6d. The Calculus of Optimization

As we know from the previous section, the goal of consumer is to Maximize Utility. There are two conditions to Maximize Utility:

1. Slope of Indifference Curve = Slope of Budget Line

In other words,

$$\boxed{MRS = P_x/P_y}$$

Brief Definitions:

*Marginal Rate of Substitution (MRS)

MRS is the maximum amount of y that a consumer is willing to give up to get one more x.

Thus,

$MRS = \triangle Y / \triangle X = dY/dX =$ Slope of IC

Or $MRS = MU_x/MU_y$ where $MU_x = \Delta U/\Delta X = \partial U/\partial X$; $MU_y = \Delta U/\Delta Y = \partial U/\partial Y$

Proof:

Given: $U(X,Y)$

Take total differential

$dU = \partial U/\partial X \cdot dX + \partial U/\partial Y \cdot dY$

$dU = MU_x \cdot dX + MU_y \cdot dY$

Along the IC, $dU = 0$

Thus, $0 = MU_x \cdot dX + MU_y \cdot dY$

$dY/dX = MU_x/MU_y$

$$\boxed{MRS = MU_x/MU_y}$$

*Slope of Budget Line (also called Budget Constraint)

In Figure 6.d.1, the blue line is the Budget Constraint. In order to find the intercept of X which represents the quantity of X, we have to divide the income (I) by the price of X (P_x). Similarly, we have to divide the income (I) by the price of Y (P_y) to find the intercept of Y which represents the quantity of Y. Thus,

The intercept of X = I/ P_x

The intercept of Y = I/ P_y

Then,

Slope of B.C.= Rise/Run

$= (I/ P_y) / (I/ P_x)$

$= (I/ P_y) \cdot (P_x /I)$

$$\boxed{\text{Slope of B.C.} = P_x/P_y}$$

2. <u>Indifference Curve (IC) must be on the Budget Line (BL)</u>

In other words,

$$\boxed{I = P_x \cdot X + P_y \cdot Y}$$

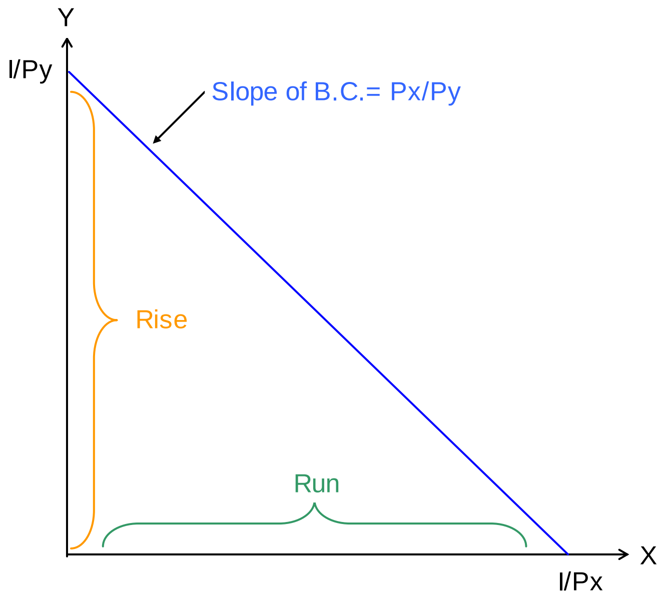

Figure 6.d.1. In order to find the slope of B.C., we have to find the intercepts of X and Y first. The intercept of X = I/ P_x; and the intercept of Y = I/ P_y. As we known, Slope is equal to rise divided by run. So, Slope of B.C. = (I/ P_y) / (I/ P_x) = P_x/P_y.

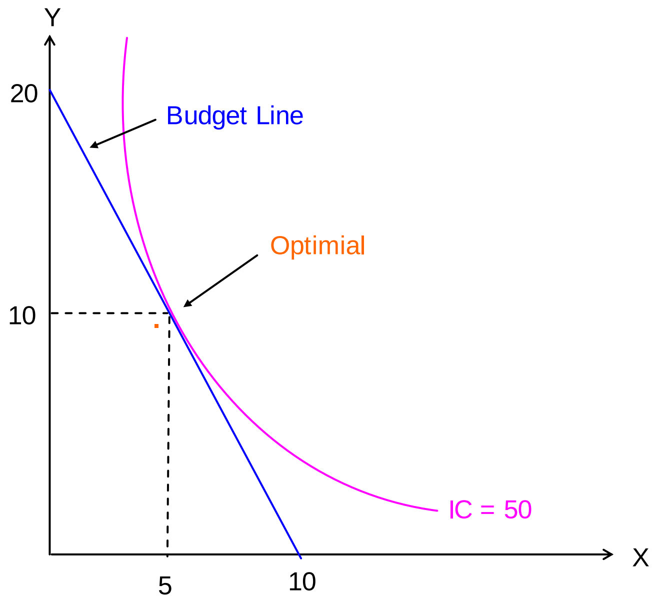

Figure 6.d.2. There are two conditions to maximize utility.

