Understanding Analysis Attempt/Solution

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Contents

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is rational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

SOLUTION

(a) AFSOC that $\sqrt{3}$ is rational, so $\exists m, n \in \mathbb{Z}$, such that

$$\sqrt{3} = \frac{m}{n},$$

where $\frac{m}{n}$ is in the lowest reduced terms. By squaring both sides, we obtain $3 = (\frac{m}{n})^2 \implies 3n^2 = m^2$. Now, we know that m^2 is a multiple of 3 and thus m must also be a multiple of 3. We can then write $m = 3k$, deriving

$$(\sqrt{3})^2 = \left(\frac{3k}{n}\right)^2$$
$$3n^2 = 9k^2$$
$$n^2 = 3k^2$$

Similar to above, we can conclude that n is a multiple of 3. However this is a contradiction since m, n are both multiples of 3 but we assumed that $\frac{m}{n}$ was in its lowest reduced term. Thus we conclude that $\sqrt{3}$ is irrational. The same proof for $\sqrt{3}$ works for $\sqrt{6}$ as well.

(b) We cannot conclude that $\sqrt{4} = \frac{m}{n}$ imply that m is a multiple of 4, as we have

$$4n^2 = m^2 \implies 2n = m,$$

preventing us from reaching our contradiction that m/n is not in its lowest terms.

Exercise 1.2.2

Show that there is no rational number r satisfying $2^r = 3$.

SOLUTION

If $r = 0$, then $2^r = 1 \neq 3$. Suppose $r = p/q$ to get $2^p = 3^q$, which is not possible as 2 and 3 share no common factors. Hence r is not rational.

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$ are all sets containing an infinite number of elements, then the intersections $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \subseteq A_3 \subseteq A_4 \dots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and non-empty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

SOLUTION

(a) False. Consider $A_n = \{n, n+1, n+2, \dots\}$, then $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(b) True. Since all A_n are nonempty, $\exists n \in \mathbb{N}$ such that $A_n = \{x\}$ for some real x. Hence $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\}$ which is empty. Since A_1 is finite, $\bigcap_{n=1}^{\infty} A_n \subseteq \{x\} \subset A_1$ is finite.

(c) False. If $A = \emptyset$, then $\emptyset = C$

(d) True. Intersection is associative as evident that both LHS and RHS implies the $x \in A, B, C$

(e) True. Drawing a Venn Diagram illustrates this.

Exercise 1.2.4

Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

SOLUTION

Consider arranging the elements of $\mathbb{N}$ in a square as such.

$$\begin{array}{cccccc}
1 & 3 & 6 & 10 & 15 & \dots \\
2 & 5 & 9 & 14 & \dots & \\
4 & 8 & 13 & \dots & & \\
7 & 12 & \dots & & & \\
11 & \dots & & & & \\
\vdots & & & & &
\end{array}$$

By letting A_i being the set of all natural numbers in the i-th row, we have satisfied the above conditions above.

Exercise 1.2.5

(De Morgan's Law) Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. THis shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \sqsubseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cup B^c$ by demonstrating inclusion both ways.

SOLUTION

(a) If $x \in (A \cap B)^c$, then $x \notin A \cap B$, so $x \notin A$ or $x \notin B$, implying $x \in A^c$ or $x \in B^c$, therfore $x \in A^c \cup B^c$.

(b) If $x \in A^c \cup B^c$, then $x \in A^c$ or $x \in B^c$, so $x \notin A$ and $x \notin B$, implying $x \notin A \cap B$, therefore $x \in (A \cap B)^c$. Since $(A \cap B)^c \subseteq A^c \cup B^c$ and $(A \cap B)^c \sqsubseteq A^c \cup B^c$, we can conclude that both sets are equal.

(c) To show that $(A \cap B)^c = A^c \cup B^c$, we need to demonstrate inclusion both ways.

  (i) If $x \in (A \cup B)^c$, then $x \notin A \cup B$, so $x \notin A$ or $x \notin B$, implying $x \in A^c$ or $x \in B^c$, therefore $x \in A^c \cup B^c$.

(ii) If $x \in A^c \cap B^c$, then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$, implying $x \notin A \cup B$, which is just $x \in (A \cup B)^c$.

Exercise 1.2.6

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

SOLUTION

(a) With both a and b having the same sign, then $|a|+|b| = |a+b|$, which satisfies $|a|+|b| \geq |a+b|$.

(b) Note that $(a+b)^2 \leq (|a|+|b|)^2$ reduces to $ab \leq |a||b|$, which is true as LHS can be negative while RHS cannot. Since squaring preserves inequality, this implies that $|a+b| \leq |a|+|b|$.

(c) Notice that $a-b = (a-c)+(c-d)+(d-b)$. Hence by triangle inequality,

$$|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c|+|c-d|+|d-b|$$

for all a, b, c and d.

(d) Since $||a|-|b|| = ||b|-|a||$, WLOG, we can assume that $|a| \geq |b|$. Then

$$||a|-|b|| = |a|-|b| = |(a-b)+b|-|b| \leq |a-b|+|b|-|b| = |a-b|$$

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0,2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbituary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbituary function g.

SOLUTION

(a) For $f(x) = x^2$, $f(A) = f([0,2]) = [0,4]$ and $f(B) = f([1,4]) = [1,16]$.

$$f(A \cap B) = f([0,2] \cap [1,4]) = f([1,2]) = [1{,}4] = [0,4] \cap [1,16] = f([1,2]) \cap f([2,4]) = f(A) \cap f(B)$$
$$f(A \cup B) = f([0,2] \cup [1,4]) = f([0,4]) = [0{,}16] = [0,4] \cup [1,16] = f([0,2]) \cup f([1,4]) = f(A) \cup f(B)$$

(b) Consider $A = [0,2]$ and $B = [-2,0]$. $f(A \cap B) = \{0\}$, but $f(A) \cap f(B) = [0,4]$.

