**An Explanation of *Fibonacci Numbers, Integer Compositions, and Nets of Antiprisms***

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## Introduction

It is easy to imagine a cube, which is a prism, two squares connected by rectangles. If one unfolds a cube into a single sheet of paper, with the edges represented by lines, this 2-dimensional outline is a net. This understanding helps with the analysis of more unfamiliar shapes. Anti-prisms are n-sided polyhedrons made of two parallel and adjacent polygons connected by triangles. Intuitively, one should assume the nets of anti-prisms are more complicated than those previously described. The properties for nets allow for formulas for the total number ($t_n$) of distinct nets, the total possible nets that can represent a specific anti-prism. Unlike prism nets, these nets will not be able to be folded in half and be symmetric, meaning anti-prisms are not bilaterally symmetric. However, they do have symmetry about its points and are represented by ($s_n$). It is possible to identify these nets using, evenly indexed Fibonacci numbers [1]. The number of possible nets and symmetric nets calculated using the combinatorial properties of anti-prisms. To imagine these polyhedrons, refer to Figures 1-3 [1].

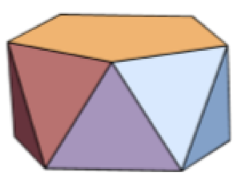


**Figure 1.** The antiprism of order $n = 5$.

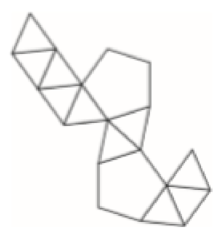


**Figure 2.** A net of the antiprism.

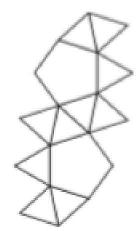


**Figure 3.** A symmetric net.

## Analysis of Nets

Though Figure 1 depicts the anti-prism as a 3-dimensional solid all convex, meaning all interior angles are less than 180, polyhedron has a net. There are no counterexamples to this claim called Durer's problem [1]. Also known as the amount of fold edges is always $2n$-1. In Figures 1 and 2 we can see $n$=5 as pentagons are 5-sided polygons and thus has 11-fold edges.

Mathematicians cleverly named the body parts of an anti-prism net. The two $n$-gons, called *heads*, that are connected by a *neck*. A neck consists of pairs of adjacent and oppositely oriented triangular faces. These pairs are created by alternating connections to the heads and define the neck size, $h$. In both Figure 1 and 2 $h$=1. There is always only one neck, but there can be triangular faces attached to the neck that form the *collar*, which has two *lapels* [1]. Firstly, looking at Figure 4, the method of numbering the faces.

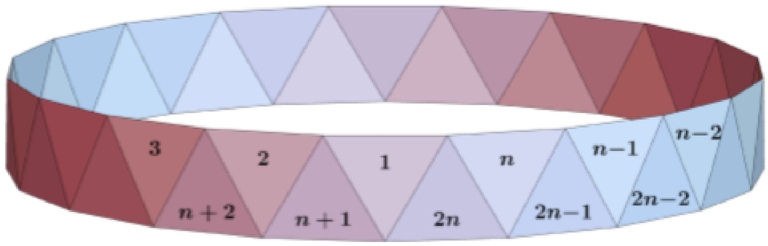
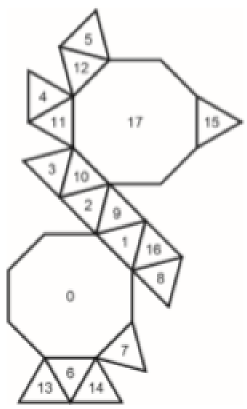
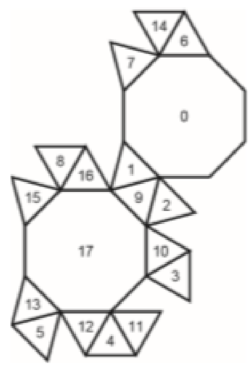



**Figure 4.** Labeling of the band of the n-antiprism.

Then considering this method nets' lapel's can be distinguished as *first lapel* and *second lapel,* who's lengths are respectively denoted $k_1$ and $k_2$. Using 0 to denote the top head, the neck face touching head 0 is always numbered 1. Hence, $k_1$ can be found counting the triangular faces connect to face 1, in the path leading away from the neck. Secondly, $k_2$ is counted the same except started from the triangular face that connects the neck with the bottom head, labeled by 2n+1.





