Symmetry in the forward and backward tenure of transient states in stationary populations

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Abstract

A perfectly stationary population is either perfectly stationary because it is of infinite size or because it is finite and deterministically repeating. Assuming that individuals in a stationary population can obtain different states through life, we take for granted that the age-state structure is also invariant over time. The population may indeed be heterogenous in the sense that different states may have different vital rates. However, everything is fixed in such a way that birth cohorts are of fixed size, and the numbers of people in each age and state are the same in each time step. Further, the distribution of state-specific tenures for individuals entering a state at a given age is also a fixed attribute, by consequence of the former requisites of stationarity.

It is by now well-established via the Brouard-Carey equality (Brouard 1989, Vaupel 2009, Rao and Carey 2014, Villavicencio and Riffe 2016) that the distributions of years-lived and years-left are identical in perfectly stationary populations.

1 An intuitive proof of transient symmetry

Theorem 1.1. Given a continuous (or infinite) stationary population and fixed transition rates, the probability that a randomly selected individual is in state s and entered the state x years ago is equal to the probability of being in state s and exiting in x years.

Proof. The proof of this statement unfolds in five intuitive steps.

1. A duration can be represented as a line segment, potentially a subset of a life-line. Points along a single within-person duration could be continuously sampled over time, used to bisect the segment, collecting two infinite sets of values: time spent in the duration and time until exiting the duration. If continuously collected, these two sets will have the same values, with time spent in ascending order and time left

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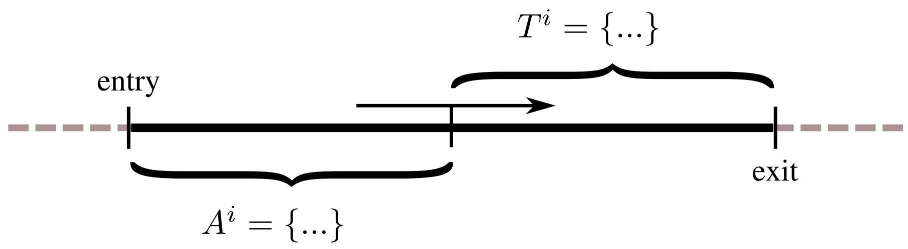
Figure 1: A lifeline of individual i showing the construction of the sets A^i and T^i.



in descending order. If sorted identically these two sets will therefore be identical. This is so by way of complements.

If the duration in question is called d^i, the date of entry is d^i_L and the date of exit is d^i_R, then the length of d^i is $d^i_R - d^i_L$ and the set of time spent values, A^i is defined as:

$$A^i = \{a \in \mathbb{R} \mid 0 \leq a \leq (d^1_R - d^1_L)\} \qquad , \tag{1}$$

and the set of time left values, T^1 is also:

$$T^i = \{a \in \mathbb{R} \mid (d^1_R - d^1_L) \geq a \geq 0\} \qquad , \tag{2}$$

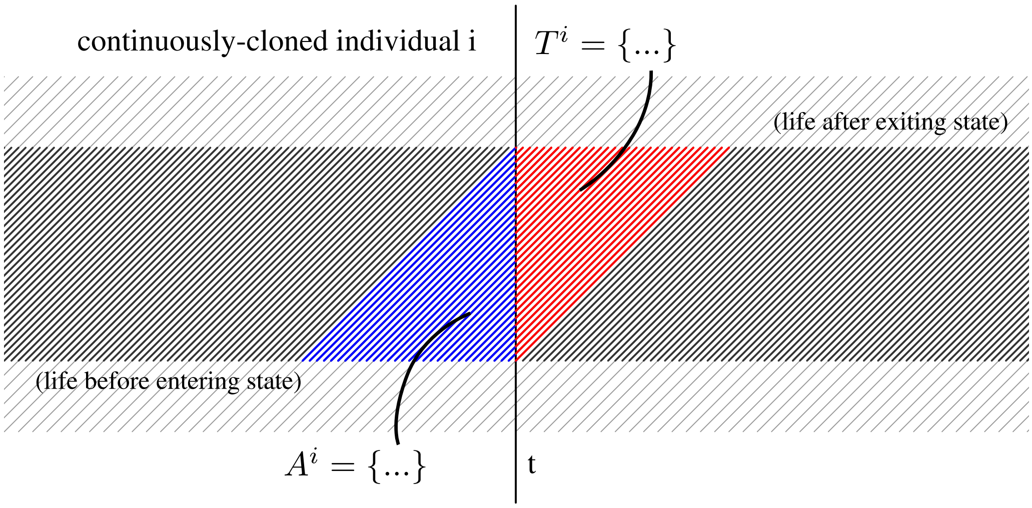
Figure 1 illustrates the construction of A^i and T^i in expressions (1) and (2). The central cutpoint moves along the duration from the time of entry until the time of exit, creating two infinite sets.

