Signal Analysis Asssignment #5

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PUL074BEX007

Problem 1

Show, if $x(t)$ is real and odd, then its Fourier series coefficients are purely imaginary and odd.

Solution:

The question suggests that the signal $x(t)$ is real and odd which means, $x(t) = -x(-t)$. We can write the Fourier series coefficients for the signal $x(t)$ as:

$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}dt \tag{1}$$

Method 1:
Rewriting Eq.(1) using the Euler's identity as,

$$\begin{aligned} C_k &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)cos(k\omega_0 t)dt - j\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)sin(k\omega_0 t)dt \\ &= 0 - j\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)sin(k\omega_0 t)dt \\ &= -j\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)sin(k\omega_0 t)dt \end{aligned}$$

Since $x(t)cos(k\omega_0 t)$ is odd the resulting integration is 0. From the above equation for C_k **we can conclude that, the fourier series coefficients are odd since $C_k = -C_{-k}$ and purely imaginary.**

Method 2:
Rewriting Eq.(1) by separating the limits of integration as,

$$\begin{aligned} C_k &= \frac{1}{T_0}\int_{-T_0/2}^{0} x(t)e^{-jk\omega_0 t}dt + \frac{1}{T_0}\int_{0}^{T_0/2} x(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T_0}\int_{0}^{T_0/2} x(-t)e^{jk\omega_0 t}dt + \frac{1}{T_0}\int_{0}^{T_0/2} x(t)e^{-jk\omega_0 t}dt \\ &= -\frac{1}{T_0}\int_{0}^{T_0/2} x(t)e^{jk\omega_0 t}dt + \frac{1}{T_0}\int_{0}^{T_0/2} x(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T_0}\int_{0}^{T_0/2} x(t)\left[e^{-jk\omega_0 t} - e^{jk\omega_0 t}\right]dt \\ &= \frac{1}{T_0}\int_{0}^{T_0/2} x(t)[-2jsin(k\omega_0 t)]dt \\ &= -2j\frac{1}{T_0}\int_{0}^{T_0/2} x(t)sin(k\omega_0 t)dt \end{aligned}$$

From this equation for C_k it is clear that **for a signal $x(t)$ that is real and odd, the fourier series coefficients are odd and purely imaginary.**

Problem 2

Plot the fourier series coefficients for the discrete time signal $x[n] = sin\left(\frac{2\pi}{N}\right)n$ for: (i) $N = 10$ (ii) $N = 15$

Solution:

(i) $N = 10$

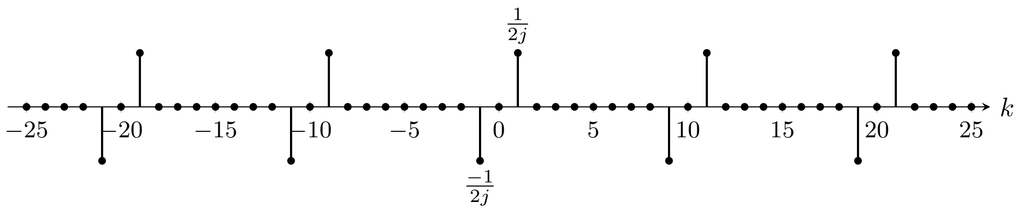

Figure 1: Plot for a_k for $N = 10$

(ii) $N = 15$

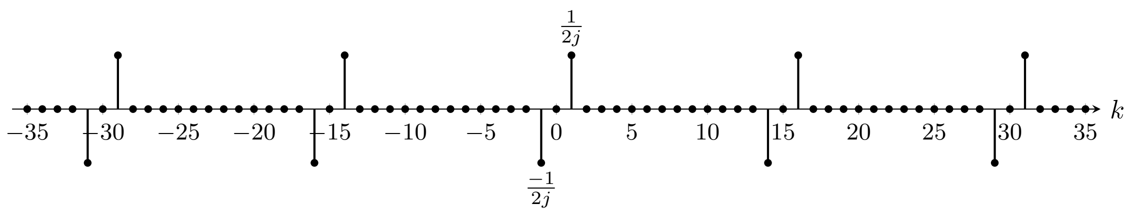

Figure 2: Plot for a_k for $N = 15$

Problem 3

Find the fourier series coefficients for the discrete-time periodic square wave.

