Filter Design Home Assignment #1

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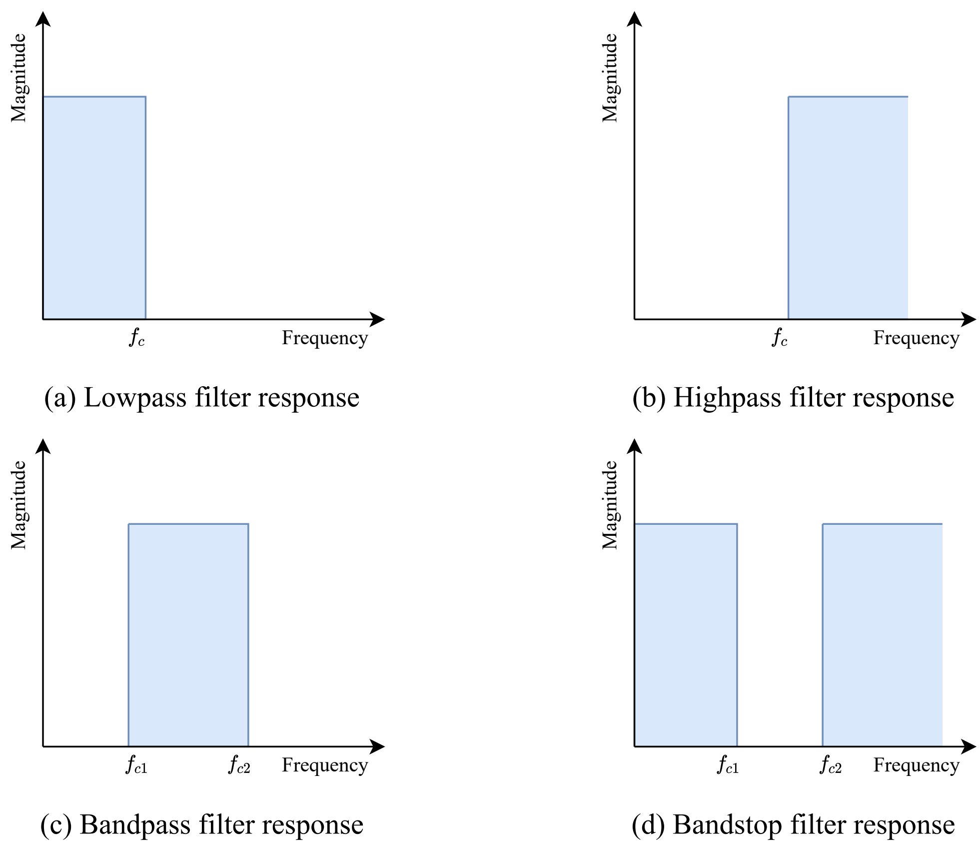
PUL074BEX007

Problem 1

What is an ideal and practical filter? Mention the applications of filters with suitable examples.

An ideal filter is one that has unity gain $(H = 1)$ in the passband and zero gain $(H = 0)$ in the stop band. In other words, the passband attenuation is zero $(\alpha = 0)$ and stopband attenuation is infinite $(\alpha \to \infty)$. This means that the transition between the passband and stopband is sharp and the response is called a brick-wall response due to its sharp shape.



(a) Lowpass filter response

(b) Highpass filter response

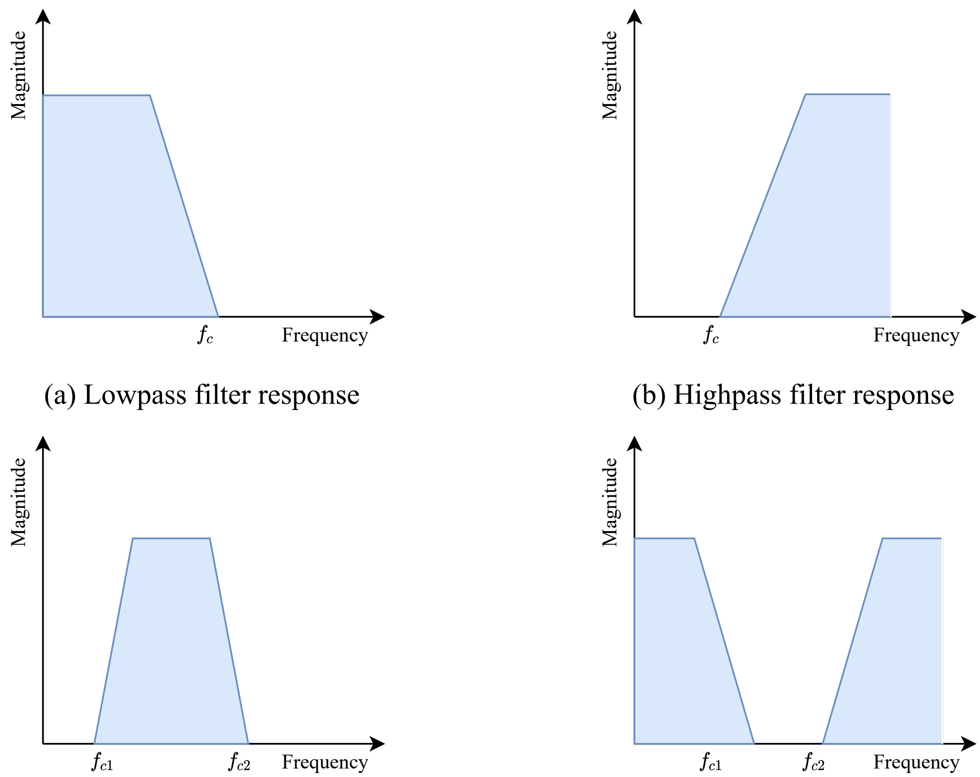
(c) Bandpass filter response

(d) Bandstop filter response

Figure 1: Ideal filter responses

The responses for different filters shown in Figure 1 are not practically achievable since the

system to produce such response need to be non-causal, which is not possible to realize.
A practical filter is one that does not have unity gain $(H \neq 1)$ in the passband and non-zero gain $(H \neq 0)$ in the stop band. In other words, the passband attenuation has some attenuation and the stopband has a small value of attenuation. The transition between the passband and stopband is not sharp, rather a gradual decay is seen in the transition band.



(a) Lowpass filter response

(b) Highpass filter response

(c) Bandpass filter response

(d) Bandstop filter response

Figure 2: Practical filter responses

However, the responses for different filters shown in Figure 2 are not the true form that they exist in. The flat passband shown in the responses can have attenuation ripples. Similarly, the transition in reality, is not a straight line and there may exist ripple in the stopband as well. For a general lowpass filter, the response takes the form shown in Figure 3.

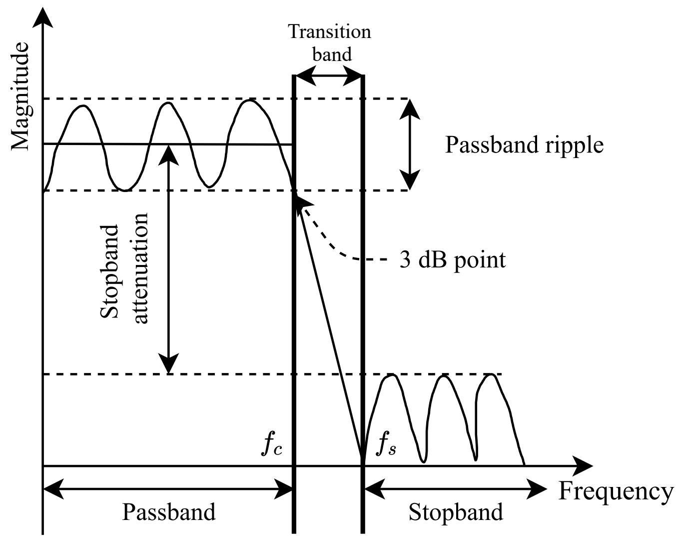


Figure 3: Actual practical response of a lowpass filter

The applications of filters are listed below:

1. Filters are an important component in frequency domain analysis of a signal.

2. Filters are used in various audio systems for pre-amplification and equalization. They are also used in multi-way loudspeakers to distribute different audio frequencies to different speakers for tone control.

3. Filters are used in image frequency rejection in radio, television and satellite receivers.

4. During telephone communication, filters are used to suppress background noise.

5. Filters are used for satellite communication where the used carrier signal is analog in nature.

Problem 2

What is normalization and denormalization? Explain its importance in filter design.

