Math Outline

- The following outline will assume you alread know:
  - functions of multiple variables
  - derivatives (including partial derivatives)
  - graphing functions
  - basic function maximization and minimizati
  - critical points and second order conditions
  - an understanting of exponents, logs, polynomials and other common operations and functional types.
  - and other Calc 1,2 concepts.

Ask if you do not know specific math concepts!

Elasticities

Let $y=f(x)$ be some arbitrary function of x that defines the corresponding y value. Also, $x>0$ and $f(x)>0$.

$\boxed{\varepsilon_{xy}=\frac{\%\Delta y}{\%\Delta x} \text{ is the elasticity of } y \text{ with respect to } x.}$

- Elasticities are a measure of the effect a change in x has on y where change is represent by %s

Common Types of elasticities are:

Elasticity of Demand: $\varepsilon_D=\frac{\Delta\% Q_D(P)}{\Delta\% P}$

Elasticity of Supply: $\varepsilon_S=\frac{\Delta\% Q_S(P)}{\Delta\% P}$

Income Elasticity: $\varepsilon_I=\frac{\Delta\% X^*(I)}{\Delta\% I}$

Cross Price Elasticity: $\varepsilon_{P_y,X^*}=\frac{\Delta\% X^*(P_y)}{\Delta\% P_y}$

And many more types of elasticities ...

Elasticities

In econ 2010 we outlined elasticities as:

$$\varepsilon_{xy} = \frac{\%\Delta y}{\%\Delta x} = \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\Delta y}{\Delta x}\frac{x}{y} = \frac{y_1 - y_2}{x_1 - x_2} \cdot \frac{\frac{x_1 + x_2}{2}}{\frac{y_1 + y_2}{2}}$$

where (x_1, y_1) and (x_2, y_2) are 2 pairs of points.

- elasticities will not be calculated this way in this class. Any answer that calculates elasticities using 2 points will be considered wrong (excluding if $f(x)$ is a line for reasons discussed later).

In econ <u>3010</u>, we will calculate elasticities with partial derivatives

$$\varepsilon_{xy} = \frac{\%\Delta y}{\%\Delta x} = \frac{\frac{\partial y}{y}}{\frac{\partial x}{x}} = \frac{\partial y}{\partial x}\frac{x}{y} = f'(x)\frac{x}{f(x)}$$

(because $y = f(x)$)

Examples: • Let $y = x^2$. $\frac{\partial y}{\partial x} = 2x$

$$\Rightarrow \varepsilon_{xy} = 2x\frac{x}{x^2} = 2$$

• Let $y = A - Bx$. $\frac{\partial y}{\partial x} = -B$

$$\Rightarrow \varepsilon_{xy} = -B \cdot \frac{x}{A - Bx}$$

(See A1 for more examples)

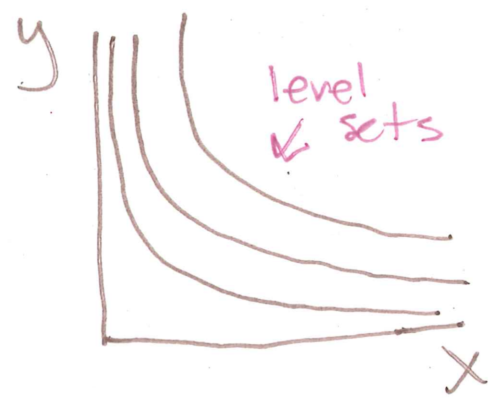
Graphs

Let $f(x,y)=z$. A level set is all of the ordered pairs of (x,y) that produce the same z value.

For example, let $f(x,y)= x\cdot y$ and let $z=10$

The level set defined by $z=10$ is the function

$$y=\frac{10}{x} \quad (\text{because } x\cdot y=10).$$



For each level set: $y=\frac{\bar{z}}{x}$ for a fixed value of z $(\bar{z})$

See the sections on utilities and production functions for the application of level sets.

- In general, $f(x,y)=\bar{z}$ defines a function of y in terms of x. A level set has slope:

$$-\frac{\frac{\partial f(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y}} = \frac{-f_x}{f_y}$$

If you are interested, this is true because the total derivative of both side yields: $f_x dx + f_y dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-f_x}{f_y}$$

$\boxed{\text{Graphs}}$ (involves future material)

As the class moves forward, level sets and their derivatives will have specific meaning.

- With utilities, $u(x,y)=\bar{u}$ is called an indifference curve and

  $-\frac{u_x}{u_y}$ is the marginal rate of substitution (MRS)

- With expenditure, $P_x x + P_y y = I$ is called a budget constraint (budget for short) and

  $\frac{-P_y}{P_x}$ is the marginal rate of transformation (MRT)

- With input costs, $P_x x + P_y y = \bar{c}$ is an iso-cost curve and

  $-\frac{P_y}{P_x}$ is the slope (not given an important name)

- With production functions, $f(x,y)=\bar{q}$ is an isoquant curve and

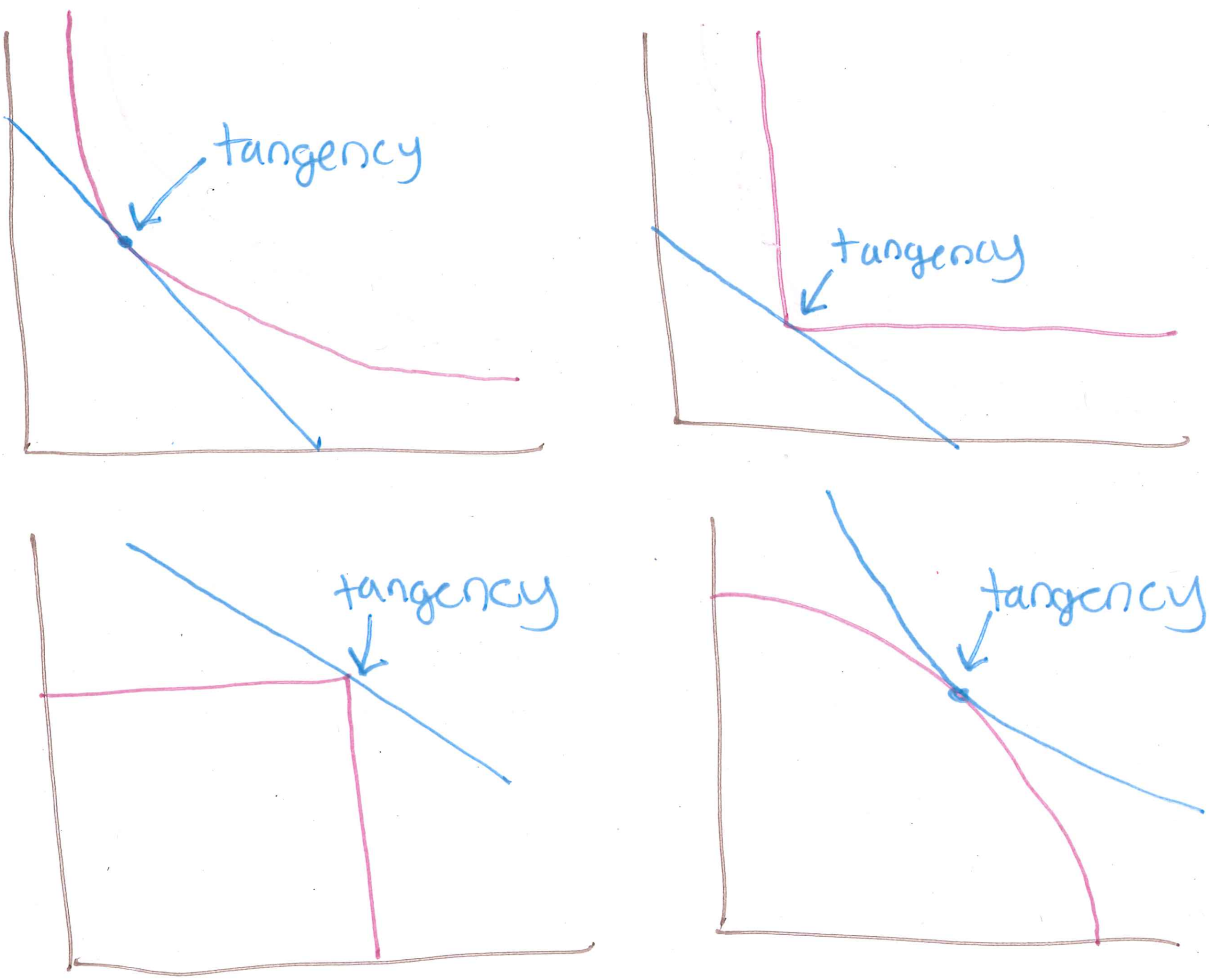
  $-\frac{f_x}{f_y}$ is the marginal rate of techical substitution (MRTS)

Tangencies

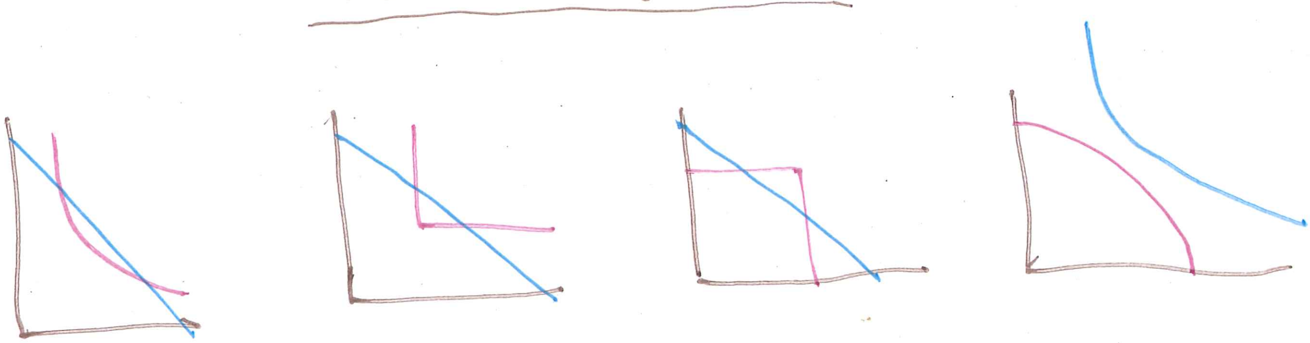
- A tangency between 2 curves is a point where the curves are equal (the same point) but do not cross eachother

Visuals:

Tangencies



Not Tangencies

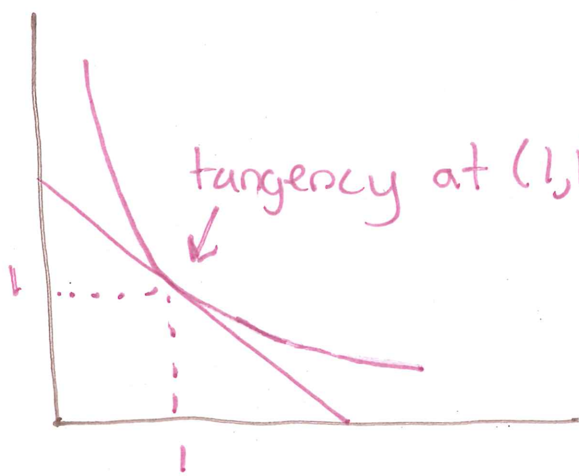


Tangencies

If two curves are tangent at a point and the slope of each curve is well-defined then the two curves have the same slope at that point.

