Understanding Analysis Attempt/Solution

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Contents

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is rational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

SOLUTION

(a) AFSOC that $\sqrt{3}$ is rational, so $\exists m, n \in \mathbb{Z}$, such that

$$\sqrt{3} = \frac{m}{n},$$

where $\frac{m}{n}$ is in the lowest reduced terms. By squaring both sides, we obtain $3 = (\frac{m}{n})^2 \implies 3n^2 = m^2$. Now, we know that m^2 is a multiple of 3 and thus m must also be a multiple of 3. We can then write $m = 3k$, deriving

$$\begin{aligned}(\sqrt{3})^2 &= \left(\frac{3k}{n}\right)^2 \\ 3n^2 &= 9k^2 \\ n^2 &= 3k^2\end{aligned}$$

Similar to above, we can conclude that n is a multiple of 3. However this is a contradiction since m, n are both multiples of 3 but we assumed that $\frac{m}{n}$ was in its lowest reduced term. Thus we conclude that $\sqrt{3}$ is irrational. The same proof for $\sqrt{3}$ works for $\sqrt{6}$ as well.

(b) We cannot conclude that $\sqrt{4} = \frac{m}{n}$ imply that m is a multiple of 4, as we have

$$4n^2 = m^2 \implies 2n = m,$$

preventing us from reaching our contradiction that m/n is not in its lowest terms.

Exercise 1.2.2

Show that there is no rational number r satisfying $2^r = 3$.

SOLUTION

If $r = 0$, then $2^r = 1 \neq 3$. Suppose $r = p/q$ to get $2^p = 3^q$, which is not possible as 2 and 3 share no common factors. Hence r is not rational.

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$ are all sets containing an infinite number of elements, then the intersections $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and non-empty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

SOLUTION

(a) False. Consider $A_n = \{n, n+1, n+2, \dots\}$, then $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(b) True. Since all A_n are nonempty, $\exists n \in \mathbb{N}$ such that $A_n = \{x\}$ for some real x. Hence $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\}$ which is empty. Since A_1 is finite, $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\} \subset A_1$ is finite.

(c) False. If $A = \emptyset$, then $\emptyset = C$

(d) True. Intersection is associative as evident that both LHS and RHS implies the $x \in A, B, C$

(e) True. Drawing a Venn Diagram illustrates this.

Exercise 1.2.4
Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

SOLUTION
Consider arranging the elements of $\mathbb{N}$ in a square as such.

$$\begin{array}{cccccc}
1 & 3 & 6 & 10 & 15 & \dots \\
2 & 5 & 9 & 14 & \dots & \\
4 & 8 & 13 & \dots & & \\
7 & 12 & \dots & & & \\
11 & \dots & & & & \\
\vdots & & & & &
\end{array}$$

By letting A_i being the set of all natural numbers in the i-th row, we have satisfied the above conditions above.

Exercise 1.2.5
(De Morgan's Law) Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. THis shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \sqsubseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cup B^c$ by demonstrating inclusion both ways.

SOLUTION

(a) If $x \in (A \cap B)^c$, then $x \notin A \cap B$, so $x \notin A$ or $x \notin B$, implying $x \in A^c$ or $x \in B^c$, therfore $x \in A^c \cup B^c$.

(b) If $x \in A^c \cup B^c$, then $x \in A^c$ or $x \in B^c$, so $x \notin A$ and $x \notin B$, implying $x \notin A \cap B$, therefore $x \in (A \cap B)^c$. Since $(A \cap B)^c \subseteq A^c \cup B^c$ and $(A \cap B)^c \sqsubseteq A^c \cup B^c$, we can conclude that both sets are equal.

(c) To show that $(A \cap B)^c = A^c \cup B^c$, we need to demonstrate inclusion both ways.

  (i) If $x \in (A \cup B)^c$, then $x \notin A \cup B$, so $x \notin A$ or $x \notin B$, implying $x \in A^c$ or $x \in B^c$, therefore $x \in A^c \cup B^c$.

(ii) If $x \in A^c \cap B^c$, then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$, implying $x \notin A \cup B$, which is just $x \in (A \cup B)^c$.

Exercise 1.2.6

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

SOLUTION

(a) With both a and b having the same sign, then $|a| + |b| = |a+b|$, which satisfies $|a| + |b| \geq |a+b|$.

(b) Note that $(a+b)^2 \leq (|a|+|b|)^2$ reduces to $ab \leq |a||b|$, which is true as LHS can be negative while RHS cannot. Since squaring preserves inequality, this implies that $|a+b| \leq |a| + |b|$.

(c) Notice that $a - b = (a-c) + (c-d) + (d-b)$. Hence by triangle inequality,

$$|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c| + |c-d| + |d-b|$$

for all a, b, c and d.

(d) Since $||a|-|b|| = ||b|-|a||$, WLOG, we can assume that $|a| \geq |b|$. Then

$$||a|-|b|| = |a| - |b| = |(a-b)+b| - |b| \leq |a-b| + |b| - |b| = |a-b|$$

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0,2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbituary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbituary function g.

SOLUTION

(a) For $f(x) = x^2$, $f(A) = f([0,2]) = [0,4]$ and $f(B) = f([1,4]) = [1,16]$.

$$f(A \cap B) = f([0,2] \cap [1,4]) = f([1,2]) = [1{,}4] = [0,4] \cap [1,16] = f([1,2]) \cap f([2,4]) = f(A) \cap f(B)$$
$$f(A \cup B) = f([0,2] \cup [1,4]) = f([0,4]) = [0{,}16] = [0,4] \cup [1,16] = f([0,2]) \cup f([1,4]) = f(A) \cup f(B)$$

(b) Consider $A = [0,2]$ and $B = [-2,0]$. $f(A \cap B) = \{0\}$, but $f(A) \cap f(B) = [0,4]$.

(c) Suppose $y \in g(A \cap B)$, then $\exists x \in A \cap B$ such that $g(x) = y$. This implies that $x \in A$ and $x \in B$, so $x \in A \cap B$, hence $y \in g(A \cap B)$. Note that contrary may not always be true as it is possible for $x_1 \in A \backslash B$ and $x_2 \in B \backslash A$ such that $g(x_1) = g(x_2)$.

(d) I conjecture that $g(A \cup B) = g(A) \cup g(B)$. To prove this, we have to show inclusion both ways:

(i) Let $y \in g(A \cup B)$, then $\exists x \in A \cup B$ such that $y = g(x)$. This implies that $x \in A$ or $x \in B$, so $y \in g(A)$ or $y \in g(B)$, hence $y \in g(A) \cup g(B)$.

(ii) Let $y \in g(A) \cup g(B)$, then $y \in g(A)$ or $y \in g(B)$, implying $x \in A$ or $x \in B$ such that $y = g(x)$. So $x \in A \cup B$, hence $y \in g(A \cup B)$.

Exercise 1.2.8
Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* $(1-1)$ if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ Give an example of each or state that the request is impossible:

(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto.

(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1.

(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

SOLUTION

(a) Let $f(x) = x + 1$, which is 1-1 but does not have a solution to $f(x) = 1$, hence not onto.

(b) Let $f(x) = 1$ for $x = 1$ and $f(x) = x - 1$ for $x > 1$, which is onto but not 1-1 as $f(1) = f(2) = 1$.

(c) Let $f(x) = n/2$ when n is even and $f(x) = -\frac{x-1}{2}$ when n is odd.

Exercise 1.2.9
Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behaviour of preimages demonstrated in (a) is completely general. Show that for an arbituary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbb{R}$.

SOLUTION

(a) For $f(x) = x^2$, $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. $f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$. Similarly, $f^{-1}(A \cup B) = f^{-1}([-1, 4]) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$.

(b) To show that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$, we have to show inclusion both ways:

(i) Let $x \in g^{-1}(A \cap B)$, so $g(x) \in A \cap B$, which implies $g(x) \in A$ and $g(x) \in B$. This shows that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, hence $x \in g^{-1}(A) \cap g^{-1}(B)$.

(ii) Let $x \in g^{-1}(A) \cap g^{-1}(B)$, so $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, then $g(x) \in A$ and $g(x) \in B$. This implies that $g(x) \in A \cap B$, so $x \in g^{-1}(A \cap B)$.

Showing $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ is obvious using Exercise 1.2.7 (d).

Exercise 1.2.10
Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

SOLUTION

(a) False. Consider the case where $a < b + \epsilon$ is true but $a = b$.

(b) False. Same reasoning as above.

(c) True. Firsly suppose $a < b + \epsilon$ for all $\epsilon > 0$. We need to show this implies $a \leq b$. We either have $a \leq b$ or $a > b$. However, $a > b$ is not possible as this implies there exists an ϵ small enough such that $a > b + \epsilon$. Secondly, suppose $a \leq b$. It is obvious that $a < b + \epsilon$ for all $\epsilon > 0$.

Exercise 1.2.11
Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this more interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $a + 1/n < b$.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.

(c) Between every two distinct real numbers there is a rational number.

SOLUTION

(a) For all $n \in \mathbb{N}$, there exists $a, b \in \mathbb{R}$ such that $a + 1/n < b$. [FALSE]

(b) For all real number $x > 0$, there exists an $n \in \mathbb{N}$ such that $x \geq 1/n$. [TRUE]

(c) There exists two real numbers $a < b$ such that if $r < b$ then $r < a$ for all $r \in \mathbb{Q}$. [FALSE]

Exercise 1.2.12
Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \dots)$ is decreasing.

SOLUTION

(a) For $n = 1$, $y_1 = 6 > -6$ (Base Case). Suppose $y_n > -6$ for some $n \in \mathbb{N}$.

$$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2(-6) - 6}{3} = -6$$

Hence, by induction, $y_n > -6$ for all $n \in \mathbb{N}$.

(b) Suppose $y_{n+1} < y_n$. The base case works as $y_2 = 2 < 6 = y_1$. Then,

$$\begin{aligned} y_{n+1} < y_n &\implies 2y_{n+1} - 6 < 2y_n - 6 \\ &\implies \frac{2y_{n+1} - 6}{3} < \frac{2y_n - 6}{3} \\ &\implies y_{n+2} < y_{n+1} \end{aligned}$$

Thus, $y_{n+1} < y_n$ is true for all $n \in \mathbb{N}$.

Exercise 1.2.13
For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \dots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbb{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

SOLUTION

(a) Using Exercise 1.2.5 as the base case. Suppose $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cup \cdots \cup A_n^c$ is true. Using associative law on the $n+1$c case,

$$\begin{aligned}((A_1 \cup \cdots \cup A_n) \cup A_{n+1})^c &= (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c \\ &= (A_1^c \cap \cdots \cap A_n^c) \cap A_{n+1}^c \\ &= A_1^c \cap \cdots \cap A_n^c \cap A_{n+1}^c\end{aligned}$$

(b) Consider $B_1 = \{1, 2, \dots\}, B_2 = \{2, 3, \dots\}, B_n = \{x : x \in \mathbb{N} \cap [n, \infty)\}$.

(c) To prove the infinite version of De Morgan's Law, we need to show inclusion both ways.

(i) Suppose $x \in (\bigcup_{i=1}^{\infty} A_i)^c$, then $x \notin \bigcup_{i=1}^{\infty} A_i$. This implies that $x \notin A_i \ \forall i \in \mathbb{N}$, so $x \in A_i^c \ \forall i \in \mathbb{N}$. Hence $x \in \bigcap_{i=1}^{\infty} A_i^c$.

(ii) Suppose $x \in \bigcap_{i=1}^{\infty} A_i^c$, then $x \in A_i^c \ \forall i \in \mathbb{N}$, so $x \notin A_i \ \forall i \in \mathbb{N}$. This implies that $x \notin \bigcup_{i=1}^{\infty} A_i$, hence $x \in (\bigcup_{i=1}^{\infty} A_i)^c$.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bound.

SOLUTION

(a) A real number i is the *greatest upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) i is a lower bound of A;

(ii) if b is any lower bound for A, then $i \geq b$.

(b) Assume $i \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then $i = \inf A$ *if and only if*, for every choice of $\epsilon > 0$, there $\exists a \in A$ satisfying $i + \epsilon > a$.

Proof Rephrasing the lemma gives us: Given that i is a lower bound, i is the greatest lower bound if and only if any number greater than i is not a lower bound.

(i) Suppose $i = \inf A$ and consider $i + \epsilon$ for an arbituarily chosen $\epsilon > 0$. Since $i + \epsilon > i$, part (ii) of the definition impliers that $i + \epsilon$ is not a lower bound for A. If this is the case, then there must be some element $a \in A$ such that $i + \epsilon > a$.

(ii) Conversely, assume i is a lower bound with the property that for any $\epsilon > 0$, $s + \epsilon$ is not a lower bound of A. Note that this implies that if b is any number more than i, then b is not an upperbound. Since we have argued that any larger number than i cannot be a lower bound, if b is some other upper bound for A, then $i \geq b$.

Exercise 1.3.2

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

SOLUTION

(a) Possible. Consider the set 0, where $\inf\{0\} = \sup\{0\} = 0$.

(b) Not possible as all finite sets must contain its supremum and infimum.

(c) Possible. Consider $A = \{r \in \mathbb{Q} \mid 1 < r \leq 2\}$.

