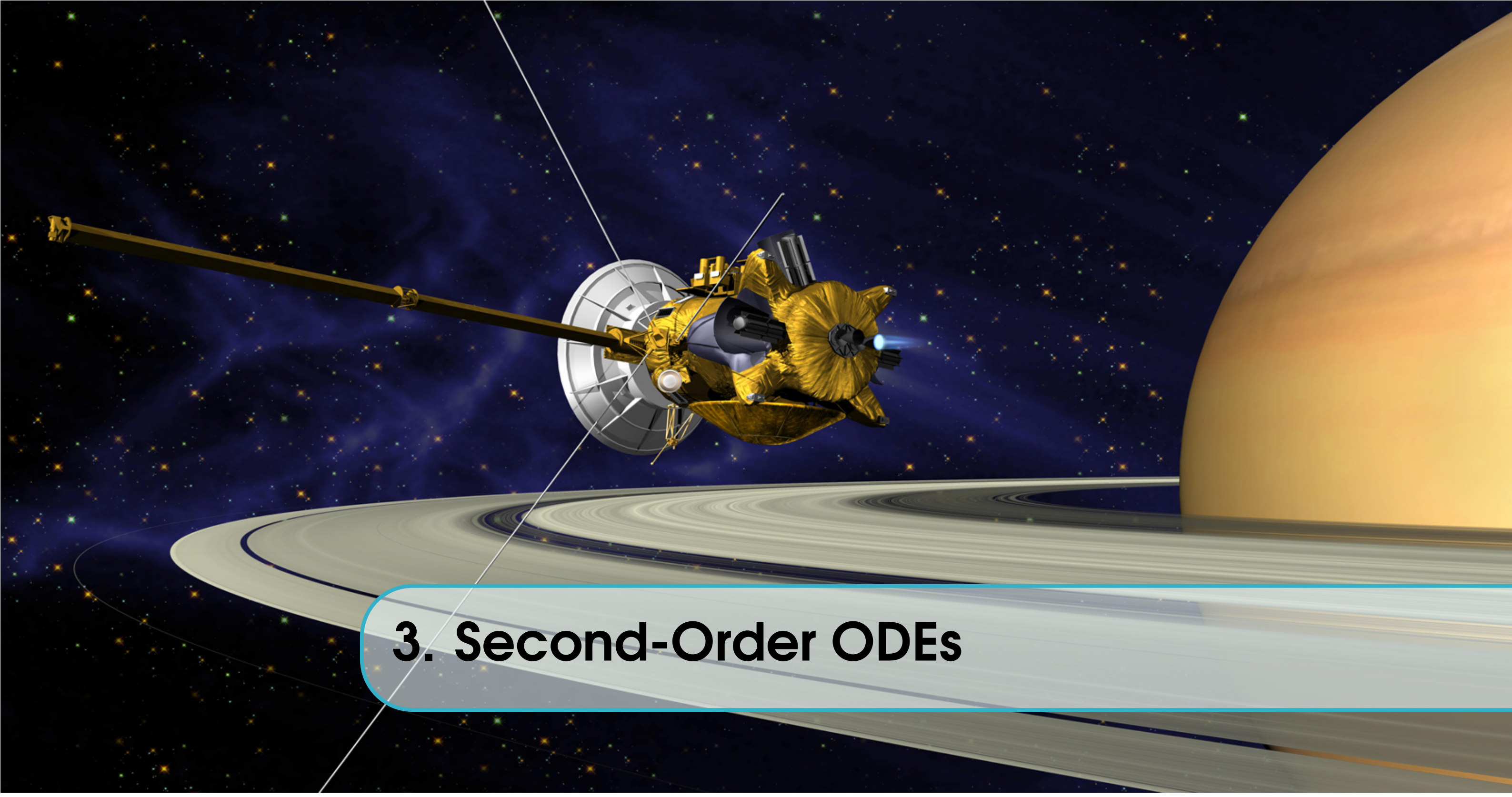
■ **Example 2.5** Continuity of $f(x,y)$ is not enough to guarantee the uniqueness. For example, consider the initial value problem

$$y' = \sqrt{|y|}, y(0) = 0$$

$f(x,y) = \sqrt{|y|}$ is continuous for all y. However,

$$y_1 = 0 \text{ and } y_2 = \frac{x^2}{4}, x\ge 0, y_2 = -\frac{x^2}{4}, x<0$$

are two solutions of the IVP.



3. Second-Order ODEs

3.1 Second-Order ODE

Similar to the first-order ODE, an equation that at most second derivatives is called second-order ODE. A second-order ODE can be written in the following form

$$F(x,y,y',y'') = 0 \tag{3.1}$$

These equations are important in engineering applications, especially in mechanical and electrical vibrations. Similar to the first-order ODE initial value problem, we have second-order ODE initial value problem which is given as follows:

$$F(x,y,y',y'') = 0, y(x_0) = y_0, y'(x_0) = y_1 \tag{3.2}$$

The initial value problem in first-order requires only one initial condition in order to eliminate the integration constant from its respective solution, whereas the solution of the second-order ODE contains two constants and hence we need at least two conditions to eliminate the constant. These constants can be initial conditions as given above. That is, initial values at x_0, the initial point on y and y' are provided. Similarly, if the second-order ODE is defined on an interval (a,b), then two conditions one on each of the boundary point can be prescribed which is called as boundary value problems.

$$F(x,y,y',y'') = 0, y(a) = c, y(b) = d \tag{3.3}$$

The condition $y(a) = c, y(b) = d$ are called boundary conditions. Apart from this, a general boundary condition can also be prescribed.

$$\begin{aligned} F(x,y,y',y'') &= 0 \\ \alpha_1 y(a) + \beta_1 y'(a) &= c \\ \alpha_2 y(b) + \beta_2 y'(b) &= d \end{aligned}$$

3.2 Second-order linear ODE

A second-order ODE is called linear if it can be written in the following form

$$y'' + p(x)y' + q(x)y = r(x) \tag{3.4}$$

and nonlinear if it cannot be written in this form. If the coefficient of y'' is not unity, then one can divide the equation by the coefficient and bring that to the standard form. If $r(x) \equiv 0$, then (3.4) is called homogeneous. That is,

$$y'' + p(x)y' + q(x)y = 0 \tag{3.5}$$

If $r(x) \neq 0$ for some x, then (3.4) is called nonhomogeneous. That is,

$$y'' + p(x)y' + q(x)y = r(x) \tag{3.6}$$

It is also represented by $L(y, y', y'')$ or $L(y)$ where L is linear in both y, y' and y''. Compactly, the homogeneous and nonhomogeneous equation can be written as follows

$$\begin{aligned} L(y) &= 0 \\ L(y) &= r(x) \end{aligned}$$

■ **Example 3.1**

$$\begin{aligned} y'' + 25y' + 36y &= x^2 \\ y''y + y'^2 &= 0 \end{aligned}$$

Similar to the solution of the first-order ODE, a function $y = h(x)$ is called a solution of a linear or nonlinear second-order ODE on some open interval (a, b) if h is defined, twice differentiable and replacing the h, h', h'' in the respective second-order ODE produces an identity.

Differential Operators: Instead of writing the equation as $y'' + p(x)y' + q(x)y = r(x)$, we can write it as $(D^2 + p(x)D + q(x)I)y = r(x)$. Here, D is called differential operator.

3.2.1 Superposition Principle or linearity principle

Theorem 3.2.1 Let y_1, y_2 be solutions of the homogeneous second-order ODE (3.5) on an interval $[a, b]$. Then the linear combination $y = c_1y_1(x) + c_2y_2(x)$ is also a solution of (3.5) on (a, b). Here, c_1, c_2 are constants.

Proof.

$$L(y) = L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2) = c_1.0 + c_2.0 = 0$$

■

Remark: A constant multiple of $y = c_1y_1(x)$ of a solution $y_1(x)$ of (3.5) is also a solution. Trivial solution is always a solution to (3.5).

Definition 3.2.1 — Linearly Dependent, Independent. Two functions $y_1(x), y_2(x)$ is said to be linearly dependent on an interval (a, b) if there exist constants c_1, c_2 not all zero such that

$$c_1y_1(x) + c_2y_2(x) = 0$$

for every x in the interval. If y_1, y_2 are not linearly dependent on the interval, it said to be linearly independent.

A solution $c_1y_1 + c_2y_2$ which contains arbitrary constants c_1, c_2 is called a general solution. If we choose a specific values for the constants, it is called particular solution.

Definition 3.2.2 — Basis. If two independent solutions $y_1(x), y_2(x)$ of (3.5) are not proportional to each other, that is, if y_1, y_2 are independent, then y_1, y_2 are called basis or fundamental solution of (3.5).

Theorem 3.2.2 Let y_1, y_2 be any two solutions of (3.5), then $\alpha y_1 + \beta y_2$ is also a solution of (3.5) for any $\alpha, \beta \in \mathbb{R}$. Suppose y_1, y_2 are independent solution of (3.5), then any solution y can be written in the form $\alpha y_1 + \beta y_2$ for some $\alpha, \beta \in \mathbb{R}$. If S denotes the set of all solutions of (3.5), then S is a linear space and $dim\ S \leq 2$.

Proof. Let y be a solution and let y_1, y_2 be linearly independent solutions of (3.5). By superposition principle, $\alpha y_1 + \beta y_2$ is also a solution. Since, y, y_1, y_2 are solutions of (3.5), they are differentiable in $[a,b]$. Therefore, their derivative exists as some point $t \in (a,b)$. (We can take even the initial value conditions x_0). Suppose that $y(t) = y_0, y'(t) = y_1$. Then

$$\begin{aligned}
\alpha y_1(t) + \beta y_2(t) &= y_0 \\
\alpha y_1'(t) + \beta y_2'(t) &= y_1
\end{aligned}$$

In matrix form

$$\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

The values of α and β can be uniquely determined if the determinant is non-zero. Define

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

This $W(x)$ for any x is called as the Wronskian. If we prove that $W(t) \neq 0$, then α, β are uniquely determined, then it proves that y can be written in the form $y = \alpha y_1 + \beta y_2$ for some $\alpha, \beta \in \mathbb{R}$. It is enough to prove that $W(t) \neq 0$ if y_1, y_2 are independent.

