Lecture Notes on Mathematics for Engineers

MA5101-Differential Equations

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This is the course material for the MA5101 course - Mathematics for engineers. It contains the differential equations part alone.

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1. Introduction

1.1 Mathematical Modeling

Let us revisit our school physics first. Do you know Newton's second law of motion? What does it say? Is it not the relationship between the force action on a body of mass and the acceleration caused by this force? If you remember your school physics, you can also remember that force is directly proportional to the acceleration. That is,

$$F = ma \tag{1.1}$$

Now, what is the meaning of acceleration? It is nothing but the name given to the velocity changing process. Isn't it? Mathematically, it is the rate of change of the velocity. What does it mean? Let us consider that we start from home or hostel at 8'O clock in the morning and reach the college which is located 10km away from your home or hostel. Have you ever noticed that, the driver of the vehicle changes the speed of the vehicle depending on the region which he travels, for example, at the school zone, driver slows the vehicle whereas on the highway he speeds up the vehicle. That means, the driver rides the vehicle at different speed at different location. How much changes was made between two different times? Let t_1 and t_2 be two different time where we would like to observe the velocity changes done by the driver and let us assume that v_1 and v_2 are the respective velocities at time t_1 and t_2. Then,

$$\Delta t = t_2 - t_1 \tag{1.2}$$

and

$$\Delta v = v_2 - v_1 \tag{1.3}$$

are the change in time and velocity respectively. Therefore, the average acceleration over a period of time is the ratio of change in velocity to the change in time. Mathematically,

$$a = \frac{\Delta v}{\Delta t} \tag{1.4}$$

If we need to find the instantaneous acceleration, then we should observe the change in velocity over an infinitesimal interval of time. In Calculus, this is achieved by taking the limit as $\Delta t \to 0$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \tag{1.5}$$

Newton's second law (1.1) can be rewritten with the help of (1.5) as follows:

$$F = m\frac{dv}{dt} \implies \frac{dv}{dt} = \frac{F}{m} \tag{1.6}$$

So, we ended up formulating a simple physical real world problems into a mathematical equation. In general, engineering or real world problem which are formulated using mathematical expression in terms of variables, functions and equations are known as mathematical modeling.

When you have attended the course Mechanical vibrations, you might have encountered the undamped free vibrations

$$m\ddot{x} + kx = 0 \tag{1.7}$$

or damped free vibrations

$$m\ddot{x} + \gamma\dot{x} + kx = 0 \tag{1.8}$$

or in general the mass-spring system as

$$m\ddot{x} + \gamma\dot{x} + kx = F(t) \tag{1.9}$$

This is another mathematical modeling in the mechanical vibrations for a mass-spring system. In fact, in a multi-degree freedom of system, this will become a system of ODEs which we will discuss in the future.

1.2 Differential Equation

What is an equation? A statement that the values of two mathematical expression are equal. $=$ sign indicates equal sign. When the equation contains derivatives of an unknown function is called differential equation. Since the function is unknown in the differential equation, our next point of interest will be searching whether any known function will satisfy the differential equation. The process of finding such a function is called method of solution finding. In this course, we will learn different ways to find a solution, its properties such as existence and uniqueness, geometrical interpretation and so on. Through out this lecture notes, we will call the unknown function as y. Since y is a function, as you aware, we need at least an independent variable and dependent variable to represent a function. In Newton's second law model, which we have discussed earlier, we have v as an unknown function which is a dependent variable depending on the independent variable t. An **ordinary differential equation (ODE)** is an equation that has one or several derivatives of an unknown function with or without the unknown function itself, which we usually call $y(x)$ if the independent variable is x. In general ODEs can be written as