1) Slope of IC = Slope of BL
2) IC must be on the BL

<u>Utility – Max Bundle</u>

Four Steps to find utility-max bundle

<u>Step One</u>:

Find MRS

<u>Step Two</u>:

Set MRS = P_x/P_y

<u>Step Three</u>:

Plug Y=… into Budget Constraint and solve for X

<u>Step Four</u>:

Find Y

<u>Step Five</u>:

Plug X&Y into U=… and solve for Maximize Utility

Sample Question:

Givens: $I = \$50$

$P_x = \$1$

$P_y = \$1$

$U = X^{0.5} Y^{0.5}$

Find utility-max bundle & Maximize Utility.

Step One: Find MRS

$MRS = MU_x / MU_y$

$= (\partial U/\partial X)/(\partial U/\partial Y)$

$= (0.5\, X^{-0.5}\, Y^{0.5}) / (0.5\, X^{0.5}\, Y^{-0.5})$

$= Y/X$

Step Two: Set $MRS = P_x/P_y$

$Y/X = 1/1$

$\boxed{Y = X}$

Step Three: Plug Y=… into Budget Constraint and solve for X

$I = P_x \cdot X + P_y \cdot Y$

$I = X + Y$

* Since Y=X & I = $50, therefore:

$50 = 2X$

$\boxed{X = 25}$

Step Four: Find Y

$X = Y$

$\boxed{\begin{array}{l} X = 25 \\ Y = 25 \end{array}}$

Thus, Consumers will maximize their utility when they buy 25 of goods X and 25 of goods Y.

Step Five: Plug X&Y into U=… and solve for Maximize Utility

Maximize Utility $= X^{0.5}\, Y^{0.5} = (25^{0.5}) \cdot (25^{0.5})$

$U = 25$

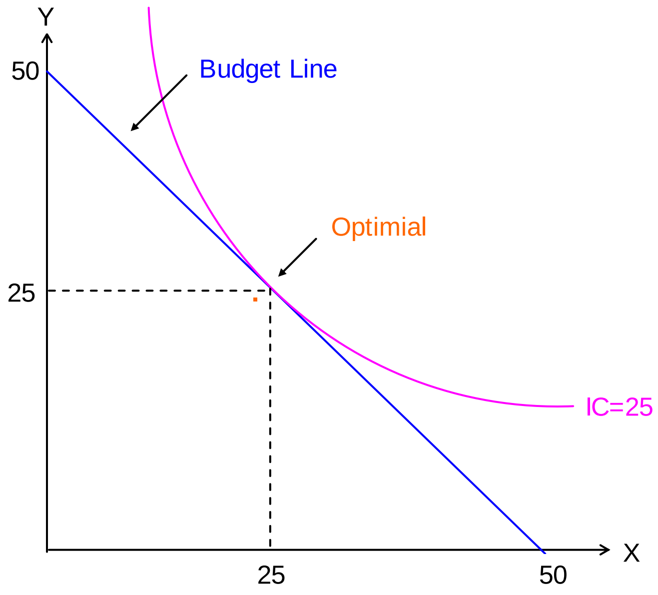

Figure 6.d.3. The maximize utility and the utility-max bundle. This graph shows that consumers will maximum utility (IC=25) when they buy 25 of goods X and 25 of goods Y.

Lagrangian Method of Constrained Optimization

Lagrangian Method of Constrained Optimization is a method of constrained optimization in which constraint is incorporated into the objective function. Objective function is a function which is being tried to maximize or minimize.

Step One: Setup Lagrangian

L = [objective function] + λ[constraint rearranged to equal zero] where λ = Lagrangian multiplier

As we want to maximize the utility, the objective function is U(x,y);

Budget Constraint: $I = P_x \bullet X + P_y \bullet Y$

Thus,

$$\boxed{L = U(x,y) + \lambda[I - P_x \bullet X - P_y \bullet Y]}$$

Step Two: First-order Conditions

(i) $\partial L/\partial X = \partial U/\partial X - \lambda P_x = 0$ Or $MU_x - \lambda P_x = 0$

$$\boxed{MU_x = \lambda P_x}$$

(ii) $\partial L/\partial Y = \partial U/\partial Y - \lambda P_y = 0$ Or $MU_y - \lambda P_y = 0$

$$\boxed{MU_y = \lambda P_y}$$

(iii) $\partial L/\partial \lambda = I - P_x \bullet X - P_y \bullet Y = 0$ Or $\boxed{I = P_x \bullet X + P_y \bullet Y}$

Step Three: Divide (i) by (ii)

$MU_x/MU_y = \lambda P_x/\lambda P_x = P_x/P_y$

$$\boxed{MRS = P_x/P_y}$$

Step Four: Plug Y=… into Budget Constraint and solve for X

Step Five: Find Y

Step Six: Plug X&Y into U=… and solve for Maximize Utility

!!! * This Method only can use for constrained optimization problem

Sample Question:

Givens: I=$100

$P_x = \$10$

$P_y = \$5$

$U = XY$

Find utility-max bundle & Maximize Utility by Lagrangian Method.

