Understanding Analysis Attempt/Solution

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Contents

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is rational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

SOLUTION

(a) AFSOC that $\sqrt{3}$ is rational, so $\exists m, n \in \mathbb{Z}$, such that

$$\sqrt{3} = \frac{m}{n},$$

where $\frac{m}{n}$ is in the lowest reduced terms. By squaring both sides, we obtain $3 = (\frac{m}{n})^2 \implies 3n^2 = m^2$. Now, we know that m^2 is a multiple of 3 and thus m must also be a multiple of 3. We can then write $m = 3k$, deriving

$$\begin{aligned}(\sqrt{3})^2 &= \left(\frac{3k}{n}\right)^2\\ 3n^2 &= 9k^2\\ n^2 &= 3k^2\end{aligned}$$

Similar to above, we can conclude that n is a multiple of 3. However this is a contradiction since m, n are both multiples of 3 but we assumed that $\frac{m}{n}$ was in its lowest reduced term. Thus we conclude that $\sqrt{3}$ is irrational. The same proof for $\sqrt{3}$ works for $\sqrt{6}$ as well.

(b) We cannot conclude that $\sqrt{4} = \frac{m}{n}$ imply that m is a multiple of 4, as we have

$$4n^2 = m^2 \implies 2n = m,$$

preventing us from reaching our contradiction that m/n is not in its lowest terms.

Exercise 1.2.2

Show that there is no rational number r satisfying $2^r = 3$.

SOLUTION

If $r = 0$, then $2^r = 1 \neq 3$. Suppose $r = p/q$ to get $2^p = 3^q$, which is not possible as 2 and 3 share no common factors. Hence r is not rational.

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$ are all sets containing an infinite number of elements, then the intersections $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and non-empty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

SOLUTION

(a) False. Consider $A_n = \{n, n+1, n+2, \dots\}$, then $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(b) True. Since all A_n are nonempty, $\exists n \in \mathbb{N}$ such that $A_n = \{x\}$ for some real x. Hence $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\}$ which is empty. Since A_1 is finite, $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\} \subset A_1$ is finite.

(c) False. If $A = \emptyset$, then $\emptyset = C$

(d) True. Intersection is associative as evident that both LHS and RHS implies the $x \in A, B, C$

(e) True. Drawing a Venn Diagram illustrates this.

Exercise 1.2.4

Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

SOLUTION

Consider arranging the elements of $\mathbb{N}$ in a square as such.

$$\begin{array}{cccccc}
1 & 3 & 6 & 10 & 15 & \dots \\
2 & 5 & 9 & 14 & \dots & \\
4 & 8 & 13 & \dots & & \\
7 & 12 & \dots & & & \\
11 & \dots & & & & \\
\vdots & & & & &
\end{array}$$

By letting A_i being the set of all natural numbers in the i-th row, we have satisfied the above conditions above.

Exercise 1.2.5

(De Morgan's Law) Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. THis shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \sqsubseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cup B^c$ by demonstrating inclusion both ways.

SOLUTION

(a) If $x \in (A \cap B)^c$, then $x \notin A \cap B$, so $x \notin A$ or $x \notin B$, implying $x \in A^c$ or $x \in B^c$, therfore $x \in A^c \cup B^c$.

(b) If $x \in A^c \cup B^c$, then $x \in A^c$ or $x \in B^c$, so $x \notin A$ and $x \notin B$, implying $x \notin A \cap B$, therefore $x \in (A \cap B)^c$. Since $(A \cap B)^c \subseteq A^c \cup B^c$ and $(A \cap B)^c \sqsubseteq A^c \cup B^c$, we can conclude that both sets are equal.

(c) To show that $(A \cap B)^c = A^c \cup B^c$, we need to demonstrate inclusion both ways.

   (i) If $x \in (A \cup B)^c$, then $x \notin A \cup B$, so $x \notin A$ or $x \notin B$, implying $x \in A^c$ or $x \in B^c$, therefore $x \in A^c \cup B^c$.

(ii) If $x \in A^c \cap B^c$, then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$, implying $x \notin A \cup B$, which is just $x \in (A \cup B)^c$.

Exercise 1.2.6

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c|+|c-d|+|d-b|$ for all a, b, c and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a-b+b$ may be useful.)