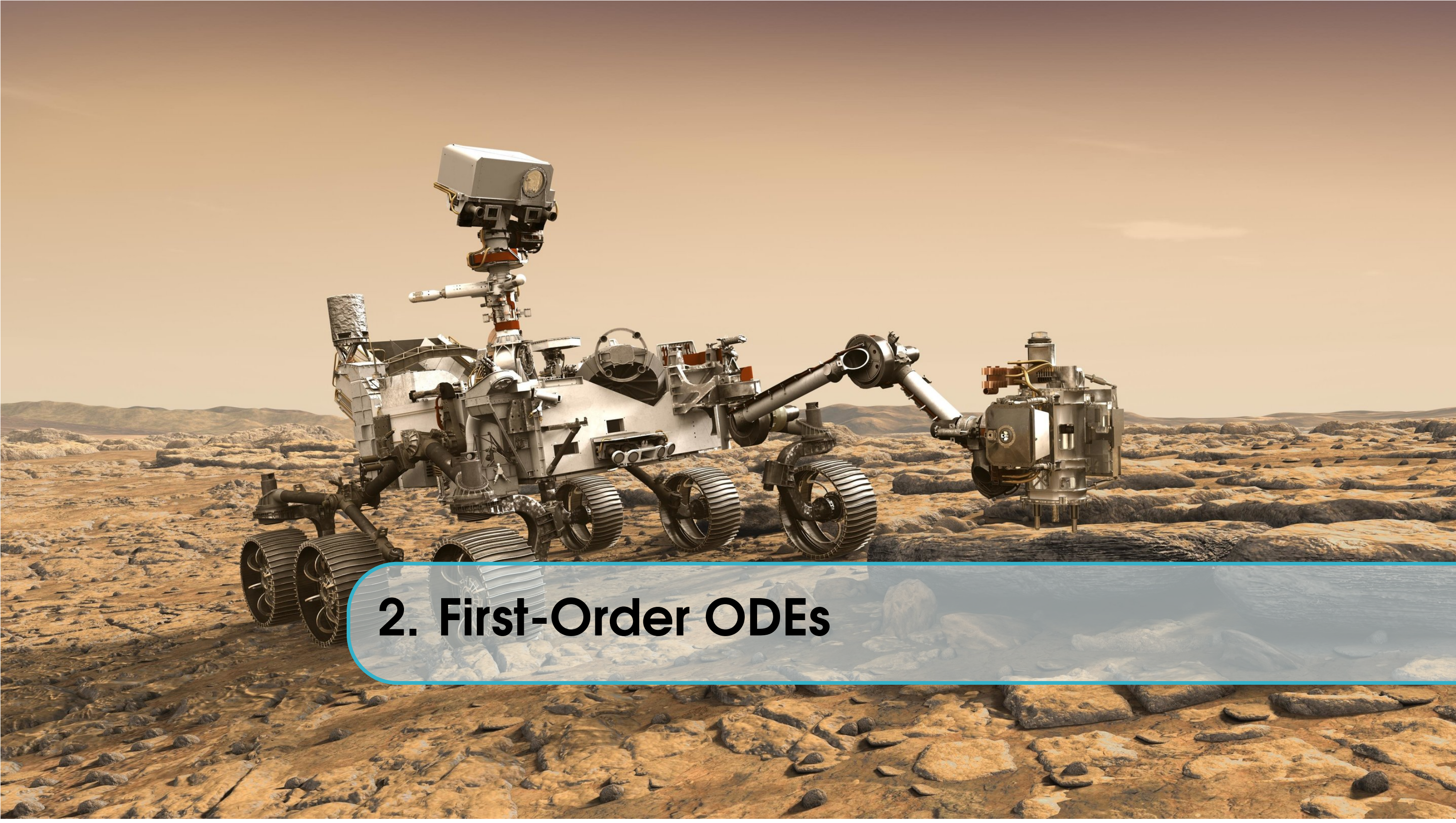
$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{1.10}$$

Why do we name it as an ordinary? Does there exist any special then? When the unknown function depends on more than one variable, then we used to obtain the derivatives with respect to one of the variable by treating the remaining variable as constant. These type of derivatives are called partial derivatives. Equation which contains partial derivatives with or without the unknown function itself is called **partial differential equation**. For instance,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1.11}$$

Here, u is an unknown function of two independent variables x and y.

An ODE (1.10) is said to be of order n if the highest derivative involved in the ODE is $y^{(n)}$. When $n = 1$, it is the first order ODE. When $n = 2$, it is the second order ODE. In this course, we will deal only first and second order ODEs. In the next chapter, we will deal with first order ODEs.



