Tutorial Sheet: MA 5101 - 26-Oct-18

Indian Institute of Technology Tirupati

Department of Mathematics

MA5101 – Mathematics for Engineers – Tutorial Sheet

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First Order ODE

Problem 1: (a) There was a murder in a hotel at room number 315 at 4:30PM. Police arrested Arjun, who was in the next room at 5:00PM. But, Arjun claims that he was not in his room for at least half an hour. The police check the water temperature of his tea kettle in his room at the instant of arrest and again 30 minutes later, obtaining the values 87℃ and 43℃, respectively. Can you investigate the case as an inspector? Is it possible to claim that Arjun is the murderer?

(Hint: Use Newton's Law of Cooling)

(b) If the temperature of a cake is 150°C when it leaves the oven and is 100°C ten minutes later, when will it reach the room temperature 20°C?

Problem 2: (a) In a hostel, there is a cylindrical water tank of diameter 2m and height 2.25m. On a fine day, when Ragu was the first person to take the shower at 7AM in the hostel, the tank was empty. After an inspection by the hostel warden, it was found that there is a circular hole in the water tank. When the hostel watchman switched off the power button of the water tank at 1AM on the previous day, it was completely filled. Without manually measuring the diameter of the hole, could you calculate the diameter of the hole?

(Hint: Torricelli's Law: $v(t) = \sqrt{2gh(t)}\,, \frac{dh}{dt} = -26.56\frac{A}{B}\sqrt{h}$)

(b) The outflow of water from a cylindrical tank with a hole at the bottom. You are asked to find the height of the water in the tank at any time if the tank has diameter 2 m, the hole has diameter 1 cm, and the initial height of the water when the hole is opened is 2.25 m. When will the tank be empty?

Problem 3: (a) One hour before a surgery, certain drug at a constant amount was injected to the patient's blood stream. Certain amount of drug is removed simultaneously to avoid over dosage of drugs which is proportional to the amount of the drug present at time *t*.

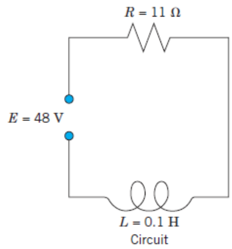
(b) It was found that hormone level of a patient varies w.r.to time. The rate of change of the hormone w.r.to time is the difference between the sinusoidal input of a 24-hour period from thyroid gland and a continuous removal rate proportional to the level. Find and solve the hormone level model?

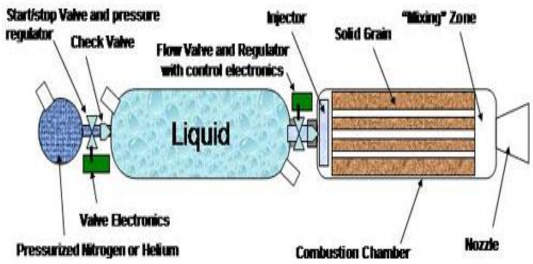
(c) In a room containing 20 cubic m^3 of air, 600 m^3 of fresh air flows in per minute, and the mixture (made practically uniform by circulating fans) is exhausted at a rate of 600 cubic metre per minute. What is the amount of fresh air at any time if there are no initial fresh air? After what time will 90% of the air be fresh?

Problem 4: (a) Model the following RL-Circuit for the current under the assumption that the initial current is zero.

(Hint: Ohm's Law: $V = I\,R$, **Kirchhoff's Voltage Law:** $Voltage\ drop + V = E, LI' + IR = E$)

(a) $R = 11\ \Omega, L = 0.1\ H, E = 110 \sin 10\ t\ V$
(b) $R = 8\ \Omega, L = 0.2\ H,, E = 100 \sin 10\, t\, V$





Problem 5: (a) A hybrid fuel tank in a rocket works on the principle of mixing two different fuel substance for combustion which in turn produces fuel supply for the throttle. The first tank contains 2 million litres of fuel in which another solid fuel substance of 0.18 million kg is dissolved. Each 50 litre of the fuel fed into the throttle after mixing contains (1 + *cos t*) kg of the dissolved solid fuel substance. The mixture is uniform and runs to the throttle at the same rate. What is the amount of solid fuel substance at any time *t*?

(b) Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank contains 1000 litres of water in which initially 100 kg of salt is dissolved. Brine runs in at a rate of 10 litre min, and each litre contains 5 kg of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 litre per min. Find the amount of salt in the tank at any time *t*.