By using these basic properties of anti-prisms, we can distinguish the symmetric nets between non-symmetric nets. This is because these properties set limitations to the possibilities of nets. There is a proven relationship between $s_n$ and $t_n$ using evenly ordered Fibonacci. This relationship can be represented as such

$$s_n=F_{2n} \tag{1}$$

and

$$t_n=\frac{1}{2}s_n(s_n+1) \tag{2}$$

for all n, "where Fj is the usual jth Fibonacci number ($F_1 = 1$, $F_2 = 1$, $F_j+2 = F_{j+1} + F_j$ for all $j \geq 1$)" [1]. Claim (1) can be proven directly and claim (2) can be proved using an induction-based proof.

**Main Results**

The focus of these claims is to identify the symmetric nets of group ($T_n$). In symmetric nets $k_1$=$k_2$, so we can denote the lapel lengths as $k$. By the properties discussed we must have

$$1 \leq h \leq n,\ 0 \leq k \leq n-1,\ 1 \leq h+k \leq n, \tag{3}$$

and each pair (h,k) must satisfy (3) in $T_n$.

Lemma 1. *For each $n \geq 1$, the number $s_n(h,k)$ of symmetric nets of the n-antiprism having neck size h and lapel size k is given by*

$$s_n(h,k) = \begin{cases} 1 & \text{if} \quad h+k=n \\ F_{2(n-h-k)} & \text{if} \quad 1 \leq h+k < n. \end{cases} \tag{4}$$

By proving Lemma 1, we imply the correctness of equation (1).

Theorem 1. *There are $F_{2n}$ symmetric nets of the n-antiprism. That is, $s_n = F_{2n}$*

$$s_n(h,*) := \sum_{k=0}^{n-h} s_n(h,k)$$

PROOF. By Lemma 1, the number of symmetric nets with neck size h, is where the last equality in (4) follows from telescoping the next sum after even Fibonacci number as so:

$$\begin{aligned} &= 1 + \sum_{k=0}^{n-h-1} F_{2(n-h-k)} \\ &= 1 + F_2 + F_4 + \cdots + F_{2(n-h)} \\ &= F_{1+2(n-h)}, \end{aligned}$$

$F_{2i}=F_{2i+1}-F_{2i-1} (i \geq 1)$

$$\begin{aligned} s_n &= \sum_{h=1}^{n} s_n(h,*) \\ &= \sum_{h=1}^{n} F_{1+2(n-h)} \\ &= F_1 + F_3 + \cdots + F_{2n-1} \\ &= F_{2n}. \end{aligned}$$

We are assuming Lemma 1 is true as it is proven by Mabry in [1]. More importantly now that Lemma 1 is assumed to be true, we can say

$$S_n(h,k)=F_{2m}=F_{2(n-h-k)}$$

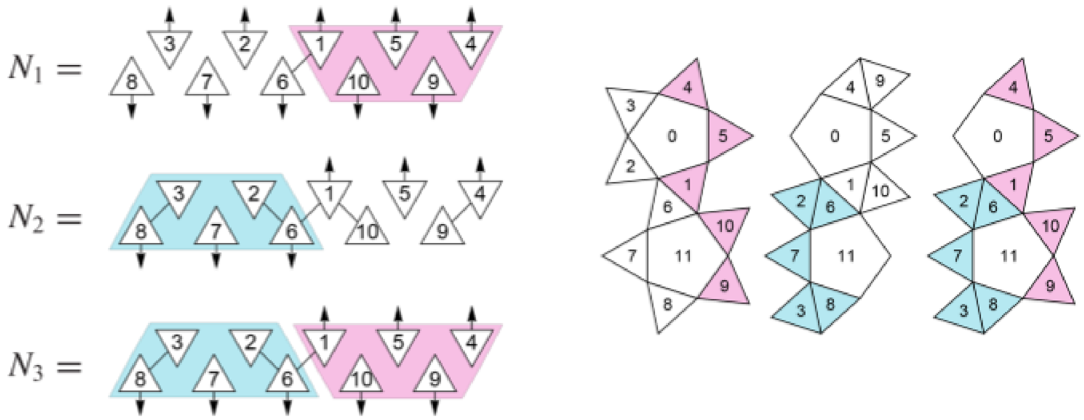
Thus, proving (1).

