

# Understanding Analysis Attempt/Solution

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# Chapter 1

## The Real Numbers

### 1.2 Some Preliminaries

#### Exercise 1.2.1

(a) Prove that  $\sqrt{3}$  is irrational. Does a similar argument work to show  $\sqrt{6}$  is rational?

(b) Where does the proof break down if we try to prove  $\sqrt{4}$  is irrational?

#### SOLUTION

(a) PROOF AFSOC that  $\sqrt{3}$  is rational, so  $\exists m, n \in \mathbb{Z}$ , such that

$$\sqrt{3} = \frac{m}{n},$$

where  $\frac{m}{n}$  is in the lowest reduced terms. By squaring both sides, we obtain  $3 = \left(\frac{m}{n}\right)^2 \implies 3n^2 = m^2$ . Now, we know that  $m^2$  is a multiple of 3 and thus  $m$  must also be a multiple of 3. We can then write  $m = 3k$ , deriving

$$\begin{aligned} (\sqrt{3})^2 &= \left(\frac{3k}{n}\right)^2 \\ 3n^2 &= 9k^2 \\ n^2 &= 3k^2 \end{aligned}$$

Similar to above, we can conclude that  $n$  is a multiple of 3. However this is a contradiction since  $m, n$  are both multiples of 3 but we assumed that  $\frac{m}{n}$  was in its lowest reduced term. Thus we conclude that  $\sqrt{3}$  is irrational.

The same proof for  $\sqrt{3}$  works for  $\sqrt{6}$  as well.

(b) We cannot conclude that  $\sqrt{4} = \frac{m}{n}$  imply that  $m$  is a multiple of 4, as we have

$$4n^2 = m^2 \implies 2n = m,$$

preventing us from reaching our contradiction that  $m/n$  is not in its lowest terms.

#### Exercise 1.2.2

Show that there is no rational number  $r$  satisfying  $2^r = 3$ .

#### SOLUTION

PROOF If  $r = 0$ , then  $2^r = 1 \neq 3$ . Suppose  $r = p/q$  to get  $2^p = 3^q$ , which is not possible as 2 and 3 share no common factors. Hence  $r$  is not rational.

#### Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

- (a) If  $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$  are all sets containing an infinite number of elements, then the intersections  $\bigcap_{n=1}^{\infty} A_n$  is infinite as well.
- (b) If  $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$  are all finite, nonempty sets of real numbers, then the intersection  $\bigcap_{n=1}^{\infty} A_n$  is finite and non-empty.
- (c)  $A \cap (B \cup C) = (A \cap B) \cup C$ .
- (d)  $A \cap (B \cap C) = (A \cap B) \cap C$ .
- (e)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

**SOLUTION**

- (a) False. Consider  $A_n = \{n, n+1, n+2, \dots\}$ , then  $\bigcap_{n=1}^{\infty} A_n = \emptyset$ .
- (b) True. Since all  $A_n$  are nonempty,  $\exists n \in \mathbb{N}$  such that  $A_n = \{x\}$  for some real  $x$ . Hence  $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\}$  which is empty. Since  $A_1$  is finite,  $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\} \subseteq A_1$  is finite.
- (c) False. If  $A = \emptyset$ , then  $\emptyset = C$
- (d) True. Intersection is associative as evident that both LHS and RHS implies the  $x \in A, B, C$
- (e) True. Drawing a Venn Diagram illustrates this.

**Exercise 1.2.4**

Produce an infinite collection of sets  $A_1, A_2, A_3, \dots$  with the property that every  $A_i$  has an infinite number of elements,  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ , and  $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$ .

**SOLUTION**

Consider arranging the elements of  $\mathbb{N}$  in a square as such.

1	3	6	10	15	...
2	5	9	14	...	
4	8	13	...		
7	12	...			
11	...				
$\vdots$					

By letting  $A_i$  being the set of all natural numbers in the  $i$ -th row, we have satisfied the above conditions above.

**Exercise 1.2.5**

**(De Morgan's Law)** Let  $A$  and  $B$  be subsets of  $\mathbb{R}$ .

- (a) If  $x \in (A \cap B)^c$ , explain why  $x \in A^c \cup B^c$ . This shows that  $(A \cap B)^c \subseteq A^c \cup B^c$ .
- (b) Prove the reverse inclusion  $(A \cap B)^c \supseteq A^c \cup B^c$ , and conclude that  $(A \cap B)^c = A^c \cup B^c$ .
- (c) Show  $(A \cup B)^c = A^c \cap B^c$  by demonstrating inclusion both ways.