Normalization is a technique used while designing filters that makes calculation and design simple to follow. Normalization simply means designing the filter with passband cutoff point (half-power frequency) at $\omega = 1$ rad/s. It is also called standardization or prototyping. The importance of normalization process can be explained better with the reactance equations as,

A normalized filter at $\omega = 1$ rad/s i.e. $f = \frac{1}{2\pi}$ has simple values of inductive and capacitive reactances as,

$$X_L = L \qquad \text{and} \qquad X_C = \frac{1}{C}$$

Any filter that isn't designed at $\omega = 1$ rad/s would have an additional $2\pi f$ factor as,

$$X_L = 2\pi f L \qquad \text{and} \qquad X_C = \frac{1}{2\pi f C}$$

From these equations for the impedances, it is clear that the normalized filter will be easier to design. However, in reality, we require filters at different frequency ranges, so normalization is incomplete design step. To achieve filters at required frequency, designer must scale the filter in frequency. A point to note is that once the filter is scaled in frequency, it may not have elemental values that are practically available, which is why impedance scaling is required to get the desired filter response with practically realizable components. Hence these two scaling procedures are called denormalization as the prototype filter is redesigned at required specifications.

To sum up, normalization and denormalization are filter design steps that assist in simpler design of filters, first at $\omega = 1$ rad/s (normalization) and then redesign with required parameters and practically realizable elements (denormalization).

Problem 3

Derive the expressions to calculate the order of given filters for lowpass specifications:

a. Butterworth b. Chebyshev c. Inverse Chebyshev

Using your expressions calculate the order for each of the filters for following lowpass specifications.

$$\alpha_{max} = 1.5 \text{ dB} \qquad \alpha_{min} = 16 \text{ dB}$$

$$\omega_p = 2000 \text{ rad/s} \qquad \omega_s = 3000 \text{ rad/s}$$

We'll derive the equations for order of different lowpass filters with specifications as,

$$\begin{cases} \text{Passband : } \omega = 0 \text{ to } \omega = \omega_p \text{ and } \alpha = \alpha_{max} \\ \text{Stopband : } \omega = \omega_s \text{ to } \infty \text{ and } \alpha = \alpha_{min} \\ \text{Transition band : } \omega_p \text{ to } \omega_s \end{cases}$$

a. Butterworth

The transfer function of a butterworth filter is given as,

$$|T(j\omega)|^2 = \frac{1}{1 + \left(\dfrac{\omega}{\omega_o}\right)^{2n}} \tag{1}$$

For attenuation α, we have,

$$\alpha = 10 \log \left(1 + \left(\frac{\omega}{\omega_o}\right)^{2n}\right)$$

$$\Rightarrow 1 + \left(\frac{\omega}{\omega_o}\right)^{2n} = 10^{\alpha/10}$$

$$\therefore \left(\frac{\omega}{\omega_o}\right)^{2n} = 10^{\alpha/10} - 1 \tag{2}$$

For passband, $\omega = \omega_p$ and $\alpha = \alpha_{max}$, so,

$$\left(\frac{\omega_p}{\omega_o}\right)^{2n} = 10^{\alpha_{max}/10} - 1 \tag{3}$$

For stopband, $\omega = \omega_s$ and $\alpha = \alpha_{min}$, so,

$$\left(\frac{\omega_s}{\omega_o}\right)^{2n} = 10^{\alpha_{min}/10} - 1 \tag{4}$$

Dividing Equation 4 by Equation 3, we get,

$$\left(\frac{\omega_s}{\omega_p}\right)^{2n} = \frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1} \tag{5}$$

Taking log on both sides on Equation 5, we get,

$$\log\left(\frac{\omega_s}{\omega_p}\right)^{2n} = \log\left(\frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1}\right)$$

$$\Rightarrow 2n\log\left(\frac{\omega_s}{\omega_p}\right) = \log\left(\frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1}\right)$$

$$\therefore n = \frac{\log\left(\frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1}\right)}{2\log\left(\frac{\omega_s}{\omega_p}\right)} \tag{6}$$

Equation 6 is the required expression for the order of butterworth filter for lowpass specifications.

Given,

$$\alpha_{max} = 1.5 \text{ dB} \qquad \alpha_{min} = 16 \text{ dB}$$

$$\omega_p = 2000 \text{ rad/s} \qquad \omega_s = 3000 \text{ rad/s}$$

Order n can be calculated using Equation 6 as,

$$n = \frac{\log\left(\frac{10^{16/10} - 1}{10^{1.5/10} - 1}\right)}{2\log\left(\frac{3000}{2000}\right)} = \frac{1.9735}{2 \times 0.1761} = 5.603 \approx 6$$

Hence the order of the butterworth lowpass filter for given specifications is $n = 6$.

b. Chebyshev

The transfer function of a chebyshev filter is given as,

$$|T(j\omega)|^2 = \frac{1}{1 + \epsilon^2 C_n^2(\omega)} \tag{7}$$

For attenuation α, we have,

$$\begin{aligned}\alpha &= 10\log\left(1+\epsilon^2 C_n^2(\omega)\right)\\ \Rightarrow 1+\epsilon^2 C_n^2(\omega) &= 10^{\alpha/10}\\ \therefore \epsilon^2 C_n^2(\omega) &= 10^{\alpha/10}-1 \end{aligned}\tag{8}$$

For passband, $\alpha = \alpha_{max}$ occurs when $C_n^2(\omega) = 1$ so,

$$\epsilon^2 = 10^{\alpha_{max}/10}-1 \tag{9}$$

For stopband, $\omega = \omega_s$ and $\alpha = \alpha_{min}$, so,

$$\begin{aligned}&\epsilon^2 C_n^2(\omega_s) = 10^{\alpha_{min}/10}-1\\ &\Rightarrow \epsilon^2\left(\cosh\left(n\cosh^{-1}\omega_s\right)\right)^2 = 10^{\alpha_{min}/10}-1\end{aligned}\tag{10}$$

Substituting value of ϵ^2 from Equation 9 into Equation 10, we get,

$$\begin{aligned}&\left(10^{\alpha_{max}/10}-1\right)\left(\cosh\left(n\cosh^{-1}\omega_s\right)\right)^2 = 10^{\alpha_{min}/10}-1\\ &\Rightarrow \cosh\left(n\cosh^{-1}\omega_s\right) = \sqrt{\frac{10^{\alpha_{min}/10}-1}{10^{\alpha_{max}/10}-1}}\\ &\Rightarrow n\cosh^{-1}\omega_s = \cosh^{-1}\sqrt{\frac{10^{\alpha_{min}/10}-1}{10^{\alpha_{max}/10}-1}}\\ &\therefore n = \frac{\cosh^{-1}\sqrt{\frac{10^{\alpha_{min}/10}-1}{10^{\alpha_{max}/10}-1}}}{\cosh^{-1}\omega_s}\end{aligned}\tag{11}$$

If ω_p is given, Equation 11 becomes,

$$n = \frac{\cosh^{-1}\sqrt{\frac{10^{\alpha_{min}/10}-1}{10^{\alpha_{max}/10}-1}}}{\cosh^{-1}\left(\frac{\omega_s}{\omega_p}\right)}\tag{12}$$

Equation 12 is the required expression for the order of chebyshev filter for lowpass specifications.

Given,

$$\alpha_{max} = 1.5 \text{ dB} \qquad \alpha_{min} = 16 \text{ dB}$$

$$\omega_p = 2000 \text{ rad/s} \qquad \omega_s = 3000 \text{ rad/s}$$

Order n can be calculated using Equation 12 as,

$$\begin{aligned} n = \frac{\cosh^{-1}\sqrt{\dfrac{10^{\alpha_{min}/10}-1}{10^{\alpha_{max}/10}-1}}}{\cosh^{-1}\left(\dfrac{\omega_s}{\omega_p}\right)} &= \frac{\cosh^{-1}\sqrt{\dfrac{10^{16/10}-1}{10^{1.5/10}-1}}}{\cosh^{-1}\left(\dfrac{3000}{2000}\right)} \\ &= \frac{\cosh^{-1} 9.699}{\cosh^{-1} 1.5} \\ &= 3.078 \approx 4 \end{aligned}$$

Hence the order of the chebyshev lowpass filter for given specifications is $n = 4$.

c. Inverse Chebyshev

The transfer function of an inverse chebyshev filter is given as,

$$|T(j\omega)|^2 = \frac{1}{1 + \dfrac{1}{\epsilon^2 C_n^2(1/\omega)}} \tag{13}$$

For attenuation α, we have,

$$\begin{aligned} \alpha &= 10 \log\left(1 + \frac{1}{\epsilon^2 C_n^2(1/\omega)}\right) \\ \Rightarrow 1 + \frac{1}{\epsilon^2 C_n^2(1/\omega)} &= 10^{\alpha/10} \\ \therefore \epsilon^2 C_n^2(1/\omega) &= \frac{1}{10^{\alpha/10}-1} \end{aligned} \tag{14}$$

For passband, $\alpha = \alpha_{max}$ and $\omega = \omega_p$ so,

$$\begin{aligned} \epsilon^2 C_n^2(1/\omega_p) &= \frac{1}{10^{\alpha_{max}/10}-1} \\ \Rightarrow \epsilon^2 \left(\cosh\left(n \cosh^{-1}(1/\omega_p)\right)\right)^2 &= \frac{1}{10^{\alpha_{max}/10}-1} \end{aligned} \tag{15}$$

