# Unit-5

## Complex Integration

### Cauchy's Theorem :-

Statement → If $f(z)$ is analytic and $f'(z)$ is continous at all the point inside and on close curve C, then

$$\oint_C F(z)dz = 0$$

### Cauchy's Integral Formula :-

If $f(z)$ is analytic and $f'(z)$ is continous and if 'a' is any point inside C, then $f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz$.

$$\Rightarrow \oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

$$= 2\pi i \left[f(z)\right]_{z=a}$$

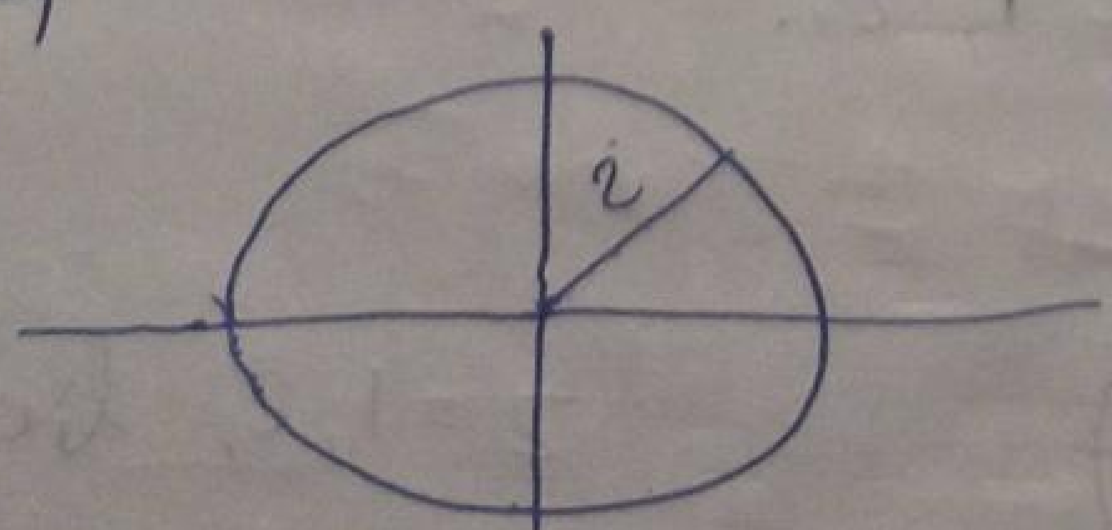
**Q** Evaluate $\oint \frac{e^{-2}}{z+1}dz$, where C is the circle $|z|=2$ and $z = \left|\frac{1}{2}\right|$.

Sol → We have $\oint \frac{e^{-2}}{z+1}dz$ ----- (1)

For pole, $z+1=0$

$z=-1$

(i) $|z|=2$.




$z=-1$, which lies inside C.

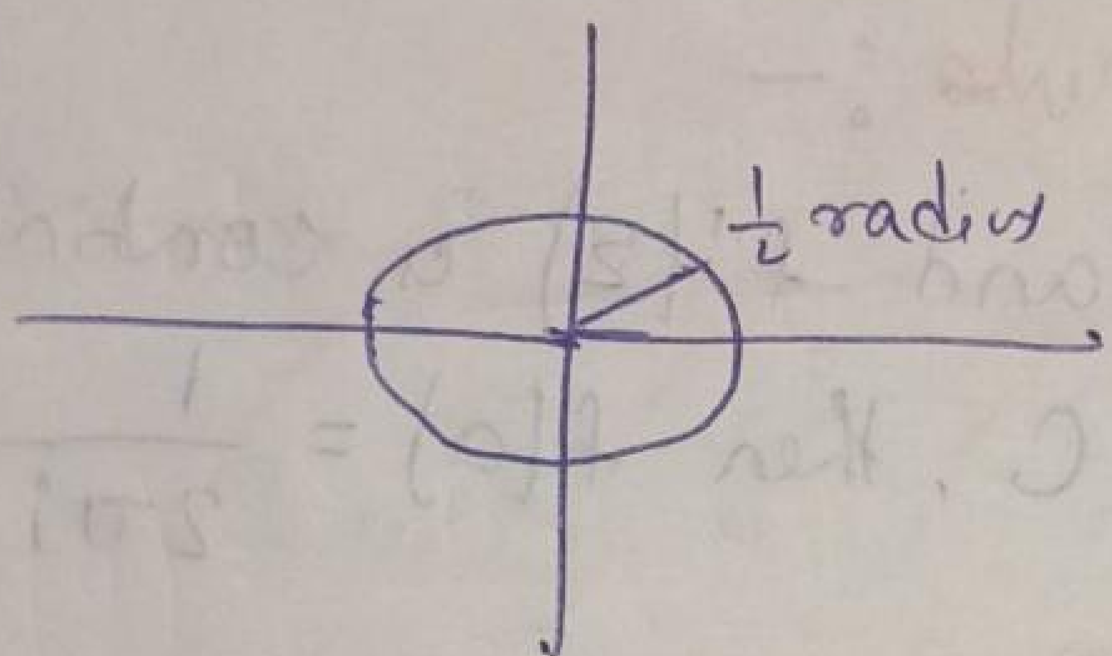
So, By Cauchy's Integral formula;

$$\oint \frac{e^{-z}}{z+1} dz = 2\pi i \left[e^{-z}\right]_{z=-1}$$

$$= 2\pi i \left[e^{1}\right]$$

$$= 2\pi e i \text{ Ans}$$

② $|z| = \frac{1}{2}$




$z = -1$ does not lie inside $C$,

By Cauchy Integral formula.

$$\oint \frac{e^{-z}}{z+1} dz = 0$$

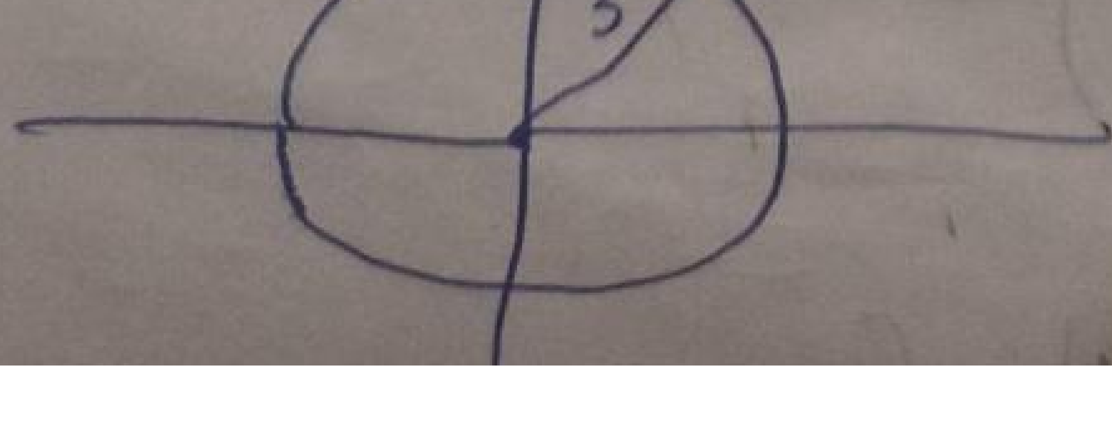
Q Evaluate $\oint \frac{\cos \pi z^2}{(z-1)(z-2)} dz$, where $C$ is circle $|z| = 3$.

Sol→ Let $I = \oint \frac{\cos \pi z^2}{(z-1)(z-2)}$

For pole. $(z-1)(z-2) = 0$

$\Rightarrow z = 1, 2$.

$|z| = 3$




$z = 1, 2$, lie inside $C$.