2. First-Order ODEs

2.1 First Order ODE

Equation that contains only first derivative y' with or without y. A first order ODE can be written in two forms: Implicit form (2.1) and Explicit form (2.2).

$$F(x,y,y') = 0 \tag{2.1}$$

Here, x is an independent variable, $y(x)$ is an unknown function to be determined and $y' = \frac{dy}{dx}$

$$y' = f(x,y) \tag{2.2}$$

Geometrically, $y'(x)$ is the slope of the curve $y(x)$. Therefore, the solution of (2.2) that passes through a point (x_0, y_0) must satisfy

$$y'(x_0) = f(x_0, y_0) \tag{2.3}$$

2.2 Initial Value Problem

Let us consider the following problem. In the morning, a cook prepared tea and poured the tea into a cup and served it to you at 7'O clock. The brewing temperature of the tea is $85°C$. If you had a phone call and forgot to drink the tea until 7.30AM, what will happen? Will the temperature of the tea be same? Assume that, the room temperature is $20°C$. Newton's law of cooling states that the rate of heat loss of an object is directly proportional to the difference between the temperature of the object and its surrounding temperature. Mathematically, it is written as

$$\frac{dT}{dt} = k(T - T_A) \tag{2.4}$$

where T and T_A represents the temperature of the tea and room respectively, k is the proportionality constant. The tea was served at 7'O clock, that is considered as the initial time for the tea. The tea was at $85°C$ at 7'O clock is treated as the initial condition. Mathematically, it is written as

$$T(0) = 85°C \tag{2.5}$$

Equations (2.4) and (2.5) constitute initial value problem. In general, an ODE with initial condition is called an initial value problem (IVP) and is given as follows, in case of explicit ODE.

$$y' = f(x,y), \quad y(x_0) = y_0 \tag{2.6}$$

2.2.1 What is a solution?

A value or values or a function which makes the equation valid. A function $h : (a,b) \to \mathbb{R}$ is said to be solution of (2.2) if h is differentiable in (a,b) and satisfies (2.2) while replacing y and y' by respectively h and h'. The graph or the curve of h is called a solution curve.

Let us consider the simple problem $y' = f(x)$. Then, it becomes a problem in integral calculus. We can obtain the solution by integration, that is

$$y = \int_{x_0}^{x} f(x)dx + c \tag{2.7}$$

where c is uniquely determined if $y(x_0) = y_0$ is specified.

2.3 Linear ODE

What is linear? Have you ever seen this term earlier? A transformation $T : \mathbb{V} \to \mathbb{W}$ is linear for any two scalars, $\alpha, \beta \in \mathbb{R}$, and any two vectors $u, v \in \mathbb{V}$ if T satisfies

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \tag{2.8}$$

Let us consider the following function

$$L(y,y') = F(x,y,y') \tag{2.9}$$

If L linear w.r.to y and y', then the ODE is said to be first-order linear ODE, otherwise, it is a first-order non-linear ODE. That is, for any $\alpha, \beta \in \mathbb{R}$ and y_1, y_2, y_1', y_2'.

$$L(\alpha y_1 + \beta y_2, y') = \alpha L(y_1, y) + \beta L(y_2, y)$$

and

$$L(y, \alpha y_1' + \beta y_2') = \alpha L(y, y_1') + \beta L(y, y_2')$$

Note: We don't say it is linear in x, we insist that it is linear w.r.to y and y' only. The general form of the first order linear ODE is

$$A_1(x)\frac{dy}{dx} + A_2(x)y = B(x) \tag{2.10}$$

provided, $A_1(x) \neq 0$. When the right hand side term $B(x) = 0$, then the ODE is said to be and ODE with null term or homogeneous ODE, on the other hand, if $B(x) \neq 0$, it is called non-homogeneous ODE. Since $A_1(x) \neq 0$, (2.10) can be re-written as