(c) Suppose $y \in g(A \cap B)$, then $\exists x \in A \cap B$ such that $g(x) = y$. This implies that $x \in A$ and $x \in B$, so $x \in A \cap B$, hence $y \in g(A \cap B)$. Note that contrary may not always be true as it is possible for $x_1 \in A \backslash B$ and $x_2 \in B \backslash A$ such that $g(x_1) = g(x_2)$.

(d) I conjecture that $g(A \cup B) = g(A) \cup g(B)$. To prove this, we have to show inclusion both ways:

(i) Let $y \in g(A \cup B)$, then $\exists x \in A \cup B$ such that $y = g(x)$. This implies that $x \in A$ or $x \in B$, so $y \in g(A)$ or $y \in g(B)$, hence $y \in g(A) \cup g(B)$.

(ii) Let $y \in g(A) \cup g(B)$, then $y \in g(A)$ or $y \in g(B)$, implying $x \in A$ or $x \in B$ such that $y = g(x)$. So $x \in A \cup B$, hence $y \in g(A \cup B)$.

Exercise 1.2.8
Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* $(1-1)$ if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ Give an example of each or state that the request is impossible:

(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto.

(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1.

(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

SOLUTION
(a) Let $f(x) = x + 1$, which is 1-1 but does not have a solution to $f(x) = 1$, hence not onto.

(b) Let $f(x) = 1$ for $x = 1$ and $f(x) = x - 1$ for $x > 1$, which is onto but not 1-1 as $f(1) = f(2) = 1$.

(c) Let $f(x) = n/2$ when n is even and $f(x) = -\frac{x-1}{2}$ when n is odd.

Exercise 1.2.9
Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behaviour of preimages demonstrated in (a) is completely general. Show that for an arbituary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbb{R}$.

SOLUTION
(a) For $f(x) = x^2$, $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. $f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$. Similarly, $f^{-1}(A \cup B) = f^{-1}([-1, 4]) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$.

(b) To show that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$, we have to show inclusion both ways:

(i) Let $x \in g^{-1}(A \cap B)$, so $g(x) \in A \cap B$, which implies $g(x) \in A$ and $g(x) \in B$. This shows that $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, hence $x \in g^{-1}(A) \cap g^{-1}(B)$.

(ii) Let $x \in g^{-1}(A) \cap g^{-1}(B)$, so $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$, then $g(x) \in A$ and $g(x) \in B$. This implies that $g(x) \in A \cap B$, so $x \in g^{-1}(A \cap B)$.

Showing $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ is obvious using Exercise 1.2.7 (d).

Exercise 1.2.10
Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

SOLUTION
(a) False. Consider the case where $a < b + \epsilon$ is true but $a = b$.

(b) False. Same reasoning as above.

(c) True. Firsly suppose $a < b + \epsilon$ for all $\epsilon > 0$. We need to show this implies $a \leq b$. We either have $a \leq b$ or $a > b$. However, $a > b$ is not possible as this implies there exists an ϵ small enough such that $a > b + \epsilon$. Secondly, suppose $a \leq b$. It is obvious that $a < b + \epsilon$ for all $\epsilon > 0$.

Exercise 1.2.11
Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this more interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $a + 1/n < b$.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.

(c) Between every two distinct real numbers there is a rational number.

SOLUTION
(a) For all $n \in \mathbb{N}$, there exists $a, b \in \mathbb{R}$ such that $a + 1/n < b$. [FALSE]

(b) For all real number $x > 0$, there exists an $n \in \mathbb{N}$ such that $x \geq 1/n$. [TRUE]

(c) There exists two real numbers $a < b$ such that if $r < b$ then $r < a$ for all $r \in \mathbb{Q}$. [FALSE]

Exercise 1.2.12
Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

SOLUTION
(a) For $n = 1$, $y_1 = 6 > -6$ (Base Case). Suppose $y_n > -6$ for some $n \in \mathbb{N}$.

$$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2(-6) - 6}{3} = -6$$

Hence, by induction, $y_n > -6$ for all $n \in \mathbb{N}$.

(b) Suppose $y_{n+1} < y_n$. The base case works as $y_2 = 2 < 6 = y_1$. Then,

$$\begin{aligned} y_{n+1} < y_n &\implies 2y_{n+1} - 6 < 2y_n - 6 \\ &\implies \frac{2y_{n+1} - 6}{3} < \frac{2y_n - 6}{3} \\ &\implies y_{n+2} < y_{n+1} \end{aligned}$$

Thus, $y_{n+1} < y_n$ is true for all $n \in \mathbb{N}$.

Exercise 1.2.13
For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbb{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

SOLUTION

(a) Using Exercise 1.2.5 as the base case. Suppose $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cup \cdots \cup A_n^c$ is true. Using associative law on the $n+1$c case,

$$\begin{aligned}((A_1 \cup \cdots \cup A_n) \cup A_{n+1})^c &= (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c \\ &= (A_1^c \cap \cdots \cap A_n^c) \cap A_{n+1}^c \\ &= A_1^c \cap \cdots \cap A_n^c \cap A_{n+1}^c\end{aligned}$$

(b) Consider $B_1 = \{1, 2, \ldots\}, B_2 = \{2, 3, \ldots\}, B_n = \{x : x \in \mathbb{N} \cap [n, \infty)\}$.

(c) To prove the infinite version of De Morgan's Law, we need to show inclusion both ways.

(i) Suppose $x \in (\bigcup_{i=1}^{\infty} A_i)^c$, then $x \notin \bigcup_{i=1}^{\infty} A_i$. This implies that $x \notin A_i \ \forall i \in \mathbb{N}$, so $x \in A_i^c \ \forall i \in \mathbb{N}$. Hence $x \in \bigcap_{i=1}^{\infty} A_i^c$.

(ii) Suppose $x \in \bigcap_{i=1}^{\infty} A_i^c$, then $x \in A_i^c \ \forall i \in \mathbb{N}$, so $x \notin A_i \ \forall i \in \mathbb{N}$. This implies that $x \notin \bigcup_{i=1}^{\infty} A_i$, hence $x \in (\bigcup_{i=1}^{\infty} A_i)^c$.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bound.