By Cauchy's Integral formula.

$$\oint \frac{\cos \pi z^2}{(z-1)(z-2)} dz = \oint \frac{\left(\frac{\cos \pi z^2}{z-2}\right)}{(z-1)} dz + \oint_c \frac{\left(\frac{\cos \pi z^2}{z-1}\right)}{(z-2)}$$

$$= 2\pi i \left[\frac{\cos \pi z^2}{z-2}\right]_{z=1} + 2\pi i \left[\frac{\cos \pi z^2}{z-1}\right]_{z=2}$$

$$= 2\pi i \left(\frac{\cos \pi}{1-2}\right) + 2\pi i \left[\frac{\cos 4\pi}{2-1}\right]$$

$$= 2\pi i \left(\frac{-1}{-1}\right] + 2\pi i \left[\frac{1}{1}\right]$$

$$= 4\pi i$$ Ans

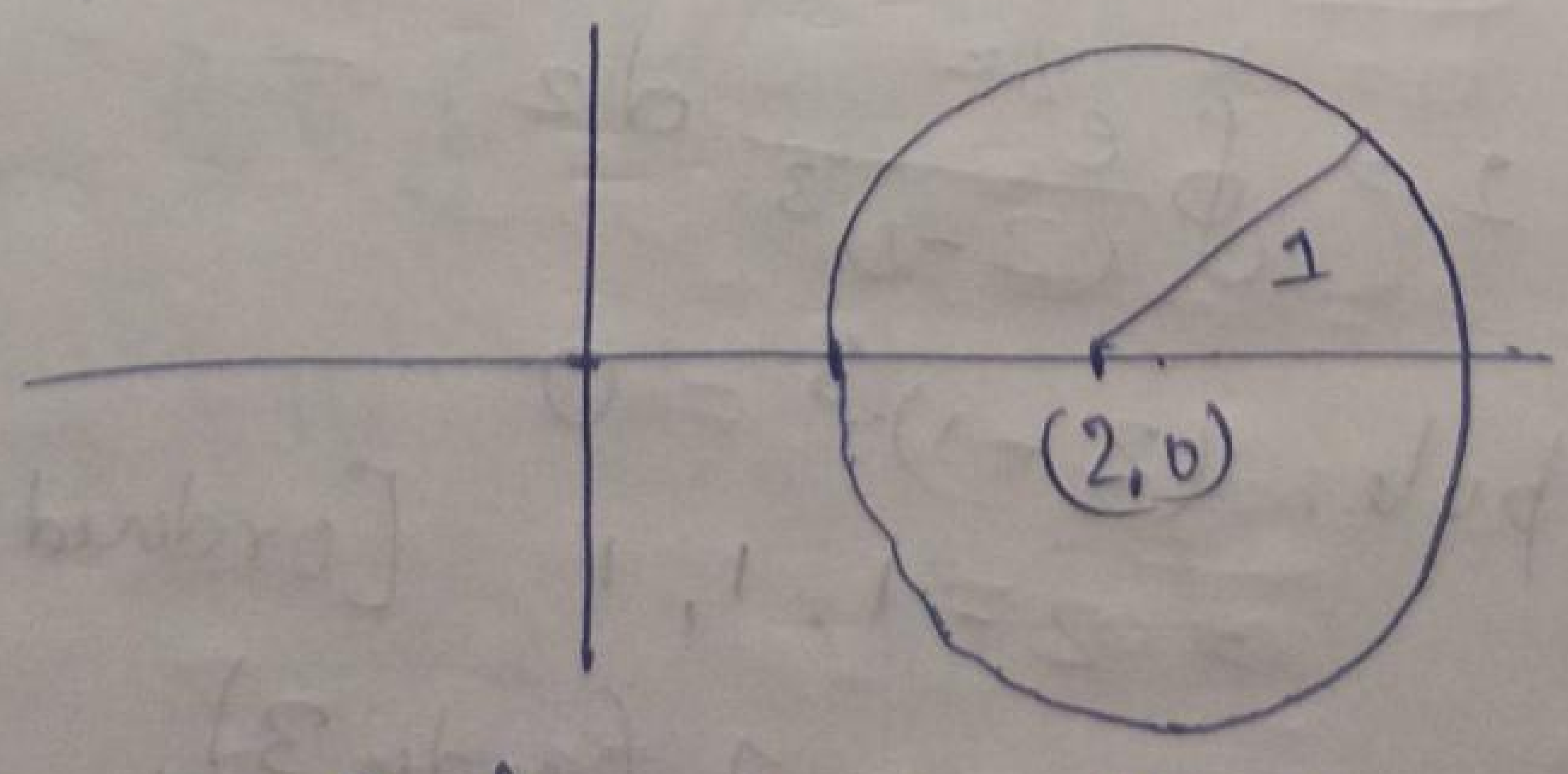
**Q** Evaluate $\oint_c \frac{dz}{z^2-2z}$ where $C$ is the circle $|z-2i|=1$

Sol → Let $I = \oint_c \frac{dz}{z^2-2z}$ — — (1)

For pole $z^2-2z=0$

$\Rightarrow z = 0, 2.$

$|z-2i| = 1$




$z=2$ lie inside the C.

and $z=0$ lie outside the C

By Cauchy Integral formula,

$$\oint_c \frac{dz}{z(z-2)} = \int \frac{(dz/z)}{(z-2)} + \oint \frac{\left(\frac{dz}{z-2}\right)}{z}$$

$$= 2\pi i \left[\frac{1}{z}\right]_{z=2} + 0$$

$$= 2\pi i \left[\frac{1}{2}\right]$$

$$= \pi i \text{ Ans}$$

## $n^{th}$ order Cauchy Integral formula

$$f^n(a) = \frac{n!}{2\pi i} \oint \frac{f(z)\,dz}{(z-a)^{n+1}}.$$

$$\oint \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(a).$$

$$= \frac{2\pi i}{n!} \left[\frac{d^n}{dz^n} f(z)\right]_{z=a}.$$

Q Evaluat $\oint_c \frac{e^{2z}}{(z-1)^3} dz$ where $C$ is $|z|=3$.

Sol→ Let $I = \oint \frac{e^{2z}}{(z-1)^3} dz$ --- ①

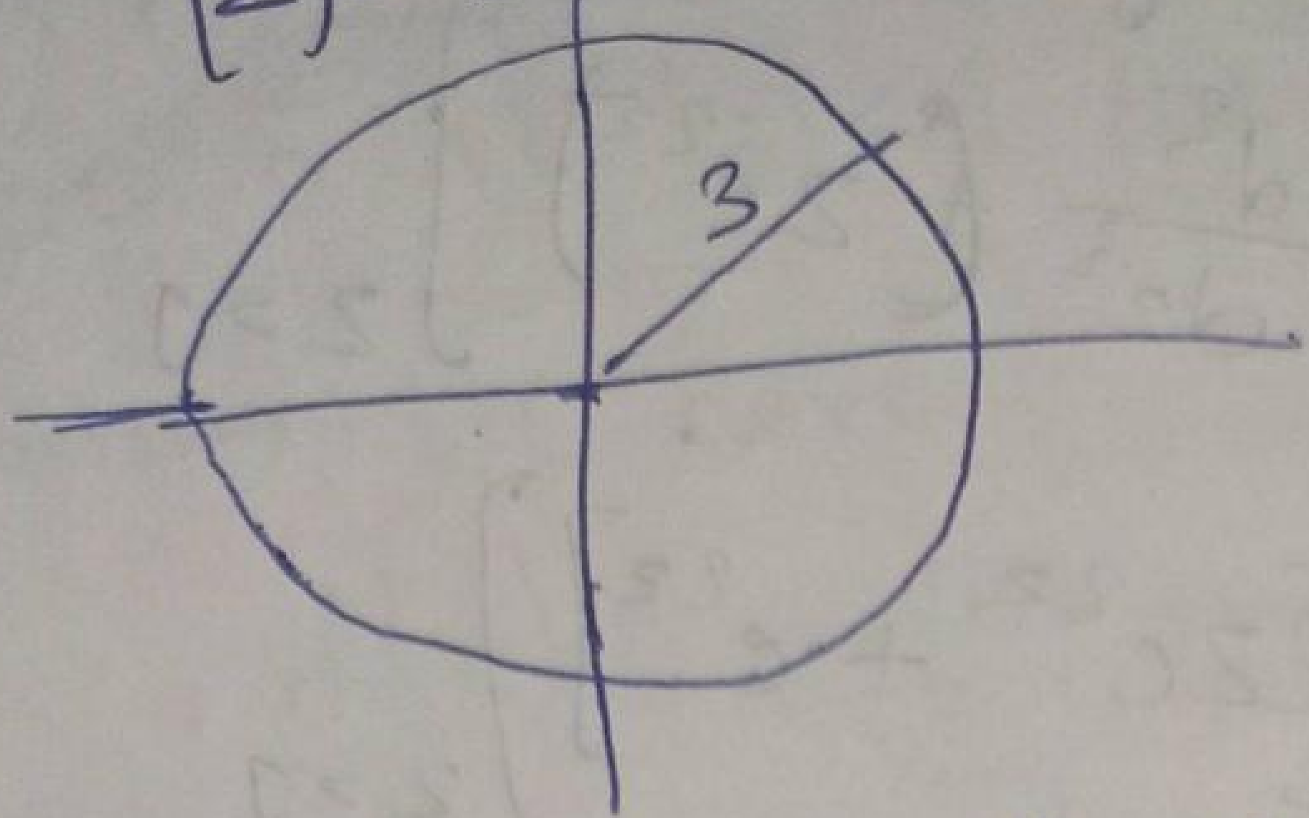
For pole, $(z-1)^3 = 0$

$= z = 1, 1, 1$ [ordered pole]

or, $z = 1$ (order 3).

to

$|z| = 3$




$z = 1$ lie inside C.

