Signal Analysis Assignment #4

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PUL074BEX007

Problem 1

Prove that the following identity holds:

$$\int_T e^{j(k-n)\omega_0 t}dt = \begin{cases} T, & \textbf{for } k = n \\ 0, & \textbf{otherwise} \end{cases}$$

Solution:

Here, we need to solve the integral on the LHS, i.e.

$$\int_T e^{j(k-n)\omega_0 t}dt \tag{1}$$

In Eq.(1) we are concerned with integrating $e^{j(k-n)\omega_0 t}$ over an interval of length T. So, we will obtain the same result over any interval of length T, say, $[0, T], [\frac{-T}{2}, \frac{T}{2}], [T, 2T]$ and so on. Choosing $[0, T]$ for simplicity, results in the integral,

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T cos[(k-n)\omega_0 t]dt + jsin[(k-n)\omega_0 t]dt \tag{2}$$

For $k = n$, the Eq.(2) becomes,

$$= \int_0^T cos(0)dt + j\int_0^T sin(0)dt$$
$$= \int_0^T dt + j.0 = T$$

For $k \neq n$, $cos[(k-n)\omega_0 t]$ and $sin[(k-n)\omega_0 t]$ are periodic sinusoidal functions with a fundamental period of $\left|\frac{T}{k-n}\right|$. Since we are integrating the Eq.(2) over an interval of length T, such that the interval is an integral number of periods of the signal, i.e. the interval is $(k-n)^{th}$ multiple of the periods of the signal. This way, the integration can be represented as a measure of the total area under the functions over the interval T, hence the integral for $k \neq n$ results in 0.
The overall evaluation of the integral represented by Eq.(1) shows that,

$$\int_T e^{j(k-n)\omega_0 t}dt = \begin{cases} T, & \text{for } k = n \\ 0, & \text{otherwise} \end{cases}$$

Problem 2

Determine the complex form of Fourier series from its trigonometric form and vice-versa.

Solution:

Complex Form from Trigonometric Form

The trigonometric form of fourier series can be written as,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n sin(n\omega_0 t) \tag{3}$$

where,

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt$$
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)cos(n\omega_0 t)dt$$
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)sin(n\omega_0 t)dt$$

From Euler's formulae, we have,

$$cos(n\omega_0 t) = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2},\ sin(n\omega_0 t) = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} = -j\left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2}\right)$$

Using this, we can reduce the Eq.(3) as,

$$\begin{aligned}
x(t) &= a_0 + \sum_{n=1}^{\infty} a_n\left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + \sum_{n=1}^{\infty} b_n\left(\frac{-je^{jn\omega_0 t} + je^{-jn\omega_0 t}}{2}\right)\\
&= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - jb_n}{2}\right)e^{jn\omega_0 t} + \sum_{n=1}^{\infty}\left(\frac{a_n + jb_n}{2}\right)e^{-jn\omega_0 t}\\
&= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - jb_n}{2}\right)e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1}\left(\frac{a_{-n} + jb_{-n}}{2}\right)e^{jn\omega_0 t}\\
&= C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} C_n^* e^{jn\omega_0 t}
\end{aligned}$$

where, $C_0 = a_0$, $C_n = \left(\frac{a_n - jb_n}{2}\right)$ and $C_n^* = \left(\frac{a_{-n}+jb_{-n}}{2}\right)$ such that C_n and C_n^* are complex conjugates. If we choose to represent all the complex fourier coefficients with C_n such that C_n is the complex conjugate of C_{-n} then, **the complex form of fourier series is given by,**

$$x(t) = \sum_{-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Trigonometric Form from Complex Form

The complex form of fourier series can be written as,

$$x(t) = \sum_{-\infty}^{\infty} C_n e^{jn\omega_0 t} \tag{4}$$

Eq.(4) can be expanded as the sum,

$$\begin{aligned}
x(t) &= C_0 + C_1 e^{j\omega_0 t} + C_2 e^{2j\omega_0 t} + \dots + C_n e^{jn\omega t} + \dots + C_{-1}e^{-j\omega_0 t} + C_{-2}e^{-2j\omega_0 t} + \dots + C_{-n}e^{-jn\omega_0 t} + \dots\\
&= C_0 + C_1[cos(\omega_0 t) + jsin(\omega_0 t)] + C_2[cos(2\omega_0 t) + jsin(2\omega_0 t)] + \dots + C_n[cos(n\omega_0 t) + jsin(n\omega_0 t)] + \dots +\\
&\quad C_{-1}[cos(\omega_0 t) - jsin(\omega_0 t)] + C_{-2}[cos(2\omega_0 t) - jsin(2\omega_0 t)] + \dots + C_{-n}[cos(n\omega_0 t) - jsin(n\omega_0 t)] + \dots\\
&= C_0 + (C_1 + C_{-1})cos(\omega_0 t) + (C_2 + C_{-2})cos(2\omega_0 t) + \dots + (C_n + C_{-n})cos(n\omega_0 t) + \dots +\\
&\quad (C_1 - C_{-1})sin(\omega_0 t) + j(C_2 - C_{-2})sin(2\omega_0 t) + \dots + j(C_n - C_{-n})sin(n\omega_0 t) + \dots\\
&= C_0 + \sum_{n=1}^{\infty}(C_n + C_{-n})cos(n\omega_0 t) + \sum_{n=1}^{\infty} j(C_n - C_{-n})sin(n\omega_0 t)
\end{aligned}$$

