Symmetry in the forward and backward tenure of transient states in stationary populations

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Abstract

We prove that the distribution of time spent is equal to the distribution of time left in transient states of multistate stationary populations.

The Brouard-Carey equality establishes that the distributions of years-lived and years-left are identical in perfectly stationary populations (Brouard 1989, Vaupel 2009, Rao and Carey 2014, Villavicencio and Riffe 2016). A perfectly stationary population is either perfectly stationary because it is of infinite size or because it is finite and deterministically repeating, and in either case vital rate schedules must be fixed and the intrinsic growth rate constant at null.

Let's say that individuals in the stationary population can obtain different states through life. If vital rates do not vary by states, then it does not matter whether state transition rates are fixed or not, for in the aggregate the population remains stationary in the traditional sense. However, if vital rates depend on one's state, transition rates must also be fixed in order for stationarity to hold in the aggregate— This is the situation that we entertain in the following. The standard set of invariant quantities of course remain: birth cohorts are of fixed size, as are death cohorts, and these are of equal size. Further, the Brouard-Carey equality still will hold in the aggregate. Under these conditions, one may take for granted that the age-state structure of the stationary population is also invariant over time. Further, the distribution of state-specific tenures for individuals entering a state at a given age is also a fixed attribute, by consequence of fixed transition and vital rate schedules.

This setup is not entirely contrived, for it corresponds with the assumptions of common multistate markov models, often used to calculate state expectancies. In the present, however, we are not bound to memoryless transition probabilities. Instead, we simply require the result of invariant age-state structure and invariant conditional age-state tenure distributions, which may result from either memoryless transitions or from arbitrarily linked dependencies. It is the fact of a fixed age-state structure on which we base the

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following observation. In essence, we end up with a fixed distribution of lifelines, each of which consists in a sequence of state-specific durations.

Under fixed vital and transition rates, and zero growth, we introduce a new theorem, which is a more general version of the Brouard-Carey equality. Namely, it turns out that if there is some state s in the population, then a randomly drawn individual from state s has equal probabilities of having entered s x years ago and exiting s in x years. This proposition is probably not intuitively true, so we prove it, and then speculate as to how this equality may be put to good use.

1 An intuitive proof of transient symmetry

Theorem 1.1. Given a continuous (or infinite) stationary population and fixed transition rates, the probability that a randomly selected individual is in state s and entered s x years ago is equal to the probability of being in state s and exiting in x years.

Proof. The proof of this statement follows five intuitive steps.

1. A duration can be represented as a line segment, potentially a subset of a life-line. Points along a single within-person duration could be continuously sampled over time, used to bisect the segment, collecting two infinite sets of values: time spent in the duration and time until exiting the duration. If continuously collected, these two sets will have the same values, with time spent in ascending order and time left in descending order. If sorted identically these two sets will therefore be identical. This is so by way of complements.

   If the duration in question is called d^i, the date of entry is d^i_L and the date of exit is d^i_R, then the length of d^i is $d^i_R - d^i_L$ and the set of time spent values, A^i is defined as:

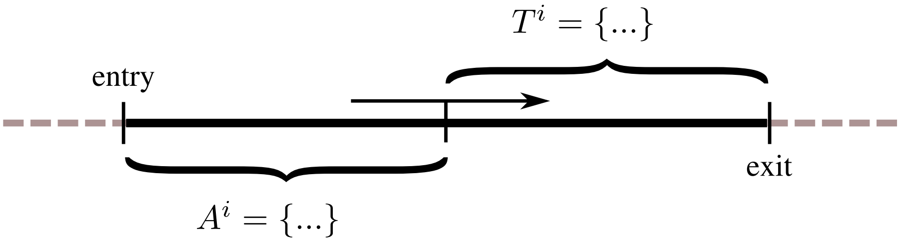
   $$A^i = \{a \in \mathbb{R} \mid 0 \leq a \leq (d^1_R - d^1_L)\} \qquad , \tag{1}$$

   and the set of time left values, T^1 is also:

   $$T^i = \{a \in \mathbb{R} \mid (d^1_R - d^1_L) \geq a \geq 0\} \qquad , \tag{2}$$

   Figure 1 illustrates the construction of A^i and T^i in expressions (1) and (2). The central cut-point moves along the duration from the time of entry until the time of exit, creating two infinite sets.