By Cauchy Integral formula

$$\oint \frac{e^{2z}}{(z-1)^3} dz = \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}(e^{2z})\right]_{z=1}$$

$$= \frac{2\pi i}{2!}\left[2\frac{d}{dz}e^{2z}\right]_{z=1}$$

$$= \frac{2\pi i}{2}\left[2 \times 2 e^{2z}\right]_{z=1}$$

$$= \frac{2\pi i}{2} \times 2^2 \times e^2$$

$$= 4\pi i e^2 \quad \text{Ans}$$

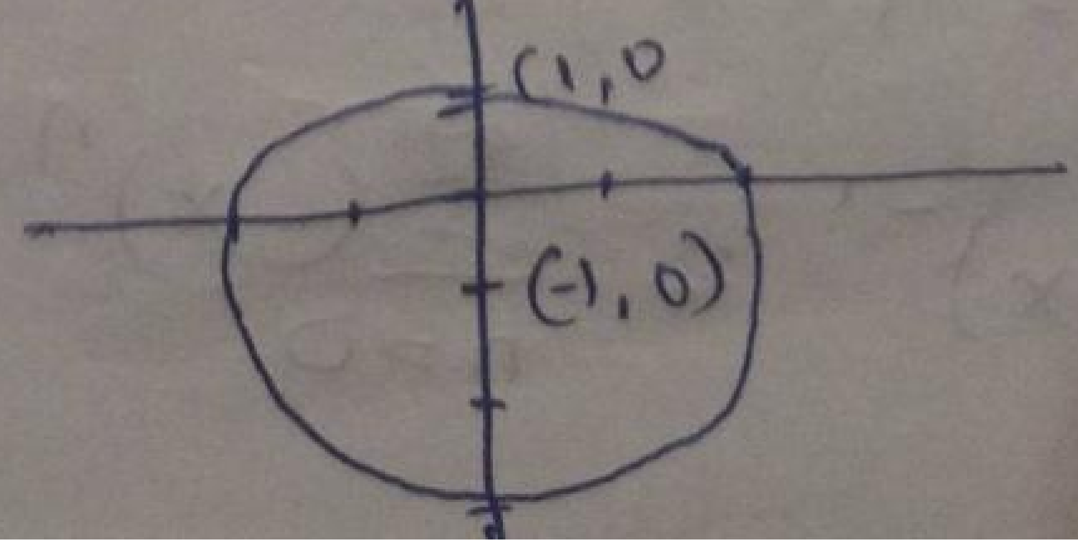
Q Evaluate $\oint \frac{z e^{2z}}{(z-1)^3} dz$ where c is circle $|z+i| = 2$

Sol→ Let $I = \oint \frac{z e^{2z}}{(z-1)^3} dz$. ---- (1)

For pole $(z-1)^3 = 1$

$\Rightarrow z = 1, 1, 1$

$|z+i| = 2$




$z = 1$ lie inside the C.

By Cauchy Integral formula.

$$\oint \frac{2e^{2z}}{(z-1)^3} = \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}\left(ze^{2z}\right)\right]_{z=1}$$

$$\Rightarrow \pi i\left[\frac{d}{dz}\left[2ze^{2z}+e^{2z}\right]\right]_{z=1}$$

$$\Rightarrow \pi i\left[z(4e^{2z})+2e^{2z}+2e^{2z}\right]_{z=1}$$

$$\Rightarrow \pi i\left[8e^2\right]$$

$$= 8\pi i e^2 \quad \text{Ans}$$

Taylor's Series :-

$$f(z) = f(a) + \frac{z-a}{1!}f'(a) + \frac{(z-a)^2}{2!}f^2(a) + \cdots$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty}\frac{(z-a)^n f^n(a)}{n!}$$

Laurent's Series :-

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=0}^{\infty} b_n(z-a)^{-n}.$$

* $(1-x)^{-1} = 1+x+x^2+x^3+\cdots$

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$$

* $(1+x)^{-1} = 1-x+x^2-x^3+\cdots$

$$(1+x)^{-1} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n.$$

Q Expand the function $f(z) = \frac{1}{(z-1)(z-2)}$ bounded by region (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$

Solution → We have.

$$f(z) = \frac{1}{(z-1)(z-2)} \quad \text{--- (1)}$$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)} \quad \text{--- (2)}$$

Case I :- when $|z|<1$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} + \frac{1}{(1-z)}$$

$$f(z) = \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

$$f(z) = \frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n.$$

The function $f(z)$ has taylor series in the region $|z|<1$ because the power of $z$ in every term is +ve.

Case II :- When $1<|z|<2$.

or $|z|>1$ & $|z|<2$.

$$f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}.$$

$$f(z) = \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

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$$f(z) = \frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$f(z) = \frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}(z)^{-n}$$

So, the function $f(z)$ has laurent's series in the region $1<|z|<2$ because the power of $z$ has both positive & negative.

Case 3 :- $|z|>2$

$$f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n.$$

This is a valid expansion in this region.

Q Expand the function $f(z) = \frac{1}{(z+1)(z+3)}$ in the power of $(z+1)$ for the range.

$0<|z+1|<2.$

Sol→ Let $(z+1) = v$

$z = v-1.$

$$f(v) = \frac{1}{v(v+2)}$$

$$= \frac{1}{u\left[2\left(1+\frac{u}{2}\right)\right]}.$$

$$= \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u}\left[1-\frac{u}{2}+\frac{u^2}{4}+\cdots\right]$$

$$= \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} + \cdots$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{(z+1)}{8} + \cdots \quad \text{Ans}$$

Q find the taylor series & ~~lore~~ lorent's series expansion for $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ in the region

(i) $|z|<2$ (ii) $2<|z|<3$ (iii) $|z|>3$.

Sol→ $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$

$$f(z) = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)} \quad \text{--- (1)}$$

Case I → when $|z|<2$

$$f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$f(z) = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{-z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}\left(\frac{-z}{3}\right)^n$$

So, the ~~series~~ function $f(z)$ has taylor series in the region $|z|<2$ because power of $z$ in every term is +ve.

Case 2 :- $2<|z|<3$ or $|z|>2$ & $|z|<3$.

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \sum_{n=0}^{\infty}\left(\frac{-2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}\left(\frac{-z}{3}\right)^n.$$

So, the function $f(z)$ has laurent's series in the region $2<|z|<3$ because power of $z$ ~~in every term~~ is both +ve & -ve.

Case 3 :- $|z|>3$.

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

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$$\Rightarrow f(z) = 1 + \frac{3}{z}\sum_{n=0}^{\infty}\left(\frac{-2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}\left(\frac{-3}{z}\right)^n$$

This is a valid ~~expansion~~ expansion in this region.

Q Expand $\log\frac{(1+z)}{(1-z)}$ at $z=0$ by using taylor series.

Sol → We have $f(z) = \log\frac{(1+z)}{(1-z)}$.

$$f(z) = \log(1+z) - \log(1-z) \quad \text{--- (1)}$$

differntiat with respect to z.

$$f'(z) = \frac{1}{(1+z)} + \frac{1}{(1-z)} \quad \text{--- (2)}$$

$$f''(z) = \frac{-1}{(1+z)^2} + \frac{1}{(1-z)^2}$$

$$f'''(z) = \frac{2}{(1+z)^4} + \frac{2}{(1-z)^2}$$

$$f^{iv}(z) = \frac{-6}{(1+z)^2} + \frac{6}{(1-z)^4}$$

at $z=0$,

$$f(z) = \log(1+0) - \log(1-0)$$
$$= 0$$

$$f'(z) = \frac{1}{1+0} + \frac{1}{1-0} = 2$$

$$f''(z) = \frac{-1}{(1+0)^2} + \frac{1}{(1-z)^2} = 0$$

$$f'''(z) = 4 \qquad f^{iv}(z) = 0$$

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By Talor Series,

$$f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

So, $f(z) = f(0) + \frac{z-0}{1!} f'(0) + \frac{(z-0)^2}{2!} f''(0) + \frac{(z-0)^3}{3!} f'''(0)$

$+ \cdots$

$\Rightarrow f(z) = 0 + z \times 2 + 0 + \frac{z^3}{3!} \times 4 + 0 + \cdots$

$f(z) = 2z + \frac{4z^3}{3!} + \cdots$ Ans

Q Expand $\frac{\sin z}{z-\pi}$ about $z = \pi$.