$$\frac{dy}{dx} + \frac{A_2(x)}{A_1(x)}y = \frac{B(x)}{A_1(x)}$$

or

$$\frac{dy}{dx} + p(x)y = r(x) \tag{2.11}$$

■ **Example 2.1** Let us consider the following problem,

$$x\frac{dy}{dx} - 3y = 0 \tag{2.12}$$

Solution: Assume that, $x = 0$ and we get

$$\frac{dy}{dx} = 3\frac{y}{x} \implies \frac{dy}{y} = 3\frac{dx}{x}$$

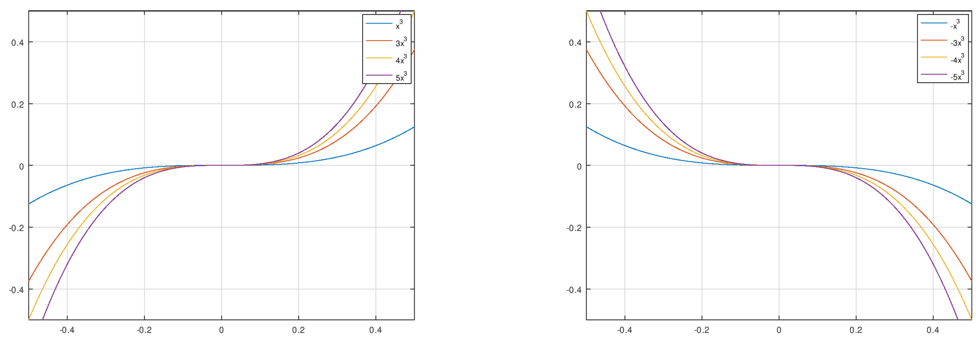
Integrating on both sides, we get

$$\begin{aligned} \int \frac{dy}{y} &= 3\int \frac{dx}{x} \\ \implies log\ y &= 3log\ x + c_1 \\ \implies log\ y &= 3log\ x + log\ c \\ \implies log\ y &= log\ (cx^3) \\ \implies y &= cx^3 \end{aligned}$$

Therefore, $y = cx^3$ is the solution to (2.12). For,

$$\frac{dy}{dx} = 3cx^2 \implies x\frac{dy}{dx} = 3cx^3 \implies \frac{dy}{dx} = 3y \implies (2.12)$$

Case 1: When we consider $y(0) = 0$, then $y = cx^3$ satisfies irrespective of the values of c. It



(a) $c > 0$ (b) $c < 0$

Figure 2.1: The graph of $y = cx^3$

means that, there are infinitely many solutions.
Case 2: When $y(0) = 1$, then there exists c satisfying $y(0) = 1$. Therefore, there exists no solution.
case 3: When $x_0 \neq 0$ and $y(x_0) = y_0$, then $c = \frac{y_0}{x_0^3}$ and there exists a unique solution, $y = \frac{y_0}{x_0^3}x^3$.

2.3.1 Simple Case

Let us consider the simple case where $p(x) = 0$. Then (2.11) becomes

$$\frac{dy}{dx} = r(x) \tag{2.13}$$

This can be solved using the fundamental theorem of calculus.

Theorem 2.3.1 **Fundamental Theorem of Calculus:** If f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x) \tag{2.14}$$

If F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a) \tag{2.15}$$

Existence: Now, for a given $r(x)$ in the simple case (2.13), with the assumption that $r(x)$ is continuous on an interval $[x_0, x]$, by using fundamental theorem of calculus, one can find $R(x) = \int_{x_0}^x r(x)dx$ such that $R'(x) = r(x)$. That is, $y = R(x)$ solves the equation (2.13). **Note:** $R(x)$ is denotes as an antiderivative of $r(x)$

Question: Does there exist any other antiderivative that satisfies (2.13)?
Yes. $y(x) = R_c(x) = R(x) + c$, where c is any constant, also satisfies (2.13).

Remark: If S is any other antidertivative that solves (2.13), then S takes the form, $S(x) = R(x) + c$.

Uniqueness: If $y(x_0) = y_0$ is the initial condition for (2.13), then it is an IVP. $y(x_0) = R(x_0) + c \implies c = y(x_0) - R(x_0)$. From the definition of integral $R(x_0) = \int_{x_0}^{x_0} r(x)dx = 0$. Hence, $c = y_0$ and there exists a unique solution $y(x) = y_0 + R(x)$ to the given IVP.