SOLUTION

(a) With both a and b having the same sign, then $|a|+|b| = |a+b|$, which satisfies $|a|+|b| \geq |a+b|$.

(b) Note that $(a+b)^2 \leq (|a|+|b|)^2$ reduces to $ab \leq |a||b|$, which is true as LHS can be negative while RHS cannot. Since squaring preserves inequality, this implies that $|a+b| \leq |a|+|b|$.

(c) Notice that $a-b = (a-c)+(c-d)+(d-b)$. Hence by triangle inequality,

$$|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c|+|c-d|+|d-b|$$

for all a, b, c and d.

(d) Since $||a|-|b|| = ||b|-|a||$, WLOG, we can assume that $|a| \geq |b|$. Then

$$||a|-|b|| = |a|-|b| = |(a-b)+b|-|b| \leq |a-b|+|b|-|b| = |a-b|$$

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0,2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbituary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbituary function g.

SOLUTION

(a) For $f(x) = x^2$, $f(A) = f([0,2]) = [0,4]$ and $f(B) = f([1,4]) = [1,16]$.

$$f(A \cap B) = f([0,2] \cap [1,4]) = f([1,2]) = [1{,}4] = [0,4] \cap [1,16] = f([1,2]) \cap f([2,4]) = f(A) \cap f(B)$$
$$f(A \cup B) = f([0,2] \cup [1,4]) = f([0,4]) = [0{,}16] = [0,4] \cup [1,16] = f([0,2]) \cup f([1,4]) = f(A) \cup f(B)$$

(b) Consider $A = [0,2]$ and $B = [-2,0]$. $f(A \cap B) = \{0\}$, but $f(A) \cap f(B) = [0,4]$.

(c) Suppose $y \in g(A \cap B)$, then $\exists x \in A \cap B$ such that $g(x) = y$. This implies that $x \in A$ and $x \in B$, so $x \in A \cap B$, hence $y \in g(A \cap B)$. Note that contrary may not always be true as it is possible for $x_1 \in A \backslash B$ and $x_2 \in B \backslash A$ such that $g(x_1) = g(x_2)$.

(d) I conjecture that $g(A \cup B) = g(A) \cup g(B)$. To prove this, we have to show inclusion both ways:

(i) Let $y \in g(A \cup B)$, then $\exists x \in A \cup B$ such that $y = g(x)$. This implies that $x \in A$ or $x \in B$, so $y \in g(A)$ or $y \in g(B)$, hence $y \in g(A) \cup g(B)$.

(ii) Let $y \in g(A) \cup g(B)$, then $y \in g(A)$ or $y \in g(B)$, implying $x \in A$ or $x \in B$ such that $y = g(x)$. So $x \in A \cup B$, hence $y \in g(A \cup B)$.

Exercise 1.2.8
Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* $(1-1)$ if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ Give an example of each or state that the request is impossible:

(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto.

(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1.

(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

SOLUTION
(a) Let $f(x) = x + 1$, which is 1-1 but does not have a solution to $f(x) = 1$, hence not onto.

(b) Let $f(x) = 1$ for $x = 1$ and $f(x) = x - 1$ for $x > 1$, which is onto but not 1-1 as $f(1) = f(2) = 1$.

(c) Let $f(x) = n/2$ when n is even and $f(x) = -\frac{x-1}{2}$ when n is odd.

Exercise 1.2.9
Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behaviour of preimages demonstrated in (a) is completely general. Show that for an arbituary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbb{R}$.

SOLUTION
(a) For $f(x) = x^2$, $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. $f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$. Similarly, $f^{-1}(A \cup B) = f^{-1}([-1, 4]) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$.

(b) To show that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$, we have to show inclusion both ways:

  (i) Let $x \in g^{-1}(A \cap B)$, so $g(x) \in A \cap B$, which implies $g(x) \in A$ and $g(x) \in B$. This shows that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, hence $x \in g^{-1}(A) \cap g^{-1}(B)$.

  (ii) Let $x \in g^{-1}(A) \cap g^{-1}(B)$, so $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, then $g(x) \in A$ and $g(x) \in B$. This implies that $g(x) \in A \cap B$, so $x \in g^{-1}(A \cap B)$.