Solution:

Considering a periodic square wave with arbitrary period N and pulse width of N_p samples for a more generic result.

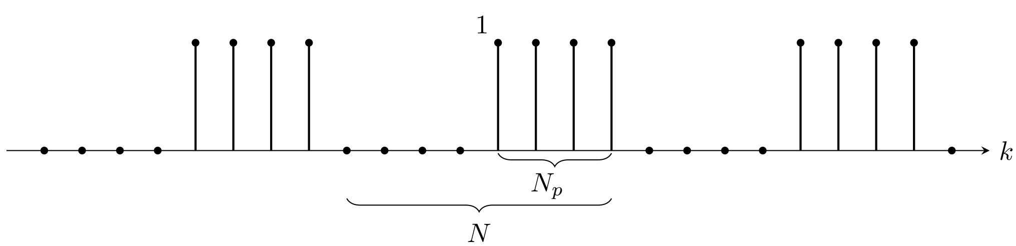

Figure 3: Plot for discrete time periodic square wave

The fourier series coefficients for any discrete time signal is given by,

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jkn\frac{2\pi}{N}} \tag{2}$$

From Fig.(3) it is clear that, the Eq.(2) is non-zero for period N_p and 0 for any other period. So, we can rewrite Eq.(2) as,

$$C_k = \frac{1}{N}\sum_{n=0}^{N_p-1} e^{-jkn\frac{2\pi}{N}}$$

For $k = 0$,

$$\begin{aligned} C_0 &= \frac{1}{N}\sum_{n=0}^{N_p-1} e^0 \\ &= \frac{N_p}{N} \end{aligned}$$

which is the average value of the signal averaged over the period N. For $k \neq 0$

$$\begin{aligned} C_k &= \frac{1}{N}\sum_{n=0}^{N_p-1} \left(e^{-jk\frac{2\pi}{N}}\right)^n \\ &= \frac{1}{N}\left[\frac{1-\left(e^{-jk\frac{2\pi}{N}}\right)^{N_p}}{1-\left(e^{-jk\frac{2\pi}{N}}\right)}\right] \\ &= \frac{1}{N}\left[\frac{e^{-jkN_p\frac{\pi}{N}}\left(e^{jkN_p\frac{\pi}{N}} - e^{-jkN_p\frac{\pi}{N}}\right)}{e^{-jk\frac{\pi}{N}}\left(e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}}\right)}\right] \\ &= \frac{1}{N}\left[\frac{e^{-jkN_p\frac{\pi}{N}}\left\{2jsin\left(kN_p\frac{\pi}{N}\right)\right\}}{e^{-jk\frac{\pi}{N}}\left\{2jsin\left(k\frac{\pi}{N}\right)\right\}}\right] \\ &= \frac{1}{N}\left[\frac{sin\left(k\frac{N_p\pi}{N}\right)}{sin\left(k\frac{\pi}{N}\right)}\right] e^{-jk(N_p-1)\frac{\pi}{N}} \end{aligned}$$

Overall, we can write,

$$C_k = \begin{cases} \frac{N_p}{N}, & k = 0 \\ \frac{1}{N}\left[\frac{sin\left(k\frac{N_p\pi}{N}\right)}{sin\left(k\frac{\pi}{N}\right)}\right] e^{-jk(N_p-1)\frac{\pi}{N}}, & k \neq 0 \end{cases} \tag{3}$$

This result suggests that the fourier series coefficient of a periodic square wave signal is a digital sinc signal.

Problem 4

State and prove DTFS properties.

Solution:

If $\mathscr{F}_S\{x[n]\}$ denotes the fourier series transformation of $x[n]$ into its fourier coefficients such that,

$$\mathscr{F}_S\{x[n]\} = c_k, \quad \text{where} \quad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} \tag{4}$$

Property 1 (Linearity). *If $\mathscr{F}_S\{x[n]\} = a_k$ and $\mathscr{F}_S\{y[n]\} = b_k$, then*

$$\mathscr{F}_S\{Ax[n] + By[n]\} = Aa_k + Bb_k$$

Proof. This can be proved based on the properties of a summation over a limit.

$$\begin{aligned}
\mathscr{F}_S\{Ax[n]+By[n]\} &= \sum_{n=0}^{N-1}(Ax[n]+By[n])e^{-(j\omega_0 kn)} \\
&= \frac{1}{N}A\sum_{n=0}^{N-1}x[n]e^{-(j\omega_0 kn)} + \frac{1}{N}B\sum_{n=0}^{N-1}y[n]e^{-(j\omega_0 kn)} \\
&= Aa_k + Bb_k
\end{aligned}$$