For stopband, $\alpha = \alpha_{min}$ occurs when $C_n^2(1/\omega) = 1$ so,

$$\epsilon^2 = \frac{1}{10^{\alpha_{min}/10} - 1} \tag{16}$$

Substituting value of ϵ^2 from Equation 16 into Equation 15, we get,

$$\frac{1}{10^{\alpha_{min}/10} - 1} \left(\cosh\left(n \cosh^{-1}\left(1/\omega_p\right)\right)\right)^2 = \frac{1}{10^{\alpha_{max}/10} - 1}$$

$$\Rightarrow \cosh\left(n \cosh^{-1}\left(1/\omega_p\right)\right) = \sqrt{\frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1}}$$

$$\Rightarrow n \cosh^{-1}\left(1/\omega_p\right) = \cosh^{-1} \sqrt{\frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1}}$$

$$\Rightarrow n = \frac{\cosh^{-1} \sqrt{\frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1}}}{\cosh^{-1}\left(1/\omega_p\right)} \tag{17}$$

If ω_s is given, Equation 17 becomes,

$$n = \frac{\cosh^{-1} \sqrt{\frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1}}}{\cosh^{-1}\left(\frac{\omega_s}{\omega_p}\right)} \tag{18}$$

Equation 18 is the required expression for the order of inverse chebyshev filter for lowpass specifications. Given,

$$\alpha_{max} = 1.5 \text{ dB} \qquad \alpha_{min} = 16 \text{ dB}$$

$$\omega_p = 2000 \text{ rad/s} \qquad \omega_s = 3000 \text{ rad/s}$$

Order n can be calculated using Equation 18 as,

$$\begin{aligned} n = \frac{\cosh^{-1} \sqrt{\frac{10^{\alpha_{min}/10} - 1}{10^{\alpha_{max}/10} - 1}}}{\cosh^{-1}\left(\frac{\omega_s}{\omega_p}\right)} &= \frac{\cosh^{-1} \sqrt{\frac{10^{16/10} - 1}{10^{1.5/10} - 1}}}{\cosh^{-1}\left(\frac{3000}{2000}\right)} \\ &= \frac{\cosh^{-1} 9.699}{\cosh^{-1} 1.5} = 3.078 \approx 4 \end{aligned}$$

Hence the order of the inverse chebyshev lowpass filter for given specifications is $n = 4$.

Problem 4

Show that the locus of poles of the Chebyshev filter is elliptic.

The transfer function of a chebyshev filter is given as,

$$|T(j\omega)|^2 = \frac{1}{1+\epsilon^2 C_n^2(\omega)} \tag{19}$$

We also have,

$$|T(j\omega)|^2 = T(s)T(-s)|_{s=j\omega} \tag{20}$$

Equating Equation 19 and Equation 20, we get,

$$T(s)T(-s) = \frac{1}{1+\epsilon^2 C_n^2\left(\frac{s}{j}\right)} \tag{21}$$

The pole locations are calculated by equating the denominator of Equation 21 as,

$$\begin{aligned} 1+\epsilon^2 C_n^2\left(\frac{s}{j}\right) &= 0 \\ \Rightarrow \epsilon^2 C_n^2\left(\frac{s}{j}\right) &= -1 \\ \Rightarrow C_n^2\left(\frac{s}{j}\right) &= \frac{j^2}{\epsilon^2} \\ \therefore C_n\left(\frac{s}{j}\right) &= \pm\frac{j}{\epsilon} \end{aligned} \tag{22}$$

For $\omega << 1$, the chebyshev polynomial $C_n(\omega)$ takes the form, $C_n(\omega) = \cos(n\cos^{-1}\omega)$, so,

$$C_n\left(\frac{s}{j}\right) = \cos\left(n\cos^{-1}\left(\frac{s}{j}\right)\right) \tag{23}$$

Since $\cos^{-1}\left(\frac{s}{j}\right)$ is a complex number, say,

$$\cos^{-1}\left(\frac{s}{j}\right) = u + jv \tag{24}$$

From Equation 23 and Equation 24, we get,

$$C_n\left(\frac{s}{j}\right) = \cos\left(n(u + jv)\right) = \cos\left(nu + jnv)\right)$$

$$\therefore C_n\left(\frac{s}{j}\right) = \cos nu \cosh nv - j \sin nu \sinh nv \tag{25}$$

Comparing Equation 22 and Equation 25, we get,

$$\cos nu \cosh nv = 0 \tag{26}$$

$$\sin nu \sinh nv = \pm\left(\frac{1}{\epsilon}\right) \tag{27}$$

For Equation 26 to be true, $\cosh nv \neq 0$ since the smallest value it can have is 1, so, the following must be true.

$$\cos nu = 0$$

$$\Rightarrow u_k = \frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n} \cdots$$

$$\therefore u_k = \frac{(2k-1)\pi}{2n} \quad [k = 1, 2, 3, \cdots, n] \tag{28}$$

For the value of u_k given by Equation 28, $\sin nu_k = \pm 1$. So from Equation 27, we get,

$$\sinh nv_k = \pm\left(\frac{1}{\epsilon}\right)$$

$$\Rightarrow nv_k = \sinh^{-1}\left(\pm\frac{1}{\epsilon}\right)$$

$$\therefore v_k = \pm\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right) = \pm a$$

Now from Equation 24 we have,

$$\cos^{-1}\left(\frac{s_k}{j}\right) = u_k + jv_k$$

$$\Rightarrow s_k = j\cos(u_k + jv_k) = j\cos\left(\frac{(2k-1)\pi}{2n} \pm ja\right)$$

$$\Rightarrow s_k = j\left[\cos\left(\frac{(2k-1)\pi}{2n}\right)\cosh a \pm j\sin\left(\frac{(2k-1)\pi}{2n}\right)\sinh a\right]$$

$$\therefore s_k = \pm\sin\left(\frac{(2k-1)\pi}{2n}\right)\sinh a + j\cos\left(\frac{(2k-1)\pi}{2n}\right)\cosh a \tag{29}$$

Consider the general equation for s_k as,

$$s_k = \sigma_k + j\beta_k \tag{30}$$

Comparing Equation 29 and Equation 30, we get,

$$\sigma_k = \pm \sin\left(\frac{(2k-1)\pi}{2n}\right) \sinh a \tag{31}$$

Considering only negative pole,

$$\begin{aligned} \sigma_k &= -\sin\left(\frac{(2k-1)\pi}{2n}\right) \sinh a \\ \therefore \frac{\sigma_k}{\sinh a} &= -\sin\left(\frac{(2k-1)\pi}{2n}\right) \end{aligned} \tag{32}$$

$$\begin{aligned} \beta_k &= \cos\left(\frac{(2k-1)\pi}{2n}\right) \cosh a \\ \therefore \frac{\beta_k}{\cosh a} &= \cos\left(\frac{(2k-1)\pi}{2n}\right) \end{aligned} \tag{33}$$

Squaring and adding Equation 32 and Equation 33, we get,

$$\begin{aligned} \left(\frac{\sigma_k}{\sinh a}\right)^2 + \left(\frac{\beta_k}{\cosh a}\right)^2 &= \sin^2\left(\frac{(2k-1)\pi}{2n}\right) + \cos^2\left(\frac{(2k-1)\pi}{2n}\right) \\ \therefore \left(\frac{\sigma_k}{\sinh a}\right)^2 + \left(\frac{\beta_k}{\cosh a}\right)^2 &= 1 \end{aligned} \tag{34}$$

Equation 34 is an equation for ellipse with,

$$\text{Major semi-axis} = \cosh a$$

$$\text{Minor semi-axis} = \sinh a$$

$$\text{Focii location} = \pm j$$

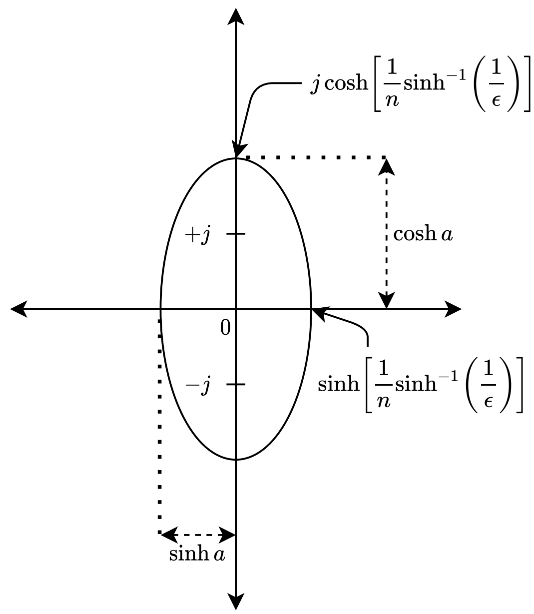


Figure 4: Pole location for chebyshev lowpass filter

Problem 5

What is a constant delay filter? Where is it important? What are the techniques of getting a filter with almost constant delay? Explain.

A constant delay filter is a filter that gives output delayed in time from input signal. The input and output signal differs by constant time hence the name constant delay filter.

