NYU

Advanced Microeconomics (MA)– Fall 2021
GA 3001.06
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Problem Set 4 – Solutions

Due October 18, 2020 at 11:59am
on Gradescope

Points: /20

Question 1. (10 points)

Consider an economy with two commodities l = 1, 2, and a production set Y = {y ∈ $\mathbb{R}^2$ | F(y) ≤0} where the transformation function is a continuously differentiable and strictly increasing function F = y_1 - f(-y_2).

In this economy, the first commodity is a non-rival non-excludable public good so that $y_1 + \omega_1 = x_{11} = x_{12} = x_1$. The second commodity, however, is a private good so that $y_2 + \omega_2 = x_{21} + x_{22}$.

There are two consumers i = 1, 2 each with a continuously differentiable utility function $U_i(x_1, x_{2i})$ defined over a consumption set $X_i \subseteq \mathbb{R}_+^{\ 2}$.

a) Write down the maximization problem that characterize a Pareto Optimal allocation.

b) Solving the first order conditions for the Pareto Problem show that at an interior Pareto Optimal allocation

$$|MRS_1(x_1, x_{21})| + |MRS_2(x_1, x_{22})| = |MRT(y_1, y_2)|$$

Where the MRT = $-\dfrac{\frac{\partial F}{\partial y_1}}{\frac{\partial F}{\partial y_2}}$

Question 2. (10 points)

Consider a pure-exchange, private-ownership economy, consisting of two consumers, denoted by i = 1, 2, who trade two commodities, x and y. Each consumer i is characterized by an endowment vector, $\omega_i \in R^2_+$, a consumption set $\Omega i \in R^2_+$ and complete, consistent and continuous preferences.

Initial endowments are given by ω_1 = (4, 2) and ω_2 = (2, 3). Individual utility functions are

$u(x_1, y_1) = x_1y_1$

$u_2(x_2, y_2) = x_2 + y_2$

a) Draw the Edgeworth Box for this economy, high light the endowment point ω and draw the indifference curves passing through it for both consumers.

b) Consider the utility functions. Do they satisfy the conditions for the First Welfare Theorem to hold? Do they satisfy the conditions for the Second Welfare Theorem to hold?

c) Find the set of Pareto Optimal allocations (interior points and, separately, boundary points). Draw it in the Edgeworth Box.

d) Let p_y = 1 and find the competitive equilibrium price and the competitive equilibrium allocation. Draw it in the Edgeworth Box.

Question 1

The PARETO PROBLEM

choose $x_1 \geq 0$; $x_{21} \geq 0$; $x_{22} \geq$; y_1 ; y_2

To maximize $U_1(x_1, x_{21})$

subject to $U_2(x_1, x_{22}) \geq \bar{U}$ UTILITY

& $y_1 - f(-y_2) \leq 0$ PRODUCTION

$x_1 \leq y_1 + \omega_1$ FEASIBILITY

$x_{21} + x_{22} \leq y_2 + \omega_2$ FEASIBILITY

If the utility functions are increasing then the solution of the Pareto Problem will be on the Frontier of the Feasible Set a

$$y_1 = f(-y_2)$$

$$x_1 = y_1 + \omega_1$$

$$x_{21} + x_{22} = y_2 + \omega_2$$

$$U_2(x_1 ; x_{22}) = \bar{U}$$

So the problem reduces to

choose $x_1 \geq 0$; $x_{21} \geq 0$; $x_{22} \geq$; y_1 ; y_2

to max $U_1(x_1, x_{21})$

s.t. $U_2(x_1, x_{22}) = \bar{U}$

$x_1 + \omega_1 = f(\omega_2 - x_{21} - x_{22})$

The Lagrangian for this problem is

$$\mathcal{L}(x_1, x_{21}, x_{22}, \lambda_1, \lambda_2) = U_1(x_1, x_{21}) - \lambda_1(\bar{U} - U_2(x_1, x_{22})) - \lambda_2(x_1 + \omega_1 - f(\omega_2 - x_{21} - x_{22}))$$

at interior solutions

$$\frac{\partial \mathcal{L}}{\partial x_1} = MU_1^1 + \lambda_1 MU_1^2 - \lambda_2 = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial x_{21}} = MU_2^1 - \lambda_2 f'(\omega_2 - x_{21} - x_{22}) = 0 \qquad (11)$$

$$\frac{\partial \mathcal{L}}{\partial x_{22}} = \lambda_1 MU_2^2 - \lambda_2 f'(\omega_2 - x_{21} - x_{22}) = 0 \qquad (111)$$

from (11) $MU_2^1 = \lambda_2 f'(\cdot)$ or $\lambda_2 = \dfrac{MU_2^1}{f'(\cdot)}$

