Advanced Microeconomics (MA)– Fall 2021
GA 3001.06
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Problem Set 2

Due October 2, 2021 at 11:59pm
on Gradescope

Points: /20

Question 1. **The Pareto Optimal Set in Pure Exchange Economies – differentiable utility (10 points)**

We are working on the concept of Pareto Optimality. When all consumers have a differentiable utility function, we can find the set of Pareto Optimal allocations of a pure exchange economy by solving the Pareto Problem:

$$\max_{x_1, x_2, \ldots, x_n} u_1(x_1)$$

subject to:

utility constraints	$u_i(x_i) \geq \bar{u}_i$	for all i=2,…,n
feasibility constraints	$\sum_{i=1}^{n} x_{1i} = \sum_{i=1}^{n} \omega_{1i}$	for all l = 1,2,…,L

Consider an Edgeworth box economy E = {(Xi, Ui)$_{i=1,2}$; $\omega \in \mathbb{R}^2_{++}$ }.

i) Show that when the two agents have identical, monotone increasing, strictly quasi-concave, continuously differentiable, and homothetic utility functions the Pareto Set is the Edgeworth Box's diagonal.
Keep in mind that when utility if homothetic, the MRS remains the same along any ray out of the origin.

ii) Suppose $U_1 = x_{11}x_{21}$ and $U_2 = x_{12} + 2\ln(x_{22})$ and $\omega = (4,9)$. Find the Pareto Set. Keep in mind that since consumer 2 has quasi-linear preferences, some PO allocations could be on the Edgeworth Box's boundary.

Question 2. **The Pareto Optimal Set in a Robinson Crusoe Economy (10 points)**

Consider an economy with one consumer and one firm. The consumer's utility is $u(x_1,x_2) = x_1^{0.5}x_2^{0.5}$
The firm's technology is represented by the production set:

$$Y = \{(-y_1; y_2) \in \mathbb{R}^2_+ | y_2 + y_1 \leq 0 \text{ } and \text{ } 2y_2 + y_1 \leq 15\}$$

i) In a graph, measuring $-y_1$ along the horizontal axis and y_2 along the vertical axis, illustrate the production set.

ii) Suppose in this economy the initial endowment is ω = (25, 0). In the same graph, illustrate the feasible set.

iii) Find the Pareto Optimal allocation(s) in this economy and illustrate it in your graph (Keep in mind that the PO allocation could be on either arm of the boundary of the feasible set, or on the kink.)

Question 1

Consider an allocation $(x_{11}, x_{21}); (x_{21}, x_{22})$ that is Pareto Optimal in the economy

$$e = \{ (U_i)_{i=1,2} ; X_i \subseteq \mathbb{R}^n_+ \ i=1,2 \}$$

where $U_1 = U_2$ & $U(.)$ is (1) monotone increasing, (2) strictly quasi-concave, (3) homothetic (4) continuosly differentiable.

Then at the P.O. allocation

$$MRS_1 (x_{11}, x_{21}) = MRS_2 (x_{21}, x_{22})$$

& since the consumers have identical preferences

$$MRS (x_{11}, x_{21}) = MRS (x_{21}, x_{22})$$

& since the utility function is homothetic

$$MRS \left(\frac{x_{21}}{x_{11}} \right) = MRS \left(\frac{x_{22}}{x_{12}} \right)$$

