1. Mass-spring systems is given by

$$my'' + cy' + ky = r(t)$$

Here $m = 1, c = 2, k = 1, r(t) = te^{-t}$

$$y'' + 2y' + y = te^{-t}$$

The auxiliary equation is

$$s^2 + 2s + 1 = 0 \Rightarrow s = -1, -1$$

The two independent solutions are

$$y_1(t) = e^{-t}, y_2(t) = te^{-t}$$

[1 Mark]

Green's function is given by

$$G(t, x) = \frac{y_1(x)y_2(t) - y_1(t)y_2(x)}{W(x)}$$

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x} \begin{vmatrix} 1 & x \\ -1 & 1 - x \end{vmatrix} = e^{-2x}(1 - x + x) = e^{-2x}$$

[1 Mark]

$$G(t, x) = \frac{e^{-x}te^{-t} - e^{-t}xe^{-x}}{e^{-2x}} = \frac{e^{-x}e^{-t}(t - x)}{e^{-2x}} = e^{x-t}(t - x)$$

[1 Mark]

The system is at rest after attaching the ball, we say it as initial value problem. That is $y(t_0) = 0$ or $y(0) = 0$

$$y_p(t) = \int_{t_0}^{t} G(t, x)r(x)dx = \int_{t_0}^{t} e^{x-t}(t - x)xe^{-x}dx$$

[1 Mark]

$$y_p(t) = e^{-t}\int_{t_0}^{t}(xt - x^2)dx = e^{-t}\left[\frac{x^2}{2}t - \frac{x^3}{3}\right]_{t_0}^{t} = e^{-t}\left[\frac{t^3}{2} - \frac{t^3}{3} - \frac{t_0^2 t}{2} + \frac{t_0^3}{3}\right] = e^{-t}\left(\frac{t^3}{6} - \frac{t_0^2 t}{2} + \frac{t_0^3}{3}\right)$$

or

$$y_p(t) = \frac{e^{-t}t^3}{6}$$

[1 Mark]

2.

$$\sin\theta \frac{d^2y}{d\theta^2} + \cos\theta \frac{dy}{d\theta} + n(n+1)\sin\theta\, y = 0$$

$$\Rightarrow \sin\theta \frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) + \frac{d}{d\theta}(\sin\theta)\frac{dy}{d\theta} + n(n+1)\sin\theta\, y = 0$$

$$\Rightarrow \frac{d}{d\theta}\left(\sin\theta \frac{dy}{d\theta}\right) + n(n+1)\sin\theta\, y = 0$$

Let $x = cos\theta$ [0.5 Marks]

$$\frac{d}{d\theta} = \frac{d}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{d}{dx} \Rightarrow \frac{dy}{d\theta} = -\sin\theta\frac{dy}{dx} \Rightarrow \sin\theta\frac{dy}{d\theta} = -\sin^2\theta\frac{dy}{dx} = -(1 - x^2)\frac{dy}{dx}$$

[0.5 Marks]

$$\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right) = -\sin\theta\frac{d}{dx}\left(-(1 - x^2)\frac{dy}{dx}\right) = \sin\theta\frac{d}{dx}\left((1 - x^2)\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right) + n(n+1)\sin\theta\, y = \sin\theta\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + n(n+1)\sin\theta\, y$$

$$\Rightarrow \frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + n(n+1)y = 0$$

Which is the Legendre's equation

[1 Marks]

The solution of this ODE is given by $P_n(\cos\theta)$

For $n = 1$

Recurrence Relation

$$P_{k+1}(x) = \frac{2k+1}{k+1}xP_k(x) - \frac{k}{k+1}P_{k-1}(x)$$

$$P_0(x) = 1 \Rightarrow P_0(\cos\theta) = 1$$

$$P_1(x) = x \Rightarrow P_1(\cos\theta) = \cos\theta$$

$$k = 1 \Rightarrow P_2(x) = \frac{3}{2}xP_1(x) - \frac{1}{2}P_0(x)$$

$$\Rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$k = 2 \Rightarrow P_3(x) = \frac{5}{3}xP_2(x) - \frac{2}{3}P_1(x) = \frac{5}{3}\left(\frac{3x^3}{2} - \frac{x}{2}\right) - \frac{2}{3}x = \frac{5x^3}{2} - \frac{x}{3}\left(\frac{5}{2} + 2\right)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

