MATHEMATICS FOR ENGINEERS

MA5101-DIFFERENTIAL EQUATIONS

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Population Dynamics

Logistic equation or Verhulst Equation

$$y' = Ay - By^2$$

- An important equation in population dynamics.
- An equation that models the evolution of populations of plants, animals or human over time t.
- If $B = 0$, exponential growth, Malthus' law
- $-By^2$ controls the growth

Population Dynamics

$$y' = Ay\left(1 - \frac{B}{A}y\right)$$

- $y < A/B \implies y' > 0$
- Small population keeps growing when $y < A/B$
- $y > A/B \implies y' < 0$
- Population decreases as long as $y > A/B$

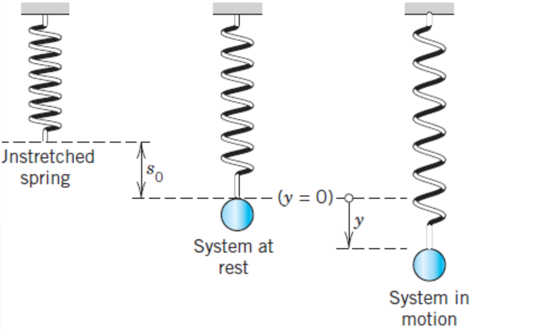
Population Dynamics

$$y' = Ay\left(1 - \frac{B}{A}y\right)$$

$$y' = f(y)$$

- Autonomous ODE
- It has constant solutions called, equilibrium solutions or equilibrium points
- Eq. Points found by solving $f(y) = 0$
- Zeros of $f(y)$ are also called us critical points
- $y = 0$ and $y = \frac{A}{B}$ are critical points

Mass-Spring System

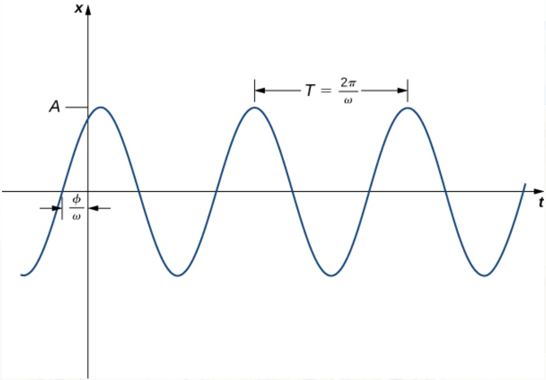


- Consider an ordinary coil spring that resists extension and compression.
- Suspend it vertically and attach a ball at its lower end
- $y = 0$: position of the ball, downward direction positive

Mass-Spring Undamped System

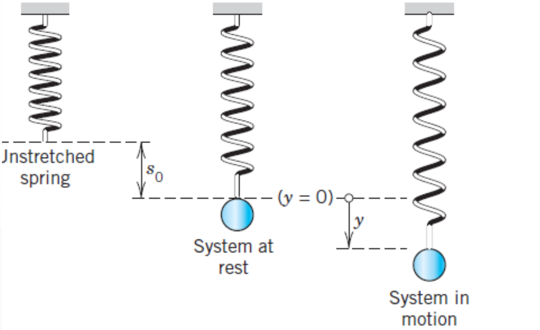
- Hooke's law $F_1 = -ky$
- Newton's second law $F = my''$
- $F = -F_1 \implies my'' + ky = 0$
- Time dependent problem, so y is a function of t
- Auxiliary equation $r^2 + \frac{k}{m} = 0 \implies r = \pm i\omega, \omega = \sqrt{\frac{k}{m}}$
- $y(t) = Acos\omega t + Bsin\omega t$
- Simple Harmonic oscillation

Undamped System



- Natural Frequency $f = \frac{\omega}{2\pi}$ of the system
- Alternative way
  $y = Ccos(\omega t - \theta), C = \sqrt{A^2 + B^2}, \theta = tan^{-1}\left(\frac{B}{A}\right)$

Mass-Spring System

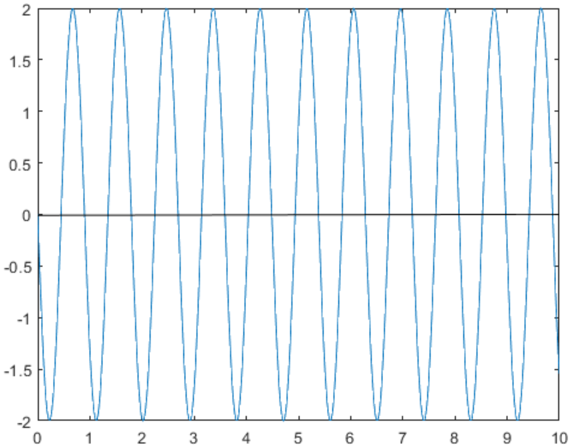


Assume an object weighing 9.8 kg stretches a spring 20 cm. Find the equation of motion if the spring is released from the equilibrium position with an upward velocity of 14 m/sec. What is the period of the motion?

