Unit-2 Vector Calculus

Recap of 12th :-

$\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$ $\qquad$ $\vec{b} = \hat{i} + \hat{j} + 3\hat{k}$.

$\vec{a} \cdot \vec{b} = 2\times1 + 3\times1 + 3\times1 = 8$

$\vec{a} \times \vec{b} = 8\hat{i} - 5\hat{j} - \hat{k}$

$\vec{a} + \vec{b} = 3\hat{i} + 4\hat{j} + 4\hat{k}$

$\vec{a} - \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

Vector :- $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{r} \cdot \vec{r} = r^2 = |r|^2 = x^2 + y^2 + z^2$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

Vector Differential operator (∇/Del) :-

We have, $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Gradient of a scalar function :-

Let $\phi(x, y, z)$ be a scalar function, then gradient of ϕ is denoted and defined by

$$\text{grad}\,\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi.$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

* By using gradient, we change scalar to vector.

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Q If $\phi = x^2y + y^2x + z^2$, find $\nabla\phi$ or grad ϕ at $(1,1,1)$.

Sol→ We are given,

$$\phi = x^2y + y^2x + z^2$$

$$\frac{\partial\phi}{\partial x} = 2xy + y^2$$

$$\frac{\partial\phi}{\partial y} = x^2 + 2yx$$

$$\frac{\partial\phi}{\partial z} = 2z$$

at $(1,1,1)$

$$\frac{\partial\phi}{\partial x} = 2\cdot1\cdot1 + 1^2 = 3$$

$$\frac{\partial\phi}{\partial y} = 1^2 + 2\cdot1\cdot1 = 3$$

$$\frac{\partial\phi}{\partial z} = 2\cdot1 = 2$$

We know, $\nabla\phi$ or grad $\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$.

$$= 3\hat{i} + 3\hat{j} + 2\hat{k}$$

↳ vector

$$\hat{i}(2xy + y^2) + \hat{j}(x^2 + 2yx) + \hat{k}(2z)$$

↳ Vector function.

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Q Find grad ϕ, if $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$.

Sol→ We are given, $\phi = x^2yz + 4xz^2$.

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2$$

At $(1,-2,-1)$

$$\frac{\partial\phi}{\partial x} = 4 + 4 = 8$$

$$\frac{\partial\phi}{\partial y} = x^2z$$

$$\frac{\partial\phi}{\partial y} = -1$$

$$\frac{\partial\phi}{\partial z} = x^2y + 8xz$$

$$\frac{\partial\phi}{\partial z} = -2 - 8 = -10$$

$$\text{Grad } \phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= 8\hat{i} - \hat{j} - 10\hat{k}$$

Unit Normal Vector :-

Let $\phi(x,y,z)$ be a scalar function, then unit Normal vector is given by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$ where $\phi \rightarrow$ scalar function.

Q Find the unit Normal vector of the surface $x^2y + 2xz = 4$ at $(2,-2,3)$.

Sol→ Let $\phi = x^2y + 2xz - 4 = 0$ ---- ①

$$\frac{\partial\phi}{\partial x} = 2xy + 2z$$

$$\frac{\partial\phi}{\partial y} = x^2$$

$$\frac{\partial\phi}{\partial z} = 2x$$

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At point $(2, -2, 3)$.

$$\frac{\partial \phi}{\partial x} = -8 + 6 = -2$$

$$\frac{\partial \phi}{\partial y} = 4$$

$$\frac{\partial \phi}{\partial z} = 4$$

$$\nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$\nabla \phi = -2\hat{i} + 4\hat{j} + 4\hat{k}$$

$$|\nabla \phi| = \sqrt{4+16+16} = \sqrt{36} = 6$$

Unit Normal vector $= \dfrac{\nabla \phi}{|\nabla \phi|}$

$$= \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{6} = \frac{-\hat{i}}{3} + \frac{2\hat{j}}{3} + \frac{2\hat{k}}{3}$$

Divergence :-

Let $\phi = \hat{i}\phi_1 + \hat{j}\phi_2 + \hat{k}\phi_3$ be a vector function, then divergence of ϕ is denoted and defined by

$$\text{div}\,\phi = \nabla \cdot \phi = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_3}{\partial z}$$

If $\text{div}\,\phi = 0$, then ϕ is called Solenoidal vector.

* Property of div is that it changes vector function to scalar ~~vec~~ function.

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Q Find div ϕ, if $\phi = \hat{i}\,2xy + \hat{j}\,y + \hat{k}\,z^2$ at $(1, 0, 1)$.

Sol→ Given $\phi = \hat{i}\,2xy + \hat{j}\,y + \hat{k}\,z^2$.

$$\text{div}\,\phi = \nabla \cdot \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\,2xy + \hat{j}\,y + \hat{k}\,z^2\right).$$

$$= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2).$$

$$= 2y + 1 + 2z$$

At point $(1, 0, 1)$.

$$\nabla \cdot \phi = 2 \cdot 0 + 1 + 2 \cdot 1 = 3 \quad \underline{\text{Ans}}$$

Q If vector is solenoidal, find n when $\phi = \hat{i}\,nx + \hat{j}\,y^2 + \hat{k}\,2z$, at $(1, 1, 1)$.

Sol→ Given $\phi = \hat{i}\,nx + \hat{j}\,y^2 + \hat{k}\,2z$.

$$\text{div}\,\phi = \frac{\partial}{\partial x}(nx) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(2z).$$

$$\text{div}\,\phi = n + 2y + 2.$$

At point $(1, 1, 1)$.

$$\text{div}\,\phi = (n + 2 \cdot 1 + 2) = n + 4$$

Now, since vector is solenoidal, $\text{div}\,\phi = 0$.

$$\Rightarrow n + 4 = 0$$

$$\Rightarrow n = -4 \quad \underline{\text{Ans}}$$

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Directional Derivative :-

Let $\phi(x, y, z)$ be scalar function, Then the directional derivative in direction of $\vec{a}$ is given by.

$$\frac{d\phi}{ds} = \text{grad}\,\phi \cdot \hat{a}.$$

directional derivative = grad $\phi \cdot \hat{a}$.

Q Find the directional derivative of $f = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.

Sol → Given $f = x^2yz + 4xz^2$.

$$\text{grad } f = \hat{i}(2xyz + 4z^2) + \hat{j}(x^2z) + \hat{k}(x^2y + 8xz)$$

at $(1, -2, -1)$,

$$\text{grad } f = \hat{i}(8) - \hat{j} - 10\hat{k}.$$

and $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$

$$|\vec{a}| = \sqrt{4+1+4} = 3.$$

$$\hat{a} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$$

The directional derivative = grad $f \cdot \hat{a}$.

$$= (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3}$$

$$= \frac{16 + 1 + 20}{3} = \frac{37}{3} \text{ Ans}$$

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Q find the directional derivative of $f = x^2 - 2y^2 + 4z^2$ at $(1, 1, -1)$ in direction of $2\hat{i} + \hat{j} - \hat{k}$.

Sol→ Given $f = x^2 - 2y^2 + 4z^2$.

$$\text{grad } f = \hat{i}(2x) + \hat{j}(-4y) + \hat{k}(8z)$$

at $(1,1,-1)$, $\text{grad } f = 2\hat{i} - 4\hat{j} - 8\hat{k}$

$$\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$$

$$|\vec{a}| = \sqrt{4+1+1} = \sqrt{6}$$

$$\hat{a} = \frac{2\hat{i} + \hat{j} - \hat{k}}{\sqrt{6}}$$

Directional derivative $= \text{grad } f \cdot \hat{a}$

$$= (2\hat{i} - 4\hat{j} - 8\hat{k}) \cdot \frac{(2\hat{i} + \hat{j} - \hat{k})}{\sqrt{6}}$$

$$= \frac{4 - 4 + 8}{\sqrt{6}} = \frac{8}{\sqrt{6}} \text{ Ans}$$

Q Find the directional derivative of $f = x^2 - y^2 + 2z^2$. at point $P(1,2,3)$ in the direction of PQ where Q is point $(5, 0, 4)$.

(i) in the direction of $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{1}$.

Sol→ Given $f = x^2 - y^2 + 2z^2$.

$$\text{grad } f = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

at $P(1,2,3)$.

$$\text{grad } f = 2\hat{i} - 4\hat{j} + 12\hat{k}$$

Also, $\vec{a} = \overrightarrow{PQ}$ = position of vector of Q − Position of vector of P.

$$= (5\hat{i} + 0\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\vec{a} = 4\hat{i} - 2\hat{j} + \hat{k}$$

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$$|\vec{a}| = \sqrt{16+4+1} = \sqrt{21}$$

$$\hat{a} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

Directional derivative = grad f . $\hat{a}$.

$$= \left(2\hat{i} - 4\hat{j} + 12\hat{k}\right) \cdot \left(\frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}\right)$$

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}} \quad \underline{\text{Ans}}$$

(i) at point (1, 1, 1).

grad $f = 2\hat{i} - 2\hat{j} + 4\hat{k}$

Let $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$ $\quad [\because \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}]$

$|\vec{a}| = \sqrt{4+9+1} = \sqrt{14}$ $\quad \hat{a} = \frac{2\hat{i} + 3\hat{j} + \hat{k}}{\sqrt{14}}$

Directional derivative $= \left(2\hat{i} - 2\hat{j} + 4\hat{k}\right) \cdot \left(\frac{2\hat{i} + 3\hat{j} + \hat{k}}{\sqrt{14}}\right)$

$$= \frac{4-6+4}{\sqrt{14}} = \frac{2}{\sqrt{14}} \quad \underline{\text{Ans}}$$

Curl of vector function :-

Let $\phi = \phi_1\hat{i} + \phi_2\hat{j} + \phi_3\hat{k}$ be a vector function, then curl of ϕ is given by.

$$\text{Curl } \phi = \nabla \times \phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_1 & \phi_2 & \phi_3 \end{vmatrix}$$

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If Curl $\phi = 0$, then vector ϕ is called irrotational vector.

Scalar potential → A vector $\vec{F}$ is said to be irrotational if $\nabla \times F = 0$. In case, we can represent

$$d\phi = \vec{F} \cdot \vec{dr} \qquad [\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$$

Here ϕ is called scalar potential.

Q Show that the vector field F given by $F = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$ is irrotational. Also find scalar potential.

Sol → Given $F = (x^2 + yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$.

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - xz) & (y^2 - xz) & (z^2 - xy) \end{vmatrix}$$

$$= \hat{i}(-x + x) - \hat{j}(-y + y) + \hat{k}(-z + z)$$

$$= 0$$

Curl F = 0

⇒ Vector is called irrotational.

For scalar potential :-

$$d\phi = F \cdot dr$$

$$d\phi = [(x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}] \cdot [dx\hat{i} + dy\hat{j} + dz\hat{k}]$$

$$d\phi = (x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz.$$

$$d\phi = (x^2 dx + y^2 dy + z^2 dz) - (yz dx + xz dy + xy dz$$

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$$= d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) + d\left(\frac{z^3}{3}\right) - d(x.yz).$$

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + C.$$ Ans

Q Show that the vector field F given by

$F = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$.

Also find scalar potential.