Sol → Let $f(z) = \frac{\sin z}{z-\pi}$ --- ①

Let $z - \pi = t$ (Let)

$$f(t) = \frac{\sin(t+\pi)}{t}$$

$$= -\frac{\sin t}{t}$$

We know, $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$

So, $-\frac{\sin t}{t} = -\frac{1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{3!} + \cdots\right]$

$$= -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \cdots$$

So, $f(z) = -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} + \cdots$ Ans

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Q Expand $ze^{-z^2}$ about $z=0$.

Sol → Let $f(z) = ze^{-z^2}$

$$= z\left[1 - \frac{z^2}{1!} + \frac{z^4}{2!} - \frac{z^6}{3!} + \cdots\right]$$

$$\left[\because e^{-t} = 1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right]$$

$$f(z) = z - \frac{z^3}{1!} + \frac{z^5}{2!} - \frac{z^7}{3!} + \cdots$$ Ans

## Zeros of analytic function $f(z)$

A zero of an analytic function $f(z)$ is a value of $z$ for which $f(z)=0$ if $f(a)=0$ and $f'(a)\neq 0$. Then, $z=a$, is called zero of $f(z)$.

Q If $f(z) = \frac{z-2}{z-3}$, find zero of $f(z)$.

Sol → for zero, Numerator $= 0$

$\Rightarrow z-2 = 0$

$\Rightarrow z = 2$ is the zero of $f(z)$

## Singularity (Pole) of analytic function $f(z)$

Those point $z_0$ at which the function $f(z)$ is not analytic is known as singular point or singularity of $f(z)$.

1) Isolated Singularity → A singular point $z=z_0$ is said to be isolated singular point of $f(z)$ if there is no other point in neighbourhood of $z_0$.

Ex. 1. find the isolated singularity of function $f(z) = \frac{z+1}{z(z-2)}$

Sol → singularity point $z(z-2) = 0$

$z = 0, 2$

Q) Is $z=2$ is isolated singularity?

⇒ No.

⇒ No isolated singularity of function $f(z)$.

2. $f(z) = \frac{z}{z-4}$

singularity. point. $z-4=0$

$z=4$

$z=4$ is isolated singularity.

2) Removable singular point :-

A singular point $z = z_0$ is said to be removable singular point of $f(z)$ if $\lim\limits_{z \to z_0} =$ Exist & finite.

Ex. Q $f(z) = \frac{\sin z}{z}$ is analytic except at $z=0$.

Sol → for, $z=0$,

$\lim\limits_{z\to 0} f(z) = \lim\limits_{z \to 0} \frac{\sin z}{z} = 1$ (finit & exist).

So, ~~f(z)~~ $z=0$, is removable singular point

Q $f(z) = \frac{\tan z}{z}$, $f(z)$ is analytic except at $z=0$

Sol → $\lim\limits_{z\to 0} \frac{\tan z}{z} = 1$ (finite & exist)

⇒ $z=0$ is removable singular point

Q $f(z) = \frac{(z^2 - a^2)}{(z+a)}$. $z=-a$ is singular point

Sol → $\lim\limits_{z \to -a} \frac{(z-a)(z+a)}{(z+a)} = -2a$ (finit)

⇒ Removable singularity.

3) Essential Singularity :- A singular point $z = z_0$ is said to be essential singular point, if it is neither an isolated singularity nor a removable singularity.

Ex → $f(z) = e^{1/z}$.

take $z = 0$

$f(z) = e^{\infty} = \infty$

$\Rightarrow$ at $z = 0$, $f(z)$ is essential singular

Ex → $f(z) = e^{1/(z+1)}$

take $z = -1$

$e^{1/0} = e^{\infty} = \infty$

$\Rightarrow$ at $z = -1$, it is essential singularity.

## Determine Residue of $f(z)$

1) Residue of $f(z)$ at $z = a$ (simple pole), then

$$\text{Res}(z = a) = \lim_{z = a} (z - a) f(z)$$

2) Residue of $f(z)$ at $z = a, a, a, \ldots$ n times (repeated pole)

then, $$\text{Res}(z = a) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)\right]_{z=a}$$

Q find residue of function $f(z) = \frac{z}{(z-1)^2 (z-2)}$

Sol → Given. $f(z) = \frac{z}{(z-1)^2 (z-2)}$

For pole, $(z-1)^2(z-2)=0$

$z = 2, 1, 1.$

Residue at $z=2$ :-

$$\text{Res}(z=2) = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}\cancel{(z-2)}\frac{z}{(z-1)^2\cancel{(z-2)}}$$

$$= \frac{2}{(2-1)^2} = 2 \quad \text{Ans}$$

Residue at $z=1$ (order 2) :-

$$\text{Res}(z=1) = \frac{1}{(2-1)!}\left[\frac{d}{dz}(z-1)^2 f(z)\right]_{z=1}$$

$$= \frac{1}{1}\left[\frac{d}{dz}\cancel{(z-1)^2}\frac{z}{\cancel{(z-1)^2}(z-2)}\right]_{z=1}$$

$$= \left[\frac{(z-2)-z}{(z-2)^2}\right]_{z=1}$$

$$= \frac{-2}{(-1)^2} = -2 \quad \text{Ans}$$

Q. Find the residue of function $f(z) = \dfrac{e^{2z}}{(z-1)^2}$.

Sol → For pole $(z-1)^2 = 0$

$\Rightarrow z = 1$ (order 2)

$$\text{Res}(z=1) = \frac{1}{(2-1)!}\left[\frac{d}{dz}(z-1)^2 f(z)\right]_{z=1}$$

$$= \frac{1}{1}\left[\frac{d}{dz}\frac{e^{2z}}{(z-1)^2}(z-1)^2\right]_{z=1}$$

$$= \left[2e^{2z}\right]_{z=1} = 2e^2 \quad \text{Ans}$$

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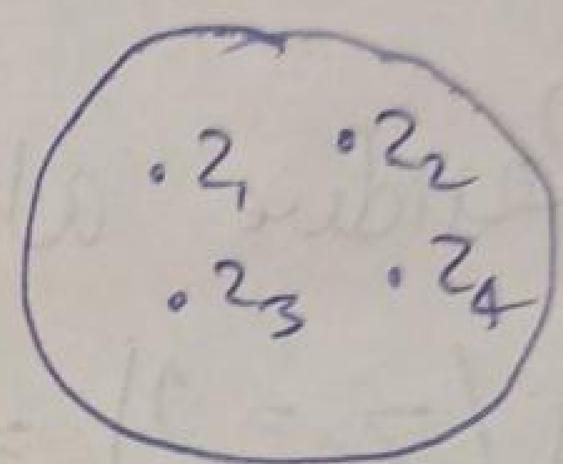
# Cauchy Residue Theorem :-

If $f(z)$ is analytic at all point inside and on simple closed curve except for a ~~pole~~ finite no. of isolated singular point (pole), $(z_1, z_2, z_3 \ldots z_n)$ within C, then

$$\oint_C f(z)\,dz = 2\pi i R.$$

where R = sum of residue of $f(z)$ at $z_1 z_2 \ldots z_n$.




**Q** Evaluate $\oint_C \frac{\cos \pi z}{z-1}\,dz$, where C is circle $|z| = 1.5$

Sol → Let $I = \oint_C \frac{\cos \pi z}{z-1}\,dz$.

Let $f(z) = \frac{\cos \pi z}{z-1}$

$I = \oint_C f(z)\,dz$

By Cauchy Residue Theorem.