& sine utility is strictly quasi-concave the MRS is a one-to-one function

&

$$\frac{x_{21}}{x_{11}} = \frac{x_{22}}{x_{12}}$$

then using feasibility

$x_{22} = \omega_2 - x_{21}$

$x_{12} = \omega_1 - x_{11}$

&

$$\frac{x_{21}}{x_{11}} = \frac{\omega_2 - x_{21}}{\omega_1 - x_{11}}$$

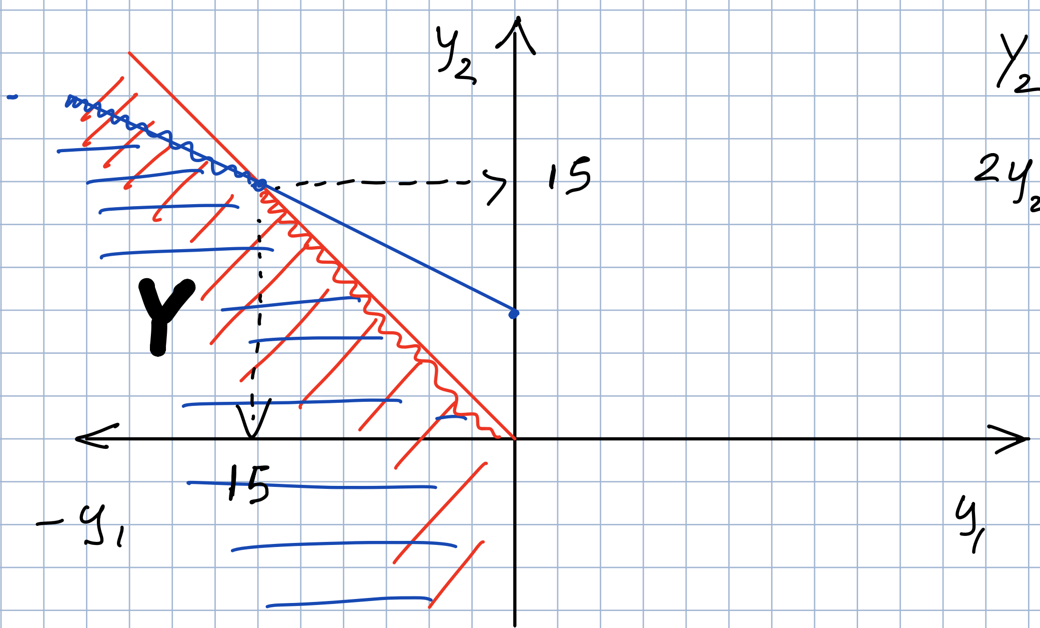
$$\Rightarrow \quad X_{21}(\omega_1 - X_{11}) = (\omega_2 - X_{21}) X_{11}$$

$$X_{21}\,\omega_1 - \cancel{X_{21} X_{11}} = \omega_2 X_{11} - \cancel{X_{21} X_{11}}$$

$$X_{21} = \frac{\omega_2}{\omega_1} X_{11} \qquad \text{Q.E.D}$$

Question 2 PRODUCTION SET

a)



$y_2 + y_1 \leq 0$

$2y_2 + y_1 \leq 15$

$\rightarrow y_2 \leq -y_1$

$\rightarrow y_2 \leq 7.5 - \frac{y_1}{2}$

$$\begin{cases} y_2 = -y_1 \\ y_2 = 7.5 - \frac{y_1}{2} \end{cases}$$

$y_2 = 7.5 + \frac{y_2}{2}$

$y_2/2 = 7.5 \quad y_2 = 15$

Feasible Set

In this economy $\omega = (25, 0)$

By feasibility $x_1 = \omega_1 + y_1 \Rightarrow -y_1 = \omega_1 - x_1$

$x_2 = \omega_2 + y_2$

$$X_2 = 0 + y_2 \leq -y_1 = \omega_1 - X_1$$

$$X_2 \leq \omega_1 - X_1$$

$$X_2 = 0 + y_2 \leq 7.5 - \frac{y_1}{2} = 7.5 + \frac{\omega_1 - X_1}{2}$$

$$X_2 = 7.5 + \frac{\omega_1}{2} - \frac{X_1}{2}$$

b)