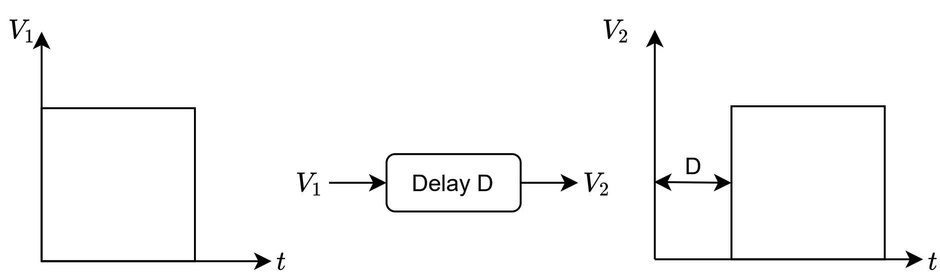


Figure 5: Constant delay filter

The relation between output and input is thus given as,

$$V_2(t) - V_1(t - D)$$

If $V_1(t) = A\sin(\omega t + \Phi)$ then $V_2(t) = A\sin(\omega t - \omega D + \Phi)$. So, the two signal differ by phase angle of $\theta = -\omega D$, i.e. $\frac{V_2}{V_1} = 1\angle - \omega D$. Hence the transfer function is given as,

$$|T| = \frac{V_2}{V_1} = e^{-j\omega D} = e^{-sD}$$

For normalized delay $D = 1$,

$$|T| = e^{-s} \tag{35}$$

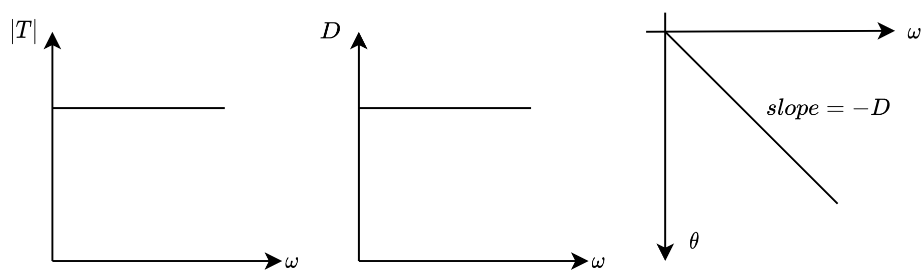


Figure 6: Requirement for constant delay and no distortion

For linear phase with negative slope and constant magnitude, the delay will be constant. Similarly, the obtained signal will be distortionless. Such filter is important in signal processing application for mostly analog video signals and other applications where a constant group delay is prioritized rather than the response.

Since the transfer function given in Equation 35 can't be realized with lumped elements, the design is approximated in quotient of polynomials. The techniques used to approximate the constant delay filters are explained in brief.

Bessel-Thomson

In this method, an all pole transfer function is assumed, using which the corresponding delay is calculated. Then the series is expanded using Taylor series and vanishing conditions for the coefficients are determined to complete the approximation.

If we assume an all pole second order transfer function as,

$$
\begin{aligned}
&T_2(s) = \frac{a_0}{s^2 + a_1 s + a_0} \\
&\Rightarrow T_2(j\omega) = \frac{a_0}{-\omega^2 + ja_1\omega + a_0} \\
&\therefore \theta = -\tan^{-1}\left(\frac{a_1\omega}{a_0 - \omega^2}\right) \\
&\because D = \frac{-d\theta}{d\omega} = \frac{d}{d\omega}\left[tan^{-1}\left(\frac{a_1\omega}{a_0 - \omega^2}\right)\right] \\
&\Rightarrow D = \frac{d\left[tan^{-1}\left(\frac{a_1\omega}{a_0 - \omega^2}\right)\right]}{d\left[\frac{a_1\omega}{a_0 - \omega^2}\right]} \times \frac{d\left[\frac{a_1\omega}{a_0 - \omega^2}\right]}{d\omega} \\
&\Rightarrow D = \frac{1}{1 + \left(\frac{a_1\omega}{a_0 - \omega^2}\right)^2} \times \frac{\left(a_0 - \omega^2\right)\frac{d(a_1\omega)}{d\omega} - a_1\omega\frac{d(a_0 - \omega^2)}{d\omega}}{(a_0 - \omega^2)^2} \\
&\Rightarrow D = \frac{(a_0 - \omega^2)^2}{(a_0 - \omega^2)^2 + (a_1\omega)^2} \times \frac{a_1(a_0 - \omega^2) - a_1\omega(-2\omega)}{(a_0 - \omega^2)^2} \\
&\Rightarrow D = \frac{a_1(a_0 - \omega^2) + 2a_1\omega^2}{(a_0 - \omega^2)^2 + (a_1\omega)^2} \\
&\Rightarrow D = \frac{a_1(a_0 + \omega^2)}{a_0^2 + (a_1^2 - 2a_0)\omega^2 + \omega^4} \\
&\Rightarrow D = \left(\frac{a_1}{a_0}\right) \times \frac{1 + \frac{\omega^2}{a_0}}{1 + \left(\frac{a_1^2}{a_0^2} - \frac{2}{a_0}\right)\omega^2 + \frac{\omega^4}{a_0^2}}
\end{aligned}
$$

Using taylor series to expand as,

$$D = \left(\frac{a_1}{a_0}\right)\left[1 + \left(\frac{1}{a_0} - \frac{a_1^2}{a_0^2} + \frac{2}{a_0}\right)\omega^2 + \cdots\right]$$

The condition for the second term to vanish is,

$$\Rightarrow \frac{3}{a_0} = \frac{a_1^2}{a_0^2}$$

$$\Rightarrow 3a_0 = a_1^2$$

$$\Rightarrow a_0 = a_1 = 3 \quad \text{[To normalize D, set } a_0 = a_1\text{]}$$

Using these values, we get,

$$T_2(s) = \frac{3}{s^2 + 3s + 3}$$

The corresponding delay is given as,

$$D_2(\omega) = \frac{\left(1 + \frac{\omega^2}{3}\right)}{\left(1 + \frac{\omega^2}{3} + \frac{\omega^4}{9}\right)} = \frac{3\omega^2 + 9}{\omega^4 + 3\omega^2 + 9}$$

Storch Method

In this method, the transfer function in equation 35 is approximated using hyperbolic expansion as,

$$T(s) = e^{-s} = \frac{1}{e^s}$$

$$\Rightarrow T(s) = \frac{1}{\sinh s + \cosh s}$$

$$\Rightarrow T(s) = \frac{\frac{1}{\sinh s}}{\frac{\sinh s + \cosh s}{\sinh s}}$$

The hyperbolic expansion is given as,

$$\cosh s = 1 + \frac{s^2}{2!} + \frac{s^4}{4!} + \frac{s^6}{6!} + \cdots$$

$$\sinh s = 1 + \frac{s^3}{3!} + \frac{s^5}{5!} + \frac{s^7}{7!} + \cdots$$

To obtain $\coth s$, we divide $\cosh s$ by $\sinh s$ then inverting and repeating the division as,

$$\coth s = \frac{1}{s} + \cfrac{1}{\cfrac{3}{s} + \cfrac{1}{\cfrac{5}{s} + \cfrac{1}{\cfrac{7}{s} + \cdots}}}$$

For n take n steps, i.e. for $n = 2$, we take two steps as,

$$\coth s = \frac{1}{s} + \frac{1}{\frac{3}{s}} = \frac{1}{s} + \frac{s}{3} = \frac{s^2+3}{3s}$$

Adding numerator and denominator, we get,

$$D_2(s) = s^2 + 3s + 3$$

The transfer function is then given as,

$$T_2(s) = \frac{D_2(0)}{D_2(s)} = \frac{3}{s^2 + 3s + 3}$$

Problem 6

What is frequency transformation? What is its importance in filter design? How can you obtain a bandstop filter from a given lowpass normalized filter? Explain with a suitable example.

Frequency transformation is a tool used to rearrange points on the j-axis to achieve different filtering characteristics. It is extensively used for obtaining different filters (highpass, bandpass, bandstop) from lowpass filter designed at normalized condition. Since direct design of such filter characteristics is not an easy task, filters are initially prototyped in lowpass nature at normalized frequency of 1 rad/s. But in reality we require filters with different characteristics with different bandwidth, and central frequency which is where frequency transformation is essential.

For a normalized lowpass filter with pass band $0 < \omega < \omega_p$, where $\omega_p = 1$ and network function is noted as $H_{LP}(\tilde{s})$. For frequency transformation to bandstop filter with bandwidth B, we use the relation,

$$\tilde{s} = \frac{Bs}{s^2 + {\Omega_0}^2}$$

The bandstop network function can then be written as,

$$H_{BS}(s) = H_{LP}\left(\frac{Bs}{s^2 + {\Omega_0}^2}\right) \tag{36}$$

Here, $B = \Omega_2 - \Omega_1$ and ${\Omega_0}^2 = \Omega_2\Omega_1$.