Problem 6: (a) In the city of Hamelin, the rat population was a big problem. It was initially assumed that the rate of change of rat population w.r.to time is equal to twice its population at any given day. Until the pied piper arrived, people killed rats and hence the growth rate of rat population decreased in proportion to the population every day. How many years will be required to have a rat-free city if the initial and 1 year population of the rat are respectively 2 million and 1.5 million?

(b) A model for the spread of contagious diseases is obtained by assuming that the rate of spread is proportional to the number of contacts between infected and non-infected persons, who are assumed to move freely among each other. $y' = k(1 - y)y$

(c) Suppose that the population of a certain kind of fish is given by the logistic equation

$$y' = (A - By)y$$

and fish are caught at a rate *Hy* proportional to *y*. Find the model
Schaefer Model:

$$y' = (A - H - By)y$$

Problem 7: On what interval does each of the following initial value problems have a unique solution?

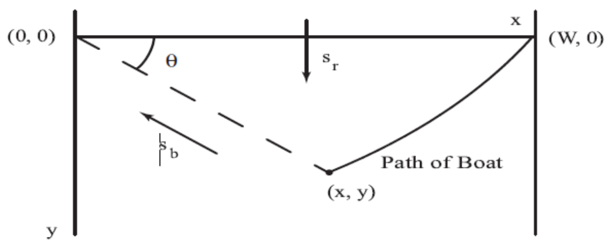
(a) $y' = \frac{y}{(x-1)(x+2)} + \frac{1}{x}, y(-1) = 2$
(b) $y' = (\cot x)y + \tan x\,, y(2) = 3$

Problem 8: (a) Experiments show that at each instant a radioactive substance decomposes and is thus decaying in time-proportional to the amount of substance present. Model it and solve it. (b) A mummy in Egypt was experimented with radio carbon dating to find its age. Approximately, when did the mummy die, if the ration of carbon ${}^{14}_{6}C$ to carbon ${}^{12}_{6}C$ in this mummy is 52.5% of that of a living organism?

Problem 9: The efficiency of the engines of subsonic airplanes depends on air pressure and is usually maximum near 10668 metre. The rate of change of air pressure is proportional to the pressure. At 5486.4 metre height, the air pressure is half of its value at the sea level. Model and solve it

Problem 10: The Gompretz model is $y' = -Ay\ln y, A > 0$, where $y(t)$ is the mass of tumour cells at time t. The declining growth rate with increasing $y > 1$ corresponds to the fact that cells in the interior of tumour may die because of insufficient oxygen and nutrients. Model and solve it.

Problem 11: Hanging Cable:
It can be shown that the curve $y(x)$ of an inextensible flexible homogeneous cable hanging between two fixed points is obtained by solving $y'' = k\sqrt{1+y'^2}$, where the constant k depends on the weight. This curve is called catenary. Find $y(x)$, if $k = 1$ and the curve passes through the fixed points $(-1,0)$ and $(1,0)$ in a vertical xy-plane.



Problem 12: Path of a Boat in a river: The y –axis and the line $x = W > 0$ represent the banks of a river. The river flows in the negative y –direction with speed s_r. A boat whose speed in still water is s_b is launched from the point $(W, 0)$. The boat is stirred so that it is always headed toward the origin. The components of the boat velocity in x- and y-direction are $\frac{dx}{dt} = -s_b cos\theta, \frac{dy}{dt} = -s_r + s_b sin\theta$. Solve for dy/dx, if $W = 0.5km, s_r = 3kmph, s_b = 3kmph$.