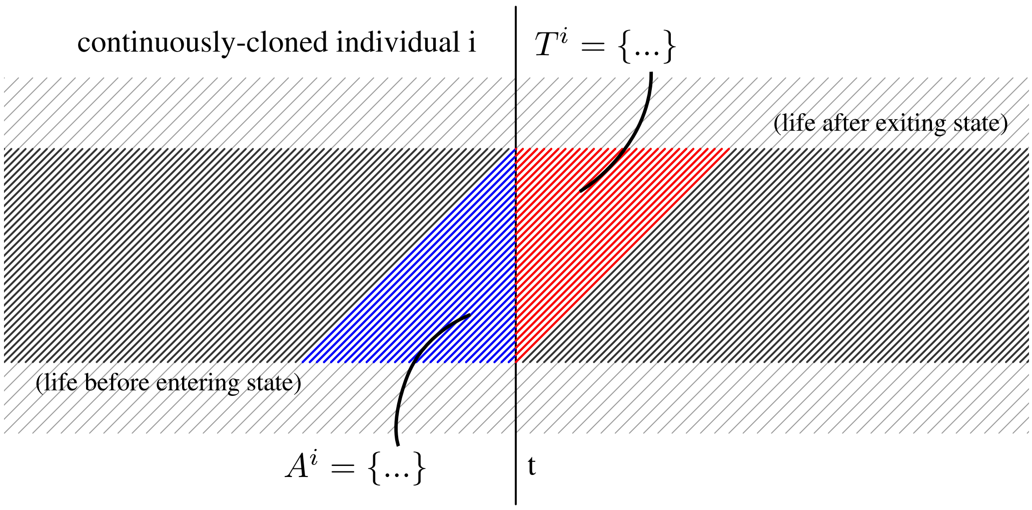
Figure 1: A lifeline of individual i showing the construction of the sets A^i and T^i.



2. If this individual is reborn in each instant, destined to relive the exact same life course as the first, we end up with an infinite number of identically aligned and identically long segments placed side-by-side. This infinite set of segments in effect forms a plane. One could in this setting take a census at a single point in time, collecting an infinite set of time spent and time left values, each from a unique cohort in sequence. It is clear that the two sets observed at a single time point will be identical to the first two sets that were observed of a single duration over its entire length.

   Figure 2 illustrates this notion with finely-spaced lifelines in a Lexis configuration, which the reader can imagine forming a continuous plane. The vertical line indicates a hypothetical census at time t of the continuously-aged population of this clones individual. At time t, the blue-highlighted segments indicate the set of time-spent values in A^i, and red-highlighted segments are the time-left elements of T^i.

Figure 2: The life of individual i repeated continuously over time. A census with followup now constructs the sets A^i and T^i with values identical to the within-individual sets.



Formally, sets A^i and T^i consist of the same values as the previous, but coming from individuals born in a continuous flow from $d_R^1 - t$ years ago until as recently as $d_L^1 - t$ years ago. This is a sort of demonstration of both period-cohort equality, but also of time spent-left equality. The blue and red triangles in Figure 2 are simple rotations of one another.

3. Assume we have a second individual from the same birth cohort that enters the same state as the first, but at a different time and for a different total duration. We could demonstrate time spent and time left equality in the same way for this individual, by sampling continuously over time. Since this individual has a different life course timing than the first individual, their sets of time-spent, A^2, and time left T^2 will be distinct. A^1 and T^1 range from 0 to d^1, whereas A^2 and T^2 range from 0 to d^2. However, their unions are identical:

$$\{A^1 \cup A^2\} = \{T^1 \cup T^2\} \tag{3}$$

4. If the second individual is perfectly clones continuously over time as in step 2, then our census at a single point in time also yields identical sets of time spent and time left values. Also from this census, the union of the first and second time-spent sets and the union of the first and second time-left sets are guaranteed to be identical, as in equation (3).

5. By induction we can keep adding durations in this state, infinitely if we please, and the union of all resultant time-spent sets and the union of time-left sets will continue being identical. Therefore the probability of selecting a particular value from the time-spent set is identical to the probability of selecting the same value from the time-left set. This constitutes a proof of symmetry of time spent and left in transient states in stationary populations.

□

2 Discussion

This approach from this proof is equally valid to prove the original statement of the Brouard-Carey equality, but it is more general. The statement and proof is so flexible as to hold for irreversible and reversible states. It is also applicable to repeatable states, whether time spent in the state is kept in cumulative fashion over spells, or whether the clock resets to zero on each entry into the state. The equality may also be conditionable in curious ways: for example, the distribution of time-spent in a state conditional on having entered at age a must also be equal to the distribution of time-left in the state, a.

Prior to proof, this equality is probably less intuitive than the Brouard-Carey equality, because state entry is not necessarily aligned on each zero. It is therefore less visible in commonly-produced plots. It might be tempting to think that due to state-varying vital rates, that the equality simply ought not hold. However, the basis of the proof is the observation that if each individual duration is symmetrical by complements, then so are aggregations of durations, irrespective of alignment.

Empirical applications of the presently-described transient tenure equality may be easy to conjure up. For example, imagine a hypothetical health state that shows no consequential or noticeable symptoms, but is medically measurable. One may take a census with regular follow-ups, until eventually the state is exited by each individual, whether by absorption into death or entry into another state. Then, if the assumption of stationarity is acceptable, one may be able to say something about onset timing in the aggregate, itself unobserved. We think that the present equality will come as good news to researchers in similar settings.

References

Brouard, N. (1989). *Mouvements et modèles de population.* Institut de formation et de recherche démographiques.

Rao, A. S. S. and J. R. Carey (2014). Generalization of Carey's equality and a theorem on stationary population. *Journal of Mathematical Biology*, 1–12.

Vaupel, J. W. (2009). Life lived and left: Carey's equality. *Demographic Research 20*, 7–10.

Villavicencio, F. and T. Riffe (2016). Symmetries between life lived and left in finite stationary populations. *Demographic Research 35*(14), 381–398.