SOLUTION

(a) A real number i is the *greatest upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) i is a lower bound of A;

(ii) if b is any lower bound for A, then $i \geq b$.

(b) Assume $i \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then $i = \inf A$ *if and only if*, for every choice of $\epsilon > 0$, there $\exists a \in A$ satisfying $i + \epsilon > a$.

PROOF Rephrasing the lemma gives us: Given that i is a lower bound, i is the greatest lower bound if and only if any number greater than i is not a lower bound.

(i) Suppose $i = \inf A$ and consider $i + \epsilon$ for an arbituarily chosen $\epsilon > 0$. Since $i + \epsilon > i$, part (ii) of the definition impliers that $i + \epsilon$ is not a lower bound for A. If this is the case, then there must be some element $a \in A$ such that $i + \epsilon > a$.

(ii) Conversely, assume i is a lower bound with the property that for any $\epsilon > 0$, $s + \epsilon$ is not a lower bound of A. Note that this implies that if b is any number more than i, then b is not an upperbound. Since we have argued that any larger number than i cannot be a lower bound, if b is some other upper bound for A, then $i \geq b$.

Exercise 1.3.2

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

SOLUTION

(a) Possible. Consider the set 0, where $\inf\{0\} = \sup\{0\} = 0$.

(b) Not possible as all finite sets must contain its supremum and infimum.

(c) Possible. Consider $A = \{r \in \mathbb{Q} \mid 1 < r \leq 2\}$.

Exercise 1.3.3

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

SOLUTION

(a) By definition, $\sup B$ is the greatest lower bound for A, meaning it equals $\inf A$.

(b) Part (a) proves the greatest lower bound exists using the least upper bound.

Exercise 1.3.4

Let $A_1, A_2, A_3, \dots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^{n} A_k)$.

(b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

SOLUTION

(a) $\sup(A_1 \cup A_2) = \sup\{\sup A_1, \sup A_2\}$ and $\sup(\bigcup_{k=1}^{n} A_k) = \sup\{\sup A_k \mid k = 1, \dots, n\}$

(b) This formula does not extend to infinity. Consider $A_k = [k, k+1]$, where $\bigcup_{k=1}^{\infty} A_k$ is unbounded.

Exercise 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Prostulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

SOLUTION

(a) The case of $c = 0$ is trivial as it implies that $cA = \{0\}$. Hence $\sup(cA) = c \sup A = 0$.

For $c > 0$, we need to show that $c \sup A$ is the lowest upper bound. Assume $c > 0$. Let $s = c \sup A$. Suppose $ca > s$, then $a > \sup A$ which is impossible, meaning that s is an upper bound on cA. Now suppose s' is an upper bound on cA and $s' < s$. Then $s'/c < s/c = \sup A$, meaning s'/c cannot bound A. Hence there $\exists a \in A$ such that $s'/c < a$, meaning $s' > ca$, thus s' cannot be an upper bound on cA, so $s = c \sup A$ is the least upper bound.

(b) $\sup(cA) = c \inf A$ for $c < 0$

Exercise 1.3.6

Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbituary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u + a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

SOLUTION

(a) By definition of supremum, $a \leq s$ and $b \leq t$. Adding both equations give $a + b \leq s + t$, hence $s + t$ is an upper bound.

(b) Since $a + b \leq u$ implies $b \leq u - a$, $u - a$ is an upper bound on b, meaning it is greater or equal to the least upper bound of t, giving $t \leq u - a$.

(c) From (a), we have shown that $s + t$ is an upper bound for $A + B$, hence it is sufficent to show that $s + t$ is the least upper bound.

Let $u = \sup(A + B)$, from (b) we have $t \leq u - a$ and $s \leq u - b$. Adding and rearranging gives $a + b \leq 2u - s - t$. Since $2u - s - t$ is an upper bound on $A + B$, it must be greater or equal to the least upper bound, giving $u \leq 2u - s - t$, implying $s + t \leq u$. Since u is the least upper bound, $s + t$ must equal u.

(d) Showing $s + t - \epsilon$ is not an upper bound for any $\epsilon > 0$ proves that it is the least upper bound by Lemma 1.3.8. Rearranging gives $(s - \epsilon/2) + (t - \epsilon/2)$ we know $\exists a > (s - \epsilon/2)$ and $b > (t - \epsilon/2)$, therefore $a + b > s + t - \epsilon$, meaning $s + t$ cannot be made smaller and thus is the least upper bound.

Exercise 1.3.7

Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

SOLUTION

Since it is given that a is an upper bound for A, we just have to show that a is the least upper bound, meaning any number lower than a would have an $a' \in A$ such that $a' > a$.

Suppose $a - \epsilon$ is also an upper bound for A for some $\epsilon > 0$. This is not possible has $a > a'$ and $a \in A$. Hence by contradiction, a is the lowest upper bound, meaning $a = \sup A$.

Exercise 1.3.8

Compute, without proofs, the suprema and inifima (if they exists) of the following sets:

(a) $\{m/m : m, n \in \mathbb{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.

(c) $\{n/(3n + 1) : n \in \mathbb{N}\}$.

(d) $\{m/(m + n) : m, n \in \mathbb{N}\}$.

SOLUTION

(a) $\inf = 0$ and $\sup = 1$

(b) $\inf = -1$ and $\sup = 1$

(c) $\inf = 1/4$ and $\sup = 1/3$

(d) $\inf = 0$ and $\sup = 1$

Exercise 1.3.9

(a) If $\sup A < \sup B$, show there exists an element that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq supB$.

SOLUTION

(a) By Lemma 1.3.8, we know there exists a b such that $(\sup B) - \epsilon < b$ for any $\epsilon > 0$. We can set ϵ to be small enough such that $\sup B - \sup A < \epsilon$, implying $\sup A < \sup B - \epsilon < b$ for some $b \in B$, thus b is an upper bound of A.

