Summary Sheet for PDE

Linear/Nonlinear: If the PDE is of the first degree in the unknown function u and its partial derivatives, then it is linear. Otherwise, nonlinear.

Homogeneous/Nonhomogeneous: If each of the PDE terms contains either u or one of its partial derivatives, then it is homogeneous. Otherwise nonhomogeneous.

Important Second-Order PDEs:

Equation	**One Dimensional**	**Two Dimensional**	**Multi-Dimensional**
Wave	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2(u_{xx} + u_{yy})$	$u_{tt} = c^2 \Delta u$
Heat	$u_t = c^2 u_{xx}$	$u_t = c^2(u_{xx} + u_{yy})$	$u_t = c^2 \Delta u$
Laplace		$u_{xx} + u_{yy} = 0$	$\Delta u = 0$
Poisson		$u_{xx} + u_{yy} = f(x, y)$	$\Delta u = f$

Here, $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$, if u is a function of more than one variable, say $x_1, x_2, x_3, .., x_n$.

Wave Equation Solution (1-D): By Variable Separation Method

$$u_{tt} = c^2 u_{xx},$$

Boundary Condition: $u(0, t) = u(L, t) = 0, t \geq 0,$

Initial Conditions: $u(x, 0) = f(x), u_t(x, 0) = g(x), 0 \leq x \leq L$

Step 1: Assume $u(x, t) = F(x)G(t)$

Step 2: Obtain u_{tt}, u_{xx}. Find two ODE's with BVP that satisfy the BCs. Solve two ODEs.

Step 3: Using the Fourier series, compose solution that satisfy both BCs and ICs.

Wave Equation Solution (1-D): d'Alemberts' Solution

Step 1: Introduce variable $v = x + ct, w = x - ct$

Step 2: Transform wave equation from (x, t) to (v, w) and obtain $u_{vw} = 0$

Step 3: Solve and use initial conditions to obtain

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

Types of PDEs

Consider the following quasilinear PDE

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} = \psi(x, y, u, u_x, u_y)$$

Depending on the discriminant it is classified into three types

Type	**Condition**
Hyperbolic	$B^2 - 4AC > 0$
Parabolic	$B^2 - 4AC = 0$
Elliptic	$B^2 - 4AC < 0$

Reduce to Canonical Form

Follow these steps to obtain Canonical Form for

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = \psi(x,y,u,u_x,u_y)$$

Step 1: Introduce the variable $\xi = \xi(x,y), \eta = \eta(x,y)$

Step 2: Transform from (x,y) to (ξ,η) and obtain $au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} = \phi(\xi,\eta,u,u_\xi,u_\eta)$

here, $a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$ $\qquad b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$

$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$ $\qquad J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$

$b^2 - 4ac = J^2(B^2 - 4AC)$ shows that if PDE is either hyperbolic or parabolic or elliptic, then its canonical form is of the same type

Type	**Cases**	**Canonical Form**
Hyperbolic	$a = c = 0$	$u_{\xi\eta} = \phi(\xi,\eta,u,u_\xi,u_\eta)$
Hyperbolic	$b = 0, c = -a$	$u_{\alpha\alpha-\beta\beta} = \phi(\alpha,\beta,u,u_\alpha,u_\beta), \alpha = \xi + \eta, \beta = \xi - \eta$
Parabolic	$a = b = 0$	$u_{\eta\eta} = \phi(\xi,\eta,u,u_\xi,u_\eta)$
Elliptic	$b = 0, c = a$	$u_{\xi\xi} + u_{\eta\eta} = \phi(\xi,\eta,u,u_\xi,u_\eta)$

(1) Step 3 for Hyperbolic Equations

$a = c = 0$: $A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0, A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0$$

$$\mu_1 = \frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \mu_2 = \frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

Along the coordinate line $\xi(x,y) = 0, d\xi = \xi_x dx + \xi_y dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = -\mu_1 \Rightarrow y = -\int \mu_1 dx + c_1 \Rightarrow \xi = y + \int \mu_1 dx$$

$$\xi = y + \int \frac{-B + \sqrt{B^2 - 4AC}}{2A} dx$$

Similarly, $\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = -\mu_2 \Rightarrow \eta = y + \int \mu_2 dx = y + \int \frac{-B-\sqrt{B^2-4AC}}{2A} dx$

(2) Step 3 for Parabolic Equations

$a = 0$: $A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$

$$\Rightarrow \mu_1 = \frac{\xi_x}{\xi_y} = \frac{-B}{2A} \Rightarrow \frac{dy}{dx} = \frac{B}{2A} \Rightarrow y = \int \frac{B}{2A} dx + c_1 \Rightarrow \xi = y - \int \frac{B}{2A} dx$$

$b = 0$: $2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 0$

$$\Rightarrow 2A\left(-\frac{B}{2A}\right)\eta_x + B\left(-\frac{B}{2A}\eta_y + \eta_x\right) + 2C\eta_y = 0$$

$\Rightarrow (B^2 - 4AC)\eta_y = 0 \Rightarrow \eta(x, y)\ is\ function\ of\ x.$ Choose $\eta = x$