Showing $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ is obvious using Exercise 1.2.7 (d).

Exercise 1.2.10
Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

SOLUTION
(a) False. Consider the case where $a < b + \epsilon$ is true but $a = b$.

(b) False. Same reasoning as above.

(c) True. Firsly suppose $a < b + \epsilon$ for all $\epsilon > 0$. We need to show this implies $a \leq b$. We either have $a \leq b$ or $a > b$. However, $a > b$ is not possible as this implies there exists an ϵ small enough such that $a > b + \epsilon$. Secondly, suppose $a \leq b$. It is obvious that $a < b + \epsilon$ for all $\epsilon > 0$.

Exercise 1.2.11
Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this more interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $a + 1/n < b$.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.

(c) Between every two distinct real numbers there is a rational number.

SOLUTION
(a) For all $n \in \mathbb{N}$, there exists $a, b \in \mathbb{R}$ such that $a + 1/n < b$. [FALSE]

(b) For all real number $x > 0$, there exists an $n \in \mathbb{N}$ such that $x \geq 1/n$. [TRUE]

(c) There exists two real numbers $a < b$ such that if $r < b$ then $r < a$ for all $r \in \mathbb{Q}$. [FALSE]

Exercise 1.2.12
Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \dots)$ is decreasing.

SOLUTION
(a) For $n = 1$, $y_1 = 6 > -6$ (Base Case). Suppose $y_n > -6$ for some $n \in \mathbb{N}$.

$$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2(-6) - 6}{3} = -6$$

Hence, by induction, $y_n > -6$ for all $n \in \mathbb{N}$.

(b) Suppose $y_{n+1} < y_n$. The base case works as $y_2 = 2 < 6 = y_1$. Then,

$$\begin{aligned} y_{n+1} < y_n &\implies 2y_{n+1} - 6 < 2y_n - 6 \\ &\implies \frac{2y_{n+1} - 6}{3} < \frac{2y_n - 6}{3} \\ &\implies y_{n+2} < y_{n+1} \end{aligned}$$

Thus, $y_{n+1} < y_n$ is true for all $n \in \mathbb{N}$.

Exercise 1.2.13
For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \dots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbb{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

SOLUTION

(a) Using Exercise 1.2.5 as the base case. Suppose $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cup \cdots \cup A_n^c$ is true. Using associative law on the $n + 1$c case,

$$\begin{aligned}((A_1 \cup \cdots \cup A_n) \cup A_{n+1})^c &= (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c \\ &= (A_1^c \cap \cdots \cap A_n^c) \cap A_{n+1}^c \\ &= A_1^c \cap \cdots \cap A_n^c \cap A_{n+1}^c\end{aligned}$$

(b) Consider $B_1 = \{1, 2, \dots\}, B_2 = \{2, 3, \dots\}, B_n = \{x : x \in \mathbb{N} \cap [n, \infty)\}$.

(c) To prove the infinite version of De Morgan's Law, we need to show inclusion both ways.

(i) Suppose $x \in (\bigcup_{i=1}^{\infty} A_i)^c$, then $x \notin \bigcup_{i=1}^{\infty} A_i$. This implies that $x \notin A_i\ \forall i \in \mathbb{N}$, so $x \in A_i^c\ \forall i \in \mathbb{N}$. Hence $x \in \bigcap_{i=1}^{\infty} A_i^c$.

(ii) Suppose $x \in \bigcap_{i=1}^{\infty} A_i^c$, then $x \in A_i^c\ \forall i \in \mathbb{N}$, so $x \notin A_i\ \forall i \in \mathbb{N}$. This implies that $x \notin \bigcup_{i=1}^{\infty} A_i$, hence $x \in (\bigcup_{i=1}^{\infty} A_i)^c$.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bound.

SOLUTION

(a) A real number i is the *greatest upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) i is a lower bound of A;

(ii) if b is any lower bound for A, then $i \geq b$.

(b) Assume $i \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then $i = \inf A$ *if and only if*, for every choice of $\epsilon > 0$, there $\exists a \in A$ satisfying $i + \epsilon > a$.

Proof Rephrasing the lemma gives us: Given that i is a lower bound, i is the greatest lower bound if and only if any number greater than i is not a lower bound.

