Undetermined Coefficients-Superposition Approach

The method to find particular solution of the second-order ODE where

$$r(x) = \{constant, polynomial\ function, exponential\ function\ e^{\alpha x}, sin\beta x, cos\beta x\}$$

or finite sums and product of these functions. Then the trial particular solutions y_p are:

$\boldsymbol{r(x)}$	$\boldsymbol{y_p}$
k	C
$k_0 + k_1 x$	$C_0 + C_1 x$
$k_0 + k_1 x + k_2 x^2$	$C_0 + C_1 x + C_1 x^2$
$k_0 + k_1 x + k_2 x^2 + \cdots + k_n x^n$	$C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$
$e^{\alpha x}$	$Ce^{\alpha x}$
$\sin \beta x$ or $cos\ \beta x$	$C_1 cos\beta x + C_2 sin\beta x$
$x^2 e^{\alpha x}$	$(C_0 + C_1 x + C_2 x^2)e^{\alpha x}$
$e^{\alpha x} sin\beta x$ or $e^{\alpha x} cos\beta x$	$e^{\alpha x}(C_1 cos\beta x + C_2 sin\beta x)$
$x^2 sin\beta x$ or $x^2 cos\beta x$	$(B_0 + B_1 x + B_2 x^2)cos\beta x + (C_0 + C_1 x + C_2 x^2)sin\beta x$
$x^2 e^{\alpha x} sin\beta x$ or $x^2 e^{\alpha x} cos\beta x$	$(B_0 + B_1 x + B_2 x^2)e^{\alpha x} cos\beta x + (C_0 + C_1 x + C_2 x^2)e^{\alpha x} sin\beta x$
$\sum_{i=0}^{n} k_i x^i\, e^{\alpha x} sin\beta x$ or $\sum_{i=0}^{n} k_i x^i\, e^{\alpha x} cos\beta x$	$\sum_{i=0}^{n} B_i x^i\, e^{\alpha x} cos\beta x + \sum_{i=0}^{n} C_i x^i\, e^{\alpha x} sin\beta x$

Assume y_p. Solve for $L(y_p) = r(x)$. Equate the coefficients and identify the constants.

Undetermined Coefficients-Annihilator Approach

$\boldsymbol{r(x)}$	$\boldsymbol{Annihilator\ A(D)}$
k	D
x^2	D^3
x^n	D^{n+1}
$k_0 + k_1 x + k_2 x^2 + \cdots + k_n x^n$	D^{n+1}
$e^{\alpha x}$	$D - \alpha$
$\sin \beta x$ or $cos\ \beta x$	$D^2 + \beta^2$
$x^n e^{\alpha x}$	$(D - \alpha)^{n+1}$
$e^{\alpha x} sin\beta x$ or $e^{\alpha x} cos\beta x$	$(D - \alpha)^2 + \beta^2$
$x^n e^{\alpha x} sin\beta x$ or $x^n e^{\alpha x} cos\beta x$	$[(D - \alpha)^2 + \beta^2]^{n+1}$

Find the roots of the $A(D)L(D) = 0$. Identify the roots of $L(D)$ and write the general solution y_h, then identify the roots of $L(D)$ and write the particular solution y_p with constants. Differentiate the particular solution and use them in $L(y_p) = r(x)$, equate the coefficients and remove constants.

Green's function: From method of variation of parameters

$$y_p = -y_1(x) \int \frac{y_2(x) r(x)}{W} dx + y_2(x) \int \frac{y_1(x) r(x)}{W} dx$$

$$y_p = \int_{x_0}^{x} \frac{-y_1(x) y_2(t) r(t)}{W} dt + \int_{x_0}^{x} \frac{y_2(x) y_1(t) r(t)}{W} dt = \int_{x_0}^{x} G(x,t) r(t) dt$$

$$G(x,t) = \frac{y_2(x) y_1(t) - y_1(x) y_2(t)}{W}$$

Green's function for IVP: From method of variation of parameters

$$y_p = \int_{x_0}^{x} G(x,t)r(t)dt\,, G(x,t) = \begin{cases} \dfrac{y_2(x)y_1(t)}{W(t)} & t \in [x_0, x] \\ \dfrac{y_1(x)y_2(t)}{W(t)} & t \in [x, x_1] \end{cases}$$

Power Series:

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

$$S_N(x) = \sum_{n=0}^{N} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots a_n(x-x_0)^N$$

A power series is **convergent** at a specified value of x if its sequence of partial sums $S_N(x)$ converges. That is, $\lim_{N\to\infty} S_N(x)$ exists.

Interval of Convergence: interval of convergence is the set of all real numbers x for which the series converges. It converges definitely at x_0.

Radius of convergence:

The radius R of the interval of convergence of a power series is called its **radius of convergence.** If $R > 0$ then a power series converges for $|x - x_0| < R$ and diverges for $|x - x_0| > R$. If the series converges only at its centerx_0, then $R = 0$. If the series converges for all x, then $R = \infty$. Recall, the absolute-value inequality is equivalent to the simultaneous inequality $x_0 - R < x < x_0 + R$. A power series may or may not converge at the endpoints of this interval.

Absolute Convergence:

Within its interval of convergence, a power series **converges absolutely.** In other words, if x is in the interval of convergence and is not an endpoint of the interval, then the series of absolute values $\sum_{n=0}^{\infty} a_n(x-x_0)^n$converges.

Ratio Test:

Suppose $a_n \neq 0$ for all n in $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and that

$$, \lim_{n\to\infty} \left|\frac{c_{n+1}(x-x_0)^{n+1}}{c_n\,(x-x_0)^n}\right| = |x-x_0| \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$$

If $L < 1$, the series converges absolutely, $L > 1$ diverges and if$L = 1$ the test is inconclusive.

Analytic at a Point: A function f is said to be analytic at a point x_0 if it can be represented by a power series in $x - x_0$ with either a positive or an infinite radius of convergence. In calculus it is seen that infinitely differentiable functions.

Ordinary and Singular Points

A point $x = x_0$ is said to be an **ordinary point** of the differential of the differential equation $y'' + p(x)y' + q(x)y = 0$ if both coefficients $p(x)$ and $q(x)$ are analytic at x_0. A point that is *not* an ordinary point of $y'' + p(x)y' + q(x)y = 0$ is said to be a **singular point** of the DE.

Regular and Irregular Singular Points:

A singular point x _ x0 is said to be a **regular singular point** of the differential equation $y'' + p(x)y' + q(x)y = 0$ if the functions $(x-x_0)p(x)$ and $(x-x_0)^2 q(x)$ are both analytic at x_0. A singular point that is not regular is said to be an **irregular singular point** of the equation.