Sol → Curl F =
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & (2y\sin x - 4) & (3xz^2+2) \end{vmatrix}$$

$$= \hat{i}(0+0) - \hat{j}(3z^2 - 3z^2) + \hat{k}(2y\cos x - 2y\cos x)$$

$$= 0$$

Curl F = 0

⇒ vector is irrotational.

for scalar potential,

$$d\phi = F \cdot dr.$$

$$d\phi = (y^2\cos x + z^3)dx + (2y\sin x - 4)dy + (3xz^2+2)dz.$$

$$d\phi = y^2\cos x\,dx + z^3dx + 2y\sin x\,dy - 4dy + 3xz^2dz + 2dz.$$

$$d\phi = (y^2\cos x\,dx + 2y\sin x\,dy) + (z^3dx + 3xz^2dz) - 4dy + 2dz$$

$$d\phi = d(y^2\sin x) + d(xz^3) - 4dy + 2dz$$

$$\phi = y^2\sin x + xz^3 - 4y + 2z + C$$ Ans

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Angle between two surface :-

Let ϕ_1 and ϕ_2 are two surfaces and let θ be angle between them, then,

$$\cos\theta = \frac{(\nabla\phi_1)\cdot(\nabla\phi_2)}{|\nabla\phi_1|\times|\nabla\phi_2|}$$

① If $\theta = 90^\circ$, then

$$(\nabla\phi_1)\cdot(\nabla\phi_2) = 0$$

$\Rightarrow$ Surface are perpendicular to each other.

② If $\theta = 0^\circ$, then

$$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2|.$$

Q Find the angle between two surface $x^2+yz=2$ & $x+2y-z=2$ at $(1,1,1)$.

Sol→ Let $\phi_1 = x^2+yz-2=0$

$\phi_2 = x+2y-z-2=0$

$$\nabla\phi_1 = \hat{i}\frac{\partial}{\partial x}(x^2+yz-2) + \hat{j}\frac{\partial}{\partial y}(x^2+yz-2) + \hat{k}(x^2+yz-2)$$

$$= \hat{i}(2x) + \hat{j}(z) + \hat{k}(y).$$

At point $(1,1,1)$

$$\nabla\phi_1 = 2\hat{i}+\hat{j}+\hat{k} \Rightarrow |\nabla\phi_1| = \sqrt{4+1+1} = \sqrt{6}$$

$$\nabla\phi_2 = \hat{i}\frac{\partial}{\partial x}(x+2y-z-2) + \hat{j}\frac{\partial}{\partial y}(x+2y-z-2) + \hat{k}\frac{\partial}{\partial z}(x+2y-z-2)$$

$$\nabla\phi_2 = \hat{i}+\hat{j}(2)-\hat{k} \Rightarrow |\nabla\phi_2| = \sqrt{1+4+1} = \sqrt{6}$$

$$\cos\theta = \frac{(2\hat{i}+\hat{j}+\hat{k})\cdot(\hat{i}+2\hat{j}-\hat{k})}{\sqrt{6}\,\sqrt{6}} = \frac{3}{2} = \frac{1}{2}$$

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$$\cos\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 60^\circ \quad \text{Ans}$$

Q Find the angle between two surface $z = x^2+y^2-3$ and $x^2+y^2+z^2=9$ at $(2,-1,2)$.

Sol $\rightarrow$ Let $\phi_1 = x^2+y^2-z-3=0$

Let $\phi_2 = x^2+y^2+z^2-9=0$

$$\nabla\phi_1 = \hat{i}\frac{\partial}{\partial x}(x^2+y^2-z-3) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2-z-3) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2-z-3)$$

$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}(-1).$$

at point $(2,-1,2)$.

$$\Rightarrow \nabla\phi_1 = 4\hat{i} - 2\hat{j} - \hat{k} \Rightarrow |\nabla\phi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$\nabla\phi_2 = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-9) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-9) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-9)$$

$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z).$$

at $(2,-1,2) = 4\hat{i} - 2\hat{j} + 4\hat{k} \Rightarrow |\nabla\phi_2| = \sqrt{16+4+16} = 6$

$$\cos\theta = \frac{(4\hat{i}-2\hat{j}-\hat{k})\cdot(4\hat{i}-2\hat{j}+4\hat{k})}{6\sqrt{21}}$$

$$\cos\theta = \frac{16+4-4}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right) \quad \text{Ans}$$

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Q Find a and b so that surface $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$

Sol → Let $\phi_1 = ax^2 - byz - (a+2)x = 0$

$\phi_2 = 4x^2y + z^3 - 4 = 0$

$$\nabla\phi_1 = \hat{i}\frac{\partial}{\partial x}\left[ax^2 - byz - (a+2)x\right] + \hat{j}\frac{\partial}{\partial x}\left[ax^2 - byz - (a+2)x\right] + \hat{k}\left[ax^2 - byz - (a+2)x\right]$$

$$= \hat{i}\left[2ax - (a+2)\right] + \hat{j}\left[-bz\right] + \hat{k}\left[-by\right].$$

at $(1,-1,2)$, $\nabla\phi_1 = (a-2)\hat{i} - 2b\hat{j} + b\hat{k}$

$$\nabla\phi_2 = \hat{i}\frac{\partial}{\partial x}(4x^2y + z^3 - 4) + \hat{j}\frac{\partial}{\partial y}(4x^2y + z^3 - 4) + \hat{k}\frac{\partial}{\partial z}(4x^2y + z^3 - 4).$$

$$= \hat{i}(8xy) + \hat{j}(4x^2) + \hat{k}(3z^2).$$

at point $(1,-1,2)$,

$$\nabla\phi_2 = -8\hat{i} + 4\hat{j} + 12\hat{k}.$$

Since both surface cut orthogonally,

$$(\nabla\phi_1)\cdot(\nabla\phi_2) = 0.$$

$\Rightarrow (a-2)(-8) - 2b(4) + 12(b) = 0$

$\Rightarrow -2a + 4 - 2b + 3b = 0$

$\Rightarrow 2a - b = 4$ ——— (1)

Since the point $(1,-1,2)$ lie on the surface.

Taking $ax^2 - byz = (a+2)x$.

$\Rightarrow a - b(-1)(2) = (a+2)$

$\Rightarrow 2b = b$

$\Rightarrow b = 1$ ——— (2)

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Putting value of eq. (2) in (1), we get

$$2a - 1 = 4$$

$$\Rightarrow a = \frac{5}{2}$$

So, $b = 1$ & $a = \frac{5}{2}$ Ans

* We have $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r^2 = x^2 + y^2 + z^2 \quad \text{——— (1)}$$

differentiating w.r.t x, y, z

$$2r\frac{dr}{dx} = 2x$$

$$\Rightarrow \boxed{\frac{dr}{dx} = \frac{x}{r}}$$

Similarly,

$$\boxed{\frac{dr}{dy} = \frac{y}{r}}$$

$$\boxed{\frac{dr}{dz} = \frac{z}{r}}$$

Q Evaluate ∇r^n.

Sol→ We have $\nabla r^n = \hat{i}\frac{\partial}{\partial x}(r^n) + \hat{j}\frac{\partial}{\partial y}(r^n) + \hat{k}\frac{\partial}{\partial z}(r^n)$ — (1)

Now, $\frac{d}{dx} r^n = n r^{n-1}\frac{dr}{dx} = n r^{n-1}\left(\frac{x}{r}\right)$

$$\frac{d}{dx}(r^n) = x n r^{n-2} \quad \text{——— (2)}$$

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Similarly, $\frac{d}{dy}(r^n) = nyr^{n-2}$ ——— (3)

$\frac{d}{dz}(r^n) = nzr^{n-2}$ ——— (4)

Putting all values of (2), (3) & (4) in eq. (1)

$$\nabla r^n = \hat{i}\, nxr^{n-2} + \hat{j}\, nyr^{n-2} + \hat{k}\, nzr^{n-2}.$$

$$= (x\hat{i} + y\hat{j} + z\hat{k})\, nr^{n-2}.$$

$$\nabla r^n = nr^{n-2}\vec{r}$$ Ans

Q Evaluate $\nabla^2 r^n$.

Sol→ We have

$$\nabla^2 r^n = \frac{\partial^2}{\partial x^2}(r^n) + \frac{\partial}{\partial y^2}(r^n) + \frac{\partial}{\partial z^2}(r^n). \quad \text{--- (1)}$$

Now, $\frac{\partial^2}{\partial x^2} r^n = \frac{\partial}{\partial x}\left(nr^{n-1}\frac{\partial r}{\partial x}\right)$

$$= \frac{\partial}{\partial x}\left(nr^{n-1}\frac{x}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(nxr^{n-2}\right)$$

$$= n\left[1 \cdot r^{n-2} + x(n-2)r^{n-3}\frac{\partial r}{\partial x}\right]$$

$$\frac{\partial^2}{\partial x^2} r^n = n\left[r^{n-2} + x^2(n-2)r^{n-4}\right] \quad \text{--- (2)}$$

Similarly, $\frac{\partial^2}{\partial y^2} r^n = n\left[r^{n-2} + y^2(n-2)r^{n-4}\right]$ ——— (3)

$$\frac{\partial^2}{\partial z^2} r^n = n\left[r^{n-2} + z^2(n-2)r^{n-4}\right] \quad \text{--- (4)}$$

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Using ②, ③ and ④, we get

$$\nabla^2 r^n = n\left[3r^{n-2} + (n-2)r^{n-4}(x^2+y^2+z^2)\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-4}(r^2)\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-2}\right]$$

$$= nr^{n-2}\left[3+n-2\right]$$

$$= nr^{n-2}(n+1)$$

$$\nabla^2 r^n = n(n+1)r^{n-2} \quad \underline{\underline{\text{Ans}}}$$

Q If $U = f(r)$, where $x = r\cos\theta$ and $y = r\sin\theta$.
Prove that $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$

Sol → We have $U = f(r)$. ——— ①

and $x = r\cos\theta$

$y = r\sin\theta$

On squaring and adding.

$$r^2 = x^2 + y^2$$

On differentiating w.r.t x and y.

$$2r\frac{dr}{\partial x} = 2x \Rightarrow \frac{dr}{dx} = \frac{x}{r}$$

Similarly, $\dfrac{dr}{dy} = \dfrac{y}{r}$.

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$$U = f(r)$$

$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x}$$

$$= f'(r)\frac{x}{r}$$

$$= f'(r)x \cdot \frac{1}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\frac{\partial r}{\partial x} \cdot \frac{x}{r} + f'(r) \cdot \frac{1}{r} \cdot 1 + f'(r)x\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\frac{x^2}{r^2} + \frac{1}{r}f'(r) - \frac{x^2}{r^3}f'(r) \quad \text{---- (2)}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{1}{r}f'(r) - \frac{y^2}{r^3}f'(r) \quad \text{---- (3)}$$

On adding (2) and (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{x^2+y^2}{r^2}\right) + \frac{2}{r}f'(r) - f'(r)\left(\frac{x^2+y^2}{r^3}\right).$$

$$= f''(r)\left(\frac{r^2}{r^2}\right) + \frac{2}{r}f'(r) - f'(r)\left(\frac{r^2}{r^3}\right).$$

$$= f''(r) + \frac{2}{r}f'(r) - \frac{1}{r}f'(r).$$

$$= f''(r) + \frac{1}{r}f'(r)$$

So, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$

Hence proved

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Q find the value of n if the vector $r^n \vec{r}$ is solenoidal.

Sol → Let $F = r^n \vec{r}$

$$= r^n (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\vec{F} = xr^n\hat{i} + yr^n\hat{j} + zr^n\hat{k}$$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(xr^n) + \frac{\partial}{\partial y}(yr^n) + \frac{\partial}{\partial z}(zr^n). \quad \text{--- (1)}$$

Now, $\frac{\partial}{\partial x}(xr^n) = r^n + xnr^{n-1}\frac{\partial r}{\partial x}$

$$\frac{\partial}{\partial x}(xr^n) = r^n + nx^2 r^{n-2} \quad \text{--- (2)}$$

Similarly, $\frac{\partial}{\partial y}(yr^n) = r^n + ny^2 r^{n-2}$ --- (3)

$$\frac{\partial}{\partial z}(zr^n) = r^n + nz^2 r^{n-2} \quad \text{--- (4)}$$

Using (2), (3) and (4), we get

$$\text{div}\,\vec{F} = 3r^n + nr^{n-2}(x^2+y^2+z^2).$$

$$= 3r^n + nr^{n-2}(r^2).$$

$$= 3r^n + nr^n$$

$$\text{div}\,\vec{F} = r^n(n+3)$$

Since vector is solenoidal $\Rightarrow \text{div}\,F = 0$

$\Rightarrow r^n(n+3) = 0$

Since $r^n \neq 0$, so, $n+3 = 0$

$\Rightarrow n = -3$ Ans

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Home Work

Q Show that the vector $f(r)\vec{r}$ is irrotational vector.