Problem 13: Solve the IVP

(a) $xy' = y + 3x^4\cos^2\left(\frac{y}{x}\right), y(1) = 0$

(b) $xy' + 4y = 8x^4, y(1) = 2$

(c) $y' + y = y^2, y(0) = -\frac{1}{3}$

(d) $(2xydx + dy)e^{x^2} = 0, y(0) = 2$

(e) $y' + 2y = 4\cos 2x, y\left(\frac{\pi}{4}\right) = 3$

(f) $y' + \frac{y}{2} = y^3, y(0) = \frac{1}{3}$

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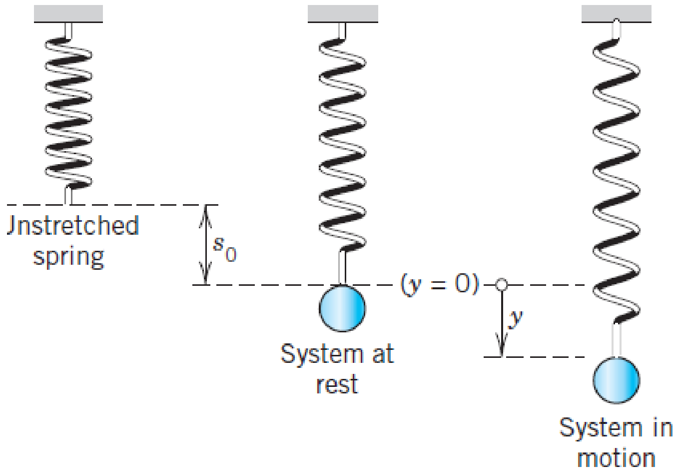
Second Order ODE



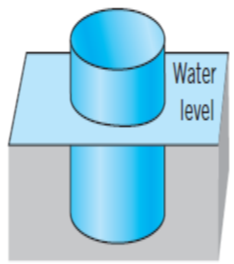
Problem 1: An ordinary coil spring resists extension as well as compression. We attach an iron ball at its lower end. When the system is at rest after attaching the iron ball, we say it as initial position. When we pull the ball down, the system experiences a force. What will be the undamped system?

Hooke's Law:

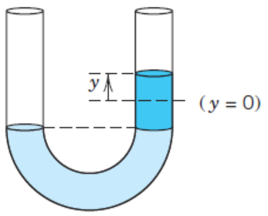
$$F = -ky$$

Newton's Second law: $F = my''$,$my'' + ky = 0$

Problem 2 (a) According to Archimedean principle buoyance force equals the weight of the water displaced by the body. A cylindrical buoy of diameter 60cm is floating in water with its axis vertical. When depressed downward in the water and released, it vibrates with period 2 sec. What is its weight?



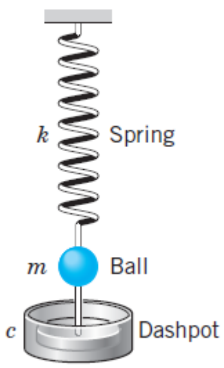
$$y'' + \omega^2 y = 0$$



(b) What is the frequency of vibration of 5 litres of water in a U-Shaped tube of diameter 4cm? (Neglect the friction).

$$y'' + \omega^2 y = 0$$

(i) $m = 10, k = 90$ (ii) $m = 1, k = 3$ (iii) $m = 9, k = -1$



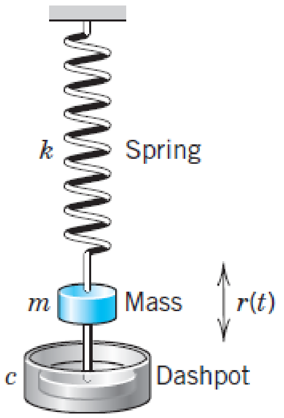
Problem 3: An ordinary coil spring resists extension as well as compression. We attach an iron ball at its lower end. When the system is at rest after attaching the iron ball, we say it as initial position. When we pull the ball down, the system experiences a force. Further, we add a damping force to the system. What will be the damped mass-spring system?

Damped Force

$$F_2 = -cy'$$

$$my'' + cy' + ky = 0$$

(a) $m = 1, k = 4.8, c = 5.76$
(b) $m = 1, k = 3, c = 2.5$
(c) $m = 2, k = 6, c = 4$

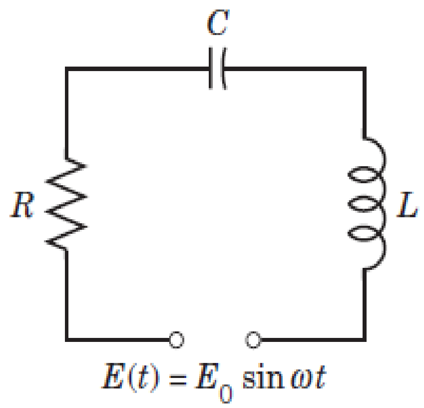


Problem 4: An ordinary coil spring resists extension as well as compression. We attach an iron mass at its lower end. When the system is at rest after attaching the iron ball, we say it as initial position. When we pull the ball down, the system experiences a force. Further, we add a damping force to the system and an additional external force. What will be the mass-spring system?