Claim: Either $W \equiv 0$ or W is never zero.

$$\begin{aligned}
W(x) &= y_1 y_2' - y_2 y_1' \\
W'(x) &= y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' \\
&= y_1 y_2'' - y_2 y_1'' \\
&= y_1(-p y_2' - q y_2) - y_2(-p y_1' - q y_1) \\
&= -p y_2' y_1 - q y_2 y_1 + p y_2 y_1' + q y_2 y_1 \\
&= -p(y_1 y_2' - y_2 y_1') \\
&= -pW \\
\implies W' &= -pW \\
\implies W &= C e^{-\int p(x)dx}
\end{aligned}$$

From this it is clear that if $C = 0$, $W \equiv 0$, and if $C \neq 0, W \neq 0$ as exponential will never be zero. Claim: $W \equiv 0$ if and only if y_1, y_2 are dependent Suppose y_1, y_2 are dependent, then either $y_1 = k y_2$ or $y_2 = l y_1$ for some k, l. Therefore, $y_1' = k y_2'$

$$W(x) = y_1 y_2' - y_2 y_1' = k(y_2 y_2' - y_2 y_2') = 0 \implies W \equiv 0$$

Conversely assume that $W \equiv 0$. Therefore, $W(x) = 0$ for all x. If $y_1 \equiv 0$ or $y_2 \equiv 0$, nothing to prove. If $y_1 \neq 0, y_2 \neq 0$. Then there exists a point x_i such that $y(x_i) \neq 0$ for some x_i. Therefore, there exists an interval $(c,d) \subset (a,b)$ such that for all $x_i \in (c,d), y_1(x_i) \neq 0$.

$$0 = \frac{W(x)}{y_1^2} = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = d\left(\frac{y_2}{y_2}\right) \implies \frac{y_2}{y_1} = k \implies y_2 = k y_1$$

This is true for all $x \in (c,d)$. By uniqueness property it is true everywhere in (a,b). Hence, if y_1, y_2 are independent, then $W(x) \neq 0$. ■

Theorem 3.2.3 dim S = 2

Proof. Let y_1, y_2 be two solutions of IVP such that

$$L(y_1) = 0, y_1(t_0) = 1, y_1'(t_0) = 0$$
$$L(y_2) = 0, y_2(t_0) = 0, y_2'(t_0) = 1$$

Then $W(x) \neq 0$ since $W(t_0) \neq 0$. Therefore, y_1, y_2 are independent. Hence $dim\ S = 2$. ■

Definition 3.2.3 — Wronskian. Suppose y_1, y_2 are differentiable, then

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

is called the Wronskian of y_1, y_2.

Definition 3.2.4 — General and Particular Solution. A general solution of (3.6) on an open interval (a,b) is a solution of the form

$$y = y_h(x) + y_p(x) \tag{3.7}$$

here $y_h = c_1 y_1 + c_2 y_2$ is a general solution to (3.5) on (a,b) and y_p is the particular solution of (3.6) on (a,b) containing no arbitrary constants. A particular solution of (3.6) on (a,b) is solution obtained from (3.7) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

Theorem 3.2.4 Let $\tilde{S}$ denote the set of all solutions of (3.6), then $\tilde{S} = S + y_p$

Proof. Let $y_h \in S$, then $y = y + y_h \in S + y_p$. We claim that $y \in \tilde{S}$ Since y_p is a particular solution of (3.6) $L(y_p) = r(x)$ Since y_h is a general solution of (3.6) $L(y_h) = 0$

$$L(y) = L(y + y_p) = L(y) + L(y_p) = 0 + r(x) = r(x)$$

Therefore, $y \in \tilde{S}$. Suppose $y \in \tilde{S}$. Then $L(y) = r(x)$. Let y_p be a particular solution of (3.6) (different from y). Then $y_h = y - y_p \in S$. For,

$$L(y - y_p) = L(y) - L(y_p) = r(x) - r(x) = 0$$

Therefore, there exists y_h such that $y = y_h + y_p \in S + y_p$. Hence the proof. ■

3.3 Reduction of Order

Suppose the solution to (3.5) takes the following form

$$y = uv$$

Then

$$y' = u'v + uv'$$
$$y'' = u''v + 2u'v' + uv''$$

Using the values of y, y', y'' in (3.5), we get

$$\begin{aligned} y'' + py' + qy &= 0 \\ (u''v + 2u'v' + uv'') + p(u'v + uv') + quv &= 0 \\ u''v + (2u' + pu)v' + (pu' + qu)v + uv'' &= 0 \end{aligned}$$

Now choose u such that v' term vanishes.

$$\begin{aligned} 2u' + pu &= 0 \\ \implies u' &= -p/2u \\ \implies u &= e^{-1/2\int p dx} \end{aligned}$$

Once the value of u is known, then we can find v

3.3.1 Reduction of Order

Let us assume that one solution of (3.5) is known to us. Let y_1 be the independent solution of (3.5). Then constant multiple of y_1 is also a solution to the (3.5). Now, let us assume that y_2 be the second independent solution to (3.5) and it takes the following form.

$$y_2 = c_1(x)y_1(x)$$

Since y_1 are y_2 are independent, one can't be a multiple of other. Therefore, c_1 is no longer constant here, but a function of x. Also, we assume that c_1 is twice differentiable. Let $u = c_1$ Then

$$\begin{aligned} y_2 &= uy_1 \\ y_2' &= u'y_1 + uy_1' \\ y_2 &= u''y_1 + 2u'y_1' + uy_1'' \\ y_2'' + py_2' + qy_2 &= 0 \\ \implies u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 &= 0 \\ \implies u''y_1 + \underbrace{(uy_1'' + puy1' + quy_1)}_{0} + (2y_1' + py_1)u' &= 0 \\ \implies u''y_1 + (2y_1' + py_1)u' &= 0 \\ \implies \frac{u''}{u'} &= -\frac{2y_1' + py_1}{y_1} \\ v = u' \implies \frac{u'}{u} &= -\frac{2y_1' + py_1}{y_1} \end{aligned}$$

Solve for v and then for u. In this case, we will obtain, y_2 as a general solution rather than an independent solution, as two integration constants will be involved, one for v and another for u.

For,

$$\begin{aligned} d\left(\frac{1}{u}\right) &= -2d\left(\frac{1}{y_1}\right) - pdx \\ ln|v| &= -2ln|y_1| - \int pdx + c^* \\ v = c_1\frac{e^{-\int pdx}}{y_1^2} \implies v &= c_1\int\frac{e^{-\int pdx}}{y_1^2}dx + c_2 \\ \implies y_2 &= y_1\left(c_1\int\frac{e^{-\int pdx}}{y_1^2}dx + c_2\right) \end{aligned}$$

Here y_1 is also part of the solution with c_2, therefore the second independent solution is obtained by setting $c_1 = 1, c_2 = 0$,

$$y_2 = y_1(x) \int \frac{e^{-\int p dx}}{y_1^2(x)} dx \tag{3.8}$$

3.4 Second-order Homogeneous Linear ODE with Constant Coefficients

We know that constant functions are continuous. For this section, let us consider that $p(x)$ and $q(x)$ in (3.5) as constants. That is, let $p(x) = p$ and $q(x) = q$. Then (3.5) becomes

$$y'' + py' + q = 0 \tag{3.9}$$

From first-order ODE, of $y' = ky$, it is easy to solve. Therefore, we assume that $y' = my$ for some m. Then $y'' = my' = m^2y$. Using y'' and y' in (3.9), we obtain that

$$m^2y + pmy + qy = 0$$
$$(m^2 + pm + q)y = 0$$

From the above equation, it is clear that either $y = 0$ or $m^2 + pm + q = 0$ satisfies the equality. Since we are interested in the non-trivial solution, we assume that $y \neq 0$. Then $m^2 + pm + q = 0$. This is a quadratic equation and it is called as an auxiliary equation or characteristic equation. The roots of the auxiliary equation is given by

$$m_1 = -\alpha + \beta, m_2 = -\alpha - \beta$$
$$\alpha = \frac{p}{2}, \beta = \frac{\sqrt{p^2 - 4q}}{2}$$

There are three possibilities: (1) The two roots are distinct, (2) Equal and (3) complex depending on $\beta > 0$ or $\beta = 0$ or $\beta < 0$ respectively. Let us discuss each cases separately to find two independent solutions of (3.9)

Case 1: Roots are real and distinct

We have assumed that $y' = my$, therefore,

$$\frac{y'}{y} = m \implies y = e^{mx}$$

Since we are interested to find only independent solutions, we have ignored that integration constant here. Since there are two distinct roots, the two independent solutions are

$$y_1 = e^{m_1x}, y_2 = e^{m_2x}$$

and the general solution is given by

$$y = c_1y_1(x) + c_2y_2(x) = c_1e^{m_1x} + c_2e^{m_2x} \tag{3.10}$$

where c_1 and c_2 are arbitrary constants.

■ **Example 3.2** Solve $2y'' - 5y' - 3y = 0$

Rearranging the equation in standard form, we obtain $y'' - \frac{5}{2}y' - \frac{3}{2}y = 0$ The auxiliary equation is

$$m^2 - \frac{5}{2}m - \frac{3}{2} = 0 \implies p = -\frac{5}{2}, q = -\frac{3}{2}$$
$$\implies \alpha = -\frac{5}{4}, \beta = \frac{\sqrt{\frac{25}{4} + 4\frac{3}{2}}}{2} = \frac{\sqrt{\frac{49}{4}}}{2} = \frac{7}{4}$$
$$\implies m_1 = \frac{5}{4} + \frac{7}{4} = 3, m_2 = \frac{5}{4} - \frac{7}{4} = -\frac{1}{2}$$
$$\implies y = c_1e^{m_1x} + c_2e^{m_2x} = c_1e^{3x} + c_2e^{-\frac{1}{2}x}$$

■

Case 2: Roots are real and equal or double roots

$$y_1 = e^{m_1 x}, y_2 = e^{m_1 x}$$

Now, y_1 and y_2 are not independent, therefore we should find an independent solution. Let us assume that one solution, $y_1 = e^{m_1 x}$ is known. Here, $m_1 = -\alpha = -\frac{p}{2} \implies p = 2\alpha$ as $p^2 - 4q = 0$. Using the method of variation of parameters, let us find the second solution. Let $y_2 = c_1(x)y_1(x) = c_1(x)e^{-\alpha x}$. Then using (3.8), we obtain that

$$y_2 = y_1(x)\int \frac{e^{-pdx}}{y_1^2(x)}dx$$
$$\implies y_2 = e^{-\alpha x}\int \frac{e^{-2\alpha x}}{e^{-2\alpha x}}dx$$
$$\implies y_2 = e^{-\alpha x}\int dx = xe^{-\alpha x} = xe^{m_1 x}$$

Therefore, the general solution is given by

$$y = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{m_1 x} + c_2 x e^{m_1 x} = (c_1 + c_2 x)e^{m_1 x} \tag{3.11}$$

■ **Example 3.3** Solve $y'' - 10y'25y = 0$
The auxiliary equation is

$$m^2 - 10m + 25 = 0 \implies p = 10, q = 25$$
$$\implies \alpha = 5, \beta = \frac{\sqrt{100 - 100}}{2} = 0$$
$$\implies m_1 = m_2 = 5$$
$$\implies y = (c_1 + c_2 x)e^{m_1 x} = (c_1 + c_2 x)e^{5x}$$