**SOLUTION**

- (a) If  $x \in (A \cap B)^c$ , then  $x \notin A \cap B$ , so  $x \notin A$  or  $x \notin B$ , implying  $x \in A^c$  or  $x \in B^c$ , therefore  $x \in A^c \cup B^c$ .
- (b) If  $x \in A^c \cup B^c$ , then  $x \in A^c$  or  $x \in B^c$ , so  $x \notin A$  and  $x \notin B$ , implying  $x \notin A \cap B$ , therefore  $x \in (A \cap B)^c$ . Since  $(A \cap B)^c \subseteq A^c \cup B^c$  and  $(A \cap B)^c \supseteq A^c \cup B^c$ , we can conclude that both sets are equal.
- (c) To show that  $(A \cup B)^c = A^c \cap B^c$ , we need to demonstrate inclusion both ways.
  - (i) If  $x \in (A \cup B)^c$ , then  $x \notin A \cup B$ , so  $x \notin A$  or  $x \notin B$ , implying  $x \in A^c$  or  $x \in B^c$ , therefore  $x \in A^c \cap B^c$ .

- (ii) If  $x \in A^c \cap B^c$ , then  $x \in A^c$  and  $x \in B^c$ , so  $x \notin A$  and  $x \notin B$ , implying  $x \notin A \cup B$ , which is just  $x \in (A \cup B)^c$ .

**Exercise 1.2.6**

- (a) Verify the triangle inequality in the special case where  $a$  and  $b$  have the same sign.  
 (b) Find an efficient proof for all the cases at once by first demonstrating  $(a + b)^2 \leq (|a| + |b|)^2$ .  
 (c) Prove  $|a - b| \leq |a - c| + |c - d| + |d - b|$  for all  $a, b, c$  and  $d$ .  
 (d) Prove  $||a| - |b|| \leq |a - b|$ . (The unremarkable identity  $a = a - b + b$  may be useful.)

**SOLUTION**

- (a) With both  $a$  and  $b$  having the same sign, then  $|a| + |b| = |a + b|$ , which satisfies  $|a| + |b| \geq |a + b|$ .  
 (b) Note that  $(a + b)^2 \leq (|a| + |b|)^2$  reduces to  $ab \leq |a||b|$ , which is true as LHS can be negative while RHS cannot. Since squaring preserves inequality, this implies that  $|a + b| \leq |a| + |b|$ .  
 (c) Notice that  $a - b = (a - c) + (c - d) + (d - b)$ . Hence by triangle inequality,

$$|a - b| = |(a - c) + (c - d) + (d - b)| \leq |a - c| + |c - d| + |d - b|$$

for all  $a, b, c$  and  $d$ .

- (d) Since  $||a| - |b|| = ||b| - |a||$ , WLOG, we can assume that  $|a| \geq |b|$ . Then

$$||a| - |b|| = |a| - |b| = |(a - b) + b| - |b| \leq |a - b| + |b| - |b| = |a - b|$$

bxGiven a function  $f$  and a subset  $A$  of its domain, let  $f(A)$  represent the range of  $f$  over the set  $A$ ; that is,  $f(A) = \{f(x) : x \in A\}$ .

- (a) Let  $f(x) = x^2$ . If  $A = [0, 2]$  (the closed interval  $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$ ) and  $B = [1, 4]$ , find  $f(A)$  and  $f(B)$ . Does  $f(A \cap B) = f(A) \cap f(B)$  in this case? Does  $f(A \cup B) = f(A) \cup f(B)$ ?  
 (b) Find two sets  $A$  and  $B$  for which  $f(A \cap B) \neq f(A) \cap f(B)$ .  
 (c) Show that, for an arbitrary function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , it is always true that  $g(A \cap B) \subseteq g(A) \cap g(B)$  for all sets  $A, B \subseteq \mathbb{R}$ .  
 (d) Form and prove a conjecture about the relationship between  $g(A \cup B)$  and  $g(A) \cup g(B)$  for an arbitrary function  $g$ .

**SOLUTION**

- (a) For  $f(x) = x^2$ ,  $f(A) = f([0, 2]) = [0, 4]$  and  $f(B) = f([1, 4]) = [1, 16]$ .

$$\begin{aligned} f(A \cap B) &= f([0, 2] \cap [1, 4]) = f([1, 2]) = [1, 4] = [0, 4] \cap [1, 16] = f([1, 2]) \cap f([2, 4]) = f(A) \cap f(B) \\ f(A \cup B) &= f([0, 2] \cup [1, 4]) = f([0, 4]) = [0, 16] = [0, 4] \cup [1, 16] = f([0, 2]) \cup f([1, 4]) = f(A) \cup f(B) \end{aligned}$$

- (b) Consider  $A = [0, 2]$  and  $B = [-2, 0]$ .  $f(A \cap B) = \{0\}$ , but  $f(A) \cap f(B) = [0, 4]$ .  
 (c) Suppose  $y \in g(A \cap B)$ , then  $\exists x \in A \cap B$  such that  $g(x) = y$ . This implies that  $x \in A$  and  $x \in B$ , so  $x \in A \cap B$ , hence  $y \in g(A \cap B)$ . Note that contrary may not always be true as it is possible for  $x_1 \in A \setminus B$  and  $x_2 \in B \setminus A$  such that  $g(x_1) = g(x_2)$ .