$$k = 3 \Rightarrow P_4(x) = \frac{7}{4}xP_3(x) - \frac{3}{4}P_2(x) = \frac{7}{8}(5x^4 - 3x^2) - \frac{3}{8}(3x^2 - 1)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \Rightarrow P_4(\cos\theta) = \frac{1}{2}(35\cos^4\theta - 30\cos^2\theta + 3)$$

$$k = 4 \Rightarrow P_5(x) = \frac{9}{5}xP_4(x) - \frac{4}{5}P_3(x) = \frac{9}{40}(35x^5 - 30x^3 + 3x) - \frac{4}{10}(5x^3 - 3x)$$

$$= \frac{1}{40}(315x^5 - 350x^3 + 75x)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \Rightarrow P_5(\cos\theta) = \frac{1}{2}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)$$

$$k = 5 \Rightarrow P_6(x) = \frac{11}{6}xP_5(x) - \frac{5}{6}P_4(x) = \frac{11}{48}(63x^6 - 70x^4 + 15x^2) - \frac{5}{48}(35x^4 - 30x^2 + 3)$$

$$= \frac{1}{48}(693x^6 - 945x^4 + 315x^2 - 15)$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$\Rightarrow P_6(\cos\theta) == \frac{1}{16}(231\cos^6\theta - 315\cos^4\theta + 105\cos^2\theta - 5)$$

3. Recurrence Relation

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x)$$

Or

$$xJ_{\nu-1}(x) + xJ_{\nu+1}(x) = 2\nu J_\nu(x)$$

[0.5 Mark]

$$\nu = \frac{1}{2} \Rightarrow J_{\frac{1}{2}} = xJ_{\frac{3}{2}} + xJ_{-\frac{1}{2}} \Rightarrow J_{\frac{3}{2}} = \frac{1}{x}J_{\frac{1}{2}} - J_{-\frac{1}{2}}$$

[0.5 Mark]

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x\,, J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

$$J_{\frac{3}{2}} = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$\nu = -\frac{1}{2} \Rightarrow J_{\frac{1}{2}} = -\frac{1}{x}J_{-\frac{1}{2}} - J_{-\frac{3}{2}} \Rightarrow J_{-\frac{3}{2}} = -\frac{1}{x}J_{-\frac{1}{2}} - J_{\frac{1}{2}}$$

$$J_{-\frac{3}{2}} = -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x$$

[0.5 Mark]

$$F(x, \sin x\,, \cos x) = J_{\frac{3}{2}} + \frac{1}{\pi}J_{-\frac{3}{2}} = \sqrt{\frac{2}{\pi x}}\sin x\left(\frac{1}{x} - \frac{1}{\pi}\right) - \sqrt{\frac{2}{\pi x}}\cos x\left(1 + \frac{1}{\pi}\frac{1}{x}\right)$$

[0.5 Mark]

$$y\left(\frac{\pi}{2}\right) = \frac{2}{\pi}\left(\frac{2}{\pi} - \frac{1}{\pi}\right) = \frac{2}{\pi^2}$$

[1 Mark]

4. $(x^2 + 1)y'' + xy' - y = 0$

Singular points at $x = \pm i$. Remaining points are ordinary points. Both are regular singular points as $(x \pm i)p(x) = \frac{x}{x \mp i}, (x \pm i)^2 q(x) = -\frac{(x \pm i)}{(x \mp i)}$

[1 Mark]

Therefore, power series solution at 0 will converge $|x| < 1$.

$$(x^2 + 1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} na_n x^n + a_1 x - \sum_{n=2}^{\infty} a_n x^n - a_0 - a_1 x$$

Replace n by $n + 2$ in second sum

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} x^n + 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} na_n x^n - \sum_{n=2}^{\infty} a_n x^n - a_0$$

$$= \sum_{n=2}^{\infty} [n(n-1)a_n + (n+2)(n+1)a_{n+2} + na_n - a_n]x^n + 2a_2 + 6a_3 x - a_0$$

$$n(n-1)a_n + (n+2)(n+1)a_{n+2} + (n-1)a_n = 0$$

[1 Mark]