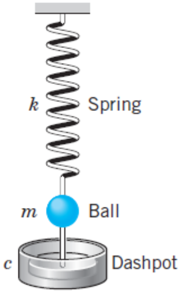
Mass-Spring Undamped System

- $W = mg = 9.8$ stretches $s = 20/100 = 1/5$
- $mg = ks \implies k = 49$
- $W = mg \implies 9.8 = m(9.8) \implies m = 1$
- $y'' + 49y = 0$ Auxiliary equation $r^2 + 49 = 0 \implies r = \pm 7i$
- $y(t) = Acos7t + Bsin7t$
- $y(0) = 0, y'(0) = -14 \implies A = 0, B = -2$
- $y(t) = -2sin7t$
- Period of motion$= \frac{2\pi}{\omega} = \frac{2\pi}{7}$

Mass-Spring Undamped System



Mass-Spring Damped System



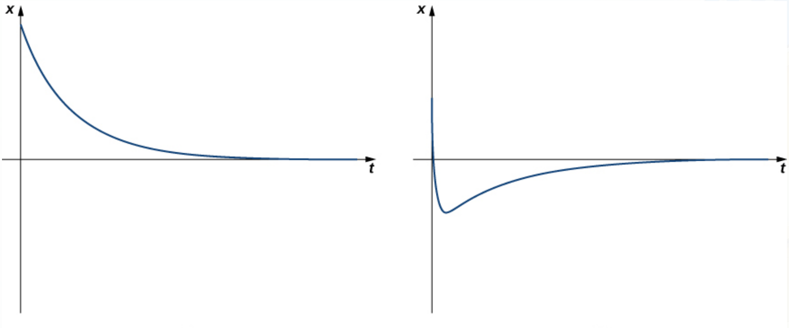
Mass-Spring Damped System

- Real world problem has always some friction in the system.
- Friction - Oscillations die slowly - Damping
- Air resistance, physical damper or dashpot
- Damping is a friction force, proportional to the velocity of the mass and acts in the opposite direction
- cy' be the damping force
- $my'' + cy' + ky = 0 \implies y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$
- Auxiliary equation: $r^2 + \frac{c}{m}r + \frac{k}{m} = 0$

Mass-Spring Damped System

- $r_1 = -\alpha + \beta, r_2 = -\alpha - \beta$
- $\alpha = \frac{c}{2m}, \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$
- Overdamping: $c^2 > 4mk$
- Critical Damping $c^2 = 4mk$
- Underdamping $c^2 < 4mk$

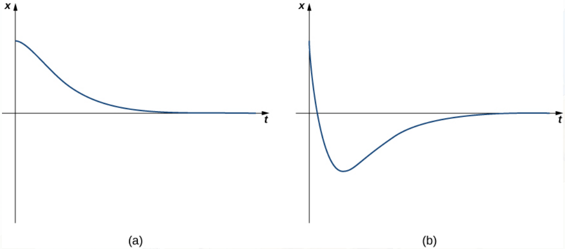
Mass-Spring Overdamped System: $c^2 > 4mk$



(a) (b)

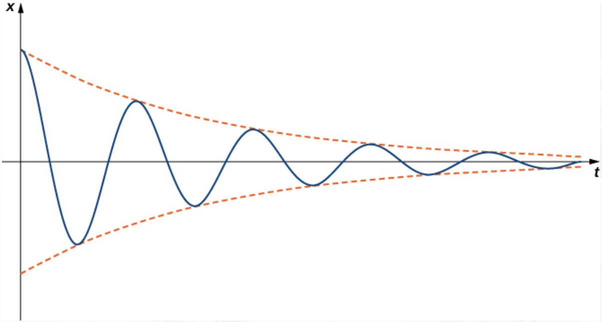
- $y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$
- Both exponents are negative. For, $\alpha > 0, \beta > 0, \beta^2 < \alpha^2$
- As $t \to \infty, y(t) \to 0$. Static equilibrium position

Mass-Spring Critically Damped System: $c^2 = 4mk$



- Similar to Overdamping, does not oscillate, however, if damping is reduced a little, it oscillates.
- Physical systems are either over or underdamped
- $\beta = 0, r_1 = r_2 = -\alpha$
- $y(y) = (c_1 + c_2 t)e^{-\alpha t}$

Mass-Spring Under Damped System: $c^2 < 4mk$



- Roots are complex. $y_1 = -\alpha + i\omega, y_2 = -\alpha - i\omega$
- $\omega = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$
- $y(t) = e^{-\alpha t}(Acos\omega t + Bcos\omega t) = Ce^{-\alpha t}Acos(\omega t - \theta)$
- Curve lies between $Ce^{-\alpha t}$ and $-Ce^{-\alpha t}$