Example: $f(x,y) = xy = 1 \Rightarrow y = \frac{1}{x}$

$g(x,y) = x+y = 2 \Rightarrow y = 2-x$



- When $x=1$, $2-x=1$
  $1/x = 1$
- when $x \neq y$, $1/x > 2-x$

$$\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{\partial}{\partial x}(2-x) = -1$$

$-\frac{1}{x^2} = -1$ when $x=1$

Thus, the slopes are the same at $(1,1)$.

- Tangency conditions are important for "constrained optimization".

Maximization

- you should all know how to solve

$$\max_x f(x) \quad \text{(the maximum of the function } f(x)\text{)}$$

Examples: $\max_x -x^2 = 0$ at the point $x=0$

$\max_x 10-x^2 = 10$ at the point $x=0$

- the argmax of a function is the point where the function is maximized ($x=0$ in the examples above)

- sometimes, we maximize a function to find the "best values", the values that maximize that function.
- other times, we only care about the value of the function at the max (and not what values maximize the function)

If a function has a derivative ($f'(x)$ exists), then $f'(x)=0$ at the point where a function is maximized.

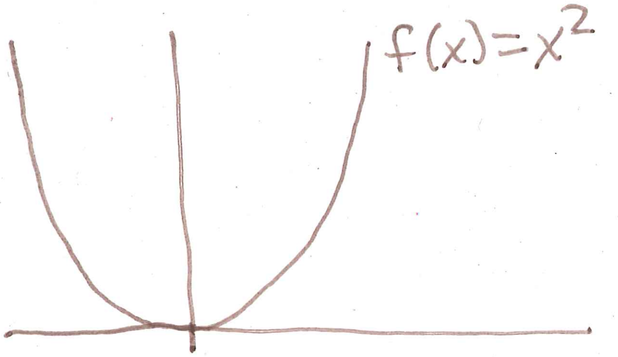
Maximization

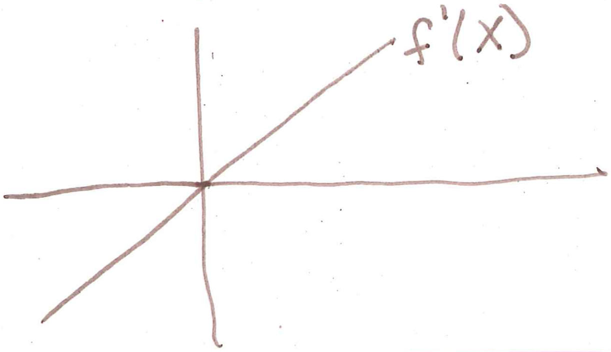
This page is a revie of what should be kno from Calc classes

In general $f'(x)=0$ only indicates a critical point. (First order condition) $\boxed{FOC}$

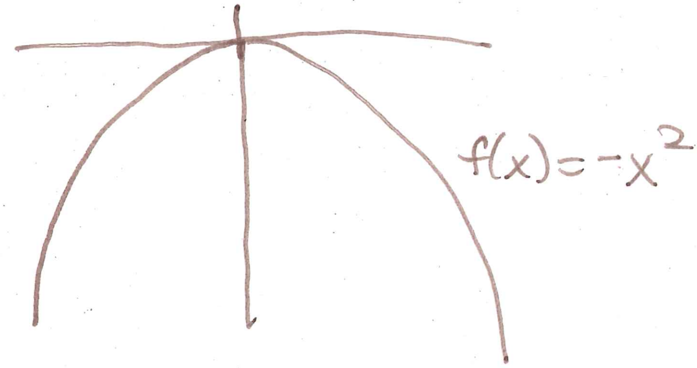
x^2, $-x^2$ and x^3 all have critical points at $x=0$. (x^3 solved in A1)

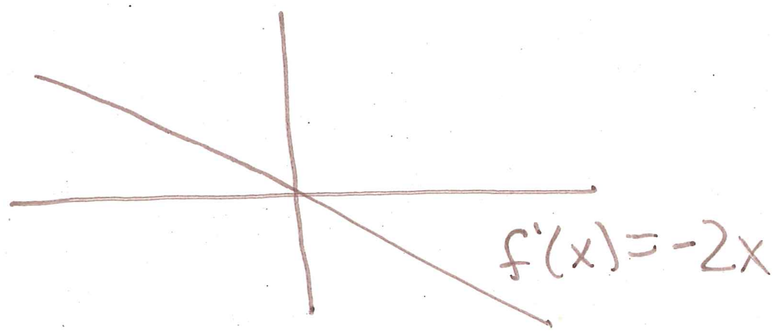
$f(x)=x^2 \Rightarrow f'(x)=2x$





$f(x)=-x^2 \Rightarrow f'(x)=-2$





min because $f''(x)>0$ for any x ($f''(x)>0$ at the critical point is a local min)	max because $f''(x)<0$ for any x ($f''(x)<0$ at the critical point is a local max)

f'' being negative or positive is called a <u>Second Order Condition</u> (SOC)

Maximization

Maximization will be used over and over in this class

2 dimensional maximization:

Let $f(x,y)$ be a function that is differentiable, concave and has a max

then: $\boxed{\max_{x,y} f(x,y)}$ $\Rightarrow$ The (x^*, y^*) pair where $f_x(x^*, y^*) = f_y(x^*, y^*) = 0$

First order conditions ($f_x = 0, f_y = 0$) are similar to $f'(x) = 0$. Second order conditions (concavity for maximization) will not be covered in this class.

$\min_{x,y} f(x,y)$ is the same but with convexity replacing the concavity requirement.

Maximization

Maximization with constraints:

$$\max_{x,y} f(x,y) : ax+by \le c$$

FOC:

$$f_x - a\lambda = 0$$

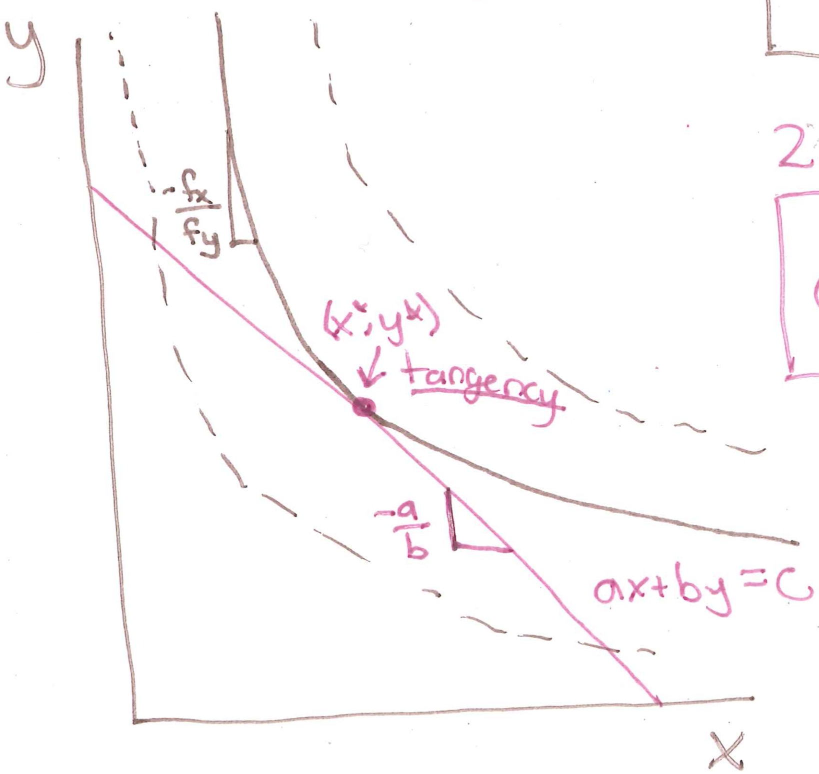
$$f_y - b\lambda = 0$$

1st Eqn

$$\Rightarrow \frac{f_x}{a} = \frac{f_y}{b} \Rightarrow \boxed{-\frac{f_x}{f_y} = -\frac{a}{b}}$$

2nd Eqn

$$\boxed{ax+by=c}$$



2 eqns, 2 unknowns
solve for x^*, y^*.
(see utility max for more info)

(This will be covered in depth during the class to build the intuition behind this)

Maximization

Minimization with constraints:

$$\min_{x,y} \; ax+by : f(x,y) \geq c$$

FOC: $a + f_x(x,y)\lambda = 0$

$b + f_y(x,y)\lambda = 0$

1st Eqn

$$\Rightarrow \frac{f_x}{a} = \frac{f_y}{b} \Rightarrow \boxed{-\frac{f_x}{f_y} = \frac{-a}{b}}$$

2nd Eqn

$$\boxed{f(x,y) = c}$$

This will be outlined in more detail in the cost minimization notes.

Consumer Theory

The building blocks of this section are:

1) Consumer Preferences
2) Budgets
3) Utility Maximization
4) Expected Value and risk preferences

<u>Key Concepts and Phrases</u> (In no particular order)

- Preferences
- Utility
- Demand Curves
- Price Consumption Curves (PCC)
- Engel Curves
- Income Consumption Curves (ICC)
- Normal Goods, Inferior Goods, Luxury Goods, Necessity Goods
- Subsitutes and compliments
- Demand, Income, ..., and other types of elasticities
- Taxes / Subsidies
- Edgeworth Boxes
- Pareto Optimality
- Trade Economies

Key Conclusion: Modelling Consumer decisions is

HARD!

Preferences

- Imagine you have the option to have either:
  A) two bananas
  B) three oranges
- Which option would you pick? What would change if option A was now three bananas?
- If you pick option A and would never pick option B, then we say that you <u>strictly prefer</u> A to B. (Denoted $A \succ B$)
- If you are willing to pick either option when given the choice, we will say that you are <u>indifferent</u> between option A and option B. (Denoted $A \sim B$)
- From now on, I will refer to these "options" as <u>bundles</u>.

Definition: Given two options (bundles) A and B,
A is strictly prefered to B ($A \succ B$) if B will never be selected when A is a possible option.

★ <u>Is this definition equivalent to the one earlier?</u>

A person (consumer) is indifferent between A and B if he/she would be willing to select either bundle when give the choice between only A and B. ($A \sim B$)

Preferences define a consumers ideal choice amoung a set of options!

