Lecture Notes for the Ordinary Differential Equation-24th October 2018

$$L(y, y', y'') = y'' + p(x)y' + qy = 0 - - - - - (1)$$

Proposition 1:

Let y_1, y_2 be any two solutions of (1), then $\alpha y_1 + \beta y_2$ is also a solution of (1), for any $\alpha, \beta \in R$. Suppose y_1, y_2 are independent solution of (1), then any solution can be written in the form $y = \alpha y_1 + \beta y_2$,for some $\alpha, \beta \in R$.

Proposition 2:

$S =$ The set of all solutions of the (1), then S is a linear space and $\dim S \leq 2$.

Proof:

Assume y_1, y_2 are independent

$$L(y_1, y_1', y_1'') = 0 = L(y_2, y_2', y_2'')$$

To prove: any solution y can be written in the form $y = \alpha y_1 + \beta y_2$

By superposition or linearity principle, $\alpha y_1 + \beta y_2$ is also a solution for (1).

y_1, y_2 and y are given to us and they are differentiable at all x in some interval $[x_0, x_1]$

In particular, it is differentiable at some x_0. Therefore,

$$y(x_0) = \alpha y_1(x_0) + \beta y_2(x_0)$$

$$y'(x_0) = \alpha {y_1}'(x_0) + \beta {y_2}'(x_0)$$

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ {y_1}'(x_0) & {y_2}'(x_0) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} y(x_0) \\ y'(x_0) \end{pmatrix}$$

To solve for $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, we need $\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ {y_1}'(x_0) & {y_2}'(x_0) \end{pmatrix}$ to be invertible.

Define

$$W(x) = det \begin{pmatrix} y_1(x) & y_2(x) \\ {y_1}'(x) & {y_2}'(x) \end{pmatrix} = y_1 y_2' - y_2 {y_1}'$$

If we prove that $W(x) \neq 0$, then α and β are uniquely determined and hence it shows that y can be written in the form $y = \alpha y_1 + \beta y_2$ for some $\alpha, \beta \in R$

To prove: $W(x) \neq 0$ if y_1, y_2 are independent

Claim 1: Either $W \equiv 0$ or W is never zero.

Proof for Claim 1:

$$W(x) = y_1 y_2' - y_2 y_1' \to W'(x) = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' = y_1 y_2'' - y_2 y''$$

$$\begin{aligned} W'(x) &= y_1(-p y_2' - q y_2) - y_2(-p y_1' - q y_1) \\ &= -p y_2' y_1 - q y_2 y_1 + p y_2 y_1' + q y_2 y_1 \\ &= -pW \end{aligned}$$

$$W(x) = Ce^{-\int p(x)dx}$$

From this, it is clear that, if $C = 0, W = 0$, and if $C \neq 0, W \neq 0$ as exponential will not be zero. Hence, the claim 1.

Claim 2: $W \equiv 0$ if and only if y_1, y_2 are dependent.

Proof for Claim 2:

Assume that y_1, y_2 are dependent. We need to prove that $W \equiv 0$. Then, either $y_1 = ky_2$ or $y_2 = ly_1$, for some k, l then $W(x) = k(y_1 y_1' - y_1 y_1') = 0 \Rightarrow W \equiv 0$.

Conversely, assume $W \equiv 0 \Rightarrow W(x) = 0, for\ all\ x$. We need to prove that y_1, y_2 are dependent. If $y_1 \equiv 0$ or $y_2 \equiv 0$, nothing to prove. (Because, if a set contains 0, then it is dependent). If $y_1 \neq 0 \neq y_2$. Then there exists a point x_i such that $y(x_i) \neq 0$. By continuity, there exists an interval $[c, d] \subset [x_0, x_1], x_i \in [c, d], y(x_i) \neq 0$.

$$0 = \frac{W(x)}{y_1^2} = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{d}{dx}\left(\frac{y_2}{y_1}\right) \to y_2 = ky_1$$

This is true in $x \in [c, d]$. By uniqueness property, it is true everywhere in $[x_0, x_1]$. Hence the claim 2. Therefore, we have also proved that $W(x) \neq 0$ if and only if y_1, y_2 are independent (You can prove this directly without Claim 2 also, I leave it as an exercise). This proves Proposition 1. From, proposition 1 and superposition principle, it is clear that S is linear space and $\dim S \leq 2$, since y_1, y_2 are the two independent solutions. This proves proposition 2.

Theorem: $\dim S = 2$

Proof: Let y_1, y_2 be solution to the IVP

$$L(y_1, y_1', y_1'') = 0, y_1(t_0) = 1, y_2(t_0) = 0$$

$$L(y_2, y_2', y_2'') = 0, y_1(t_0) = 0, y_2(t_0) = 1$$

$W(t) \neq 0, W(t_0) \neq 0$. Therefore, y_1, y_2 are independent. Hence dim S = 2.