This can be rearranged such that **the fourier series in trigonometric form is given by,**

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n sin(n\omega_0 t)$$

where,

$$a_0 = C_0$$
$$a_n = (C_n + C_{-n})$$
$$b_n = j(C_n - C_{-n})$$

Problem 3

Plot the graph of the magnitude and phase of a_k for the example below and interpret.

Solution:

Here, $x(t) = 1 + sin(\omega_0 t) + 2cos(\omega_0 t) + cos\left(2\omega_0 t + \frac{\pi}{4}\right)$, which has the fundamental frequency ω_0. The signal $x(t)$ can be expanded directly in terms of its complex exponentials for easier approach to determine the complex fourier coefficients.
Using the euler's formulae, we get,

$$\begin{aligned} x(t) &= 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \frac{\pi}{4})} - e^{-j(2\omega_0 t + \frac{\pi}{4})}] \\ &= 1 + \left(1 + \frac{1}{2j}\right) e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right) e^{-j\omega_0 t} + \left(\frac{1}{2} e^{j(\frac{\pi}{4})}\right) e^{2j\omega_0 t} + \left(\frac{1}{2} e^{-j(\frac{\pi}{4})}\right) e^{-2j\omega_0 t} \end{aligned}$$

From this, we can determine the fourier series coefficients as,

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right) = \left(1 - \frac{1}{2}j\right),$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right) = \left(1 + \frac{1}{2}j\right),$$

$$a_2 = \frac{1}{2} e^{j(\frac{\pi}{4})} = \frac{\sqrt{2}}{4}(1 + j),$$
$$a_{-2} = \frac{1}{2} e^{-j(\frac{\pi}{4})} = \frac{\sqrt{2}}{4}(1 - j),$$
$$a_k = 0, |k| > 2$$

To plot the magnitude and phase of a_k, the following table will be used,

Value / a_k for	Magnitude ($\lvert a_k \rvert$)	Phase($\measuredangle a_k$)
$k = 0$	1	0
$k = 1$	1.118033989	-0.463647609
$k = -1$		0.463647609
$k = 2$	0.5	0.7853981634
$k = -2$		-0.7853981634
$\lvert k \rvert > 2$	0	0

Table 1: Magnitude and Phase calculations for a_k

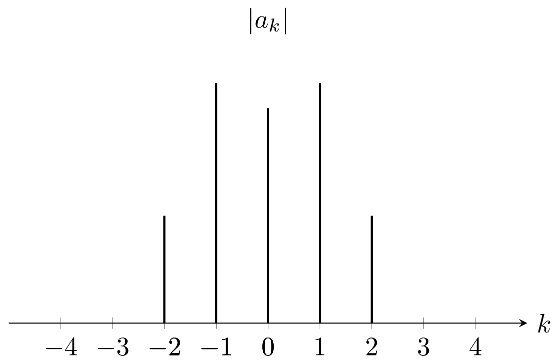

Figure 1: Plot for magnitude of a_k

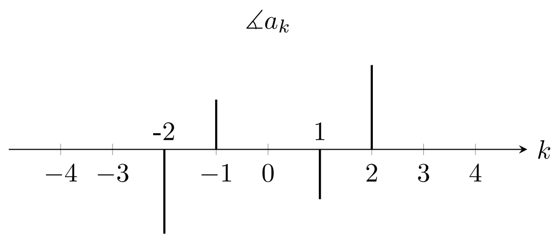

Figure 2: Plot for phase of a_k

From Figure(1) and (2), viz. the magnitude and phase plots of a_k, i.e. the complex coefficients of fourier series, we can see that the plots are symmetrically distributed. We can draw out the following properties of the fourier series spectrum a_k,

Property 1

If $x(t)$ is a real-valued periodic signal, then, $a_k = {a_{-k}}^*$, i.e. the fourier coefficients follow **conjugate symmetry**. This is evident from the values of a_k determined above.

Property 2

If $x(t) = x(-t)$, i.e. the signal has even-symmetry about the origin, then, $a_k = a_{-k}$. This is true since the fourier coefficients of even signals are real-valued and the fourier expansion of a real-valued fourier coefficient results in only the cosine terms, which is the simplest form of an even signal.

Property 3

If $x(t) = -x(-t)$, i.e. the signal has odd-symmetry about the origin, then, $a_k = -a_{-k}$. This is true since the fourier coefficients of odd signals are purely imaginary and the fourier expansion of a purely imaginary fourier coefficient results in only the sine terms, which is the simplest form of an odd signal.