MATHEMATICS FOR ENGINEERS

MA5101-DIFFERENTIAL EQUATIONS

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Outline

First Order ODE

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$$F(x, y, y') = 0 \quad \textit{Implicit}$$
$$y' = f(x, y) \quad \textit{Explicit}$$

First Order Linear ODE

First Order Linear ODE

$$y' + p(x)y = r(x)$$

Solutions of First Order Linear ODE

1st Order Linear ODE

$$y' + p(x)y = r(x)$$

Solutions

$$\begin{aligned} y &= \int r(x)dx + c, \quad \textit{if} \qquad && p(x) = 0, r(x) \neq 0 \\ y &= ce^{-\int p(x)dx}, \quad \textit{if} \qquad && p(x) \neq 0, r(x) = 0 \\ y &= e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}dx + c\right), \quad \textit{if} \qquad && p(x) \neq 0, r(x) \neq 0 \end{aligned}$$

Solutions of First Order Linear ODE: IVP

1st Order Linear ODE: IVP

$$y' + p(x)y = r(x), y(x_0) = y_0$$

Solutions

$$\begin{aligned}
y = \int_{x_0}^{x} r(x)dx + y_0, &\quad if \qquad p(x) = 0, r(x) \neq 0 \\
y = y_0 e^{-\int p(x)dx}, &\quad if \qquad p(x) \neq 0, r(x) = 0 \\
y = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} dx + y_0\right), &\quad if \qquad p(x) \neq 0, r(x) \neq 0
\end{aligned}$$

Exact ODE

Exact ODE

$$u(x, y) \implies du = \frac{\partial u}{\partial x} du + \frac{\partial u}{\partial y} dy$$

$$M(x, y)dx + N(x, y)dy = 0$$

Theorem

Suppose $M, N \in C^1(D), D = (a, b) \times (c, d)$. Then there exists ϕ such that

$$M = \frac{\partial \phi}{\partial x}, N = \frac{\partial \phi}{\partial y}$$

if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Integrating Factor

Exact ODE

$$M(x,y)dx + N(x,y)dy = 0$$

Theorem

Suppose $M(x,y)dx + N(x,y)dy = 0$ and if μ is an integrating factor such that

$$\frac{1}{\mu}\frac{d\mu}{dx} = R(x)$$

$$R(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

then

$$\mu = e^{\int R(x)dx}$$

Existence Theorem

Theorem

Consider the following initial value problem

$$y' = f(x, y), y(x_0) = y_0.$$

Let $f(x, y)$ be continuous at all points (x, y) in some rectangle $R : |x - x_0| < a, |y - y_0| < b$ and bounded, that is there is a number K such that $|f(x, y)| \leq K$ for all $(x, y) \in R$. Then the initial value problem has at least one solution $y(x)$. This solution exists at least for all x in the sub-interval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $\alpha = min\{a, b/K\}$.

Uniqueness Theorem

Theorem

Consider the following initial value problem

$$y' = f(x, y), y(x_0) = y_0.$$

Let $f(x, y)$ and f_y be continuous at all points (x, y) in some rectangle $R : |x - x_0| < a, |y - y_0| < b$ and bounded, that is there is a number K such that $|f(x, y)| \leq K$ and $|f_y(x, y)| \leq M$ for all $(x, y) \in R$. Then the initial value problem has at most one solution $y(x)$. This solution exists at least for all x in the sub-interval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $\alpha = min\{a, b/K\}$.

Remark: Weaker condition

The condition $|f_y(x, y)| \leq M$ can be replaced by a weaker condition or Lipschitz condition: $|f(x, y_1) - f(x, y_2)| \leq M|y_2 - y_1|$ for all $(x, y_1), (x, y_2) \in R$.

Picard Iteration

Picard's Existence and Uniqueness theorem

Consider the following initial value problem

$$y' = f(x, y), y(x_0) = y_0.$$

and the following Picard iteration

$$\begin{aligned} y_0(x) &= y_0 \\ y_n(x) = y_0 + \int_{x_0}^{x} & f(t, y_{n-1}(t))dt \end{aligned}$$

Suppose $f(x, y)$ satisfy all the conditions of Existence and Uniqueness theorem, then the iteration $y_n(x)$ converges to the unique solution of the IVP.

Solutions of First Order Linear ODE

1st Order Linear ODE

$$y' + p(x)y = r(x)$$

Bernoulli Equation

$$\begin{aligned} y' + p(x)y &= r(x)y^a \\ u(x) &= y^{1-a} \\ u' = (1-a)y^{-a}y' &= (1-a)y^{-a}(ry^a - py) \\ u' + (1-a)pu &= (1-a)r \end{aligned}$$

Thank you for your attention