Theorem 2. There are precisely $s_n(s_n+1)/2$ nets of the n-antiprism. That is, $t_n = F_{2n}(F_{2n}+1)/2$

The purpose of this formula allows for the chance to look at pairs of symmetric nets. These two are used construct a unique net in $T_n$.

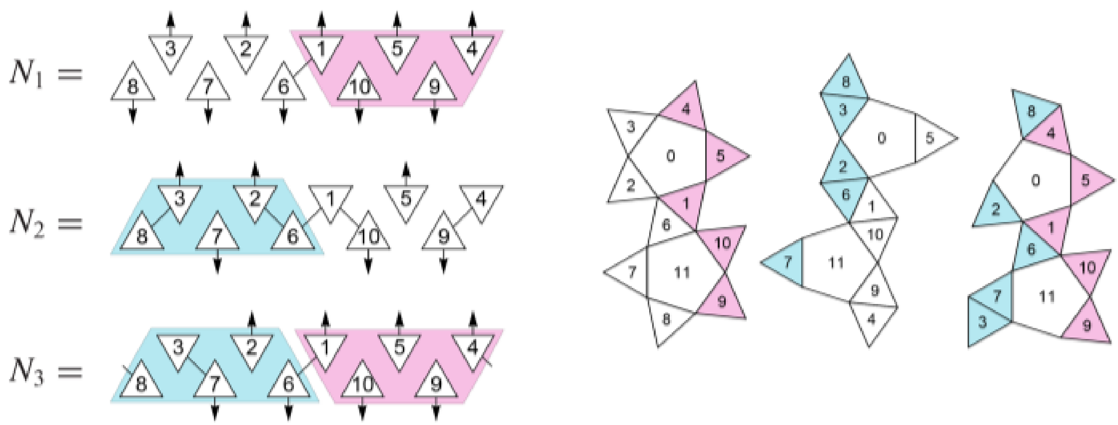
PROOF of Theorem 2. Let $N_1, N_2 \in S_n$ such that $N_1$ and $N_2$ are not necessarily distinct and can be used to construct $EN_3 \in T_n$. Let V be an injective function such that, $V: S_n \rightarrow T_n$ and $V^{-1}: T_n \rightarrow S_n$.

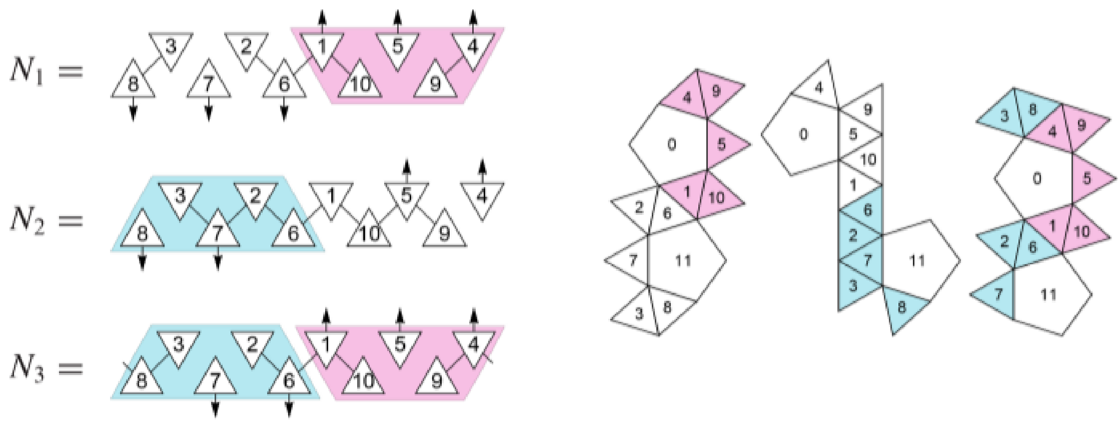
Case 1: $N_1, N_2 \in S_n$, such that, $h_1=h_2$. As we can see in Figure 5, the right side of $N_1$ and the left side $N_2$ are both use to create $N_3$. By Lemma (Arrow-gram criteria) we can see how their respective arrow-grams are redrawn about their necks. Hence, $N_3 \in S_n$ if and only if $N_1=N_2$.