Method 1: For solving $L(y, y', y'') = 0$

$$y = uv$$

$$y' = u'v + uv', y'' = u''v + 2u'v' + uv''$$

$$y'' + p(x)y' + qy = u''v + 2u'v' + uv'' + p(u'v + uv') + quv = 0$$
$$\Rightarrow uv'' + (2u' + pu)v' + (u'' + pu' + qu)v = 0$$

Choose u so that v' term vanishes

$$2u' + pu = 0 \to u = e^{-\frac{1}{2}\int p\, dx}$$

Since u is known to us, we can use these terms to solve for v.

Example: $y'' + 2ty' + (1 + t^2)y = 0$

$$p = 2t, q = 1 + t^2$$

$$y = uv \Rightarrow u = e^{-\frac{1}{2}\int p\,dt} = e^{-\frac{1}{2}\int 2t\,dt} = e^{-\frac{t^2}{2}}$$

$$u = e^{-\frac{t^2}{2}}$$
$$\Rightarrow u' = -te^{-\frac{t^2}{2}}$$
$$\Rightarrow u'' = -e^{-\frac{t^2}{2}} + t^2 e^{-\frac{t^2}{2}}$$
$$\Rightarrow uv'' + (2u' + pu)v' + (u'' + pu' + qu)v = 0$$
$$2u' + pu = -2te^{-\frac{t^2}{2}} + 2te^{-\frac{t^2}{2}} = 0$$

$$u'' + pu' + qu = -e^{-\frac{t^2}{2}} + t^2 e^{-\frac{t^2}{2}} + 2t\left(-te^{-\frac{t^2}{2}}\right) + (1 + t^2)e^{-\frac{t^2}{2}} = 0$$

$$uv'' + (2u' + pu)v' + (u'' + pu' + qu)v = 0 \Rightarrow uv'' = 0$$

$$uv'' = 0 \Rightarrow v'' = 0\ (u \neq 0\ known) \Rightarrow v' = c_1 \Rightarrow v = c_1 t + c_2$$

Therefore,

$$y = (c_1 t + c_2)e^{-\frac{t^2}{2}}$$

Exercise: Are $y_1 = e^{-\frac{t^2}{2}}, y_2 = te^{-\frac{t^2}{2}}$ independent? Check

Method 2: Order of Reduction

If one solution is known, then it is possible to find the second solution. Assume y_1 is known. Then ky_1 is also a solution. But not independent, so take $y_2 = c(x)y_1$

Compute y_2', y_2'' and solve $Ly_2 = 0$

$$y = cy_1$$

$$y' = c'y_1 + cy_1',$$
$$y'' = c''y_1 + 2c'y_1' + cy_1''$$

$$y'' + py' + qy = c''y_1 + 2c'y_1' + cy_1'' + p(c'y_1 + cy_1') + qcy_1 = 0$$
$$c''y_1 + (2y_1' + py_1)c' + (y_1'' + py_1 + qy_1)c = 0$$

Since y_1 is a solution, $y_1'' + py_1 + qy_1 = 0$

Therefore,

$$c''y_1 + c'(2y_1' + py_1) = 0$$

$$\Rightarrow \frac{c''}{c'} = -\frac{2y_1' + py_1}{y_1} \Rightarrow \frac{v'}{v} = -\frac{2y_1'}{y_1} - p$$

Solve for v, then solve for $v = c'$

Example: $t^2 y'' + ty' - y = 0$

Solution: By trial and error, $y_1 = t$ is a solution, for, $y_1' = 1, y_1'' = 0 \Rightarrow 0 + t - t = 0$

Assume $y_2 = ct$

Then,

$$y_2' = c + c't$$
$$\Rightarrow y_2'' = c' + c''t + c' = c''t + 2c'$$

$$t^2y'' + ty' - y = 0 \Rightarrow t^2(c''t + 2c') + t(c + c't) - ct = 0$$

$$\Rightarrow c''t^3 + 2c't^2 + ct + c't^2 - ct = 0$$
$$\Rightarrow c''t^3 + 3c't^2 = 0$$
$$\Rightarrow c'' = -\frac{3c'}{t} \Rightarrow \frac{c''}{c'} = -\frac{3}{t} \Rightarrow \frac{v'}{v} = -\frac{3}{t} \Rightarrow \log v = -3\log t \Rightarrow v = \frac{1}{t^3}$$

$$c' = \frac{1}{t^3} \Rightarrow c = -\frac{1}{2t^2}$$

$$y_2 = -\frac{1}{2t^2}t = -\frac{1}{2t}$$

Or simply

$$y_2 = \frac{1}{t}$$

Exercise: Are $y_1 = t, y_2 = \frac{1}{t}$ independent? Check.