(b) Consider the sets $A = (-\infty, 1]$ and $B = (-\infty, 1)$. No $b \in B$ is an upperbound since $1 \in A$ and $1 > b$.

Exercise 1.3.10
(Cut Property) The *Cut Property* of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists a $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbb{R}$ is replaced by $\mathbb{Q}$.

SOLUTION

(a) If $c = \sup A = \inf B$, it is obvious that $a \leq c \leq b$ and we are done. Hence we need to show, by contradiction, that $\sup A < \inf B$ and $\sup A > \inf B$ is false.

  (i) AFSOC $\sup A < \inf B$. We can choose $c = \frac{\sup A + \inf B}{2}$, which satisfies $\sup A < c < \inf B$. Hence it is obvious that $c \notin A$ and $c \notin B$, so $c \notin A \cup B \neq \mathbb{R}$ which is a contradiction.

  (ii) AFSOC $\sup A > \inf B$. We can find a such that $a > b$ by subtracting $\epsilon > 0$ and using the definition of supremum and infimum similar to Lemma 1.3.8. Thus creating a contradiction.

  Since both alternatives are impossible, $\sup A = \inf B$.

(b) If E is finite or has a maximum element, that is $\sup E$ and we are done.

  Consider the case where E has not maximum element (for example, $\{-1/n : n \in \mathbb{N}\}$). Let B be the sets of all upper bounds of E and let $A = B^c$. It can be said $E \cap B = \emptyset$ otherwise E has a maximum element. Thus $E \subseteq A$.

  By the Cut Property, there exists c such that $a \leq c \leq b$ for all $a \in A$ and $b \in B$. Since c is an upper bound on A and $E \subseteq A$, c is also an upper bound on E. And since $c \leq b$, c is the lowest upper bound. Therefore, $c = \sup E$.

(c) Consider $A = \{r \in \mathbb{Q} \mid r^2 < 2 \text{ or } r < 0\}$, $B = A^c$ does not satisfy the cut property in $\mathbb{Q}$ as $\sqrt{2} \notin \mathbb{Q}$.

Exercise 1.3.11
Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

SOLUTION

(a) True. We know $a \leq \sup A$, $a \leq \sup B$ since $A \subseteq B$. Since $\sup A$ is the least upper bound on A, we have $\sup A \leq \sup B$.

(b) True. Let $c = \frac{\sup A + \inf B}{2}$, $c > \sup A$ implies $a < c$ and $c < \inf B$ implies $c < b$, giving $a < c < b$.

(c) False. Consider $A = (-\infty, 1)$ and $B = (1, \infty)$, $a < 1 < b$ but $\sup A \not< \inf B$ as $\sup A = \inf B = 1$.

1.4 Consequences of Completeness

Exercise 1.4.1
Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s+t$ and st?

SOLUTION

(a) This is trivial. Since $a, b \in \mathbb{Q}$, they can be expressed as a fraction of two integers. Let $a = m/n$ and $b = x/y$ where $m, n, x, y \in \mathbb{Z}$, then $a + b = \frac{my+xn}{ny}$ and $ab = \frac{mx}{ny}$, which are fractions with integer numerators and denominators, hence $a + b$ and ab are elements of $\mathbb{Q}$.

(b) AFSOC that $a + t \in \mathbb{Q}$ and $at \in \mathbb{Q}$. Let $a + t = \alpha$ and $at = \beta$, so $t = \alpha - a$ and $t = \beta/a$. Since $\alpha, \beta, -a, 1/a \in \mathbb{Q}$, using part (a) gives use $t \in \mathbb{Q}$ which is a contradiction. Hence $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$.

(c) $\mathbb{I}$ is not closed under addition or multiplication. Consider $\sqrt{2} + (-\sqrt{2}) = 0$ and $\sqrt{2} \cdot \sqrt{2} = 2$.

Exercise 1.4.2
Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upperbound for A. SHow $s = \sup A$.

SOLUTION
We can rephrase Lemma 1.3.8 using the archimedean property.

(i) AFSOC $s < \sup A$, then there exists an n such that $s + 1/n < \sup A$, contradicting $\sup A$ being the least upper bound.

(ii) AFSOC $s > \sup A$, then there exists an n such that $s - 1/n > \sup A$ where $s - 1/n$ is not an upper bound, contradicting $\sup A$ being an upper bound.

Hence, we can conclude that $\sup A = s$.

Exercise 1.4.3
Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

SOLUTION
AFSOC the $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$, so $0 < x < 1/n$ for all $n \in \mathbb{N}$ which is impossible by archimedean property.

Exercise 1.4.4
Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$.

SOLUTION
To show that $\sup T = b$, it needs to satify both conditions of supremum:

(i) Since $x \leq b$ for all $x \in [a, b]$, $y \leq b$ for all $y \in T$ as $T \subseteq [a, b]$.

(ii) AFSOC b' is also an upper bound such that $b' < b$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists an $\alpha \in \mathbb{Q} \cap [b', b] \subseteq T$. This implies there exists $t \in T$ satisfying $b' < t$, which is a contradiction.

Exercise 1.4.5
Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

SOLUTION
Recall that **Corollary 1.4.4** states that *Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.*

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $t \in \mathbb{Q}$ that is between any two real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$, with $a < b$. Hence, $a - \sqrt{2} < t < b - \sqrt{2}$, meaning $a < t + \sqrt{2} < b$. By Exercise 1.4.1, $t + \sqrt{2} \in \mathbb{I}$ and we are done.

Exercise 1.4.6
Recall that a set B is *dense* in $\mathbb{R}$ if can element B can be found between any two real numbers $a < b$. WHich of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2.

(c) The set of all rational numbers p/q with $10|p| \geq q$.

SOLUTION

(a) Not dense since we cannot make $|p|/q < 1/10$.

(b) Dense.

(c) Not dense since we cannot make $|p|/q < 1/10$.