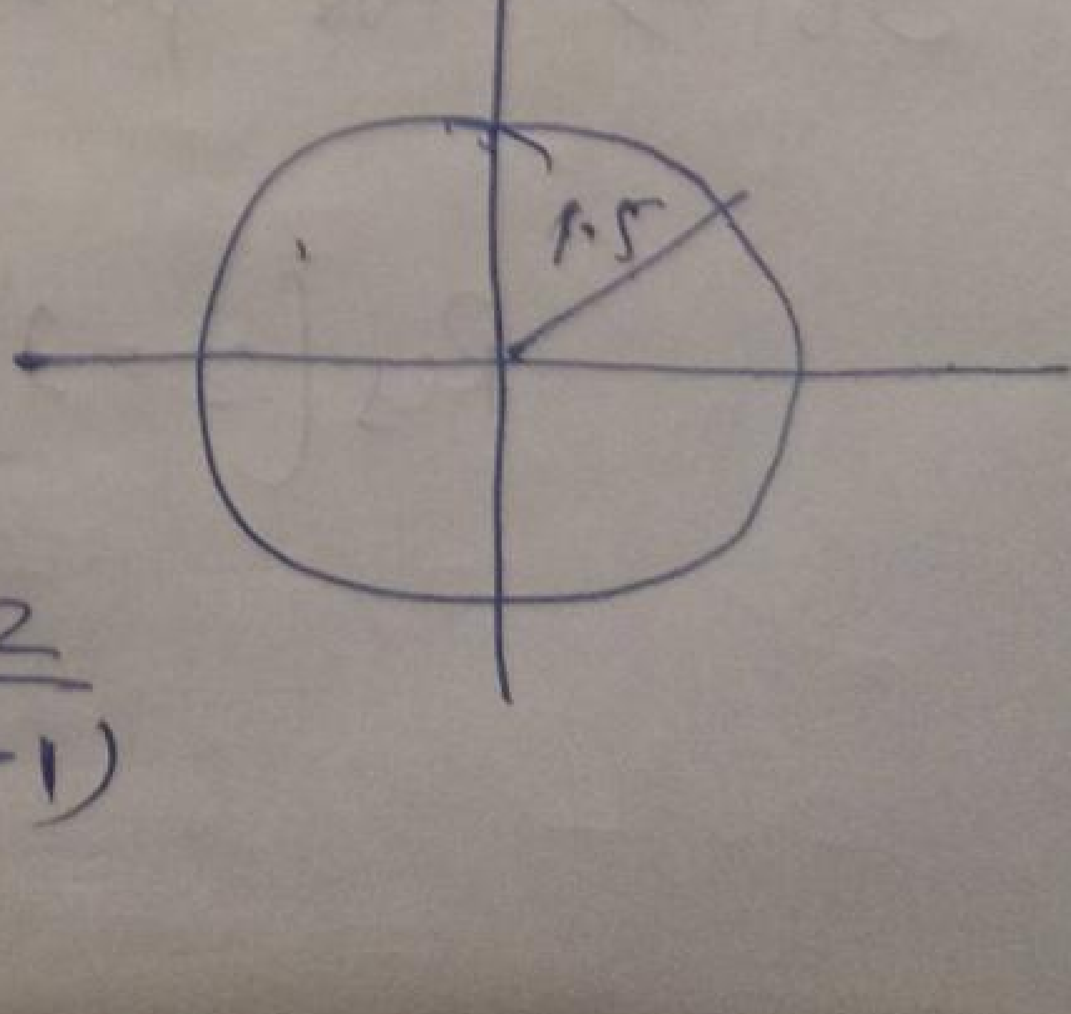
$I = \oint_C f(z)\,dz = 2\pi i R.$

$\Rightarrow I = 2\pi i R$ ——— (2)

For pole, $(z-1) = 0$

$z = 1$

which lie inside ~~lies~~ C.




So, $\text{Res}(z=1) = \lim_{z \to 1} (z-1) \frac{\cos \pi z}{(z-1)}$

$R = -1$

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Putting value of R in eq. 1

$I = 2\pi i(-1)$

$I = -2\pi i$ Ans

Q Evaluate $\oint_C \frac{z^2}{(z-1)^2(z-2)} dz$, when C is circle $|z| = \frac{1}{2}$

Sol→ Let $I = \oint_C \frac{z^2}{(z-1)^2(z-2)}$

let $f(z) = \frac{z^2}{(z-1)^2(z-2)}$. $\Rightarrow I = \oint f(z) dz$

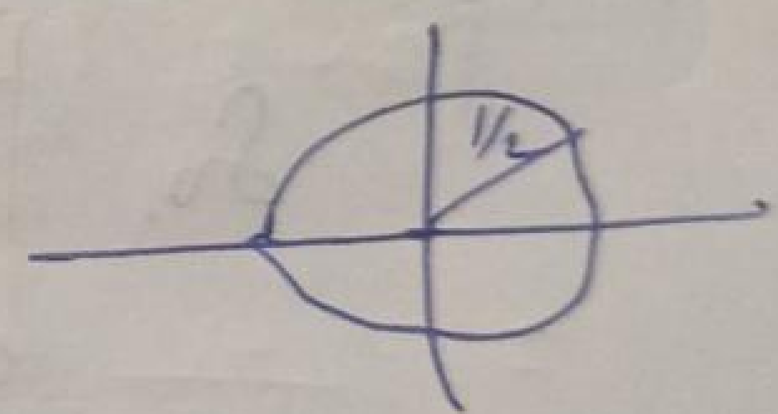
By Cauchy Residue theorem.

$I = 2\pi i R$.

For pole, $(z-1)^2(z-2) = 0$

$\Rightarrow z = 2, 1, 1$.

which lie outside C.




$\Rightarrow$ The pole does not lie inside C.

Hence residue is 0.

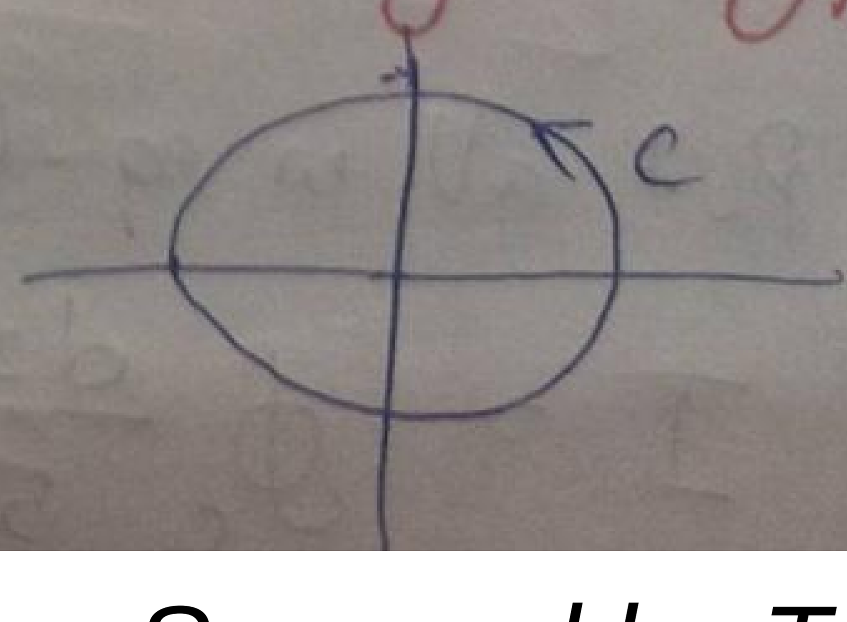
$\Rightarrow R = 0$

Put value of R in eq 1.

$I = 0$ Ans

* Determine the Integration of the type :-

$\int_0^{2\pi} f(\sin\theta, \cos\theta) d\theta$

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Let $|z| = 1$.

$z = e^{i\theta}$

$dz = ie^{i\theta}\, d\theta$

$dz = iz\, d\theta$

$$\boxed{d\theta = \frac{dz}{iz}}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2-1}{2iz}$$

So, $\boxed{\sin\theta = \frac{z^2-1}{2zi}}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2+1}{2z}$$

So, $\boxed{\cos\theta = \frac{z^2+1}{2z}}$

**Q** Evaluat $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$

Sol → Let $I = \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$ — — ①

Let $|z| = 1$

$d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{z^2+1}{2z}$

Put all in eq. ①

$$I = \oint_C \frac{dz/iz}{5+3\left(\frac{z^2+1}{2z}\right)}$$

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$$I = \oint \frac{dz/iz}{\frac{10z+3z^2+3}{2z}}$$

$$I = \frac{2}{i}\oint \frac{dz}{3z^2+10z+3}$$

$$I = \frac{2}{i}\oint f(z)\,dz \quad , \text{ where } f(z) = \frac{1}{3z^2+10z+3}$$

By Cauchy's residue Theorem.