Step One: Setup Lagrangian

$L = U(x,y) + \lambda[I - P_x \bullet X - P_y \bullet Y]$

$L = XY + \lambda[100 - 10X - 5Y]$

Step Two: First-order Conditions

(i) $\partial L/\partial X = \partial U/\partial X - \lambda P_x = 0$

$Y - 10\lambda = 0$

$Y = 10\lambda$

(ii) $\partial L/\partial Y = \partial U/\partial Y - \lambda P_y = 0$

$X - 5\lambda = 0$

$X = 5\lambda$

(iii) $\partial L/\partial \lambda = I - P_x \bullet X - P_y \bullet Y = 0$

$100 - 10X - 5Y = 0$

Step Three: Divide (i) by (ii)

$Y/X = 10\lambda/5\lambda$

$Y/X = 2$

$Y = 2X$

Step Four: Plug Y=… into Budget Constraint and solve for X

$$I = P_x \bullet X + P_y \bullet Y$$

$$100 = 10X + 5Y$$

$$100 = 10X + 5(2X)$$

$$100 = 20X$$

$$X = 5$$

Step Five: Find Y

$$Y = 2X$$

$$Y = 2(5)$$

$$\boxed{\begin{array}{l} Y = 10 \\ X = 5 \end{array}}$$

Thus, Consumers will maximize their utility when they buy 5 of goods X and 10 of goods Y.

Step Six: Plug X&Y into U=… and solve for Maximize Utility

$$U = XY$$

$$\text{Maximize Utility} = 5 \bullet 10$$

$$U = 50$$

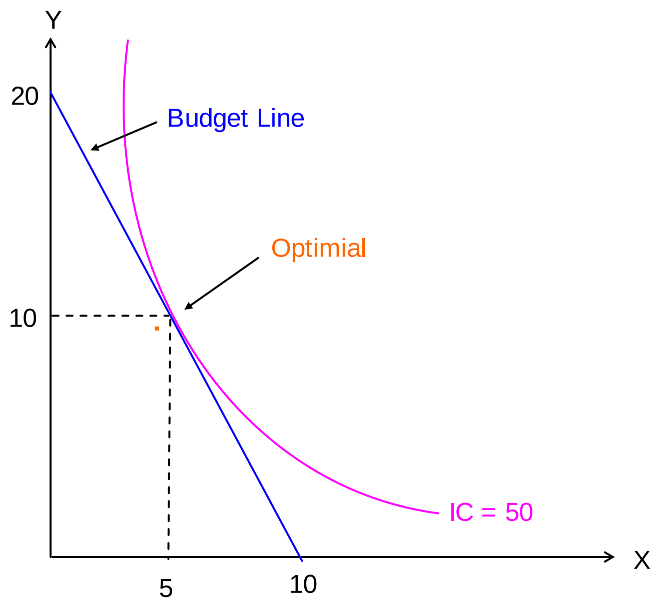

Figure 6.d. 4. Result of utility-max bundle & Maximize Utility from Lagrangian Method of Constrained Optimization. This graph shows that consumers will maximum utility (IC=50) when they buy 5 of goods X and 10 of goods Y.

Practice Problems

1. Given MRS = $\frac{4y}{5x}$, I = \$300, P_x = \$4, P_y = \$2. Find the optimum.
2. Given U = $x^{0.5}y^{0.5}$, I = \$100, P_x = \$1, P_y = \$2. Find the optimum.
3. Given U=xy, I = \$250, P_x = \$5, P_y = \$10. Find the optimum.

Answers

1. Optimum at ($\frac{100}{3}$, $\frac{250}{3}$)

   Work:

$$MRS = \frac{4y}{5x} = \frac{4}{2}$$

$$4y = 10x \rightarrow 2y = 5x$$

$$4x + 2y = 300$$

$$4x + 5x = 300$$

$$x = \frac{300}{9} = \frac{100}{3}$$

$$y = \frac{250}{3}$$

2. Optimum at (20,40)

   Work:

$$U = x^{0.5}y^{0.5}$$

$$MRS = 0.5x^{-0.5}y^{0.5} / 0.5x^{0.5}y^{-0.5} = \frac{y}{x}$$

$$\frac{y}{x} = \frac{2}{1}$$

$$y = 2x$$

$$x + 2y = 100$$

$$x + 2(2y) = 100$$

$$5x = 100$$

$$x = 20$$

$$y = 40$$

3. Optimum at (50,25)

   Work:

First, remember to set BC = 0.

$$5x + 10y = 250$$

$$x + 2y – 50 = 0$$

Then, plug it into the Lagrangian.

$$L = xy + \lambda\,(x + 2y - 50)$$

$$\frac{\partial L}{\partial x} = y + \lambda = 0$$

$$\frac{\partial L}{\partial y} = x + 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x + 2y - 100 = 0$$

$$\frac{y}{x} = \frac{1}{2}$$

$$2y = x$$

$$2y + 2y = 100$$

$$y = 25$$

$$x = 50$$