Exercise 1.4.7
Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

SOLUTION
Recall $T = \{t \in \mathbb{R} : t^2 < 2\}$ and $\alpha = \sup T$. AFSOC $\alpha^2 > 2$, we will show that there exists an $n \in \mathbb{N}$ such that $(\alpha - 1/n)^2 > 2$, contradicting the assumption that α is the least upper bound.

Using $(\alpha - 1/n)^2 > 2$, we can find $n \in \mathbb{N}$ such that $(\alpha^2 - 1/n) > 2$.

$$2 < (\alpha - 1/n)^2 = \alpha^2 - \frac{2\alpha}{n} + 1/n^2 < \alpha^2 - \frac{2\alpha - 1}{n}$$

Then

$$2 < \alpha^2 - \frac{2\alpha - 1}{n} \implies n(2 - \alpha^2) < 1 - 2\alpha$$

Since $2 - \alpha^2 < 0$, dividing reverses the inequality, giving

$$n > \frac{1 - 2\alpha}{2 - \alpha^2}$$

Hence we can pick $n \in \mathbb{N}$ such that $(\alpha^2 - 1/n) > 2$, so α is the the least upper bound which is a contradiction. Hence it is not possible for $\alpha^2 > 2$.

Exercise 1.4.8
Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B \neq \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \dots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \dots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \dots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

SOLUTION

(a) Consider $A = (0,1) \cap \mathbb{Q}$ and $B = (0,1) \cap \mathbb{I}$. $\sup A = \sup B = 1$ and $1 \in A, B$.

(b) Consider $J_n = (-1/n, 1/n)$, then $\bigcap_{n=1}^{\infty} J_n = \{0\}$ which is nonempty and contains a finite number of elements.

(c) Consider $L_n = [n, \infty)$, then $\bigcap_{n=1}^{\infty} L_n = \emptyset$.

(d) Impossible. Let $j_n = \bigcap_{k=1}^{n} I_k$. Since $\bigcap_{n=1}^{N} I_n \neq \emptyset$, we have $J_n \neq \emptyset$. Note that J_n is the intersection of closed intervals, making it a closed interval. With $I_{n+1} \cap J_n \subseteq J_n$, we have $J_{n+1} \subseteq J_n$. Since J_n is a series of nested bounded closed intervals, by Nested Interval Property, $\bigcap_{n=1}^{\infty} J_n = \bigcap_{n=1}^{\infty}(\bigcap_{k=1}^{n} I_k) = \bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

1.5 Cardinality

Exercise 1.5.1
Finish the following proof for Theorem 1.5.7.

Assume B is a countable set. Thus, there exists $f : \mathbb{N} \to B$, which is 1-1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from $\mathbb{N}$ onto A.

SOLUTION
Let $n_k = min\{n \in \mathbb{N} : f(n) \in A, n \notin \{n_1, n_2, \dots n_{k-1}\}\}$ and $g(k) = f(n_k)$. Since $g : \mathbb{N} \to A$ is 1-1 and onto, A is countable.

Exercise 1.5.2
Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbb{R}$ is uncountable, and then find the flow in the following erroneous proof that $\mathbb{Q}$ is uncountable:

Assume, for contradiction, that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3, \dots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} \neq \emptyset$. This contradiction implies $\mathbb{Q}$ must therefore be uncountable.

SOLUTION
The Nested Interval Property only applies to $\mathbb{R}$ and not $\mathbb{Q}$.

Exercise 1.5.3
Use the following proofs for the statements in Theorem 1.5.8.

(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \backslash A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)

Now, explain how that more general statement in (i) follows.

(b) Explain why induction cannot be used to prove part (ii) of Theorem 1.5.8 from part(i).

(c) Show how arranging $\mathbb{N}$ into the two-dimensional array

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \dots \\ 2 & 5 & 9 & 14 & \dots & \\ 4 & 8 & 13 & \dots & & \\ 7 & 12 & \dots & & & \\ \vdots & & & & & \end{array}$$

leads to a proof of Theorem 1.5.8 (ii).

SOLUTION

(a) Let $B = \{b_1, b_2, b_3, \dots\}$ and $C = \{c_1, c_2, c_3, \}$ be countable, disjoint sets. We can define a function $g : \mathbb{N} \to B \cup C$ following a similar method of mapping $\mathbb{N}$ onto $\mathbb{Z}$, listing them as follows:

$$B \cup C = \{b_1, c_1, b_2, c_2, \dots\}$$

Implying that $B \cup C$ is countable. By letting $B = A_1$ and $C = A_2 \backslash A_1$, we can show that $A_1 \cup A_2$ is also countable.

By using induction, suppose $A_1 \cup \dots \cup A_n$ is countable, $(A_1 \cup \dot{\cup} A_n) \cup A_{n+1}$ is the unions of two countable sets, which as proven above is also countable.

(b) Induction only shows something for each $n \in \mathbb{N}$, it does not apply in the infinite case. For example, $\bigcap_{k=1}^{n} [k, \infty) \neq \emptyset$ is true for all $n \in \mathbb{N}$, but the infinite case $\bigcap_{k=1}^{\infty} [k, \infty) \neq \emptyset$ is false.

(c) By rearranging $\mathbb{N}$ as in (c) gives us disjoint sets $C_n = \{k \in \mathbb{N} : k \text{ is in the n-th row}\}$, such that $\bigcup_{n=1}^{\infty} C_n = \mathbb{N}$. Let B_n be disjoint sets, constructed as $B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $\ldots$, $B_n = A_n \backslash (A_1 \cup \ldots A_{n-1})$. If we can find a function $f : C \to B$ that is bijective, we can show:

$$f(\mathbb{N}) = f(\bigcup_{n=1}^{\infty} C_n) = \bigcup_{n=1}^{\infty} f_n(C_n) = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

Let $f_n : C_n \to B_n$ be bijective since B_n is countable. Define $f : \mathbb{N} \to \bigcup_{n=1}^{\infty} B_n$ as

$$f(n) = \begin{cases} f_1(n) & \text{if } n \in C_1 \\ f_2(n) & \text{if } n \in C_2 \\ \vdots \end{cases}$$

We must now show that f is bijective:

(i) Since each C_n is disjoint and each f_n is 1-1, $f(n_1) = f(n_2) \implies n_1 = n_2$, meaning f is 1-1.