Sol → Let $f(r)\vec{r} = F$

$$F = f(r)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$F = xf(r)\hat{i} + yf(r)\hat{j} + zf(r)\hat{k}$$

$$\text{Curl } F = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$\text{Curl } F = \hat{i}\left[\frac{\partial}{\partial y}(zf(r)) - \frac{\partial}{\partial z}(yf(r))\right] - \hat{j}\left[\frac{\partial}{\partial x}(zf(r)) - \frac{\partial}{\partial z}(xf(r))\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(yf(r)) - \frac{\partial}{\partial y}(xf(r))\right] \quad \text{--- (1)}$$

Now, $\frac{\partial}{\partial x}[yf(r)]$

$$= 0 + y\frac{\partial}{\partial x}f(r)$$

$$= 0 + y\, f'(r)\frac{dr}{dx}$$

$$\frac{\partial}{\partial x}(yf(r)) = \frac{xy\, f'(r)}{r} \quad \text{--- (2)}$$

Similarly, $\frac{\partial}{\partial x}(zf(r)) = \frac{xz\, f'(r)}{r}$ --- (3)

$$\frac{\partial}{\partial y}(zf(r)) = \frac{zy\, f'(r)}{r} \quad \text{--- (4)}$$

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$$\frac{\partial}{\partial y}\left(x f(r)\right) = \frac{xy f'(r)}{r} \quad \text{--- (5)}$$

$$\frac{\partial}{\partial z}\left(x f(r)\right) = \frac{zx f'(r)}{r} \quad \text{--- (6)}$$

$$\frac{d}{\partial z}\left(y f(r)\right) = \frac{zy f'(r)}{r} \quad \text{--- (7)}$$

Using (2), (3), (4), (5), (6), (7) in eq (1), we get

$$\text{Curl} F = \hat{i}\left(\frac{zy f'(r)}{r} - \frac{zy f'(r)}{r}\right) + \hat{j}\left(\frac{xz f'(r)}{r} - \frac{zx f'(r)}{r}\right)$$
$$+ \hat{k}\left(\frac{yx f'(r)}{r} - \frac{xy f'(r)}{r}\right)$$

$$\text{Curl} F = \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

$$\text{Curl} F = 0$$

Since $\text{Curl} F = 0$

$\Rightarrow$ F is irrotational.

So, the vector $f(r)\vec{r}$ is irrotational <u>vector</u>.

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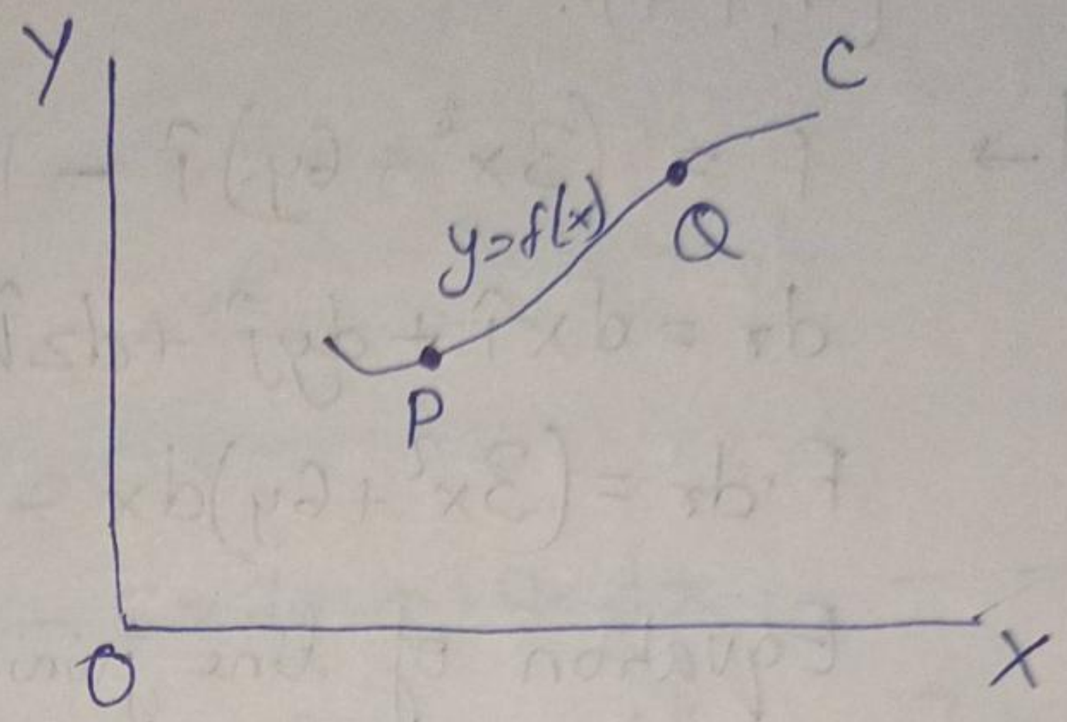
Line Integral :-

Let C be a curve and P & Q be two point on the curve then the integral under P to Q is called Line Integral.



$$= \int_C F \cdot dr$$

$$= \int_P^Q F \cdot dr$$

Total Work done $= \int_C F \cdot dr$

Q ~~Find~~ Evaluate $\int_C F \cdot dr$, where $F = x^2 \hat{i} + y^3 \hat{j}$ and C the curve $y = x^2$ in xy plane from $(0,0)$ to $(1,1)$.

Sol→ $F = x^2 \hat{i} + y^3 \hat{j}$.

$dr = dx\hat{i} + dy\hat{j}$

$F \cdot dr = x^2 dx + y^3 dy$. ——— (1)

Given :- $y = x^2$
$dy = 2x\,dx$ ——— (2)

using (2) in (1)

$F \cdot dr = x^2 dx + x^6 (2x\,dx)$

$F \cdot dr = (x^2 + 2x^7)\,dx$ ——— (3)

Now, $\int_C F \cdot dr = \int_{(0,0)}^{(1,1)} (x^2 + 2x^7)\,dx$

$= \int_0^1 (x^2 + 2x^7)\,dx = \left| \frac{x^3}{3} + \frac{x^8}{4} \right|_0^1$

$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ Ans

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Q If $F = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$. Evaluate $\int_C F \cdot dr$, where C is straight line joining (0, 0, 0) to (1, 1, 1).

Sol→ $F = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$.

$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$.

$F \cdot dr = (3x^2 + 6y)dx - 14yz\,dy + 20xz^2 dz$. — ①

Equation of line joining (0, 0, 0) to (1, 1, 1)

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

$$= \frac{x - 0}{1 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{1 - 0} = t \text{ (say)}$$

$$= \left.\begin{array}{lll} x = t, & y = t, & z = t \\ dx = dt & dy = dt & dz = dt \end{array}\right] \text{--- ②}$$

Using ② in ①.

$F \cdot dr = (20t^3 - 11t^2 + 6t)\,dt$

Now, $\int_C F \cdot dr = \int_0^1 (20t^3 - 11t^2 + 6t)\,dt$

$$= \left| 5t^4 - \frac{11}{3}t^3 + \frac{6t^2}{2} \right|_0^1$$

$$= \left| 5t^4 - \frac{11}{3}t^3 + 3t^2 \right|_0^1$$

$$= \frac{15 - 11 + 9}{3} = \frac{13}{3} \text{ Ans}$$

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Q Find the total work done in moving a particle in a field given by $F = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$. along the curve

$x = t^2+1,\ y = 2t^2,\ z = t^3$

from $t=1$ to $t=2$.

Sol→ $F = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$.

$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$.

$F \cdot dr = 3xy\,dx - 5z\,dy + 10x\,dz$ --- (1)

Given $x = t^2+1,\ y = 2t^2,\ z = t^3$

$dx = 2t\,dt \quad dy = 4t\,dt \quad dz = 3t^2\,dt$ --- (2)

Using (2) in (1).

$F \cdot dr = 3(t^2+1)(2t^2)(2t\,dt) - 5(t^3)(4t\,dt) + 10(t^2+1)(3t^2\,dt)$

$= 12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2$

$F \cdot dr = (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$.

Now, $\int_c F \cdot dr = \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$

$= \left| \frac{12t^6}{6} + \frac{10t^5}{5} + \frac{12t^4}{4} + \frac{30t^3}{3} \right|_1^2$

$= \left| 2t^6 + 2t^5 + 3t^4 + 10t^3 \right|_1^2$

$= (2\times 64 + 2\times 32 + 3\times 16 + 10\times 8) - (17)$

$= 320 - 17$

$= 303$ Ans

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Q. Evaluate $\int_C F \cdot dr$ given that $F = z\hat{i} + x\hat{j} + y\hat{k}$ and C being the arc of the curve $r = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$ from $t=0$ to $t = 2\pi$.

Sol $\rightarrow$ $F = z\hat{i} + x\hat{j} + y\hat{k}$.

$$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$F \cdot dr = z\,dx + x\,dy + y\,dz \quad \text{--- (1)}$$

and $r = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$.

On comparing with $r = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\left.\begin{array}{lll} x = \cos t & y = \sin t & z = t \\ dx = -\sin t\,dt & dy = \cos t\,dt & dz = dt \end{array}\right] \text{--- (2)}$$

Putting (2) in (1).

$$F \cdot dr = -t\sin t\,dt + \cos t \cdot \cos t\,dt + \sin t\,dt$$

$$F \cdot dr = (\cos^2 t + \sin t - t\sin t)\,dt$$

Now, $\int_C F \cdot dr = \int_0^{2\pi} (\cos^2 t + \sin t - t\sin t)\,dt$

$$= \int_0^{2\pi} \left[\left(\frac{1+\cos 2t}{2}\right) + \sin t - t\sin t\right] dt$$

$$= \int_0^{2\pi} \left[\frac{1+\cos 2t}{2} + \sin t - t\sin t\right] dt$$

$$= \left[\frac{t}{2} + \frac{\sin 2t}{4} - \cos t - [-t\cos t + \sin t]\right]_0^{2\pi}$$

$$= \left(\frac{2\pi}{2} + 0 - \cancel{1} + 2\pi - 0\right) - (0 + 0 - \cancel{1} + 0 + 0) = 3\pi \text{ Ans}$$

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Surface Integral :-

Type→ Evaluate $\iint_S f\cdot\hat{n}\, d\gamma$

$d\gamma$ — $dxdy$, $dydz$, $dzdx$.

$$= \iint f\cdot\hat{n}\,\frac{dxdy}{|\hat{k}\cdot\hat{n}|} \Rightarrow xy \text{ plane}$$

$$= \iint f\cdot\hat{n}\,\frac{dydz}{|\hat{i}\cdot\hat{n}|} \Rightarrow yz \text{ plane}$$

$$= \iint f\cdot\hat{n}\,\frac{dzdx}{|\hat{j}\cdot\hat{n}|} \Rightarrow zx \text{ plane}$$

Q Evaluate $\iint_S \vec{A}\cdot\hat{n}\, dS$ where $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of $2x+y+2z=6$ in first octant.