(i) Suppose $i = \inf A$ and consider $i + \epsilon$ for an arbituarily chosen $\epsilon > 0$. Since $i + \epsilon > i$, part (ii) of the definition impliers that $i + \epsilon$ is not a lower bound for A. If this is the case, then there must be some element $a \in A$ such that $i + \epsilon > a$.

(ii) Conversely, assume i is a lower bound with the property that for any $\epsilon > 0$, $s + \epsilon$ is not a lower bound of A. Note that this implies that if b is any number more than i, then b is not an upperbound. Since we have argued that any larger number than i cannot be a lower bound, if b is some other upper bound for A, then $i \geq b$.

Exercise 1.3.2

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

SOLUTION

(a) Possible. Consider the set 0, where $\inf\{0\} = \sup\{0\} = 0$.

(b) Not possible as all finite sets must contain its supremum and infimum.

(c) Possible. Consider $A = \{r \in \mathbb{Q} \mid 1 < r \leq 2\}$.

Exercise 1.3.3

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

SOLUTION

(a) By definition, $\sup B$ is the greatest lower bound for A, meaning it equals $\inf A$.

(b) Part (a) proves the greatest lower bound exists using the least upper bound.

Exercise 1.3.4

Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^{n} A_k)$.

(b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

SOLUTION

(a) $\sup(A_1 \cup A_2) = \sup\{\sup A_1, \sup A_2\}$ and $\sup(\bigcup_{k=1}^{n} A_k) = \sup\{\sup A_k \mid k = 1, \ldots, n\}$

(b) This formula does not extend to infinity. Consider $A_k = [k, k+1]$, where $\bigcup_{k=1}^{\infty} A_k$ is unbounded.

Exercise 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Prostulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

SOLUTION

(a) The case of $c = 0$ is trivial as it implies that $cA = \{0\}$. Hence $\sup(cA) = c \sup A = 0$.

For $c > 0$, we need to show that $c \sup A$ is the lowest upper bound. Assume $c > 0$. Let $s = c \sup A$. Suppose $ca > s$, then $a > \sup A$ which is impossible, meaning that s is an upper bound on cA. Now suppose s' is an upper bound on cA and $s' < s$. Then $s'/c < s/c = \sup A$, meaning s'/c cannot bound A. Hence there $\exists a \in A$ such that $s'/c < a$, meaning $s' > ca$, thus s' cannot be an upper bound on cA, so $s = c \sup A$ is the least upper bound.

(b) $\sup(cA) = c \inf A$ for $c < 0$

Exercise 1.3.6

Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbituary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u + a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

SOLUTION

(a) By definition of supremum, $a \leq s$ and $b \leq t$. Adding both equations give $a + b \leq s + t$, hence $s + t$ is an upper bound.

(b) Since $a + b \leq u$ implies $b \leq u - a$, $u - a$ is an upper bound on b, meaning it is greater or equal to the least upper bound of t, giving $t \leq u - a$.

(c) From (a), we have shown that $s + t$ is an upper bound for $A + B$, hence it is sufficent to show that $s + t$ is the least upper bound.

Let $u = \sup(A + B)$, from (b) we have $t \leq u - a$ and $s \leq u - b$. Adding and rearranging gives $a + b \leq 2u - s - t$. Since $2u - s - t$ is an upper bound on $A + B$, it must be greater or equal to the least upper bound, giving $u \leq 2u - s - t$, implying $s + t \leq u$. Since u is the least upper bound, $s + t$ must equal u.

(d) Showing $s + t - \epsilon$ is not an upper bound for any $\epsilon > 0$ proves that it is the least upper bound by Lemma 1.3.8. Rearranging gives $(s - \epsilon/2) + (t - \epsilon/2)$ we know $\exists a > (s - \epsilon/2)$ and $b > (t - \epsilon/2)$, therefore $a + b > s + t - \epsilon$, meaning $s + t$ cannot be made smaller and thus is the least upper bound.

Exercise 1.3.7

Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

SOLUTION

Since it is given that a is an upper bound for A, we just have to show that a is the least upper bound, meaning any number lower than a would have an $a' \in A$ such that $a' > a$.