■

Case 3: Roots are complex
When $p^2 - 4q < 0$, we have

$$m_1 = -\alpha + i\beta, m_2 = -\alpha - i\beta, \beta = \frac{\sqrt{4q - p^2}}{2}$$
$$Y_1 = e^{m_1 x}, Y_2 = e^{m_2 x}$$
$$Y_1 = e^{(-\alpha + i\beta)x}, Y_2 = e^{(-\alpha - i\beta)x}$$

The general solution is given by

$$y = C_1 Y_1(x) + C_2 Y_2(x) = C_1 e^{(-\alpha + i\beta)x} + C_2 x e^{(-\alpha - i\beta)x} \tag{3.12}$$

Here constants, Y_1, Y_2, C_1 and C_2 are chosen for a purpose (instead of lower case, upper case) From Euler, we know that

$$e^{ix} = cosx + isinx, e^{-ix} = cosx - isinx,$$

Therefore, (3.12) becomes

$$y = e^{-\alpha x}(C_1 e^{i\beta x} + C_2 x e^{-i\beta x})$$
$$y = e^{-\alpha x}(C_1(cos\beta x + isin\beta x) + C_2 x(cos\beta x - isin\beta x)$$
$$y = e^{-\alpha x}((C_1 + C_2)cos\beta x + (C_1 - C_2)isin\beta x)$$

Therefore, the two independent solutions are given by $y_1 = e^{-\alpha x} cos\beta x, y_2 = e^{-\alpha x} sin\beta x$

$$y = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{-\alpha x} cos\beta x + c_2 e^{-\alpha x} sin\beta x = e^{-\alpha x}(c_1 cos\beta x + c_2 sin\beta x) \tag{3.13}$$

■ **Example 3.4** Solve $y'' + 4y' + 7y = 0$
The auxiliary equation is

$$m^2 + 4m + 7 = 0 \implies p = 4, q = 7$$
$$\implies p^2 - 4q = -12 < 0 \implies \alpha = 2, \beta = \frac{\sqrt{16-28}}{2} = \sqrt{3}$$
$$\implies y = e^{-\alpha x}(c_1 cos\beta x + c_2 sin\beta x) = e^{-2x}(c_1 cos\sqrt{3}x + c_2 sin\sqrt{3}x)$$

■

3.5 Second-order Nonhomogeneous Linear ODE with Constant Coefficients

In this section, we will see three different ways to find the solution of the second-order nonhomogeneous linear ODE with constant coefficients. That is $p(x)$ and $q(x)$ are constants whereas $r(x)$ need not be constant in (3.6).

$$y'' + py' + qy = r(x) \tag{3.14}$$

There are two ways we can find a particular solution y_p for (3.14). The first way is undetermined coefficients and the second one is variation of parameters. In the first way, there are two approaches available. (1) Superposition approach and (2) Annihilator approach.

3.5.1 Method of Undetermined Coefficients: Superposition Approach

The basic idea behind the superposition approach is that we guess the particular solution and try to solve (3.14). For example, constant function, polynomials, exponential for which we can guess the particular solution. The following table shows the list of particular solutions for the corresponding $r(x)$.

$r(x)$	y_p
k	C
$k_0 + k_1 x$	$C_0 + C_1 x$
$k_0 + k_1 x + k_2 x^2$	$C_0 + C_1 x + C_2 x^2$
$k_0 + k_1 x + k_2 x^2 + \cdot + k_n x^n$	$C_0 + C_1 x + C_2 x^2 + \cdot C_n x^n$
$e^{\alpha x}$	$Ce^{\alpha x}$
$sin\beta x$ or $cos\beta x$	$C_1 cos\beta x + C_2 sin\beta x$
xe^{ax}	$(C_0 + C_1 x)e^{ax}$
$x^2 e^{ax}$	$(C_0 + C_1 x + C_2 x^2)e^{ax}$
$e^{ax} sin\beta x$ or $e^{ax} cos\beta x$	$e^{ax}(C_1 cos\beta x + C_2 sin\beta x)$
$x^2 sin\beta x$ or $x^2 cos\beta x$	$(B_0 + B_1 x + B_2 x^2)cos\beta x + (C_0 + C_1 x + C_2 x^2)sin\beta x)$
$\sum_{i=0}^{n} k_i x^i sin\beta x$ or $\sum_{i=0}^{n} k_i x^i cos\beta x$	$\sum_{i=0}^{n} B_i x^i cos\beta x + \sum_{i=0}^{n} C_i x^i sin\beta x)$

Rules For choosing y_p

1. **Basic Rule:** If $r(x)$ is one of the functions in the first column of the table, then choose y_p from the respective line of the second column in the table. Determine the coefficients by calculating the derivatives of y_p and substituting in the given ODE.

2. **Modification Rule**: If a term in your choice of y_p happens to be a solution of the homogeneous ODE corresponding to (3.14), multiply this term by x (or by x^2 if this solution corresponds to a double root of the auxiliary equation of the homogeneous ODE).
3. **Sum Rule:** If $r(x)$ is a sum of functions in the first column of the table, choose for y_p the sum of the functions in the corresponding lines of the second column.

Example 3.5 Solve $y'' + 4y' - 2y = 2x^2 - 4x + 6$
The auxiliary equation for the homogeneous ODE is

$$\begin{aligned} m^2 + 4m - 2 = 0 &\implies p = 4, q = -2 \\ &\implies \alpha = 2, \beta = \frac{\sqrt{16+8}}{2} = \sqrt{6} \\ \implies m_1 = -2 + \sqrt{6}, m_2 = -2 - \sqrt{6} &\implies y = c_1 e^{-(2-\sqrt{6})x} + c_2 e^{-(2+\sqrt{6})x} \end{aligned}$$

The particular solution takes the following form

$$\begin{aligned} y_p &= A_0 + A_1 x + A_2 x^2 \\ \implies y_p' &= A_1 + 2A_2 x \ \ \& \ \ y_p'' = 2A_2 \\ \implies 2A_2 + 4(A_1 + 2A_2 x) - 2(A_0 + A_1 x + A_2 x^2) &= 2x^2 - 4x + 6 \\ \implies -2A_2 = 2 &\implies A_2 = -1 \\ \implies 8A_2 - 2A_1 = -4 &\implies A_1 = -2 \\ \implies 2A_2 + 4A_1 - 2A_0 = 6 &\implies A_0 = -8 \\ \implies y_p &= -8 - 2x - x^2 \\ \implies y = c_1 e^{-(2-\sqrt{6})x} + c_2 e^{-(2+\sqrt{6})x} &- 8 - 2x - x^2 \end{aligned}$$

■

Example 3.6 Solve $y'' - 4y' + 4y = 5t^3 - 2 - t^2 e^{2t} + 4e^{2t} \cos t$
The auxiliary equation for the homogeneous ODE is

$$\begin{aligned} m^2 - 4m + 4 = 0 &\implies p = +4, q = 4 \\ \implies \alpha = 2, \beta = 0 &\implies m_1 = m_2 = 2 \\ &\implies y = (c_1 + c_2 t)e^{2t} \end{aligned}$$

Since $r(t)$ contains sum of functions of the first column of the table, therefore, we apply the sum rule. The particular solution is given by as follows. Since e^{2t} is a solution of the homogeneous ODE, we have chosen up to degree 4 for the polynomial for e^{2t} terms

$$\begin{aligned} y_p = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + (B_0 + B_1 t + B_2 t^2 + B_3 t^3 + B_4 t^4)e^{2t} + e^{2t}(C_0 \cos t + C_1 \sin t) \\ \implies y_p' = A_1 + 2A_2 t + 3A_3 t^2 \\ +(2B_0 + B_1 + (2B_1 + 2B_2)t + (2B_2 + 3B_3)t^2 + (2B_3 + 4B_4)t^3 + B_4 t^4)e^{2t} \\ +e^{2t}((2C_0 + C_1)\cos t + (2C_1 - C_0)\sin t) \\ \implies y_p'' = 2A_2 + 6A_3 t \\ +(4B_0 + 4B_1 + 2B_2 + (4B_1 + 8B_2 + 6B_3)t + (4B_2 + 12B_3 + 12B_4)t^2 + (4B_3 + 12B_4)t^3 + 2B_4 t^4)e^{2t} \\ +e^{2t}((3C_0 + 4C_1)\cos t + (3C_1 - 4C_0)\sin t) \end{aligned}$$

$$\implies y_p'' = 2A_2 + 6A_3 t$$
$$+(4B_0 + 4B_1 + 2B_2 + (4B_1 + 8B_2 + 6B_3)t + (4B_2 + 12B_3 + 12B_4)t^2 + (4B_3 + 12B_4)t^3 + 2B_4 t^4)e^{2t}$$
$$+e^{2t}((3C_0 + 4C_1)\cos t + (3C_1 - 4C_0)\sin t)$$
$$-4y_p' = -4A_1 - 8A_2 t - 12A_3 t^2 + (-8B_0 - 4B_1 + (-8B_1 - 8B_2)t - 8B_2 t^2)e^{2t}$$
$$+e^{2t}((-8C_0 - 4C_1)\cos t + (-8C_1 + 4C_0)\sin t)$$
$$\implies 4y_p = 4A_0 + 4A_1 t + 4A_2 t^2 + 4A_3 t^3 + (4B_0 + 4B_1 t + 4B_2 t^2)e^{2t}$$
$$+e^{2t}(4C_0 \cos t + 4C_1 \sin t)$$
$$\implies (2A_2 - 4A_1 + 4A_0) + (6A_3 - 8A_2 + 4A_1)t + (-12A_3 + 5A_2)t^2 + 4A_3 t^3$$
$$+(2B_2 + (6B_3)t + (12B_3 + 12B_4)t^2 + (4B_3 + 12B_4)t^3 + 2B_4 t^4)e^{2t}$$
$$+e^{2t}\cos t(-C_0 \cos t - C_1 \sin t)$$

Equating the coefficient of t, t^2, t^3 and constants, we get

$$\implies A_3 = \frac{5}{4}, -12A_3 + 4A_2 = 0 \implies A_2 = \frac{15}{4}$$
$$\implies 6A_3 - 8A_2 + 4A_1 = 0 \implies A_1 = \frac{45}{8}$$
$$\implies 2A_2 - 4A_1 + 4A_0 = -2 \implies A_0 = \frac{13}{4}$$

Equating the coefficients of $e^{2t}, te^{2t}, t^2e^{2t}$ and t^3e^{2t}, we get

$$B_0 = B_1 = B_2 = B_3 = 0, 12B_4 = -1 \implies B_4 = -\frac{1}{12}$$

Equating the coefficients of $e^{2t}\cos t$ and $e^{2t}\sin t$, we get

$$C_0 = -4, C_1 = 0$$

Therefore, the particular solution is

$$y_p = \frac{13}{4} + \frac{45}{8}t + \frac{15}{4}t^2 + \frac{5}{4}t^3 - \frac{1}{12}t^4 e^{2t} - 4e^{2t}\cos t$$

Therefore, the general solution is

$$y = (c_1 + c_2 t)e^{2t} + \frac{13}{4} + \frac{45}{8}t + \frac{15}{4}t^2 + \frac{5}{4}t^3 - \frac{1}{12}t^4 e^{2t} - 4e^{2t}\cos t$$

■

3.5.2 Method of Undetermined Coefficients: Annihilator Approach

If A is a linear differential operator with constant coefficients and r is sufficiently differentiable function such that

$$A(r(x)) = 0$$

then A is said to be an annihilator of the function. For example, a constant function $y = k$ is annihilated by D as $Dk = 0$. The function $y = x$ is annihilated by D^2. The annihilator of k and $y = x$ are respectively $A(D) = D$ and $A(D) = D^2$. The following table shows the list of annihilators. To solve the for $L(y) = r(x)$, we need to follow the following steps.