2.4 Homogeneous ODE

Let us now consider (2.11) with the assumption that $r(x) = 0$. Then we get

$$\frac{dy}{dx} + p(x)y = 0 \tag{2.16}$$

In order to solve (2.16), we need to revise our knowledge on the separable ODEs.

2.4.1 Revisit: Separable ODE

A class of ODEs can be reduced to the seperable equation form after algebraic manipulations,

$$g(y)y' = f(x) \tag{2.17}$$

Here, we assume that f and g are continuous. Integrating on both sides w.r.to x, we obtain

$$\int g(y)y'dx = \int f(x)dx + c \implies \int g(y)dy = \int f(x)dx + c$$

Since f and g are continuous, both integrals exist. By evaluating the integrals we obtain the general solution of (2.17). This method of solving ODEs is called the method of separating variables.

■ **Example 2.2** Solve the following IVP

$$y' = (x+y-2)^2, y(0) = 2$$

Solution: Let $v = x + y - 2$, then

$$\begin{aligned} v' &= 1 + y' \\ \implies y' &= v' - 1 \end{aligned}$$

$$and y(0) = 2 \implies v(0) = 0$$

Replacing y' and $(x+y-2)$ by $1-v'$ and v in the given equation, we obtain

$$\begin{aligned} v'-1 &= v^2 \\ \implies v' &= 1+v^2 \\ \implies \frac{dv}{dx} &= 1+v^2 \\ \implies \frac{dv}{1+v^2} &= dx \end{aligned}$$

Integrating on both sides

$$\begin{aligned} \int \frac{1}{1+v^2} dv &= \int dx + c \\ \implies tan^{-1}(v) &= x+c \\ \implies v &= tan(x+c) \\ v(0)=0 \implies tan(c) &= 0 \\ \implies c &= n\pi \\ \implies x+y-2 &= tan(x+n\pi) \\ \implies y &= tan(x+n\pi)-x+2 \end{aligned}$$

■ **Example 2.3** Solve the Newtons law of cooling problem for the tea cub which is explained at section 2.2.
Solution: Given,

$$\frac{dT}{dt} = k(T-T_A), T(0) = 85°C, T_A = 20°C$$

After rearrangement, we obtain

$$\begin{aligned} \frac{dT}{T-20} &= kdt \\ \implies \int \frac{dT}{T-20} &= k\int dt \\ \implies ln|T-20| &= kt+c_1 \\ \implies |T-20| &= e^{kt+c_1} \\ \implies |T-20| &= ce^{kt}, c = e^{c_1} \\ \implies T-20 &= ce^{kt} \text{(Why?!)} \\ \implies T &= 20+ce^{kt} \\ T(0) = 85 \implies 85 &= 20+ce^0 \\ \implies c = 65 \implies T &= 20+65e^{kt} \end{aligned}$$

To find k, we need an additional condition, let us assume that $T(1) = 40°C$. That is, the temperature of the tea in the tea cup at 8AM is $40°C$. Then,

$$\begin{aligned} T(1) = 40 = 20+65e^k \implies e^k &= \frac{20}{65} \\ \implies k &= ln(4/13) \approx -1.178 \\ \implies T &= 20+65e^{-1.178t} \end{aligned}$$

When the time is 7.30AM, then $t = 0.5$, and we obtain the temperature of the tea cup at time as $T(0.5) = 20+65e^{-1.178/2} \approx 56°C$

2.4.2 Solution of Homogeneous ODE

We want to solve (2.16) in some interval. Now, (2.16) can be rewritten as follows

$$\frac{dy}{y} = -p(x)dx \implies \int \frac{dy}{y} = -\int p(x)dx$$

Upon integration, we obtain

$$ln|y| = -\int p(x)dx + c_1$$

Taking exponents on both sides, we obtain

$$y = ce^{-\int p(x)dx}, c = \pm e^{c_1} \text{ depending on } y > 0 \text{ or } y < 0$$