Exercise 1.3.3

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

SOLUTION

(a) By definition, $\sup B$ is the greatest lower bound for A, meaning it equals $\inf A$.

(b) Part (a) proves the greatest lower bound exists using the least upper bound.

Exercise 1.3.4

Let $A_1, A_2, A_3, \dots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^{n} A_k)$.

(b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

SOLUTION

(a) $\sup(A_1 \cup A_2) = \sup\{\sup A_1, \sup A_2\}$ and $\sup(\bigcup_{k=1}^{n} A_k) = \sup\{\sup A_k \mid k = 1, \dots, n\}$

(b) This formula does not extend to infinity. Consider $A_k = [k, k+1]$, where $\bigcup_{k=1}^{\infty} A_k$ is unbounded.

Exercise 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Prostulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

SOLUTION

(a) The case of $c = 0$ is trivial as it implies that $cA = \{0\}$. Hence $\sup(cA) = c \sup A = 0$.

For $c > 0$, we need to show that $c \sup A$ is the lowest upper bound. Assume $c > 0$. Let $s = c \sup A$. Suppose $ca > s$, then $a > \sup A$ which is impossible, meaning that s is an upper bound on cA. Now suppose s' is an upper bound on cA and $s' < s$. Then $s'/c < s/c = \sup A$, meaning s'/c cannot bound A. Hence there $\exists a \in A$ such that $s'/c < a$, meaning $s' > ca$, thus s' cannot be an upper bound on cA, so $s = c \sup A$ is the least upper bound.

(b) $\sup(cA) = c \inf A$ for $c < 0$

Exercise 1.3.6

Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbituary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u + a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

SOLUTION

(a) By definition of supremum, $a \leq s$ and $b \leq t$. Adding both equations give $a + b \leq s + t$, hence $s + t$ is an upper bound.

(b) Since $a + b \leq u$ implies $b \leq u - a$, $u - a$ is an upper bound on b, meaning it is greater or equal to the least upper bound of t, giving $t \leq u - a$.

(c) From (a), we have shown that $s + t$ is an upper bound for $A + B$, hence it is sufficent to show that $s + t$ is the least upper bound.

Let $u = \sup(A + B)$, from (b) we have $t \leq u - a$ and $s \leq u - b$. Adding and rearranging gives $a + b \leq 2u - s - t$. Since $2u - s - t$ is an upper bound on $A + B$, it must be greater or equal to the least upper bound, giving $u \leq 2u - s - t$, implying $s + t \leq u$. Since u is the least upper bound, $s + t$ must equal u.

(d) Showing $s+t-\epsilon$ is not an upper bound for any $\epsilon > 0$ proves that it is the least upper bound by Lemma 1.3.8. Rearranging gives $(s - \epsilon/2) + (t - \epsilon/2)$ we know $\exists a > (s - \epsilon/2)$ and $b > (t - \epsilon/2)$, therefore $a + b > s + t - \epsilon$, meaning $s + t$ cannot be made smaller and thus is the least upper bound.

Exercise 1.3.7

Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

SOLUTION

Since it is given that a is an upper bound for A, we just have to show that a is the least upper bound, meaning any number lower than a would have an $a' \in A$ such that $a' > a$.

Suppose $a - \epsilon$ is also an upper bound for A for some $\epsilon > 0$. This is not possible has $a > a'$ and $a \in A$. Hence by contradiction, a is the lowest upper bound, meaning $a = \sup A$.

Exercise 1.3.8

Compute, without proofs, the suprema and inifima (if they exists) of the following sets:

(a) $\{m/m : m, n \in \mathbb{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.

(c) $\{n/(3n + 1) : n \in \mathbb{N}\}$.

(d) $\{m/(m + n) : m, n \in \mathbb{N}\}$.

SOLUTION

(a) $\inf = 0$ and $\sup = 1$

(b) $\inf = -1$ and $\sup = 1$

(c) $\inf = 1/4$ and $\sup = 1/3$

(d) $\inf = 0$ and $\sup = 1$

Exercise 1.3.9

(a) If $\sup A < \sup B$, show there exists an element that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq supB$.

SOLUTION

(a) By Lemma 1.3.8, we know there exists a b such that $(\sup B) - \epsilon < b$ for any $\epsilon > 0$. We can set ϵ to be small enough such that $\sup B - \sup A < \epsilon$, implying $\sup A < \sup B - \epsilon < b$ for some $b \in B$, thus b is an upper bound of A.

(b) Consider the sets $A = (-\infty, 1]$ and $B = (-\infty, 1)$. No $b \in B$ is an upperbound since $1 \in A$ and $1 > b$.

Exercise 1.3.10
(Cut Property) The *Cut Property* of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists a $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbb{R}$ is replaced by $\mathbb{Q}$.

SOLUTION

(a) If $c = \sup A = \inf B$, it is obvious that $a \leq c \leq b$ and we are done. Hence we need to show, by contradiction, that $\sup A < \inf B$ and $\sup A > \inf B$ is false.

  (i) AFSOC $\sup A < \inf B$. We can choose $c = \frac{\sup A + \inf B}{2}$, which satisfies $\sup A < c < \inf B$. Hence it is obvious that $c \notin A$ and $c \notin B$, so $c \notin A \cup B \neq \mathbb{R}$ which is a contradiction.

  (ii) AFSOC $\sup A > \inf B$. We can find a such that $a > b$ by subtracting $\epsilon > 0$ and using the definition of supremum and infimum similar to Lemma 1.3.8. Thus creating a contradiction.

  Since both alternatives are impossible, $\sup A = \inf B$.

(b) If E is finite or has a maximum element, that is $\sup E$ and we are done.

  Consider the case where E has not maximum element (for example, $\{-1/n : n \in \mathbb{N}\}$). Let B be the sets of all upper bounds of E and let $A = B^c$. It can be said $E \cap B = \emptyset$ otherwise E has a maximum element. Thus $E \subseteq A$.

  By the Cut Property, there exists c such that $a \leq c \leq b$ for all $a \in A$ and $b \in B$. Since c is an upper bound on A and $E \subseteq A$, c is also an upper bound on E. And since $c \leq b$, c is the lowest upper bound. Therefore, $c = \sup E$.

(c) Consider $A = \{r \in \mathbb{Q} \mid r^2 < 2 \text{ or } r < 0\}$, $B = A^c$ does not satisfy the cut property in $\mathbb{Q}$ as $\sqrt{2} \notin \mathbb{Q}$.

Exercise 1.3.11
Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

SOLUTION

(a) True. We know $a \leq \sup A$, $a \leq \sup B$ since $A \subseteq B$. Since $\sup A$ is the least upper bound on A, we have $\sup A \leq \sup B$.

(b) True. Let $c = \frac{\sup A + \inf B}{2}$, $c > \sup A$ implies $a < c$ and $c < \inf B$ implies $c < b$, giving $a < c < b$.

(c) False. Consider $A = (-\infty, 1)$ and $B = (1, \infty)$, $a < 1 < b$ but $\sup A \not< \inf B$ as $\sup A = \inf B = 1$.

1.4 Consequences of Completeness

Exercise 1.4.1
Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s+t$ and st?

SOLUTION

(a) This is trivial. Since $a, b \in \mathbb{Q}$, they can be expressed as a fraction of two integers. Let $a = m/n$ and $b = x/y$ where $m, n, x, y \in \mathbb{Z}$, then $a + b = \frac{my+xn}{ny}$ and $ab = \frac{mx}{ny}$, which are fractions with integer numerators and denominators, hence $a + b$ and ab are elements of $\mathbb{Q}$.

(b) AFSOC that $a + t \in \mathbb{Q}$ and $at \in \mathbb{Q}$. Let $a + t = \alpha$ and $at = \beta$, so $t = \alpha - a$ and $t = \beta/a$. Since $\alpha, \beta, -a, 1/a \in \mathbb{Q}$, using part (a) gives use $t \in \mathbb{Q}$ which is a contradiction. Hence $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$.

(c) $\mathbb{I}$ is not closed under addition or multiplication. Consider $\sqrt{2} + (-\sqrt{2}) = 0$ and $\sqrt{2} \cdot \sqrt{2} = 2$.

Exercise 1.4.2
Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upperbound for A. SHow $s = \sup A$.

SOLUTION
We can rephrase Lemma 1.3.8 using the archimedean property.

(i) AFSOC $s < \sup A$, then there exists an n such that $s + 1/n < \sup A$, contradicting $\sup A$ being the least upper bound.

(ii) AFSOC $s > \sup A$, then there exists an n such that $s - 1/n > \sup A$ where $s - 1/n$ is not an upper bound, contradicting $\sup A$ being an upper bound.

Hence, we can conclude that $\sup A = s$.

Exercise 1.4.3
Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

SOLUTION
AFSOC the $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$, so $0 < x < 1/n$ for all $n \in \mathbb{N}$ which is impossible by archimedean property.

Exercise 1.4.4
Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.

SOLUTION
To show that $\sup T = b$, it needs to satify both conditions of supremum:

(i) Since $x \leq b$ for all $x \in [a, b]$, $y \leq b$ for all $y \in T$ as $T \subseteq [a, b]$.

(ii) AFSOC b' is also an upper bound such that $b' < b$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists an $\alpha \in \mathbb{Q} \cap [b', b] \subseteq T$. This implies there exists $t \in T$ satisfying $b' < t$, which is a contradiction.

Exercise 1.4.5
Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

SOLUTION
Recall that **Corollary 1.4.4** states that *Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.*

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $t \in \mathbb{Q}$ that is between any two real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$, with $a < b$. Hence, $a - \sqrt{2} < t < b - \sqrt{2}$, meaning $a < t + \sqrt{2} < b$. By Exercise 1.4.1, $t + \sqrt{2} \in \mathbb{I}$ and we are done.

Exercise 1.4.6

Recall that a set B is *dense* in $\mathbb{R}$ if can element B can be found between any two real numbers $a < b$. WHich of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2.

(c) The set of all rational numbers p/q with $10|p| \geq q$.

SOLUTION

(a) Not dense since we cannot make $|p|/q < 1/10$.

(b) Dense.

(c) Not dense since we cannot make $|p|/q < 1/10$.

Exercise 1.4.7

Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

SOLUTION

Recall $T = \{t \in \mathbb{R} : t^2 < 2\}$ and $\alpha = \sup T$. AFSOC $\alpha^2 > 2$, we will show that there exists an $n \in \mathbb{N}$ such that $(\alpha - 1/n)^2 > 2$, contradicting the assumption that α is the least upper bound.

Using $(\alpha - 1/n)^2 > 2$, we can find $n \in \mathbb{N}$ such that $(\alpha^2 - 1/n) > 2$.

$$2 < (\alpha - 1/n)^2 = \alpha^2 - \frac{2\alpha}{n} + 1/n^2 < \alpha^2 - \frac{2\alpha - 1}{n}$$

Then

$$2 < \alpha^2 - \frac{2\alpha - 1}{n} \implies n(2 - \alpha^2) < 1 - 2\alpha$$

Since $2 - \alpha^2 < 0$, dividing reverses the inequality, giving

$$n > \frac{1 - 2\alpha}{2 - \alpha^2}$$

Hence we can pick $n \in \mathbb{N}$ such that $(\alpha^2 - 1/n) > 2$, so α is the the least upper bound which is a contradiction. Hence it is not possible for $\alpha^2 > 2$.

Exercise 1.4.8

Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B \neq \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

SOLUTION

(a) Consider $A = (0, 1) \cap \mathbb{Q}$ and $B = (0, 1) \cap \mathbb{I}$. $\sup A = \sup B = 1$ and $1 \in A, B$.

(b) Consider $J_n = (-1/n, 1/n)$, then $\bigcap_{n=1}^{\infty} J_n = \{0\}$ which is nonempty and contains a finite number of elements.

(c) Consider $L_n = [n, \infty)$, then $\bigcap_{n=1}^{\infty} L_n = \emptyset$.

(d) Impossible. Let $j_n = \bigcap_{k=1}^{n} I_k$. Since $\bigcap_{n=1}^{N} I_n \neq \emptyset$, we have $J_n \neq \emptyset$. Note that J_n is the intersection of closed intervals, making it a closed interval. With $I_{n+1} \cap J_n \subseteq J_n$, we have $J_{n+1} \subseteq J_n$. Since J_n is a series of nested bounded closed intervals, by Nested Interval Property, $\bigcap_{n=1}^{\infty} J_n = \bigcap_{n=1}^{\infty} (\bigcap_{k=1}^{n} I_k) = \bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

1.5 Cardinality

Exercise 1.5.1
Finish the following proof for Theorem 1.5.7.

Assume B is a countable set. Thus, there exists $f : \mathbb{N} \to B$, which is 1-1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from $\mathbb{N}$ onto A.

SOLUTION
Let $n_k = min\{n \in \mathbb{N} : f(n) \in A, n \notin \{n_1, n_2, \dots n_{k-1}\}\}$ and $g(k) = f(n_k)$. Since $g : \mathbb{N} \to A$ is 1-1 and onto, A is countable.

Exercise 1.5.2
Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbb{R}$ is uncountable, and then find the flow in the following erroneous proof that $\mathbb{Q}$ is uncountable:

Assume, for contradiction, that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3, \dots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} \neq \emptyset$. This contradiction implies $\mathbb{Q}$ must therefore be uncountable.

