Lyx - Practice Assignment

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1 Abstract

This report presents a study of various split-step fourier methods applied to the problem of wave propagation down a dispersive, optical fibre. It is found that the algorithms that centre the dispersion and the nonlinearity computations about each other converge much better, i.e., the cumulative error scales as Δz^4. Of such algorithms, the algorithm that takes a first half step (and last half step) of dispersion is found to be superior to the algorithm that starts and ends with nonlinearity half steps.

2 Introduction

This report presents four different algorithms to solve the following equation

$$\frac{\partial A}{\partial z} + i\frac{\partial^2 A}{\partial t^2} = |A|^2 A \qquad (1)$$

which appears in the theory of optical waveguides. In this equation, z represents position in the “wave frame”, and both z and t have been scaled to eliminate the constants usually present. Here, A is the complex amplitude of the transverse Electric field of the wave.

Equation (1) is a nonlinear partial differential equation (PDE) of the hyperbolic, and is therefore difficult to solve. In the absence of the nonlinear term (the term on the right), Eq. (1) is a wave equation that can be solved by fourier methods. In the absence of the time derivitive term, Eq. (1) reduces to an ordinary differential equation, though a nonlinear one. Such are easy to solve using a standard differential equation solver such as the fifth order, adaptive step, Runge Kutta solver.

To make progress, we consider solving the equation over a very small step in z. It is easy to show that the two terms are additive in their effect in this limit. That is to say, first solve the linear equation (which represents “dispersion”) from z to $z+dz$. Use that as initial condition and solve the nonlinear ODE from z to $z+dz$. In the limit of dz going to zero, this is equivalent to solving the full equation exactly.

In this study, we try four different ways of carrying out this approach:

1. Perform the following steps, using the result of each as the initial condition of the next.
   (a) Solve the linear equation from z to $z+dz$.
   (b) Solve the nonlinear ODE from z to $z+dz$.
   (c) Solve the linear equation from $z+dz$ to $z+2dz$.
   (d) etc.
2. Perform the following steps:
   (a) Solve the nonlinear ODE from z to $z+dz$. Save the result.
   (b) Solve the linear equation from z to $z+dz$. Save the result.
   (c) Average the results of (a) and (b) to obtain the initial condition for the next step.
   (d) etc.
3. Perform the following steps, using the result of each as the initial condition of the next.
   (a) Take a step of $dz/2$ first, on which the linear equation is solved.
   (b) Now solve the nonlinear ODE from z to $z+dz$.
   (c) Solve the linear equation from $z+dz/2$ to $z+3dz/2$.
   (d) Solve the ODE from $z+dz$ to $z+2dz$.
   (e) etc.
   (f) Solve the linear equation from $L-dz/2$ to L.

Results and Discussion

The four algorithms described in the Introduction were implemented in C. The simulation was carried out for a fixed length of waveguide, using different dz. As this particular choice of coefficients is theoretically tractable, the exact solution was also known.

Table 1 presents the error in each algorithm, for different dz values. Figure 1 presents the same data in a log-log plot. Each series was also fitted to a power law fit of the form

$$adz^b$$

and the corresponding best fit a and b were obtained. These are indicated in the legend in the figure.