The transformation for the different circuit elements can be calculated as:

Inductor

The impedance of an inductor in the normalized lowpass filter is given as,

$$|Z_L|_{LP} = \tilde{s}.L_i$$

For easier solution, using the relation for admittance of inductor as,

$$|Y_L|_{LP} = \frac{1}{\tilde{s}.L_i}$$

After transformation, the admittance becomes,

$$|Y_L|_{BS} = \frac{1}{\left(\frac{Bs}{s^2+{\Omega_0}^2}\right) L_i} = \frac{s^2 + {\Omega_0}^2}{BsL_i}$$

$$\therefore |Y_L|_{BS} = s\left(\frac{1}{BL_i}\right) + \frac{1}{s\left(\frac{BL_i}{{\Omega_0}^2}\right)} \tag{37}$$

Capacitor

The impedance of a capacitor in the normalized lowpass filter is given as,

$$|Z_C|_{LP} = \frac{1}{\tilde{s}.C_i}$$

After transformation, the impedance becomes,

$$|Z_C|_{BS} = \frac{1}{\left(\frac{Bs}{s^2+{\Omega_0}^2}\right)C_i} = \frac{s^2+{\Omega_0}^2}{BsC_i}$$

$$\therefore |Z_C|_{BS} = s\left(\frac{1}{BC_i}\right) + \frac{1}{s\left(\frac{BC_i}{{\Omega_0}^2}\right)} \tag{38}$$

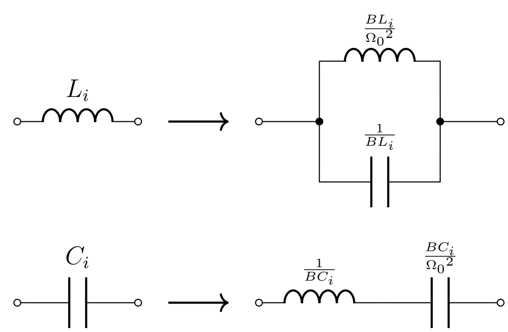


Figure 7: Lowpass to bandstop transformation

Example

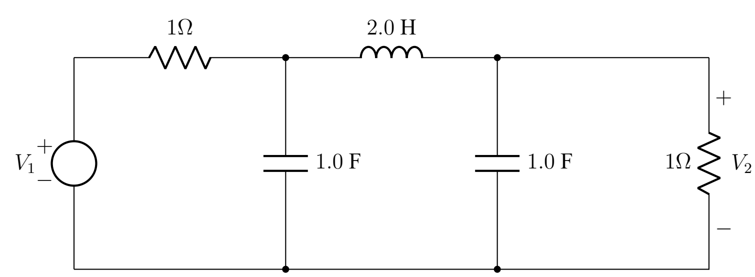


Figure 8: Normalized lowpass filter

Here,

Half power frequency of given normalized filter $(\omega) = 1$ rad/s

Required center frequency for bandstop filter $(\Omega_0) = 40000$ rad/s

Required bandwidth for bandstop filter $(B) = 4000$ rad/s

While using frequency transformation to design a bandstop filter from a normalized lowpass filter, the inductor in the original circuit is converted to a parallel LC combination and a capacitor is converted to a series LC combination. The relation is given in Equation 37 and 38.

Inductance in given normalized filter $(L_{LP}) = 2$ H

Capacitance in given normalized filter $(C_{LP}) = 1$ F

For mapped elements in bandstop filter arising due to inductance in normalized filter:

Parallel inductance $(L^p_{BS}) = \dfrac{BL_{LP}}{{\Omega_0}^2} = \dfrac{4000 \times 2}{40000^2} = 5 \times 10^{-6}$ H

Parallel capacitance $(C^p_{BS}) = \dfrac{1}{BL_{LP}} = \dfrac{1}{4000 \times 2} = 1.25 \times 10^{-4}$ F

For mapped elements in bandstop filter arising due to capacitance in normalized filter:

Series inductance $(L^s_{BS}) = \dfrac{1}{BC_{LP}} = \dfrac{1}{4000 \times 1} = 2.5 \times 10^{-4}$ H

Series capacitance $(C^s_{BS}) = \dfrac{BC_{LP}}{{\Omega_0}^2} = \dfrac{4000 \times 1}{40000^2} = 2.5 \times 10^{-6}$ F

Since the elemental values are not practically realizable, using the impedance scaling factor $K_m = 4000$, we get,

$R_{new} = K_m.R_{old} = 4000 \times 1 = 4000\ \Omega = 4$ KΩ

$L^s_{new} = K_m.L^s_{old} = 4000 \times 2.5 \times 10^{-4} = 5 \times 10^{-3}$ H $= 1$ H

$L^p_{new} = K_m.L^p_{old} = 4000 \times 5 \times 10^{-6} = 5 \times 10^{-3}$ H $= 20$ mH

$C^s_{new} = \dfrac{C^s_{old}}{K_m} = \dfrac{2.5 \times 10^{-6}}{4000} = 0.625$ nF

$C^p_{new} = \dfrac{C^p_{old}}{K_m} = \dfrac{1.25 \times 10^{-4}}{4000} = 31.25$ nF

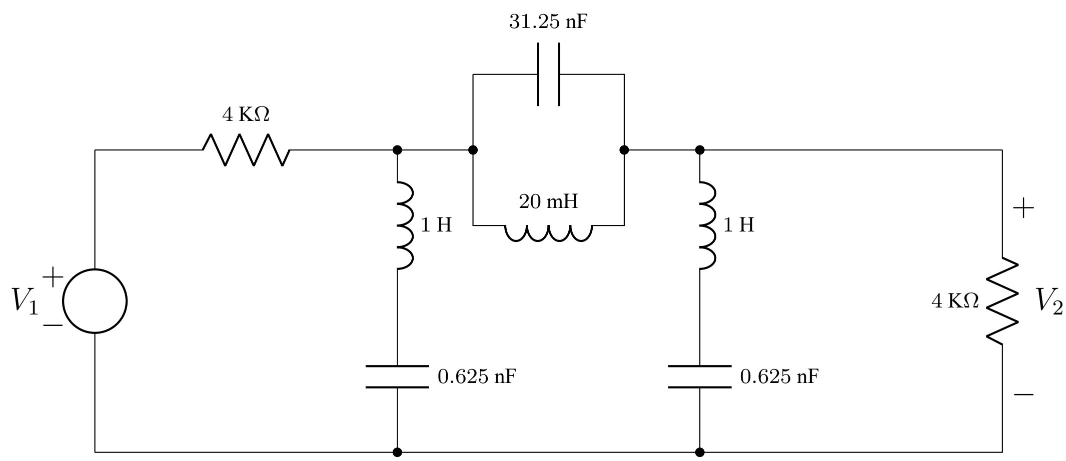


Figure 9: Final circuit for bandstop filter designed from normalized lowpass filter

Problem 7

What are the required properties to check whether the given function is lossless oneport network function or not? Explain with examples.

The properties that are required to check if a given function is lossless oneport network function are,

1. The driving point function, (impedance function or admittance function), $Z(s)$ or $Y(s)$ is the ratio of even polynomial to odd polynomial degree or vice versa.
2. There must be either a pole or zero at origin and infinity.
3. The highest degree of numerator and denominator must differ by at most unity. Similar condition is necessary for the lowest power.
4. The succeeding powers of both numerator and denominator polynomials must differ by two.
5. All the poles and zeros must lie on the imaginary axis and must alternate.

6. The reciprocal of a lossless circuit is also lossless.

An important point to note is that if the poles and zeros lie on the imaginary axis and they alternate, then it is the sufficient condition for the function to be lossless oneport network. Let us assume two driving point functions as,

$$Z_1(s) = \frac{5(s^2+1)(s^2+9)}{(s^2+4)(s^2+12)} \quad Z_2(s) = \frac{2(s^2+1)(s^2+9)}{s(s^2+4)}$$

Here, $Z_1(s)$ is not the ratio of even polynomial to odd polynomial, so it is simply rejected and hence is not a valid lossless function. However, $Z_2(s)$ is a ratio of even polynomial to odd polynomial, so it may be valid lossless function. We need to check the poles and zero plot as,

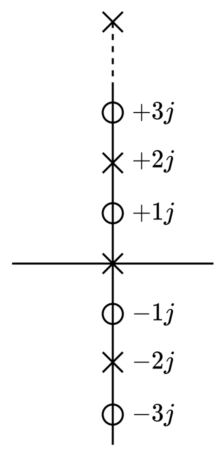


Figure 10: Pole and zero location for $Z_2(s)$

Problem 8

What are the zeros of transmission? How can they be realized in a network? Explain with suitable examples.

As the name suggests, zeros of transmission are the frequency group at which a considered two-port network's output is zero despite having a non-zero, finite input. Such network condition is

said be zero transmission. To realize zeros of transmission in a network, combination of network elements are employed such that input is prevented in reaching output by either shortening or opening the available transmission paths. Another way of attaining zero transmission is by cancelling out the different path signals such that zero output is observed.
For the admittance function given as,

$$Y(s) = \frac{2K_i s}{s^2 + \omega_i^2}$$

At pole frequency $s = j\omega_i$, $Y(s) = \infty$ and $Z(s) = 0$. Short circuit occurs in the short arm. For the impedance function given as,

$$Z(s) = \frac{2K_i s}{s^2 + \omega_i^2}$$

At pole frequency $s = j\omega_i$, $Z(s) = \infty$ and $Y(s) = 0$. Open circuit occurs in the series arm.