Suppose $a - \epsilon$ is also an upper bound for A for some $\epsilon > 0$. This is not possible has $a > a'$ and $a \in A$. Hence by contradiction, a is the lowest upper bound, meaning $a = \sup A$.

Exercise 1.3.8

Compute, without proofs, the suprema and inifima (if they exists) of the following sets:

(a) $\{m/m : m, n \in \mathbb{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.

(c) $\{n/(3n + 1) : n \in \mathbb{N}\}$.

(d) $\{m/(m + n) : m, n \in \mathbb{N}\}$.

SOLUTION

(a) $\inf = 0$ and $\sup = 1$

(b) $\inf = -1$ and $\sup = 1$

(c) $\inf = 1/4$ and $\sup = 1/3$

(d) $\inf = 0$ and $\sup = 1$

Exercise 1.3.9

(a) If $\sup A < \sup B$, show there exists an element that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq supB$.

SOLUTION

(a) By Lemma 1.3.8, we know there exists a b such that $(\sup B) - \epsilon < b$ for any $\epsilon > 0$. We can set ϵ to be small enough such that $\sup B - \sup A < \epsilon$, implying $\sup A < \sup B - \epsilon < b$ for some $b \in B$, thus b is an upper bound of A.

(b) Consider the sets $A = (-\infty, 1]$ and $B = (-\infty, 1)$. No $b \in B$ is an upperbound since $1 \in A$ and $1 > b$.

Exercise 1.3.10
(Cut Property) The *Cut Property* of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists a $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbb{R}$ is replaced by $\mathbb{Q}$.

SOLUTION

(a) If $c = \sup A = \inf B$, it is obvious that $a \leq c \leq b$ and we are done. Hence we need to show, by contradiction, that $\sup A < \inf B$ and $\sup A > \inf B$ is false.

  (i) AFSOC $\sup A < \inf B$. We can choose $c = \frac{\sup A + \inf B}{2}$, which satisfies $\sup A < c < \inf B$. Hence it is obvious that $c \notin A$ and $c \notin B$, so $c \notin A \cup B \neq \mathbb{R}$ which is a contradiction.

  (ii) AFSOC $\sup A > \inf B$. We can find a such that $a > b$ by subtracting $\epsilon > 0$ and using the definition of supremum and infimum similar to Lemma 1.3.8. Thus creating a contradiction.

  Since both alternatives are impossible, $\sup A = \inf B$.

(b) If E is finite or has a maximum element, that is $\sup E$ and we are done.

  Consider the case where E has not maximum element (for example, $\{-1/n : n \in \mathbb{N}\}$). Let B be the sets of all upper bounds of E and let $A = B^c$. It can be said $E \cap B = \emptyset$ otherwise E has a maximum element. Thus $E \subseteq A$.

  By the Cut Property, there exists c such that $a \leq c \leq b$ for all $a \in A$ and $b \in B$. Since c is an upper bound on A and $E \subseteq A$, c is also an upper bound on E. And since $c \leq b$, c is the lowest upper bound. Therefore, $c = \sup E$.

(c) Consider $A = \{r \in \mathbb{Q} \mid r^2 < 2 \text{ or } r < 0\}$, $B = A^c$ does not satisfy the cut property in $\mathbb{Q}$ as $\sqrt{2} \notin \mathbb{Q}$.

Exercise 1.3.11
Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

SOLUTION

(a) True. We know $a \leq \sup A$, $a \leq \sup B$ since $A \subseteq B$. Since $\sup A$ is the least upper bound on A, we have $\sup A \leq \sup B$.

(b) True. Let $c = \frac{\sup A + \inf B}{2}$, $c > \sup A$ implies $a < c$ and $c < \inf B$ implies $c < b$, giving $a < c < b$.

(c) False. Consider $A = (-\infty, 1)$ and $B = (1, \infty)$, $a < 1 < b$ but $\sup A \not< \inf B$ as $\sup A = \inf B = 1$.

1.4 Consequences of Completeness

Exercise 1.4.1
Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a+b$ are elements of $\mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a+t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s+t$ and st?