$$(n+1)(n-1)a_n + (n+2)(n+1)a_{n+2} = 0$$

$$(n-1)a_n + (n+2)a_{n+2} = 0$$

$$a_{n+2} = \frac{1-n}{n+2} a_n$$

$$a_2 = \frac{a_0}{2}, a_3 = 0$$

Odd terms are zero after except $n = 1$

$$y_1(x) = a_1 x$$

$$a_4(x) = -\frac{1}{4} a_2 = -\frac{1}{2^2 2!} a_0$$

$$a_6 = -\frac{3}{6} a_4 = \frac{3}{2.4.6} a_0 = \frac{1.3}{2^3 3!} a_0$$

$$a_{2n} = \frac{(-1)^{n-1}\big(1.3.5 \ldots (2n-3)\big)}{2^n n!}$$

$$y_2(x) = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\big(1.3.5 \ldots (2n-3)\big)}{2^n n!} x^{2n}$$

[1 Mark]

Second Problem

$$3xy'' + y' - y = 0$$

$x = 0$ is a singular point and it is a regular singular point, remaining points are ordinary points. It satisfies the conditions of Frobenius theorem, therefore, we apply the Frobenius method.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= 3 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} (n+r)(3n+3r-3+1)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= x^r \left[\sum_{n=0}^{\infty} (n+r)(3n+3r-2)a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \right]$$

$$= x^r \left[r(3r-2)x^{-1} + \sum_{n=1}^{\infty} (n+r)(3n+3r-2)a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \right]$$

Replace n by $n+1$ in first sum

$$= x^r \left[r(3r-2)a_0 x^{-1} + \sum_{n=0}^{\infty} (n+r+1)(3n+3r+1)a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n \right]$$

$$= x^r \left[r(3r-2)a_0 x^{-1} + \sum_{n=0}^{\infty} [(n+r+1)(3n+3r+1)a_{n+1} - a_n] x^n \right]$$

Indicial equation $r(3r-2) = 0 \Rightarrow r = 0, \frac{2}{3}$

[1 Mark];

$$a_{n+1} = \frac{a_n}{(n+r+1)(3n+3r+1)}$$

$r = 0,$

$$a_{n+1} = \frac{a_n}{(n+1)(3n+1)}$$

[1 Mark]

$r = \frac{2}{3},$

$$a_{n+1} = \frac{a_n}{\left(n+\frac{2}{3}+1\right)3(n+1)} = \frac{a_n}{(3n+5)(n+1)}$$

[1 Mark]

For $r = 0$

$$a_1 = \frac{a_0}{1.1}$$

$$a_2 = \frac{a_1}{2.4} = \frac{a_0}{2!\,1.4}$$

$$a_n = \frac{a_0}{n!\,1.4.7\ldots(3n-2)}$$

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{a_0}{n!\,1.4.7\ldots(3n-2)} x^n$$

For $r = \frac{2}{3}$

$$a_1 = \frac{a_0}{5.1}$$

$$a_2 = \frac{a_1}{2.8} = \frac{a_0}{2!\,5.8}$$

$$a_n = \frac{a_0}{n!\,5.8.11\ldots(3n+2)}$$

$$y_2(x) = x^{2/3} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!\,5.8.11\ldots(3n+2)} x^n \right]$$

It converges for all x

5.

a. $u_{xx} + x^2 u_{yy} = 0$

It is elliptic everywhere except $x = 0$. At $x = 0$, it is parabolic.

[0.5 Marks]

For elliptic region

$$A = 1, B = 0, C = x^2$$

$$\sqrt{4AC - B^2} = 2x$$

$$\xi = y + \int -\frac{B}{2A}dx = y$$

[0.5 Mark]

$$\eta = \int \frac{\sqrt{4AC - B^2}}{2A}dx = \int xdx = \frac{x^2}{2}$$

[0.5 Mark]

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$
$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$
$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$
$$\xi_x = 0, \xi_{xx} = 0, \xi_y = 1\,, \xi_{yy} = 0$$
$$\eta_x = x, \eta_{xx} = 1, \eta_y = 0, \eta_{yy} = 0$$

[1 Mark]

$$u_{xx} = x^2 u_{\eta\eta} + u_\eta = 2\eta u_{\eta\eta} + u_\eta$$
$$u_{yy} = u_{\xi\xi}$$
$$x^2 u_{yy} = 2\eta u_{\xi\xi}$$
$$u_{xx} + x^2 u_{yy} = 0 \Rightarrow 2\eta\left(u_{\xi\xi} + u_{\eta\eta}\right) + u_\eta = 0$$
$$\Rightarrow \left(u_{\xi\xi} + u_{\eta\eta}\right) = -\frac{u_\eta}{2\eta}$$

b. $u_{xx} - xu_{yy} = 0$

This problem is hyperbolic for $x > 0$, elliptic for $x < 0$, parabolic for $x = 0$

$$A = 1, B = 0, C = -x$$
$$B^2 - 4AC = 4x$$

[0.5 Marks]