Problem 9

What is zero shifting by partial removal of a pole? Explain with a suitable example.

The partial removal of a pole means the removal of some fraction of the total network, removing which would not destroy the positive, real nature of the function.
A lossless function in partial form is given as,

$$Z(s) = Hs + \frac{K_o}{s} + \sum_{i=1}^{n} \frac{2K_i s}{s^2 + \omega_i^2} \tag{39}$$

Weakening pole at infinity

The term Hs in Equation 39 is contributed by pole at infinity. Since we are concerned with partial removal of this pole, we subtract a K_p fraction of Hs from $Z(s)$ as,

$$Z_1(s) = Z(s) - K_p Hs \quad [K_p < 1]$$

Since the zeros of $Z_1(s)$ are still located on the $j\omega$ axis, we substitute $s = j\omega$ to find them as,

$$X_1(\omega) = X(\omega) - K_p H\omega \quad [\because Z_1(j\omega) = jX_1(\omega)]$$

The zeros of $X_1(\omega)$ are the values of ω satisfying

$$X(\omega) = K_p H\omega$$

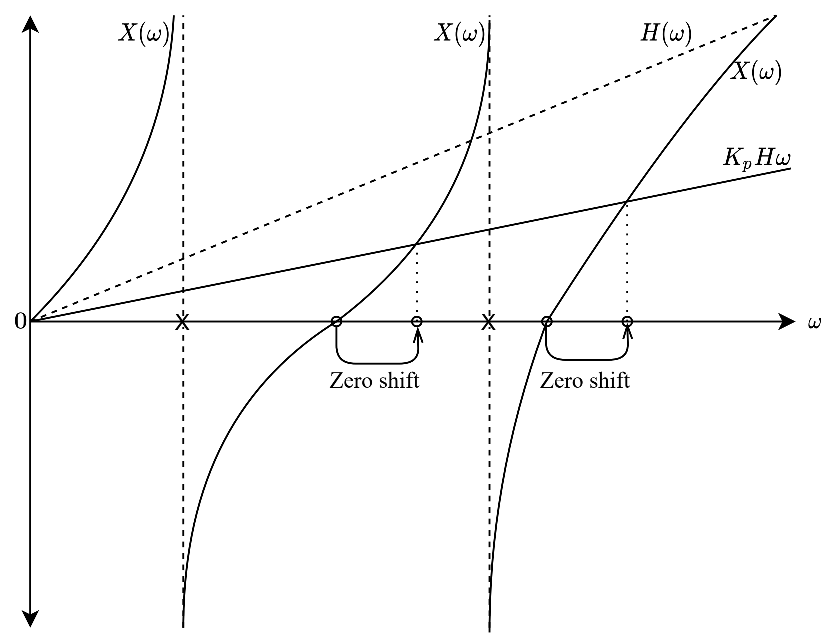


Figure 11: Zero shifting by weakening pole at infinity

Weakening pole at origin

The term $\frac{K_o}{s}$ in Equation 39 is contributed by pole at origin. Since we are concerned with partial removal of this pole, we subtract a K_p fraction of $\frac{K_o}{s}$ from $Z(s)$ as,

$$Z_2(s) = Z(s) - K_p \frac{K_o}{s} \quad [K_p < 1]$$

Similarly, the zeros of this function can be located by the intersection of $X(\omega)$ and $-K_p\left(\frac{K_o}{\omega}\right)$ with ω as they must satisfy,

$$X(\omega) = -K_p\left(\frac{K_o}{\omega}\right)$$

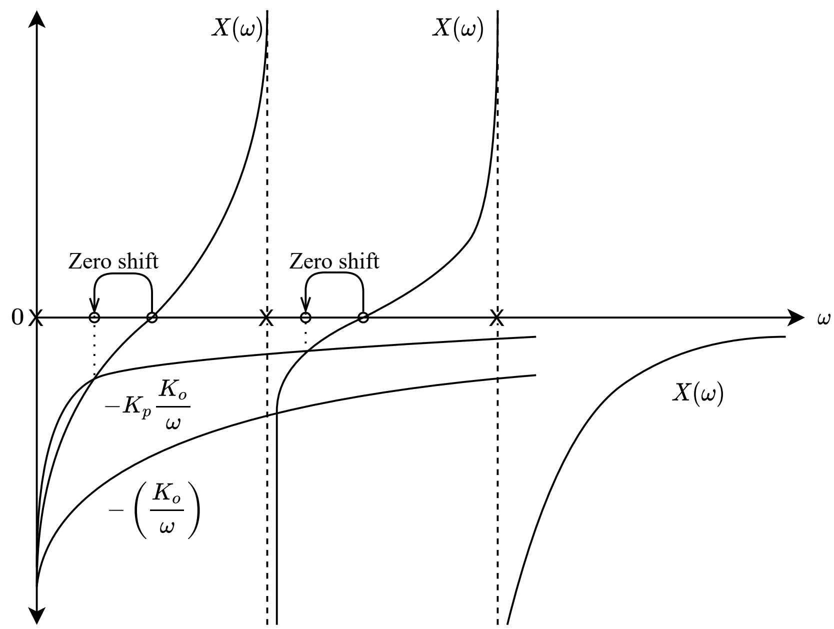


Figure 12: Zero shifting by weakening pole at origin

Weakening pole at complex conjugate pair

The term $\dfrac{2K_i s}{s^2+\omega_i^2}$ in Equation 39 is contributed by pole at a finite non-zero location. Since we are concerned with partial removal of this pole, we subtract a K_p fraction of $\dfrac{2K_i s}{s^2+\omega_i^2}$ from $Z(s)$ as,

$$Z_3(s) = Z(s) - K_p\frac{2K_i s}{s^2+\omega_i^2} \quad [K_p < 1]$$

Similarly, the zeros of this function can be located by the intersection of $X(\omega)$ and $-K_p\left(\frac{2K_i s}{\omega^2-\omega_i^2}\right)$ with ω as they must satisfy,

$$X(\omega) = -K_p\left(\frac{2K_i s}{\omega^2-\omega_i^2}\right)$$

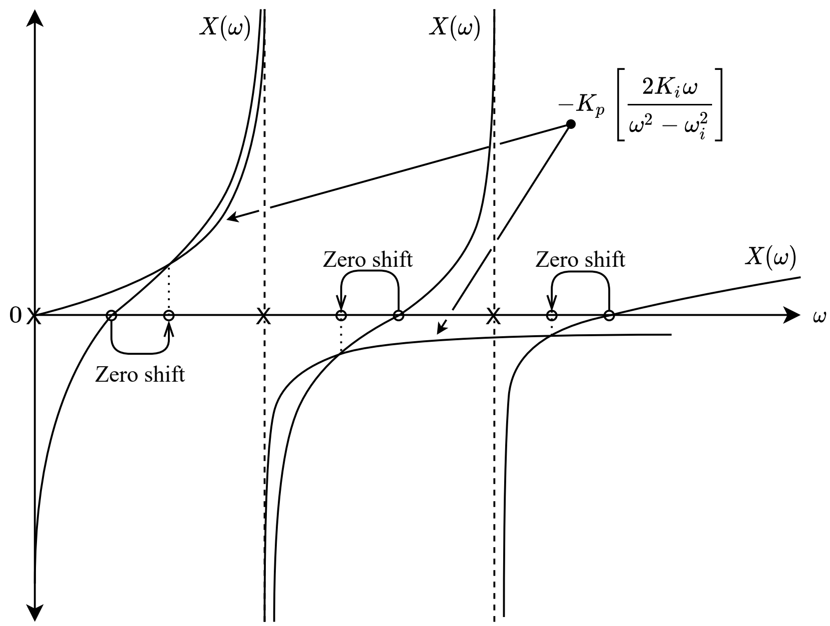


Figure 13: Zero shifting by weakening pole at complex conjugate pair

To summarize,

1. The partial removal of a pole shifts the zero towards that pole.
2. The zero shift amount depends on the value of K_p and proximity of zero to that pole.
3. The partial removal of pole at origin doesn't affect zero at infinity and vice versa.
4. A zero can never be shifted beyond an adjacent pole, rather it can be shifted a fraction of that distance.

Problem 10

What are the advantages of active filter over passive filter? Design an active filter having a pole at 1000 and a zero at 4000 with a dc gain of 4 using non-inverting configuration.

The advantages of active filter over a passive filter are compared on the following basis:

1. Loading effect: Active filters show no loading effect while passive filters have loading effect. A key advantage of no loading effect on active filters is that higher order filters can be obtained by cascading lower order active filters.

2. Gain: Unlike passive filters where the gain can be at most 1, active filters can have any value of gain.

3. Size and weight: Active filters have smaller size and weight since bulky inductors are not used and hence fabrication on smaller chips is possible.