1. Choose the correct $A(D)$ such that $A(r(x)) = 0$
2. Find the roots of $A(D)L(D) = 0$
3. Identify the roots of $L(D)$ and write the homogeneous solution part as y_h
4. Write the roots of $A(D)$ and write the particular solution y_p with constants
5. Differentiate the particular solution and use them in $L(y_p) = r(x)$.
6. Equate the coefficients and find the values of constants
7. Write the general solution as $y = y_h + y_p$

$r(x)$	$A(D)$
k	d
x	D^2
x^2	D^3
x^n	D^{n+1}
$k_0 + k_1x + k_2x^2 + \cdot + k_nx^n$	D^{n+1}
$sin\beta x$ or $cos\beta x$	$D^2 + \beta^2$
$e^{\alpha x}$	$D - \alpha$
$x^n e^{\alpha x}$	$(D-\alpha)^{n+1}$
$e^{\alpha x} sin\beta x$ or $e^{\alpha x} cos\beta x$	$(D-\alpha)^2 + \beta^2$
$x^n e^{\alpha x} sin\beta x$ or $x^n e^{\alpha x} cos\beta x$	$[(D-\alpha)^2 + \beta^2]^{n+1}$

■ **Example 3.7** Find a differential operator that annihilates the given function
(a) $1 - 5x^2 + 8x^3$, (b) e^{-3x}

$$\begin{aligned} D^4x^3 = 0 \implies D^4(1 - 5x^2 + 8x^3) &= 0 \\ (D+3)e^{-3x} &= 0 \end{aligned}$$

■

■ **Example 3.8** Solve $y'' + 5y' + 6y = 2e^{-x}$

$$\begin{aligned} L(D) &= D^2 + 5D + 6 \\ A(D) &= (D+1) \\ A(D)L(D) = 0 \implies (m+1)(m^2+5m+6) = 0 \implies m &= -1, -3, -2 \\ \implies y_h = Ae^{-3x} + Be^{-2x}, y_p &= C_1e^{-x} \\ \implies L(y_p) = C_1(1-5+6)e^{-x} \implies 2C_1e^{-x} = 2e^{-x} \implies C_1 &= 1 \\ \implies y_p = Ae^{-3x} + Be^{-2x} &+ e^{-x} \end{aligned}$$

■

■ **Example 3.9** Solve $y'' + y' + y = 2x\sin x$

$$\begin{aligned}
L(D) &= D^2 + D + 1 \\
A(D) &= (D^2+1)^2 \\
A(D)L(D) = 0 \implies & (m^+1)^2(m^2+m+1) = 0 \\
\implies & m = -i, i, -i, i, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2} \\
\implies & y_h = e^{\frac{-x}{2}}(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x) \\
\implies & y_p = (a+bx)\cos x + (c+dx)\sin x \\
\implies & y_p' = (c+dx+b)\cos x - (a+bx-d)\sin x \\
\implies & y_p'' = (2d-a-bx)\cos x - (c+dx+2b)\sin x \\
\implies L(y_p) = (b+c+2d+dx)\cos x - (a+bx-d+2b)\cos x &= 2x\sin x \\
\implies & b+c+2d = 0, d = 0, b = -2, a+2b-d = 0 \\
\implies & c = 2, a = 4 \\
\implies & y = e^{\frac{-x}{2}}(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x) + (4-2x)\cos x + 2\sin x
\end{aligned}$$

■

3.6 Euler-Cauchy Equation

Euler-Cauchy equations are ODEs of the form

$$x^2y'' + axy' + by = 0 \tag{3.15}$$

where a and b are constants. Use $y = x^m$, then

$$\begin{aligned}
m(m-1)x^m + max^m + bx^m &= 0 \\
m^2 + (a-1)m + b &= 0
\end{aligned}$$

Solving this equation for m, we obtain two roots say m_1 and m_2. Depending on the nature of the roots, we obtain the solutions as follows Case 1: Real and distinct roots. That is $m_1 \neq m_2$. Then the general solution is given by

$$y(x) = c_1x^{m_1} + c_2x^{m_2}$$

■ **Example 3.10** Solve $x^2y'' + 6xy' + 6y = 0$

$$a = 6, b = 6 \implies m^2 + 5m + 6 = 0 \implies m = -2, -3 \implies y = c_1x^{-2} + c_2x^{-3}$$

■

Case 2: Real double root. That is $m_1 = m_2 = m = \frac{1-a}{2}$. Then one solution is given by

$$y_1(x) = x^{(1-a)/2}$$

The second solution is obtained using (3.8)

$$\begin{aligned}
y_2 &= x^{(1-a)/2}\int \frac{e^{-\int \frac{a}{x}dx}}{x^{1-a}}dx = x^{(1-a)/2}\int \frac{e^{-a\ln x}}{x^{1-a}}dx = x^{(1-a)/2}\int \frac{x^a}{x^{1-a}}dx \\
&= x^{(1-a)/2}\int \frac{1}{x}dx = x^{(1-a)/2}\ln x = x^m \ln x
\end{aligned}$$

The general solution is given by

$$y = (c_1 + c_2\ln x)x^m \tag{3.16}$$

■ **Example 3.11** Solve $x^2y'' + 7xy' + 9y = 0$

$$a = 7, b = 9 \implies m^2 + 6m + 9 = 0 \implies m = -3, -3 \implies y = (c_1 + c_2 \ln x)x^{-3}$$

■

Case 3: Complex roots. That is $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$. Then solution is given by

$$\begin{aligned} y &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= x^{\alpha}(c_1 x^{i\beta} + c_2 x^{-i\beta}) \\ &= x^{\alpha}(c_1 (e^{\ln x})^{i\beta} + c_2 (e^{\ln x})^{-i\beta}) \\ &= x^{\alpha}(c_1 e^{i\beta \ln x} + c_2 e^{-i\beta \ln x}) \\ &= x^{\alpha}((c_1 + c_2)\cos(\beta \ln x) + (c_1 - c_2) i \sin(\beta \ln x) \end{aligned}$$

Therefore, the general solution for complex roots is given by

$$y = x^{\alpha}(A\cos(\beta \ln x) + B\sin(\beta \ln x) \tag{3.17}$$

■ **Example 3.12** Solve $x^2y'' + 3xy' + 3y = 0$

$$\begin{aligned} a = 3, b = 3 \implies m^2 + 2m + 3 = 0 &\implies m_1 = -1 + i\sqrt{2}, m_2 = -1 - i\sqrt{2} \\ &\implies y = \frac{1}{x}(A\cos(\sqrt{2}\ln x) + B\sin(\sqrt{2}\ln x) \end{aligned}$$

■

3.7 Variation of Parameters

Suppose solution of the second-order homogeneous linear ODE (3.6) is given by

$$y_h = c_1 y_1(x) + c_2 y_2(x)$$

where y_1, y_2 are two independent solutions and c_1, c_2 are constants. Let us change c_1 and c_2 as parameters and obtain the solution to the non-homogeneous linear ODE. Since we have variable parameters, this method is called variation of parameters. Therefore, we have the particular solution of (3.6) as

$$y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$$

where c_1 and c_2 are function of x. Now

$$\begin{aligned} y_p &= c_1 y_1 + c_2 y_2 \\ y_p' &= c_1' y_1 + c_2' y_2 + c_1 y_1' + c_2 y_2' \\ y_p'' &= c_1'' y_1 + c_2'' y_2 + c_1' y_1' + c_2' y_2' + c_1' y_1' + c_2 y_2' + c_1 y_1'' + c_2 y_2'' \end{aligned}$$

$$\begin{aligned} y_p'' + p(x)y_p' + q(x)y_p = c_1 &\overbrace{[y_1'' + p(x)y_1' + q(x)y_1]}^{0} + c_2 \overbrace{[y_2'' + p(x)y_2' + q(x)y_2]}0 \\ &+ c_1'' y_1 + c_1' y_1' + c_2 y_2'' + c_2' y_2' + p(x)[c_1 y_1' + c_2 y_2'] + c_1' y_1' + c_2' y_2' \\ &= \frac{d}{dx}[c_1' y_1 + c_2' y_2] + + p(x)[c_1 y_1' + c_2 y_2'] + c_1' y_1' + c_2' y_2' \end{aligned}$$

If we assume $c_1'y_1 + c_2'y_2 = 0$, then first two terms of the above equation vanishes and we left with $c_1'y_1' + c_2'y_2' = r(x)$. Therefore, we have

$$\begin{aligned} c_1'y_1 + c_2'y_2 &= 0 \\ c_1'y_1' + c_2'y_2' &= r(x) \end{aligned}$$

Then the solution of c_1 and c_2 are obtained using Cramer's rule as

$$c_1' = \frac{-y_2 r(x)}{W} \quad c_2' = \frac{y_1 r(x)}{W} \tag{3.18}$$

where W denotes the Wronskian of y_1 and y_2. Since y_1 and y_2 are linearly independent, Wronskian is non-zero and hence we obtain the particular solution by solving 3.18.