SOLUTION

(a) This is trivial. Since $a, b \in \mathbb{Q}$, they can be expressed as a fraction of two integers. Let $a = m/n$ and $b = x/y$ where $m, n, x, y \in \mathbb{Z}$, then $a + b = \frac{my+xn}{ny}$ and $ab = \frac{mx}{ny}$, which are fractions with integer numerators and denominators, hence $a+b$ and ab are elements of $\mathbb{Q}$.

(b) AFSOC that $a + t \in \mathbb{Q}$ and $at \in \mathbb{Q}$. Let $a + t = \alpha$ and $at = \beta$, so $t = \alpha - a$ and $t = \beta/a$. Since $\alpha, \beta, -a, 1/a \in \mathbb{Q}$, using part (a) gives use $t \in \mathbb{Q}$ which is a contradiction. Hence $a+t \in \mathbb{I}$ and $at \in \mathbb{I}$.

(c) $\mathbb{I}$ is not closed under addition or multiplication. Consider $\sqrt{2} + (-\sqrt{2}) = 0$ and $\sqrt{2} \cdot \sqrt{2} = 2$.

Exercise 1.4.2
Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upperbound for A. SHow $s = \sup A$.

SOLUTION
We can rephrase Lemma 1.3.8 using the archimedean property.

(i) AFSOC $s < \sup A$, then there exists an n such that $s + 1/n < \sup A$, contradicting $\sup A$ being the least upper bound.

(ii) AFSOC $s > \sup A$, then there exists an n such that $s - 1/n > \sup A$ where $s - 1/n$ is not an upper bound, contradicting $\sup A$ being an upper bound.

Hence, we can conclude that $\sup A = s$.

Exercise 1.4.3
Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

SOLUTION
AFSOC the $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$, so $0 < x < 1/n$ for all $n \in \mathbb{N}$ which is impossible by archimedean property.

Exercise 1.4.4
Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.

SOLUTION
To show that $\sup T = b$, it needs to satify both conditions of supremum:

(i) Since $x \leq b$ for all $x \in [a, b]$, $y \leq b$ for all $y \in T$ as $T \subseteq [a, b]$.

(ii) AFSOC b' is also an upper bound such that $b' < b$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists an $\alpha \in \mathbb{Q} \cap [b', b] \subseteq T$. This implies there exists $t \in T$ satisfying $b' < t$, which is a contradiction.

Exercise 1.4.5
Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

SOLUTION
Recall that **Corollary 1.4.4** states that *Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.*

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $t \in \mathbb{Q}$ that is between any two real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$, with $a < b$. Hence, $a - \sqrt{2} < t < b - \sqrt{2}$, meaning $a < t + \sqrt{2} < b$. By Exercise 1.4.1, $t + \sqrt{2} \in \mathbb{I}$ and we are done.

Exercise 1.4.6
Recall that a set B is *dense* in $\mathbb{R}$ if can element B can be found between any two real numbers $a < b$. WHich of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2.

(c) The set of all rational numbers p/q with $10|p| \geq q$.

SOLUTION

(a) Not dense since we cannot make $|p|/q < 1/10$.

(b) Dense.

(c) Not dense since we cannot make $|p|/q < 1/10$.

Exercise 1.4.7
Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

SOLUTION
Recall $T = \{t \in \mathbb{R} : t^2 < 2\}$ and $\alpha = \sup T$. AFSOC $\alpha^2 > 2$, we will show that there exists an $n \in \mathbb{N}$ such that $(\alpha - 1/n)^2 > 2$, contradicting the assumption that α is the least upper bound.

Using $(\alpha - 1/n)^2 > 2$, we can find $n \in \mathbb{N}$ such that $(\alpha^2 - 1/n) > 2$.

$$2 < (\alpha - 1/n)^2 = \alpha^2 - \frac{2\alpha}{n} + 1/n^2 < \alpha^2 - \frac{2\alpha - 1}{n}$$

Then

$$2 < \alpha^2 - \frac{2\alpha - 1}{n} \implies n(2 - \alpha^2) < 1 - 2\alpha$$

Since $2 - \alpha^2 < 0$, dividing reverses the inequality, giving

$$n > \frac{1 - 2\alpha}{2 - \alpha^2}$$

Hence we can pick $n \in \mathbb{N}$ such that $(\alpha^2 - 1/n) > 2$, so α is the the least upper bound which is a contradiction. Hence it is not possible for $\alpha^2 > 2$.