Preferences

- Time to introduce sets! (see math notes)
- Let $\mathbb{A}$ be the set of all possible bundles a consumer could ever have.
- We say preferences are complete if $\forall A, B \in \mathbb{A}$,

$$A \prec B \text{ or } A \sim B \text{ or } A \succ B$$

- Completeness is the concept that a consumer has an opinion about everything.
- We say preferences are transitive iff

$$(A \sim B, B \sim C) \Rightarrow A \sim C$$

$$(A \succ B, B \succ C) \Rightarrow A \succ C$$

In plain terms: If you prefer bundle A to B and you prefer B to C, then you prefer A to B. (similar statement for indifference

- Weak Preference (denoted $A \succsim B$) means either $A \sim B$ or $A \succsim B$.

Good Practice: Show that the definition for transitivity is equivalent to the following statement:

$$(A \succsim B, B \succsim C) \Rightarrow A \succsim C$$

See the cyclic vote problem for more on this topic

- We say preferences are reflexive if $A \sim A$ for any bundle A.

Completeness + Transitivity + Reflexivity will always be assumed unless otherwise stated

Preferences

For simplicity, let's assume bundes consist of varying amounts of two types of goods.

- Let x_1 denote the amount of good 1
- Let x_2 denote the amount of good 2
- Let (x_1, x_2) denote the bundle.

Let (x_1^*, x_2^*) be a fixed bundle. Let

$$IC^* = \{(x_1, x_2) : (x_1, x_2) \sim (x_1^*, x_2^*)\}$$

An **Indifference Curve** is the set of all bundles that a consumer is indiffent to when compared to a specific bundle (x_1^*, x_2^*).

Completeness + Reflexity $\Rightarrow$ all bundles belong to atleast one indifference curve

\+ Transitivity $\Rightarrow$ each bundle belongs to exactly one indifference curve

Why is this true? Try to come up with an intuitive explanation of these statements

Preferences

Time to introduce Utility Functions!

- Imagine you wanted to give each bundle a score where higher scores correspond to bundles that are strictly prefered to bundles with a lower score. Can this "scoring method" be used to represent preferences
  - The answer is yes!
- Let $u(x_1, x_2)$ be a function where

$$(x_1^*, x_2^*) \succ (\hat{x}_1, \hat{x}_2) \Leftrightarrow u(x_1^*, x_2^*) > u(\hat{x}_1, \hat{x}_2)$$

and

$$(x_1^*, x_2^*) \sim (\hat{x}_1, \hat{x}_2) \Leftrightarrow u(x_1^*, x_2^*) = u(\hat{x}_1, \hat{x}_2)$$

(for any bundles (x_1^*, x_2^*) and $(\hat{x}_1, \hat{x}_2)$)

$u(\cdot)$ is a utility function where $u(x_1^*, x_2^*)$ is the utility level assigned to the bundle (x_1^*, x_2^*)

Can different utility function represent the same preferences? Try to answer this before looking ahead.

Moving forward, assume $x_1, x_2 \geq 0$.

Preferences

Utility Function Examples:

(i) $u(x_1, x_2) = x_1 + x_2$
(ii) $u(x_1, x_2) = \min\{x_1, x_2\}$
(iii) $u(x_1, x_2) = \max\{x_1, x_2\}$
(iv) $u(x_1, x_2) = x_1^{1/2} + x_2^{1/2}$
(V) $u(x_1, x_2) = x_1^2 + x_2^2$
(vi) $u(x_1, x_2) = \ln(x_1) + \ln(x_2)$
(vii) $u(x_1, x_2) = x_1 x_2$ (viii) $u(x_1, x_2) = x_1^{1/2} + x_2$

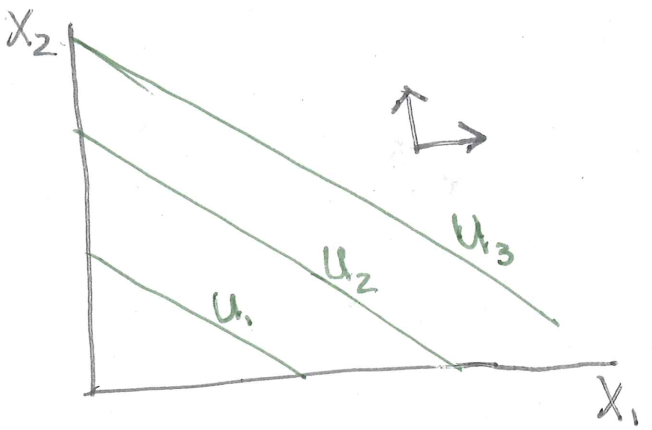
Note that indifference curves can now be defined by a fixed utility level. The curve will be defined by $u(x_1, x_2) = \bar{u}$ for some $\bar{u}$ value.

An $\boxed{\text{indifference map}}$ is a graph with a few representative indifference curves and arrows representing the direction of preference.

The following pages show indifference maps for the functions listed above.

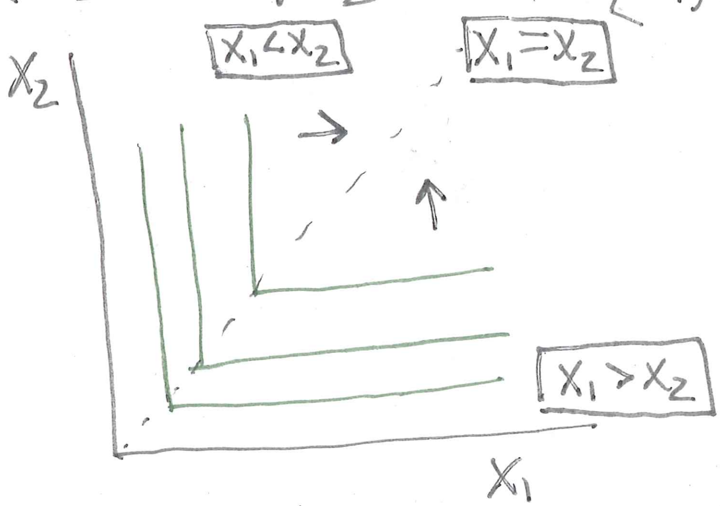
Preferences

(i) $u(x_1, x_2) = x_1 + x_2$ $\qquad \bar{u} = x_1 + x_2 \Rightarrow x_2 = \bar{u} - x_1$



Linear utilities represent 2 goods that are perfect substitutes. Why?
(subsitutable goods will be more formally discussed later

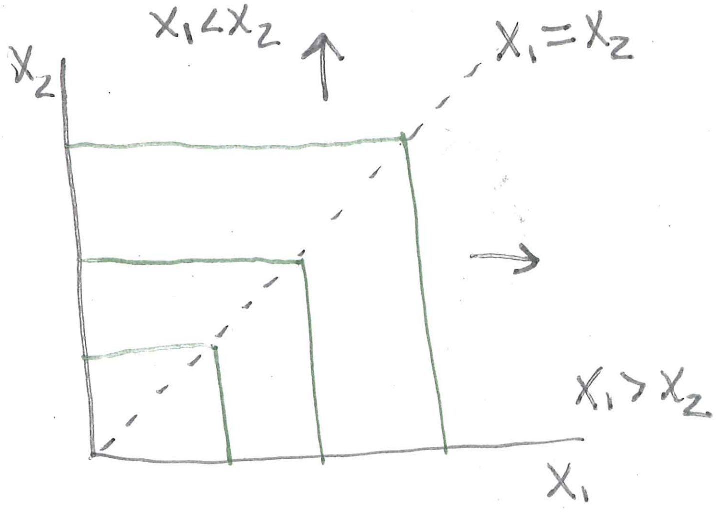
(ii) $u(x_1, x_2) = \min\{x_1, x_2\}$ $\qquad \bar{u} = \min\{x_1, x_2\}$



$\Rightarrow$ if $x_1 > x_2$, $x_2 = \bar{u}$
if $x_1 \leq x_2$, $x_1 = \bar{u}$

Min utilities represent 2 goods that are perfect compliments. Why?
(formally discussed later)

(iii) $u(x_1, x_2) = \max\{x_1, x_2\}$ $\qquad \bar{u} = \max\{x_1, x_2\}$

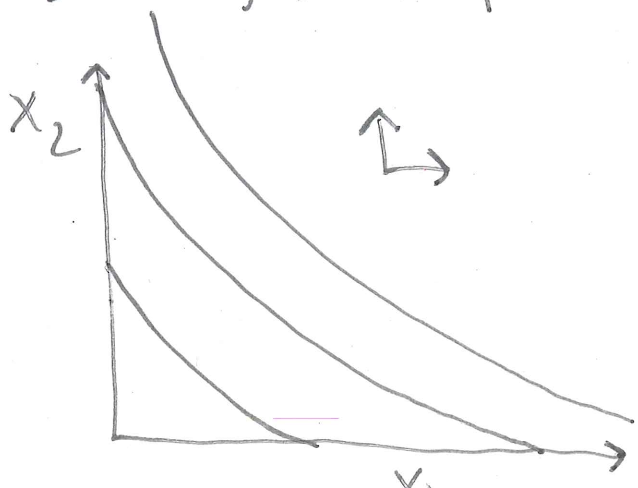


$\Rightarrow$ if $x_1 > x_2$, then $x_1 = \bar{u}$
if $x_1 \leq x_2$, then $x_2 = \bar{u}$

Does this function represen perfect subsitutes?

Preferences

(iv) $u(x_1, x_2) = x_1^{1/2} + x_2^{1/2}$



$x_1^{1/2} + x_2^{1/2} = \bar{u}$

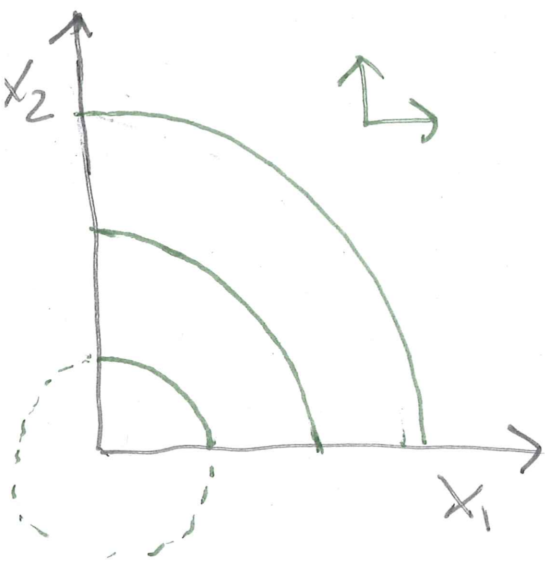
$\Rightarrow x_2 = (\bar{u} - x_1^{1/2})^2$

$x_2 = 0$ if $x_1 = \bar{u}^2$

$x_1 = 0$ if $x_2 = \bar{u}^2$

Why does the I map look this way?