■ **Example 3.13** Solve $x^2y'' + xy' - 9y = 48x^5$ This equation is similar to Euler-Cauchy equation, here

$$\begin{aligned} a = 1, b = -9, r(x) = 48x^3 \implies m^2 - 9 = 0 \implies m = 3, -3 \\ \implies y_h = c_1x^3 + c_2x^{-3} \implies y_1 = x^3, y_2 = x^{-3} \\ \implies y_2' = -3x^{-4}, y_1' = 3x^2 \\ \implies W = -\frac{3}{x} - \frac{3}{x} = -\frac{6}{x} \\ c_1' = \frac{-y_2 r(x)}{W} = 8x \implies c_1 = 4x^2 \\ c_2' = \frac{y_1 r(x)}{W} = -8x^7 \implies c_2 = -x^8 \\ \implies y_p = 4x^5 - x^5 = 3x^5 \\ \implies y = c_1x^3 + c_2x^{-3} + 3x^5 \end{aligned}$$

■

3.8 Green's Function

From method of variation of parameters, we observed that the particular solution of (3.6) is given by

$$\begin{aligned} y_p &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ y_p(x) &= -y_1(x)\int_{x_0}^{x} \frac{-y_2(t)r(t)}{W(t)}dt + y_2(x)\int_{x_0}^{x} \frac{y_1(t)r(t)}{W(t)}dt \\ y_p(x) &= \int_{x_0}^{x} \frac{-y_2(t)y_1(x)}{W(t)}r(t)dt + \int_{x_0}^{x} \frac{y_1(t)y_2(x)}{W(t)}r(t)dt \end{aligned}$$

$$y_p(x) = \int_{x_0}^{x} G(x,t)r(t)dt \tag{3.19}$$

where the function $G(x,t)$ is called the Green's function for (3.6) and is given by

$$G(x,t) = \frac{y_1(t)y_2(x) - y_2(t)y_1(x)}{W(t)} \tag{3.20}$$

The above problem gives the Green's function for an initial value problem. The following Green's function works for the boundary value problem.

$$G(x,t) = \begin{cases} \frac{y_1(t)y_2(x)}{W(t)} & a \le t \le x \\ \frac{y_2(t)y_1(x)}{W(t)} & x \le t \le b \end{cases}$$

Example 3.14 Solve the IVP $y''+4y=x, y(0)=0, y'(0)=0$ using Green's function

$$\begin{aligned}
m^2+4m=0 &\implies m=2i,-2i\\
&\implies y_1=\cos 2x, y_2=\sin 2x\\
&\implies y_2'=2\cos 2x, y_1'=-2\cos 2x\\
&\implies W=2(\cos^2 2x+\sin^2 2x)=2\\
G(x,t) &= \frac{\cos 2t\sin 2x-\cos 2x\sin 2t}{2}\\
\implies y_p=\frac{1}{2}\left(\int_0^x tcos2t\sin 2xdt-\int_0^x t\cos 2x\sin 2tdt\right)\\
\implies y_p=\frac{1}{2}\left(\sin 2x\int_0^x tcos2tdt-\cos 2x\int_0^x t\sin 2tdt\right)\\
&\implies y_p=\frac{x}{4}-\frac{\sin 2x}{8}\\
&\implies y=A\cos 2x+B\sin 2x+\frac{x}{4}-\frac{\sin 2x}{8}
\end{aligned}$$

■

Example 3.15 Solve the BVP $y''+4y=3, y'(0)=0, y(\pi/2)=0$ using Green's function

$$\begin{aligned}
m^2+4m=0 &\implies m=2i,-2i\\
&\implies y_1=\cos 2x, y_2=\sin 2x\\
&\implies y_2'=2\cos 2x, y_1'=-2\cos 2x\\
&\implies W=2(\cos^2 2x+\sin^2 2x)=2
\end{aligned}$$

$$G(x,t)=\begin{cases}\frac{\cos 2t\sin 2x}{2} & 0\le t\le x\\ \frac{\cos 2x\sin 2t}{2} & x\le t\le \pi/2\end{cases}$$

$$\begin{aligned}
&\implies y_p=\int_0^{\pi/2} G(x,t)r(t)dt\\
&\implies y_p=\frac{3}{2}\left(\sin 2x\int_0^x cos2tdt+\cos 2x\int_x^{\pi/2}\sin 2tdt\right)\\
&\implies y_p=\frac{3}{4}-\frac{3\cos 2x}{4}\\
&\implies y=A\cos 2x+B\sin 2x+\frac{3}{4}-\frac{3\cos 2x}{4}
\end{aligned}$$

■

4. Power-Series, Legendre, Bessel

4.1 Power Series Method

When the linear ODE contains variable coefficients, then power series method is very helpful to solve the problem. The solution of the ODE is in the form of power series.

Definition 4.1.1 — Power Series. A power series (in powers of $(x-x_0)$ is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m(x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \tag{4.1}$$

Here x is a variable. $a_0, a_1, a_2, \cdots$ are constants, called the coefficient of the series. x_0 is a constant, called the center of the series.

If $x_0 = 0$, then we obtain the power series in power of x.

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x f^2 + \cdots \tag{4.2}$$

The n^{th} partial sum of (4.1) is

$$s_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n$$

where $n = 0, 1, \cdots$. The remainder of (4.1) is given by

$$R_n(x) = a_{n+1}(x-x_0)^{n+1} + a_{n+2}(x-x_0)^{n+2} + \cdots$$

If for some x_1, the sequence $s_1(x), s_2(x), \cdots$ converges, say

$$\lim_{n\longrightarrow\infty} s_n(x_1) = s(x_1)$$

then the series (4.1) is called convergent at $x = x_1$, the number $s(x_1)$ is called the value or sum of (4.1) at x_1, and we write

$$s(x_1) = \sum_{m=0}^{\infty} a_m(x-x_0)^m$$

If the sequence $s_1(x), s_2(x), \cdots$ does not converge at $x = x_1$, we say (4.1) diverges at $x = x_1$

Remarks:

- If we choose $x = x_0$, the series (4.1) reduces to a single term a_0, and hence the series converges at x_0
- If there are values other than x_0 at which (4.1) converges, these values form an interval. This interval is called the interval of convergence.
- If the interval of Convergence is $I = |x - x_0| < R$, the (4.1) converges for all $x \in I$ and diverges for all $|x - x_0| > R$.
- The quantity R is called the radius of convergence.
- If the series converges for all x, we set $R = \infty$

To find the radius of convergence, we use the following ratio and root tests (provided these limits exist and are not zero.)

$$\frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \tag{4.3}$$

$$\frac{1}{R} = \lim_{n \to \infty} \sqrt[m]{|a_m|} \tag{4.4}$$

Example 4.1 Find the radius and interval of convergence of the following power series

$$1. \sum_{m=0}^{\infty} (m+1)mx^m \quad 2. \sum_{m=0}^{\infty} x^m \quad 3. \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{m} x^m \quad 4. \sum_{m=0}^{\infty} \frac{5^m}{m!} x^m$$

$$1. \frac{1}{R} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m \to \infty} \left| \frac{(m+2)(m+1)}{(m+1)m} \right| = \lim_{m \to \infty} \left| 1 + \frac{2}{m} \right| = 1$$
$$\Longrightarrow R = 1, I = (-1, 1)$$
$$2. \frac{1}{R} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m \to \infty} \left| \frac{1}{1} \right| = 1$$
$$\Longrightarrow R = 1, I = (-1, 1)$$
$$3. \frac{1}{R} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m \to \infty} \left| \frac{(-1)^{m+1} m}{(-1)^m (m+1)} \right| = \lim_{m \to \infty} \left| \frac{m+1-1}{m+1} \right| = 1$$
$$\Longrightarrow R = 1, I = (-1, 1)$$
$$4. \frac{1}{R} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m \to \infty} \left| \frac{5^{m+1} m!}{5^m (m+1)!} \right| = \lim_{m \to \infty} \left| \frac{5}{m+1} \right| = 0$$
$$\Longrightarrow R = \infty, I = (-\infty, \infty)$$

■

1. **Termwise Addition:** Two power series may be added term by term. If

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m \text{ and } \sum_{m=0}^{\infty} b_m (x - x_0)^m$$

have positive radii of convergence and their sums are $f(x)$ and $g(x)$, then

$$\sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m$$

converges and represents $f(x) + g(x)$ for each x that lies in the interior of the convergence interval common to each of the two given series.

2. **Termwise Multiplication:** Two power series may be multiplied term by term. If

$$\sum_{m=0}^{\infty} a_m(x-x_0)^m \text{ and } \sum_{m=0}^{\infty} b_m(x-x_0)^m$$

have positive radii of convergence and their sums are $f(x)$ and $g(x)$, then product of these two series

$$\left(\sum_{m=0}^{\infty}(a_m)(x-x_0)^m\right)\left(\sum_{m=0}^{\infty}(b_m)(x-x_0)^m\right) = \sum_{m=0}^{\infty}(a_0b_m + a_1b_{m-1} + \cdots a_mb_0)(x-x_0)^m$$

converges and represents $f(x)g(x)$ for each x that lies in the interior of the convergence interval common to each of the two given series.

3. **Termwise Differentiation:** A power series may be differentiated term by term. If

$$y(x) = \sum_{m=0}^{\infty} a_m(x-x_0)^m$$

converges for $|x-x_0| < R, R > 0$, then the series obtained by differentiating term by term also converges for those x and represents the derivative y' of y for those x

$$y'(x) = \sum_{m=1}^{\infty} ma_m(x-x_0)^{m-1}$$

4. **Vanishing of all coefficients:** If a power series has a positive radius of convergence and the sum is identically zero throughout its interval of convergence, then each coefficient of the series must be zero.

■ **Example 4.2** Find the power series solution for the following ODEs

$$1. y' + y = 0 \quad 2. y' - 5y = 0 \quad 3. (1+x)y' = y \quad 4. y' = -2xy$$

Consider

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$
$$y'(x) = \sum_{m=1}^{\infty} ma_m x^{m-1}$$

1. Substitute the values of y and y' in the respective equation and use the property termwise addition and vanishing all coefficients of the series

$$\sum_{m=1}^{\infty} ma_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m = 0$$
$$\sum_{m=0}^{\infty}((m+1)a_{m+1} + a_m)x^m = 0$$
$$\implies a_{m+1} = -a_m$$
$$\implies a_1 = -a_0, a_2 = \frac{a_0}{2!}, a_3 = \frac{-a_0}{3!}, a_m = \frac{(-1)^m a_0}{m!}$$
$$y = a_0\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{m!} x^m = e^{-x}$$

2.

$$\sum_{m=1}^{\infty} m a_m x^{m-1} - 5\sum_{m=0}^{\infty} a_m x^m = 0$$
$$\sum_{m=0}^{\infty} ((m+1)a_{m+1} - 5a_m)x^m = 0$$
$$\implies a_{m+1} = 5a_m$$
$$\implies a_1 = 5a_0, a_2 = \frac{5^2 a_0}{2!}, a_3 = \frac{5^3 a_0}{3!}, a_m = \frac{5^m a_0}{m!}$$
$$y = a_0(1 + 5x + \frac{5^2 x^2}{2!} + \frac{5^3 x^3}{3!} + \cdots) = a_0 \sum_{m=0}^{\infty} \frac{5^m a_0}{m!} x^m = e^{5x}$$