(ii) SInce any $b \in \bigcup_{n=1}^{\infty} B_n$ as $b \in B_n$ for some n, $b = f_n(x)$ has a solution since f_n is onto. Letting $x = f_n^{-1}(b)$, we have $f(x) = f_n(x) = b$ as $f_n^{-1}(b) \in C_n$, meaning f is onto.

By (i) and (ii), f is bijective, so $\bigcup_{n=1}^{\infty} B_n$ is countable, implying that $\bigcup_{n=1}^{\infty} A_n$ is also countable, completing the proof.

Exercise 1.5.4

(a) Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

SOLUTION

(a) Example 1.5.4 provides the function $f(x) = \frac{x}{x^2-1}$ which takes the interval $(-1, 1)$ onto $\mathbb{R}$ in a 1-1 fashion. If we shift f to the interval of (a, b), we get:

$$g(x) = f(\frac{2x-1}{b-a} - a)$$

which maps (a, b) onto $\mathbb{R}$ in a 1-1 fashion.

(b) To show that $(a, \infty) \sim \mathbb{R}$, we need to find a function h that maps $\mathbb{R}$ onto (a, ∞). Consider the function $h(x) = e^x + a$, which maps $\mathbb{R}$ onto (a, ∞) in a 1-1 fashion. Hence we are done.

(c) Define $f : [0, 1) \to (0, 1)$ as

$$f(x) = \begin{cases} 1/2 & \text{if } x = 0 \\ 1/4 & \text{if } x = 1/2 \\ 1/8 & \text{if } x = 1/4 \\ \vdots \\ x & \text{otherwise} \end{cases}$$

To show that both $[0, 1)$ and $(0, 1)$ have the both cardinality, we need to prove that f is 1-1 and onto.

We start by showing that $y = f(x)$ has exactly one solution for all $y \in (0, 1)$.

If $y = 1/2^n$, then the only solution is $y = f(1/2^{n-1})$ (or $x = 0$ in the special case $n = 1$).

Otherwise, the only solution is $y = f(y)$.

Exercise 1.5.5

(a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are hwat is meant by saying that $\sim$ is an *equivalence relation*.

SOLUTION

(a) The identity function $f(x) = x$ is trivially bijective.

(b) If $f : A \to B$ is bijective, then the inverse $f^{-1} : B \to A$ is also bijective.

(c) If $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$ is a bijection, thus $A \sim C$.

Exercise 1.5.6

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

SOLUTION

(a) Consider $I_1 = (1, 2)$, $I_2 = (2, 3)$, $\ldots$, $I_n = (n, n + 1)$.

(b) No such collection exists. Let A denote this set.

For any nonempty interval I_n, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find an $r \in \mathbb{Q}$ such that $r \in I_n$. Assigning each $I \in A$ a rational number $r \in I$ proves that $I \subseteq \mathbb{Q}$, thus I is countable.

Exercise 1.5.7

Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1-1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy)

(b) Use the fact that every real number has a decimal expansion to produce a 1-1 function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999 . . . .)

The Schröder-Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

SOLUTION

(a) Consider $f(x) = (\frac{1}{4}x, \frac{1}{3})$.

(b) Let $g : S \to (0, 1)$ be a function that interweaves decimals in the representation without trailing nines, padding with zeros if necessary. $g(0.12, 0.34) = 0.1324$, $g(0.1\bar{9}, 0.2) = g(0.2, 0.2) = 0.22$, $g(0.4, 0.89) = 0.4809$, $g(0.6, 0.\bar{7}) = 0.67\overline{07}$, etc.

To prove that g is 1-1, we need show that there does not exists any solutions for $g(x_1, y_1) = g(x_2, y_2)$. Every real number can be written with tow representations, one with trailing 9's and one without. However, $g(x, y) = 0.d_1d_2d_3 \ldots \bar{9}$ is impossible as it would imply both x and y has trailing 9's, which contradicts the definition of g. Therefore, $g(s)$ is unique and so g is 1-1.

g is not onto since $g(x, y) = 0.1$ has no solutions.

Exercise 1.5.8

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B will always give a sum of 2 or less. Show B must be finite or countable.

SOLUTION

It is obvious that $B \cap [a, 2)$ must be finite for some $a \in (0, 2)$, otherwise, we can choose $\lceil \frac{2}{a} \rceil$ number of elements from $B \cap [a, 2)$ would result in a sum of more than 2. Since B is the countable union of finite sets $\bigcup_{n=1}^{\infty} B \cap [\frac{1}{n}, 2)$, B must be countable or finite.

Exercise 1.5.9
A real number $x \in \mathbb{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, \dots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_n x^n + a_{n-1}^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$ and $\sqrt{3} + \sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbb{N}$ and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

SOLUTION
(a) The first two are trivial as they are roots of $x^2 - 2 = 0$ and $x^3 - 2 = 0$. For $\sqrt{3} + \sqrt{2}$, we can start working backwards:

$$\begin{aligned} x = \sqrt{3} + \sqrt{2} &\implies x^2 = 5 + 2\sqrt{6} \\ &\implies (x^2 - 5)^2 - 24 = 0, \end{aligned}$$

thus giving a polynomial, hence $\sqrt{2} + \sqrt{3}$ is algebraic.

(b) We need to show that $A_n \sim \mathbb{Z}^n \sim \mathbb{N}^n \sim \mathbb{N}$ and hence countable.

(i) $A_n \sim \mathbb{Z}^n$ since choosing n integer coefficients for a polynomial of degree n is equivilant to an ordered list of n integers.

(ii) $\mathbb{Z}^n \sim \mathbb{N}^n$ since $f : \mathbb{N}^n \to \mathbb{Z}^n$ is just the piecewise application of $g : \mathbb{N} \to \mathbb{Z}$.