Here,

Pole location $(p) = 1000$

Zero location $(z) = 4000$

DC gain $(K) = 4$

The transfer function can be written as,

$$T(s) = K\left(\frac{s+z}{s+p}\right)$$

We have, for realization of $T(s)$ using non inverting configuration shown in Figure 14,

$$1 + \frac{Z_f}{Z_{in}} = K\left(\frac{s+z}{s+p}\right) \tag{40}$$

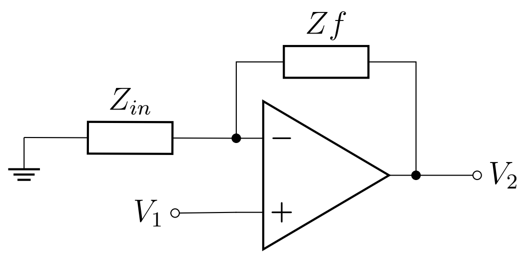


Figure 14: Non-inverting configuration

Substituting values in Equation 40, we get,

$$
\begin{aligned}
1+\frac{Z_f}{Z_{in}} &= 4\left(\frac{s+4000}{s+1000}\right)\\
\Rightarrow \frac{Z_f}{Z_{in}} &= \frac{4s+16000}{s+1000}-1\\
\Rightarrow \frac{Z_f}{Z_{in}} &= \frac{4s+16000-s-1000}{s+1000}\\
\Rightarrow \frac{Z_f}{Z_{in}} &= \frac{3s+15000}{s+1000}\\
\Rightarrow \frac{Z_f}{Z_{in}} &= \frac{3+\dfrac{15000}{s}}{1+\dfrac{1000}{s}}\\
\Rightarrow \frac{Z_f}{Z_{in}} &= \frac{3+\dfrac{1}{66.67\times10^{-6}s}}{1+\dfrac{1}{1\times10^{-3}s}}
\end{aligned}
$$

Comparing equivalent terms, we get,

$$
Z_f = 3+\frac{1}{66.67\times10^{-6}s} \qquad Z_{in} = 1+\frac{1}{1\times10^{-3}s}
$$

This can be realized with,

$$
R_f = 3\,\Omega \qquad C_f = 66.67\ \mu\text{F}
$$

$$
R_{in} = 1\,\Omega \qquad C_{in} = 1\ \text{mF}
$$

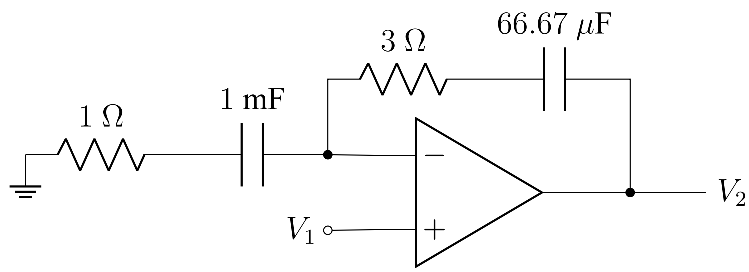


Figure 15: Designed active filter in non-inverting configuration

These elements can be scaled to practically realizable values as per necessity.

Problem 11

What is the importance of sensitivity analysis in filter design? Explain the sensitivity of the passive filter.

The importance of sensitivity analysis in filter design are noted as,

1. Sensitivity is used in prediction of behavior of device with change in parameters.
2. It is used to figure out the extent that errors in elements contribute to entire operation of the network.
3. It is used to select circuit components and filters for various applications.
4. Prediction of the difference between a practical filter and nominal design considerations is guided by sensitivity analysis.
5. Tradeoff for conflicting requirements for low cost and precision are based on sensitivity analysis.

For the sensitivity of passive filter, let us consider a RLC network as,

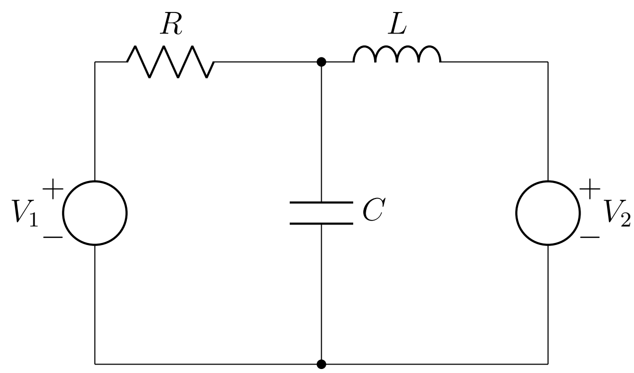


Figure 16: Passive RLC network

The transfer function of the network shown in Figure 16 is given as,

$$\begin{aligned}
T(s) &= \frac{V_2}{V_1} \\
\Rightarrow T(s) &= \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} \\
\Rightarrow T(s) &= \frac{1}{s^2LC + sRC + 1} \\
\therefore T(s) &= \frac{\frac{1}{LC}}{s^2 + s\frac{R}{L} + \frac{1}{LC}}
\end{aligned} \tag{41}$$

Also, the general equation for a lowpass filter is given as,

$$T(s) = \frac{H\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \tag{42}$$

Comparing Equation 41 and Equation 42, we get,

$$\begin{aligned}
\omega_0^2 = \frac{1}{LC} &\Rightarrow \omega_0 = \sqrt{\frac{1}{LC}} = L^{-1/2}C^{-1/2} \\
\frac{\omega_0}{Q} = \frac{R}{L} &\Rightarrow Q = \frac{1}{R}\sqrt{\frac{L}{C}} = R^{-1}L^{1/2}C^{-1/2}
\end{aligned}$$

So, we can calculate the following,

$$
\begin{aligned}
S_L^{\omega_0} &= \left(\frac{L}{\omega_0}\right)\left(\frac{\delta\omega_0}{\delta L}\right) = \left(\frac{L}{\omega_0}\right)\left(\frac{\delta(L^{-1/2}C^{-1/2})}{\delta L}\right) \\
&= \left(\frac{L}{\omega_0}\right)\left(-\frac{1}{2}L^{-3/2}\right)C^{-1/2} \\
&= \left(-\frac{1}{2}\right)\frac{L.L^{-3/2}C^{-1/2}}{\omega_0} \\
&= \left(-\frac{1}{2}\right)\left(\frac{L^{-1/2}L^{-1/2}}{L^{-1/2}C^{-1/2}}\right) = -\frac{1}{2} \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
S_C^{\omega_0} &= \left(\frac{C}{\omega_0}\right)\left(\frac{\delta\omega_0}{\delta C}\right) = \left(\frac{C}{\omega_0}\right)\left(\frac{\delta(L^{-1/2}C^{-1/2})}{\delta C}\right) \\
&= \left(\frac{C}{\omega_0}\right)\left(-\frac{1}{2}C^{-3/2}\right)L^{-1/2} \\
&= \left(-\frac{1}{2}\right)\frac{L^{-1/2}C.C^{-3/2}}{\omega_0} \\
&= \left(-\frac{1}{2}\right)\left(\frac{L^{-1/2}C^{-1/2}}{L^{-1/2}C^{-1/2}}\right) = -\frac{1}{2} \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
S_R^{Q} &= \left(\frac{R}{Q}\right)\left(\frac{\delta Q}{\delta R}\right) = \left(\frac{R}{Q}\right)\left(\frac{\delta(R^{-1}L^{1/2}C^{-1/2})}{\delta Q}\right) \\
&= \left(\frac{R}{Q}\right)\left(-R^{-2}\right)L^{1/2}C^{-1/2} \\
&= (-1)\,\frac{R.R^{-2}L^{1/2}C^{-1/2}}{Q} \\
&= (-1)\left(\frac{R^{-1}L^{1/2}C^{-1/2}}{R^{-1}L^{1/2}C^{-1/2}}\right) = -1 \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
S_L^{Q} &= \left(\frac{L}{Q}\right)\left(\frac{\delta Q}{\delta L}\right) = \left(\frac{L}{Q}\right)\left(\frac{\delta(R^{-1}L^{1/2}C^{-1/2})}{\delta Q}\right) \\
&= \left(\frac{L}{Q}\right)\left(\frac{1}{2}L^{-1/2}\right)R^{-1}C^{-1/2} \\
&= \left(\frac{1}{2}\right)\frac{L.L^{-1/2}R^{-1}C^{-1/2}}{Q} \\
&= \left(\frac{1}{2}\right)\left(\frac{R^{-1}L^{1/2}C^{-1/2}}{R^{-1}L^{1/2}C^{-1/2}}\right) = \frac{1}{2} \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
S_C^Q &= \left(\frac{C}{Q}\right)\left(\frac{\delta Q}{\delta C}\right) = \left(\frac{C}{Q}\right)\left(\frac{\delta(R^{-1}L^{1/2}C^{-1/2})}{\delta Q}\right) \\
&= \left(\frac{C}{Q}\right)\left(-\frac{1}{2}C^{-3/2}\right)R^{-1}L^{1/2} \\
&= \left(-\frac{1}{2}\right)\frac{C.C^{-3/2}R^{-1}L^{1/2}}{Q} \\
&= \left(-\frac{1}{2}\right)\left(\frac{R^{-1}L^{1/2}C^{-1/2}}{R^{-1}L^{1/2}C^{-1/2}}\right) = -\frac{1}{2} \qquad (47)
\end{aligned}
$$

Problem 12

What is a switched capacitor filter? How can the summer, inverting integrator, lossy integrator and non-inverting integrator be realized using the switched capacitor? Explain with suitable diagrams and necessary derivations.