3.

$$(1+x)\sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^m = 0$$
$$\sum_{m=1}^{\infty} m a_m x^{m-1} + \sum_{m=1}^{\infty} m a_m x^m - \sum_{m=0}^{\infty} a_m x^m = 0$$
$$\sum_{m=1}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} (m-1) a_m x^m = 0$$
$$\sum_{m=0}^{\infty} ((m+1)a_{m+1} + (m-1)a_m)x^m = 0$$
$$\implies a_{m+1} = -\frac{m-1}{m+1} a_m$$
$$\implies a_1 = a_0, a_2 = 0, a_3 = a_4 = \cdots = 0 y = a_0(1+x)$$

4.

$$\sum_{m=1}^{\infty} m a_m x^{m-1} + 2x \sum_{m=0}^{\infty} a_m x^m = 0$$
$$\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=0}^{\infty} 2a_m x^{m+1} = 0$$
$$\text{Equating coefficient of } x^{2m-1}, 2m a_{2m} + 2a_{2m-2} = 0 \implies a_{2m} = -\frac{a_{2m-2}}{m}$$
$$\text{Equating coefficient of } x^{2m}, 2m + 1 a_{2m+1} + 2a_{2m-1} = 0 \implies a_{2m+1} = 0$$
$$\implies a_{2m} = (-1)^m \frac{1}{m!}$$
$$y = a_0(1 - x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} - \frac{x^8}{4!} + \cdots)$$
$$y = a_0 e^{-x^2}$$

■

Definition 4.1.2 — Analytic. A function f is said to be anaytic at a point x_0 if it can be represented by a power series

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m \tag{4.5}$$

with either a positive or infinite radius of convergence. If the function is analytic at all points Here x is a variable. $a_0, a_1, a_2, \cdots$ are constants, called the coefficient of the series. x_0 is a constant, called the center of the series.

Definition 4.1.3 — Ordinary and Singular Points. A point $x = x_0$ is said to be an ordinary point of the differential equation

$$y'' + p(x)y' + q(x)y = r(x) \tag{4.6}$$

if both coefficients $p(x)$ and $q(x)$ of (4.6) are analytic at x_0. A point that is not an ordinary point of the differential equation (4.6) is said to be a singular point of the ODE.

Theorem 4.1.1 — Existence of Power Series Solution. If p, q and r in (4.6) are analytic at $x = x_0$, then every solution of (4.6) is analytic at $x = x_0$ and can be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.

Definition 4.1.4 — Regular and irregular Singular Points. A singular point $x = x_0$ is said to be a regular singular point of the differential equation (4.6) if the functions $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are both (4.6) analytic at x_0. A point that is not regular is said to be an irregular singular point of the ODE.

■ **Example 4.3** Identify ordinary, singular, regular singular and irregular singular points for the following ODES.

$$1.(x^2-4)^2y'' + 3(x-2)y' + 5y = 0 \quad 2.x^3y'' + 4x^2y' + 3y = 0$$
$$3.3xy'' + y' - y = 0 \quad 4.(x^2+x-6)y'' + (x+3)y' + (x-2)y = 0$$

1. $x = 2$ and $x = -2$ are singular points. Remaining points are ordinary points.

$$p(x) = \frac{3(x-2)}{(x^2-4)^2}, q(x) = \frac{5}{(x^2-4)^2}$$

$$(x-2)p(x) = \frac{3(x-2)^2}{(x^2-4)^2} = \frac{3}{(x+2)^2}, (x-2)^2q(x) = \frac{5(x-2)^2}{(x^2-4)^2} = \frac{5}{(x+2)^2}$$

Both $(x-2)p(x)$ and $(x-2)^2q(x)$ are analytic at $x = 2$. Therefore, 2 is a regular singular point

$$(x+2)p(x) = \frac{3(x^2-4)}{(x^2-4)^2} = \frac{3}{(x-2)(x+2)}, (x+2)^2q(x) = \frac{5(x+2)^2}{(x^2-4)^2} = \frac{5}{(x-2)^2}$$

Although $(x+2)^2q(x)$ is analytic at $x = -2$, $(x+2)p(x)$ is not analytic at $x = -2$. Therefore, -2 is an irregular singular point.

2. $x = 0$ is a singular point. Remaining points are ordinary points.

$$p(x) = \frac{4}{x}, q(x) = \frac{3}{x^3} \implies xp(x) = 4, x^2q(x) = \frac{3}{x}$$

Although $xp(x)$ is analytic at $x = 0$, $x^2q(x)$ is not analytic at $x = 0$. Therefore, 0 is an irregular singular point.

3. $x = 0$ is a singular point. Remaining points are ordinary points.

$$p(x) = \frac{1}{3x}, q(x) = \frac{-1}{3x} \implies xp(x) = \frac{1}{3}, x^2q(x) = \frac{-x}{3}$$

Both $xp(x)$ and $x^2q(x)$ is analytic at $x = 0$. Therefore, 0 is a regular singular point.

4. $x=-3$ and $x=2$ are singular points. Remaining points are ordinary points.

$$p(x)=\frac{1}{x-2}, q(x)=\frac{1}{x+3}$$

$$(x+3)p(x)=\frac{x+3}{x-2}, (x+3)^2q(x)=x+3$$

Both $(x+3)p(x)$ and $(x+3)^2q(x)$ are analytic at $x=-3$. Therefore, -3 is a regular singular point

$$(x-2)p(x)=1, (x-2)^2q(x)=\frac{(x-2)^2}{x+3}$$

Both $(x-2)p(x)$ and $(x-2)^2q(x)$ are analytic at $x=2$. Therefore, 2 is a regular singular point

■

Theorem 4.1.2 — Frobenius Theorem. Let $b(x)$ and $c(x)$ be any functions that are analytic at $x=0$. Then the ODE

$$y''+\frac{b(x)}{x}y'+\frac{c(x)}{x^2}y=0 \tag{4.7}$$

has at least one solution that can be written in the form

$$y(x)=x^r\sum_{m=0}^{\infty}a_mx^m \tag{4.8}$$

where the exponent r may be any real or complex number and r is chosen so that $a_0\neq 0$. The ODE (4.7) has a second solution (such that these two solutions are linearly independent) that may be similar to (4.8) (with a different r and different coefficients) or may contain a logarithmic term.

4.1.1 Indicial Equation

Since $b(x)$ and $c(x)$ are analytic, we have

$$b(x)=\sum_{m=0}^{\infty}b_mx^m$$

$$c(x)=\sum_{m=0}^{\infty}c_mx^m$$

Let us substitute (4.8) in (4.7) we obtain

$$y'(x)=\sum_{m=0}^{\infty}(m+r)a_mx^{m+r-1}=x^{r-1}(ra_0+(r+1)a_1x+\cdots)$$

$$y''(x)=\sum_{m=0}^{\infty}(m+r)(m+r-1)a_mx^{m+r-2}=x^{r-2}(r(r-1)a_0+(r+1)ra_1x+\cdots)$$

$$x^r[r(r-1)a_0+\cdots]+(\sum_{m=0}^{\infty}b_mx^m)x^r(ra_0+\cdots)+(\sum_{m=0}^{\infty}c_mx^m)x^r(a_0+a_1x+\cdots)=0$$

By equating the coefficient of smallest power x^r to zero, we obtain the indicial equation

$$r(r-1)+b_0r+c_0=0 \tag{4.9}$$

The Frobenius method gives the basis of solutions. One of the solution is always of the form (4.8), where r is a root of (4.9).

Theorem 4.1.3 Suppose $b(x)$ and $c(x)$ of (4.7) are analytic at $x=0$. Let r_1 and r_2 be the roots of the indicial equation (4.9). Then we have the following three cases
Case 1: Distinct roots not by differing by an integer. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \cdots) \tag{4.10}$$

and

$$y_2(x) = x^{r_2}(A_0 + A_1x + A_2x^2 + \cdots) \tag{4.11}$$

with coefficients obtained successively with $r = r_1$ and $r = r_2$ respectively.
Case 2: Double root $r_1 = r_2 = r$. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \cdots), r = \frac{1}{2}(1 - b_0) \tag{4.12}$$

and

$$y_2(x) = y_1(x)\ln x + x^r(A_0 + A_1x + A_2x^2 + \cdots) \tag{4.13}$$

Case 3: Roots differing by an integer. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \cdots) \tag{4.14}$$

and

$$y_2(x) = ky_1(x)\ln x + x^{r_2}(A_0 + A_1x + A_2x^2 + \cdots) \tag{4.15}$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero.

■ **Example 4.4** Use Frobenius method to find two independent solutions of the following ODE

$$1.x(x-1)y'' + (3x-1)y' + y = 0 \quad 2.(x+2)^2y'' + (x+2)y' - y = 0$$
$$3.(x^2 - x)y'' - xy' + y = 0 \quad 4.x^2y'' + 6xy' + (4x^2 + 6)y = 0$$

1. Let us find the $b(x)$ and $c(x)$

$$b(x) = \frac{3x-1}{x(x-1)}, c(x) = \frac{1}{x(x-1)}$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_mx^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_mx^{m+r-1}$$
$$+3\sum_{m=0}^{\infty}(m+r)a_mx^{m+r} - \sum_{m=0}^{\infty}(m+r)a_mx^{m+r-1} + \sum_{m=0}^{\infty}a_mx^{m+r} = 0$$

The indicial equation is

$$-r(r-1) - r = 0 \implies r^2 = 0 \implies r = 0(\text{double root})$$

Using $r = 0$, in our expression and equating the coefficient of x^m to zero, we obtain

$$m(m-1)a_m - (m+1)ma_{m+1} + 3ma_m - (m+1)a_{m+1} + a_m = 0 \implies a_{m+1} = a_m$$
$$a_0 = 1 \implies y_1(x) = \sum_{m=0}^{\infty}x^m = \frac{1}{1-x}$$

Using the order of reduction, let us find the second solution. Using (3.8), we obtain

$$y_2 = y_1(x)\int \frac{e^{-\int p dx}}{y_1^2(x)}dx$$

$$\int pdx = \int \frac{3x-1}{x(x-1)}dx = \int \left(\frac{2}{x-1}+\frac{1}{x}\right)dx = \ln(x-1)^2 x$$

$$y_2 = y_1(x)\int \frac{e^{-\int p dx}}{y_1^2(x)}dx = \frac{1}{1-x}\int \frac{1}{x}dx = \frac{\ln x}{1-x}$$