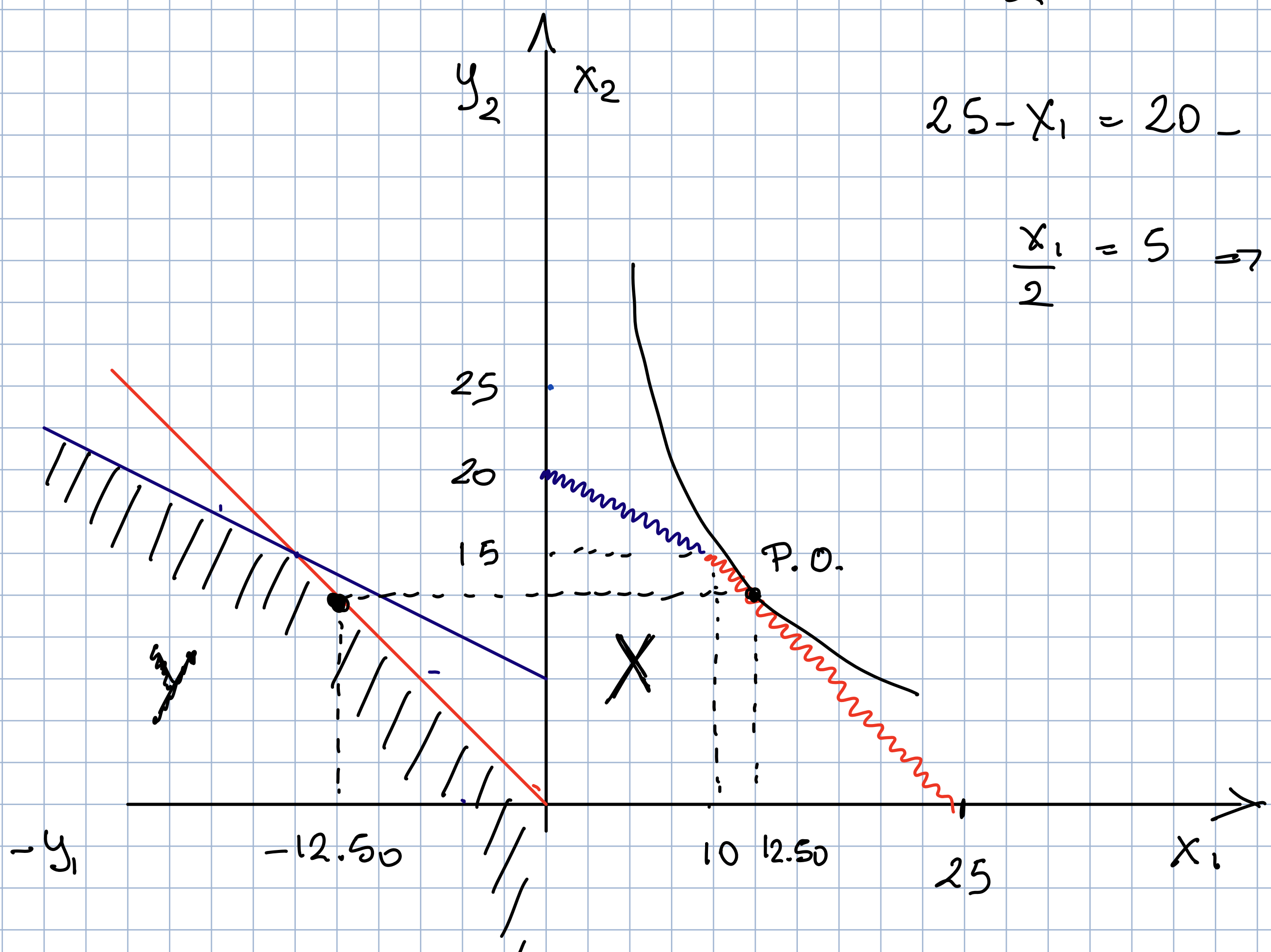
Given $\omega_1 = 25$

$$\begin{cases} X_2 \leq 25 - X_1 \\ X_2 \leq 7.5 + 12.5 - \dfrac{X_1}{2} = 20 - X_1/2 \end{cases}$$

$$25 - X_1 = 20 - \frac{X_1}{2}$$

$$\frac{X_1}{2} = 5 \Rightarrow X_1^* = 10$$

$$X_2^* = 15$$



c) Pareto Optimal allocation:

The P.O. allocation(s) solves

$$\max \quad x_1^{0.5} x_2^{0.5} \quad \text{s.t.} \quad x \in \text{Feasible set}$$

Since $U(\cdot)$ is Cobb Douglas $\quad x_1 \geq 0 \quad x_2 \geq 0$

The P.O. will be

an interior solution

$$\begin{cases} x_2 \leq 25 - x_1 & \text{if } x_1 \geq 10 \\ x_2 \leq 20 - \dfrac{x_1}{2} & 0 \leq x_1 \leq 10 \end{cases}$$

if $\quad x_1 \in [0, 10]$

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{0.5} x_2^{0.5} - \lambda\left(x_2 - 20 + \frac{x_1}{2}\right)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0.5\, x_1^{-0.5} x_2^{0.5} - \lambda/2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0.5\, x_1^{0.5} x_2^{-0.5} - \lambda = 0$$

$$\Rightarrow \frac{x_2}{x_1} = \frac{1}{2}$$

$$x_2^* = 0.5\, x_1^\circ$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\left(x_2 - 20 + \frac{x_1}{2}\right) = 0$$

$$x_2^\circ = 20 - \frac{x_1^\circ}{2} \qquad 0.5 x_1^\circ = 20 - 0.5 x_1^\circ$$

$$x_1^\circ = 20 > 10$$

not admissible

if $\quad x_1 \in [10, \infty)$

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{0.5} x_2^{0.5} - \lambda(x_2 - 25 + x_1)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0.5\, x_1^{-0.5} x_2^{0.5} - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0.5\, x_1^{0.5} x_2^{-0.5} - \lambda = 0$$

$$\Rightarrow \frac{x_2}{x_1} = 1$$

$$x_1^* = x_2^*$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\left(x_2 - 25 + x_1\right) = 0$$

$$x_2^* = 25 + x_2^* \qquad x_2^* = 12.5 = x_1^* > 10$$

Admissible

$$-y_1 = 25 - 12.5 = \quad -y_1 = 12.50$$

$$y_2 = 12.50$$