**Indian Institute of Technology Tirupati**

**Department of Mathematics**

**MA5101 – Mathematics for Engineers – Summary and Tutorial Sheet-PDE**

1. Laplace Equation
   a. Verify that the potential $u = \frac{c}{r}, r = \sqrt{x^2 + y^2 + z^2}$ satisfies Laplace equation in spherical coordinates.
   b. Verify that the potential $u = \frac{c}{r} + k, r = \sqrt{x^2 + y^2 + z^2}$ satisfies Laplace equation in spherical coordinates.
   c. Verify that $u = c\,ln\,r + k, r = \sqrt{x^2 + y^2}$ is a solution of Laplace equation in cylindrical coordinates
   d. Find electric potential between coaxial cylinders of radii
2. Wave Equation in Polar Coordinate
   a. Show that using $u = F(r,\theta)G(t)$ in wave equation in polar coordinate gives an ODE and a PDE
   $$\ddot{G} + \lambda^2 G = 0, \lambda = ck$$
   $$F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\theta\theta} + k^2 F = 0$$
   b. Show that the using $F = W(r)Q(\theta)$ gives
   $$Q'' + n^2 Q = 0$$
   $$r^2 W'' + rW' + (k^2 r^2 - n^2)W = 0$$
3. Solve the following wave equation for $u(x,y,t)$
   $$u_{tt} = (u_{xx} + u_{yy})$$
   $$u(x,b,t) = u(0,y,t) = u(a,y,t) = u(x,0,t) = 0$$
   $$u(x,y,0) = \sin\frac{6\pi x}{a}\sin\frac{2\pi y}{b}, u_t(x,y,0) = 0, 0 \le x \le a, 0 \le y \le b$$
4. Solve the Heat equation (1-D) for the following initial condition $(0 \le x \le L)$
   a. $u(x,0) = x,$
   b. $u(x,0) = 1,$
   c. $u(x,0) = \cos 2x$
5. Find the canonical form of the following PDE
   a. $u_{xx} + 4u_{yy} = 0$
   b. $u_{xx} - 16u_{yy} = 0$
   c. $u_{xx} + 2u_{xy} + u_{yy} = 0$
   d. $u_{xx} - 2u_{xy} + u_{yy} = 0$
   e. $u_{xx} + 2u_{xy} + 10u_{yy} = 0$
   f. $u_{xx} - 4u_{xy} + 5u_{yy} = 0$
   g. $u_{xx} - xu_{yy} = 0$
   h. $u_{xx} + x^2 u_{yy} = 0$
   i. $c^2 u_{xx} = u_t$
   j. $u_{xx} + yu_{yy} = 0$
   k. $u_{xy} + u_x + u_y = 2x$

l. $u_{xx} - 6u_{xy} + 12u_{yy} + 4u_x - u = \sin(xy)$

6. Verify that the following $u(x,t)$ satisfy the wave equation (1-D)
   a. $u = x^2 + t^2$
   b. $u = \cos 4t \sin 2x$
   c. $u = \sin at \sin bx$
7. Verify that the following $u(x,t)$ satisfy the Heat equation (1-D)
   a. $u = e^{-t} \sin x$
   b. $u = e^{-\omega^2 c^2 t} cos\omega x$
   c. $u = e^{-\pi^2 t} cos25x$
8. Verify that the following $u(x,t)$ satisfy the Laplace equation (2-D)
   a. $u = e^x cosy$
   b. $u = \tan^{-1}\frac{y}{x}$
   c. $u = \cos y \sinh x$
   d. $u = \frac{x}{x^2+y^2}$
9. Verify that the following $u(x,t)$ satisfy the Poisson equation (2-D) with $f(x,y)$
   a. $u = \frac{y}{x}, f = \frac{2y}{x^3}$
   b. $u = \sin xy, f = (x^2 + y^2)\sin xy$
10. Solve the one-dimensional wave equation with the following initial conditions

$$u_{tt} = u_{xx}, u(0,t) = 0, u(\pi,t) = 0$$

$$u(x,0) = 0, u_t(x,0) = \begin{cases} 0.01x & x \in \left[0, \frac{\pi}{2}\right] \\ 0.01(\pi - x) & x \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$$

11. Solve the one-dimensional wave equation with the following initial conditions

$$u_{tt} = u_{xx}, u(0,t) = 0, u(1,t) = 0$$

   a. $u(x,0) = k \sin 3\pi x, u_t(x,0) = 0$
   b. $u(x,0) = kx(1-x), u_t(x,0) = 0$
   c. $u(x,0) = kx^2(1-x), u_t(x,0) = 0$
   d. $u(x,0) = \begin{cases} 2x(1-x) & x \in \left[0, \frac{1}{2}\right] \\ 0 & x \in \left[\frac{1}{2}, 1\right] \end{cases}, u_t(x,0) = 0$
12. Obtain the solution of the heat equation in integral form for the following conditions

$$u_t = u_{xx}, u(0,t) = 0, u(1,t) = 0$$

   a. $u(x,0) = \frac{1}{1+x^2}$
   b. $u(x,0) = \frac{\sin x}{x}$
   c. $u(x,0) = \begin{cases} 1 & |x| < 1 \\ 0 & Otherwise \end{cases}$
   d. $u(x,0) = \begin{cases} |x| & |x| < 1 \\ 0 & Otherwise \end{cases}$
   e. $u(x,0) = \begin{cases} x & |x| < 1 \\ 0 & Otherwise \end{cases}$