Summary Sheet for First and Second Order Linear ODE

First Order ODE

1. Implicit Form:
$$F(x,y,y') = 0 \tag{1}$$

2. Explicit Form:
$$y' = f(x,y) \tag{2}$$

3. Solution: A function $y = h(x)$ is called a solution of given ODE (1) on some interval $a < x < b$ if $h(x)$ is defined and differentiable in that interval and is such that the equation becomes an identity if y and y' are replaced by h and h' respectively. The curve corresponds to h is called a solution curve.

4. General Solution: A solution which contains an arbitrary constant is called a general solution of the ODE

5. Particular Solution: If we choose a specific constant in the general solution, it is called a particular solution of the ODE.

6. Initial Value Problem (IVP): An ODE in the explicit form (2), with initial condition $y(x_0) = y_0$
$$y' = f(x,y) \quad y(x_0) = y_0 \tag{3}$$

7. Separable ODE: $g(y)dy = f(x)dx$

8. Exact ODE:
$$M(x,y)dx + N(x,y)dy = 0 \tag{4}$$
is called an exact ODE if the differential form $M(x,y)dx + N(x,y)dy$ is exact, that is, there exists some u such that
$$\frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial u}{\partial y} = N \tag{5}$$

9. Theorem on Exact ODE: Suppose $M, N \in C^1(D), D = (a,b) \times (c,d)$. Then there exists ϕ such that
$$M = \frac{\partial \phi}{\partial x}, N = \frac{\partial \phi}{\partial y}$$
if and only if
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

10. Integrating Factor (I.F.): If an ODE is of the form (4),then μ is said to be the integrating factor if $\mu M(x,y)dx + \mu N(x,y)dy$ is exact.

11. Theorem on I.F. Suppose $Mdx + Ndy = 0$ and if μ is an integrating factor such that

$$\frac{1}{\mu}\frac{d\mu}{dx} = R(x)$$
$$R(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$
$$\implies \mu = e^{\int R(x)dx}$$

12. First Order Linear ODE(FLODE): A first-order ODE is said to be linear if it can be brought to the form

$$y' + p(x)y = r(x) \tag{6}$$

Otherwise, it is called first order nonlinear ODE.

13. Homogeneous, nonhomogeneous: If $r(x) \equiv 0$ in (6), it is called homogeneous. Otherwise, it is called nonhomogeneous.

14. Solution for FLODE:

$$\boxed{y(x) = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} r(x)dx + c\right)} \tag{7}$$

15. Solution for FLODE IVP:

$$y(x) = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} r(x)dx + y_0\right) \tag{8}$$

16. Bernoulli Equation:

$$y' + p(x)y = r(x)y^a$$
$$u = y^{1-a} \implies u' + (1-a)pu = (1-a)r$$

17. Existence Theorem: Consider (3). Let $f(x,y)$ be continuous at all points (x,y) in some rectangle $R : |x - x_0| < a, |y - y_0| < b$ and bounded, that is there is a number K such that $|f(x,y)| \leq K$ for all $(x,y) \in R$. Then the initial value problem has at least one solution $y(x)$. This solution exists at least for all x in the sub-interval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $\alpha = min\{a, b/K\}$.

18. Uniqueness Theorem: Consider (3). Let $f(x,y)$ and f_y be continuous at all points (x,y) in some rectangle $R : |x - x_0| < a, |y - y_0| < b$ and bounded, that is there is a number K such that $|f(x,y)| \leq K$ and $|f_y(x,y)| \leq M$ for all $(x,y) \in R$. Then the initial value problem has at most one solution $y(x)$. This solution exists at least for all x in the sub-interval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $\alpha = min\{a, b/K\}$.
Remark: The condition $|f_y(x,y)| \leq M$ can be replaced by a weaker condition or Lipschitz condition: $|f(x,y_1) - f(x,y_2)| \leq M|y_2 - y_1|$ for all $(x,y_1),(x,y_2) \in R$.

Second Order ODE

1. Second-order Linear ODE (SLODE): A second-order ODE is called linear if it can be written in the form
$$y'' + p(x)y' + q(x)y = r(x) \tag{9}$$

2. Homogeneous and nonhomogeneous: If $r(x) \equiv 0$ in (9), then it is called second-order homogeneous linear ODE (SHLODE) (10), otherwise, it is called nonhomogeneous ODE (SNHLODE).
$$y'' + p(x)y' + q(x)y = 0 \tag{10}$$

3. Superposition Principle or Linearity Principle: If y_1 and y_2 are any two solutions of the SHLODE (10) on an interval $[a, b]$, then any linear combination of y_1 and y_2, say $\alpha y_1 + \beta y_2$, for any $\alpha, \beta \in \mathbb{R}$, is also a solution of (10) in $[a, b]$

4. Initial Value Problem (IVP):
$$y'' + p(x)y' + q(x)y = 0 \quad y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{11}$$

5. Boundary Value Problem (BVP):
$$y'' + p(x)y' + q(x)y = 0, x \in [x_0, x_1], \quad y(x_0) = y_0, \quad y(x_1) = y_1 \tag{12}$$

6. General and Particular Solution: A solution $c_1y_1 + c_2y_2$ which contains arbitrary constants c_1 and c_2 is called a general solution. If we choose specific values for the constants, it is called particular solution.