(v) $u(x_1, x_2) = x_1^2 + x_2^2$



$x_1^2 + x_2^2 = \bar{u}$

$\Rightarrow$ IC are "circles"

(vi) $u(x_1, x_2) = \ln(x_1) + \ln(x_2)$

$\bar{u} = \ln(x_1) + \ln(x_2)$

$\bar{u} = \ln(x_1 x_2)$

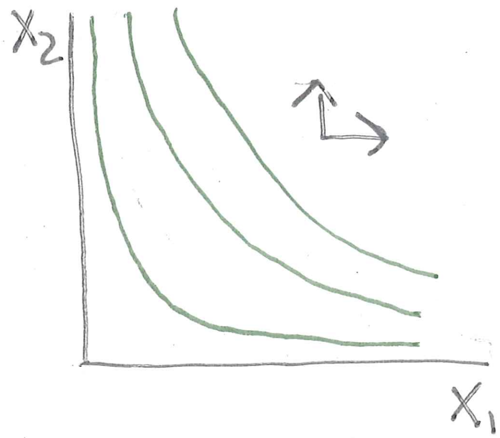
$e^{\bar{u}} = x_1 x_2$

$x_2 = e^{\bar{u}} / x_1$

Is (vii) the same indifference map?

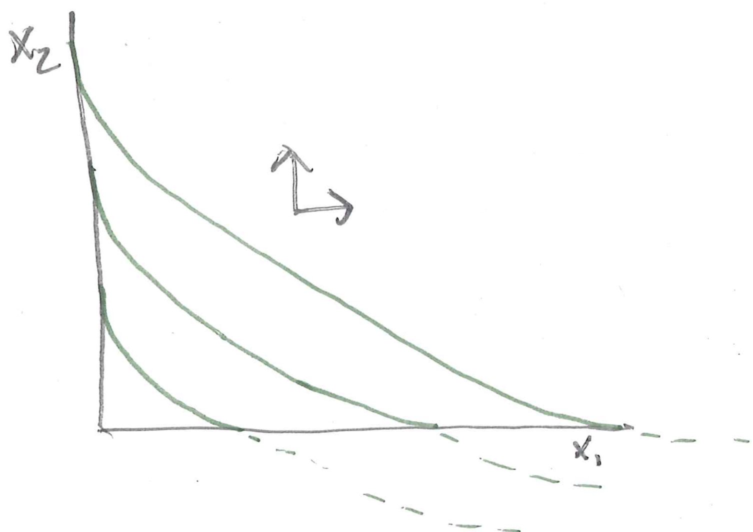
Preferences

(vii) $u(x_1, x_2) = x_1 x_2$ $\qquad \bar{u} = x_1 x_2 \Rightarrow x_2 = \frac{\bar{u}}{x_1}$



(viii) $u(x_1, x_2) = x_1^{1/2} + x_2$ $\qquad \bar{u} = x_1^{1/2} + x_2 \Rightarrow x_2 = \bar{u} - x_1^{1/2}$

ICs are shifted vertically but the shape does not change.



Now let's define some families of functions

Preferences

- Cobb Douglas: Any utility of the form
  $u(x_1, x_2) = x_1^a x_2^b$ for $a,b>0$
- (Perfect Sub) Linear: Any utility of the form
  $u(x_1, x_2) = ax_1 + bx_2$ for $a,b>0$
- Quasi-Linear: Any utility of the form
  $u(x_1, x_2) = f(x_1) + x_2$ where $f: \mathbb{R} \to \mathbb{R}$
- Perfect Compliments: Any utility of the form
  $u(x_1, x_2) = \min\{ax_1, bx_2\}$ for $a,b>0$

The work so far gives rise to a few natural questions:

1. When do two different utility function represent the same underlying preferences?
2. What implicit assumptions are we making with the utilities we have seen thus far?
3. Can indifference curves slope up?
4. Can indifference curves be thick?
5. Can indifference curves cross?

Preferences

Two utility functions u_A, u_B represent the same preferences if and only if

$$u_A(x_1^*, x_2^*) > u_A(\hat{x}_1, \hat{x}_2) \iff u_B(x_1^*, x_2^*) > u_B(\hat{x}_1, \hat{x}_2)$$

and $$u_A(x_1^*, x_2^*) = u_A(\hat{x}_1, \hat{x}_2) \iff u_B(x_1^*, x_2^*) = u_B(\hat{x}_1, \hat{x}_2)$$

for any two arbitrary bundles (* and ^)

Is this a helpful addition to this definition?

Example The following utilities all represent the same preferences:

- $u_1(x_1, x_2) = x_1 x_2$
- $u_2(x_1, x_2) = \ln(x_1) + \ln(x_2)$
- $u_3(x_1, x_2) = \left(\ln(x_1) + \ln(x_2) - \frac{1}{2}\ln(x_1 x_2)\right)^3 \cdot 100 + 3$

Suppose $u_1(x_1^*, x_2^*) > u_1(\hat{x}_1, \hat{x}_2)$.

$$\iff x_1^* x_2^* > \hat{x}_1 \hat{x}_2$$

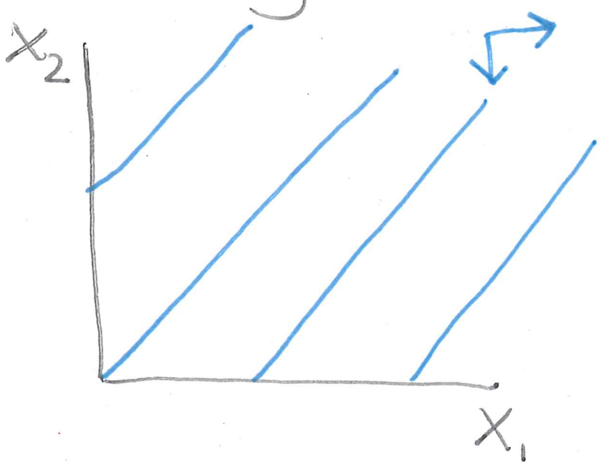
$$\iff \ln(x_1^*) + \ln(x_1^*) > \ln(\hat{x}_1) + \ln(\hat{x}_2) \quad (\ln(\cdot) \text{ is a strictly increasing functio})$$

$$\iff u_2(x_1^*, x_2^*) > u_2(\hat{x}_1, \hat{x}_2)$$

Show u_1 and u_3 represent the same preferences!

Preferences

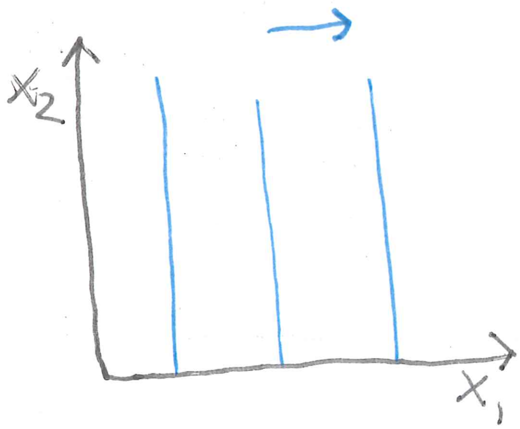
Let $u(x_1, x_2) = x_1 - x_2$. Is this function a valid utility function? Yes!



Having more x_1 increases utility (is strictly prefered)

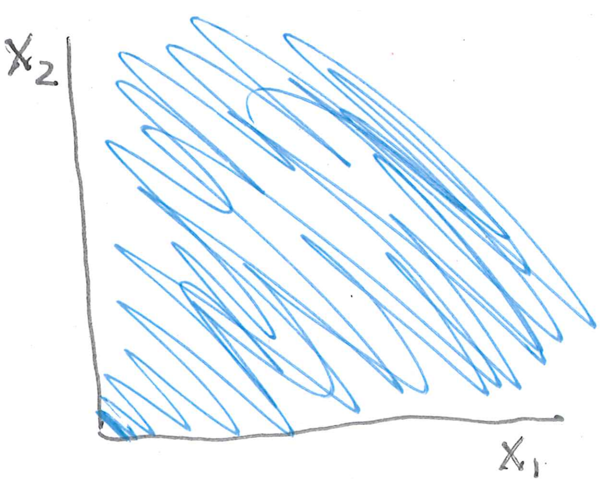
Have more x_2 decreases utility. x_2 is "bad"

Let $u(x_1, x_2) = x_1$



Utility does not change with x_2. The consumer is indifferent over any two values of x_2 with the same paired x_1.

Let $u(x_1, x_2) = 10$



The "indifference curve" is any possible x_1, x_2 pair. The consumer is indifferent over all options.

Preferences

- Preferences are monotonic given
  - $x_1^* \geq \hat{x}_1$ and $x_2^* \geq \hat{x}_2 \Rightarrow (x_1^*, x_2^*) \succeq (\hat{x}_1, \hat{x}_2)$
    $\Leftrightarrow u(x_1^*, x_2^*) \geq u(\hat{x}_1, \hat{x}_2)$
  - (strict) $x_1^* > \hat{x}_2$ and $x_2^* > \hat{x}_2 \Rightarrow (x_1^*, x_2^*) \succ (\hat{x}_1, \hat{x}_2)$
    $\Leftrightarrow u(x_1^*, x_2^*) > u(\hat{x}_1, \hat{x}_2)$

Basically, monotonicity $\Rightarrow$ More is better
strict monotonicity $\Rightarrow$ More is always better

A quick aside: the definition of "strict" is not as extreme as possible because I make no stateme about $x_1^* > \hat{x}_1$ and $x_2^* = \hat{x}_2 \Rightarrow u(x_1^*, x_2^*) > u(\hat{x}_1, \hat{x}_2)$
- I don't use this stronger version because $\min\{x_1, x_2\}$ fails this requirement.

Unless otherwise stated, assume strict monotonici

The marginal rate of substitution (MRS) is the slope of an indifference curve. Mathematically

$$MRS = \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}}$$

(some classes write $MRS = -(\,)$ instead becaus the slope of an IC is negative)

(See Math section for more details)

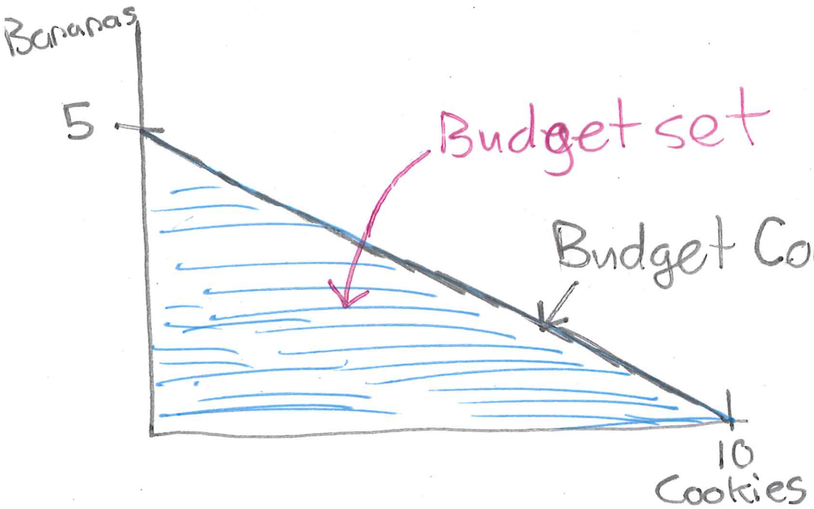
Budgets

- We have discussed what consumers want, now let's discuss what they can afford.