(iii) $\mathbb{N}^n \sim \mathbb{N}$ since it is the intersection of finite sets $\bigcup_{k=1}^{\infty} \{(x_1, \dots, x_n) : x_1 + \cdots + x_n = k\}$.

(c) The set of algebraic numbers is countable as it is the union of countable sets $\bigcup_{n=1}^{\infty} A_n$. The set of transcendental numbers is uncountable, otherwise, it implies that the set of real numbers is countable, which is not possible.

Exercise 1.5.10
(a) Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is countable.

(b) Now let A be the set of all $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

SOLUTION
(a) AFSOC a does not exists, so $C \cap [a, 1]$ is countable for all $a \in (0, 1)$. Therefore $C = \bigcup_{n=1}^{\infty} C \cap [1/n, 1]$ is also countable, which is a contradiction. Hence, there exist a $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

(b) Since $\alpha = \sup A$, for any $\epsilon > 0$, $C \cap [\alpha + \epsilon, 1]$ is countable. Therefore, $C \cap [\alpha, 1] = \alpha \cup \bigcup_{n=1}^{\infty} C \cap [1/n, 1]$ is countable.

(c) No. Consider $C = \{1/n : n \in \mathbb{N}\}$. It has $C \cap [\alpha, 1]$ finite for every α, but $C \cap [0, 1]$ is infinite.

Exercise 1.5.11
(Schröder-Bernstein Theorem) Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$.

The strategy is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' and A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

(b) Set $A_1 = X \backslash g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is a similar collection in Y.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

(d) Let $A' = X \backslash A$ and $B' = Y \backslash B$. Show g maps B' onto A'.

SOLUTION

(a) $f : A \to B$ and $g : B' \to A'$ are bijective, therefore we can define

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in A' \end{cases}$$

which is bijective.

(b) If $A_1 = \emptyset$, we are done as it implies $g(Y) = X$ so $X \sim Y$

Since $A_n \subseteq X$, then $f(A_n) \subseteq Y$, hence $A_n \cap A_1 = \emptyset$ for all $n \in \mathbb{N}$ as $A_1 = X \backslash g(Y)$.

Note that $g \circ f$ is bijective implies that $g(f(A \cap B)) = g(f(A)) \cap g(f(B))$. We will show that by inclusion both ways. Let h be an arbitrary bijective function.

(i) Let $y \in h(A \cap B)$, so there exists $x \in A \cap B$ such that $h(x) = y$, then $x \in A$ and $x \in B$. This implies that $h(x) \in h(A)$ and $h(x) \in h(B)$, hence $y = h(x) \in h(A) \cap h(B)$.

(ii) Let $y \in h(A) \cap h(B)$, so $y \in h(A)$ and $y \in h(B)$, then there exist $x \in A$ and $x \in B$ such that $h(x) = y$. This implies $x \in A \cap B$, hence $y \in h(A \cap B)$.

Now to show pairwise disjointment for all subsets, consider $i, j \in \mathbb{N}$, with $i < j$.

$$A_j \cap A_i = g(f(A_j \cap A_i)) = g(f(A_j)) \cap g(f(A_i)) = A_{j-1} \cap A_{i-1} = \cdots = A_{j-i+1} \cap A_1 = \emptyset$$

Similarly, $\{f(A_n) : n \in \mathbb{N}\}$ is also pairwise disjoint as

$$A_j \cap A_i = \emptyset \implies f(A_j) \cap f(A_i) = f(A_j \cap A_i) = f(\emptyset) = \emptyset$$

(c) $f(A) = B$ as $f(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f(A_n) = B$. ($B$ is defined to be the range of f.)

(d) To show that $g(B') = A'$, we will show inclusion both ways:

(i) $g(B') \subseteq A'$. AFSOC $g(b') \in A$, then since $A_1 \cap g(Y) = \emptyset$, $g(b') \notin A$, implying $g(b') \in \bigcup_{n=2}^{\infty} A_n = g(B)$, meaning $\exists b \in B$ such that $g(b) = g(b')$, contradicting the fact that g is 1-1.

(ii) $g(B') \supseteq A'$. AFSOC $\exists a' \in A'$, with $a' \notin g(B')$. Since $A' \subseteq g(Y)$, we have $a' \in g(B)$, since $a' \notin g(B')$, and $g(B) \subseteq A$ contradicting $a' \in A$.

1.6 Cantor's Theorem

Exercise 1.6.1

Show that $(0, 1)$ is uncountable if and only if $\mathbb{R}$ is uncountable.

SOLUTION

In Exercise 1.5.4 (a), we have found a bijection $f : (0, 1) \to \mathbb{R}$. Suppose $g : (0, 1) \to \mathbb{N}$ is some map, we must show g is bijective if and only if $(g \circ f) : \mathbb{R} \to \mathbb{N}$ is bijective. This is true as if g is bijective, then $(g \circ f)$ is bijective as it is the composition of bijective functions. Similarly, if $(g \circ f)$ is bijective, then $(g \circ f) \circ f^{-1} = g$ is bijective.

This proof works as $(0, 1)$ is uncountable, implies that g is not bijective, then $(g \circ f) : \mathbb{R} \to \mathbb{N}$ is not bijective, implying that $\mathbb{R}$ is not countable and hence uncountable.

Exercise 1.6.2

(a) Explain why the real number $x = .b_1b_2b_3b_4\ldots$ cannot be $f(1)$.

(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbb{N}$.

(c) Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

SOLUTION

(a) Since $b_1 \neq a_{11}$, the first digit of $f(1)$ differ from x, hence cannot be equal.

(b) Since $b_n \neq a_{nn}$, the n-th digit of $f(n)$ differ from x, hence cannot be equal.

(c) Since x is not in the list, it is a contradiction.

Exercise 1.6.3

Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.

(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.

(b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as $.5$ or as $.4999\ldots$. Doesn't this cause some problems?

