Understanding Analysis Attempt/Solution

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Contents

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is rational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

SOLUTION

(a) PROOF AFSOC that $\sqrt{3}$ is rational, so $\exists m, n \in \mathbb{Z}$, such that

$$\sqrt{3} = \frac{m}{n},$$

where $\frac{m}{n}$ is in the lowest reduced terms. By squaring both sides, we obtain $3 = (\frac{m}{n})^2 \implies 3n^2 = m^2$. Now, we know that m^2 is a multiple of 3 and thus m must also be a multiple of 3. We can then write $m = 3k$, deriving

$$\begin{aligned} (\sqrt{3})^2 &= \left(\frac{3k}{n}\right)^2 \\ 3n^2 &= 9k^2 \\ n^2 &= 3k^2 \end{aligned}$$

Similar to above, we can conclude that n is a multiple of 3. However this is a contradiction since m, n are both multiples of 3 but we assumed that $\frac{m}{n}$ was in its lowest reduced term. Thus we conclude that $\sqrt{3}$ is irrational.

The same proof for $\sqrt{3}$ works for $\sqrt{6}$ as well.

(b) We cannot conclude that $\sqrt{4} = \frac{m}{n}$ imply that m is a multiple of 4, as we have

$$4n^2 = m^2 \implies 2n = m,$$

preventing us from reaching our contradiction that m/n is not in its lowest terms.

Exercise 1.2.2

Show that there is no rational number r satisfying $2^r = 3$.

SOLUTION

PROOF If $r = 0$, then $2^r = 1 \neq 3$. Suppose $r = p/q$ to get $2^p = 3^q$, which is not possible as 2 and 3 share no common factors. Hence r is not rational.

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$ are all sets containing an infinite number of elements, then the intersections $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and non-empty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

SOLUTION

(a) False. Consider $A_n = \{n, n+1, n+2, \dots\}$, then $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(b) True. Since all A_n are nonempty, $\exists n \in \mathbb{N}$ such that $A_n = \{x\}$ for some real x. Hence $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\}$ which is empty. Since A_1 is finite, $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\} \subset A_1$ is finite.

(c) False. If $A = \emptyset$, then $\emptyset = C$

(d) True. Intersection is associative as evident that both LHS and RHS implies the $x \in A, B, C$

(e) True. Drawing a Venn Diagram illustrates this.

Exercise 1.2.4

Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

SOLUTION

Consider arranging the elements of $\mathbb{N}$ in a square as such.

$$\begin{array}{cccccc}
1 & 3 & 6 & 10 & 15 & \dots \\
2 & 5 & 9 & 14 & \dots & \\
4 & 8 & 13 & \dots & & \\
7 & 12 & \dots & & & \\
11 & \dots & & & & \\
\vdots & & & & &
\end{array}$$

By letting A_i being the set of all natural numbers in the i-th row, we have satisfied the above conditions above.

Exercise 1.2.5

(De Morgan's Law) Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. THis shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \sqsubseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cup B^c$ by demonstrating inclusion both ways.

SOLUTION

(a) If $x \in (A \cap B)^c$, then $x \notin A \cap B$, so $x \notin A$ or $x \notin B$, implying $x \in A^c$ or $x \in B^c$, therfore $x \in A^c \cup B^c$.

(b) If $x \in A^c \cup B^c$, then $x \in A^c$ or $x \in B^c$, so $x \notin A$ and $x \notin B$, implying $x \notin A \cap B$, therefore $x \in (A \cap B)^c$. Since $(A \cap B)^c \subseteq A^c \cup B^c$ and $(A \cap B)^c \sqsubseteq A^c \cup B^c$, we can conclude that both sets are equal.

(c) To show that $(A \cap B)^c = A^c \cup B^c$, we need to demonstrate inclusion both ways.

  (i) If $x \in (A \cup B)^c$, then $x \notin A \cup B$, so $x \notin A$ or $x \notin B$, implying $x \in A^c$ or $x \in B^c$, therefore $x \in A^c \cup B^c$.

(ii) If $x \in A^c \cap B^c$, then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$, implying $x \notin A \cup B$, which is just $x \in (A \cup B)^c$.

Exercise 1.2.6

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c|+|c-d|+|d-b|$ for all a, b, c and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a-b+b$ may be useful.)

SOLUTION

(a) With both a and b having the same sign, then $|a|+|b| = |a+b|$, which satisfies $|a|+|b| \geq |a+b|$.

(b) Note that $(a+b)^2 \leq (|a|+|b|)^2$ reduces to $ab \leq |a||b|$, which is true as LHS can be negative while RHS cannot. Since squaring preserves inequality, this implies that $|a+b| \leq |a|+|b|$.

(c) Notice that $a-b = (a-c)+(c-d)+(d-b)$. Hence by triangle inequality,

$$|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c|+|c-d|+|d-b|$$

for all a, b, c and d.

(d) Since $||a|-|b|| = ||b|-|a||$, WLOG, we can assume that $|a| \geq |b|$. Then

$$||a|-|b|| = |a|-|b| = |(a-b)+b|-|b| \leq |a-b|+|b|-|b| = |a-b|$$

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0,2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbituary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbituary function g.

