

Understanding Analysis Attempt/Solution

pand4monium

November 12, 2025

Contents

1 The Real Numbers	1
1.2 Some Preliminaries	1

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

- (a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is rational?
(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

SOLUTION

- (a) PROOF AFSOC that $\sqrt{3}$ is rational, so $\exists m, n \in \mathbb{Z}$, such that

$$\sqrt{3} = \frac{m}{n},$$

where $\frac{m}{n}$ is in the lowest reduced terms. By squaring both sides, we obtain $3 = (\frac{m}{n})^2 \implies 3n^2 = m^2$. Now, we know that m^2 is a multiple of 3 and thus m must also be a multiple of 3. We can then write $m = 3k$, deriving

$$\begin{aligned}(\sqrt{3})^2 &= \left(\frac{3k}{n}\right)^2 \\ 3n^2 &= 9k^2 \\ n^2 &= 3k^2\end{aligned}$$

Similar to above, we can conclude that n is a multiple of 3. However this is a contradiction since m, n are both multiples of 3 but we assumed that $\frac{m}{n}$ was in its lowest reduced term. Thus we conclude that $\sqrt{3}$ is irrational.

The same proof for $\sqrt{3}$ works for $\sqrt{6}$ as well.

- (b) We cannot conclude that $\sqrt{4} = \frac{m}{n}$ imply that m is a multiple of 4, as we have

$$4n^2 = m^2 \implies 2n = m,$$

preventing us from reaching our contradiction that m/n is not in its lowest terms.

Exercise 1.2.2

Show that there is no rational number r satisfying $2^r = 3$.

SOLUTION

PROOF If $r = 0$, then $2^r = 1 \neq 3$. Suppose $r = p/q$ to get $2^p = 3^q$, which is not possible as 2 and 3 share no common factors. Hence r is not rational.

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

- (a) If $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$ are all sets containing an infinite number of elements, then the intersections $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.
- (b) If $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and non-empty.
- (c) $A \cap (B \cup C) = (A \cap B) \cup C$.
- (d) $A \cap (B \cap C) = (A \cap B) \cap C$.
- (e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

SOLUTION

- (a) False. Consider $A_n = \{n, n+1, n+2, \dots\}$, then $\bigcap_{n=1}^{\infty} A_n = \emptyset$.
- (b) True. Since all A_n are nonempty, $\exists n \in \mathbb{N}$ such that $A_n = \{x\}$ for some real x . Hence $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\}$ which is empty. Since A_1 is finite, $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\} \subset A_1$ is finite.
- (c) False. If $A = \emptyset$, then $\emptyset = C$
- (d) True. Intersection is associative as evident that both LHS and RHS implies the $x \in A, B, C$
- (e) True. Drawing a Venn Diagram illustrates this.

Exercise 1.2.4

Produce an infinite collection of sets A_1, A_2, A_3, \dots with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

SOLUTION

Consider arranging the elements of \mathbb{N} in a square as such.

1	3	6	10	15	...
2	5	9	14	...	
4	8	13	...		
7	12	...			
11	...				
					⋮

By letting A_i being the set of all natural numbers in the i -th row, we have satisfied the above conditions above.

Exercise 1.2.5

(De Morgan's Law) Let A and B be subsets of \mathbb{R} .

- (a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.
- (b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.
- (c) Show $(A \cup B)^c = A^c \cup B^c$ by demonstrating inclusion both ways.

SOLUTION

- (a) If $x \in (A \cap B)^c$, then $x \notin A \cap B$, so $x \notin A$ or $x \notin B$, implying $x \in A^c$ or $x \in B^c$, therefore $x \in A^c \cup B^c$.
- (b) If $x \in A^c \cup B^c$, then $x \in A^c$ or $x \in B^c$, so $x \notin A$ and $x \notin B$, implying $x \notin A \cap B$, therefore $x \in (A \cap B)^c$. Since $(A \cap B)^c \subseteq A^c \cup B^c$ and $(A \cap B)^c \supseteq A^c \cup B^c$, we can conclude that both sets are equal.
- (c) To show that $(A \cap B)^c = A^c \cup B^c$, we need to demonstrate inclusion both ways.
- (i) If $x \in (A \cup B)^c$, then $x \notin A \cup B$, so $x \notin A$ or $x \notin B$, implying $x \in A^c$ or $x \in B^c$, therefore $x \in A^c \cup B^c$.