For hyperbolic region

$$\xi = y + \int \frac{-B + \sqrt{B^2 - 4AC}}{2A}dx = y + \int \sqrt{x}dx = y + \frac{2}{3}x^{\frac{3}{2}}$$

[0.5 Mark]

$$\eta = y + \int \frac{-B - \sqrt{B^2 - 4AC}}{2A}dx = y - \int \sqrt{x}dx = y - \frac{2}{3}x^{\frac{3}{2}}$$
$$\xi - \eta = \frac{4}{3}x^{\frac{3}{2}}$$

[0.5 Mark]

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$
$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$
$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$
$$\xi_x = \sqrt{x}, \xi_{xx} = \frac{1}{2\sqrt{x}}, \xi_y = 1\,, \xi_{yy} = 0$$
$$\eta_x = -\sqrt{x}, \eta_{xx} = -\frac{1}{2\sqrt{x}}, \eta_y = 1, \eta_{yy} = 0$$

[1 Mark]

$$u_{xx} = xu_{\xi\xi} - 2xu_{\xi\eta} + xu_{\eta\eta} + \frac{1}{2\sqrt{x}}u_\xi - \frac{1}{2\sqrt{x}}u_\eta$$
$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$
$$-xu_{yy} = -x(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$$u_{xx} - xu_{yy} = 0 \Rightarrow \frac{1}{2\sqrt{x}}\left[\left(u_\xi - u_\eta\right) + 2x^{\frac{3}{2}}\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta}\right)\right] = 0$$

$$\Rightarrow \left(u_\xi - u_\eta\right) + \frac{3}{2}(\xi - \eta)\left(-4u_{\xi\eta}\right) = 0$$

$$u_{\xi\eta} = \frac{\left(u_\xi - u_\eta\right)}{6(\xi - \eta)}$$

c. $u_{xx} - x^2 u_{yy} = 0$

This problem is hyperbolic for all $x \neq 0$, and parabolic for $x = 0$

$$A = 1, B = 0, C = -x^2$$

$$B^2 - 4AC = 4x^2$$

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \pm x$$

[0.5 Marks]

For hyperbolic region

$$\xi = y + \int \frac{-B + \sqrt{B^2 - 4AC}}{2A} dx = y + \int x dx = y + \frac{x^2}{2}$$

[0.5 Mark]

$$\eta = y + \int \frac{-B - \sqrt{B^2 - 4AC}}{2A} dx = y - \int x dx = y - \frac{x^2}{2}$$

$$\xi - \eta = x^2$$

[0.5 Mark]

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$

$$\xi_x = x, \xi_{xx} = 1, \xi_y = 1\ , \xi_{yy} = 0$$

$$\eta_x = -x, \eta_{xx} = -1, \eta_y = 1, \eta_{yy} = 0$$

[1 Mark]

$$u_{xx} = x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} + u_\xi - u_\eta$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$-x^2 u_{yy} = -x^2\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right)$$

$$u_{xx} - x^2 u_{yy} = 0 \Rightarrow \left[\left(u_\xi - u_\eta\right) + x^2\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta}\right)\right] = 0$$

$$\Rightarrow \left(u_\xi - u_\eta\right) + (\xi - \eta)\left(-4u_{\xi\eta}\right) = 0$$

$$u_{\xi\eta} = \frac{\left(u_\xi - u_\eta\right)}{4(\xi - \eta)}$$

6. Using separating variables

$$u(x,t) = F(x)G(t)$$

$$F'' - kF = 0$$

$$\ddot{G} - 4kG = 0$$

$$u_x(0,t) = F'(0)G(t) = 0, F'(0) = 0$$

For $k = 0$

$$F(x) = ax + b$$

$$F'(0) = 0 \Rightarrow a = 0 \Rightarrow F(x) = b$$

$$\ddot{G} = 0 \Rightarrow G = ct + d$$

Therefore,

$$u(x,t) = b(ct + d)$$
$$u(x,0) = x^2 e^{-x} \Rightarrow bd = x^2 e^{-x}$$
$$u_t(x,0) = \frac{x^2}{\sqrt{x^2 + 4}} \Rightarrow bc = \frac{x^2}{\sqrt{x^2 + 4}}$$