2. Let $t=(x+2)$. Then

$$\frac{dt}{dx}=1, \frac{dx}{dt}=1\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt} \implies \frac{dy}{dx}=\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2}=\frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\frac{dx}{dt}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt}=\frac{d^2y}{dx^2}$$

$$\implies (x+2)^2y''+(x+2)y'-y=0 \implies t^2y''+ty'-y=0$$

$$b(t)=\frac{1}{t}, c(t)=\frac{-1}{t^2}$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m t^{m+r}+\sum_{m=0}^{\infty}(m+r)a_m t^{m+r}-\sum_{m=0}^{\infty}a_m t^{m+r}=0$$

The indicial equation is obtained by equating the coefficient of x^r to zero

$$r(r-1)+r-1=0 \implies r_1=1, r_2=-1$$

Using $r=1$, in our expression and equating the coefficient of t^{m+1} to zero, we obtain

$$\sum_{m=0}^{\infty}(m+1)ma_m t^{m+1}+\sum_{m=0}^{\infty}(m+1)a_m t^{m+1}-\sum_{m=0}^{\infty}a_m t^{m+1}=0$$

$$\implies a_1=a_2=\cdots=0$$

$$a_0=1 \implies y_1(t)=t \implies y_1(x)=x+2$$

Using $r=-1$, in our expression and equating the coefficient of t^{m-1} to zero, we obtain

$$\sum_{m=0}^{\infty}(m-1)(m-2)A_m t^{m-1}+\sum_{m=0}^{\infty}(m-1)A_m t^{m-1}-\sum_{m=0}^{\infty}A_m t^{m-1}=0$$

$$\implies A_1=A_2=\cdots=0$$

$$A_0=1 \implies y_2(t)=\frac{1}{t} \implies y_2(x)=\frac{1}{x+2}$$

3.

$$b(x)=\frac{-x}{x(x-1)}=-\frac{1}{x-1}, c(x)=\frac{1}{x(x-1)}$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r}-\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1}$$
$$-\sum_{m=0}^{\infty}(m+r)a_m x^{m+r}+\sum_{m=0}^{\infty}a_m x^{m+r}=0$$

$$\sum_{m=0}^{\infty}[(m+r)(m+r-1)-(m+r)+1]a_m x^{m+r}-\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1}=0$$
$$\sum_{m=0}^{\infty}(m+r-1)^2 a_m x^{m+r}-\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1}=0$$

The indicial equation is obtained from the coefficient of x^{r-1}

$$r(r-1)=0 \implies r_1=0, r_2=1$$

Using $r=1$, in our expression and equating the coefficient of x^{m+1} to zero, we obtain

$$\begin{aligned} m^2 a_m-(m+2)(m+1)a_{m+1}=0 \implies a_{m+1} &= \frac{m^2}{(m+2)(m+1)}a_m \\ &\implies a_1=a_2=\cdots=0 \\ a_0=1 &\implies y_1(x)=x \end{aligned}$$

Using the order of reduction, let us find the second solution. Using (3.8), we obtain

$$\begin{aligned} y_2 &= y_1(x)\int \frac{e^{-\int p dx}}{y_1^2(x)}dx \\ -\int p dx &= \int \frac{1}{(x-1)}dx=\ln(x-1) \\ y_2=y_1(x)\int \frac{e^{-\int p dx}}{y_1^2(x)}dx &= x\int \frac{x-1}{x^2}dx = x\ln x+1 \end{aligned}$$

4.

$$\begin{aligned} b(x) &= \frac{6}{x}, c(x)=4+\frac{6}{x^2} \\ \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} &+ 6\sum_{m=0}^{\infty}(m+r)a_m x^{m+r} \\ &+4\sum_{m=0}^{\infty}a_m x^{m+r+2}+6\sum_{m=0}^{\infty}a_m x^{m+r}=0 \end{aligned}$$

$$\sum_{m=0}^{\infty}[(m+r)(m+r-1)+6(m+r)+6]a_m x^{m+r}+\sum_{m=0}^{\infty}4a_m x^{m+r+2}=0$$
$$\sum_{m=0}^{\infty}[(m+r)(m+r+5)+6]a_m x^{m+r}+4\sum_{m=0}^{\infty}a_m x^{m+r+2}=0$$

The indicial equation is obtained from the coefficient of x^r

$$r(r+5)+6=0 \implies r_1=-2, r_2=-3$$

Using $r=-2$, in our expression and equating the coefficient of x^{m-2} to zero, we obtain

$$((m-2)(m+3)+6)a_m+4a_{m-2}=0$$
$$\implies a_m=\frac{-4}{m^2+m}a_{m-2} \text{ and } a_1=a_3=\cdots=0$$

$$a_0 = 1 \implies y_1(x) = \frac{1}{x^2} - \frac{2}{3} + \frac{2}{15}x^2 - \frac{4}{315}x^4 + \cdots = \frac{1}{2}\frac{\sin 2x}{x^3}$$

Using $r = -3$, in our expression and equating the coefficient of x^{m-3} to zero, we obtain

$$((m-3)(m+2)+6)A_m + 4A_{m-3} = 0$$
$$\implies a_m = \frac{-4}{m^2 - m}A_{m-3} \text{ and } A_0 = A_2 = \cdots = 0$$

$$A_1 = 1 \implies y_2(x) = \frac{1}{x^3} - \frac{2}{x} + \frac{2}{3}x - \frac{4}{45}x^3 + \cdots = \frac{\cos 2x}{x^3}$$

■

4.2 Legendre's Equation

One of the most important ODE in physics is the Legendre's equation. It is given by

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{4.16}$$

This equation contains a parameter n, therefore, it is a family of ODEs. Any solution of (4.16) is called a Legendre function. (4.16) has a regular singular point at $x = 1$, $x = -1$. At $x = 0$, both $p(x)$ and $q(x)$ are analytic. Therefore, using the power series method we can find a solution of the form

$$y = \sum_{m=0}^{\infty} a_m x^m$$

Applying this power series in (4.16), we obtain

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$
$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$
$$\sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s - 2\sum_{s=1}^{\infty} s a_s x^s + n(n+1)\sum_{s=0}^{\infty} a_s x^s = 0$$
$$2a_2 + n(n+1)a_0 + 6a_3 + [-2+n(n+1)]a_1 x$$
$$+\sum_{s=2}^{\infty}[(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + n(n+1)]a_s]x^s = 0$$

By equating the coefficient of x^s to zero, we obtain

$$(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + n(n+1)]a_s = 0$$
$$a_{s+2} = -\frac{[-s(s-1) - 2s + n(n+1)]}{(s+2)(s+1)}a_s$$
$$a_{s+2} = -\frac{[-s^2 - s + n^2 + n]}{(s+2)(s+1)}a_s$$
$$a_{s+2} = -\frac{[n^2 - s^2 + n - s]}{(s+2)(s+1)}a_s$$
$$a_{s+2} = -\frac{(n-s)(n+s) + n - s}{(s+2)(s+1)}a_s$$

We obtained the following recurrence relation or recursion formula.

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s, s = 0,1,\cdots \tag{4.17}$$

Now, we can obtain the coefficient of x^s in terms of a_0 and a_1.

$$\begin{aligned} a_2 &= -\frac{n(n+1)}{2!}a_0 \\ a_4 &= -\frac{(n-2)(n+3)}{4.3}a_2 \\ &= \frac{(n-2)n(n+1)(n+3)}{4.3.2}a_0 \end{aligned}$$

$$\begin{aligned} a_3 &= -\frac{(n-1)(n+2)}{3!}a_1 \\ a_5 &= -\frac{(n-3)(n+4)}{5.4}a_3 \\ &= \frac{(n-3))(n-1)(n+2)(n+4)}{5!}a_1 \end{aligned}$$

$$\begin{aligned} y_1(x) &= 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \cdots \\ y_2(x) &= x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3))(n-1)(n+2)(n+4)}{5!}x^5 - \cdots \end{aligned}$$

Then $y = a_0y_1(x) + a_1y_2(x)$ is the general solution of the Legendre Equation. This series converges for $|x| < 1$. Since the power of y_1 and y_2 are different, they are independent. For $s = n$, the values of a_{s+2} becomes zero and hence the remaining terms are. Therefore, we obtain a polynomial solution. Since y_1 and y_2 are solutions of the Legendre equation, any constant multiple of y_1 and y_2 are also a solution of it. Let us choose the constant as follows.

$$a_n = \frac{(2n)!}{2^n(n!)^2}$$

The recurrence relation (4.17) can also be written as

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)}a_n = -\frac{n(n-1)}{2(2n-1)}.\frac{(2n)!}{2^n(n!)^2} = -\frac{(2n-2)!}{2^n(n-1)!(n-2)!}$$

By repeating this, we obtain

$$\begin{aligned} a_{n-4} &= \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!} \\ a_{n-2m} &= (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} \end{aligned}$$

Using this values in either $y_1(x)$ or $y_2(x)$ depending on n, we obtain a polynomial solution as follows.

$$P_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}$$

where $[n/2]$ denotes the integer function.

Appendices

A. Solutions For Chapter 1

■ **Solution A.1** Solution to 1.1

$$
\begin{aligned}
\frac{dT}{dt} &= k(T - T_A) \\
T(0.5) &= 87, T(1) = 43 \\
\text{or } \ T(0) &= 87, T(0.5) = 43
\end{aligned}
$$

By separation of variables and integrating, we obtain

$$
\begin{aligned}
\frac{dT}{T - T_A} &= kdt \\
\int \frac{dT}{T - T_A} &= k \int dt \\
ln|T - T_A| &= kt + c_1 \\
T(t) &= T_A + ce^{kt}, (c = e^{c_1})
\end{aligned}
$$

We have two unknowns k and c (Of course T_A is also an unknown, but this can be estimated). The room temperature can be between $15°C$ and $25°C$, let us take the average of them, that is $T_A = 20°C$
Case 1:

$$
\begin{aligned}
T(1) = 43 \implies 43 = 20 + ce^{k} \implies ce^{k} = 23 \implies c = e^{-k}(23) \\
T(0.5) = 87 \implies 87 = 20 + ce^{0.5k} \implies 67 = e^{-k}(23)e^{0.5k} \\
\implies \frac{67}{23} = e^{-0.5k} \implies e^{k/2} \approx 0.3433 \implies \frac{k}{2} \approx -1.0692 \implies k \approx -2.138 \\
c = 195.17 \implies T(t) = 20 + 195.17e^{-2.138t}
\end{aligned}
$$

When $t = 0$, we have

$$
T(0) = T_A + ce^{0} = T_A + c = 215.17
$$

This is impossible as the water boils at $100°C$. Therefore Arjun is not a criminal and he might have been in his room just 10 minutes before, as $T(0.4) = 102°C$.
Case 2: If we assume that $T(0) = 87, T(0.5) = 43$, then

$$\begin{aligned} T(0) = 87 \implies 67 = c \implies T(0.5) = 43 \implies 23 = 67e^{0.5k} \implies \frac{67}{23} = e^{-0.5k} \\ \implies e^{k/2} \approx 0.3433 \implies \frac{k}{2} \approx -1.0692 \implies k \approx -2.138 \\ \implies T(t) = 20 + 67 * e^{-2.138t} \implies T(-0.5) = 20 + 67 * e^{2.135/2} \approx 215.13 \end{aligned}$$

For both cases, we have obtained that an impossible temperature for the water in the kettle. If we assume that, the hotel is in Russia or the place where the temperature is around $-40°C$, then also, the temperature will be around $150°C$. But, a kettle switches off after the water reaches the boiling temperature. You can vary T_A and find the values. The ranges of T at 4.30 PM for the room temperature $[-40, 40]$ is $[154, 776]$, which is not our desired range. Also, you can vary t and find Arjun's arrival time in his room.