SOLUTION
The Nested Interval Property only applies to $\mathbb{R}$ and not $\mathbb{Q}$.

Exercise 1.5.3
Use the following proofs for the statements in Theorem 1.5.8.

(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \backslash A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)

Now, explain how that more general statement in (i) follows.

(b) Explain why induction cannot be used to prove part (ii) of Theorem 1.5.8 from part(i).

(c) Show how arranging $\mathbb{N}$ into the two-dimensional array

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \dots \\ 2 & 5 & 9 & 14 & \dots & \\ 4 & 8 & 13 & \dots & & \\ 7 & 12 & \dots & & & \\ \vdots & & & & & \end{array}$$

leads to a proof of Theorem 1.5.8 (ii).

SOLUTION

(a) Let $B = \{b_1, b_2, b_3, \dots\}$ and $C = \{c_1, c_2, c_3, \dot{\}}$ be countable, disjoint sets. We can define a function $g : \mathbb{N} \to B \cup C$ following a similar method of mapping $\mathbb{N}$ onto $\mathbb{Z}$, listing them as follows:

$$B \cup C = \{b_1, c_1, b_2, c_2, \dots\}$$

Implying that $B \cup C$ is countable. By letting $B = A_1$ and $C = A_2 \backslash A_1$, we can show that $A_1 \cup A_2$ is also countable.

By using induction, suppose $A_1 \cup \dots \cup A_n$ is countable, $(A_1 \dot{\cup} A_n) \cup A_{n+1}$ is the unions of two countable sets, which as proven above is also countable.

(b) Induction only shows something for each $n \in \mathbb{N}$, it does not apply in the infinite case. For example, $\bigcap_{k=1}^{n} [k, \infty) \neq \emptyset$ is true for all $n \in \mathbb{N}$, but the infinite case $\bigcap_{k=1}^{\infty} [k, \infty) \neq \emptyset$ is false.

(c) By rearranging $\mathbb{N}$ as in (c) gives us disjoint sets $C_n = \{k \in \mathbb{N} : k \text{ is in the n-th row}\}$, such that $\bigcup_{n=1}^{\infty} C_n = \mathbb{N}$. Let B_n be disjoint sets, constructed as $B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $\ldots$, $B_n = A_n \backslash (A_1 \cup \ldots A_{n-1})$. If we can find a function $f : C \to B$ that is bijective, we can show:

$$f(\mathbb{N}) = f(\bigcup_{n=1}^{\infty} C_n) = \bigcup_{n=1}^{\infty} f_n(C_n) = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

Let $f_n : C_n \to B_n$ be bijective since B_n is countable. Define $f : \mathbb{N} \to \bigcup_{n=1}^{\infty} B_n$ as

$$f(n) = \begin{cases} f_1(n) & \text{if } n \in C_1 \\ f_2(n) & \text{if } n \in C_2 \\ \vdots & \end{cases}$$

We must now show that f is bijective:

(i) Since each C_n is disjoint and each f_n is 1-1, $f(n_1) = f(n_2) \implies n_1 = n_2$, meaning f is 1-1.

(ii) SInce any $b \in \bigcup_{n=1}^{\infty} B_n$ as $b \in B_n$ for some n, $b = f_n(x)$ has a solution since f_n is onto. Letting $x = f_n^{-1}(b)$, we have $f(x) = f_n(x) = b$ as $f_n^{-1}(b) \in C_n$, meaning f is onto.

By (i) and (ii), f is bijective, so $\bigcup_{n=1}^{\infty} B_n$ is countable, implying that $\bigcup_{n=1}^{\infty} A_n$ is also countable, completing the proof.

Exercise 1.5.4

(a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

SOLUTION

(a) Example 1.5.4 provides the function $f(x) = \frac{x}{x^2-1}$ which takes the interval $(-1, 1)$ onto $\mathbb{R}$ in a 1-1 fashion. If we shift f to the interval of (a, b), we get:

$$g(x) = f(\frac{2x-1}{b-a} - a)$$

which maps (a, b) onto $\mathbb{R}$ in a 1-1 fashion.

(b) To show that $(a, \infty) \sim \mathbb{R}$, we need to find a function h that maps $\mathbb{R}$ onto (a, ∞). Consider the function $h(x) = e^x + a$, which maps $\mathbb{R}$ onto (a, ∞) in a 1-1 fashion. Hence we are done.

(c) Define $f : [0, 1) \to (0, 1)$ as

$$f(x) = \begin{cases} 1/2 & \text{if } x = 0 \\ 1/4 & \text{if } x = 1/2 \\ 1/8 & \text{if } x = 1/4 \\ \vdots & \\ x & \text{otherwise} \end{cases}$$

To show that both $[0, 1)$ and $(0, 1)$ have the both cardinality, we need to prove that f is 1-1 and onto.

We start by showing that $y = f(x)$ has exactly one solution for all $y \in (0, 1)$.

If $y = 1/2^n$, then the only solution is $y = f(1/2^{n-1})$ (or $x = 0$ in the special case $n = 1$).

Otherwise, the only solution is $y = f(y)$.

Exercise 1.5.5

(a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are hwat is meant by saying that $\sim$ is an *equivalence relation*.

SOLUTION

(a) The identity function $f(x) = x$ is trivially bijective.

(b) If $f : A \to B$ is bijective, then the inverse $f^{-1} : B \to A$ is also bijective.

(c) If $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$ is a bijection, thus $A \sim C$.

Exercise 1.5.6

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

SOLUTION

(a) Consider $I_1 = (1, 2)$, $I_2 = (2, 3)$, $\ldots$, $I_n = (n, n + 1)$.

(b) No such collection exists. Let A denote this set.

For any nonempty interval I_n, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find an $r \in \mathbb{Q}$ such that $r \in I_n$. Assigning each $I \in A$ a rational number $r \in I$ proves that $I \subseteq \mathbb{Q}$, thus I is countable.

Exercise 1.5.7

Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1-1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy)

(b) Use the fact that every real number has a decimal expansion to produce a 1-1 function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999 . . . .)

The Schröder-Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

SOLUTION

(a) Consider $f(x) = (\frac{1}{4}x, \frac{1}{3})$.

(b) Let $g : S \to (0, 1)$ be a function that interweaves decimals in the representation without trailing nines, padding with zeros if necessary. $g(0.12, 0.34) = 0.1324$, $g(0.1\bar{9}, 0.2) = g(0.2, 0.2) = 0.22$, $g(0.4, 0.89) = 0.4809$, $g(0.6, 0.\bar{7}) = 0.67\overline{07}$, etc.

To prove that g is 1-1, we need show that there does not exists any solutions for $g(x_1, y_1) = g(x_2, y_2)$. Every real number can be written with tow representations, one with trailing 9's and one without. However, $g(x, y) = 0.d_1d_2d_3 \ldots \bar{9}$ is impossible as it would imply both x and y has trailing 9's, which contradicts the definition of g. Therefore, $g(s)$ is unique and so g is 1-1.

g is not onto since $g(x, y) = 0.1$ has no solutions.

Exercise 1.5.8

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B will always give a sum of 2 or less. Show B must be finite or countable.

SOLUTION

It is obvious that $B \cap [a, 2)$ must be finite for some $a \in (0, 2)$, otherwise, we can choose $\lceil \frac{2}{a} \rceil$ number of elements from $B \cap [a, 2)$ would result in a sum of more than 2. Since B is the countable union of finite sets $\bigcup_{n=1}^{\infty} B \cap [\frac{1}{n}, 2)$, B must be countable or finite.

Exercise 1.5.9
A real number $x \in \mathbb{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_n x^n + a_{n-1}^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$ and $\sqrt{3}+\sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbb{N}$ and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

SOLUTION

(a) The first two are trivial as they are roots of $x^2 - 2 = 0$ and $x^3 - 2 = 0$. For $\sqrt{3}+\sqrt{2}$, we can start working backwards:

$$\begin{aligned} x = \sqrt{3}+\sqrt{2} &\implies x^2 = 5 + 2\sqrt{6} \\ &\implies (x^2-5)^2 - 24 = 0, \end{aligned}$$

thus giving a polynomial, hence $\sqrt{2}+\sqrt{3}$ is algebraic.

(b) We need to show that $A_n \sim \mathbb{Z}^n \sim \mathbb{N}^n \sim \mathbb{N}$ and hence countable.

(i) $A_n \sim \mathbb{Z}^n$ since choosing n integer coefficients for a polynomial of degree n is equivilant to an ordered list of n integers.

(ii) $\mathbb{Z}^n \sim \mathbb{N}^n$ since $f : \mathbb{N}^n \to \mathbb{Z}^n$ is just the piecewise application of $g : \mathbb{N} \to \mathbb{Z}$.

(iii) $\mathbb{N}^n \sim \mathbb{N}$ since it is the intersection of finite sets $\bigcup_{k=1}^{\infty} \{(x_1, \ldots, x_n) : x_1 + \cdots + x_n = k\}$.

(c) The set of algebraic numbers is countable as it is the union of countable sets $\bigcup_{n=1}^{\infty} A_n$. The set of transcendental numbers is uncountable, otherwise, it implies that the set of real numbers is countable, which is not possible.

Exercise 1.5.10

(a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is countable.

(b) Now let A be the set of all $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

SOLUTION

(a) AFSOC a does not exists, so $C \cap [a,1]$ is countable for all $a \in (0,1)$. Therefore $C = \bigcup_{n=1}^{\infty} C \cap [1/n, 1]$ is also countable, which is a contradiction. Hence, there exist a $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Since $\alpha = \sup A$, for any $\epsilon > 0$, $C \cap [\alpha+\epsilon, 1]$ is countable. Therefore, $C \cap [\alpha, 1] = \alpha \cup \bigcup_{n=1}^{\infty} C \cap [1/n, 1]$ is countable.

(c) No. Consider $C = \{1/n : n \in \mathbb{N}\}$. It has $C \cap [\alpha, 1]$ finite for every α, but $C \cap [0,1]$ is infinite.

Exercise 1.5.11
(Schröder-Bernstein Theorem) Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$.

The strategy is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' and A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

(b) Set $A_1 = X \backslash g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is a similar collection in Y.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

(d) Let $A' = X \backslash A$ and $B' = Y \backslash B$. Show g maps B' onto A'.

SOLUTION

(a) $f : A \to B$ and $g : B' \to A'$ are bijective, therefore we can define

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in A' \end{cases}$$

which is bijective.

(b) If $A_1 = \emptyset$, we are done as it implies $g(Y) = X$ so $X \sim Y$

Since $A_n \subseteq X$, then $f(A_n) \subseteq Y$, hence $A_n \cap A_1 = \emptyset$ for all $n \in \mathbb{N}$ as $A_1 = X \backslash g(Y)$.

Note that $g \circ f$ is bijective implies that $g(f(A \cap B)) = g(f(A)) \cap g(f(B))$. We will show that by inclusion both ways. Let h be an arbitrary bijective function.

(i) Let $y \in h(A \cap B)$, so there exists $x \in A \cap B$ such that $h(x) = y$, then $x \in A$ and $x \in B$. This implies that $h(x) \in h(A)$ and $h(x) \in h(B)$, hence $y = h(x) \in h(A) \cap h(B)$.

(ii) Let $y \in h(A) \cap h(B)$, so $y \in h(A)$ and $y \in h(B)$, then there exist $x \in A$ and $x \in B$ such that $h(x) = y$. This implies $x \in A \cap B$, hence $y \in h(A \cap B)$.

Now to show pairwise disjointment for all subsets, consider $i, j \in \mathbb{N}$, with $i < j$.

$$A_j \cap A_i = g(f(A_j \cap A_i)) = g(f(A_j)) \cap g(f(A_i)) = A_{j-1} \cap A_{i-1} = \cdots = A_{j-i+1} \cap A_1 = \emptyset$$

Similarly, $\{f(A_n) : n \in \mathbb{N}\}$ is also pairwise disjoint as

$$A_j \cap A_i = \emptyset \implies f(A_j) \cap f(A_i) = f(A_j \cap A_i) = f(\emptyset) = \emptyset$$

(c) $f(A) = B$ as $f(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f(A_n) = B$. ($B$ is defined to be the range of f.)

(d) To show that $g(B') = A'$, we will show inclusion both ways:

(i) $g(B') \subseteq A'$. AFSOC $g(b') \in A$, then since $A_1 \cap g(Y) = \emptyset$, $g(b') \notin A$, implying $g(b') \in \bigcup_{n=2}^{\infty} A_n = g(B)$, meaning $\exists b \in B$ such that $g(b) = g(b')$, contradicting the fact that g is 1-1.

(ii) $g(B') \supseteq A'$. AFSOC $\exists a' \in A'$, with $a' \notin g(B')$. Since $A' \subseteq g(Y)$, we have $a' \in g(B)$, since $a' \notin g(B')$, and $g(B) \subseteq A$ contradicting $a' \in A$.

1.6 Cantor's Theorem

Exercise 1.6.1
Show that $(0, 1)$ is uncountable if and only if $\mathbb{R}$ is uncountable.