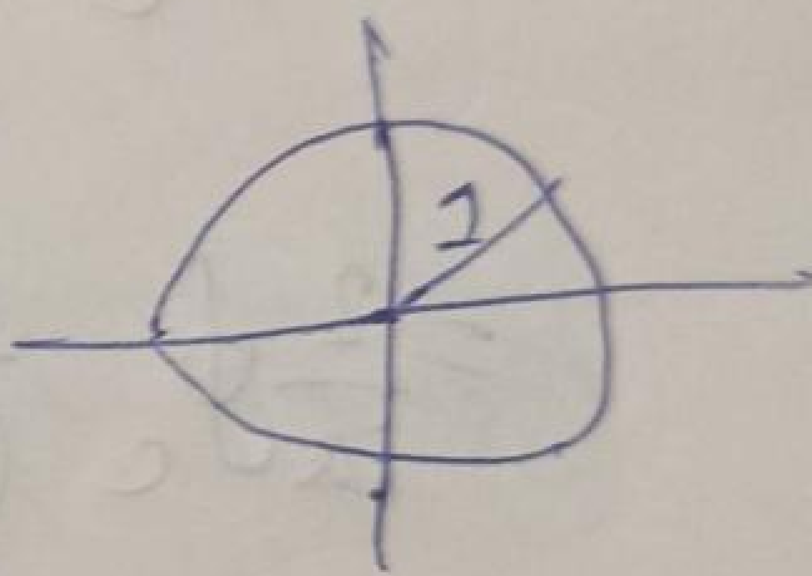
$$I = \frac{2}{\cancel{i}}\, 2\pi \cancel{i} R = 4\pi R \quad --- \quad (2)$$

for pole $3z^2+10z+3 = 0$

$$\Rightarrow z = -3, \; -\frac{1}{3}$$

$\Rightarrow z = -3$ lie out side the C.

$z = -\frac{1}{3}$ lie inside the C.




Residue at $z = -\frac{1}{3}$

$$Res\left(z=-\tfrac{1}{3}\right) = \lim_{z\to -\frac{1}{3}} \left(z+\tfrac{1}{3}\right) f(z)$$

$$= \lim_{z\to -\frac{1}{3}} \left(z+\tfrac{1}{3}\right)\frac{1}{(z+3)(3z+1)}$$

$$= \lim_{z\to -\frac{1}{3}} \frac{\cancel{(3z+1)}}{3(z+3)\cancel{(3z+1)}}$$

$$= \frac{1}{3\left(-\frac{1}{3}+3\right)} = \frac{1}{8}$$

So, $R = \frac{1}{8}$ , Putting it in eq. (2)

$$I = 4\pi R \Rightarrow I = 4\pi \times \frac{1}{8} = \frac{\pi}{2} \quad \text{Ans}$$

Q Show that $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}$.

Sol → Let $I = \int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$ —— (1)

Let $|z| = 1$, $d\theta = dz/iz$, $\cos\theta = \frac{z^2+1}{2z}$

Putting all these in eq. (1)

$$I = \oint_c \frac{dz/iz}{1+a\left[\frac{z^2+1}{2z}\right]}$$

$$= \oint_c \frac{dz/iz}{\frac{2z+az^2+a}{2z}}$$

$$= \frac{2}{ia}\oint_c \frac{dz}{\left(z^2+\frac{2z}{a}+1\right)}$$

$$= \frac{2}{ia}\oint_c f(z)\,dz, \text{ where } f(z) = \frac{1}{z^2+\frac{2z}{a}+1}$$

By Cauchy Residue theorem.

$$I = \frac{2}{ia} 2\pi i R$$

$$\Rightarrow I = \frac{4}{a}\pi R \quad \text{—— (2)}$$

For pole, $z^2+\frac{2z}{a}+1 = 0$

$$z = \frac{-\left(\frac{2}{a}\right) \pm \sqrt{\frac{4}{a^2}-4}}{2}$$

$$= \frac{-\frac{2}{a} \pm 2\sqrt{\frac{1}{a^2}-1}}{2}$$

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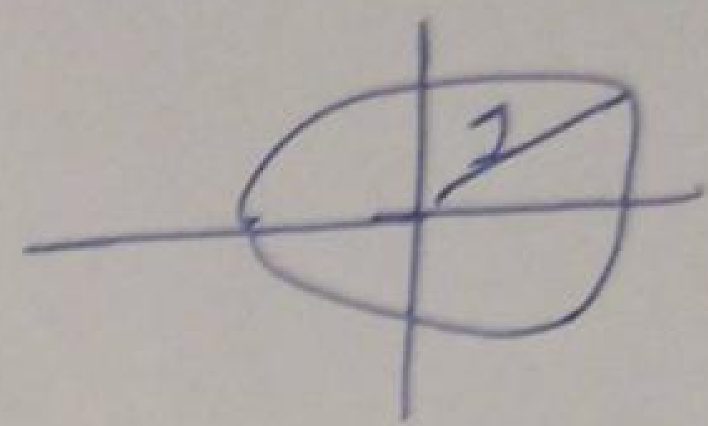
$$\Rightarrow \frac{-1}{a} \pm \sqrt{\frac{1-a^2}{a^2}} \Rightarrow \frac{-1}{a} \pm \frac{\sqrt{1-a^2}}{a}$$

$$\Rightarrow \left[\left(\frac{-1}{a}\right) + \frac{\sqrt{1-a^2}}{a}\right] = \alpha$$

$$\left[\left(\frac{-1}{a}\right) - \frac{\sqrt{1-a^2}}{a}\right] = \beta .$$

$\Rightarrow$ $z = \alpha$ lie inside C &

$z = \beta$ lie outside C.



Residue at $z = \alpha$.

$$\text{Res}(z=\alpha) = \lim_{z \to \alpha} (z-\alpha) f(z)$$

$$= \lim_{z \to \alpha} (z-\alpha) \frac{1}{(z-\alpha)(z-\beta)}$$

$$= \frac{1}{\alpha - \beta}$$

$$\Rightarrow R = \frac{1}{\alpha-\beta} \quad --- \quad (3)$$

Putting it in eq. (2).

$$I = \frac{4}{a}\pi\left(\frac{1}{\alpha-\beta}\right)$$

$$= \frac{4}{a}\pi\left[\frac{1}{\frac{-1}{a} + \frac{\sqrt{1-a^2}}{a} + \frac{1}{a} + \frac{\sqrt{1-a^2}}{a}}\right]$$

$$= \frac{\cancel{4}^{2}}{\cancel{a}}\pi \times \frac{\cancel{a}}{\cancel{2}\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}$$

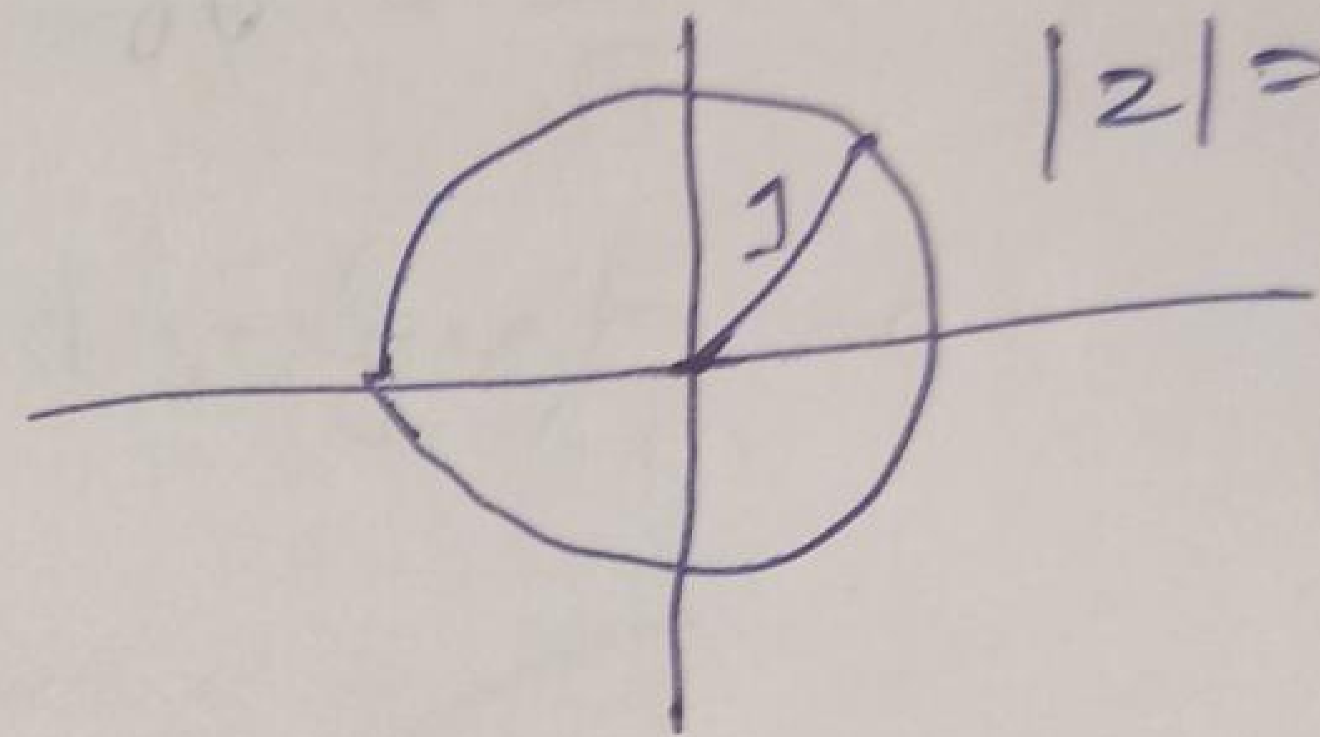
Hence proved

Q Evaluat $\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta$

Sol→ Let $I = \int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta$

$\Rightarrow I = \text{Real part of } \int_0^{2\pi} \frac{e^{i2\theta}}{5-4\cos\theta} d\theta.$