substituting in (111) $\lambda_1 MU_2^2 - MU_2^1 = 0$

$$\Rightarrow \lambda_1 = \frac{MU_2^1}{MU_2^2} \qquad (1V)$$

substituting (1V) in (1)

$$MU_1^1 + MU_1^2 \cdot \frac{MU_2^1}{MU_2^2} - \lambda_2 = 0$$

$$MU_1^1 + MU_2^1 \cdot |MRS^2| - \lambda_2 = 0$$

$$MU_1^1 + MU_2^1 = \lambda_2 = \frac{MU_2^1}{f'(\cdot)}$$

deviding both sides by MU_2^1

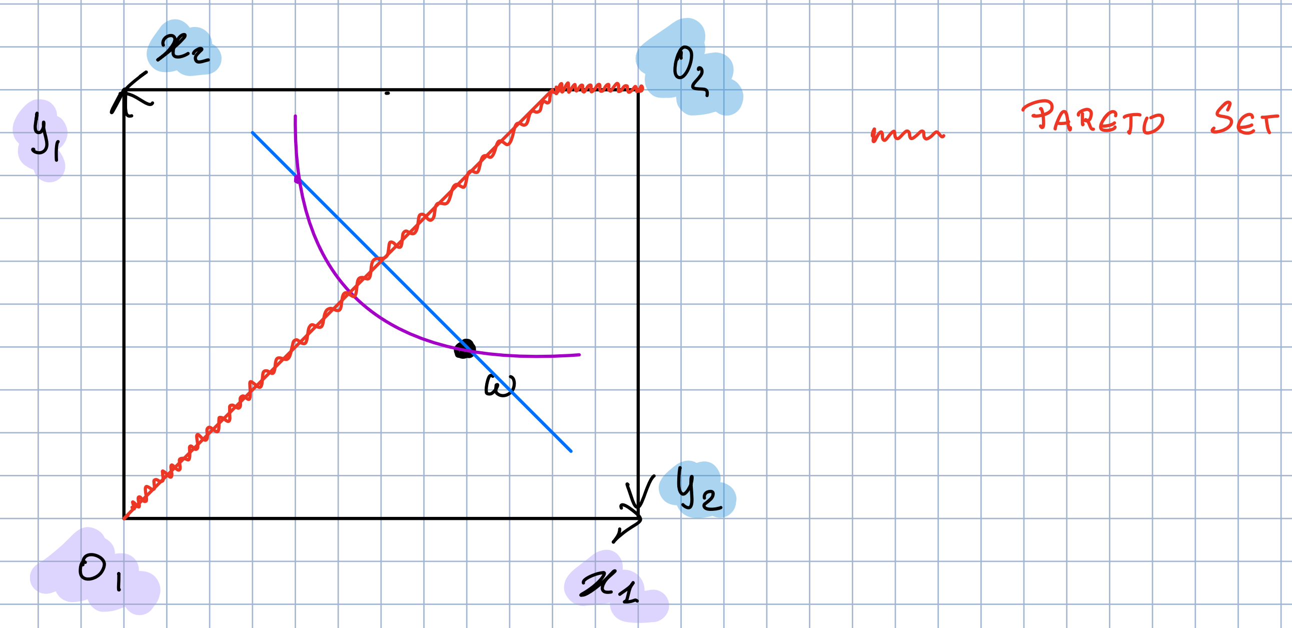
$$\frac{MU_1^1}{MU_2^1} + \frac{\cancel{MU_2^1}}{\cancel{MU_2^1}} \cdot |MRS^2| = \frac{\cancel{MU_2^1}}{f'(\cdot)} \cdot \frac{1}{\cancel{MU_2^1}}$$

$$|MRS^1| + |MRS^2| = \frac{1}{f'(\cdot)} = |MRT|$$

$$|MRT| = \frac{\frac{\partial F}{\partial y_1}}{\frac{\partial F}{\partial y_2}} = \frac{1}{f'(\cdot)}$$

Question 2

a)



b) (1) First Welfare Theorem

Condition : LNS

Both function satisfy LNS

$\forall\ \varepsilon > 0$ consider $(\tilde{x}, \tilde{y}) = (x_0 + \frac{\varepsilon}{2}; y_0) \in N_\varepsilon(x_0, y_0)$

$$U_1(\tilde{x},\tilde{y}) = \left(x_0 + \frac{\varepsilon}{2}\right) y_0 = x_0 y_0 + \frac{\varepsilon}{2} y_0 > x_0 y_0 = U(x_0, y_0)$$

$$U_2(\tilde{x},\tilde{y}) = x_0 + \frac{\varepsilon}{2} + y_0 > x_0 + y_0 = U_2(x_0, y_0)$$

Q.E.D.