□

Property 2 (Time Shifting)**.** *Shifting a signal $x[n]$ by n_0 results in a phase shift of fourier coefficients such that,*

$$\mathscr{F}_S\{x[n-n_0]\} = c_k e^{-(j\omega_0 kn_0)}$$

Proof.

$$\begin{aligned}
\mathscr{F}_S\{(x\left[n-n_0\right])\} &= \frac{1}{N}\sum_{n=0}^{N-1}x\left[n-n_0\right]e^{-(j\omega_0 kn)} \\
&= \frac{1}{N}\sum_{n-n_0=0}^{N-n_0-1}x\left[n-n_0\right]e^{-(j\omega_0 k(n-n_0))}e^{-(j\omega_0 kn_0)} \\
&= \frac{1}{N}\sum_{n-n_0=0}^{N-n_0-1}x[\tilde{n}]e^{-(j\omega_0 k\tilde{n})}e^{-(j\omega_0 kn_0)} \quad , where \quad \tilde{n}=n-n_0 \\
&= c_k e^{-(j\omega_0 kn_0)}
\end{aligned}$$

If $c_k = |c_k|\, e^{j\angle(c_k)}$ then, the magnitude is given by,

$$\left|c_k e^{-(j\omega_0 kn_0)}\right| = |c_k|\left|e^{-(j\omega_0 kn_0)}\right| = |c_k| \tag{5}$$

Likewise, the phase is given by,

$$\angle\left(c_k e^{-(j\omega_0 n_0 k)}\right) = \angle\left(c_k\right) - \omega_0 n_0 k \tag{6}$$

Eq.(6) clearly shows that the shift of n_0 in the signal $x[n]$ results in a phase shift for the fourier coefficients. □

Property 3 (Frequency Shifting)**.** *The frequency shift of a signal $x[n]$ results in an equivalent shift in its fourier coefficient spectrum such that,*

$$\mathscr{F}_S\{x[n]e^{jm\frac{2\pi}{N}n}\} = c_{k-m}$$

Proof. From Eq.(4) we can write,

$$\begin{aligned}
\mathscr{F}_S\{x[n]e^{jm\frac{2\pi}{N}n}\} &= \frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{jm\frac{2\pi}{N}n}e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}n(k-m)} \\
&= c_{k-m}
\end{aligned}$$

□

Property 4 (Conjugation)**.** *The fourier coefficient for a conjugate of a signal $x[n]$ is also an equivalent conjugate such that,*

$$\mathscr{F}_S\{x^*[n]\} = c^*_{-k}$$

Proof. To prove this, we start from the Eq.(4)

$$\begin{aligned}
c_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n} \\
\Rightarrow \quad c_k^* &= \frac{1}{N}\sum_{n=0}^{N-1} x^*[n]e^{jk\frac{2\pi}{N}n} \\
\Rightarrow \quad c_{-k}^* &= \frac{1}{N}\sum_{n=0}^{N-1} x^*[n]e^{-jk\frac{2\pi}{N}n} \\
\Rightarrow \quad c_{-k}^* &= \mathscr{F}_S\{x^*[n]\}
\end{aligned}$$

□

Property 5 (Time Reversal)**.** *The time reversal of a signal $x[n]$ results in an equivalent reversal in its fourier coefficient spectrum such that,*

$$\mathscr{F}_S\{x[-n]\} = c_{-k}$$

Proof. To prove this, we start off as,

$$\begin{aligned}
\mathscr{F}_S\{x[-n]\} &= \frac{1}{N}\sum_{n=0}^{N-1} x[-n]e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N}\sum_{\tilde{n}=0}^{N-1} x[\tilde{n}]e^{jk\frac{2\pi}{N}\tilde{n}} \quad \textit{where}, \quad \tilde{n} = -n, \\
&= \frac{1}{N}\sum_{\tilde{n}=0}^{N-1} x[\tilde{n}]e^{-j(-k)\frac{2\pi}{N}\tilde{n}} \\
&= c_{-k}
\end{aligned}$$