[1 Mark]

Assume $k = \mu^2$

$$F(x) = Ae^{\mu x} + Be^{-\mu x}$$
$$F'(x) = A\mu e^{\mu x} - \mu B e^{-\mu x}$$
$$F'(0) = 0 \Rightarrow \mu(A - B) = 0 \Rightarrow \mu = 0 \; or \; A = B$$

$\mu = 0 \Rightarrow k = 0$. So, let assume $\mu \neq 0$

$$F(x) = A(e^{\mu x} + e^{-\mu x})$$
$$\ddot{G} - 4\mu^2 G = 0 \Rightarrow G = Ce^{2\mu t} + De^{-2\mu t}$$
$$u(x,t) = A(e^{\mu x} + e^{-\mu x})(Ce^{2\mu t} + De^{-2\mu t})$$
$$u(x,0) = ACe^{\mu x} + ADe^{-\mu x} = x^2 e^{-x}$$
$$u_t(x,0) = 2AC\mu e^{\mu x} - 2AD\mu e^{-\mu x} = \frac{x^2}{\sqrt{x^2 + 4}}$$
$$AC = \frac{e^{-\mu x}}{3\mu}\left[\frac{x^2}{\sqrt{x^2 + 4}} - 2\mu x^2 e^{-x}\right]$$
$$AD = -\frac{e^{\mu x}}{3\mu}\left[\frac{x^2}{\sqrt{x^2 + 4}} - 2\mu x^2 e^{-x}\right]$$

Assume $k = -p^2$

$$F(x) = A\cos\, px + B\sin px$$
$$F'(x) = -pA\sin px + p\,B\cos px$$
$$F'(0) = 0 \Rightarrow pB = 0$$

Either $p = 0$ or $B = 0$

$p = 0 \Rightarrow k = 0$. So, let assume $p \neq 0$

$$F(x) = A\cos px$$

$$\ddot{G} + 4p^2 G = 0 \Rightarrow G = C\cos 2pt + D\sin 2pt$$

$$u_p(x,t) = \cos px\left(C_p \cos 2pt + D_p \sin 2pt\right)$$

For all p. Here we assumed that $A = 1$

Since it is true for any p, we get a Fourier integral representation of the solution which is written as

$$u(x,t) = \int_0^\infty \cos px\left(C_p \cos 2pt + D_p \sin 2pt\right) dp$$
$$u(x,0) = x^2 e^{-x} \Rightarrow \int_0^\infty \cos px\, C_p\, dp = x^2 e^{-x} \Rightarrow C_p = \frac{2}{\pi}\int_0^\infty \cos pv\, v^2 e^{-v}\, dv$$

$$u_t(x,0) = \frac{x^2}{\sqrt{x^2+4}} \Rightarrow \int_0^\infty \cos px\, 2pD_p\, dp = \frac{x^2}{\sqrt{x^2+4}} \Rightarrow D_p = \frac{2}{p}\int_0^\infty \cos pv \frac{v^2}{\sqrt{v^2+4}}\, dv$$

However, the solution diverges as $x \to \infty$

7. $x = r\cos\theta\sin\phi\,, y = r\sin\theta\sin\phi\,, z = r\cos\phi$

   Method 1:

$$x_r = \cos\theta\sin\phi = \frac{x}{r}, y_r = \sin\theta\sin\phi = \frac{y}{r}, z_r = \cos\phi = \frac{z}{r}$$
$$x_\theta = -r\sin\theta\sin\phi = -y, y_\theta = r\cos\theta\sin\phi = x, z_\theta = 0$$
$$x_\phi = r\cos\theta\cos\phi\,, y_\phi = r\sin\theta\cos\phi\,, z_\phi = -r\sin\phi$$
$$u_r = u_x x_r + u_y y_r + u_z z_r$$
$$ru_r = u_x r\cos\theta\sin\phi + u_y r\sin\theta\sin\phi + u_z r\cos\phi$$
$$u_\theta = u_x x_\theta + u_y y_\theta$$
$$u_\theta = -u_x r\sin\theta\sin\phi + u_y r\cos\theta\sin\phi$$
$$u_\phi = u_x x_\phi + u_y y_\phi + u_z z_\phi$$
$$u_\phi = u_x r\cos\theta\cos\phi + u_y r\sin\theta\cos\phi - u_z r\sin\phi$$