Sol→ $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$

Given $2x+y+2z=6$ ——— (1)

Let $\phi = 2x+y+2z-6=0$ ——— (2)

$$\nabla\phi = 2\hat{i}+\hat{j}+2\hat{k}$$

$$|\nabla\phi| = \sqrt{4+1+4} = \sqrt{9} = 3.$$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}.$$

Now, $\vec{A}\cdot\hat{n} = \frac{1}{3}\left[2(x+y^2) - 2x + 4yz\right]$

$$\vec{A}\cdot\hat{n} = \frac{2}{3}\left[x+y^2-x+2y\left(\frac{6-2x-y}{2}\right)\right]$$

[from eq. (i)]

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$$\Rightarrow \vec{A}\cdot\hat{n} = \frac{2}{3}\left[y^2 + 6y - 2xy - y^2\right]$$

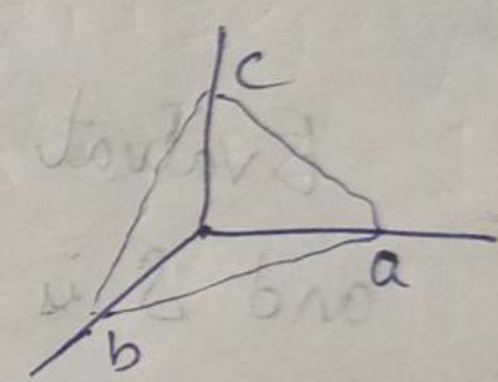
$$\Rightarrow \vec{A}\cdot\hat{n} = \frac{2}{3}\left[2y(3-x)\right]$$

$$\vec{A}\cdot\hat{n} = \frac{4y}{3}(3-x)$$

and $|\hat{k}\cdot\hat{n}| = \frac{2}{3}$.

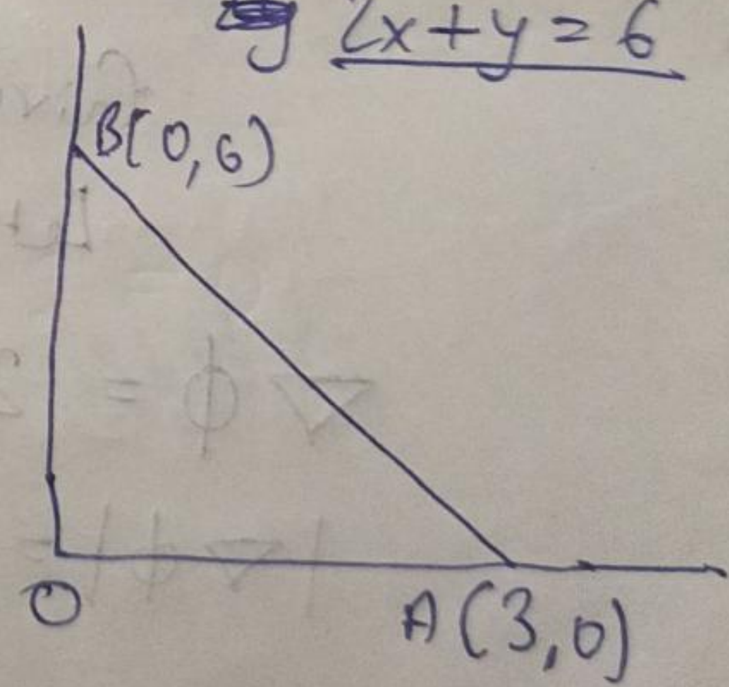
We have $\iint_S \vec{A}\cdot\hat{n}\,dr = \iint \frac{A\cdot\hat{n}\,dx\,dy}{|\hat{k}\cdot\hat{n}|}$

$$= \iint \frac{\frac{4y}{3}(3-x)\,dx\,dy}{(2/3)}$$



$$= 2\iint y(3-x)\,dy\,dx$$

$2x+y=6$



$$= 2\int_0^3 \int_0^{6-2x} y(3-x)\,dy\,dx$$

$$= 2\int_0^3 \left|\frac{y^2}{2}\right|_0^{6-2x}(3-x)\,dx$$

$$= \int_0^3 (6-2x)^2(3-x)$$

$$= 2\int_0^3 (3-x)^3 = 4\left|\frac{(3-x)^4}{4(-1)}\right|_0^3$$

$$= 0 - \frac{(3)^4}{(-1)} = 3^4 = 81 \text{ Ans}$$

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Q Evaluate $\iint \vec{A}\cdot\hat{n}\,dr$ where $A = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$, S is the surface of the cyclinder $x^2+y^2=16$ included in first octant between $z=0$ to $z=5$.

Sol→ $A = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$

and $x^2+y^2 = 16 = 0$ — — ①

Let $\phi = x^2+y^2-16=0$ — — ②

$\nabla\phi = 2x\hat{i} + 2y\hat{j}$

$|\nabla\phi| = \sqrt{4x^2+y^2} = 2\sqrt{x^2+y^2}$

$= 2\sqrt{16} = 8$

$\therefore \hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2x\hat{i}+2y\hat{j}}{8} = \dfrac{x\hat{i}+y\hat{j}}{4}$

$\vec{A}\cdot\hat{n} = \dfrac{1}{4}\left[xz + yx\right]$

$= \dfrac{x}{4}(z+y)$

$|\hat{j}\cdot\hat{n}| = \dfrac{x}{4}$

Now, $\int \vec{A}\cdot\hat{n}\,ds = \iint \dfrac{\cancel{x}}{\cancel{4}}(z+y)\dfrac{dz\,dy}{\left(\frac{\cancel{x}}{\cancel{4}}\right)}$

$\Rightarrow \int_0^5\int_0^4 (z+y)\,dz\,dy$

$= \int_0^5 \left| zy + \dfrac{y^2}{2} \right|_0^4$

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$$\Rightarrow \int_0^5 (4z+8)$$

$$= \left|\frac{4z^2}{2} + 8z\right|_0^5$$

$$= [2\times 25 + 8\times 5 - (0+0)]$$

$$= 90 \quad \underline{\underline{\text{Ans}}}$$

Home-Work

Q Evaluate $\iint F\cdot\hat{n}\,ds$ where $F = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is the part of the surface $x^2+y^2+z^2=1$ that lie in the first octant.

Solution $\rightarrow$ $F = yz\hat{i} + zx\hat{j} + xy\hat{k}$

and $x^2+y^2+z^2 = 1$

Let $\phi = x^2+y^2+z^2-1 = 0$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2+4z^2}$$

$$= 2\sqrt{(x^2+y^2+z^2)} = 2\sqrt{1}$$

$$= 2$$

$$\therefore |\nabla\phi| = 2.$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2} = x\hat{i}+y\hat{j}+z\hat{k}.$$

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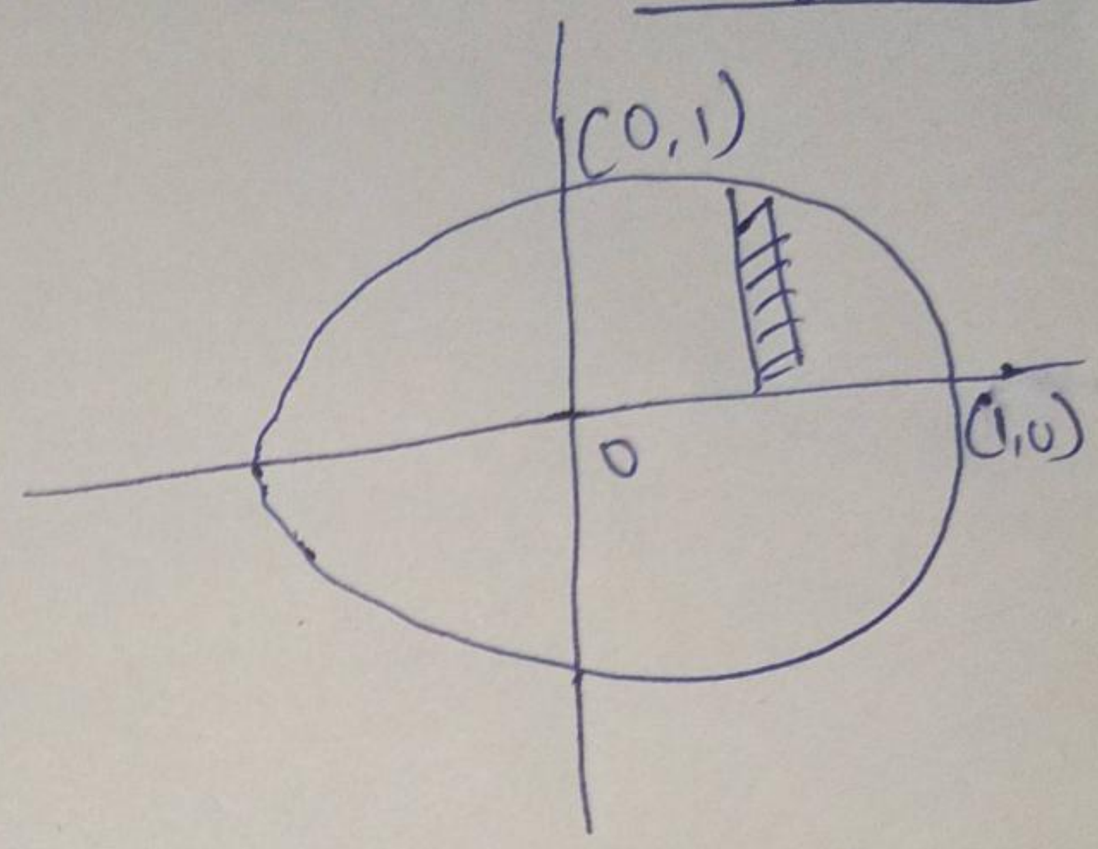
$\vec{F}\cdot\hat{n} = xyz + xyz + xyz$

$\vec{F}\cdot\hat{n} = 3xyz$

$|\hat{k}\cdot\hat{n}| = z$

$x^2+y^2=1$



$x = 0, \text{ to } 1$

$y = 0 \text{ to } \sqrt{1-x^2}$

Now, $\iint \frac{\vec{F}\cdot\hat{n}\,dx\,dy}{|\hat{k}\cdot\hat{n}|}$

$$= \iint \frac{3xyz}{z}\,dx\,dy$$

$$= 3\iint xy\,dx\,dy$$

$$= 3\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dx\,dy$$

$$= 3\int_0^1 x\left|\frac{y^2}{2}\right|_0^{\sqrt{1-x^2}} dx$$

$$= \frac{3}{2}\int_0^1 x\left(\sqrt{1-x^2}\right)^2 dx = \frac{3}{2}\int_0^1 x(1-x^2)\,dx$$

$$= \frac{3}{2}\left|\frac{x^2}{2} - \frac{x^4}{4}\right|_0^1$$

$$= \frac{3}{2}\left[\frac{1}{2} - \frac{1}{4} - (0-0)\right]$$

$$= \frac{3}{2}\left[\frac{2-1}{4}\right] = \frac{3}{2}\times\frac{1}{4}$$

$$= \frac{3}{8} \text{ Ans}$$

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Green Theorem :-

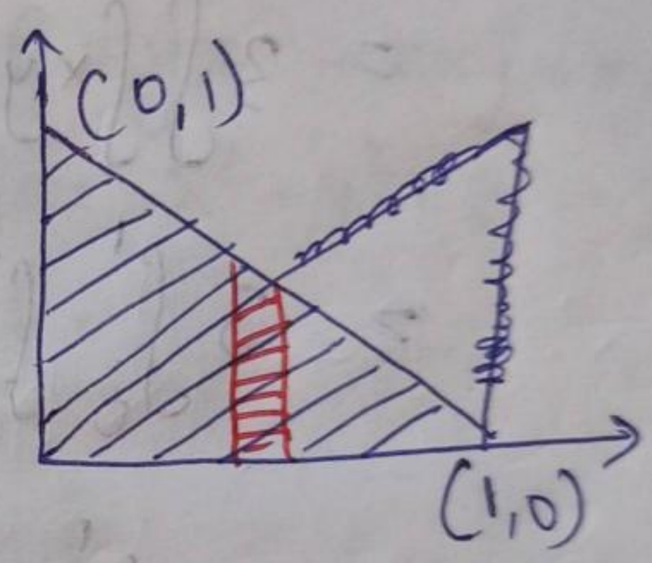
$$\oint_C (M dx + N dy) = \iint_S \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx dy.$$

where C is the closed curve and S is the surface integral.