- (ii) If $x \in A^c \cap B^c$, then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$, implying $x \notin A \cup B$, which is just $x \in (A \cup B)^c$.

Exercise 1.2.6

- (a) Verify the triangle inequality in the special case where a and b have the same sign.
- (b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.
- (c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c and d .
- (d) Prove $\|a|-|b\| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

SOLUTION

- (a) With both a and b having the same sign, then $|a| + |b| = |a+b|$, which satisfies $|a| + |b| \geq |a+b|$.
- (b) Note that $(a+b)^2 \leq (|a|+|b|)^2$ reduces to $ab \leq |a||b|$, which is true as LHS can be negative while RHS cannot. Since squaring preserves inequality, this implies that $|a+b| \leq |a| + |b|$.
- (c) Notice that $a-b = (a-c) + (c-d) + (d-b)$. Hence by triangle inequality,

$$|a-b| = |(a-c) + (c-d) + (d-b)| \leq |a-c| + |c-d| + |d-b|$$

for all a, b, c and d .

- (d) Since $\|a|-|b\| = \|b|-|a\|$, WLOG, we can assume that $|a| \geq |b|$. Then

$$\|a|-|b\| = |a|-|b| = |(a-b)+b|-|b| \leq |a-b| + |b| - |b| = |a-b|$$

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A ; that is, $f(A) = \{f(x) : x \in A\}$.

- (a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?
- (b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
- (c) Show that, for an arbitrary function $g : \mathbb{R} \rightarrow \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.
- (d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g .

SOLUTION

- (a) For $f(x) = x^2$, $f(A) = f([0, 2]) = [0, 4]$ and $f(B) = f([1, 4]) = [1, 16]$.

$$\begin{aligned} f(A \cap B) &= f([0, 2] \cap [1, 4]) = f([1, 2]) = [1, 4] = [0, 4] \cap [1, 16] = f([1, 2]) \cap f([2, 4]) = f(A) \cap f(B) \\ f(A \cup B) &= f([0, 2] \cup [1, 4]) = f([0, 4]) = [0, 16] = [0, 4] \cup [1, 16] = f([0, 2]) \cup f([1, 4]) = f(A) \cup f(B) \end{aligned}$$

- (b) Consider $A = [0, 2]$ and $B = [-2, 0]$. $f(A \cap B) = \{0\}$, but $f(A) \cap f(B) = [0, 4]$.
- (c) Suppose $y \in g(A \cap B)$, then $\exists x \in A \cap B$ such that $g(x) = y$. This implies that $x \in A$ and $x \in B$, so $x \in A \cap B$, hence $y \in g(A \cap B)$. Note that contrary may not always be true as it is possible for $x_1 \in A \setminus B$ and $x_2 \in B \setminus A$ such that $g(x_1) = g(x_2)$.
- (d) I conjecture that $g(A \cup B) = g(A) \cup g(B)$. To prove this, we have to show inclusion both ways:
 - (i) Let $y \in g(A \cup B)$, then $\exists x \in A \cup B$ such that $y = g(x)$. This implies that $x \in A$ or $x \in B$, so $y \in g(A)$ or $y \in g(B)$, hence $y \in g(A) \cup g(B)$.
 - (ii) Let $y \in g(A) \cup g(B)$, then $y \in g(A)$ or $y \in g(B)$, implying $x \in A$ or $x \in B$ such that $y = g(x)$. So $x \in A \cup B$, hence $y \in g(A \cup B)$.