1.5 Marks

$$x_{rr} = 0, y_{rr} = 0, z_{rr} = 0$$
$$x_{\theta\theta} = -y_\theta = -x, y_{\theta\theta} = x_\theta = -y, z_{\theta\theta} = 0$$
$$x_{\phi\phi} = -r\cos\theta\sin\phi = -x, y_{\phi\phi} = -r\sin\theta\sin\phi = -y, z_{\phi\phi} = -r\cos\phi = -z$$
$$u_{rr} = \left(u_x x_r + u_y y_r + u_z z_r\right)_r$$
$$u_{rr} = u_{xx}x_r^2 + u_{yy}y_r^2 + u_{zz}z_r^2 + 2u_{xy}x_r y_r + 2u_{xz}x_r z_r + 2u_{yz}y_r z_r + u_x x_{rr} + u_y y_{rr} + u_z z_{rr}$$
$$u_{\theta\theta} = u_{xx}x_\theta^2 + u_{yy}y_\theta^2 + 2u_{xy}x_\theta y_\theta + u_x x_{\theta\theta} + u_y y_{\theta\theta}$$
$$u_{\phi\phi} = u_{xx}x_\phi^2 + u_{yy}y_\phi^2 + u_{zz}z_\phi^2 + 2u_{xy}x_\phi y_\phi + 2u_{xz}x_\phi z_\phi + 2u_{yz}y_\phi z_\phi + u_x x_{\phi\phi} + u_y y_{\phi\phi} + u_z z_{\phi\phi}$$
$$u_{rr} = u_{xx}(\cos^2\theta\sin^2\phi) + u_{yy}(\sin^2\theta\sin^2\phi) + u_{zz}\cos^2\phi + 2u_{xy}(\cos\theta\sin\theta\sin^2\phi) + 2u_{xz}\cos\theta\sin\phi\cos\phi + 2u_{yz}\sin\theta\sin\phi\cos\phi$$

1 Mark

$$u_{\theta\theta} = u_{xx}r^2\sin^2\theta\sin^2\phi + u_{yy}r^2\cos^2\theta\sin^2\phi - 2u_{xy}r^2\cos\theta\sin\theta\sin^2\phi - u_x r\cos\theta\sin\phi - u_y r\sin\theta\sin\phi$$

1 Mark

$$u_{\phi\phi} = u_{xx}r^2\cos^2\theta\cos^2\phi + u_{yy}r^2\sin^2\theta\cos^2\phi + u_{zz}r^2\sin^2\phi + 2u_{xy}r^2\cos\theta\sin\theta\cos^2\phi - 2u_{xz}r^2\cos\theta\cos\phi\sin\phi - 2u_{yz}r^2\sin\theta\cos\phi\sin\phi - u_x r\cos\theta\sin\phi - u_y r\sin\theta\sin\phi - u_z r\cos\phi$$

1 Mark

$$r^2u_{rr} + u_{\phi\phi} = u_{xx}(r^2\cos^2\theta\,(\sin^2\phi + \cos^2\phi)) + u_{yy}(\mathrm{r}^2\sin^2\theta\,(\sin^2\phi + \cos^2\phi)) + u_{zz}\,\mathrm{r}^2(\cos^2\phi + \sin^2\phi) + 2u_{xy}\big(r^2\cos\theta\sin\theta\,(\sin^2\phi + \cos^2\phi)\big) + 2u_{xz}\,\mathrm{r}^2\cos\theta\sin\phi\cos\phi - 2u_{xz}\,\mathrm{r}^2\cos\theta\sin\phi\cos\phi + 2u_{yz}r^2\sin\theta\sin\phi\cos\phi - 2u_{yz}r^2\sin\theta\cos\phi\sin\phi - u_x r\cos\theta\sin\phi - u_y r\sin\theta\sin\phi - u_z r\cos\phi$$

$$r^2u_{rr} + u_{\phi\phi} = u_{xx}r^2\cos^2\theta + u_{yy}\,\mathrm{r}^2\sin^2\theta + u_{zz}r^2 + 2u_{xy}r^2\cos\theta\sin\theta - u_x r\cos\theta\sin\phi - u_y r\sin\theta\sin\phi - u_z r\cos\phi$$