Monetary Constraints

Suppose you have \$10 and are deciding how many cookies and bananas to purchase. Cookies cost \$1 and bananas cost \$2. Assuming you can buy fractions of products, what set of goods can you buy?

$$\text{Budget Set} = \{(c,b) : 1 \cdot c + 2 \cdot b \leq 10\}$$



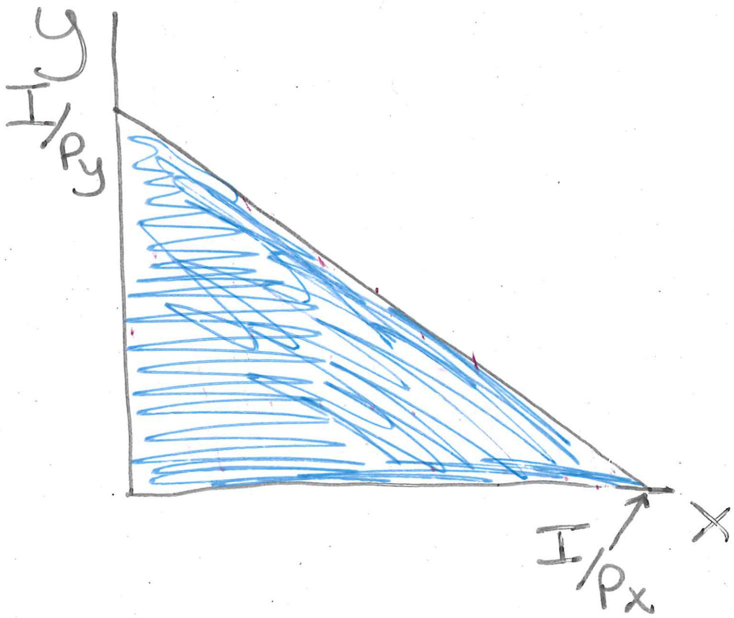
Budget Constraint is the line defined by $c + 2b = 10$

In general, the budget set is the set of all bundles that a consumer can afford given their financial situation.

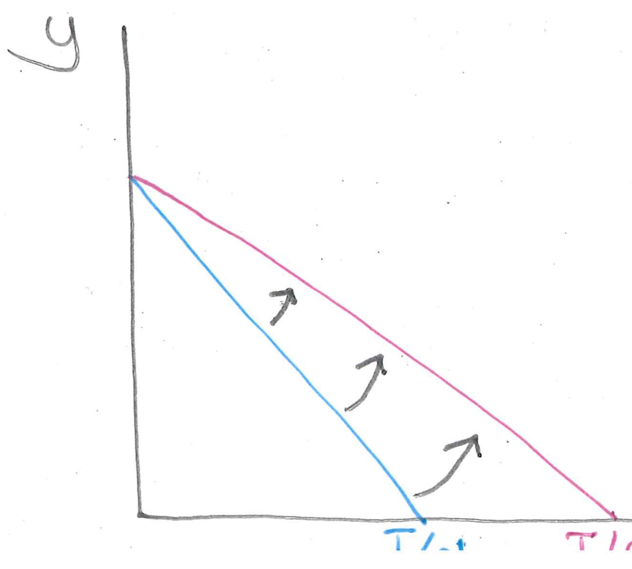
Budgets

Assume there are 2 goods, good x and good y. Assume each unit of x cost P_x and unit of y costs P_y. Assume you have income I.

- The Budget Set $= \{(x,y): P_x x + P_y y \leq I\}$
- The Budget Constraint is the line $P_x x + P_y y = I$.



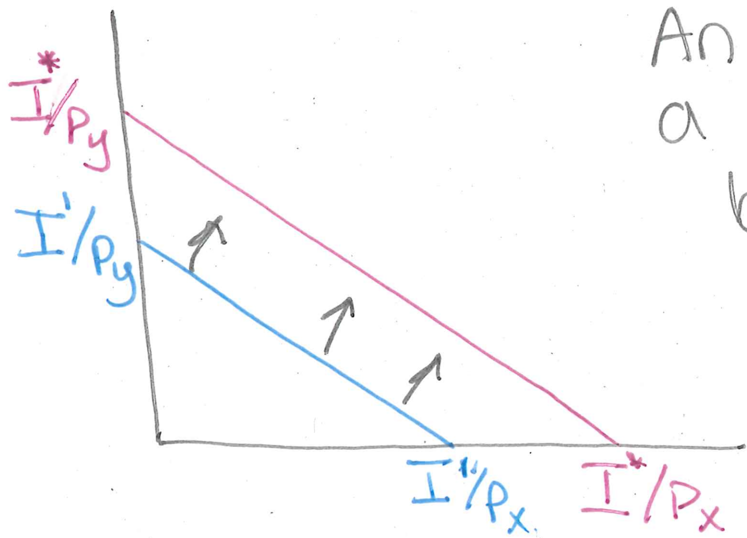
What does a price increase do to the budget set.



- When the price of good x decreases from P_x' to P_x^*, the budget set pivots out.
- Since P_y is fixed, both budgets include the point $(0, I/P_y)$.

Budgets

What does an income change look like? Suppose income increases for I' to I^*.



An income change causes a parallel shift in the budget constraint.

What would taxes, gifts and other changes mean for our budget sets? See A1 and A1 practice questions for a plethora of examples.

- <u>A very important note</u>: Constraints are not always monetary. Time and other limited rescurces also create constraints!
- The slope of the budget constraint $P_x X + P_y Y = I$ is called the marginal rate of transformation where $MRT = -\frac{P_x}{P_y}$

Utility Maximization

- So we know how to model what a consumer <u>desires</u> and we know how to represent what a consumer can <u>afford</u>. What will they choos

We will assume consumers maximize their utility subject to their constraints.

<u>The common Utility Max Problem</u>

$$\boxed{\max_{x,y} \; u(x,y) : P_x x + P_y y \leq I}$$

Given utility $u(x,y)$, prices P_x and P_y for each unit of x and y respectively, and income I; a consumer solves the problem above!

Assuming strict monotonicity of preferences, this problem similifies to

$$\boxed{\max_{x,y} \; u(x,y) : P_x x + P_y y = I}$$

Let's solve a "nice" utility max problem.

$\boxed{\text{Utility Maximization}}$

$$\max_{x,y} u(x,y) : P_x x + P_y y \leq I$$

$$\mathcal{L}(x,y;\lambda) = u(x,y) + \lambda(I - P_x x - P_y y)$$

(MRS = MRT)

$$\Rightarrow \mathcal{L}_x = u_x - \lambda P_x = 0 \quad \Rightarrow \frac{u_x}{P_x} = \frac{u_y}{P_y}$$

$$\mathcal{L}_y = u_y - \lambda P_y = 0$$

This is the tangency condition $\Rightarrow -\frac{P_x}{P_y} = -\frac{u_x}{u_y}$

Since P_x, P_y are exogenous and $u(x,y)$ is a function of 2 variables

$$\frac{u_x}{P_x} = \frac{u_y}{P_x} \text{ and } P_x x + P_y y = I$$

are 2 equations with 2 unknowns (x, y).

Example: If $u(x,y) = xy$, then $u_x = y$, $u_y = x$

$$\Rightarrow \frac{y}{P_x} = \frac{x}{P_y} \Rightarrow P_x x = P_y y \Rightarrow P_x x + P_x x = I$$

$$\Rightarrow x^* = \frac{I}{2P_x}. \text{ Similarly } y^* = \frac{I}{2P_y}$$

Important Note: Not all utilities provide "nice" frameworks. The math above does not

Utility Maximization

Why does the earlier math not work for all problems?

- Let $U(x,y) = x^2 + y^2$. Let $P_x = P_y = 1$. Let $I = 2$

$$\max_{x,y} x^2 + y^2 : x + y = 2$$

$$U_x = 2x, \; U_y = 2y$$

- $\frac{U_x}{P_x} = 2x = 2y = \frac{U_y}{P_y}$

$\Rightarrow x = y \Rightarrow x = y = 1$ (using $x + y = 2$).

The "math" tells use that $x = y = \underline{1}$ maximizes utility subject to the budget. Is this the actual answer?

Take a minute and try to show that $x = y = 1$ is not the bundle that maximizes utility subject to the budget constraint

Maximizing Utility

Showing a bundle is not the answer only requires finding a better alterative.

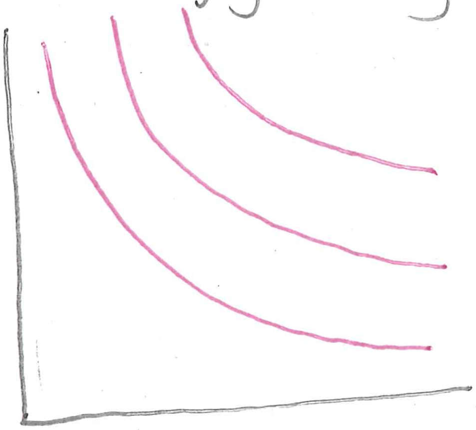
Let $x=0$ $y=2$. $x+y\le 2$ so this is the budge set.