Q Verify Green theorem in the plane for $\oint_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$ where C is the boundary of curve $x=0, y=0$ & $x+y=1$.

Sol→ Let $I = \oint_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$. --- ①

and $x=0, y=0$ & $x+y=1$



We have limit

$x = 0$ to 1

$y = 0$ to $1-x$

We know by Green theorem

$$\oint_C (M dx + N dy) = \iint_S \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx dy \quad --- ②$$

On Comparing ① and ②,

$N = (4y - 6xy) \Rightarrow \frac{dN}{dx} = -6y$

$M = (3x^2 - 8y^2) \Rightarrow \frac{dM}{dy} = -16y$

Now, $\frac{dN}{dx} - \frac{dM}{dy} = 10y$

Taking RHS of equation ②

$$\iint_R \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx dy = \int_0^1 \int_0^{1-x} 10y \, dx dy$$

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$$= \int_0^1 \left| \frac{10y^2}{2} \right|_0^{1-x} dx\, dy$$

$$= 5 \int_0^1 (1-x)^2$$

$$= 5 \left| \frac{(1-x)^3}{-3} \right|_0^1$$

$$= 5 \left(0 + \frac{1}{3} \right)$$

$$= \frac{5}{3} \quad \text{--- (3)}$$

Taking LHS of eq^n (2)

$$\oint_C M dx + N dy = \int_{OA} (M dx + N dy) + \int_{AB} (M dx + N dy) + \int_{BO} (M dx + N dy) \quad \text{--- (4)}$$

Along OA :- we have

$x = 0$ to 1 $(0 < x < 1)$

$y = 0$ to 0 (constant)

$dy = 0$

Now, $\int_{OA} (M dx + N dy) = \int_0^1 (3x^2 - 8x(0)^2) dx + (4y - 6xy) \times 0$

$$= \int_0^1 3x^2 dx = \left| x^3 \right|_0^1 = 1 \quad \text{--- (5)}$$

Along AB :- we have

$x + y = 1 \Rightarrow y = 1 - x$

$dy = -dx$

Now $\int_{AB} M dx + N dy = \int_1^0 \left[3x^2 - 8(1-x)^2 \right] dx - \left[4(1-x) - 6x(1-x) \right] dx$

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$$= \int_1^0 (-11x^2 + 26x - 12)\,dx$$

$$= \left[\frac{-11x^3}{3} + 13x^2 - 12x\right]_1^0$$

$$= \left[0 - \left(\frac{-11 + 39 - 36}{3}\right)\right]$$

$$= \frac{8}{3} \quad \text{——— (6)}$$

Along BO, we have $x = 0$ to 0.

$dx = 0$

Now, $\oint (M\,dx + N\,dy) = \int_0^1 9y\,dy$

$$= \left[2y^2\right]_0^1$$

$$= -2 \quad \text{——— (7)}$$

Putting (5), (6), (7) in eq. (4)

$$\oint (M\,dx + N\,dy) = 1 + \frac{8}{3} - 2 = \frac{5}{3} \quad \text{——— (8)}$$

from (3) & (8)

$$\oint (M\,dx + N\,dy) = \iint_S \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\,dy$$

Hence Green theorem is verified.

CT-2 Question type (MCQ) → Evaluate $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the boundary between $x = 0$, $y = 0$ & $x + y = 1$

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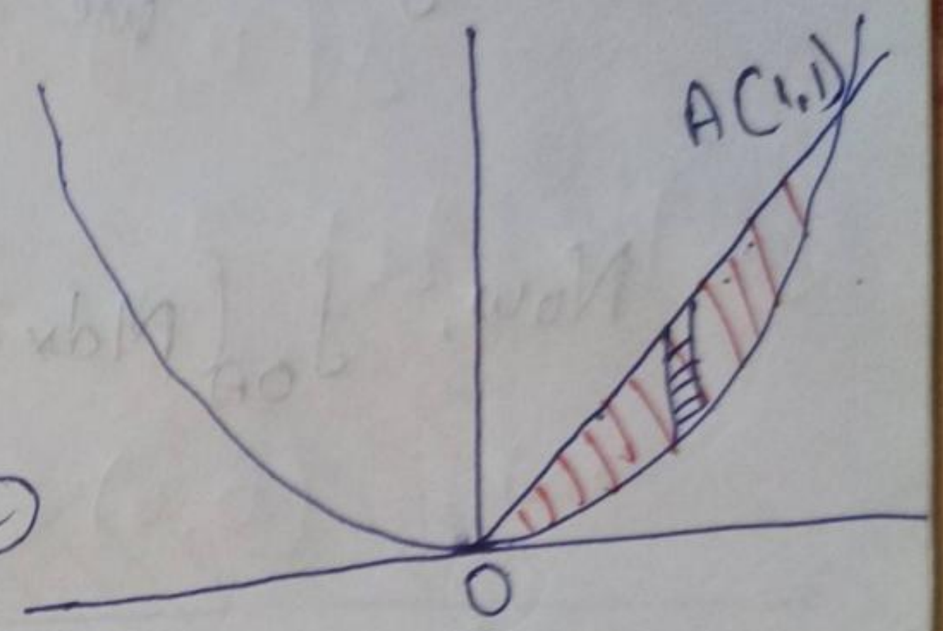
Q Verify Green's theorem in plane for $\oint_C (xy+y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $y=x$ & $y=x^2$.

Sol → Let $I = \oint_C (xy+y^2)\,dx + x^2 dy$ —— ①

and $y=x$ and $y=x^2$



We know by green theorem.

$$\oint (M dx + N dy) = \iint_S \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\,dy \quad \text{—— ②}$$

Comparing ① and ②, we get.

$$M = xy + y^2 \Rightarrow \frac{dM}{dy} = x + 2y$$

$$N = x^2 \Rightarrow \frac{dN}{dx} = 2x.$$

$$\frac{dN}{dx} - \frac{dM}{dy} = x - 2y$$

Taking RHS of eq. ②,

$$\iint_S \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\,dy = \int_0^1 \int_{x^2}^{x} (x-2y)\,dx\,dy$$

$$= \int_0^1 \left| \left(xy - \frac{2y^2}{2}\right) \right|_{x^2}^{x} dx$$

$$= \int_0^1 (x \cdot x - x^2) - (x \cdot x^2 - x^4)\,dx$$

$$= \int_0^1 x^4 - x^3$$

$$= \left| \frac{x^5}{5} - \frac{x^4}{5} \right|_0^1 = \frac{4-5}{20} = \frac{-1}{20}. \quad \text{—— ③}$$

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Taking LHS of eq^n (2).

$$\oint (M\,dx + N\,dy) = \int_{OA} (M\,dx + N\,dy) + \int_{AO} (M\,dx + N\,dy). \quad \text{--- (4)}$$

Along the line OA, we have, $y = x^2$

$$\Rightarrow dy = 2x\,dx$$

Now, $\int_{OA} (M\,dx + N\,dy) = \int_0^1 (x \cdot x^2 + x^4)\,dx + x^2(2x\,dx)$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx$$

$$= \left| \frac{3x^4}{4} + \frac{x^5}{5} \right|_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{15+4}{20} = \frac{19}{20} \quad \text{--- (5)}$$

Along AO :- we have $y = x \Rightarrow dy = dx$

Now, $\int_{AO} (M\,dx + N\,dy) = \int_1^0 (x \cdot x + x^2)\,dx + x^2\,dx$

$$= \int_1^0 3x^2\,dx$$

$$= \left| x^3 \right|_1^0$$

$$= 0 - 1 = -1 \quad \text{--- (6)}$$

Putting (5) & (6) in eq. (4)

$$\oint (M\,dx + N\,dy) = \frac{19}{20} - 1 = \frac{19-20}{20} = \frac{-1}{20} \quad \text{--- (7)}$$

from (7) and (3),

$$\oint (M\,dx + N\,dy) = \iint_S \left(\frac{dN}{dx} - \frac{dM}{dy} \right) dx\,dy$$

Hence Green theorem is verified.

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Q Evaluate $\oint_c (x^2 y\,dx + x^2 dy)$ where C is the boundary of the triangle whose vertices $(0,0)$, $(1,0)$, $(1,1)$

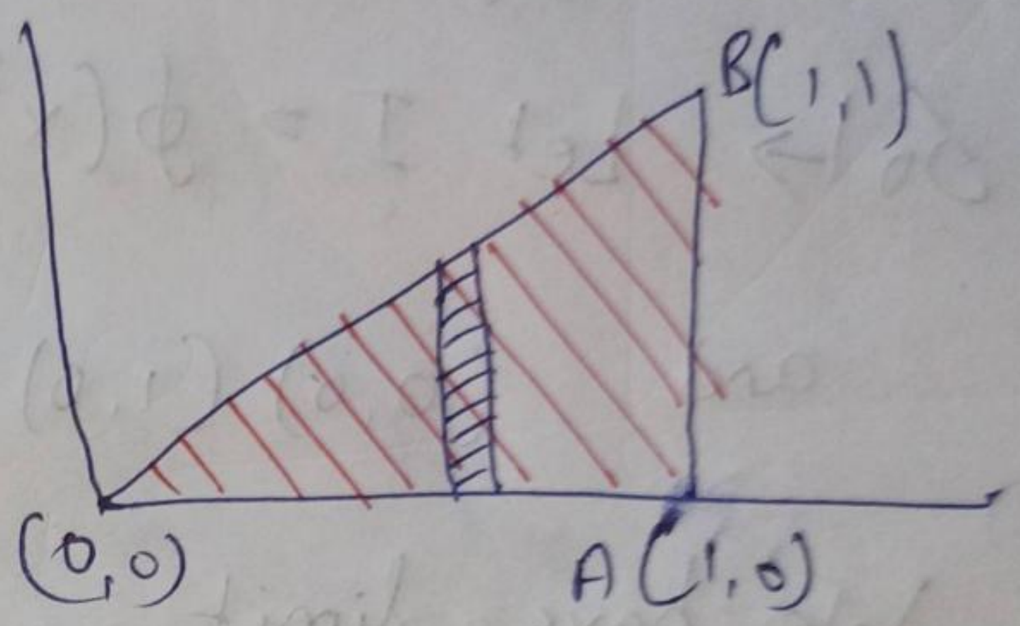
Sol→ Let $I = \oint_c (x^2 y\,dx + x^2 dy)$ --- (1)

Given $(0,0)$, $(1,0)$ $(1,1)$.

We have limit

$x = 0$ to 1

$y = 0$ to x.



From Green theorem

$$\oint (M\,dx + N\,dy) = \iint_S \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\,dy \quad \text{--- (2)}$$

from (1) and (2).

$M = x^2 y \Rightarrow \frac{dM}{dy} = x^2$

$N = x^2 \Rightarrow \frac{dN}{dx} = 2x$

$$\left(\frac{dN}{dx} - \frac{dM}{dy}\right) = 2x - x^2$$

From RHS of eq. (2).

$$\iint_S \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\,dy = \int_0^1 \int_0^x (2x - x^2)\,dx\,dy.$$

$$= \int_0^1 \left|(2x - x^2)y\right|_0^x$$

$$= \int_0^1 (2x - x^2)x\,dx = \int_0^1 (2x^2 - x^3)\,dx$$

$$= \left|\frac{2x^3}{3} - \frac{x^4}{4}\right|_0^1 = \frac{8-3}{12} = \frac{5}{12} \text{ Ans}$$

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Home - Work

Q Verify Green theorem in the plane $\oint_C (x^2+2xy)\,dx + (y^2+x^3y)\,dy$ where C is a square with vertices P(0,0), Q(1,0), R(1,1) and S(0,1).