SOLUTION
In Exercise 1.5.4 (a), we have found a bijection $f : (0, 1) \to \mathbb{R}$. Suppose $g : (0, 1) \to \mathbb{N}$ is some map, we must show g is bijective if and only if $(g \circ f) : \mathbb{R} \to \mathbb{N}$ is bijective. This is true as if g is bijective, then $(g \circ f)$ is bijective as it is the composition of bijective functions. Similarly, if $(g \circ f)$ is bijective, then $(g \circ f) \circ f^{-1} = g$ is bijective.

This proof works as $(0, 1)$ is uncountable, implies that g is not bijective, then $(g \circ f) : \mathbb{R} \to \mathbb{N}$ is not bijective, implying that $\mathbb{R}$ is not countable and hence uncountable.

Exercise 1.6.2

(a) Explain why the real number $x = .b_1b_2b_3b_4 \dots$ cannot be $f(1)$.

(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbb{N}$.

(c) Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

SOLUTION

(a) Since $b_1 \neq a_{11}$, the first digit of $f(1)$ differ from x, hence cannot be equal.

(b) Since $b_n \neq a_{nn}$, the n-th digit of $f(n)$ differ from x, hence cannot be equal.

(c) Since x is not in the list, it is a contradiction.

Exercise 1.6.3
Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.

(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.

(b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as .5 or as .4999 . . . . Doesn't this cause some problems?

SOLUTION

(a) False, since the constructed number has aninfinte number of decimals, it is irrational.

(b) No, since changing the n-th digit would still result in a different number.

Exercise 1.6.4
Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \dots) : a_n = 0 or 1\}.$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \dots)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, 1, \dots)$.
Give a rigorous argument showing S is uncountable.

SOLUTION
Similar to the proof in Exercise 1.6.1, suppose there exists a function $f : \mathbb{N} \to S$ that is 1-1 and onto. For each $n \in \mathbb{N}$, $f(n)$ is an element in S, represented as:

$$f(n) = (a_{n1}, a_{n2}, a_{n3}, \dots)$$

where $a_{nm} = 0$ or 1 for each $m, n \in \mathbb{N}$. We construct a $s \in S$, with $s = (b_1, b_2, b_3, \dots)$, where $b_n = 0$ if $a_nn = 1$ and $b_n = 1$ otherwise. Since the n-th digit of sequence s differs from $f(n)$ for all $n \in \mathbb{N}$, $s \notin S$, which is a contradiction. Hence S is not countable, therefore uncountable.

Alternatively, we can define $g : S \to \bar{S}$ where $g(a_1, a_2, a_3, \dots) = .a_1a_2a_3 \dots$ which is trivially bijective, and $h : \mathbb{R}_2 \to \mathbb{R}$ which converts a number in base 2 to base 10 and is clearly bijective. Hence $h \circ g : S \to \mathbb{R}$ is bijective and hence shows that $S \sim \mathbb{R}$ so S is uncountable.

Exercise 1.6.5

(a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)

(b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

SOLUTION

(a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

(b) There are n elements, with 2 possible states: included or excluded, implying that there are 2^n elements.

Exercise 1.6.6

(a) Using the particular set $A = \{a, b, c\}$, exhibit two different 1-1 mappings from A into $P(A)$.

(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a $1-1$ map $g : C \to P(C)$.

(c) Explain *why*, in parts (a) and (b), it is impossible to construct mappings that are *onto*.

SOLUTION

(a) Consider $f(x) = \{x\}$ and $f(x) = \{x\}$ for $x \neq b, \{x, c\}$ for $x = b$.

(b) Let $g(x) = \{x\}$.

(c) $|P(C)| > |C|$.

Exercise 1.6.7

Return to the particular functions constructed in Exercise 1.6.6 and construct the subset B that results using the preceding rule. In each case, note that B is not in the range of the function used.

SOLUTION

(i) For $A = \{a, b, c\}$, $B_A = \{b\}$

(ii) For $C = \{1, 2, 3, 4\}$, $B_C = \emptyset$

Exercise 1.6.8

(a) First, show that the case $a' \in B$ leads to a contradiction.

(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

SOLUTION

(a) AFSOC $a' \in B$. This means $a' \notin f(a')$, then $a' \notin B$ which is a contratiction.

(b) AFSOC $a' \notin B$. This means $a' \in f(a')$, then $a' \in B$, which is a contradiction.

Exercise 1.6.9

Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling arguement showing that $P(\mathbb{N}) \sim \mathbb{R}$.

SOLUTION

We can make a function $f : P(\mathbb{N}) \to S$, with S following the definition given in Exercise 1.6.4, being a set consisting of all sequences of 0's and 1's.

$$f(P_N) = (a_1, a_2, a_3, \dots) \text{ with } a_n = 0 \text{ if } a_n \notin P_N \text{ and 1 otherwise}$$

f is clearly onto as every element of S can be mapped to by its definiton. To show that is it injective, AFSOC that there exists $X, Y \in P(\mathbb{N})$, where $X \neq Y$, such that $f(X) = f(Y)$. There will exists $n \in \mathbb{N}$ where $n \in X$ but $n \notin Y$. By definiton, the n-th element of $f(X)$ is 1, but for $f(Y)$ is 0, which is a contradiction.

Using Exercise 1.6.4, $S \sim \mathbb{R}$, hence $P(\mathbb{N}) \sim \mathbb{R}$.

Exercise 1.6.10

As a final exercise, answer each of the following by establishing a 1-1 correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ countable or uncountable?

(b) Is the set of all functions from $\mathbb{N}$ to $\{0, 1\}$ countable or uncountable?

(c) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an *antichain* if no element of A is a subset of any other element of A. Does $P(\mathbb{N})$ contain an uncountable anitchain?

SOLUTION

(a) Both elements of $\{0, 1\}$ can be mapped to an element in $\mathbb{N}$, hence we can find a bijective function to show that this set is the same as $\mathbb{N}^2$ hence countable.

(b) As shown in Exercise 1.6.9, this set maps to $S \sim \mathbb{R}$ which is uncountable.

(c) Let $\mathcal{A}$ be an antichain of $P(\mathbb{N})$ and let $\mathcal{A}_l$ be the sets in $\mathcal{A}$ of size l. For finite l, $\mathcal{A}_l$ is countable since $\mathcal{A}_l \subseteq \mathbb{N}^l$ is countable. Hence the countable union $\bigcup_{l=0}^{\infty} \mathcal{A}_l = \mathcal{A}$ is countable.

If l is infinite, only countably many elements of $\mathcal{A}$ can be finite, while there will be uncountably many that myst be infinite, hence forming an uncountable antichain $\mathcal{A}$.

Chapter 2

Sequences and Series

2.2 The Limit of a Sequence

Exercise 2.2.1
What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent dequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

SOLUTION

(i) Consider the sequence $x_n = \sin x$. (x_n) verconges to 0 as we can choose $\epsilon = 1$ such that $|x_n - 0| < \epsilon$ for any value of $n \in \mathbb{N}$.

(ii) There is not vercongent sequence that diverges. THe proof is obvious and left as an exercise for the reader.

(iii) The sequence in (i) verconges to $x \in \mathbb{R}$ by choosing $\epsilon = x + 2$.

(iv) The definition of verconges described a bounded sequence.

Exercise 2.2.2
Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin\left(n^2\right)}{\sqrt[3]{n}}$.

SOLUTION

(a) For $\epsilon > 0$, we can choose $N = 1/\epsilon$, such that for all $n > N$,

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{-3}{5(5n+4)}\right| = \frac{3}{5(5n+4)} < \frac{1}{n} < \epsilon \implies n > \frac{1}{\epsilon}$$

(b) For $\epsilon > 0$, we can choose $N = 2/\epsilon$, such that for all $n > N$,

$$\left|\frac{2n^2}{n^3+3} - 0\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} < \epsilon \implies n > \frac{2}{\epsilon}$$

(c) For $\epsilon > 0$, we can choose $N = 1/\epsilon^3$, such that for all $n > N$,

$$\left|\frac{\sin(n^2)}{\sqrt[3]{n}} - 0\right| = \frac{\sin(n^2)}{n^{1/3}} \leq \frac{1}{n^{1/3}} \implies n > \frac{1}{\epsilon^3}$$

Exercise 2.2.3
Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

SOLUTION
(a) Find a college in the United States that has every student under seven feet.

(b) Find a college in the United States that have every professor giving at least one grade lower than a B.

(c) Find a student in each college in the United States that is under six feet

Exercise 2.2.4
Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of onesthat does not converge to one.

(b) A sequence with an infinite number of oes that converges to a limit not equals to one.

(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

SOLUTION
(a) Consider $x_n = (-1)^n$.

(b) Impossible. Suppose $\lim x_n = a \neq 1$. Since there are infinite ones, we cannot find $N > 0$ for $\epsilon < |1 - a|$.

(c) Consider the sequence $(1, 2, 1, 1, 3, 1, 1, 1, \dots)$.

Exercise 2.2.5
Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$.

(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following the Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

SOLUTION
(a) For all $n > 5$, we have $[[5/n]] = 0$, hence $\lim a_n = 0$.

(b) The insides converges to $4/3$ from above, so $\lim a_n = 1$.
Some sequences eventually reach their limit, meaning that N no longer has to increase.

Exercise 2.2.6
Prove Theorem 2.2.7. To get started assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

SOLUTION

Theorem 2.2.7 (Uniqueness of Limits). The limit of a sequence, when it exists, must be unique.

AFSOC that $\lim a_n = a$ and $\lim a_n = b$, such that $a \neq b$. Then for $\epsilon > 0$, choosing $n > N_a$ implies $|a_n - a| < \epsilon/2$ and choosing $n > N_b$ implies $|a_n - b| < \epsilon/2$. Let $N = max(N_a, N_b)$, hence $|a_n - a| < \epsilon/2$ and $|a_n - b| < \epsilon/2$. By triangle inequality,

$$\epsilon = \epsilon/2 + \epsilon/2 > |a_n - a| + |a_n - b| = |a - a_n| + |a_n - b| \leq |a - b|$$

showing that $a = \lim a = b$, which is a contradiction.

Exercise 2.2.7

Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *Frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such taht $a_n \in A$.

- (a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?
- (b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?
- (c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?
- (d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

SOLUTION

(a) $(-1)^n$ is frequently but not eventually in $\{1\}$.

(b) Eventually implies frequently, but the converse is not true.

(c) $(x_n) \to x$ *if and only if* x_n is eventually in any ϵ-neighborhood around x.

(d) (x_n) is frequently in $(1.9, 2.1)$ but not always eventually (considering $x_n = 2(-1)^n$).

Exercise 2.2.8

For some additional practive with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $(0, 1, 0, 1, 0, 1, \dots)$ zero heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counterexample.

(d) From the logical negation of the above definition. That is, complete the sentence: A sequence is *not zero-heavy* if ....

SOLUTION

(a) The sequence is zero-heavy, choosing $M = 1$.

(b) Yes. If there were a finite number of zeros, with the last zero at x_K, then choosing $N > K$ would lead to a contradiction.

(c) No, consider $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \dots)$ where the gap between the 0's grows to infinity.

(d) A sequence is not zero-heavy if for all $M \in \mathbb{N}$, there exists some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $N \leq n \leq N + M$, $x_n \neq 0$.

2.3 The Algebraic and Order Limit Theorem

Exercise 2.3.1
Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

SOLUTION
(a) For $\epsilon^2 > 0$, there exists $N > 0$, such that for all $n > N$, $x_n < \epsilon^2$, implying $\sqrt{x_n} < \epsilon$.

(b) We want to show that $|\sqrt{x_n} - \sqrt{x}| < \epsilon$. Multiplying by $(\sqrt{x_n} + \sqrt{x})$ gives us $|x_n - x| < (\sqrt{x_n} + \sqrt{x})\epsilon$. Since (x_n) converges, it is bounded, giving $|x_n| \leq M$, then $\sqrt{x_n} \leq \sqrt{M}$.

$$|x_n - x| < (\sqrt{x_n} + \sqrt{x})\epsilon \leq (M + \sqrt{x})\epsilon$$

Setting $|x_n - x| < (M + \sqrt{x})\epsilon$ for $n > N$, we get

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{M + \sqrt{x}} = \epsilon$$

thus completing the proof.

Exercise 2.3.2
Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $(\frac{2x_n - 1}{3}) \to 1$;

(b) $(1/x_n) \to 1/2$;

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

SOLUTION
(a) We have $|\frac{2x_n-1}{3} - 1| = |\frac{2}{3}x_n - \frac{4}{3}| = \frac{2}{3}|x_n - 2|$. Setting $|x_n - 2| < \frac{3}{2}\epsilon$ for all $n > N$, we get $|\frac{2x_n-1}{3} - 1| < \epsilon$.

(b) We set N such that $|x_n - 2| <$. Since x_n is at least 1, we can bound $|1/x_n| \leq 1$, giving

$$|1/x_n - 1/2| = \frac{|2 - x_n|}{|2x_n|} \leq \frac{|x_n - 2|}{2} \leq \frac{\epsilon}{2} < \epsilon.$$

Exercise 2.3.3
(Squeeze Theorem) Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

SOLUTION
Let $\epsilon > 0$, set $N > 0$ so that $|x_n - l| < \epsilon/3$ and $|z_n - l| < \epsilon/3$. By triangle inequality, we get

$$|z_n - x_n| \leq |z_n - l| + |l - x_n| < 2\epsilon/3$$

Since $z_n \geq y_n \geq x_n$, $|y_n - x_n| = y_n - x_n \leq z_n - x_n$. So by triangle inequality,

$$|y_n - l| \leq |y_n - x_n| + |x_n - l| \leq |z_n - x_n| + |x_n - l| < 2\epsilon/3 + \epsilon/3 = \epsilon$$

Exercise 2.3.4
Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim \left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$.