□

Property 6 (Multiplication)**.** *Signal multiplication in time domain results in a DT circular convolution in the frequency domain. If $x[n]$ and $y[n]$ are two signals with fourier coefficients a_k and b_k respectively, the signal multiplication $z[n] = x[n]y[n]$ has the fourier series coefficients c_k such that,*

$$c_k = \sum_{l=0}^{N-1} a_l b_{k-l}$$

Proof.

$$\begin{aligned}
c_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]y[n]e^{-(j\omega_0 kn)} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{l=0}^{N-1} a_l e^{j\omega_0 ln} y[n]e^{-(j\omega_0 kn)} \\
&= \sum_{l=0}^{N-1} a_l \left(\frac{1}{N}\sum_{n=0}^{N-1} y[n]e^{-(j\omega_0(k-l)n)}\right) \\
&= \sum_{l=0}^{N-1} a_l b_{k-l}
\end{aligned}$$

□

Property 7 (Periodic Convolution)**.** *The periodic convolution of signals in the time domain results in the multiplication of their fourier coefficients. If If $x[n]$ and $y[n]$ are two signals with fourier coefficients a_k and b_k respectively, then the periodic convolution $\sum_{m=0}^{N-1} x[m]y[n-m]$ has the fourier coefficient c_k such that*

$$c_k = Na_kb_k$$

Proof. To prove this, we start off as,

$$\begin{aligned}
c_k =& \mathscr{F}_S\{\sum_{m=0}^{N-1} x[m]y[n-m]\} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x[m]y[n-m]e^{-\left(j\frac{2\pi}{N}kn\right)} \\
&= N\left[\left\{\frac{1}{N}\sum_{m=0}^{N-1} x[m]e^{-\left(j\frac{2\pi}{N}km\right)}\right\}\left\{\frac{1}{N}\sum_{n-m=0}^{N-m-1} y[n-m]e^{-\left(j\frac{2\pi}{N}k(n-m)\right)}\right\}\right] \\
&= Na_kb_k
\end{aligned}$$

□

Property 8 (First Difference)**.** *For a discrete time signal $x[n]$,*

$$\mathscr{F}_S\{x[n]-x[n-1]\} = c_k\left(1-e^{-jk\frac{2\pi}{N}}\right)$$

Proof. To prove the first difference property, Linearity and Time Shifting properties are essential.

$$\begin{aligned}
\mathscr{F}_S\{x[n]-x[n-1]\} &= \frac{1}{N}\sum_{n=0}^{N-1}\left(x[n]-x[n-1]\right)e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n} - \frac{1}{N}\sum_{n=0}^{N-1} x[n-1]e^{-jk\frac{2\pi}{N}n} \\
&= c_k - \frac{1}{N}\sum_{n-1=0}^{N-1} x[n-1]e^{-jk\frac{2\pi}{N}(n-1)}e^{-jk\frac{2\pi}{N}} \\
&= c_k - e^{-jk\frac{2\pi}{N}}\frac{1}{N}\sum_{\tilde{n}=0}^{N-1} x[\tilde{n}]e^{-jk\frac{2\pi}{N}(\tilde{n})} \quad where, \tilde{n} = n-1 \\
&= c_k - c_ke^{-jk\frac{2\pi}{N}} \\
&= c_k\left(1-e^{-jk\frac{2\pi}{N}}\right)
\end{aligned}$$

□

Property 9 (Duality)**.** *The duality theorem states that, for a signal $x[n]$ such that $x[n] = \sum_{k=0}^{N-1} c_ke^{jk\frac{2\pi}{N}n}$, the signal shows dual property for role change of n and k as,*

$$\mathscr{F}_S\{c_n\} = \frac{1}{N}x[-k]$$

Proof.

$$\begin{aligned}
\mathscr{F}_S\{c_n\} &= \frac{1}{N}\sum_{n=0}^{N-1} c_ne^{-\left(j\frac{2\pi}{N}kn\right)} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} c_ne^{\left(j\frac{2\pi}{N}(-k)n\right)} \\
&= \frac{1}{N}x[-k]
\end{aligned}$$

□

Property 10 (Symmetry Properties)**.** *A signal $x[n]$ has various symmetry properties for odd, even and real valued conditions as,*

Symmetry Property 10.1 (Conjugate symmetry for real signal)**.** *For a signal $x[n]$ such that $x[n] = x^*[n]$, we have,*

$$c_k = c_{-k}^*$$

Proof.

