Solution of the problem:

1. Infinitely many solutions
2. Unique Solution
3. No Solution
4. Trivial Solution

$$\frac{dy}{dx} = r(x)$$

Existence: For given $r(x)$, continuous on an interval $[x_0, x]$, using fundamental theorem of calculus, $R(x) = \int_{x_0}^{x} r(t)dt$, then $y = R(x)$ solves the above equation

How many antiderivatives?

For any constant, $R_c(x) = R(x) + c$ is also a solution.

Uniqueness of IVP?

Solution of Homogeneous ODE:

$$\frac{dy}{dx} + p(x)y = 0$$

$$y = ce^{-\int pdx}$$

Existence for Homogeneous ODE:

Uniqueness for Homogeneous ODE (IVP):

Non-homogeneous ODE:

$$\frac{dy}{dx} + p(x)y = r(x)$$

Solution to Non-homogeneous ODE:

$$y = e^{-\int pdx}(\int e^{-\int pdx} rdx + c)$$

Existence:

Under what conditions does an initial value problem of the form $y' = f(x,y), y(x_0) = y_0$ have at least one solution (hence one or several solutions)?

Existence Theorem:
If $f(x,y)$ is continuous at all points (x,y) in the rectangle $R: |x - x_0| < a, |y - y_0| < b$ and bounded in R, that is, $|f(x,y)| \le K$ for some K for all $(x,y) \in R$. Then the IVP has at least one solution $y(x)$. This solution exists at least for all x in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $\alpha = \min\{a, \frac{b}{K}\}$

Uniqueness:
Under what conditions does that problem have at most one solution (hence excluding the case that is has more than one solution)?

Uniqueness Theorem:
Let f and its partial derivative $\frac{\partial f}{\partial y}$ be continuous for all (x,y) in the rectangle $R: |x - x_0| < a, |y - y_0| < b$ bounded in R, that is, $|f(x,y)| \le K, |f_y(x,y)| \le M$ for some K and M for all $(x,y) \in R$. Then the IVP has at most one solution $y(x)$. This solution exists at least for all x in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $\alpha = \min\{a, \frac{b}{K}\}$

Lipschitz condition:
Instead of $|f_y(x,y)| \le M$, $|f(x,y_2) - f(x,y_1)| \le M|y_2 - y_1|$

If the function is $u(x,y)$, then its total differential is given by

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

A first order ODE $M(x,y) + N(x,y)y' = 0$ is exact differential equation if there exists a ϕ such that $M(x,y) + N(x,y)y' = \phi'(x,y) = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$, that is $M = \frac{\partial \phi}{\partial x}, N = \frac{\partial \phi}{\partial y}$

Question: Given two functions $M(x,y), N(x,y)$, does there exists a function ϕ such that $M = \frac{\partial \phi}{\partial x}, N = \frac{\partial \phi}{\partial y}$

Theorem:

Suppose $M, N \in C^1(D), D: (a,b) \times (c,d)$. Then there exists ϕ such that $M = \frac{\partial \phi}{\partial x}, N = \frac{\partial \phi}{\partial y}$ if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

A first order ODE $M(x,y) + N(x,y)y' = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solve the IVP:

$$y' = \sqrt{|y|}, y(0) = 0$$

Picard Iteration for IVP:

$$y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big) dt$$

Let $y_0(x) = y_0$. Then

$$y_1(x) = y_0 + \int_{x_0}^{x} f\big(t, y_0(t)\big) dt$$

and

$$y_2(x) = y_0 + \int_{x_0}^{x} f\big(t, y_1(t)\big) dt$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f\big(t, y_{n-1}(t)\big) dt$$

Theorem:

If $f(x, y)$ satisfies existence and uniqueness theorem of IVP, then $y_n(x) \to y(x)$ as $n \to \infty$ uniquely.

Bernoulli Equation:

$$y' + p(x)y = g(x)y^a$$

Convert Bernoulli Equation to linear ODE:

$u = y^a$, then $u' + (1 - a)pu = (1 - a)g$

Logistic Equation or Verhulst Equation:

$$y' = Ay - By^2$$

or

$$P' = rP\left(1 - \frac{P}{K}\right)$$

r: rate, P: Population, K: Carrying capacity