7. Basis: If the solutions of (10) are not proportional to each other, that is, if y_1 and y_2 are independent, then y_1, y_2 are called basis or fundamental system of (10).

8. Proposition: Let y_1, y_2 be any two solutions of (10), then $\alpha y_1 + \beta y_2$ is also a solution of (10), for any $\alpha, \beta \in \mathbb{R}$. Suppose y_1, y_2 are independent solution of (10), then any solution can be written in the form $y = \alpha y_1 + \beta y_2$,for some $\alpha, \beta \in \mathbb{R}$. If S denotes the set of all solutions of (10), then S is a linear space and $dim\ S \leq 2$.

9. Wronskian: Wronskian of two solutions y_1 and y_2 are defined as follows:
$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1^{'}(x) & y_2^{'}(x) \end{vmatrix} = y_1y_2^{'} - y_2y_1^{'} \tag{13}$$

10. Theorem:

    (a) If $p(x)$ and $q(x)$ of (10) are continuous in an open interval $[a, b]$, then two solutions y_1 and y_2 of (10) are linearly independent if and only if $W(y_1, y_2) \neq 0$ at some $x_0 \in [a, b]$.

    (b) $W \equiv 0$ or W is never zero.

(c) $W \equiv 0$ if and only if y_1 and y_2 are dependent.

(d) $dim\ S = 2$.

11. Existence and Uniqueness Theorem: If $p(x)$ and $q(x)$ of (10) are continuous in an open interval $[a, b]$, then the IVP (11) has a unique solution $y(x)$ in the interval $[a, b]$.

12. Method to find Solution: For the given SHLODE (10), assume the solution of this format $y = uv$. Step 1: Find u such that v' term vanishes, then solve for v using u from (14)
$$u = e^{-\frac{1}{2}\int p dx} \tag{14}$$

13. Reduction of Order Method: Assume that one solution y_1 of (10) is known.Use $y_2(x) = c_1(x)y_1$. Compute y_2', y_2'' and solve $Ly_2 = 0$. Assume $v = c'$ and solve (15) for v, then solve for c
$$\frac{c''}{c'} = -\frac{2y_1'}{y_1} - p \tag{15}$$

14. Constant Coefficients: If $p(x), q(x)$ are constant in (10) and if r_1 and r_2 are the roots of the auxiliary or characteristic equation $\boldsymbol{r^2 + pr + q = 0}$, then general solution of (10) is given as follows:
$$y(x) = \begin{cases} c_1e^{r_1x} + c_2e^{r_2x} & \text{if } r_1, r_2 \text{ are real and } r_1 \neq r_2 \\ (c_1 + c_2x)e^{r_1x} & \text{if } r_1, r_2 \text{ are real and } r_1 = r_2 \\ e^{\frac{-\alpha}{2}x}(Acos\beta x + Bsin\beta x) & \text{if } r_1, r_2 \text{ are complex and } r_1 = \frac{\alpha}{2} + i\beta, r_2 = \frac{\alpha}{2} - i\beta \end{cases}$$

15. Euler-Cauchy Equation: An ODE of the form
$$x^2y'' + axy' + by = 0 \tag{16}$$
is called Euler-Cauchy equation, where a and b are constants. Assume $y = x^m$, auxiliary equation, $m^2 + (a - 1)m + b = 0$ and the general solution $y = c_1x^{m_1} + c_2x^{m_2}$

16. General and Particular solution of SNHLODE: A general solution of SNHLODE (9) in an open interval $[a, b]$ is of the form $\boldsymbol{y(x) = y_h(x) + y_p(x)}$ where y_h is a general solution of (10) on $[a, b]$ and y_p is any solution of (10) without any arbitrary constants. If specific values are prescribed for c_1 and c_2, then the solution is called a particular solution.

17. Method of undetermined coefficients:
$$y_p(x) = \begin{cases} Ce^{\gamma x} & \text{if } r(x) = ke^{rx} \\ \sum_{i=0}^{n} K_ix^i & \text{if } r(x) = kx^n \\ e^{\alpha x}(Acos\omega x + Bsin\omega x) & \text{if } r(x) = ke^{\alpha x}cos\omega x, ke^{\alpha x}sin\omega x \end{cases}$$

18. Method of Variation of Parameters: Lagrange method gives a particular solution of (9) in the following form
$$y_p(x) = -y_1\int \frac{y_2r}{W}dx + y_2\int \frac{y_1r}{W}dx \tag{17}$$