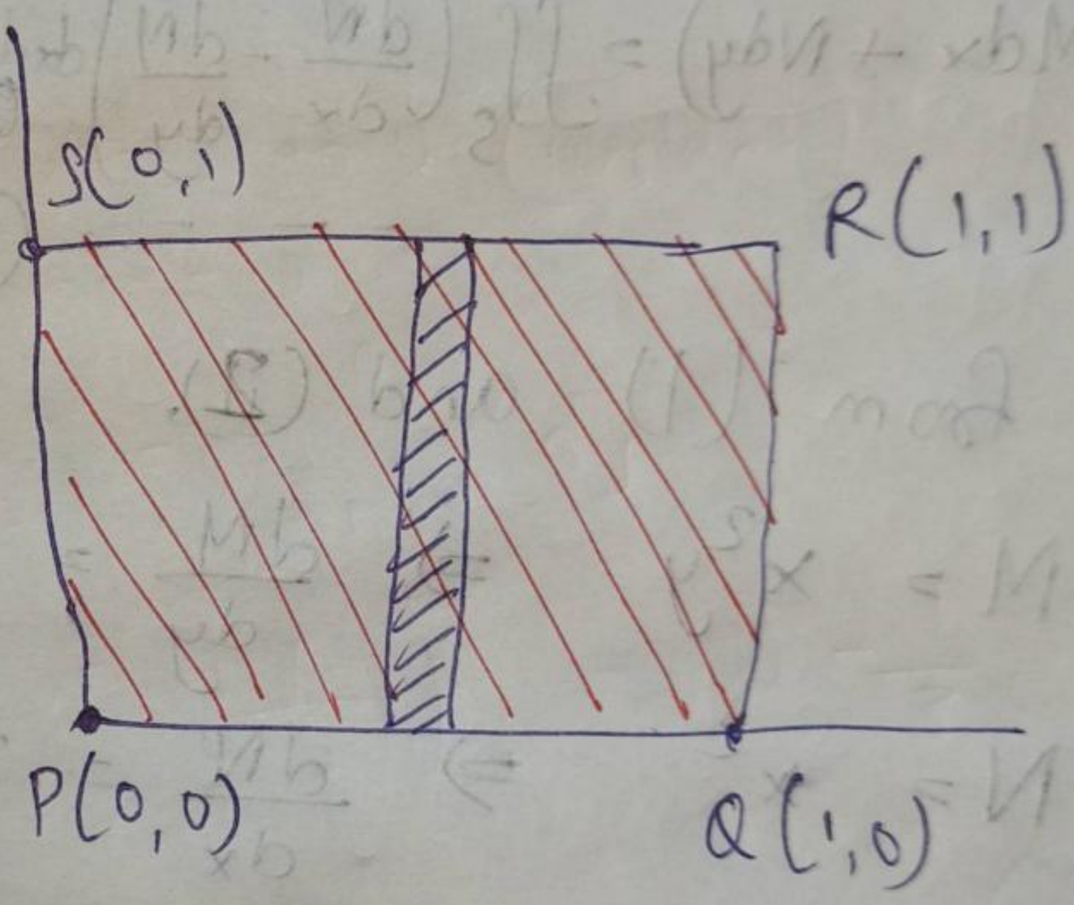
Sol→ Let $I = \oint (x^2+2xy)\,dx + (y^2+x^3y)\,dy$ --- ①

and (0,0) (1,0) (1,1) (0,1)

We have limit

$x = 0$ to 1

$y = 0$ to 1



from Green's theorem

$$\oint_C (M\,dx + N\,dy) = \iint_S \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\,dy \quad \text{--- ②}$$

On Comparing ① and ②,

$M = x^2 + 2xy \Rightarrow \frac{dM}{dy} = 2x$

$N = y^2 + x^3y \Rightarrow \frac{dN}{dx} = 3x^2y$

$\frac{dN}{dx} - \frac{dM}{dy} = 3x^2y - 2x$

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By taking RHS of eq. ②

$$\iint_S \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\,dy = \int_0^1\int_0^1 (3x^2y - 2x - 2y)\,dx\,dy$$

$$= \int_0^1 \left| \frac{3x^3}{3}y - \frac{2x^2}{2} \right|_0^1 dy$$

$$= \int_0^1 (3y - 1)\,dy$$

$$= \left| \frac{3y^2}{3\times 2} - 1 \right|_0^1$$

$$= \frac{3}{3\times 2} - 1 = \frac{-1}{2} \quad \text{--- ③}$$

Now, taking LHS of eq. ②

$$\oint (Mdx + Ndy) = \int_{PQ} (Mdx + Ndy) + \int_{QR} (Mdx + Ndy) + \int_{RS} (Mdx + Ndy) + \int_{SP} (Mdx + Ndy) \quad \text{--- ④}$$

Along PQ, we have $y = 0$ to 0 (constant)

$dy = 0$

Now, $\int_{PQ} (Mdx + Ndy) = \int_0^1 x^2\,dx = \frac{x^3}{3} = \frac{1}{3}$ --- ⑤

Along QR, we have $x = 1$ to 1 (constant)

$dx = 0$.

Now, $\int_{QR} (Mdx + Ndy) = \int_0^1 (y^2 + y)\,dy = \left| \frac{y^3}{3} + \frac{y^2}{2} \right|_0^1$

$= \frac{1}{3} + \frac{1}{2}$. --- ⑥

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Along RS. we have $y = 1$ to 1 (constant)

$$dy = 0.$$

Now, $\int_{RS} (M dx + N dy) = \int_1^0 (x^2 + 2x)$

$$= \left[\frac{x^3}{3} + x^2\right]_1^0$$

$$= 0 - \left(\frac{1}{3} + 1\right)$$

$$= \frac{-1}{3} - 1 \quad \text{---} \quad (7)$$

Along SP, we have $x = 0$ to 0 (constant)

$$dx = 0$$

Now, $\int_{SP} (M dx + N dy) = \int_1^0 y^2 dy$

$$= \left[\frac{y^3}{3}\right]_1^0 = 0 - \frac{1}{3}$$

$$= \frac{-1}{3} \quad \text{---} \quad (8)$$

Putting (5), (6), (7), (8) in eq. (4).

$$\oint (M dx + N dy) = \frac{1}{3} + \frac{1}{3} + \frac{1}{2} - \frac{1}{3} - 1 - \frac{1}{3}$$

$$= \frac{1}{2} - 1$$

$$= \frac{-1}{2} \quad \text{---} \quad (9)$$

from (9) and (3)

$$\oint (M dx + N dy) = \iint_R \left(\frac{dN}{dx} - \frac{dN}{dy}\right) dx\, dy$$

Hence Green Theorem is verified

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Stoke's Theorem :-

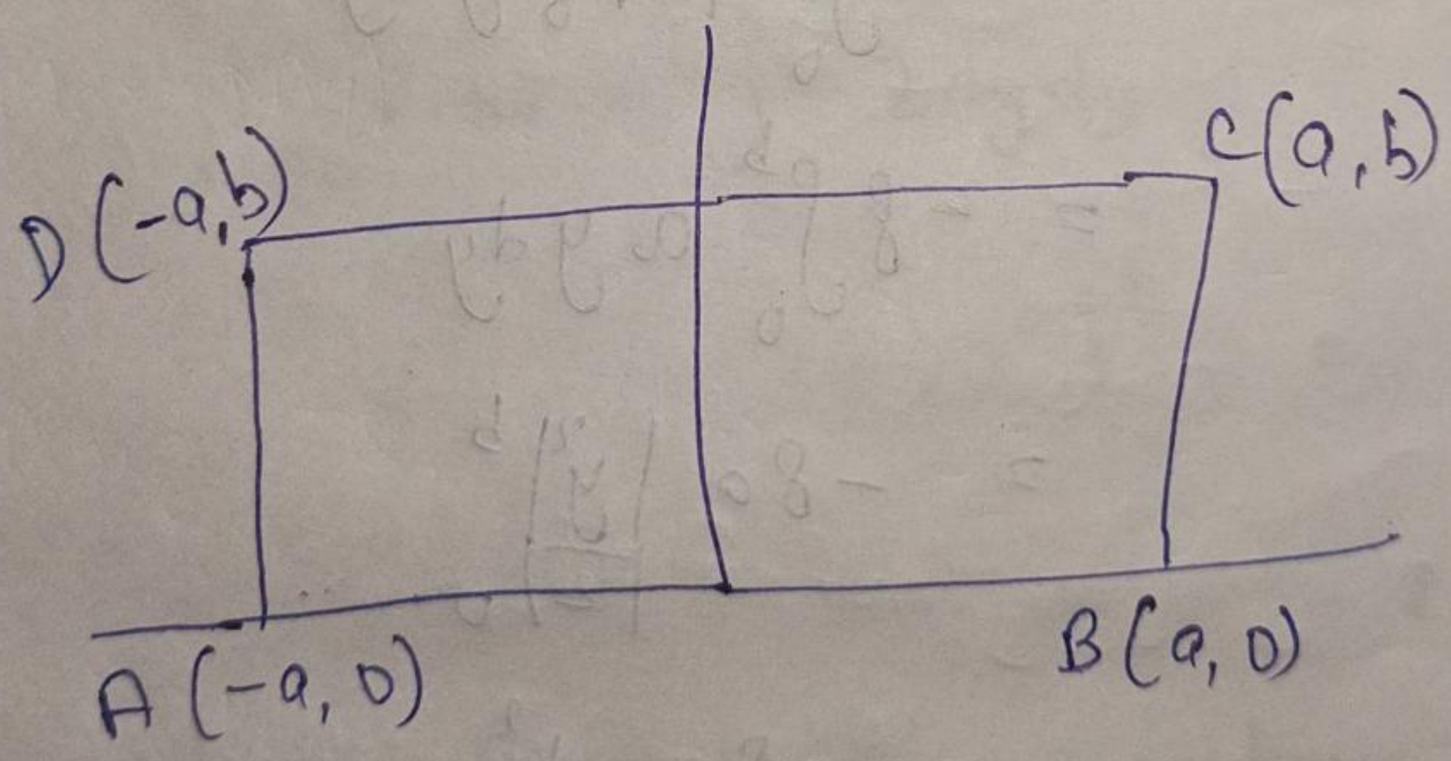
If S is the open surface bounded by a ~~clo~~ closed curve C and $F = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any vector point. then,

$$\oint_C F \cdot dr = \iint_S \text{Curl } F \cdot \hat{n}\, dS,$$ where $\hat{n}$ is the unit Normal Vector.

Q Verify Stoke's theorem for vector $\vec{f} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by $x = \pm a,\ y = 0,\ y = b$.

Sol→ Given $f = (x^2+y^2)\hat{i} - 2xy\hat{j}$

$x = \pm a,\ y = 0,\ y = b.$



$$\text{Curl } f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \hat{i}[0+0] - \hat{j}[0+0] + \hat{k}[-2y-2y]$$

$$\text{Curl } f = -4y\hat{k} \quad \text{---} \quad (1)$$

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$\hat{n} = \hat{k}$

By stokes theorem.

$$\oint_C f \cdot dr = \iint_C \text{Curl } f \cdot \hat{n} \, d$$

Taking RHS.

$$\iint_S \text{Curl } f \cdot \hat{n} \, dS = \int_0^b \int_{-a}^a (-4y\hat{k}) \cdot \hat{k} \, dx dy$$

$$= \int_0^b \int_{-a}^a -4y \, dx dy$$

$$= -8 \int_0^b \left(\int_0^a dx \right) y \, dy$$

$$= -8 \int_0^b |x|_0^a \, y \, dy$$

$$= -8 \int_0^b a y \, dy$$

$$= -8a \left| \frac{y^2}{2} \right|_0^b$$

$$= -8a \cdot \frac{b^2}{2}$$

$$= -4ab^2 \quad \text{--- (2)}$$

Now. $f \cdot dr = (x^2 + y^2) dx - 2xy \, dy$ --- (3)

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Taking LHS of eq. (i).

$$\oint F\cdot dr = \int_{AB} F\cdot dr + \int_{BC} F\cdot dr + \int_{CD} F\cdot dr + \int_{DA} F\cdot dr \quad \text{--- (4)}$$

Along AB: we have $x = -a$ to a.