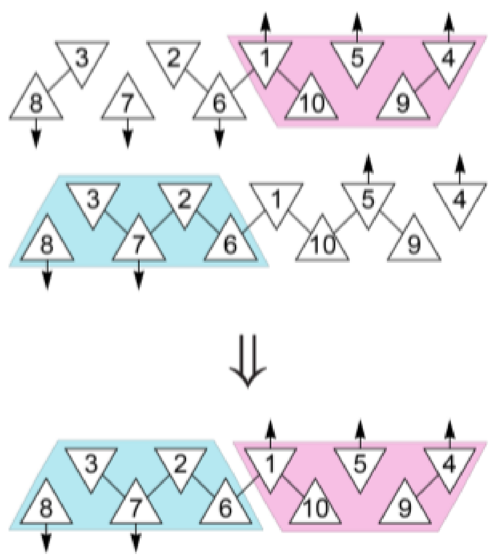

Because all arrow-gram criteria are met $N_3 \in T_n$; however, $N_3$ is not symmetric.

Case 2: $N_1, N_2 \in S_n$, such that, $h_1 \neq h_2$. Assume $h_2 > h_1$ and thus $\Delta h = h_2 - h_1$. $\Delta h$ can either be odd Figure 6 or even Figure 7. However, when the two sides are connected it doesn't resemble Case 1. When $\Delta h$ is odd an illegal arrow-gram is created and when $\Delta h$ is even the results are not unique.

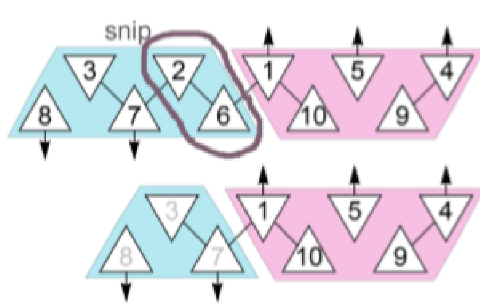




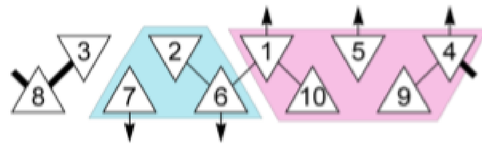

To achieve the desired net, the arrow-gram will need to be compressed and shifted. To begin we will examine when Δh is even, and the opposite actions can be applied when Δh is odd. In Figure 8, step 1 illustrates the two-to-one technique used in Case 1.



We will now shift and compress the side with the long neck parts (the left side), starting at the center. Thus, the neck size has been changed and now the net has Δh too few edges and vertices. However, the remainder of the 'long-neck' side will be shifted from left to right (and the opposite for when Δh is odd). See Figure 9.



In the final step, the remainder's structure will stay and the snipped band will be replaced. The vertices, in this case {3,8}, will be moved to the far-left to create a wrapped-edge from the left to the right. Note that in any case a net with a wraparound edge is asymmetric and in fact $N_3$ is such. See Figure 10.




Hence, $N_1$, $N_2 \in S_n$ and $N_3 \in T_n$ but $N_3 \notin S_n$.

To prove $V^{-1}$: $T_{n\rightarrow}$ $S_n$, we can repeat case 1 and 2 starting from the end and working backwards. This proves the cardinality of $T_n$ and $S_n$ are equal and the theorem follows.

## References

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