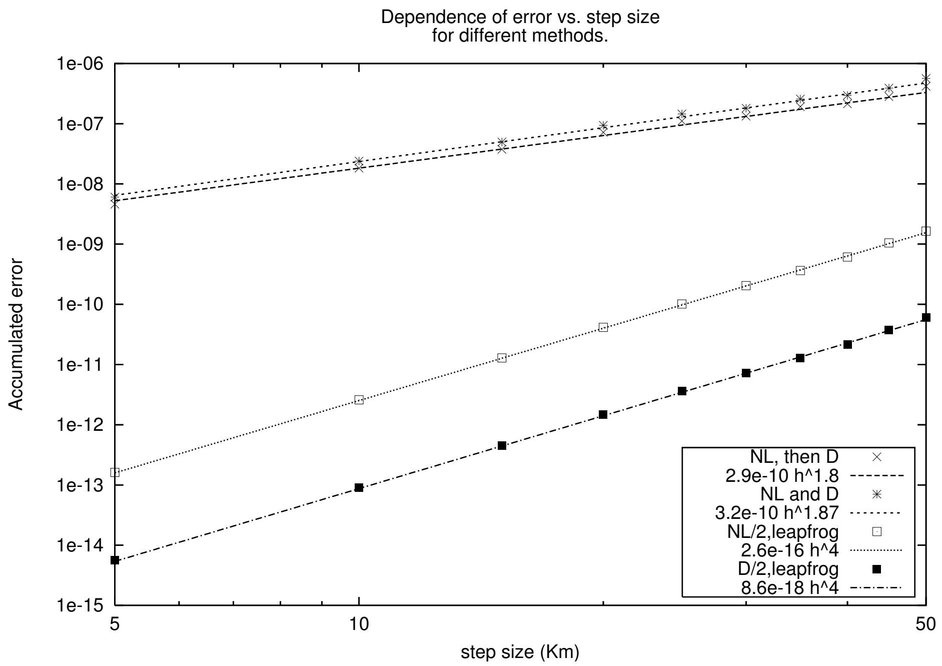
All the algorithms performed better as dz is reduced. This was expected, since the algorithms are all exact in the limit of dz going to zero. It is clear from the results, however, that the first two algorithms are inferior in that the error improves slowly as dz is decreased. The third and fourth algorithms improved as dz^4, while the first two improved only as dz^2. This is a clear indication that the interleaving of nonlinearity and dispersion manages error in a much better fashion.

While both algorithms (3) and (4) scaled the same, there was a clear two orders of improvement in error when using

Table 1: Cumulative error due to different split-step algorithms for an optical link of 100 Km. Algorithm 1 calculates the contribution from nonlinearity (NL) first, and then uses the result to compute the contribution from dispersion (D). Algorithm 2 calculates NL and D based on the original signal. Algorithm 3 computes NL for a half step, and then computes D and NL, each centered about the other, with a final half step of NL. Algorithm 4 computes D for a half step, and then computes NL and D, each centered about the other, with a final step of D.

dz	ε_1	ε_2	ε_3	ε_4
10	1.824×10^{-8}	2.393×10^{-8}	2.589×10^{-12}	9.099×10^{-14}
20	7.17×10^{-8}	9.396×10^{-8}	4.150×10^{-11}	1.467×10^{-12}
30	1.341×10^{-8}	1.812×10^{-7}	2.052×10^{-10}	7.260×10^{-12}
40	2.137×10^{-7}	2.959×10^{-7}	6.111×10^{-10}	2.143×10^{-11}
50	4.206×10^{-7}	5.651×10^{-7}	1.643×10^{-9}	6.069×10^{-11}
60	4.229×10^{-7}	6.063×10^{-7}	2.662×10^{-9}	8.902×10^{-11}
70	4.930×10^{-7}	7.524×10^{-7}	5.306×10^{-9}	1.768×10^{-10}
80	6.545×10^{-7}	1.000×10^{-6}	1.033×10^{-8}	3.636×10^{-10}
90	9.507×10^{-7}	1.500×10^{-6}	1.813×10^{-8}	7.003×10^{-10}
100	1.500×10^{-6}	2.200×10^{-6}	2.830×10^{-8}	1.235×10^{-9}

Figure 1: Scaling of Error for different split-step algorithms. The points correspond to the data in Table 1. The lines correspond to best power-law fits as indicated in the legends. As can be seen, algorithms 1 and 2 scale as dz^2 , while algorithms 3 and 4 scale as dz^4 .



algorithm (4). The difference between the two algorithms is merely that the third algorithm uses a first and last half-step of nonlinearity, while the fourth algorithm uses a first and last half-step of dispersion. It is clear that the nonlinear term is a delicate term that is susceptible to error if it is carried out in a " non-centered" manner. Thus it always has to be sandwitched between dispersion computations.

To conclude, it is clear that not all split-step fourier methods are equal. Great care has to be taken to implement the algorithm correctly. The reward is an enormous reduction in computational time. For example, if an accuracy of 10^{-12} is required, algorithms 1 and 2 would require extremely small steps of the order of 0.06 Km to hope to achieve the result. Algorithm 3 would require a step size of between 5 and 10 Km. But algorithm 4 would require a step size of 20 Km, thus reducing the computation by a further factor of 3.