External Force:

$$my'' + cy' + ky = r(t)$$

(a) $2y'' + 4y' + 6.5y = \cos(1.5t)$
(b) $y'' + 16y = 56 \cos t$



Problem 5: Model the following RLC-Circuit and solve for I

Ohm's Resistor		R
Inductor		L
Capacitor		C

Ohm's Resistance	ohms (Ω)	RI
Inductance	henrys (H)	$L\frac{dI}{dt}$
Capacitance	farads (F)	Q/C

$$LI' + RI + \frac{I}{C} = E(t)$$

$$I = Q' \Rightarrow LQ'' + RQ' + \frac{Q}{C} = E(t)$$

Upon differentiation I w.r.to t, $LI'' + RI' + \frac{I}{C} = E'(t)$

(a) $R = 11\ \Omega, L = 0.1\ H, C = 10^{-2} F, E = 110 \cos 50\, t\ \ V$
(b) $R = 8\ \Omega, L = 0.2\ H, C = 12.5 \times 10^{-3} F, E = 100 \sin 10\, t\, V$
(c) $R = 18\ \Omega, L = 1\ H, C = \frac{1}{250} F, E = 250(cost + \sin t)V$
(d) $R = 16\ \Omega, L = 2.0\ H, C = \frac{1}{200} F, E = 120 \sin 50\, t\, V$
(e) $R = 24\ \Omega, L = 1.2\ H, C = \frac{1}{90} F, E = 220 \sin 5\, t\, V$
(f) $R = 20\Omega, L = 0.4\ H, C = 4 \times 10^{-4} F, E = 0\, V, Q(0) = 2 \times 10^{-3} F, Q'(0) = 0A$

Problem 6: Let P be a particle of mass m, acted upon by an elastic force of attraction $F = -ky$, where y is the position vector, k is the coefficient of elasticity. Describe the motion of the particle.

Problem 7: Shock Absorber. What is the smallest value of the damping constant of a shock absorber in the suspension of a wheel of a car (consisting of a spring and an absorber) that will provide (theoretically) an oscillation free ride if the mass of the car is $2000\ kg$ and the spring constant is $4500\frac{kg}{s^2}$

Problem 8: Reduce to First order and Solve

(a) $yy'' = 3y'^2$
(b) $x^2y'' + xy' - 4y = 0, y_1 = x^2$

Problem 9: Solve the following

(a) $y'' + 2k^2y' + k^4y = 0$
(b) $10y'' - 32y' + 25.6y = 0$
(c) $4y'' - 4y' - 3y = 0$
(d) $y'' - 2y' - 3y = 0, y(-1) = e, y(-1) = -e/4$
(e) $x^2y'' = 2y$
(f) $5x^2y'' + 23xy' + 16.2y = 0$
(g) $x^2y'' + 2xy' - 6y = 0, y(1) = 0.5, y'(1) = 1.5$

Problem 10: Find the Wronskian and show linear independence

(a) $2x, \frac{1}{4x}$ (b) x^3, x^2 (c) $e^{-x}cos\omega x, e^{-x}sin\omega x$

Problem 11: Find a second order homogeneous linear ODE for which the given functions are solutions. Show linear independence by Wronskian. Solve the IVP

(a) $\cos 5x, \sin 5x, y(0) = 3, y'(0) = -5$
(b) $x^2, x^2 \ln x, y(1) = 4, y'(1) = 6$
(c) $1, e^{3x}, y(0) = 2, y'(0) = -1$

Problem 12: Solve the following non-homogeneous linear ODE

(a) $y'' + 5y' + 6y = 2e^{-x}$
(b) $10y'' + 50y' + 57.6y = \cos x$
(c) $y'' + y' + y = 2x \sin x$
(d) $y'' + 6y' + 9y = e^{-x} \cos 2x, y(0) = 1, y'(0) = -1$
(e) $y'' - 4y' + 4y = 5t^3 - 2 - t^2e^{2t} + 4e^{2t} \cos t$

Problem 13: Solve by using variation of Parameters

(a) $y'' + 4y = \cos 2x$
(b) $x^2y'' + xy' - 9y = 48x^5$
(c) $(x^2D^2 - 4xD + 6I)y = 21x^{-4}$