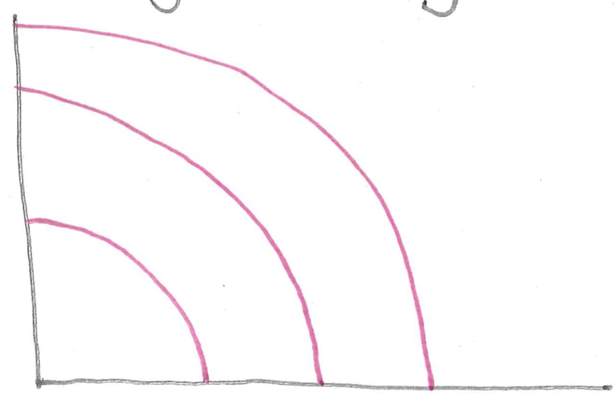
$u(0,2) = 0^2+2^2 = 4 \Rightarrow x^*=1, y^*=1$ is no the answer

$u(1,1) = 1^2+1^2 = 2$

What went wrong? Lets compare

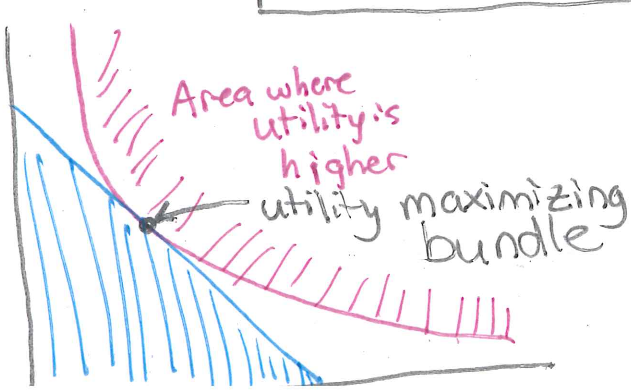
$u(x,y)=xy$ | $u(x,y)=x^2+y^2$

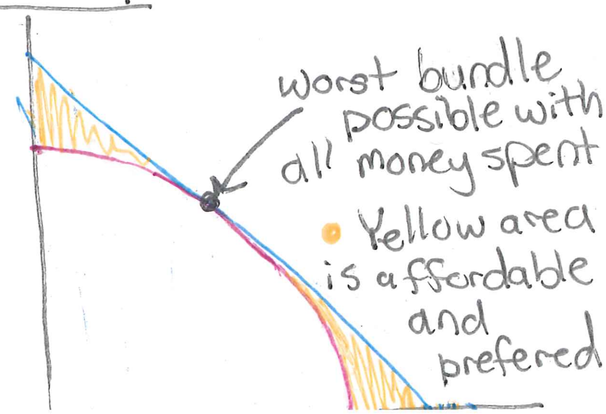




I.C are convex! | I.C are concave!

2 simple graphs show why convexity is good and concavity is bad





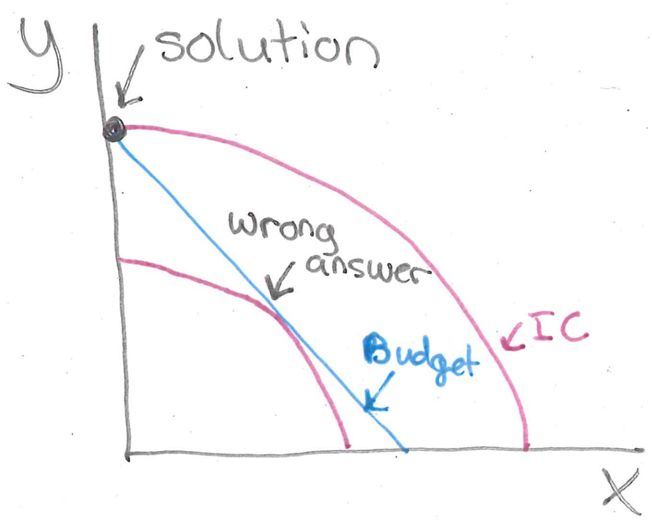
Utility Maximization

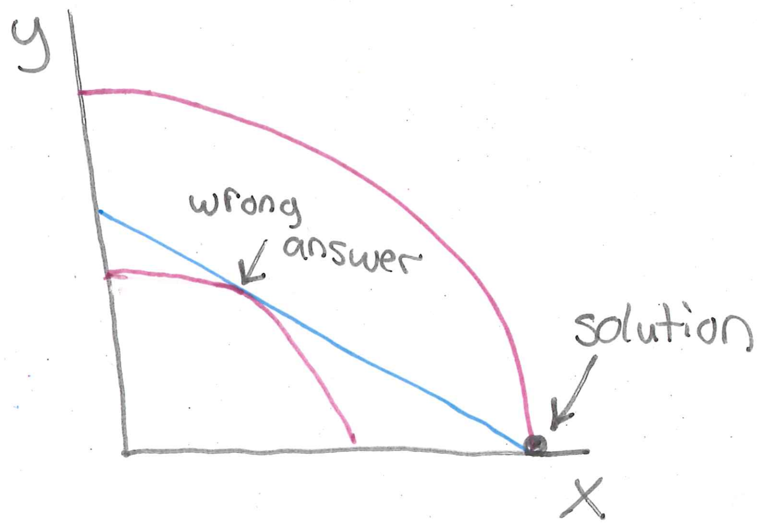
Corner Solutions
vs
Interior Solutions

A corner solution is when

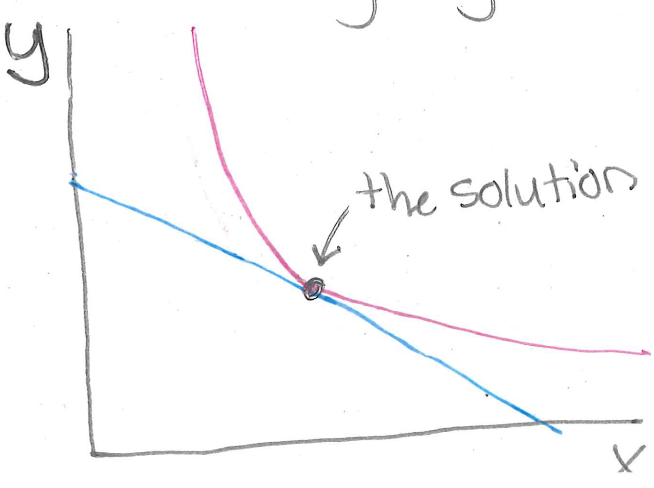
$$\max_{x,y} U(x,y) : P_x X + P_y y = I$$

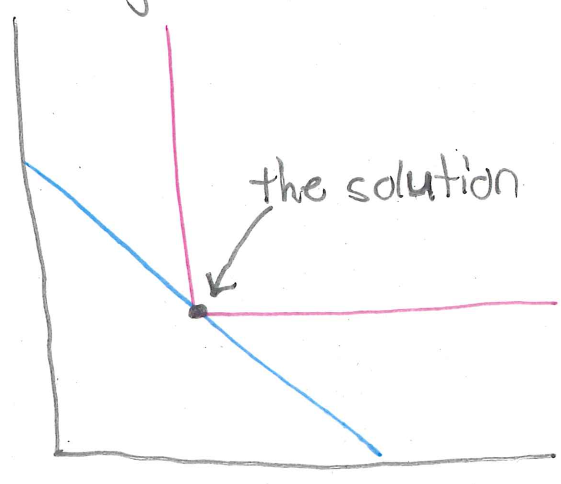
has a solution where either x or y is 0.





An interior solution is where $x^*, y^* > 0$ maximize utility given the budget.





Utility Maximization

Let's solve a few utility max problems:
(for the general version of utilities see A1 answers)

$U(x,y) = xy \Rightarrow \max_{x,y} \; xy : P_x x + P_y y \leq I$
(solved earlier)

$$\frac{U_x}{P_x} = \frac{y}{P_x} \qquad \frac{U_y}{P_y} = \frac{x}{P_y}$$

$$\Rightarrow \frac{y}{P_x} = \frac{x}{P_y} \Rightarrow P_x x = P_y y$$

$\Rightarrow$ optimal consumers spend the same amount on both goods

$$\Rightarrow x^* = \frac{I}{2P_x}, \quad y^* = \frac{I}{2P_y}.$$

Note: x^* does not change with P_y and y^* does not change with P_x.
(we will return to this property in a few minutes)

The solution above works because
$xy = \bar{U} \Rightarrow y = \bar{U}/x$ which is a convex function.

$\boxed{\text{Utility Maximization}}$

$$u(x,y) = x+y \Rightarrow \max_{x,y} \; x+y : P_x x + P_y y$$

$$\frac{U_x}{P_x} = \frac{1}{P_x}, \quad \frac{U_y}{P_y} = \frac{1}{P_y}$$

Remember x and y are what we want to solve for. P_x and P_y are values we do not get to choose when maximizing utility.

A common mistake: Do not assume $\frac{1}{P_x} = \frac{1}{P_y}$

Time for some intuition. U_x (sometimes denote MU_x) is the marginal utility with respect to good x. Loosely, it is the utility gained from a extra unit of good x

- Also, - P_x is the price of good x

- Thus, $\frac{U_x}{P_x}$ is the marginal increase in utility given one more dollar spent on good x.
- This is often refered to as <u>bang per buck</u>.

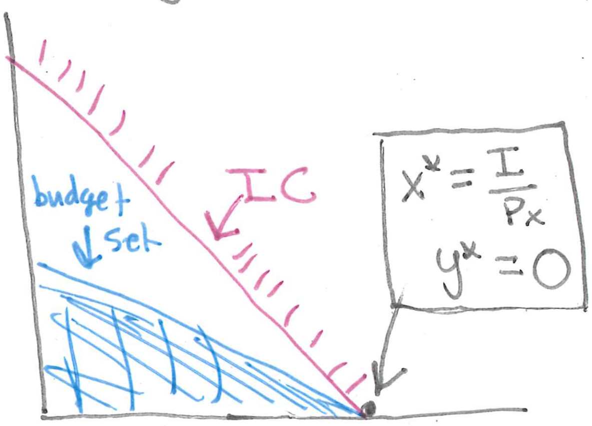
When $\frac{U_x}{P_x} > \frac{U_y}{P_y}$ a consumer is better off decreasing spending on good y and increasing spending on good x because the per $ improvement is better (see A2 for more)

Utility Maximization

Since $\frac{u_x}{P_x} = \frac{1}{P_x}$ and $\frac{u_y}{P_y} = \frac{1}{P_y}$, there are 3 distinct cases for us to consider.

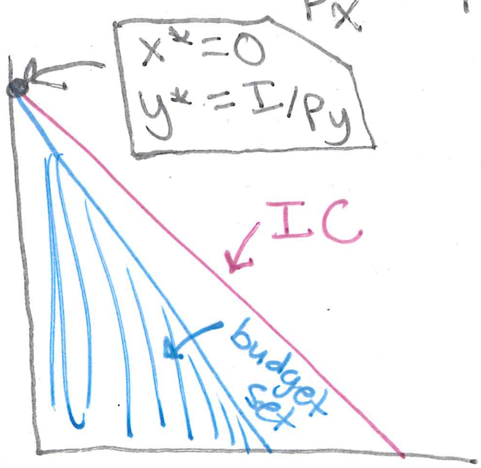
Case 1: $\frac{1}{P_x} > \frac{1}{P_y}$ $(P_x < P_y)$

In this case x is always a better option because spending less on good y and more on good x always increases utility.



This all relies on $\frac{u_x}{P_x} > \frac{u_y}{P_y}$ regardless of the value of x and y.

Case 2: $\frac{1}{P_x} < \frac{1}{P_y}$ $(P_x > P_y)$



Same argument with x and y switched.