SOLUTION

(a) By ALT,

$$\begin{aligned}\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) &= \frac{\lim(1+2a_n)}{\lim(1+3a_n-4a_n^2)} \\ &= \frac{1+2\lim a_n}{1+3\lim a_n-4(\lim a_n)^2} \\ &= 1\end{aligned}$$

(b) By ALT,

$$\begin{aligned}\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) &= \lim\left(\frac{a_n^2+4a_n}{a_n}\right) \\ &= \lim(a_n+4) = 4+\lim a_n = 4\end{aligned}$$

(c) By ALT,

$$\begin{aligned}\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \lim\left(\frac{2+3a_n}{1+5a_n}\right) \\ &= \frac{\lim(2+3a_n)}{\lim(1+5a_n)} \\ &= \frac{2+3\lim a_n}{1+5\lim a_n} = 2\end{aligned}$$

Exercise 2.3.5

Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

SOLUTION

Obviously, if $\lim x_n = \lim y_n = l$, then for $\epsilon > 0$, we can set $N > 0$ such that $|x_n - l| < \epsilon$ and $|y_n - l| < \epsilon$ for $n > N$, thus for all $n > 2N$, $|z_n - l| < \epsilon$ will be true.

Conversely, if $(z_n) \to l$, for $\epsilon > 0$, we can set $2N > 0$ such that for all $n > 2N$, $|z_n - l| < \epsilon$, implying that for $n > N$, $|x_n - l| < \epsilon$ and $|y_n - l| < \epsilon$ will also be true.

Exercise 2.3.6

Consider the sequence given by $b_n = n - \sqrt{n^2+2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

SOLUTION

Let $(b_n) \to b$. To remove the radical from the term, consider:

$$(n-\sqrt{n^2+2n})(n+\sqrt{n^2+2n}) = n^2-(n^2+2n) = -2n$$

By ALT,

$$b = \lim\left(\frac{-2n}{n+\sqrt{n^2+2n}}\right) = \lim\frac{-2}{1+\sqrt{1+2/n}} = \frac{-2}{1+\sqrt{1+2\lim(1/n)}} = \frac{-2}{1+\sqrt{1+0}} = -1$$

thus showing that the limit exists and $(b_n) \to 1$.

Exercise 2.3.7

Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

SOLUTION

(a) Consider $x_n = n$ and $y_n = -n$ but $x_n + y_n = 0$ for all $n \in \mathbb{N}$.

(b) Impossible. By ALT, $\lim y_n = \lim(x_n + y_n) - \lim x_n$ must converge, which is a contradiction.

(c) Consider $b_n = 1/n$, then $1/b_n = n$, so $(1/b_n) \to \infty$.

(d) Impossible. Let $|b_n| \leq M$ and $|a_n - b_n| \leq N$. By triangle inequality,

$$|a_n| \leq |a_n - b_n| + |b_n| \leq M + N$$

, hence is also bounded, which is a contradiction.

(e) Consider $a_n = 1/n$ and $b_n = n$, then $a_n b_n = 1$.

Exercise 2.3.8
Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

SOLUTION

(a) Applying the Algebraic Limit Theorem repeatedly results in $(x_n^d) \to x^d$, meaning

$$\lim p(x_n) = \lim \sum_{i=0}^{d} k_i x_n^i = \sum_{i=0}^{d} k_i (\lim x_n)^i = p(\lim x_n) = p(x)$$

(b) Consider the sequence (x_n), where $x_n = 1/n$, and let f be defined as

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $(x_n) \to x$ but $f(x_n) \to 1 \neq f(x) = 0$.

Exercise 2.3.9

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

SOLUTION

(a) Since (a_n) is bounded, let $|a_n| \leq M_a$. Consider

$$|a_n b_n| = |a_n||b_n| \leq M|b_n|$$

For $\epsilon > 0$, we can set $N > 0$ such that $|b_n| < \epsilon/M$, giving us $|a_n b_n| < \epsilon$, hence $\lim(a_n b_n) = 0$.

ALT does not hold as (a_n) does not converge.

(b) If (a_n) converges, then (a_nb_n) converges by ALT. Otherwise, (a_nb_n) does not converge.

(c) Shown in (a), $\lim(a_nb_n) = 0 = ab$ for $b = 0$, proving part (iii) of the ALT.

Exercise 2.3.10
Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then $(b_n) \to b$.

SOLUTION

(a) False. Consider $a_n = n$ and $b_n = -n$, then $\lim(a_n - b_n) = 0$ but $\infty = \lim a_n \neq \lim b_n = -\infty$.

(b) True. Since $|b_n - b| < \epsilon$, then without loss of generality, we assume $|b_n| > |b|$, so

$$||b_n| - |b|| = |b_n| - |b| = |b_n - b + b| - |b| \leq |b_n - b| + |b| - |b| = |b_n - b| < \epsilon$$

(c) True. By ALT, $\lim b_n = \lim a_n + \lim(b_n - a_n) = a + 0 = a$.

(d) True. Note that $0 \leq |b_n - b| \leq a_n$. Since $(a_n) \to 0$, by Squeeze Theorem in Exercise 2.3.3, $\lim |b_n - b| = 0$, hence $\lim b_n = b$.

Exercise 2.3.11
(Cesaro Means).

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

SOLUTION

(a) Let $D = \sup\{|x_n - x| : n \in \mathbb{N}\}$ and $0 < \epsilon < D$, we have

$$|y_n - x| = \left|\frac{x_1 + x_2 + \cdots + x_n}{n} - \frac{xn}{n}\right| \leq \left|\frac{|x_1 - x| + |x_2 - x| + \cdots + |x_n - x|}{n}\right| \leq D$$

Set $N_1 > 0$ such that $|x_n - x| < \epsilon/2$ for $n > N_1$, giving

$$|y_n - x| = \left|\frac{|x_1 - x| + \ldots |x_{N_1} - x| + \cdots + |x_n - x|}{n}\right| \leq \left|\frac{N_1D + (n - N_1)\epsilon/2}{n}\right|$$

Set $N_2 > 0$ such that for all $n > N_2$,

$$0 < \frac{N_1(D - \epsilon/2)}{n} < \epsilon/2$$

Therefore

$$|y_n - x| \leq \left|\frac{N_1(D - \epsilon/2)}{n} + \epsilon/2\right| \leq \left|\frac{N_1(D - \epsilon/2)}{n}\right| + \epsilon/2 < \epsilon$$

Setting $N = max\{N_1, N_2\}$ completes the proof.

(b) $x_n = (-1)^n$ diverges but $(y_n) \to 0$.

Exercise 2.3.12
A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.

(c) If every a_n is rational, then a is rational.

SOLUTION
(a) True. For all $n \in \mathbb{N}$, $a_n \geq \sup B$, by Order Limit Theorem, $a = \lim a_n \geq \sup B$.

(b) True. If $a \in (0, 1)$, then there would exists an ϵ-neighborhood inside $(0, 1)$ that a_n would have to fall in, contradicting the fact the $a \notin (0, 1)$.

(c) False. Consider the sequence $(1, 1.4, 1.41, 1.414, \dots) \to \sqrt{2}$.

Exercise 2.3.13
(Iterated Limits). Given a doubly indexed array a_{mn} where $m, n \in \mathbb{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m + n)$ and compute the *iterated* limits

$$\lim_{n\to\infty} \left(\lim_{m\to\infty} a_{mn} \right) \quad \text{and} \quad \lim_{m\to\infty} \left(\lim_{n\to\infty} a_{mn} \right).$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

(b) Let $a_{mn} = 1/(m + n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2 + n^2)$.

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbb{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exists, then all three limits must be equal.

SOLUTION
(a)

$$\lim_{n\to\infty} \left(\lim_{m\to\infty} a_{mn} \right) = 1 \quad \text{and} \quad \lim_{m\to\infty} \left(\lim_{n\to\infty} a_{mn} \right) = 0$$

(b) For $a_{mn} = a/(m + n)$, both iterated limits exists and equal to $\lim_{m,n\to\infty} a_{mn} = 0$.

For $a_{mn} = mn/(m^2 + n^2)$, both iterated limits equal to zero. However, $\lim_{m,n\to\infty} a_{mn}$ does not exists.

(c) Consider $b_{mn} = 1/\min\{m, n\}$. WLOG, we can fix m and as n approaches infinity, $n > m$ causing the limit to be $1/m$. To ensure that the limit does not exists, we can introduce the term $(-1)^{n+m}$, giving

$$a_{mn} = \frac{(-1)^{m+n}}{\min\{m, n\}}$$

This works as the inner limits of the iterated limits are not defined since it is oscillating between $1/m$ and $-1/m$. However, if both $m, n \geq N$ for some $N > 0$, the limit equals zero as $0 < \lim |a_{mn}| \leq \lim |1/(m + n)| \leq \lim |1/2N| = 0$.

(d) For $\epsilon > 0$, we can choose $N > 0$ such that for all $m, n > N_0$, $|a_{mn} - a| < \epsilon/2$. For each fixed $m \in \mathbb{N}$, we can choose $N_m > 0$ such that for $n > N_m$, $|a_{mn} - b_m| < \epsilon/2$. By triangle inequality,

$$|b_m - a| = |b_m - a_{mn} + a_{mn} - a| \leq |b_m - a_{mn}| + |a_{mn} - a| \leq \epsilon/2 + \epsilon/2 = \epsilon$$

Hence we can choose $N = \max\{N_0, N_m\}$ and we are done.

(e) Let $b_m = \lim_{n\to\infty}(a_{mn})$, $c_n = \lim_{m\to\infty}(a_{mn})$, and $a = \lim_{m,n\to\infty}(a_{mn})$. In (d), we have shown that $(b_m) \to a$. A similar argument shows $(c_n) \to a$. Thus all three limits are equal to a.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{a - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

SOLUTION

(a) Calculating the first few terms, we get $(3, 1, 1/3, \dots)$. I conjecture that (x_n) is decreasing.

Suppose $x_{n+1} < x_n$, then

$$4 - x_n > 4 - x_{n+1} \implies \frac{1}{4 - x_n} < \frac{1}{4 - x_{n+1}} \implies x_{n+2} < x_{n+1}$$

Thus, (x_n) is decreasing. Since $x_n < 4$ for all $n \in \mathbb{N}$, $4 - x_n > 0$ and hence $x_{n+1} > 0$. Since (x_n) is decreasing and bounded below, (x_n) converges.

(b) By definition of limits, changing the index of the sequence does not change its limit.

(c) Since $x = \lim x_n = \lim x_{n+1}$,

$$x_{n+1} = \frac{1}{4 - x_n} \implies x = \frac{1}{4 - x} \implies x^2 - 4x + 1 = 0 \implies x = 2 \pm \sqrt{3}$$

Since $x_n < 3$, then $x < 3$, hence $x \neq 2 + \sqrt{3}$ and $x = 2 - \sqrt{3}$.

Exercise 2.4.2

(a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$.

What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

SOLUTION

(a) The argument only works if the series converges, which does not in this case.

(b) Yes it can as $0 < y_n < 3$ and (y_n) is increasing (Proving this is left as an exercise for the reader).

Exercise 2.4.3

(a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$$

converge? If so, find the limit.

SOLUTION

(a) Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$, clearly $x_2 > x_1$. Suppose $x_{n+1} > x_n$, then

$$2 + x_{n+1} > 2 + x_n \implies \sqrt{2 + x_{n+1}} > \sqrt{2 + x_n} \implies x_{n+2} > x_{n+1}$$

showing that x_n is monotonically increasing.

Next we show that $x_n \le 2$, clearing $x_1 = \sqrt{2} \le 2$. Suppose $x_n \le 2$, then

$$2 + x_n \le 4 \implies \sqrt{2 + x_n} \le 4 \implies x_{n+1} \le 4$$

Since (x_n) is monotone and bounded, the monotone convergence theorem tells us that $(x_n) \to x$. Equating the limits on both sides gives

$$x = \sqrt{2 + x} \implies x^2 - x - 2 = 0 \implies x = \frac{1}{2} \pm \frac{3}{2}$$

Since $x > 0$, we have $x = 2$.

(b) Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$, clearly $x_2 > x_1$. Suppose $x_{n+1} > x_n$, then

$$2x_{n+1} > 2x_n \implies \sqrt{2x_{n+1}} > \sqrt{2x_n} \implies x_{n+2} > x_{n+1}$$

showing that (x_n) is monotonically increase. Next we show that $x_n \le 2$, which is clearly true for x_1. Suppose $x_n \le 2$, then

$$2x_n \le 4 \implies x_{n+1} = \sqrt{2x_n} \le 2$$

Hence, x_n is bounded. By monotone convergence theorem, the sequence converges, hence $(x_n) \to x$. Taking limits on both sides,

$$x = \sqrt{2x} \implies x^2 - 2x = 0 \implies x = 1 \pm 1$$

Since $x > 0$, then $x = 2$.

