Second order ODE:

$$f(t, y, y', y'') = 0$$

Second order linear ODE (homogeneous):

$$L(y, y', y'') = 0$$

General Second order linear ODE:

$$y'' + p(x)y' + q(x)y = r(x)$$

Superposition Principle:

If y_1, y_2 are solution of the homogeneous equation, that is

$$L(y_1, y_1', y_1'') = y_1'' + p(x)y_1' + q(x)y_1 = 0$$

$$L(y_2, y_2', y_2'') = y_2'' + p(x)y_2' + q(x)y_2 = 0$$

Then, $y = \alpha y_1 + \beta y_2$ also satisfies $L(y, y', y'') = 0$.

Existence and uniqueness:

Assume p, q, r continuous in $[x_0, x_1]$, then there exists a unique solution to the IVP: $L(y, y', y'') = r, y(x_0) = y_0, y(x_1) = y_1$.

Second Order to System:

$$y' = v$$

$$v' = y'' = -py' - qy + r = -pv - qy + r$$

$$\underbrace{\begin{pmatrix} y \\ v \end{pmatrix}'}_{X'} = \underbrace{\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y \\ v \end{pmatrix}}_{X} + \underbrace{\begin{pmatrix} 0 \\ r \end{pmatrix}}_{B}$$

Initial Value Conditions:

$$\underbrace{\begin{pmatrix} y \\ v \end{pmatrix}'}_{X'} = \underbrace{\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y \\ v \end{pmatrix}}_{X} + \underbrace{\begin{pmatrix} 0 \\ r \end{pmatrix}}_{B}$$

$$y(x_0) = y_0, v(x_0) = y'(x_0) = y_1$$

Boundary Value Problem:

$L = r(x)$ in an interval $[a, b], y(a), y(b)$ can be prescribed.

Homogeneous LSDE:

Not possible to find solution always

Proposition:

Let y_1, y_2 be any two solutions of HLSDE, then $\alpha y_1 + \beta y_2$ is also a solution of HLSDE, for any $\alpha, \beta \in R$. Suppose y_1, y_2 are independent solution of HLSDE, then any solution can be written in the form $y = \alpha y_1 + \beta y_2$,for some $\alpha, \beta \in R$.

Proposition:

$S =$ The set of all solutions of the HSLDE, then S is a linear space and $\dim S \leq 2$.

Proposition: $W \equiv 0$ if and only if y_1, y_2 are dependent.

Theorem: $\dim S = 2$

Method 1: For solving $L(y, y', y'') = 0$

$$y = uv$$

$$y' = u'v + uv', y'' = u''v + 2u'v' + uv''$$

$$y'' + p(x)y' + q = u''v + 2u'v' + uv'' + p(u'v + uv') + quv = 0$$

Choose u so that v' term vanishes

$$2u' + pu = 0 \to u = e^{-\frac{1}{2}\int p\, dx}$$

Method 2: Order of Reduction

If one solution is known, then it is possible to find the second solution. Assume y_1 is known. Then ky_1 is also a solution. But not independent, so take $y_2 = c(x)y_1$

Compute y_2', y_2'' and solve $Ly_2 = 0$

$$c''(x)y_1 + c'(x)(2y_1' + py_1) = 0$$

$$\frac{c''}{c'} = -\frac{2y_1' + py_1}{y_1} \to \frac{v'}{v} = -\frac{2y_1'}{y_1} - p$$

Solve for v, then solve for $v = c'$