(3) Step 3 for Elliptic Equations

$a = c \Rightarrow a - c = 0 \Rightarrow A(\xi_x^2 - \eta_x^2) + B(\xi_x\xi_y - \eta_x\eta_y) + C(\xi_y^2 - \eta_y^2) = 0$

$b = 0 \Rightarrow 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$

$$A\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right)^2 + B\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right) + C = 0$$

$$\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Let $\alpha = \xi(x, y) + i\eta(x, y), \beta = \xi(x, y) - i\eta(x, y),$

$$\mu_1 = -\frac{\alpha_x}{\alpha_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \mu_2 = -\frac{\beta_x}{\beta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

Along the coordinate line $\alpha(x, y) = 0, d\alpha = \alpha_x dx + \alpha_y dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{\alpha_x}{\alpha_y} = -\mu_1 \Rightarrow y = -\int \mu_1 dx + c_1 \Rightarrow \alpha = y + \int \mu_1 dx$$

Similarly, $\frac{dy}{dx} = -\frac{\beta_x}{\beta_y} = -\mu_2 \Rightarrow \beta = y + \int \mu_2 dx$

$$\xi = \frac{\alpha + \beta}{2} = y + \int \frac{-B}{2A} dx, \eta = \frac{\alpha - \beta}{2i} = \int \frac{\sqrt{4AC - B^2}}{2A} dx$$

Step 4: Using the ξ and η obtained from step 3, transform the given PDE to canonical form

Heat Equation Solution (1-D): By Variable Separation Method

$$u_t = c^2 u_{xx},$$

BC: $u(0, t) = u(L, t) = 0, t \geq 0$, IC: $u(x, 0) = f(x), 0 \leq x \leq L$

Step 1: Assume $u(x, t) = F(x)G(t)$

Step 2: Obtain u_t, u_{xx}. Find two ODE's with BVP that satisfy the BCs. Solve two ODEs.

Step 3: Using the Fourier series, compose solution that satisfy both BCs and ICs.

Laplace Equation Solution (2-D): By Variable Separation Method

$$u_{xx} + u_{yy} = 0\ \ on\ a\ rectangle\ R = [0, a] \times [0, b]$$

Boundary Condition: $u(x, b) = f(x), u(0, y) = u(a, y) = u(x, 0) = 0$

Step 1: Assume $u(x, t) = F(x)G(y)$

Step 2: Obtain u_{xx}, u_{yy}. Find two ODE's with BVP that satisfy the BCs. Solve two ODEs.

Step 3: Using the Fourier series, compose solution that satisfy both BCs and ICs.

Wave Equation Solution (2-D): By Variable Separation Method

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

Boundary Condition: $u(x, b, t) = u(0, y, t) = u(a, y, t) = u(x, 0, t) = 0$

Initial Conditions: $u(x,y,0) = f(x,y), u_t(x,y,0) = g(x,y), 0 \leq x \leq a, 0 \leq y \leq b$

Step 1: Assume $u(x,t) = F(x,y)G(t), F(x,y) = H(x)Q(y)$

Step 2: Obtain u_{tt}, u_{xx}, u_{yy} and $F\ddot{G} = c^2(F_{xx}G + F_{yy}G)$

Step 2(a): Solve $\ddot{G} + \lambda^2 G = 0, \lambda = cv$

Step 2(b): Solve the Helmholtz equation $F_{xx} + F_{yy} + v^2 F = 0$ using $F(x,y) = H(x)Q(y)$

Step 2(c): Obtain $H'' + k^2 H = 0, Q'' + p^2 Q = 0, p^2 = v^2 - k^2$

Step 3: Using the boundary conditions, obtain $F_{mn}(x,y)$

Step 4: Using Double Fourier series, obtain the solution that satisfy both BCs and ICs.

Laplace equation in Polar Coordinates

$$r = \sqrt{x^2 + y^2}, tan\theta = \frac{y}{x}$$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Wave Equation Solution (2-DPolar Coordinate): By Variable Separation Method

$$u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta})$$

BC: $u(R,t) = 0$, IC: $u(r,0) = f(r), u_t(r,0) = g(r)$

Step 1: Assume $u(r,t) = W(r)G(t)$

Step 2: Obtain u_{tt}, u_{rr} and $\frac{\ddot{G}}{c^2 G} = \frac{1}{W}\left(W'' + \frac{1}{r}W'\right) = -k^2$

Step 2(a): Solve $\ddot{G} + \lambda^2 G = 0, \lambda = ck$

Step 2(b): Solve the Helmholtz equation $W'' + \frac{1}{r}W' + k^2 W = 0$. Bessel Equation

Step 3: Using the boundary conditions, $W(r)$

Step 4: Using Fourier-Bessel Series, obtain the solution

Laplace equation in Cylindrical Coordinates

$$x = rcos\theta, y = rsin\theta, z = z$$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$$

Laplace equation in Spherical Coordinates.

$$x = rcos\theta sin\phi, y = rsin\theta sin\phi, z = rcos\phi$$

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\phi\phi} + \frac{cot\phi}{r^2}u_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}$$