Case 3: $1/P_x = 1/P_y$ This is asked in the assignment

Utility Maximization

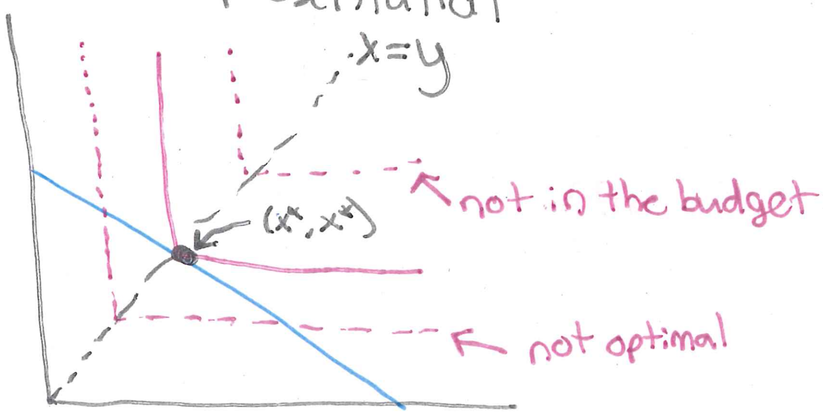
$u(x,y) = \min\{x,y\} \Rightarrow \max_{x,y} \min\{x,y\} : P_x x + P_y y \leq I$

Recall: $x > y \Rightarrow u(x,y) = y \Rightarrow u(x,x) = u(x,y)$
$x < y \Rightarrow u(x,y) = x \Rightarrow u(y,y) = u(x,y)$

Having extra x $(x>y)$ does not improve utility $(x>y)$
Having extra y $(y>x)$ does not improve utility $(y>x)$

Since every extra unit of a good costs money, the optimal bundle must be when $x=y$.

Visual Representation



The fact that $x=y$ is the tangency condition

Since $x=y$, $P_x x + P_y y = I \Rightarrow P_x x + P_y x = I$

$$\Rightarrow x = \frac{I}{P_x + P_y}$$

$$y = \frac{I}{P_x + P_y}$$

We call min utilities perfect compliments because $x=y$ no matter what P_x/P_y equa

The optimal ratio does not depend on prices

Utility Maximization

When solving a utility maximization problem, the optimal bundle (x^*, y^*) will depend on P_x, P_y, I which are the exogenous aspects of the problem.

$$\max_{x,y}\ u(x,y) : P_x x + P_y y \leq I \Rightarrow \boxed{\begin{array}{l} x^*(P_x, P_y, I) \\ y^*(P_x, P_y, I) \end{array}}$$

For example: $u(x,y) = xy \Rightarrow x^* = \dfrac{I}{2P_x}$

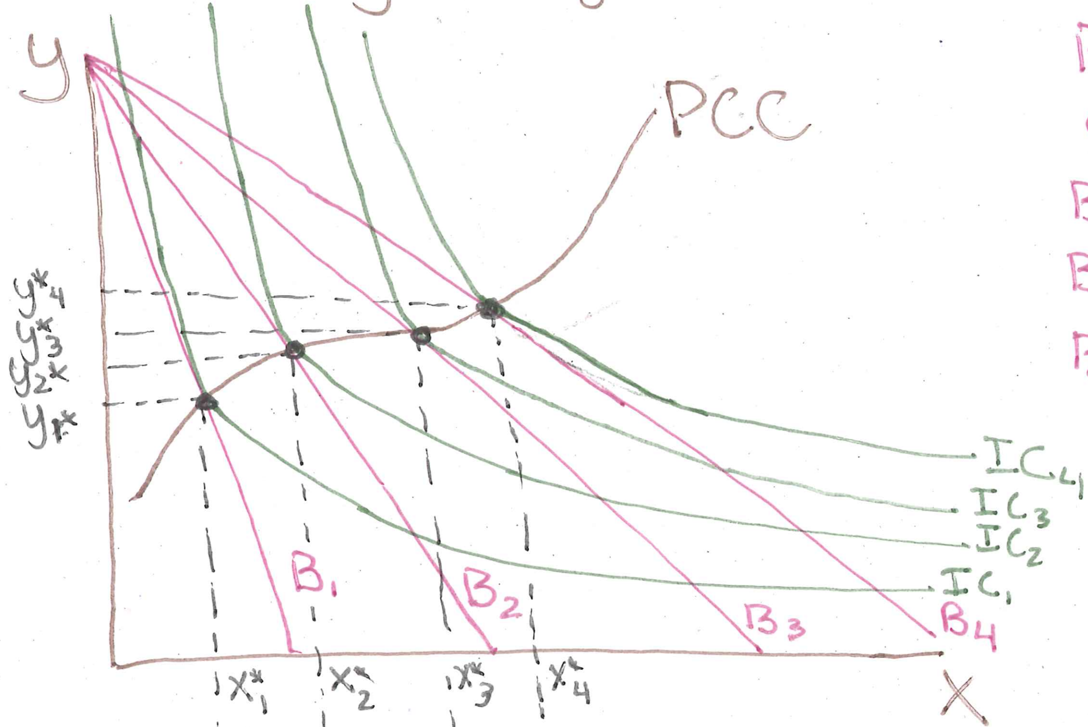
$$y^* = \frac{I}{2P_y}$$

$$u(x,y) = \min\{x,y\} \Rightarrow \boxed{\begin{array}{l} x^* = \dfrac{I}{P_x+P_y} \\ y^* = \dfrac{I}{P_x+P_y} \end{array}}$$

The rest of this section will cover how to analyze the effect price and income changes have on x^* and y^*.

Utility Maximization

he Price Consumption Curve (PCC) is the curve formed by (x^*, y^*) as either P_x or P_y changes.



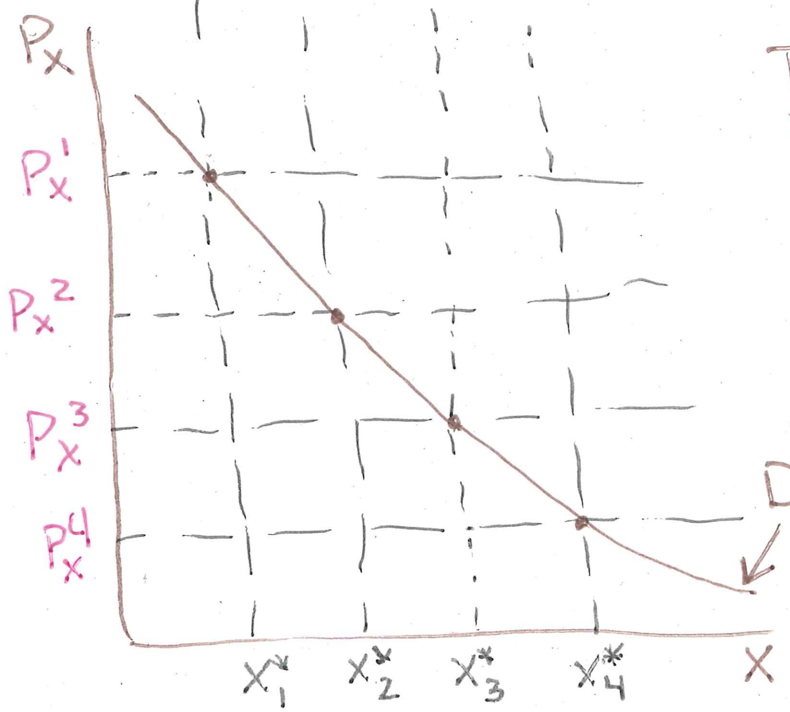
B_1 is the budget given P_x^1

$B_2 \rightarrow P_x^2$

$B_3 \rightarrow P_x^3$

$B_4 \rightarrow P_x^4$

$P_x^1 > P_x^2 > P_x^3 > P_x^4$

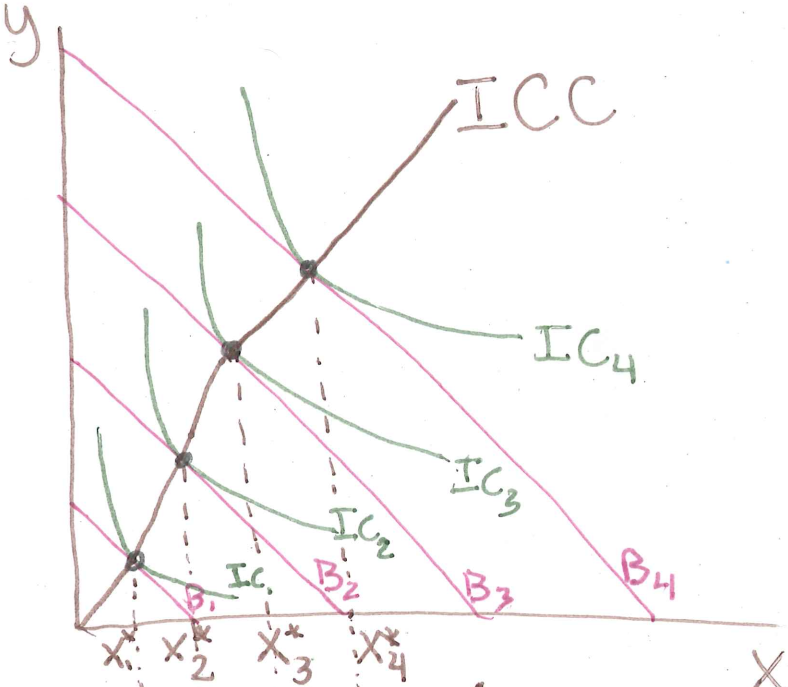


The Demand curve for good x is the curve formed by graphing x^* as a function of P_x.

The elasticity of demand $\varepsilon_D = \dfrac{\Delta\% x^*}{\Delta\% P_x}$

Utility Maximization

The Income Consumption Curve (ICC) is the curve formed by (x^*, y^*) as I changes.



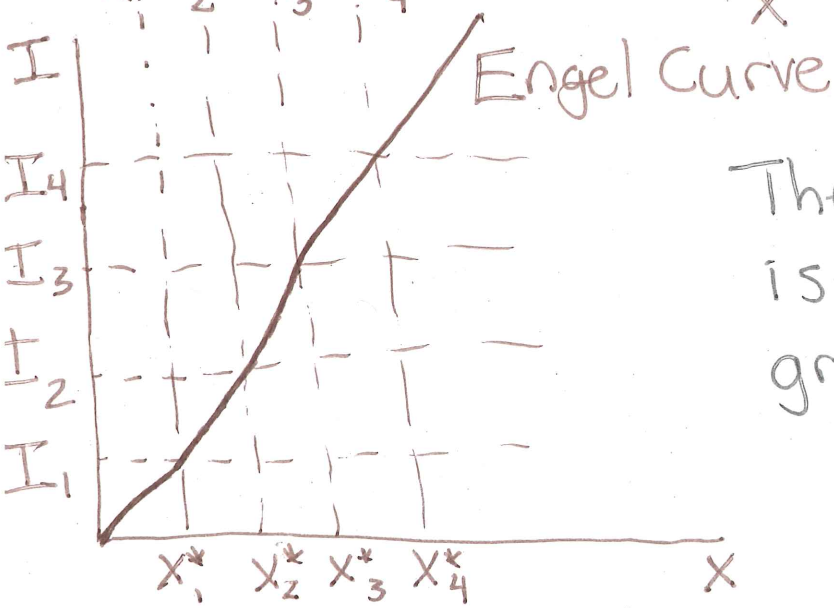
B_1 is the budget when $I = I'$

$B_2 \to I_2$

$B_3 \to I_3$

$B_4 \to I_4$

$I_1 < I_2 < I_3 < I_4$



The Engel Curve for good x is the curve formed by graphing x^* as a function of I.