$y = 0 \Rightarrow dy = 0$

$$\Rightarrow \int_{AB} F\cdot dr = \int_{-a}^{a} x^2\,dx \quad [\text{using } 3]$$

$$= \left|\frac{x^3}{3}\right|_{-a}^{a} = \frac{a^3}{3} + \frac{a^3}{3} = \frac{2a^3}{3} \quad \text{--- (5)}$$

Along BC:- We have $x = a$

$dx = 0$

$y = 0$ to b.

$$\int_{BC} F\cdot dr = \int_{0}^{b} -2ay\,dy$$

$$= -2a\left|\frac{y^2}{2}\right|_{0}^{b} = -2a\frac{b^2}{2}$$

$$= -ab^2 \quad \text{--- (6)}$$

Along CD:- we have, $y = b$

$dy = 0$

$x = -a$ to a

$$\int_{CD} = \int_{+a}^{-a} (x^2 + b^2)\,dx = \left|\frac{x^3}{3} + b^2x\right|_{+a}^{-a}$$

$$\Rightarrow \frac{-a^3}{3} - ab^2 + \left(\frac{-a^3}{3} - ab^2\right) = \frac{-2a^3}{3} - 2ab^2 \quad \text{--- (7)}$$

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Along DA :- we have $x = -a$.

$dx = 0$

$y = b$ to 0.

$$\int_{DA} F \cdot dr = \int_b^0 (-2(-a)y)\,dy$$

$$= 2a \left| \frac{y^2}{2} \right|_b^0$$

$$= 2a \left[0 - \frac{b^2}{2} \right]$$

$$= -ab^2 \quad \text{———} \quad (8)$$

Putting values of (5), (6), (7), (8) in (4)

$$\oint F \cdot dr = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2$$

$$= -4ab^2 \quad \text{———} \quad (9)$$

from (2) and (9).

$$\oint F \cdot dr = \iint_S \text{Curl} F \cdot \hat{n}\, dS.$$

Hence Stoke's theorem is verified.

Q Evaluate $\oint_C (xy\,dx + xy^2\,dy)$ by Stoke's theorem where C is the ~~curve~~ square in xy-plane with vertices $(1,0)$ $(-1,0)$ $(0,1)$ $(0,-1)$.

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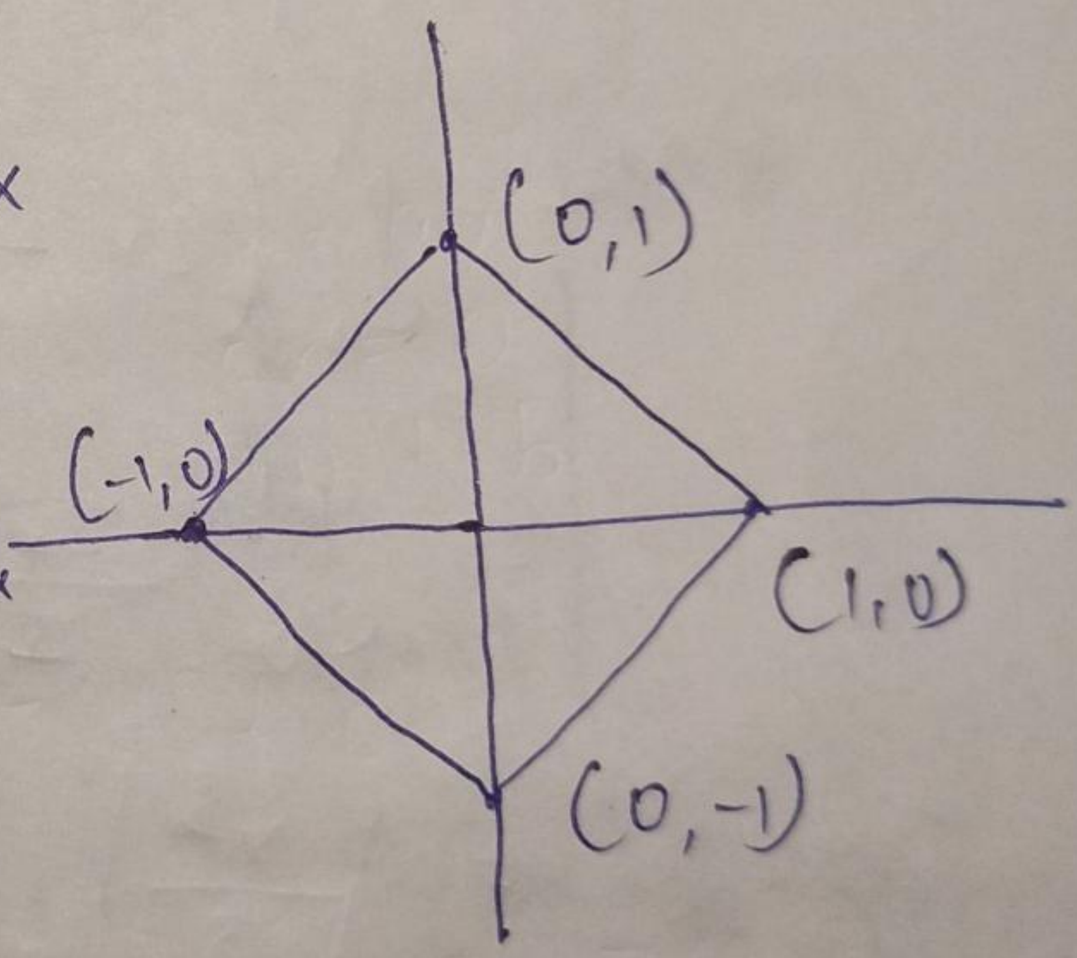
Sol → We have $F = xy\hat{i} + xy^2\hat{j}$

$$\text{Curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(y^2 - x).$$

$$\iint_C \text{Curl } F \cdot \hat{n}\, d\theta = \int_{-1}^{1}\int_{-1}^{1} (y^2 - x)\hat{k}\cdot\hat{k}\, dx\, dy$$

$$= \int_{-1}^{1} \left| \frac{y^3}{3} - xy \right|_{-1}^{1} dx$$



$$= \int_{-1}^{1} \left(\frac{1}{3} - x - \left(\frac{-1}{3} + x \right) \right) dx$$

$$= \int_{-1}^{1} \frac{2}{3} - 2x$$

$$= \left| \frac{2}{3}x - x^2 \right|_{-1}^{1} = \left[\frac{2}{3} - 1 - \left(-\frac{2}{3} - 1 \right) \right]$$

$$= \frac{2+2}{3} = \frac{4}{3} \quad \text{Ans.}$$

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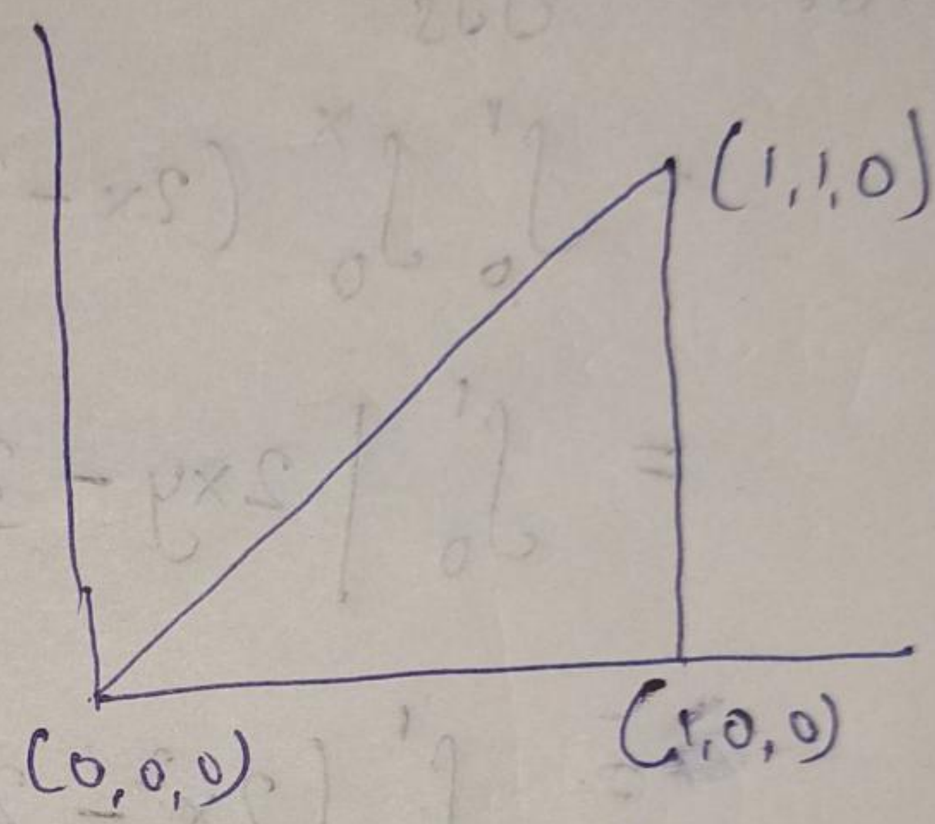
Home - Work

Q Evaluate $\oint F \cdot dr$ by stoke's theorem where $F = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ where c be the boundary of triangle $(0,0,0)$ $(1,0,0)$ $(1,1,0)$.

Sol→ We have $F = y^2\hat{i} + x^2\hat{j} - (x+y)\hat{k}$.



$$\text{Curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \hat{i}[0+0] - \hat{j}[-1+0] + \hat{k}[2x - 2y]$$

$$= \hat{j} + (2x - 2y)\hat{k}$$

Equation of the triangular surface.

$$\begin{vmatrix} x-0 & y-0 & z-0 \\ 1-0 & 0-0 & 0-0 \\ 1-0 & 1-0 & 0-0 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix}$$

$$\Rightarrow z = 0$$

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$\because$ Eq. of surface is $z=0$.

$\therefore \hat{n} = \hat{k}$ [$\because z=0$, so triangular plane lies in $x-y$ plane]

By stoke's theorem.

$$\oint F \cdot dr = \iint_S \text{curl}\vec{F} \cdot \hat{n}\, dS$$

$$= \int_0^1 \int_0^x (2x-2y)\, dx\, dy$$

$$= \int_0^1 \left| 2xy - \frac{2y^2}{2} \right|_0^x dx$$

$$= \int_0^1 \left(2x^2 - \frac{2x^2}{2} \right) dx$$

$$= \int_0^1 \frac{2x^2}{2} dx = \int_0^1 x^2 dx$$

$$= \left| \frac{x^3}{3} \right|_0^1$$

$$= \left(\frac{1}{3} - 0 \right) = \frac{1}{3} \text{ Ans}$$

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Gauss - Divergence Theorem :-

$$\iint_S F \cdot \hat{n}\, dS = \iiint_V \operatorname{div} F \, dV.$$

Q Evaluate $\iint (x^3 - yz)\, dy\, dz - 2x^2 y\, dz\, dx + z\, dx\, dy$. over the surface of a cube bounded by the coordinate plane $x = y = z = a$.