Exercise 2.4.4

(a) In Section 1.4, we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbb{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC

These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as out starting axiom for building a proper theory of the real numbers.

SOLUTION

(a) MCT tells us $(1/n)$ converges, obviously it must converge to zero, giving us $|1/n - 0| = 1/n < \epsilon$ for any $\epsilon > 0$, which is the Archimedean Property.

(b) We have $I_n = [a_n, b_n]$ with $a_n \leq b_n$ since $I_n \neq \emptyset$. Since $I_{n+1} \subseteq I_n$, we must have $b_{n+1} \leq b_n$ and $a_{n+1} \geq a_n$, the MCT tells us that $(a_n) \to a$ and $(b_n) \to b$. By Order Limit Theorem, we have $a \leq b$, hence $a \in I_n$ for all $n \in \mathbb{N}$, meaning $a \in \bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Exercise 2.4.5
(Calculating Square Roots) Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove thath $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

SOLUTION

(a) Clearing $x_1 \geq 2$. For any n,

$$\begin{aligned} x_{n+1}^2 &= \left[\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right]^2 \\ &= \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) \\ &= \frac{1}{4}\left(x_n^2 - 4 + \frac{4}{x_n^2}\right) + 2 \\ &= \frac{1}{4}\left(x_n + \frac{2}{x_n}\right) + 2 \geq 2 \end{aligned}$$

Now we show $x_n - x_{n+1} \geq 0$.

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= \frac{1}{2}x_n - \frac{1}{x_n} \geq 0 \because x_n \geq 0 \end{aligned}$$

MCT tells us that $(x_n) \to x$. By taking limits on both sides,

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right) \implies \frac{1}{2}x^2 = 1 \implies x = \pm\sqrt{2}$$

Since $x_n \geq 0$, $x = \sqrt{2}$.

(b)

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

Proof of this is similar to (a) and left as an exercise for the reader.

Exercise 2.4.6
(Arithmetic-Geometric Mean)

(a) Explain why $\sqrt{xy} \leq (x + y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

SOLUTION

(a) We have

$$\sqrt{xy} \le \frac{x+y}{2} \iff 4xy \le x^2 + 2xy + y^2 \iff 0 \le (x-y)^2$$

(b) As shown above, if $\lim x_n = x$ and $\lim y_n = y$, then $x_n = y_n$ is the only fixed point, hence $x = y$. Thus we only need to show both sequences converge.

The inequality $0 \le x_n \le y_n$ is always true. It is obvious both terms will always be positive. Suppose $x_n \le y_n$, then by (a), $x_{n+1} \le y_{n+1}$ true.

To continue, $x_n \le y_n$ implies $(x_n + y_n)/2 = y_{n+1} \le y_n$. Similarly, $\sqrt{x_n y_n} = x_{n+1} \ge x_n$. This means both sequences are monotone and bounded by each other, hence Monotone Convergence Theorem tells us both sequences converge.

Exercise 2.4.7

(Limit Superior) Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \ge n\}$ converges.

(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \le \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ *if and only if* $\lim a_n$ exists. In this case, all three share the same value.

SOLUTION

(a) (y_n) is decreasing, hence MCT tells us it converges.

(b) The *limit inferior* of (a_n), or $\liminf a_n$, is defined by $\liminf a_n = \lim x_n$, where $x_n = \inf\{a_k : k \ge n\}$.

Since (x_n) is increasing, MCT tells us it converges, hence the limit inferior exists.

(c) Obviously, $\inf\{a_k : k \ge n\} \le \sup\{a_k : k \ge n\}$, so Order Limit Theorem implies $\liminf a_n \le \limsup a_n$.

The equality is strict when the series does not converge, for example $a_n = (-1)^n$.

(d) $(\Rightarrow)$ If $\liminf a_n = \limsup a_n$, then Squeeze Theorem implies that a_n converges to the same value, since $\inf\{a_{k \ge n}\} \le a_n \le \sup\{a_{k \ge n}\}$.

$(\Leftarrow)$ Assume that $\lim a_n = a$. For $\epsilon > 0$, we can find $N > 0$ such that for all $k > N$, $|a_k - a| < \epsilon$. This means

$$a - \epsilon < x_N \le x_k \le y_k \le y_N < a + \epsilon$$

implying that both x_n and y_n converge to a.

Exercise 2.4.8

For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{(b)} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad \text{(c)} \sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$$

(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

SOLUTION

(a) This is a geometric series, allowing us to use the usal trick to determine s_n.

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} \\ \frac{1}{2}s_n &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n+1}} \\ \frac{1}{2}s_n - s_n &= \frac{1}{2^{n+1}} - 1 \implies s_n = \frac{2^{-n-1}-1}{2^{-1}-1} \end{aligned}$$

As n tends to infinity,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = -1 + \sum_{n=1}^{\infty} \frac{1}{2^n} = -1 + \lim_{n\to\infty} \frac{2^{-n-1}-1}{2^{-1}-1} = 1$$

(b) We can use partial fractions to get

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

which gives a telescoping sum

$$s_n = 1 - \frac{1}{n+1}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

(c) Similar to (b), we can use telescoping sum since

$$\log\left(\frac{n+1}{n}\right) = \log(n+1) - \log(n)$$

Hence, we get

$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) = \log(n+1)$$

which does not converge.

Exercise 2.4.9

Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

SOLUTION

The conditions given state that (b_n) is decreasing and $b_n \geq 0$ for all $n \in \mathbb{N}$. Suppose $\sum_{n=1}^{\infty} 2^n b_{2^n}$ diverges, let $s_k = \sum_{n=1}^{k} n_n$ and $t_k = \sum_{n=1}^{k} 2^n b_{2^n}$, then

$$\begin{aligned} s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + \cdots + (b_{2^{k-1}+1} + \cdots + b_{2^k}) \\ &\geq b_1 + b_2 + (b_4 + b_4) + \cdots + 2^{k-1} b_2^k \end{aligned}$$

We define t'_k to be our new series $b_1 + b_2 + 2b_4 + \cdots + 2^{k-1} b_{2^k}$. Then,

$$t'_k = \frac{1}{2}(b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_k) + \frac{1}{2}b_1 = \frac{1}{2}t_k + \frac{1}{2}b_1$$

Therefore,

$$s_{2^k} \geq t'_k \geq \frac{1}{2}t_k$$

By Order Limit Theorem, since t_k diverges, s_n must also diverge.

Exercise 2.4.10
(Infinite Products). A close relative of infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \dots$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \dots b_m$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty}(1+a_n) = (1+a_1)(1+a_2)(1+a_3)\dots, \quad \text{where } a_n \geq 0$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1_x \leq 3^x$ for positive x will be useful in one direction.)

SOLUTION

(a) This is a telescoping product as most terms cancel each other

$$p_m = \prod_{n=1}^{m}\left(1+\frac{1}{n}\right) = \prod_{n=1}^{m}\frac{n+1}{n} = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\dots\cdot\frac{m+1}{m} = m+1$$

therefore, (p_m) diverges.

For the case $a_n = 1/n^2$,

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right) = \frac{2}{1}\cdot\frac{5}{4}\cdot\frac{10}{9}\cdot\dots$$

I conjecture that the series converges as the growth seems slower.

(b) Let p_m be the series of partial product and s_n be the series of partial sums.

($\Rightarrow$) We will show this using induction. It is obvious that $p_1 \leq s_1 + 1$. By distributing the terms, we get $p_2 = (1+a_1)(1+a_2) = 1 + a_1 a_2 + a_1 + a_2 > a_1 + a_2 + 1 = s_2 + 1$, hence $p_2 > s_2 + 1$. Suppose $p_n > s_n + 1$, then

$$p_{n+1} = (1+a_{n+1})p_n > (1+a_{n+1})(s_n+1) = s_n + s_n a_{n+1} + a_{n+1} + 1 > s_{n+1} + 1$$

By Order Limit Theorem, if (p_m) converge, (s_n) converges.

($\Leftarrow$) Using the inequality $1 + x \leq 3^x$, we get

$$p_n = (1+a_1)(1+a_2)\dots(1+a_n) = 3^{a_1}\cdot 3^{a_2}\cdot\dots\cdot 3^{a_n} = 3^{s_n}$$

Since s_n converges, it is bounded, hence $s_n \leq M$. Hence we get

$$p_n \leq 3^{s_n} \leq 3^M$$

Monotone Convergence Theorem tells us that (p_m) converges as it is monotone increasing (since $(1 + a_n) \geq 1$) and bounded.

2.5 Subsequences and the Bolzano-Weierstrass Theorem

Exercise 2.5.1
Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$.

(d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$, and no subsequences converging to points outside of this set.

SOLUTION

(a) Impossible, the Bolzano-Weierstrass theorem tells us a convergent subsequence of the subsequence exists, and that sub-sub-sequence is also a subsequence of the original sequence.

(b) $(1+1/n) \to 1$ and $(1/n) \to 0$. By interweaving the terms, we get $(1/2, 3/2, 1/3, 4/3, \dots)$ that contains subsequences that converge to both 0 and 1.

(c) Consider the sequence where there is an infinite number of terms for each element in the set

$$(1, 1/2, 1, 1/2, 1/3, 1, 1/2, 1/3, 1/4, \dots)$$

(d) Impossible, the sequence must converge to zero which is not in the set.

Let $\epsilon > 0$ be arbituary, pick $N > 0$ large enough such that $1/n < \epsilon/2$ for $n > N$. We can find a subsequence $(b_m) \to 1/n$ meaning $|b_m - 1/n| < \epsilon/2$ for some m. By triangle inequality, we get

$$|b_m - 0| \leq |b_n - 1/n| + |1/n - 0| < \epsilon/2 + \epsilon/2 = \epsilon$$

therefore we have found a number b_m in the sequence a_m with $|b_m| < \epsilon$. Repeating this process for any ϵ allows us to construct a subsequence that converges to zero.

Exercise 2.5.2
Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

SOLUTION

(a) True, removing the first term gives us the proper subsequence $(x_2, x_3, \dots)$ which converges. This implies that (x_n) converges to the same value, since changing the index of the terms does not change the limit behaviour of the sequence.

(b) True, the contrapositive of the statement is "If (x_n) converges, then every subsequence converges to the same value", which is true.

(c) True. By BWT, the bounded sequence (x_n) has a convergent subsequence as it is bounded. Let (a_n) be one such subsequence with limit a. Since (x_n) does not converge to a, there exists $\epsilon > 0$ such that there are infinitely many terms of (x_n) that lie outside the interval $(a - \epsilon, a + \epsilon)$. Let (b_n) be the sequence of those terms, and since it is bounded, BWT tells us we can find another convergent subsequence (c_n) with limit of c. Since the terms of (c_n) lie outside the ϵ-neighbourhood of a, $c \neq a$. Hence we have found two subsequences that converge to different limits.

(d) True. The subsequence (x_{n_k}) converges, means that it is bounded $|x_{n_k}| \leq M$. Suppose (x_n) is increasing, then x_n is bounded since we can pick a k so that $n_k > n$, we have $x_n \leq x_{n_k} \leq M$, thus converging. A similar argument can done if (x_n) is decreasing by taking the negative of each terms as a new sequence.

Exercise 2.5.3

(a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \dots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \dots$$

leads to a series that also converges to L.

(b) Compare this result to the example discussed at the end of Section 2.1 where the infinite addition was not to be associative. Why doesn't our proof in (a) apply to this example?

SOLUTION

(a) Let s_n be the original partial sums and s'_m be the regrouping. Since (s'_m) is a subsequence of (s_n), $(s_n) \to L$ implies $(s'_n) \to L$.

(b) The converse is not true. (s'_m) converging does not imply that (s_n) converges.

Exercise 2.5.4

The Bolzano-Weierstrass Theorem is extremely important, and so is the strategy employed in the proof. TO gain some more experience with this technique, assume the Nested Interval Property is true an duse it to provide a proof of the Axiom of Completeness. TO prevent the argument from being circular, assume also that $(1/2^n) \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

SOLUTION

Let A be a bounded set. We are going to conduct a binary search for $\sup A$ and then use NIP to prove the limit exists.

Let M be an upper bound on A, and pick any $L \in A$ as our starting lower bound for $\sup A$. We define $I_1 = [L, M]$. Doing a binary search, gives us $I_{n+1} \subseteq I_n$ with length proportional to $(1/2)^n$. Applying NIP gives us $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. As the length goes to zero, there exists a single $s \in \bigcap_{n=1}^{\infty} I_n$, which must be the least upper bound since,

(i) $s \geq L_n$ implies s is an upper bound

(ii) $s \leq M_n$ implies s is the least upper bound

The assumption that $(1/2^n) \to 0$ is necessary because the Archimedian Property was proved using the Axiom of Completeness.

Exercise 2.5.5

Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

SOLUTION

Suppose (a_n) does not converge to a. For some $\epsilon > 0$ wee can find a subsequence (b_n) satisfying $|b_n - a| \geq \epsilon$ for all $n \in \mathbb{N}$. Since (a_n) is bounded, so must (b_n). BWT tells us there must be a convergent subsequence of (b_n) that converges to b. Evidently, $b \neq a$ as $|b_n - a| \geq \epsilon$, leading to a contradiction as not every convergent subsequence of (a_n) converges to the same limit.