Income Elasticity $\varepsilon_I = \dfrac{\%\Delta X^*}{\%\Delta I}$

Cross Price Elasticity $\varepsilon_{P_y, X^*} = \dfrac{\%\Delta X^*}{\%\Delta P_y}$

Utility Maximization

Given $X^*(P_x, P_y, I)$, there are 3 elasticities to discuss.

$$\varepsilon_D = \frac{\%\Delta X^*}{\%\Delta P_x} = \frac{\partial X^*}{\partial P_x}\frac{P_x}{X}$$

Since $X>0, P_x>0$ and $\frac{\partial X^*}{\partial P_x}<0$, $\boxed{\varepsilon_D<0}$

Remember for any elasticity ε,

$|\varepsilon|<1 \Rightarrow$ inelastic

$|\varepsilon|=1 \Rightarrow$ unit elastic

$|\varepsilon|>1 \Rightarrow$ elastic

$\varepsilon=0 \Rightarrow$ Perfectly inelastic, $|\varepsilon|=\infty \Rightarrow$ Perfectly elastic

$$\varepsilon_I = \frac{\%\Delta X^*}{\%\Delta I} = \frac{\partial X^*}{\partial I}\frac{I}{X^*}$$

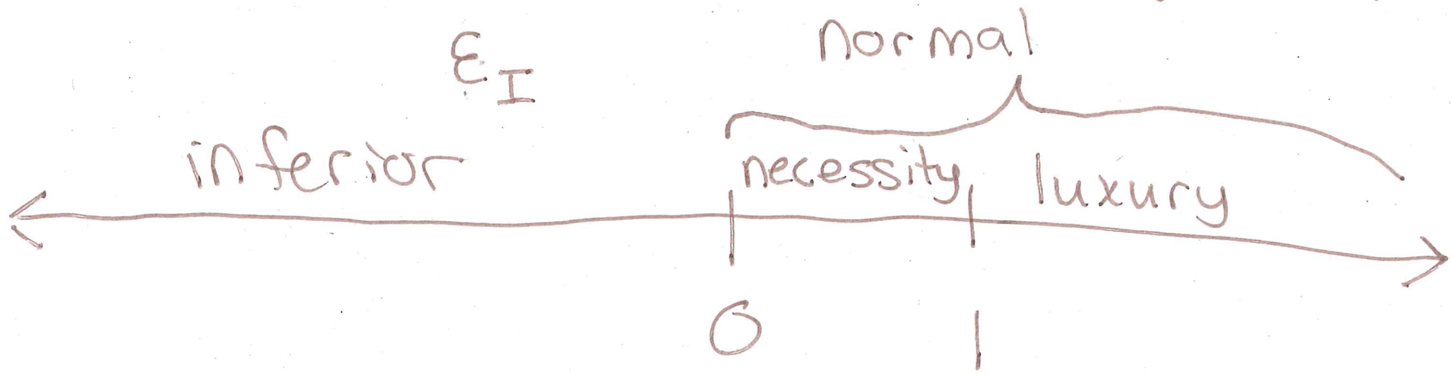
When $\varepsilon_I<0$, we call good x an inferior good because an income increase causes a decrease in consumption.

When $\varepsilon_I>0$, we call good x a normal good because an income increases causes a consumption increase

Utility Maximization

When $0 < \varepsilon_I < 1$, good x is a necessity good because consumption of x increases by a %Δ that is less than the % change in I (inelastic)

When $\varepsilon_I > 1$, good x is a luxury good because consumption changes by a %Δ that is greater than the % change in I.



Cross Price Elasticity: $\frac{\%\Delta X}{\%\Delta P_y}$

When $\frac{\Delta\% X}{\Delta\% P_y} > 0 \Rightarrow$ x and y are <u>substitutes</u> because an increase in the price of good y causes consumption of good x to increase

When $\frac{\Delta\% X}{\Delta\% P_y} < 0 \Rightarrow$ x and y are <u>complements</u> because ... an increase ... causes a decrease

Utility Maximization

- Two goods are substitutes if a price increase in one good causes an increase in consumption of the other good.

  $u(x,y) = ax + by$ means good x and good y are <u>perfect substitutes</u>

  x and y are perfect subsitutes because whichever good is relatively cheaper will be the only good a consumer consumes.

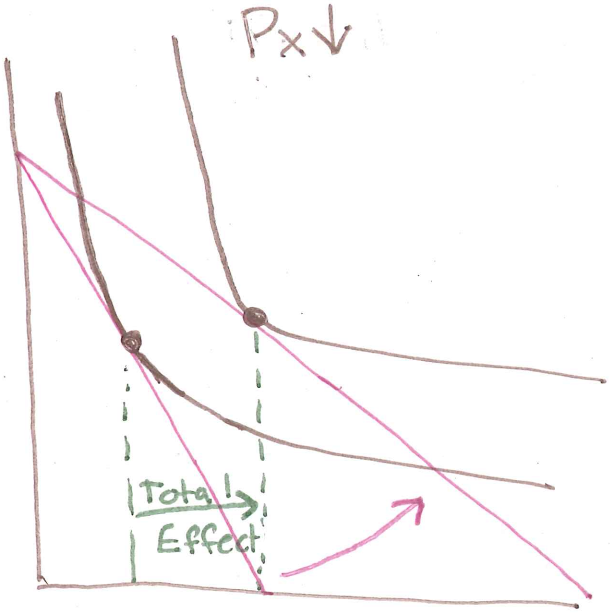
  See A2 for more

- Two goods are complements if a price increase in one good causes a decrease in the consumption of the other good.

  $u(x,y) = \min\{ax, by\}$ because no matter how much the price of one good increases by the two goods will still be consumed at the same ratio.

  See A2

Utility Maximization



When the price of good X decreases,

1. $\frac{-P_x}{P_y}$ changes
2. The budget set increases to include more bundles

The goal: How much of the total change in the consumption of good x is the result of purely the change in the price ratio?

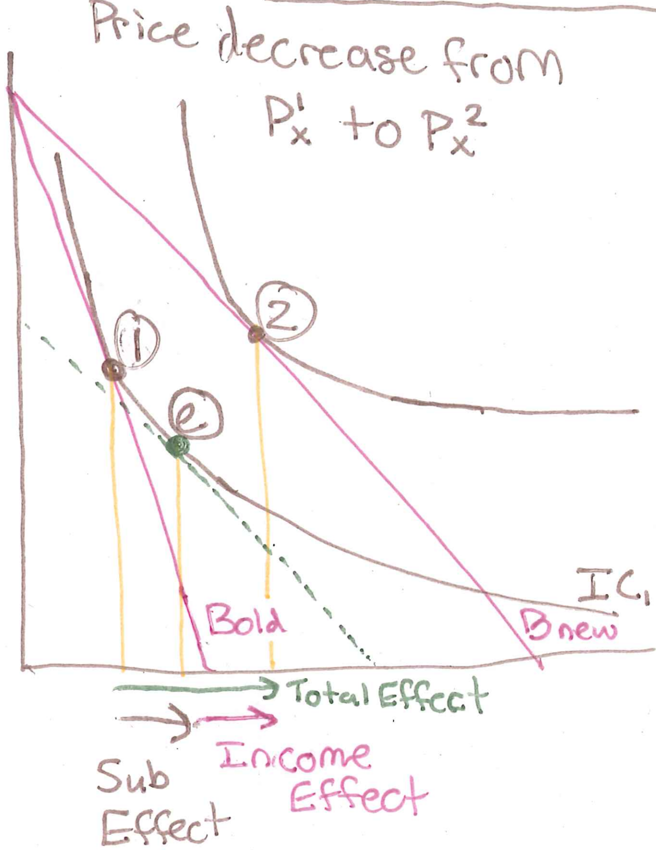
We will call this part the **substitution effect**

How much of the total change in the consumpt of good x is the result of the increase in a consumers "real income"?

We call this effect the income ~~effect~~ effect.

This will be explained indetail in class.

Utility Maximization



The (SE) Substitution effect is the change in x^* due to the change in $\frac{P_x}{P_y}$ holding the original utility level constant.

Math: $SE = x^e - x_1$

where x_1 is the original consumption of good x.

$x_1 = x^*(P_x^1, P_y, I)$

$x^e = x(P_x^2, P_y, I^e)$

where I^e is the income such that

$u^*(P_x^2, P_y, I^e) = u^*(P_x^1, P_y, I)$

(u^* is the maximized value of utility.

x^e is the hardest of the 3 values to calculate.

The Income Effect is the change in x^* that results from the income increase holding the price change constant.

$IE = x_2 - x^e$

where $x_2 = x^*(P_x^2, P_y, I$

See A2 and A2 practice for more details!

Note: $IE + SE = TE$ (total effect)

Utility Maximization

Extensions of Utility Maximization:

① Labor - Leisure Model

- Let $U(l,x)$ be a consumer's utility function given l amount of leisure and x amount of good x.
- A consumer has 16 hours at most to work or relax (leisure) during a day.

$\Rightarrow$ $n+l \leq 16$ where n is the # of hours worked.

- A consumer earns w per hour and has no outside income so their budget constraint is: $w \cdot n \geq P_x X$

$$\Rightarrow \max_{l,X} U(l,x): \begin{array}{l} n+l \leq 16 \\ w \cdot n \geq P_x X \end{array}$$

Assuming more is better (which is always assumed unless otherwise stated)

$n+l=16$ and $wn=P_x X \Rightarrow w(16-l)=P_x X$

$\Rightarrow 16w = wl + P_x X$

Utility Maximization

$$\Rightarrow \max_{\ell, x} u(\ell, x) : w\ell + P_x x = 16w$$

Since you already know how to solve:

$$\max_{x,y} u(x,y) : P_x x + P_y y = 16w$$

you can solve the labor-leisure problem where $P_y = w$, $y = \ell$ and $I = 16w$.
(see A3 for more details)

2 Utility maximization with endowments.

Instead of having income I, assume consumers have x^e and y^e as an endowment that can either be sold (at P_x and P_y respectively) or can be consumed. Any extra x or y past x^e and y^e must be paid for by selling the other good.

$$\Rightarrow \max_{x,y} u(x,y) : P_x x + P_y y \leq P_x x^e + P_y y^e$$

(try to solve this model on