Sol → Let $I = \iint (x^3 - yz)\, dy\, dz - 2x^2 y\, dz\, dx - z\, dx\, dy$. — (1)

We know, $dS = \hat{i}\, dy\, dz + \hat{j}\, dx\, dz + \hat{k}\, dx\, dy$.

By Gauss - Divergence theorem.

$$\iint \left(F_1\, dy\, dx + F_2\, dx\, dz + F_3\, dx\, dy\right) = \iiint \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\, dy\, dz \quad \text{--- (2)}$$

On comparing (1) and (2).

$F_1 = x^3 - yz$

$F_2 = -2x^2 y$

$F_3 = z$

Putting values of $F_1\ F_2\ F_3$ in eq. (2), and integrate under the limit.

$$\iint (x^3 - yz)\, dz\, dy - 2x^2 y\, dz\, dx + z\, dx\, dy = \iiint (3x^2 - 2x^2 + 1)\, dx\, dy\, dz.$$

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$$= \int_0^a\int_0^a\int_0^a (x^2+1)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^a \left|\frac{x^3}{3}+x\right|_0^a dy\,dz$$

$$= \int_0^a\int_0^a \left(\frac{a^3}{3}+a\right).dy\,dz$$

$$= \left(\frac{a^3}{3}+a\right)\int_0^a |y|_0^a\, dz$$

$$= a\left(\frac{a^3}{3}+a\right)|z|_0^a$$

$$= a^2\left(\frac{a^3}{3}+a\right) \text{ ——— } \textcircled{8}$$

Ans

Q Verify Gauss Divergence theorem for $F=(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}$ taking over the rectangular parallelopiped

$0\le x\le a,\ 0\le y\le b\quad 0\le z\le c.$

Sol → Given $F=(x^2-yz)\hat{i}+(y^2-xz)\hat{j}+(z^2-xy)\hat{k}$

$$\operatorname{div} F = 2x+2y+2z.$$

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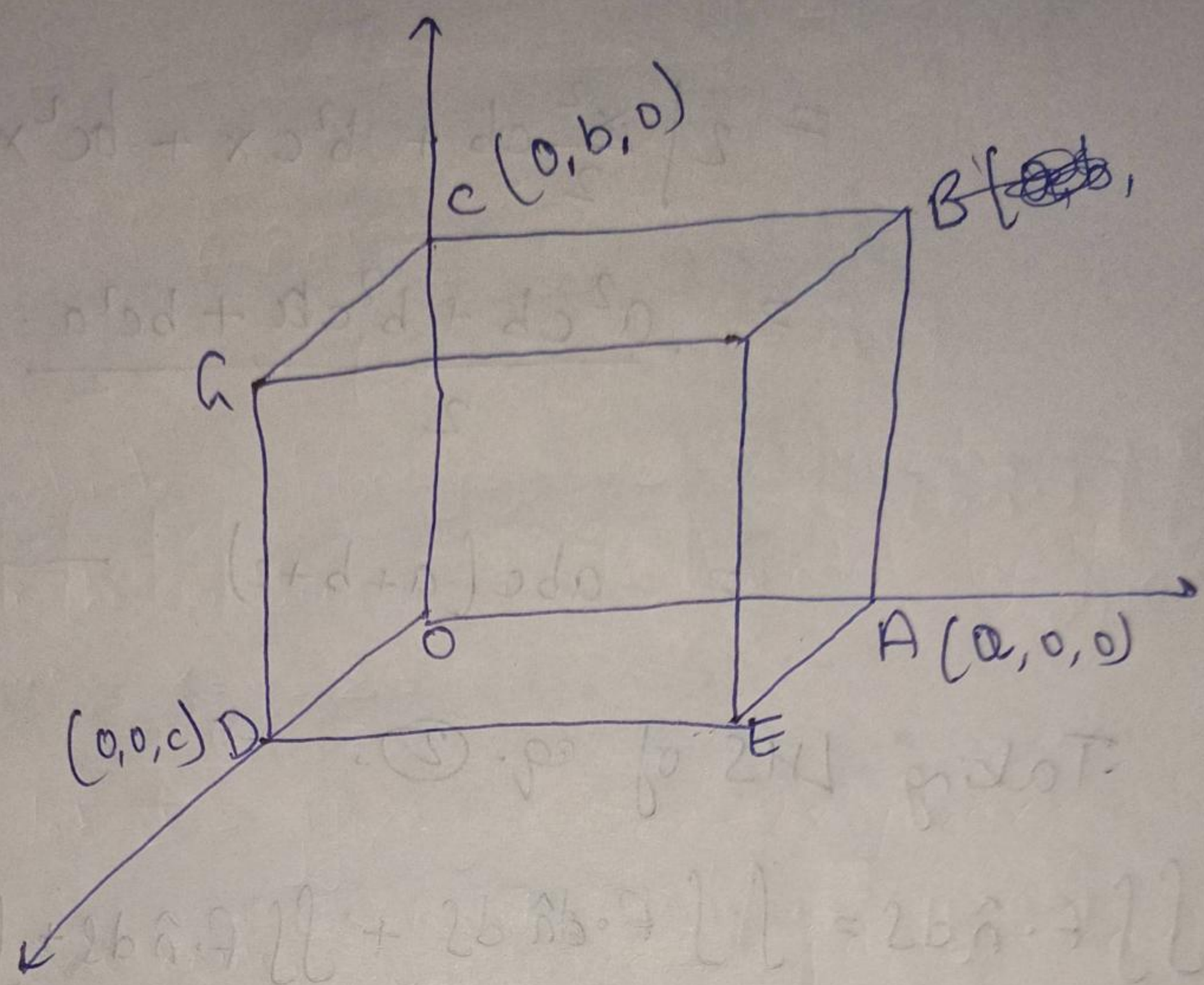

By Gauss Divergence Theorem.

$$\iint f\cdot\hat{n}\, ds = \iiint \operatorname{div} f\, dv \quad --- \quad (1)$$

Taking RHS of eq (i).

$$\iiint \operatorname{div} f\, dv = \int_0^a\int_0^b\int_0^c (2x+2y+2z)\, dx\,dy\,dz$$

$$= 2\int_0^a\int_0^b \left| xz+yz+\frac{z^2}{2}\right|_0^c dx\,dy$$

$$= 2\int_0^a\int_0^b \left(cx+yc+\frac{c^2}{2}\right) dx\,dy$$

$$= 2\int_0^a \left(cxy+\frac{y^2c}{2}+\frac{c^2y}{2}\right)_0^b dx$$

$$= 2\int_0^a \left(bcx+\frac{b^2c}{2}+\frac{c^2b}{2}\right) dx$$

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$$= 2\left[\frac{x^2}{2}cb + b^2cx + bc^2x\right]_0^a$$

$$= \frac{a^2cb + b^2ca + bc^2a}{2} \times 2$$

$$= abc(a+b+c) \quad \text{--- (2)}$$

Taking LHS of eq. (2).

$$\iint_{ABEFA} F\cdot\hat{n}\,ds = \iint F\cdot\hat{n}\,ds + \iint_{OCGDO} F\cdot\hat{n}\,ds + \iint_{CGFBC} F\cdot\hat{n}\,ds$$

$$+ \iint_{ODEAO} F\cdot\hat{n}\,ds + \iint_{DEFGD} F\cdot\hat{n}\,ds + \iint_{OABCO} F\cdot\hat{n}\,ds \quad \text{--- (3)}$$

Along the phase ABFEA :-

we have $x = a$, $\hat{n} = \hat{i}$.

$$\vec{F}\cdot\hat{n} = (x^2 - yz) = (a^2 - yz)$$

Now, $$\iint_{ABFEA} F\cdot\hat{n}\,ds = \int_0^b\int_0^c (a^2 - yz)\,dy\,dz$$

$$= \int_0^b \left[a^2z - \frac{yz^2}{2}\right]_0^c dy$$

$$= \int_0^b a^2c - \frac{yc^2}{2}\,dy$$

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$$= \int_0^b \left(a^2 c - \frac{yc^2}{2}\right) dy$$

$$= \left[a^2 c y - \frac{y^2 c^2}{4}\right]_0^b$$

$$= \left(bca^2 - \frac{b^2 c^2}{4}\right) \quad \text{--- (4)}$$

Along the phase. OCGDO :-

$x = 0$, and $\hat{n} = -\hat{i}$

$$f \cdot \hat{n} = -(x^2 - yz) = yz$$

$$\iint_{OCGDO} f \cdot \hat{n}\, ds = \int_0^b \int_0^c yz \, dz\, dy = \int_0^b \left[\frac{yz^2}{2}\right]_0^c dy$$

$$\int_0^b \left(\frac{yc^2}{2}\right) dy = \left[\frac{y^2 c^2}{4}\right]_0^b$$

$$= \frac{b^2 c^2}{4} \quad \text{--- (5)}$$

Along the phase CGFBC :-

we have $y = b$, & $\hat{n} = \hat{j}$

$$f \cdot \hat{n}\, ds = (y^2 - xz) = (b^2 - xz).$$

$$\int_0^a \int_0^c (b^2 - xz)\, dx\, dz = \int_0^c \left[b^2 x - \frac{x^2 z}{2}\right]_0^a = \int_0^c \left(ab^2 - \frac{a^2 z}{2}\right)$$

$$= \left[ab^2 z - \frac{a^2 z^2}{4}\right]_0^c = \left(ab^2 c - \frac{a^2 c^2}{4}\right)$$

--- (6)

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Along the phase ODEAO

we have $y = 0$.

$$\hat{n} = -\hat{j}$$

$$F \cdot \hat{n} = -(y^2 - xz) = xz$$

$$\iint_{ODEAO} F \cdot \hat{n}\, dS = \int_0^a \int_0^c xz\, dx\, dz = \left|\frac{x^2}{2}\right|_0^a \left|\frac{z^2}{2}\right|_0^c$$

$$= \frac{a^2 c^2}{4} \quad \text{--- (7)}$$

Along the phase, DEFGD :-

we have $z = c$ and $\hat{n} = \hat{k}$.

$$F \cdot \hat{n} = (z^2 - xy) = (c^2 - xy).$$

$$\iint_{DEFGD} F \cdot \hat{n}\, dS = \int_0^a \int_0^b (c^2 - xy)\, dx\, dy$$

$$= \int_0^a \left(c^2 y - \frac{xy^2}{2}\right)_0^b dx = \int_0^a \left(c^2 b - \frac{xb^2}{2}\right) dx$$

$$= \left| c^2 bx - \frac{x^2 b^2}{4} \right|_0^a$$

$$= \left(c^2 ba - \frac{a^2 b^2}{4}\right) \quad \text{--- (8)}$$

Along the phase OABCO,

we have $z = 0$ $\hat{n} = -\hat{k}$

$$F \cdot \hat{n} = -(z^2 - xy) = -xy$$

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$$\iint_{OABCO} F\cdot\hat{n}\,dS = \int_0^a\int_0^b xy\,dx\,dy$$

$$= \left|\frac{x^2}{2}\right|_0^a \left|\frac{y^2}{2}\right|_0^b$$

$$= \frac{a^2}{2}\times\frac{b^2}{2}$$

$$= \frac{a^2b^2}{4} \quad \text{---} \quad (9)$$

Putting value of (4), (5), (6), (7), (8), (9) in (3).

we get

$$\iint F\cdot\hat{n}\,dS = bca^2 - \frac{b^2c^2}{4} + \frac{b^2c^2}{4} + ab^2c - \frac{a^2c^2}{4}$$

$$+ \frac{a^2c^2}{4} + c^2ba - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

$$= bca^2 + b^2ac + c^2ba$$

$$= abc(a+b+c) \quad \text{---} \quad (10)$$

from eq. (10) and (2).

$$\iint F\cdot\hat{n}\,dS = \iiint \operatorname{div} F\,dV$$

Hence Gauss Divergence theorem is verified