Let $|z| = 1$



$|z|=1$

$z = e^{i\theta}$

$d\theta = dz/i\theta$

$\cos\theta = \frac{z^2+1}{2z}$

Putting these all in above eq. we get

$I = \text{Real part of } \oint \frac{z^2 dz/iz}{5-4\left(\frac{z^2+1}{2z}\right)}$

$= \text{R.P of } \oint \frac{z^2 dz/iz}{\frac{5z-2z^2-2}{z}}$

$= \text{R.P of } \frac{-1}{i} \oint \frac{z^2 dz}{2z^2-5z+2}$

$\Rightarrow \text{R.P of } \frac{-1}{i} \oint f(z) dz$, where

$f(z) = \frac{z^2}{2z^2-5z+2}$

By Cauchy residue theorem.

$I =$ R.P of $\frac{-1}{i} \times 2\pi i R$

$I =$ Real part of $-2\pi R$ - - - (1)

Now, For pole, $2z^2 - 5z + 2 = 0$

$\Rightarrow 2z(z-2) - 1(z-2) = 0$

$\Rightarrow z = \frac{1}{2}, 2$.

$z = \frac{1}{2}$ lie inside pole and $z = 2$ lie outside pole

Now, Residue at $z = \frac{1}{2}$.

$$Res\left(z = \tfrac{1}{2}\right) = \lim_{z \to \frac{1}{2}} \left(z - \tfrac{1}{2}\right) \frac{z^2}{(z-2)(2z-1)}$$

$$= \lim_{z \to \frac{1}{2}} \frac{(2z-1)}{2} \times \frac{z^2}{(z-2)(2z-1)}$$

$$= \frac{1}{2} \times \frac{\frac{1}{4}}{\frac{-3}{12}} = \frac{-1}{12}$$

So, $R = \frac{-1}{12}$

Putting it in eq. (1)

$I =$ Real part of $-2\pi R =$ Real part of $\frac{\pi}{6}$

$= \pi/6$.

So. $\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta} = \pi/6$ Ans

Q Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta$

Sol → Let $I = \int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2(5+4\cos\theta)}\, d\theta$$

$$= \text{R.P of } \frac{1}{2}\int_0^{2\pi} \frac{1-e^{2i\theta}}{5+4\cos\theta}\, d\theta$$

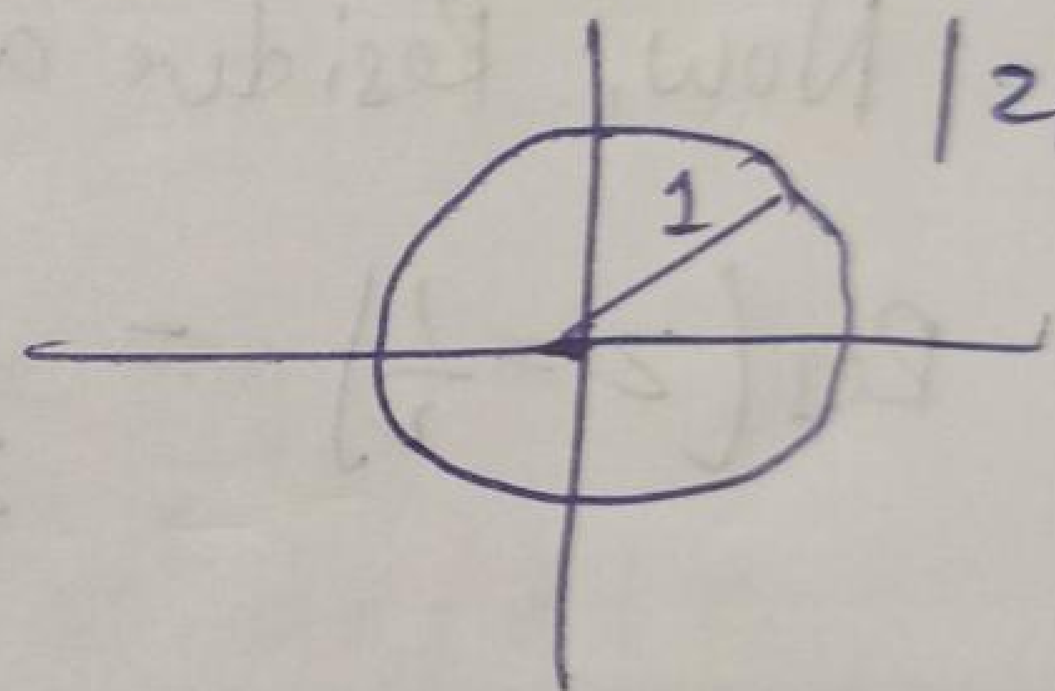
Let $|z| = 0$,

$d\theta = dz/iz$

$z = e^{i\theta}$

$\cos\theta = \frac{z^2+1}{2z}$

$|z| = 1$




$$I = \text{R.P of } \frac{1}{2} \oint_c \frac{(1-z^2)\, dz/iz}{5+4\left(\frac{z^2+1}{2z}\right)}.$$

$$= \text{R.P of } \frac{1}{2i} \oint_c \frac{(1-z^2)}{5z+2z^2+2}\, dz$$

$$= \text{Real part of } \frac{1}{2i} \oint f(z)\, dz$$

where $f(z) = \frac{1-z^2}{5z+2z^2+2}$

By Cauchy Residue theorem.

$$I = R.P \text{ of } \frac{1}{2i} \times 2\pi i R$$

$$I = R.P \text{ of } \pi R \quad ---- (1)$$

For pole, $2z^2 + 5z + 2 = 0$

$$\Rightarrow z = -2, -\frac{1}{2}$$

$\Rightarrow z = -\frac{1}{2}$ lie inside C and $z = -2$ lie outside C.

$\Rightarrow$ Residue at $z = -\frac{1}{2}$

$$Res\left(z = -\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z)$$

$$= \lim_{z \to -\frac{1}{2}} \frac{(2z+1)}{2} \times \frac{1 - z^2}{(2z+1)(z+2)}$$

$$= \frac{1}{2} \times \frac{3}{4} \times \frac{2}{3}$$

$$= \frac{1}{4}$$

$\Rightarrow R = \frac{1}{4}$, Putting it in eq. (1), we get

$$I = R.P \text{ of } \pi R \Rightarrow R.P \text{ of } \pi \frac{1}{4}$$

$$= \pi/4$$

So, $\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta} d\theta = \pi/4$ Ans