SOLUTION

(a) False, since the constructed number has aninfinte number of decimals, it is irrational.

(b) No, since changing the n-th digit would still result in a different number.

Exercise 1.6.4

Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 or 1\}.$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \ldots)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, 1, \ldots)$.

Give a rigorous argument showing S is uncountable.

SOLUTION

Similar to the proof in Exercise 1.6.1, suppose there exists a function $f : \mathbb{N} \to S$ that is 1-1 and onto. For each $n \in \mathbb{N}$, $f(n)$ is an element in S, represented as:

$$f(n) = (a_{n1}, a_{n2}, a_{n3}, \ldots)$$

where $a_{nm} = 0$ or 1 for each $m, n \in \mathbb{N}$. We construct a $s \in S$, with $s = (b_1, b_2, b_3, \ldots)$, where $b_n = 0$ if $a_n n = 1$ and $b_n = 1$ otherwise. Since the n-th digit of sequence s differs from $f(n)$ for all $n \in \mathbb{N}$, $s \notin S$, which is a contradiction. Hence S is not countable, therefore uncountable.

Alternatively, we can define $g : S \to \bar{S}$ where $g(a_1, a_2, a_3, \ldots) = .a_1a_2a_3\ldots$ which is trivially bijective, and $h : \mathbb{R}_2 \to \mathbb{R}$ which converts a number in base 2 to base 10 and is clearly bijective. Hence $h \circ g : S \to \mathbb{R}$ is bijective and hence shows that $S \sim \mathbb{R}$ so S is uncountable.

Exercise 1.6.5

(a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)

(b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

SOLUTION

(a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

(b) There are n elements, with 2 possible states: included or excluded, implying that there are 2^n elements.

Exercise 1.6.6

(a) Using the particular set $A = \{a, b, c\}$, exhibit two different 1-1 mappings from A into $P(A)$.

(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a $1-1$ map $g : C \to P(C)$.

(c) Explain *why*, in parts (a) and (b), it is impossible to construct mappings that are *onto*.

SOLUTION

(a) Consider $f(x) = \{x\}$ and $f(x) = \{x\}$ for $x \neq b, \{x, c\}$ for $x = b$.

(b) Let $g(x) = \{x\}$.

(c) $|P(C)| > |C|$.

Exercise 1.6.7

Return to the particular functions constructed in Exercise 1.6.6 and construct the subset B that results using the preceding rule. In each case, note that B is not in the range of the function used.

SOLUTION

(i) For $A = \{a, b, c\}$, $B_A = \{b\}$

(ii) For $C = \{1, 2, 3, 4\}$, $B_C = \emptyset$

Exercise 1.6.8

(a) First, show that the case $a' \in B$ leads to a contradiction.

(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

SOLUTION

(a) AFSOC $a' \in B$. This means $a' \notin f(a')$, then $a' \notin B$ which is a contratiction.

(b) AFSOC $a' \notin B$. This means $a' \in f(a')$, then $a' \in B$, which is a contradiction.

Exercise 1.6.9

Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling arguement showing that $P(\mathbb{N}) \sim \mathbb{R}$.

SOLUTION

We can make a function $f : P(\mathbb{N}) \to S$, with S following the definition given in Exercise 1.6.4, being a set consisting of all sequences of 0's and 1's.

$$f(P_N) = (a_1, a_2, a_3, \dots) \text{ with } a_n = 0 \text{ if } a_n \notin P_N \text{ and } 1 \text{ otherwise}$$

f is clearly onto as every element of S can be mapped to by its definiton. To show that is it injective, AFSOC that there exists $X, Y \in P(\mathbb{N})$, where $X \neq Y$, such that $f(X) = f(Y)$. There will exists $n \in \mathbb{N}$ where $n \in X$ but $n \notin Y$. By definiton, the n-th element of $f(X)$ is 1, but for $f(Y)$ is 0, which is a contradiction.

Using Exercise 1.6.4, $S \sim \mathbb{R}$, hence $P(\mathbb{N}) \sim \mathbb{R}$.

Exercise 1.6.10

As a final exercise, answer each of the following by establishing a 1-1 correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0,1\}$ to $\mathbb{N}$ countable or uncountable?

(b) Is the set of all functions from $\mathbb{N}$ to $\{0,1\}$ countable or uncountable?

(c) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an *antichain* if no element of A is a subset of any other element of A. Does $P(\mathbb{N})$ contain an uncountable anitchain?

SOLUTION

(a) Both elements of $\{0,1\}$ can be mapped to an element in $\mathbb{N}$, hence we can find a bijective function to show that this set is the same as $\mathbb{N}^2$ hence countable.

(b) As shown in Exercise 1.6.9, this set maps to $S \sim \mathbb{R}$ which is uncountable.

(c) Let $\mathcal{A}$ be an antichain of $P(\mathbb{N})$ and let $\mathcal{A}_l$ be the sets in $\mathcal{A}$ of size l. For finite l, $\mathcal{A}_l$ is countable since $\mathcal{A}_l \subseteq \mathbb{N}^l$ is countable. Hence the countable union $\bigcup_{l=0}^{\infty} \mathcal{A}_l = \mathcal{A}$ is countable.

If l is infinite, only countably many elements of $\mathcal{A}$ can be finite, while there will be uncountably many that myst be infinite, hence forming an uncountable antichain $\mathcal{A}$.

Chapter 2

Sequences and Series

2.2 The Limit of a Sequence

Exercise 2.2.1
What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent dequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

SOLUTION

(i) Consider the sequence $x_n = \sin x$. (x_n) verconges to 0 as we can choose $\epsilon = 1$ such that $|x_n - 0| < \epsilon$ for any value of $n \in \mathbb{N}$.

(ii) There is not vercongent sequence that diverges. THe proof is obvious and left as an exercise for the reader.

(iii) The sequence in (i) verconges to $x \in \mathbb{R}$ by choosing $\epsilon = x + 2$.

(iv) The definition of verconges described a bounded sequence.