Exercise 2.5.6

Use a similar strategy to the one in Example 2.5.3 to show that $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

SOLUTION

To show that the limit exists, we need to apply the Monotone Convergence Theorem. Consider the three cases

(i) Case 1: $b > 1$, then $b^{1/n}$ is decreasing as $\frac{1}{n+1} < \frac{1}{n}$ implies $\frac{\ln b}{n+1} < \frac{\ln b}{n}$ hence $b^{1/(n+1)} < b^{1/n}$. Also, $b^{1/n} \iff b > 1^n$, hence bounded.

(ii) Case 2: $b < 1$, then MCT applies for similar reason as Case 1.

(iii) Case 3: $b = 1$. This case is trivial.

Monotone Convergence Theorem applies, hence we know the limit exists, then $\lim b^{1/n} = c$. Consider the subsequence:

$$b^{\frac{1}{2n}} = \sqrt{b^{1/n}}$$

Using Exercise 2.3.1, we know $\lim \sqrt{b^{1/n}} = \sqrt{c}$ and since subsequences converge to the same limit:

$$c = \sqrt{c}$$

Hence $c = 0$ or 1. For $b > 1$, the only possible limit is $c = 1$. For $b < 1$, the increasing sequence is bounded by $b^{1/n} \geq b$. Hence, for $0 < b < 1$, $c = 1$. For the case when $b = 0$, $c = 0$ is trivial.

Exercise 2.5.7
Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n)$ if and only if $-1 < b < 1$.

SOLUTION
($\Rightarrow$) Given $\lim(b^n) = 0$, AFSOC $|b| \geq 1$. Then $\lim b^n \neq b$. (Diverges for $b \neq 1$)

($\Leftarrow$) If $|b| < 1$, the $|b^n| < 1$, thus b^n is bounded. By Exercise 2.5.3, it is decreasing and monotone convergence theorem applies. To find the limit, equating the terms gives us $b^{n+1} = b^n$, thus $b = 0$ or $b = 1$. Since b is strictly decreasing, we have $b = 0$.

Exercise 2.5.8
Another way to prove the Bolzano-Weierstrass Theorem is to show that every sequence contains a monotone subsequence. A useful device in this endeavor is the notion of *peak term*. Given a sequence (x_n), a particular x_m is a peak term if no later term in the sequence exceeds it; i.e., if $x_m \geq x_n$ for all $n \geq m$.

(a) Find examples of sequences with zero, one and two peak terms. Find an example of a sequnce with infinitely many peak terms that is not monotone.

(b) Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzano-Weierstrass Theorem.

SOLUTION

(a) $(1, 2, 3, \dots)$, $(1, 1/2, 2/3, \dots)$ and $(2, 1, 1/2, 2/3 \dots)$ has zero, one and two peak terms respectively. The sequence $(1, -1/2, 1/3, -1/4, \dots)$ has infinitely many peak terms, but is not montone.

(b) If there is an infinite number of peak terms. Each peak term is, by definition, lower than the last. We can take the subsequence of peak terms, which is strictly increasing, to find a monotone subsequence.

If there is a finite number of peak term, let the last peak term be at the index k. Consider the subsequence of terms after the last peak term. Since there is no peak terms after x_k, then $x_m \geq x_n$ for any $m > n > k$. Hence, the subsequence is monotone increasing.

If the original sequence is bounded, and we can find a monotone subsequence, MCT tells us that this subsequence converges, proving BWT.

Exercise 2.5.9
Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.)

SOLUTION
For every $\epsilon > 0$, there exists an $x \in S$ with $x > s - \epsilon$, implying $|s - x| < \epsilon$. There for we get can close to $s = \sup S$. We can hence pick $x_n \in S$ such that $|x_n - s| < 1/n$ for all $n \in \mathbb{N}$. Hence picking $N > 1/\epsilon$, we get $|x_n - s| < \epsilon$ for all $n > N$.

2.6 The Cauchy Criterion

Exercise 2.6.1
Supply a proof for Theorem 2.6.2.

SOLUTION
Suppose $(x_n) \to x$, hence for $\epsilon > 0$, $|x - x_n| < \epsilon/2$ when $n > N$. Similarly, $|x_m - x| < \epsilon/2$ for $m > N$. By Triangle Inequality,

$$|x_n - x_m| \leq |x_n - x| + |x - x_m| < \epsilon/2 + \epsilon/2 = \epsilon$$

Exercise 2.6.2
Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

(c) A divergent monotone sequence with a Cauchy subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

SOLUTION
(a) $x_n = (-1)^n/n$ converges and hence Cauchy but not monotone.

(b) Impossible. Cauchy sequences must converge and hence bounded.

(c) Impossible. A Cauchy subsequence will converge, implying that it is bounded. Since the sequence is montone, it is also bounded, thus MCT tells us it converges.

(d) $(2, 1/2, 3, 1/3, \dots)$ has the subsequence $(1/2, 1/3, \dots)$ which is Cauchy.

Exercise 2.6.3
If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(x_n y_n)$.

SOLUTION
(a) $|(x_n + y_n) - (x_m + y_m)| = |(x_n - x_m) + (y_n - y_m)| \leq |x_n - x_m| + |y_n - y_m| < \epsilon/2 + \epsilon/2 = \epsilon$.

(b) Let $|x_n| \leq M_x$ and $|y_n| \leq M_y$ for all $n \in \mathbb{N}$.

$$\begin{aligned}
|x_n y_n - x_m y_m| &= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\
&\leq |x_n(y_n - y_m)| + |y_m(x_n - x_m)| = |x_n||y_n - y_m| + |y_m||x_n - x_m| \\
&\leq M_x|y_n - y_m| + M_y|x_n - x_m| = M_x(\frac{\epsilon}{2M_x}) + M_y(\frac{\epsilon}{2M_y}) = \epsilon
\end{aligned}$$

After setting $|x_n - x_m| < \epsilon/(2M_y)$ and $|y_n - y_m| < \epsilon/(2M_x)$.

Exercise 2.6.4
Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

(a) $c_n = |a_n - b_n|$

(b) $c_n = (-1)^n a_n$

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

SOLUTION

(a) By Triangle Inequality and that $||x| - |y|| \leq |x - y|$ as proven in earlier Exercises,

$$\begin{aligned}|c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \\ &\leq |a_n - b_n - a_m + b_m| = |a_n - a_m + b_m - b_n| \\ &\leq |a_n - a_m| + |b_m - b_n| < \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

(b) No. If $a_n = 1$, then $(-1)^n$ diverges and thus not Cauchy.

(c) No. If $a_n = 1 - (-1)^n/n$, then c_n oscillate between 0 and 1, thus not Cauchy.

Exercise 2.6.5

Consider the following (invented) definition: A sequence (s_n) is *pseudo-Cauchy* if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$.

Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

SOLUTION

(i) False. Let $s_n = \sum_{i=1}^{n} 1/i$. For any $\epsilon > 0$, we can choose $N > 1/\epsilon$, such that when $n > N$, $|s_{n+1} - s_n| = 1/(n+1) < \epsilon$. However, $(s_n) \to \infty$.

(ii) For $\epsilon > 0$, if $n > N$, then

$$|(x_{n+1} + y_{n+1}) - (x_n + y_n)| = |(x_{n+1} - x_n) + (y_{n+1} - y_n)| \leq |x_{n+1} - x_n| + |y_{n+1} - y_n| < \epsilon/2 + \epsilon/2 = \epsilon$$

Exercise 2.6.6

Let's call a sequence (a_n) *quasi-increasing* if for all $\epsilon > 0$ there exists an N such that whenever $n > m \geq N$ it follows that $a_n > a_m - \epsilon$.

(a) Give an example of a sequence that is quasi-increasing but not monotone or eventually monotone.

(b) Give an example of a quasi-increasing sequence that is divergent and not monotone or eventually monotone.

(c) Is there an analogue of the Monotone Convergence Theorem for quasi increasing sequences? Giva an example of a bounded, quasi-increasing sequence that doesn't converge, or prove that no such sequence exists.

SOLUTION

Before we begin, we can rearrage the terms in the definition, giving $a_m - a_n < \epsilon$. Intuitively, the sequence must either decrease or converge. We will prove this later.

(a) Let $x_n = (-1)^n/n$. Then $a_m - a_n = (-1)^m/m - (-1)^n/n \leq 1/m + 1/n < 2/N < \epsilon$ after picking some $N > 2/\epsilon$. Thus it is quasi-increasing.

(b) Let $y_n = (2, 2 - 1/2, 3, 3 - 1/3, \dots)$. For $\epsilon > 0$, set $N > 1/\epsilon$. For $n > m \geq N$ consider to cases. We have $a_n > a_m$ as long as $a_n \neq m - 1/m$. If $a_n = m - 1/m$, then $a_n > a_m - \epsilon$.

(c) Suppose (a_n) is quasi-increasing and bounded. By BWT, we have a convergent subsequence $(a_{n_k}) \to a$. For $\epsilon > 0$, let N_1 be large enough such that $n_k > N_1$ implies $|a_{n_k} - a| < \epsilon/2$. Let $N_2 > N_1$ be large enough that that by the quasi-increasing property of (a_n), for $n > m > N_2$ it follows that $a_n > a_m - \epsilon/2$. Finally, let $N = n_{k_1} > N_2$.

Now for any $n > N$, we can choose $M = n_{k_2} > n$, then

$$a_n > a_N - \epsilon/2 > a - \epsilon$$

and

$$a_n - \epsilon/2 < a_M < a + \epsilon/2 \implies a_n < a + \epsilon$$

and hence $|a_n - a| < \epsilon$, therefor $(a_n) \to a$.

Exercise 2.6.7

Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano-Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, AoC and MCT are all equivalent.

(b) Use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.

(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

SOLUTION

(a) Suppose (x_n) is increasing and bounded, BW tells us there exists a convergent subsequence $(x_{n_k}) \to x$. We will show that $(x_n) \to x$. Note the $x_k \le x_{n_k}$ implies $x_n \le x$ by Order Limit Theorem.

Pick K such that for $k \ge K$, we have $|x_{n_k} - x| < \epsilon$. Since (x_n) is increasing and $x_n \le x$, every $n \ge n_K$ satisfies $|x_n - x| < \epsilon$ as well. Thus (x_n) converges, completing the proof.

(b) We are going to use the Cauchy Criterion to replace NIP in the proof of BW. We have I_0 being of length M and $I_{n+1} \subseteq I_n$ and being half of its length with $a_{n_k} \in I_k$. We will show a_{n_k} is Cauchy.

The length of I_k is $M(1/2)^{k-1}$, so $|a_{n_k} - a_{n_j}| < M(1/2)^{N-1}$ for $k, j \ge N$, implying that (a_{n_k}) converges. This uses the Archimedean Property to conclude $M(1/2)^{N-1} \in \mathbb{Q}$ can be made smaller than any $\epsilon > 0$.

(c) The Archimedean Property is true for $\mathbb{Q}$ does not prove AoC which is only true for $\mathbb{R}$.

2.7 Properties of Infinite Series

Exercise 2.7.1

Proving tha Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).) Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

SOLUTION

(a) Let $N \in \mathbb{N}$ be even and let $n > N$. Since the series is alternating, we have

$$s_N \le s_n \le s_{N+1}$$

A similar statement can be made if N is odd. Obviously, $|s_{N+1} - s_N| = |a_{N+1}|$ can be made as small as we want by increasing N. Hence we can set N large enough so that $|a_N| < \epsilon/2$, giving

$$|s_m - s_n| \le |s_m - s_N| + |s_n - s_N| < \epsilon/2 + \epsilon/2 = \epsilon$$

showing that (s_n) is Cauchy, hence converges.

(b) Let $I_1 = [s_2, s_1]$ and in general, $I_n = [s_{2n}, s_{2n-1}]$. Since a_n is decreasing and it is an alternating series, we get $I_{n+1} \subseteq I_n$. Considering the length of I_n, which equals to the length $|s_{2n} - s_{2n-1}| = a_{2n}$. If we set N large enough, we can get $|a_{2n}| < \epsilon$. By NIP, there exists $x \in \bigcap_{n=1}^{\infty} I_n$. By order limit theorem, $\lim s_{2n} \leq x \leq \lim s_{2n-1}$, hence $\lim s_n = x$, showing that the series converges.

(c) We can regroup the elements in the partial sum. Consider

$$s_{2n+1} = a_1 - (a_2 - a_3) - \cdots - (a_{2n} - a_{2n+1}) \leq a_1$$

Thus $s_{2n+1} \to s$ by MCT. To show this hold true for (s_{2n}), note that $s_{2n} = s2n + 1 - a_{2n+1}$ with $(a_{2n+1}) \to 0$, by triangle inequality

$$|s_{2n} - s| \leq |s_{2n} - s2n + 1| + |s_{2n+1} - s| < \epsilon/2 + \epsilon/2 = \epsilon$$

And we are done.

Exercise 2.7.2
Decide whether each of the following series converges or diverges:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \dots$

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \dots$

(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2}$

SOLUTION

(a)

$$0 < \frac{1}{2^n + n} \leq \frac{1}{2^n} \implies 0 < \sum_{n=1}^{\infty} \frac{1}{2^n + n} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Hence converges by comparison test.

(b) By Theorem 2.7.19, if $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right|$ converges, so does $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$. By comparison test,

$$0 < \sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right| < \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

hence the series converges.