Note that, here we used the time in hours. If we change the time in seconds, values of k and c will change. However, the conclusion will not change. Therefore, one should know the dimension analysis to solve the problem.

■ **Solution A.2** Solution to 1.3

$$\begin{aligned} \frac{dh}{dt} = -\frac{A}{\pi} 2.656\sqrt{h} \\ h(0) = 2.25, h(21600) = 0 \end{aligned}$$

Let us assume that hole is in circular form and its radius is r, then its area is $A = \pi * r^2$. Therefore $\frac{A}{\pi} = r^2$

$$\begin{aligned} \frac{dh}{\sqrt{h}} = -2.656r^2 dt \implies 2\sqrt{h} = -2.656r^2 t + c_1 \implies h = (-1.328r^2 t + c)^2 \\ h(0) = 2.25 \implies c^2 = 2.25 \implies c = 1.5 \implies h(t) = (1.5 - 1.328r^2 t)^2 \\ h(21600) = 0 \implies 1.5 - 1.328r^2 * 21600 = 0 \implies 28684.8r^2 = 1.5 \\ \implies r^2 \approx 0.00723m \implies r \approx 0.723cm == 7.23mm \end{aligned}$$

Diameter of the hole is approximately $14.46mm$

■ **Solution A.3** Solution to 1.5

$$\frac{dy}{dt} = A - ky$$

$$\begin{aligned} u = A - ky \implies \frac{du}{dt} = -k\frac{dy}{dt} \implies \frac{dy}{dt} = -\frac{1}{k}\frac{du}{dt} \implies \frac{du}{dt} = -ku \\ \frac{du}{u} = -kdt \implies ln|u| = -kt + c_1 \implies u = ce^{-kt} \\ A - ky = ce^{-kt} \implies ky = A - ce^{-kt} \implies y = \frac{1}{k}(A - ce^{-kt}) \end{aligned}$$

■ **Solution A.4** Solution to 1.6 It is a non-homogeneous ODE, therefore both general and particular solution should be found.

$$y' + Ky = A + Bsin\omega t$$

Comparing it with

$$\begin{aligned}
y' + p(t)y = r(t) &\implies p(t) = K, r(t) = A + Bsin\omega t\\
y' + p(t)y = r(t) &\implies y = e^{\int -p(t)dt}\left(\int e^{\int p(t)dt} r(t)dt + c\right)\\
\int pdt = \int Kdt = Kt &\implies y = e^{-Kt}\left(\int e^{Kt}(A + Bsin\omega t)dt + c\right)\\
&\implies y = e^{-Kt}\left(\int e^{Kt}Adt + \int e^{Kt}Bsin\omega tdt + c\right)\\
&\implies y = e^{-Kt}\left(\frac{A}{K}e^{Kt} + \frac{B}{K^2+\omega^2}e^{Kt}(Ksin\omega t - \omega cos\omega t + c\right)\\
&\implies y = e^{-Kt}\left(\frac{A}{K}e^{Kt} + \frac{B}{K^2+(\frac{\pi}{12})^2}e^{Kt}\left(Ksin\frac{\pi}{12}t - \frac{\pi}{12}cos\frac{\pi}{12}t\right) + c\right)\\
&\implies y = \frac{A}{K} + \frac{B}{K^2+(\frac{\pi}{12})^2}\left(Ksin\frac{\pi}{12}t - \frac{\pi}{12}cos\frac{\pi}{12}t\right) + ce^{-Kt}
\end{aligned}$$

When $t \to \infty$, the last term tends to zero (since K is positive) irrespective of the value of c. Except the last term, remaining terms are called the steady-state solution as they contain constant and periodic terms. y is a transient solution as it models from rest to the steady state.

■ **Solution A.5** Solution to 1.11

$$\begin{aligned}
I' + \frac{R}{L}I = \frac{E}{L} &\implies p = \frac{R}{L}, r = \frac{E}{L}\\
y' + p(t)y = r(t) &\implies y = e^{\int -p(t)dt}\left(\int e^{\int p(t)dt} r(t)dt + c\right)\\
&\implies I = e^{\int -\frac{R}{L}dt}\left(\int e^{\int \frac{R}{L}dt}\frac{E}{L}dt + c\right)\\
&\implies I = e^{-\frac{R}{L}t}\left(\frac{E}{L}\frac{e^{\frac{R}{L}t}}{\frac{R}{L}} + c\right)\\
&\implies I = \frac{E}{R} + ce^{-\frac{R}{L}t}\\
I(0) = 0 &\implies c = -\frac{E}{R}
\end{aligned}$$

$$\implies I = \frac{E}{R}(1 - e^{-\frac{R}{L}t})$$

For the given problem, it is given that $E = 48, R = 11, \frac{R}{L} = 110$. Therefore,

$$I = \frac{48}{11}(1 - e^{-110t})$$

If $E = Bsin\omega t$, then

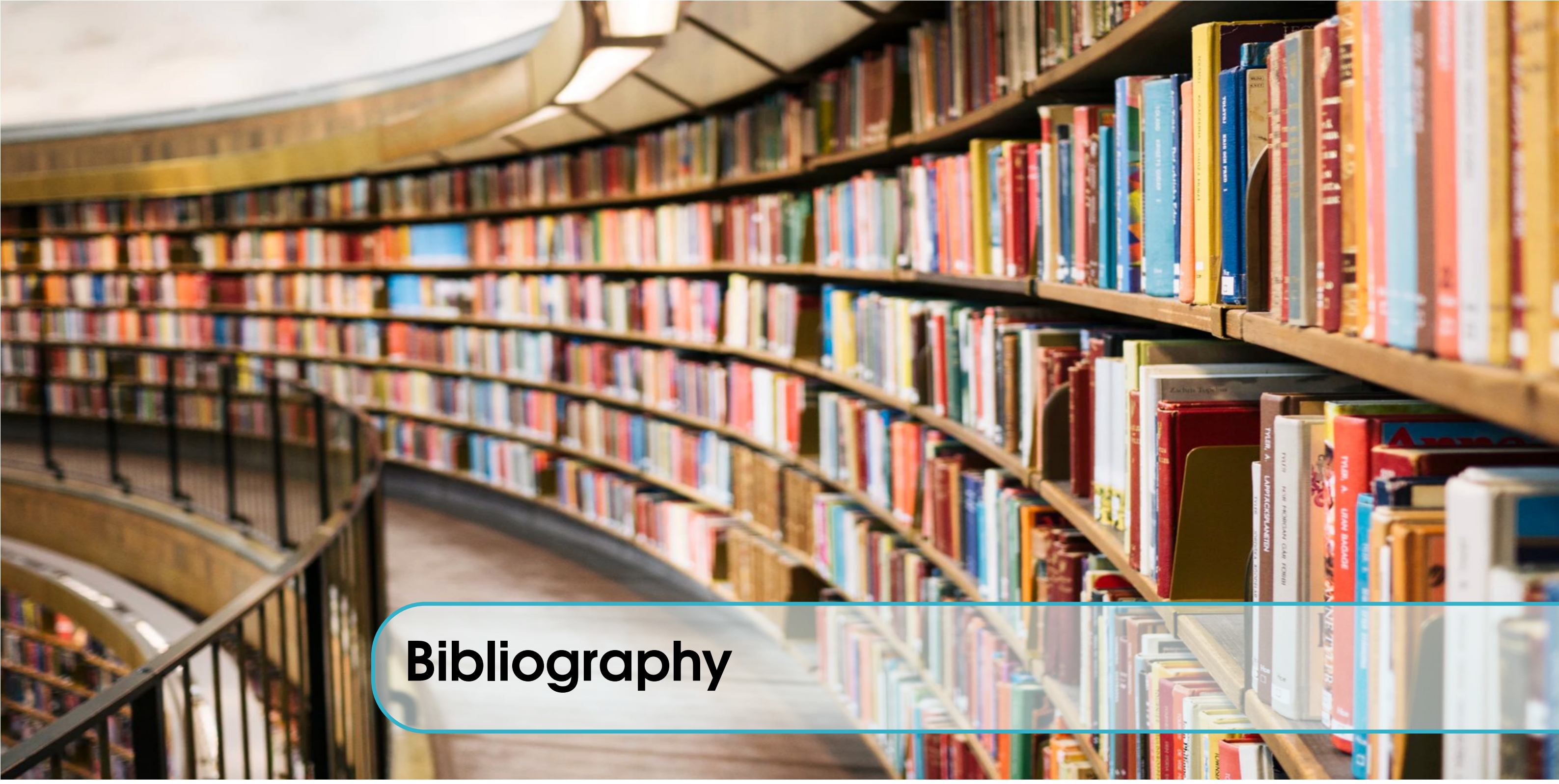
$$\begin{aligned}
y &= e^{-\frac{R}{L}t}\left(\int e^{\frac{R}{L}t}\frac{B}{L}sin\omega tdt + c\right)\\
&\implies I = e^{-\frac{R}{L}t}\left(\frac{\frac{B}{L}}{(\frac{R}{L})^2+\omega^2}e^{\frac{R}{L}t}(Bsin\omega - \omega cos\omega t) + c\right)\\
&\implies I = \frac{BL}{R^2+(\omega L)^2}(Bsin\omega t - \omega cos\omega t) + ce^{-\frac{R}{L}t}\\
I(0) = 0 &\implies I = \frac{BL}{R^2+(\omega L)^2}(Bsin\omega t - \omega cos\omega t + \omega e^{-\frac{R}{L}t})
\end{aligned}$$

Solution A.6 Solution to 1.10

$$y' = k(1-y)y \implies y' = ky - ky^2$$

This is Bernoulli equation type with $a = 2$

$$\begin{aligned} u = y^{1-a} = y^{-1} &\implies u' = -y^{-2}y' = -y^{-2}k(y-y^2) = -ky^{-1} + k = k - ku \\ u' + ku = k &\implies u = ce^{-kt} + 1 \\ y = \frac{1}{u} &\implies y = \frac{1}{ce^{-kt}+1} \end{aligned}$$



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