SOLUTION

(a) For $f(x) = x^2$, $f(A) = f([0,2]) = [0,4]$ and $f(B) = f([1,4]) = [1,16]$.

$$f(A \cap B) = f([0,2] \cap [1,4]) = f([1,2]) = [1{,}4] = [0,4] \cap [1,16] = f([1,2]) \cap f([2,4]) = f(A) \cap f(B)$$
$$f(A \cup B) = f([0,2] \cup [1,4]) = f([0,4]) = [0{,}16] = [0,4] \cup [1,16] = f([0,2]) \cup f([1,4]) = f(A) \cup f(B)$$

(b) Consider $A = [0,2]$ and $B = [-2,0]$. $f(A \cap B) = \{0\}$, but $f(A) \cap f(B) = [0,4]$.

(c) Suppose $y \in g(A \cap B)$, then $\exists x \in A \cap B$ such that $g(x) = y$. This implies that $x \in A$ and $x \in B$, so $x \in A \cap B$, hence $y \in g(A \cap B)$. Note that contrary may not always be true as it is possible for $x_1 \in A \backslash B$ and $x_2 \in B \backslash A$ such that $g(x_1) = g(x_2)$.

(d) I conjecture that $g(A \cup B) = g(A) \cup g(B)$. To prove this, we have to show inclusion both ways:

(i) Let $y \in g(A \cup B)$, then $\exists x \in A \cup B$ such that $y = g(x)$. This implies that $x \in A$ or $x \in B$, so $y \in g(A)$ or $y \in g(B)$, hence $y \in g(A) \cup g(B)$.

(ii) Let $y \in g(A) \cup g(B)$, then $y \in g(A)$ or $y \in g(B)$, implying $x \in A$ or $x \in B$ such that $y = g(x)$. So $x \in A \cup B$, hence $y \in g(A \cup B)$.

Exercise 1.2.8
Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* $(1-1)$ if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ Give an example of each or state that the request is impossible:

(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto.

(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1.

(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

SOLUTION
(a) Let $f(x) = x + 1$, which is 1-1 but does not have a solution to $f(x) = 1$, hence not onto.

(b) Let $f(x) = 1$ for $x = 1$ and $f(x) = x - 1$ for $x > 1$, which is onto but not 1-1 as $f(1) = f(2) = 1$.

(c) Let $f(x) = n/2$ when n is even and $f(x) = -\frac{x-1}{2}$ when n is odd.

Exercise 1.2.9
Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behaviour of preimages demonstrated in (a) is completely general. Show that for an arbituary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbb{R}$.

SOLUTION
(a) For $f(x) = x^2$, $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. $f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$. Similarly, $f^{-1}(A \cup B) = f^{-1}([-1, 4]) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$.

(b) To show that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$, we have to show inclusion both ways:

(i) Let $x \in g^{-1}(A \cap B)$, so $g(x) \in A \cap B$, which implies $g(x) \in A$ and $g(x) \in B$. This shows that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, hence $x \in g^{-1}(A) \cap g^{-1}(B)$.

(ii) Let $x \in g^{-1}(A) \cap g^{-1}(B)$, so $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, then $g(x) \in A$ and $g(x) \in B$. This implies that $g(x) \in A \cap B$, so $x \in g^{-1}(A \cap B)$.

Showing $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ is obvious using Exercise 1.2.7 (d).

Exercise 1.2.10
Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

SOLUTION
(a) False. Consider the case where $a < b + \epsilon$ is true but $a = b$.

(b) False. Same reasoning as above.

(c) True. Firsly suppose $a < b + \epsilon$ for all $\epsilon > 0$. We need to show this implies $a \leq b$. We either have $a \leq b$ or $a > b$. However, $a > b$ is not possible as this implies there exists an ϵ small enough such that $a > b + \epsilon$. Secondly, suppose $a \leq b$. It is obvious that $a < b + \epsilon$ for all $\epsilon > 0$.

Exercise 1.2.11
Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this more interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $a + 1/n < b$.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.

(c) Between every two distinct real numbers there is a rational number.

SOLUTION
(a) For all $n \in \mathbb{N}$, there exists $a, b \in \mathbb{R}$ such that $a + 1/n < b$. [FALSE]

(b) For all real number $x > 0$, there exists an $n \in \mathbb{N}$ such that $x \geq 1/n$. [TRUE]

(c) There exists two real numbers $a < b$ such that if $r < b$ then $r < a$ for all $r \in \mathbb{Q}$. [FALSE]

Exercise 1.2.12
Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \dots)$ is decreasing.

SOLUTION
(a) For $n = 1$, $y_1 = 6 > -6$ (Base Case). Suppose $y_n > -6$ for some $n \in \mathbb{N}$.

$$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2(-6) - 6}{3} = -6$$

Hence, by induction, $y_n > -6$ for all $n \in \mathbb{N}$.

(b) Suppose $y_{n+1} < y_n$. The base case works as $y_2 = 2 < 6 = y_1$. Then,

$$\begin{aligned} y_{n+1} < y_n &\implies 2y_{n+1} - 6 < 2y_n - 6 \\ &\implies \frac{2y_{n+1} - 6}{3} < \frac{2y_n - 6}{3} \\ &\implies y_{n+2} < y_{n+1} \end{aligned}$$

Thus, $y_{n+1} < y_n$ is true for all $n \in \mathbb{N}$.