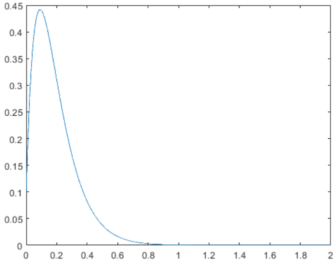
Motorbike Suspension System

A herohonda bike weighs 109 kg, and we assume a rider weight of 67.4 kg. When the rider mounts the bike, the suspension compresses 9.8 cm., then comes to rest at equilibrium.The suspension system provides damping equal to 360 times the instantaneous vertical velocity of the bike (and rider).

Motorbike System

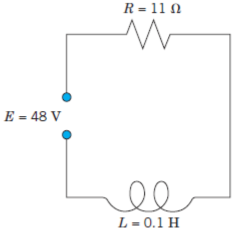
- $W = mg = 109 + 67.4 = 176.4$ stretches
  $s = 9.8/100 = 0.098$
- $mg = ks \implies 176.4 = 0.098k \implies k = 1800$
- $W = mg \implies 176.4 = m(9.8) \implies m = 18$
- $18y'' + 360y' + 1800y = 0$ Auxiliary equation
  $r^2 + 20r + 100 = 0 \implies r = -10$ (double root)
- $y(t) = (c_1 + c_2 t)e^{-10t}$
- When the bike was in the air before contacting the ground, the wheel was hanging freely and the spring was uncompressed. Therefore, the wheel is 0.098m below the equilibrium position
- $y(0) = 0.098 \implies 0.098 = c_1$

Motorbike System



- If the bike hits the ground with velocity, 10m/sec downward, then $y'(0) = 10$
- $y' = c_2 e^{-10t} - 10(c_1 + c_2 t)e^{-10t}$
- $10 = c_2 - 10c_1 \implies c_2 10.98$
- $y(t) = (0.098 + 10.98t)e^{-10t}$

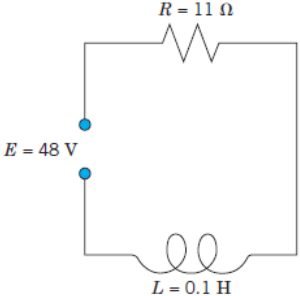
RLC Circuit



Circuit

- A current I in the circuit causes a voltage drop RI across the resistor (Ohm's law) and a voltage drop LI' across the conductor,
- sum of these two voltage drops equals the EMF (Kirchhoff's Voltage Law)

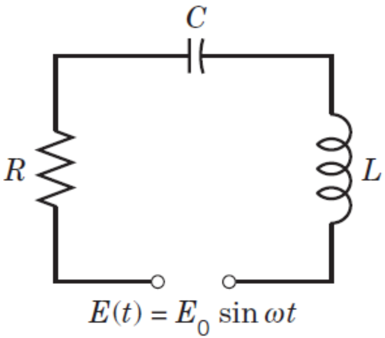
RLC Circuit



Circuit

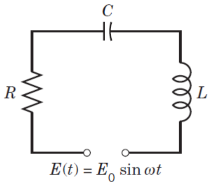
$$I' + \frac{R}{L} I = \frac{E}{L}$$

RLC Circuit



- An RLC-circuit is obtained from an RL-circuit by adding a capacitor
- the voltage drop Q/C across the capacitor

RLC Circuit



$$LI' + RI + \frac{Q}{C} = E(t)$$

$$I = Q' \implies Q = \int I$$

$$LI'' + RI' + \frac{I}{C} = E'(t)$$

RLC Circuit

$$E'(t) = E_0 \omega cos\omega t$$

$$\begin{aligned} A(D) &= D^2 + \omega^2 \\ \implies I_p &= Acos\omega t + Bsin\omega t \\ \implies I_p' &= \omega(-Asin\omega t + Bcos\omega t) \\ \implies I_p'' &= -\omega^2(Acos\omega t + Bsin\omega t) \\ LI_p'' + RI_p' + \frac{I_p}{C} &= E_0 cos\omega t \\ L\omega^2(-A) + R\omega B + \frac{A}{C} &= E_0\omega \\ L\omega^2(-B) + R\omega(-A) + \frac{b}{C} &= 0 \end{aligned}$$

RLC Circuit

$$\text{Reactance}\quad S = \omega L - \frac{1}{\omega C}$$

$$-Sa + Rb = E_0$$

$$-Ra - Sb = 0$$

$$A = \frac{-E_0 S}{R^2 + S^2}$$

$$B = \frac{E_0 R}{R^2 + S^2}$$

$$\text{Impedance} = \sqrt{R^2 + S^2}$$

Thank you for your attention