$$\begin{aligned}
c_k &= \mathscr{F}_S\{x[n]\} \\
\Rightarrow c_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} \\
\Rightarrow c_k^* &= \frac{1}{N}\sum_{n=0}^{N-1} x^*[n]e^{\left(j\frac{2\pi}{N}kn\right)} \\
\Rightarrow c_k^* &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-\left(j\frac{2\pi}{N}(-k)n\right)} \\
\Rightarrow c_k^* &= c_{-k} \\
\Rightarrow c_k &= c_{-k}^*
\end{aligned}$$

□

Symmetry Property 10.2 (Real and even signal)**.** *For a signal $x[n]$ such that $x[n] = x[-n]$, the coefficients are real and even.*

Proof.

$$\begin{aligned}
c_k &= \frac{1}{N}\sum_{n=0}^{N} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} \\
&= \frac{1}{N}\sum_{n=0}^{N/2} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} + \frac{1}{N}\sum_{N/2}^{N} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} \\
&= \frac{1}{N}\sum_{-n=0}^{N/2} x[-n]e^{\left(j\frac{2\pi}{N}kn\right)} + \frac{1}{N}\sum_{N/2}^{N} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} \\
&= \frac{1}{N}\sum_{n=0}^{N/2} x[n]e^{\left(j\frac{2\pi}{N}kn\right)} + \frac{1}{N}\sum_{N/2}^{N} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} \\
&= \frac{1}{N}\sum_{n=0}^{N} x[n]\left\{e^{\left(j\frac{2\pi}{N}kn\right)} + e^{-\left(j\frac{2\pi}{N}kn\right)}\right\} \\
&= \frac{2}{N}\sum_{n=0}^{N} x[n]\left\{cos\left(k\frac{2\pi}{N}n\right)\right\}
\end{aligned}$$

The above equation is indeed a real and even signal since there's no imaginary term and cosine signal is an even signal.

□

Symmetry Property 10.3 (Real and odd signal)**.** *For a signal $x[n]$ such that $x[n] = -x[-n]$, the coefficients are purely imaginary and odd.*

Proof.

$$
\begin{aligned}
c_k &= \frac{1}{N}\sum_{n=0}^{N} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} \\
&= \frac{1}{N}\sum_{n=0}^{N/2} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} + \frac{1}{N}\sum_{N/2}^{N} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} \\
&= \frac{1}{N}\sum_{-n=0}^{N/2} x[-n]e^{\left(j\frac{2\pi}{N}kn\right)} + \frac{1}{N}\sum_{N/2}^{N} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} \\
&= -\frac{1}{N}\sum_{n=0}^{N/2} x[n]e^{\left(j\frac{2\pi}{N}kn\right)} + \frac{1}{N}\sum_{N/2}^{N} x[n]e^{-\left(j\frac{2\pi}{N}kn\right)} \\
&= -\frac{1}{N}\sum_{n=0}^{N} x[n]\left\{e^{\left(j\frac{2\pi}{N}kn\right)} - e^{-\left(j\frac{2\pi}{N}kn\right)}\right\} \\
&= -\frac{2j}{N}\sum_{n=0}^{N} x[n]\left\{sin\left(k\frac{2\pi}{N}n\right)\right\}
\end{aligned}
$$

The above equation for c_k is purely imaginary and is indeed an odd signal due to the sine function. □

Problem 5

Prove the parseval's relation for discrete time periodic signal.

Solution:

Parseval's relation for a discrete time periodic signal states that the average power in a periodic signal, say, $x[n]$ is equal to the sum of the average powers in all of its harmonic components.
Mathematically,

$$
\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2 \tag{7}
$$

Proof. To prove the Eq.(7) we'll use the relation for c_k from Eq.(4)

$$
\begin{aligned}
\sum_{k=0}^{N-1} |c_k|^2 &= \sum_{k=0}^{N-1} c_k c_k^* \\
&= \frac{1}{N}\sum_{k=0}^{N-1} c_k \sum_{n=0}^{N-1} x^*[n]e^{\left(j\frac{2\pi}{N}kn\right)} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x^*[n]\left\{\sum_{k=0}^{N-1} c_k e^{\left(j\frac{2\pi}{N}kn\right)}\right\} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x^*[n]x[n] \\
&= \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2
\end{aligned}
$$

□