A switched capacitor filter is a filter where a number of capacitors are periodically switched back and forth among a number of terminals to replace the use of resistance in such circuits. Since a resistor occupies large space in MOS technology, the fabrication is made easier with switched capacitor replacements. The charging and discharging of the capacitor acts as an equivalent resistor.

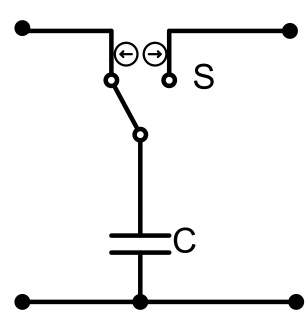


Figure 17: Switched capacitor as a replacement for resistor

Summer

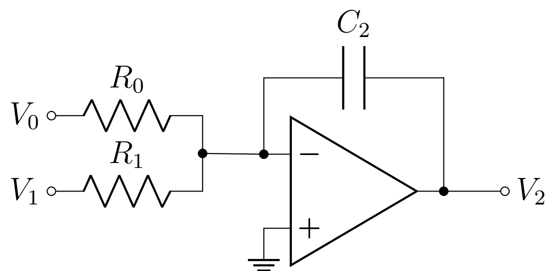


Figure 18: Summer with resistors

The output voltage of the circuit in Figure 18 is given as,

$$V_2 = -\frac{1}{R_0 C_2 s} V_0 - \frac{1}{R_1 C_2 s} V_1 \tag{48}$$

Replacing each resistor with an equivalent switched capacitor gives,

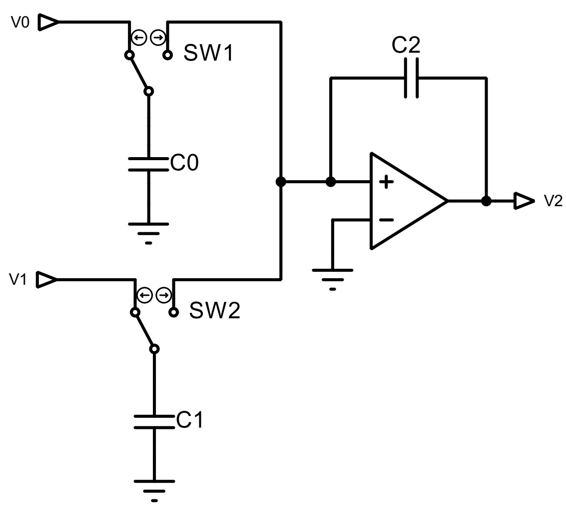


Figure 19: Summer with switched capacitor

The output voltage of the circuit in Figure 19 is given as,

$$V_2 = -f\frac{C_0}{C_2 s} V_0 - f\frac{C_1}{C_2 s} V_1 \tag{49}$$

Inverting Integrator

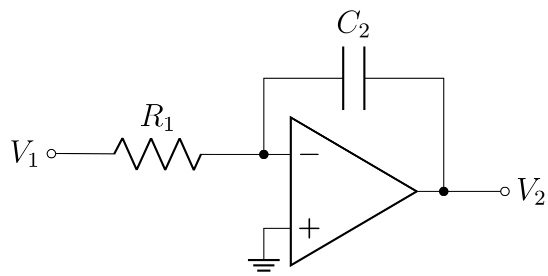


Figure 20: Inverting integrator with resistors

The output voltage of the circuit in Figure 20 is given as,

$$V_2 = -\frac{1}{R_1 C_2 s} V_1 \tag{50}$$

Replacing each resistor with an equivalent switched capacitor gives,

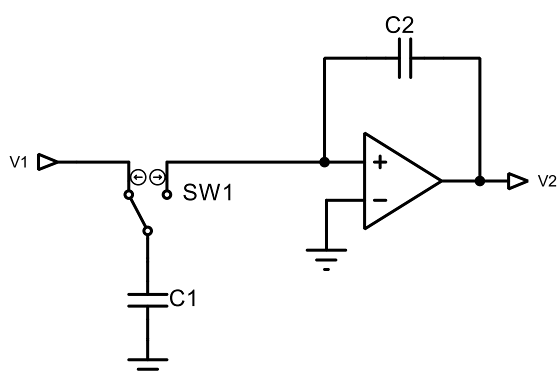


Figure 21: Inverting integrator with switched capacitor

The output voltage of the circuit in Figure 21 is given as,

$$V_2 = -f \frac{C_1}{C_2 s} V_1 \tag{51}$$

Lossy integrator

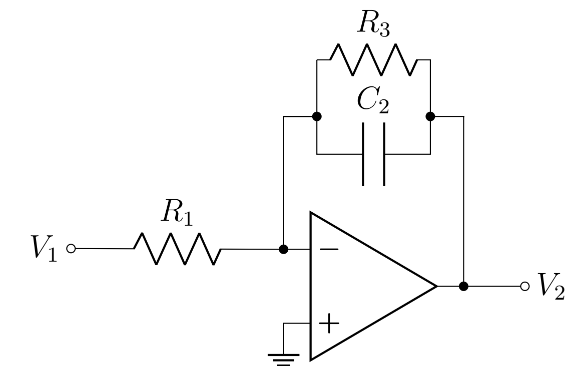


Figure 22: Lossy integrator with resistors

The output voltage of the circuit in Figure 22 is given as,

$$V_2 = \frac{-\dfrac{1}{C_2 s + \dfrac{1}{R_3}}}{R_1} V_1 = -\frac{1}{R_1\left(C_2 s + \dfrac{1}{R_3}\right)} V_1 \tag{52}$$

Replacing each resistor with an equivalent switched capacitor gives,

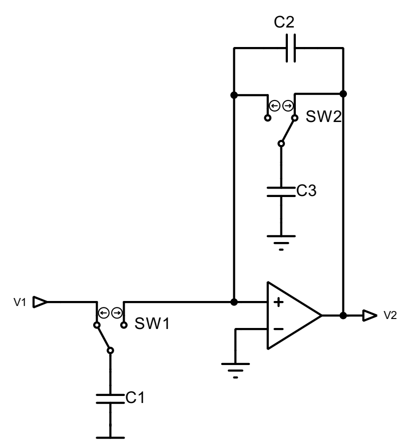


Figure 23: Lossy integrator with switched capacitor

The output voltage of the circuit in Figure 23 is given as,

$$V_2 = -f \frac{C_1}{C_2 s + f C_3} V_1 \tag{53}$$

Non-inverting Integrator

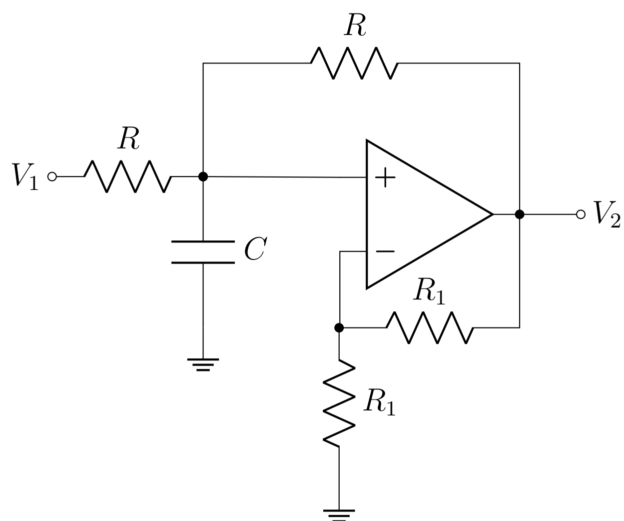


Figure 24: Non-inverting integrator with resistors

Replacing each resistor with an equivalent switched capacitor gives,

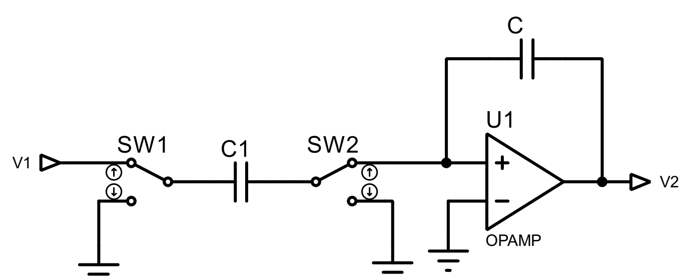


Figure 25: Non-inverting integrator with switched capacitor

The output voltage of the circuit in Figure 25 is given as,

$$V_2 = -f\frac{C_1}{C}\frac{1}{s}V_1 \tag{54}$$