Advanced Microeconomics (MA)– Fall 2020
GA 3001.11
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Problem Set 3

Due October 9, 2020 at 11:59pm
on Gradescope

Points: /20

Question 1. (10 points)

Consider a pure exchange economy with two consumers and two goods. The initial endowment is ω = (x=8,y=8) and the two consumers have preferences over good x and good y summarized by the utility functions:

$U_1(x_1, y_1) = x_1 y_1$
$U_2(x_2, y_2) = y_2 + 2\ln(x_2)$

a. Find the set of pareto optimal allocations.

Suppose that both consumers have an initial endowment $\omega i = (x_i=4, y_i=4)$ for i = 1,2.

b. Call p_X and p_Y the prices of good x and y and write consumer 1's Utility Maximization Problem.

c. Write and solve the first order conditions for consumer 1's utility maximization problem and find this consumer's offer curve (Hint: the functions $x_1^*(p_X, p_Y)$ and $y_1^*(p_X, p_Y)$ that solve consumer 1's UMP).

d. Write the Utility Maximization Problem of consumer 2.

e. Find this consumer's offer curve.

f. Impose a market clearing condition and find the competitive equilibrium prices.

g. Find the competitive equilibrium allocation of x and y.

h. Is the allocation from part f. Pareto Optimal (hint: is it one of the points in the Pareto set you found in part a.)?

Question 2. (10 points)

Consider a private-ownership, competitive economy with two consumers (N = 2), one firm (J = 1) that can use labor time to produce a consumption good. The firm's technology is represented by the production set:

$$Y = \{(-L_1 - L_2, y) \in \mathbb{R}_+^2 |\ y \leq\ 2(-L_1 - L_2)\}$$

Where $-L_1$ is the amount of labor supplied by the first consumer and $-L_2$ is the amount of labor supplied by the second consumer.

The consumers have identical preferences summarized by the utility function U (l, c) = ln(c) + ln(l).

Consumer 1 has an initial endowment of $\omega_1 = (l_1 = 1, c_1 = 0)$ and owns the firm, Consumer 2 has an initial endowment of $\omega_2 = (l_2 = 2, c_2 = 0)$.

a. Normalize the price of the consumption good to 1 and call w_1 and w_2 the wage rates paid to consumer 1 and to consumer 2 respectively. Then, write the profit maximization problem of the firm.

b. Notice that the firm would not want to produce a finite amount of consumption good unless

$2 = w_1 = w_2$

At these prices, is the firm earning a profit?

c. Write the Utility Maximization Problem of consumer 1 and solving the first order conditions find consumer 1's offer curve.

d. Write the Utility Maximization Problem of consumer 1 and solving the first order conditions find consumer 2's offer curve.

e. You found the competitive equilibrium prices solving the problem of the firm. Now substitute these prices in the consumers' offer curves and find the consumers' optimal amount of leisure time and consumption.

f. How many units of consumption good does the firm produce in equilibrium?

Question 1

a) Pareto Optimality

The Pareto Optimal allocations solve the Pareto Problem

choose $(x_1, y_1)(x_2, y_2)$ to

maximize $x_1 y_1$

subject to $y_2 + 2\ln(x_2) \geq \bar{U}$

$x_1 \geq 0 \quad y_1 \geq 0 \qquad x_2 \geq 0 \quad y_2 \geq 0$

$x_1 + x_2 = 8 \qquad x_2 = 8 - x_1$

$y_1 + y_2 = 8 \qquad y_2 = 8 - y_1$

Since u_1 is Cobb-Douglas at the P.O.

$x_1 > 0$; $y_1 > 0$. Since $MU_{x_2} = \dfrac{2}{x_2}$ & $\lim_{x_2 \to 0} MU_{x_2} \to \infty$

$$\mathcal{L}(x_1, y_1, \lambda) = x_1 y_1 - \lambda(\bar{u} - (8 - y_1) - 2\ln(8 - x_1)) + \mu(8 - y_1)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = y_1 - \lambda \frac{2}{8 - x_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_1} = x_1 - \lambda + \mu = 0 \qquad (8 - y_1) \geq 0 \qquad \mu(8 - y_1) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(\bar{u} - (8 - y_1) - 2\ln(8 - x_1)) = 0$$

First we check for interior solutions:

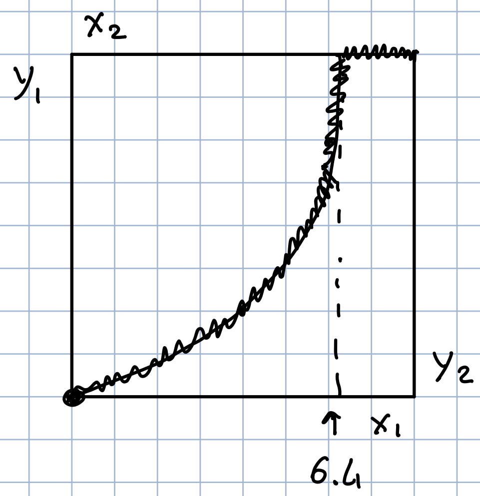
$$y_1 < 8$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = y_1 = \lambda \frac{2}{8-x_1}$$

$$\frac{\partial \mathcal{L}}{\partial y_1} = x_1 = \lambda$$

$$\Rightarrow \quad y_1 = \frac{2x_1}{8-x_1}$$

$$y_1 = \frac{2x_1}{8-x_1} < 8$$



$$2x_1 < 64 - 8x_1$$

$$10x_1 < 64$$

$$x_1 < 6.4$$

Interior P.O. $\left(x_1 \in [0, 6.4) ; y_1 = \frac{2x_1}{8-x_1} ; x_2 = 8 - x_1 ; y_2 = 8 - y_1 \right)$

Next we check for boundary solutions $y_1 = 8$ & $y_2 = 0$

$$\frac{\partial \mathcal{L}}{\partial x_1} = y_1 - \lambda \frac{2}{8-x_1} = 8 - \lambda \frac{2}{8-x_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_1} = x_1 - \lambda - \mu = 0 \qquad \lambda = (x_1 - \mu)$$

$$8 - (x_1 - \mu) \cdot \frac{2}{8-x_1} = 0$$

$$\mu \frac{2}{8-x_1} = \frac{2x_1}{8-x_1} - 8$$

$$\mu = x_1 - \frac{8(8-x_1)}{2} = x_1 - 32 + 4x_1 = 5x_1 - 32$$

$$\mu \geq 0 \qquad 5x_1 - 32 \geq 0 \qquad x_1 \geq \frac{32}{5} = 6.4$$

Boundary P.O. $\left[x_1 \in [6.4, 8) \quad y_1 = 8 ; x_2 = 8 - x_1 ; y_2 = 0 \right]$

b) Now we are interested in the way consumer 1 & 2 behave in the market

Consumer 1 would observe the prices P_x, P_y and would choose

x_1, y_1 to maximize $x_1 y_1$

subject to $P_x x_1 + P_y y_1 = 4P_x + 4P_y$

c)

Since the consumer has Cobb-Douglas preferences we do not need to worry about corner solutions.

$$\mathcal{L}(x_1, y_1, \lambda) = x_1 y_1 - \lambda(P_x x_1 + P_y y_1 - 4P_x - 4P_y)$$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= y_1 - \lambda P_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y_1} &= x_1 - \lambda P_y = 0 \end{aligned}\right\} \Rightarrow \frac{y_1}{x_1} = \frac{P_x}{P_y} \Rightarrow P_x x_1 = P_y y_1$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(P_x x_1 + P_y y_1 - 4P_x - 4P_y) = 0$$

$$2P_x x_1 = 4P_x + 4P_y \Rightarrow x_1^* = \frac{4P_x + 4P_y}{2P_x}$$

$$x_1^* = 2 + 2\frac{P_y}{P_x}$$

$$y_1^* = 2 + 2\frac{P_x}{P_y}$$

d) Now let's consider Consumer 2

choose $x_2 \geq 0$ $y_2 \geq 0$ to

$$\max \quad y_2 + 2\ln(x_2) \qquad \text{s.t.} \quad P_x x_2 + P_y y_2 = 4P_x + 4P_y$$

$$\mathcal{L}(x_2, y_2, \lambda) = y_2 + 2\ln(x_2) - \lambda(P_x x_2 + P_y y_2 - 4P_x - 4P_y) + \mu_x x_2 + \mu_y y_2$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{2}{x_2} - \lambda P_x + \mu_x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_2} = 1 - \lambda P_y + \mu_y = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(P_x x_2 + P_y y_2 - 4P_x - 4P_y) = 0$$

e) Interior solution : $\mu_x = 0$ $\mu_y = 0$

$$\left.\begin{array}{l} \dfrac{2}{x_2} - \lambda P_x = 0 \\ 1 - \lambda P_y = 0 \end{array}\right\} \quad \frac{2}{x_2} = \frac{P_x}{P_y}$$

$$P_x x_2 = 2P_y \qquad x_2 = 2\frac{P_y}{P_x}$$

$$2P_y + P_y y_2 = 4P_x + 4P_y$$

$$y_2 = \frac{4P_x + 2P_y}{P_y}$$

$$y_2 = 4P_x/P_y + 2$$

f) Market Clearing

$$\left(2 + 2\frac{P_y}{P_x} - 4\right) + \left(2\frac{P_y}{P_x} - 4\right) = 0$$

$$\Rightarrow 4\frac{P_y}{P_x} = 6 \qquad \frac{P_y}{P_x}^* = \frac{3}{2}$$

g)

$$x_1^* = 2 + 2\cdot\frac{3}{2} = 5 \qquad y_1^* = 2 + 2\cdot\frac{2}{3} = \frac{10}{3}$$

$$x_2^* = 2\cdot\frac{3}{2} = 3 \qquad y_2^* = 4\cdot\frac{2}{3} + 2 = \frac{14}{3}$$

$$\text{c.e.} \quad \left(\frac{P_y}{P_x}\right)^* = \frac{3}{2} \quad \left(x_1^* = 5\,;\ y_1^* = \frac{10}{3}\right) \quad \left(x_2^* = 3\,;\ y_2^* = \frac{14}{3}\right)$$

h) Is the c.e. allocation Pareto Optimal?