$$r^2u_{rr} + u_{\phi\phi} + ru_r = u_{xx}r^2\cos^2\theta + u_{yy}\,\mathrm{r}^2\sin^2\theta + u_{zz}r^2 + 2u_{xy}r^2\cos\theta\sin\theta$$

$$u_{\theta\theta} = \sin^2\phi\left(u_{xx}r^2\sin^2\theta + u_{yy}r^2\cos^2\theta - 2u_{xy}r^2\cos\theta\sin\theta - \frac{u_x r\cos\theta}{\sin\phi} - \frac{u_y r\sin\theta}{\sin\phi}\right)$$

$$\frac{1}{\sin^2\phi}u_{\theta\theta} = \left(u_{xx}r^2\sin^2\theta + u_{yy}r^2\cos^2\theta - 2u_{xy}r^2\cos\theta\sin\theta - \frac{u_x r\cos\theta}{\sin\phi} - \frac{u_y r\sin\theta}{\sin\phi}\right)$$

$$\begin{aligned} r^2u_{rr} + u_{\phi\phi} + ru_r + \frac{1}{\sin^2\phi}u_{\theta\theta} \\ &= u_{xx}r^2(\cos^2\theta + \sin^2\theta) + u_{yy}\,\mathrm{r}^2(\sin^2\theta + \cos^2\theta) + u_{zz}r^2 + 2u_{xy}r^2\cos\theta\sin\theta \\ &- 2u_{xy}r^2\cos\theta\sin\theta - \frac{u_x r\cos\theta}{\sin\phi} - \frac{u_y r\sin\theta}{\sin\phi} \end{aligned}$$

0.5 Mark

$$r^2u_{rr} + u_{\phi\phi} + ru_r + \frac{1}{\sin^2\phi}u_{\theta\theta} = r^2(u_{xx} + u_{yy} + u_{zz}) - \frac{u_x r\cos\theta}{\sin\phi} - \frac{u_y r\sin\theta}{\sin\phi}$$

$$r^2u_{rr} + u_{\phi\phi} + \frac{1}{\sin^2\phi}u_{\theta\theta} + ru_r = -\frac{u_x r\cos\theta}{\sin\phi} - \frac{u_y r\sin\theta}{\sin\phi}$$

$$u_\phi = u_x r\cos\theta\cos\phi + u_y r\sin\theta\cos\phi - u_z r\sin\phi$$

$$ru_r = u_x r\cos\theta\sin\phi + u_y r\sin\theta\sin\phi + u_z r\cos\phi$$

$$\frac{\cos\phi}{\sin\phi}u_\phi = \frac{u_x r\cos\theta\cos^2\phi}{\sin\phi} + \frac{u_y r\sin\theta\cos^2\phi}{\sin\phi} - u_z r\cos\ \phi$$

$$\frac{\cos\phi}{\sin\phi}u_\phi + ru_r = \frac{u_x r\cos\theta\cos^2\phi}{\sin\phi} + \frac{u_y r\sin\theta\cos^2\phi}{\sin\phi} + \frac{u_x r\cos\theta\sin^2\phi}{\sin\phi} + \frac{u_y r\sin\theta\sin^2\phi}{\sin\phi}$$

$$\frac{\cos\phi}{\sin\phi}u_\phi + ru_r = \frac{u_x r\cos\theta\,(\cos^2\phi + \sin^2\phi)}{\sin\phi} + \frac{u_y r\sin\theta\,(\cos^2\phi + \sin^2\phi)}{\sin\phi}$$

$$\frac{\cos\phi}{\sin\phi}u_\phi + ru_r = \frac{u_x r\cos\theta}{\sin\phi} + \frac{u_y r\sin\theta}{\sin\phi}$$

0.5 Mark

$$r^2u_{rr} + u_{\phi\phi} + \frac{1}{\sin^2\phi}u_{\theta\theta} + ru_r = -\left(\frac{\cos\phi}{\sin\phi}u_\phi + ru_r\right)$$

$$r^2u_{rr} + u_{\phi\phi} + \frac{1}{\sin^2\phi}u_{\theta\theta} + 2ru_r + \frac{\cos\phi}{\sin\phi}u_\phi = 0$$

0.5 Mark