(ii) Second Welfare Theorem

Conditions: a) LNS (see above)

b) convex preferences

U_1 & U_2 quasi concave

$U_1 = x \cdot y$

The boundary of a generic UCS is $y = \frac{\bar{U}}{x}$

This is a convex function $y' = -\frac{U}{x^2}$

$$y'' = \frac{2U}{x^3} > 0 \quad \text{for } x > 0$$

hence all UCS are convex sets and U is quasi-concave.

$U_2 = x + y$ here we can use the definition of quasi-concavity

Let a & b be two baskets such that $U_2(a) \geq U_2(b)$

Then $U(\lambda a + (1-\lambda) b) = \lambda x_a + \lambda y_a + (1-\lambda) x_b + (1-\lambda) y_b$

$$\geq \lambda x_b + \lambda y_b + (1-\lambda) x_b + (1-\lambda) y_b$$

$$= x_b + y_b = \min [U_2(a); U_2(b)]$$

c) The Pareto Optimal allocations solve the PARETO PROBLEM :

$$\max \ x_1 y_1 \quad \text{s.t.} \quad x_2 + y_2 \geq \overline{U}_2$$

$$\& \quad x_1 + x_2 \leq 4+2 = 6$$

$$y_1 + y_2 \leq 2+3 = 5$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad y_1 \geq 0 \quad y_2 \geq 0$$

Since U_1 & U_2 are LNS at all P.O. allocations

$$x_1 + x_2 = 6 \quad \& \quad y_1 + y_2 = 5$$

& the problem reduces to:

$$\max \ x_1 y_1 \quad \text{s.t.} \quad 6 - x_1 + 5 - y_1 \geq U$$

$$x_1 \geq 0 \quad x_1 \leq 6$$

$$y_1 \geq 0 \quad y_1 \leq 5$$

$$\mathcal{L}(x, y_1) = x_1 y_1 - \lambda(U - 11 + x_1 + y_1) + \mu_1 x_1 - \mu_2 (x_1 - 6) + \mu_3 y_1 - \mu_4 (y_1 - 5)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = y_1 - \lambda + \mu_1 - \mu_2 = 0 \qquad x_1 \geq 0 \ \ \mu_1 \geq 0 \quad x_1 \leq 6 \ \ \mu_2 \geq 0$$

$$\frac{\partial \mathcal{L}}{\partial y_1} = x_1 - \lambda + \mu_3 - \mu_4 = 0 \qquad y_1 \geq 0 \ \ \mu_3 \geq 0 \quad y_1 \geq 0 \ \ \mu_4 \geq 0$$

Interior solutions

$x_1 > 0$; $y_1 > 0$; $x_1 < 6$; $y_1 < 5$

$\mu_1 = 0$ $\quad \mu_3 = 0$ $\quad \mu_2 = 0$ $\quad \mu_4 = 0$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= y_1 - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y_1} &= x_1 - \lambda = 0 \end{aligned}\right\} \Rightarrow x_1^* = y_1^*$$

hence $\left[x_1^* \in (0,5) ; \; y_1^* = x_1^* ; \; x_2^* = 6 - x_1^* ; \; y_2^* = 5 - x_1^* \right]$

Boundaries :

(i) Since U_1 & U_2 are strictly increasing in x & y

$[(0,0) ; (6,5)]$ & $[(6,5) ; (0,0)]$ are P.O.

(ii) Since U_1 is Cobb Douglas no allocation where

$x_1 = 0$ & $y_1 > 0$ or $x_1 > 0$ & $y_1 = 0$ is P.O.

(iii) Let's show that allocations where

$x_1 = 6$ $y_1 < 5$ are NOT Pareto Optimal

$$\frac{\partial \mathcal{L}}{\partial x_1} = y_1 - \lambda - \mu_3 = 0$$

$\mu_1 = 0$ $\mu_2 = 0$ $\mu_4 = 0$

$\mu_3 \geq 0$

$$\frac{\partial \mathcal{L}}{\partial y_1} = x_1 - \lambda = 0 \Rightarrow 6 = \lambda$$

contraddiction

$$\Rightarrow y_1 - 6 = \mu_3$$

& since $y_1 < 5 \Rightarrow \mu_3 < 5 - 6 = -1 < 0$

not acceptable

Let's show that allocations where

$5 \le x_1 < 6$ & $y_1 = 5$ are Pareto Optimal

$$\frac{\partial \mathcal{L}}{\partial x_1} = y_1 - \lambda = 0 \qquad \mu_1 = 0 \quad \mu_3 = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_1} = x_1 - \lambda - \mu_4 = 0 \qquad \mu_3 = 0 \quad \mu_4 \geq 0$$