(c) By comparison test,

$$\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1} < \sum_{n=1}^{\infty}(-1)^n \frac{n+1}{2n}$$

thus does diverging.

(d) Grouping terms give

$$\frac{1}{n} + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \geq \frac{1}{n}$$

By comparison test

$$\sum_{n=1}^{\infty} \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n} > \sum_{n=1}^{\infty} \frac{1}{3n-2} > 1 + \sum_{n=2}^{\infty} \frac{1}{3n-3} = 1 + \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges, hence the series diverges.

(e) Splitting the series gives

$$s_{2n} = \sum_{k=1}^{\infty} \frac{1}{2k-1} + \sum_{k=1}^{\infty} \frac{1}{(2n)^2} > 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} + \frac{\pi^2}{24}$$

which diverges by comparison test.

Exercise 2.7.3

(a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.

SOLUTION

Suppose $a_n, b_n \geq 0$, $a_n \leq b_n$ and define $s_n = a_1 + \cdots + a_n$, $t_n = b_1 + \cdots + b_n$.

(a) We have $|a_m + \ldots a_n| \leq |b_m + \cdots + a_n| < \epsilon$, implying that $\sum_{n=1}^{\infty} a_n$ converges by Cauchy criterion. The other direction is similar, with $|b_m + \cdots + b_n| \geq |a_m + \cdots + a_n| > \epsilon$, showing that (s_n) diverges implies (t_n) mst also diverge.

(b) Since $(t_n) \to t$. This implies that s_n is bounded since $s_n \leq t_n < t$. With $a_n \geq 0$, MCT tells us that (s_n) converges. The other direction can be proven by contradiction, assuming that (t_n) is bounded tells us (s_n) is bounded by MCT, which contradicts the fact that (s_n) diverges.

Exercise 2.7.4

Give an example of each or explain why the request is impossible, referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such taht $\sum x_n y_n$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but $\sum y_n$ diverges.

(d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum(-1)^n x_n$ diverges.

SOLUTION

(a) Let $x_n = y_n = 1/n$, hence $\sum x_n = \sum y_n = \sum 1/n$ which diverges, but $\sum x_n y_n = \sum 1/n^2 = \pi^2/6$ which converges.

(b) Let $x_n = (-1)^n 1/n$, then $\sum x_n$ converges by Alternating Series Test, and $y_n = (-1)^n$ which is bounded. Hence $\sum x_n y_n = \sum 1/n$ which diverges.

(c) Impossible. By Algebraic Limit Theorem, $\sum y_n = \sum(x_n + y_n) - \sum x_n$ must converge.

(d) Let

$$x_n = \begin{cases} \frac{1}{n} & \text{if n is even} \\ 0 & \text{otherwise} \end{cases}$$

then $\sum(-1)^n x_n = \sum 1/(2n)$ which diverges by Comparison Test with harmonic series.

Exercise 2.7.5

Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7.

SOLUTION

Corollary 2.4.7 states that the series $\sum_{n=1}^{\infty} 1/n^p$ *converges if and only if* $p > 1$.
Using the Cauchy Condensation Test, the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if the series

$$\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=0}^{\infty} 2^{n-np} = \sum_{n=0}^{\infty} \left(2^{1-p}\right)^n$$

converges. This is a geometric series that converges if and only if $2^{1-p} < 1 \implies p > 1$.

Exercise 2.7.6
Let's say that a series *subverges* if the sequence of partial sums contains a subsequence that converges. Consider the (invented) definition for a moment, and then decide which of the following statements are valid propositions about subvergent series:

(a) If (a_n) is bounded, then $\sum a_n$ subverges.

(b) All convergent series are subvergent.

(c) If $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.

(d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

SOLUTION

(a) False. Let $a_n = 1$, then $\sum a_n = n$ which does not subverge.

(b) True. Every subsequence will converge to the same limit.

(c) True. Since $|a_n| \geq 0$, then $s_n = \sum_{i=1}^{n} |a_i|$ is increasing and contains a converging subsequence. Hence $s_n \leq M$ is bounded. By triangle inequality, $|\sum a_n| \leq \sum |a_n| = s_n \leq M$, then $\sum a_n$ is bounded. By BWT, we can find a convergent subsequence, thus subverging.

(d) False. Consider the sequence $(1, -1, 2, -2, \dots)$ which every even partial sum converges to zero, but the series does not converge.

Exercise 2.7.7

(a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

SOLUTION

(a) For $\epsilon > 0$, we get $na_n \in (l - \epsilon, l + \epsilon)$ implying $a_n > (l - \epsilon)(1/n)$. Using comparison text, we get $\sum a_n > (l - \epsilon) \sum 1/n$ which diverges, thus $\sum a_n$ diverges.

(b) Similarly, we get $n^2 a_n \in (l - \epsilon, l + \epsilon)$ so $a_n < \frac{l+\epsilon}{n^2}$. By comparison test, $\sum a_n < (l + \epsilon) \sum \frac{1}{n^2}$, thus converging.

Exercise 2.7.8
Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

SOLUTION

(a) True. Since $\sum |a_n|$ converges, there exists $N > 0$ such that for $n > N$, $0 < a_n^2 < |a_n| < 1$. By comparison test, $\sum a_n^2$ also converges absolutely.

(b) False. Let $a_n = b_n = (-1)^n/\sqrt{n}$, with $\sum a_n$ converging by Alternating Series Test, but $\sum a_n b_n = \sum 1/n$ which diverges.

(c) True. Suppose $\sum n^2 a_n$ converges, since $(n^2 a_n) \to 0$, we have $|n^2 a_n| < \epsilon$ for $n > N$, implying $|a_n| < 1/n^2$. By comparison test, $\sum |a_n| < \sum 1/n^2$ converges, implying $\sum a_n$ converges absolutely, which is a contradiction. Therefore $\sum n^2 a_n$ must diverge.

Exercise 2.7.9
(Ratio Test) Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n|r'$.

(b) Why does $|a_N| \sum (r')^n$ converge?

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

SOLUTION

(a) We are given

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - r \right| < \epsilon$$

Since $1 > r' > r$, we can set $\epsilon = r - r'$ meaning the neighbourhood

$$\left| \frac{a_{n+1}}{a_n} \right| \in (r - \epsilon, r + \epsilon) = (2r - r', r')$$

which is all less than r' meaning

$$\left| \frac{a_{n+1}}{a_n} \right| \leq r' \implies |a_{n+1}| \leq |a_n|r'$$

(b) With $|a_N|$ being a constant and $\sum (r')^n$ converging as it is a geometric series with $r' < 1$, then $|a_N| \sum (r')^n$ converges.

(c) To show that $\sum |a_n|$ converges, it is sufficient to show that $\sum_{k=N}^{\infty} |a_k|$ converges as $\sum_{k=1}^{N-1} |a_k|$ is the sum of finite terms, hence a constant.

Let N be large enough such that for $n > N$, we have $|a_{n+1}| \leq |a_n|r'$. Applying this multiple times gives us $|a_n| \leq (r')^{n-N}|a_N|$ which gives,

$$|a_N| + |a_{N+1}| + \cdots + |a_n| \leq |a_N| + r'|a_N| + \cdots + (r')^{n-N}|a_N|$$

Factoring gives us

$$\sum_{k=N}^{n} |a_k| \leq |a_N| \sum_{k=0}^{n-1} (r')^k$$

which converges as $n \to \infty$ by part (b). Hence implying that the sum converges absolutely.

Exercise 2.7.10

(Infinite Products) Review Exercise 2.4.10 about infinite products and then answer the following questions:

(a) Does $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots$ converge?

(b) The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots$ certainly converges. (Why?) Does it converge to zero?

(c) In 1655, John Wallis famously derived the formula

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots = \frac{\pi}{2}$$

Show that the left side of this identity at least converges to something. (A complete proof of this result is taken up in Section 8.3)

SOLUTION

(a) Using Exercise 2.4.10,

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots = \prod_{n=0}^{\infty} \frac{2^n + 1}{2^n} = \prod_{n=0}^{\infty} \left(1 + \frac{1}{2^n}\right)$$

converges as $\sum 2^{-n}$ converges by geometric series.

(b) Each term is positive and less than one, causing the product to be monotone decreasing and bounded below by zero, converging by MCT.

Let $p_n = \prod_{n=1}^{\infty} \frac{2n-1}{2n}$ and $(p_n) \to p$. Suppose $p \neq 0$, then $(1/p_n) \to p^{-1}$.

$$p_n^{-1} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} = \prod_{n=1}^{\infty} (1 + \frac{1}{2n-1})$$

Since $\sum 1/(2n-1)$ diverges by comparison test with the harmonic series, $(1/p_n)$ diverges. This is a contradiction, showing that $p = 0$.

(c) Rewriting the terms as a partial fraction,

$$\frac{(2n)(2n)}{(2n-1)(2n+1)} = \frac{4n^2}{4n^2-1} = 1 + \frac{1}{4n^2-1}$$

By Exercise 2.4.10, $\sum 1/(4n^2-1)$ converges by comparison test with $\sum 1/n^2$. Hence the product converges.

Exercise 2.7.11
Find an example of two series $\sum a_n$ and $\sum b_n$ both of which diverge for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are strictly positive and decreasing.

SOLUTION
Let $m_n = \min\{a_n, b_n\}$. Intuitively, (m_n) must take an infinite amount of (a_n) and (b_n) terms, as removing the finite terms would imply that either $\sum a_n$ or $\sum b_n$ would converge.

We will begin by trying to obtain $m_n = 1/2^n$ to ensure that $\sum m_n$ converges. To have $\sum a_n$ and $\sum b_n$ diverging, we can simply repeat values when it is the greater value of the two.

n	1	2	3	$[4, 4+8=12)$	$[12, 12+2^{11})$	$[12+2^{11}, 12+2^{11}+2^{12+2^1 1})$	$\dots$
m_n	$1/2$	$1/4$	$1/8$	$1/2^n$	$1/2^n$	$1/2^n$	$\dots$
$\min\{a_n, b_n\}$	a_n	b_n	b_n	a_n	b_n	a_n	$\dots$
a_n	$1/2$	$1/2$	$1/2$	$1/2^n$	$1/2^{11}$	$1/2^n$	$\dots$
b_n	1	$1/4$	$1/8$	$1/8$	$1/2^n$	$1/2^{12+2^{11}}$	$\dots$

Now we show that this construction satisfies the conditions. It is obvious that $\sum \min\{a_n, b_n\}$ converges as stated above. For $\sum a_n$ and $\sum b_n$, we have to group the terms as shown above. Let $k_1 = 1$, $k_n = k_{n-1} + 2^{k_{n-1}-1}$ Then

$$a_n = \begin{cases} 1/2^n & \text{if } n \in [k_{2p-1}, k_{2p}) \\ 1/2^{k_{2p}-1} & \text{if } n \in [k_{2p}, k_{2p+1}) \end{cases} \text{ and } b_n = \begin{cases} 1/2^n & \text{if } n \in [k_{2p}, k_{2p+1}) \\ 1/2^{k_{2p-1}-1} & \text{if } n \in [k_{2p-1}, k_{2p}) \end{cases}$$

Hence both $\sum a_n$ and $\sum b_n$ diverges as the first case has its blocks sum to one, causing the sum to approach infinity.

Exercise 2.7.12
(Summation-by-parts) Let (x_n) and (y_n) by sequences, let $s_n = x_1 + x_2 + \cdots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1})$$

SOLUTION

$$\begin{aligned}\sum_{j=m}^{n} x_j y_j &= \sum_{j=m}^{n} (s_j - s_{j-1}) y_j \\ &= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n} s_{j-1} y_j \\ &= \sum_{j=m}^{n} s_j y_j - \sum_{j=m+1}^{n+1} s_j y_{j+1} \\ &= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n} s_j y_{j+1} - s_{m-1} y_m + s_n y_{n+1} \\ &= s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1})\end{aligned}$$

Exercise 2.7.13
(Abel's Test) Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Use Exercise 2.7.12 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1})$$

where $s_n = x_1 + x_2 + \ldots x_n$.

(b) Use the Comparison Test to argue that $\sum_{k=1}^{\infty} s_n (y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

SOLUTION

(a) From Exercise 2.7.12,

$$\begin{aligned}\sum_{k=1}^{n} x_k y_k &= s_n y_{n+1} - s_0 y_1 + \sum_{k=1}^{n} s_j (y_k - y_{k+1}) \\ &= s_n y_{n+1} + \sum_{k=1}^{n} s_j (y_k - y_{k+1})\end{aligned}$$

(b) Since $\sum x_k$ converges, we can bound $s_n \leq M$. By Comparison Test, we get

$$\begin{aligned}\sum_{k=1}^{\infty} |s_n (y_k - y_{k+1})| &\leq |M| \sum_{k=1}^{\infty} (y_k - y_{k+1}) \\ &\leq |M| (y_1 - \lim y_n) = |M| (y_1 - y)\end{aligned}$$

where $(y_n) \to y$.

Since both $s_n y_{n+1}$ and $\sum s_k (y_k - y_{k+1})$ converge, then $\sum_{k=1}^{\infty} x_k y_k$ also converge by algebraic limit theorem.

2.8 Double Summations and Products of Infinite Series