$$x_1^* = 5 \Rightarrow y_1^* = \frac{2x_1}{8-x_1} = \frac{2\cdot 5}{8-5} = \frac{10}{3}$$

Yes.

Question 2

Now we consider a production economy with two consumers and one firm.

a) First we focus on the problem of the firm. The firm is a price taker and chooses the amount of labor 1 $(-L_1)$ & labor 2 $(-L_2)$ to hire to maximize profit:

$$P\cdot y + W_1 L_1 + W_2 L_2$$

where $-L_1 \geq 0 \qquad -L_2 \geq 0$

& $\quad y = 2(-L_1 - L_2)$

Profit : $-2L_1 - 2L_2 + W_1 L_1 + W_2 L_2$

b)

For the firm to produce a non zero & finite amount of y, the firm must be at an interior solution

Interior

$$\frac{\partial \mathcal{L}}{\partial L_1} = -2 + W_1 = 0 \Rightarrow W_1 = 2$$

$W_1 = W_2 = 2$

$$\frac{\partial \mathcal{L}}{\partial L_2} = -2 + W_2 = 0 \Rightarrow W_2 = 2$$

At these prices, the firm breaks even

c) In equilibrium, the consumers earn no profit.

The UMP of consumer 1 is:

choose e_1 & c_1 to

$$\max \ \ln(c_1) + \ln(e_1)$$
$$\text{s.t.} \quad c_1 = W_1(1 - e_1)$$

Since the consumer's $\underset{e \to 0}{MRS} \to \infty$ $\underset{c \to 0}{MRS} \to 0$

the solution to the consumer's problem is an interior solution.

$$\mathcal{L}(e_1, c_1, \lambda) = \ln c_1 + \ln e_1 - \lambda(c_1 - W_1(1 - e_1))$$

F.O.C.

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial e} &= \frac{1}{e_1} - \lambda w_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial c} &= \frac{1}{c_1} - \lambda = 0 \end{aligned}\right\} \qquad \frac{c_1}{e_1} = w_1$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\left(c_1 - w_1(1 - e_1)\right) = 0 \qquad \Rightarrow c_1 + w_1 e_1 = w_1$$

$$2 e_1 w_1 = w_1 \qquad e_1^* = 1/2$$

$$c_1^* = \frac{w_1}{2}$$

d) The UMP of Consumer 2 is

choose e_2, c_2 to $\max \ln(c_2) + \ln(e_2)$

s.t. $c_2 = w_2(2 - e_2)$

$$\mathcal{L}(e, c, \lambda) = \ln e_2 + \ln c_2 - \lambda\left(c_2 - w_2(2 - e_2)\right)$$

F.O.C.

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial e} &= \frac{1}{e_2} - \lambda w_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial c} &= \frac{1}{c_2} - \lambda = 0 \end{aligned}\right\} \qquad \frac{c_2}{e_2} = w_2 \Rightarrow c_2 = w_2 e_2$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\left(c_2 - w_2(2 - e_2)\right) = 0 \qquad \Rightarrow c_2 + w_2 e_2 = 2 w_2$$

$$2 w_2 e_2 = 2 w_2 \qquad e_2^* = 1$$

$$c_2^* = w_2$$

e) We found earlier that in this economy the equilibrium prices are $w_1 = w_2 = 2$.
We substitute these prices in Consumer 1's & Consumer 2's offer curves and we find:

$c_1^* = 1$ $\quad \ell_1^* = 1/2$ $\quad c_2^* = 2$ $\quad \ell_2^* = 1$

f) In equilibrium, the consumers supply labor to the firm in the following amounts:

$$-L_1^* = \omega_1 - \ell_1^* = 1 - 1/2 = 1/2$$

$$-L_2^* = \underbrace{\omega_2}_{\text{initial endowment}} - \ell_2^* = 2 - 1 = 1$$

& the firm produces

$$y^* = 2(-L_1 - L_2) = 2(1/2 + 1) = 3 \text{ units}$$

The C.E. then is:

prices : $p^* = 1$ $\quad w_1^* = 2$ $\quad w_2^* = 2$

consumption allocation : $(c_1^* = 1/2, \ell_1^* = \frac{1}{2})$; $(c_2^* = 1, \ell_2^* = 1)$

production allocation : $-L_1^* = 1/2$, $-L_2^* = 1$, $y^* = 3$