at $y_1 = 5$

$$5 - \lambda = 0 \implies \lambda = 5$$

$$x_1 - 5 = \mu_4 \qquad \mu_4 = x_1 - 5 \geq 0$$

$$\implies x_1 \geq 5 \quad \text{acceptable}$$

hence the Pareto Set is

$$\left[x_1^* \in [0,5) ; \ y_1^* = x_1^* \ ; \ x_2^* = 6 - x_1^* \ ; \ y_2' = 5 - y_1'\right] \cup$$

$$\left[x_1 \in [5,6], \ y_1^* = 5 \ ; \ x_2' = 6 - x_1^* \ ; \ y_2^* = 0\right]$$

d) We are looking for the competitive equilibrium assuming it exists.

Route 1

Find consumer 1's offer curve

Find consumer 2's offer curve

impose market clearing conditions

Route 2

Since all C.E. are P.O.,

find the P.O. allocation that is on the budget constraint for consumer 1

Route 1

consumer 1 chooses $x_1 \geq 0$, $y_1 \geq 0$

to max $x_1 y_1$ s.t. $P_x x_1 + y_1 \leq 6P_x + 5$

Since the utility function is LNS & Cobb-Douglas

on the consumer's offer curve:

$x_1^* > 0$, $y_1^* > 0$ & $P_x x_1^* + y_1^* = 4P_x + 2$

$$\mathcal{L}(x_1; y_1; \lambda) = x_1 y_1 - \lambda (P_x x_1 + y_1 - 4P_x - 2)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = y_1 - \lambda P_x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_1} = x_1 - \lambda = 0 \quad \Rightarrow \quad \lambda = x_1$$

$$y_1 - P_x x_1 = 0 \qquad y_1 = P_x x_1$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -[P_x x_1 + y_1 - 4P_x - 2] = 0 \qquad 2P_x x_1 = 4P_x + 2$$

$$x_1^* = \frac{4P_x + 2}{2P_x}$$

$$y_1^* = \frac{4P_x + 2}{2}$$

consumer 2 chooses $x_2 \geq 0$, $y_2 \geq 0$

to max $x_2 + y_2$ s.t. $P_x x_2 + y_2 \leq 2P_x + 3$

since U_2 is LNS at the consumer's offer curve:

$$p_x x_2 + y_2 = 2p_x + 3$$

$$\mathcal{L}(x_2, y_2) = x_2 + y_2 - \lambda(p_x x_2 + y_2 - 2p_x - 3) + \mu_1 x_2 + \mu_2 y_2$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 - \lambda p_x + \mu_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_2} = 1 - \lambda + \mu_2 = 0$$

a) Interior solutions

$x_2 > 0 \quad y_2 > 0 \qquad \mu_1 = 0 \quad \mu_2 = 0$

$$\left.\begin{aligned} 1 - \lambda p_x &= 0 \\ 1 - \lambda &= 0 \end{aligned}\right\} \Rightarrow \quad \lambda = 1$$

$p_x = 1$

$x_2^* + y_2^* = 5$

b) Corner solutions

b.1 $\quad x_2 > 0 \quad y_2 = 0$

$1 - \lambda p_x = 0$

$1 - \lambda + \mu_2 = 0 \qquad \lambda = 1 + \mu_2 > 1$

$$p_x = \frac{1}{\lambda} = \frac{1}{1+\mu_2} < 1$$

$$x_2 = 2 + \frac{3}{p_x}$$

$y_2 = 0$

b.2 $\quad x_2 = 0 \quad y_2 > 0$

$$1 - \lambda p_x + \mu_1 = 0 \quad \Rightarrow \quad 1 - p_x + \mu_1 = 0 \quad p_x > 1$$

$$1 - \lambda = 0 \quad \Rightarrow \lambda = 1$$

$$x_2 = 0$$

$$y_2 = 2p_x + 3$$

Equilibrium :

CASE 1

$p_x = 1$

$$x_1^* = \frac{4+2}{2 \cdot 1} = 6/2 = 3$$

$$y_1^* = \frac{4+2}{3} = 6/2 = 3$$

$$x_2^* + y_2^* = 5$$

market clearing

$$x_1^* - \omega_1 + x_2^* - \omega_2 = 0$$

$$3 - 4 + 5 - y_2^* - 2 = 0$$

$$y_2^* = 3 - 4 + 5 - 2 = 2 \geq 0$$

$$x_2^* = 5 - 2 = 3$$

ACCEPTABLE

hence C.E. $\Rightarrow$ $p_x = 1$ $\quad x_1^* = 3 \,;\, y_1^* = 3 \quad x_2^* = 3 \,;\, y_2^* = 2$