2. If this individual is reborn in each instant, destined to relive the exact same life course as the first, we end up with an infinite number of identically aligned and identically long segments placed side-by-side. This infinite set of segments in effect forms a plane. One could in this setting take a census at a single point in time, collecting an infinite set of time spent and time left values, each from a unique cohort in sequence. It is clear that the two sets observed at a single time point will be identical to the first two sets that were observed of a single duration over its entire length.

   Figure 2 illustrates this notion with finely-spaced lifelines in a Lexis configuration, which the reader can imagine forming a continuos plane. The vertical line indicates a hypothetical census at time t of the continuously-aged population of this clones individual. At time t, the blue-highlighted segments indicate the set of time-spent values in A^i, and red-highlighted segments are the time-left elements of T^i.

   Formally, sets A^i and T^i consist of the same values as the previous, but coming from individuals born in a continuous flow from $d^1_R - t$ years ago until as recently as $d^1_L - t$ years ago. This is a sort of demonstration of both period-cohort equality, but also of time spent-left equality. The blue and red triangles in Figure 2 are simple rotations of one another.

Figure 2: The life of individual i repeated continuosly over time. A census with followup now constructs the sets A^i and T^i with values identical to the within-individual sets.



3. Assume we have a second individual from the same birth cohort that enters the same state as the first, but at a different time and for a different total duration. We could demonstrate time spent and time left equality in the same way for this individual, by sampling continously over time. Since this individual has a different lifecourse timing than the first individual, their sets of time-spent, A^2, and time left T^2 will be distinct. A^1 and T^1 range from 0 to d^1, whereas A^2 and T^2 range from 0 to d^2. However, their unions are identical:

$$\{A^1 \cup A^2\} = \{T^1 \cup T^2\} \tag{3}$$

4. If the second individual is perfectly clones continously over time as in step 2, then our census at a single point in time also yields identical sets of time spent and time left values. Also from this census, the union of the first and second time-spent sets and the union of the first and second time-left sets are guaranteed to be identical, as in equation (3).

5. By induction we can keep adding durations in this state, infinitely if we please, and the union of all resultant time-spent sets and the union of time-left sets will continue being identical. Therefore the probability of selecting a particular value from the time-spent set is identical to the probability of selecting the same value from the time-left set. This constitutes a proof of symmetry of time spent and left in transient states in stationary populations.

□

2 Discussion

This approach from this proof is equally valid to proove the original statement of the Brouard-Carey equality, but it is more general. The statement and proof is so flexible as to hold for irreversible and reversible states. It is also applicable to repeatible states, whether time spent in the state is kept in cumulative fashion over spells, or whether the clock resets to zero on each entry into the state. The equality may also be conditionable in curious ways: for example, the distribution of time-spent in a state conditional on having entered at age a must also be equal to the distribution of time-left in the state, a.

Prior to proof, this equality is probably less intuitive than the Brouard-Carey equality, because state entry is not necessarily aligned on each zero. It is therefore less visible in commonly-produced plots. It might be tempting to think that due to state-varying vital rates, that the equality simply ought not hold. However, the basis of the proof is the observation that if each individual duration is symmetrical by complements, then so are aggregations of durations, irrespective of alignment.

Empirical applications of the presently-described transient tenure equality may be easy to conjure up. For example, imagine a hypothetical health state that shows no consequential or noticeable symptoms, but is medically measureable. One may take a census with regular follow-ups, until eventually the state is exited by each individual, whether by absorbtion into death or entry into another state. Then, if the assumption of stationarity is acceptible, one may be able to say something about onset timing in the aggregate, itself unobserved. We think that the present equality will come as good news to researchers in similar settings.

References

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