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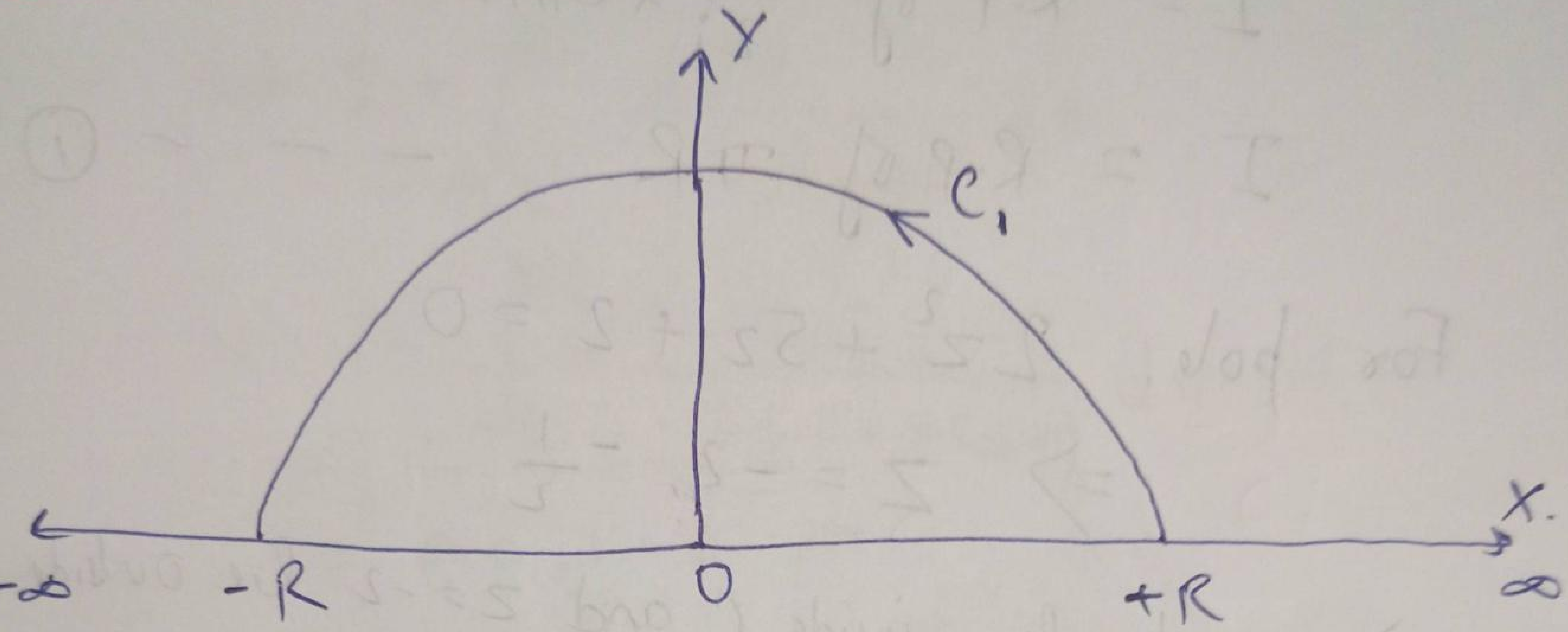
## The Integral of the type $\int_{-\infty}^{\infty} f(x)\,dx$.




$$\oint_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx.$$

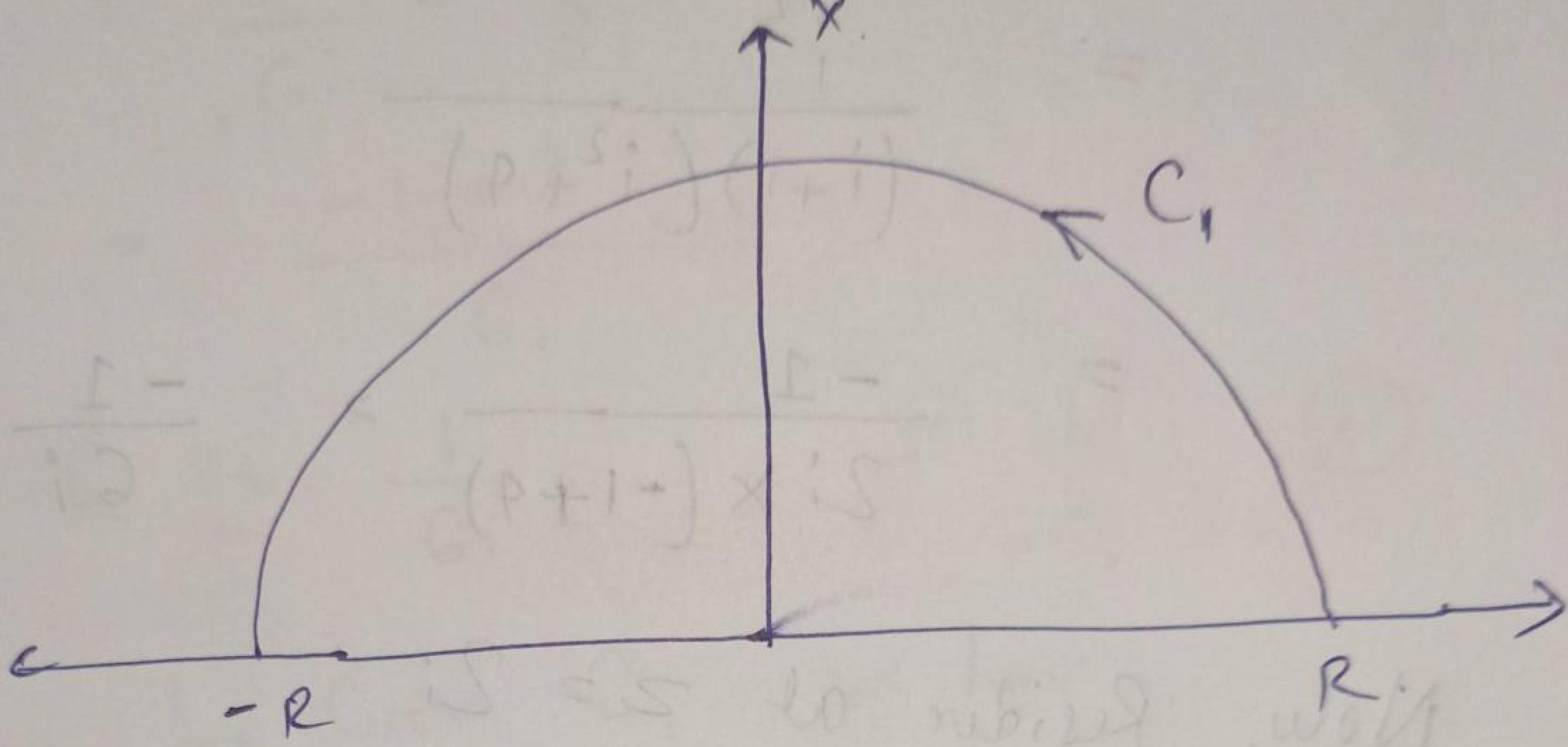
Closed (under $\oint_C$) Not Closed (under $\int_{C_1}$)

**Cauchy's lemma :-** If $f(z)$ is continous function such that $\lim\limits_{z\to\infty} z f(z) = 0$ on the upper semicircle $C_1$, then $\int_{C_1} f(z)\,dz = 0$.

$|z| = R$ as $R \longrightarrow \infty$.

**Q** Evaluat $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}$ using Contour integration.

Sol→ Let $I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx.$




Let $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$

Now, $\oint_c f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx$ ---- (1)

For pole, $(z^2+1)(z^2+4) = 0$

$\Rightarrow z = \pm i, \pm 2i$.

$z = +i, +2i$ lie inside and $z = -i, -2i$ lie outside.

Now, Residue at $z = +i$

$R_1(z=i) = \lim_{z\to i} (z-i) f(z)$.

$= \lim_{z\to i} (z-i) \frac{z^2}{(z^2+1)(z^2+4)}$

$$= \lim_{z \to i} (z-i) \frac{z^2}{(z-i)(z+i)(z^2+4)}$$

$$= \frac{i^2}{(i+i)(i^2+4)}$$

$$= \frac{-1}{2i \times (-1+4)} = \frac{-1}{6i}$$

Now, Residue at $z = 2i$

$$R_2(z=2i) = \lim_{z \to 2i} \frac{z^2 (z-2i)}{(z+1)(z^2+4)}$$

$$= \lim_{z \to 2i} \frac{(z-2i)\, z^2}{(z^2+1)(z-2i)(z+2i)}$$

$$= \frac{(2i)^2}{(z^2+1)(2i+2i)}$$

$$= \frac{4i^2}{(4i^2+1)(4i)} = \frac{i^2}{(-4+1)i}$$

$$= \frac{-1}{-3 \times i}$$

$$= \frac{1}{3i}$$

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$$R = R_1 + R_2$$

$$= \frac{-1}{6i} + \frac{1}{3i}$$

$$= \frac{-1+2}{6i}$$

$$R = \frac{1}{6i} \quad \text{---} \quad (2)$$

Now, for curve $C_1$, we have

$$\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} z \cdot \frac{z^2}{(z^2+1)(z^2+4)}$$

$$= \lim_{z\to\infty} \frac{z^3}{z^2\left(1+\frac{1}{z^2}\right) z^2\left(1+\frac{4}{z^2}\right)}$$

$$= \lim_{z\to\infty} \frac{1}{z\left(1+\frac{1}{z^2}\right)\left(1+\frac{4}{z^2}\right)}$$

$$= \frac{1}{\infty} = 0.$$

Thus, $\int_{C_1} f(z)\,dz = 0 \quad \text{---} \quad (3)$

Putting value of (3) in eq. (1).

$$\oint_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{+R} f(x)\,dx.$$

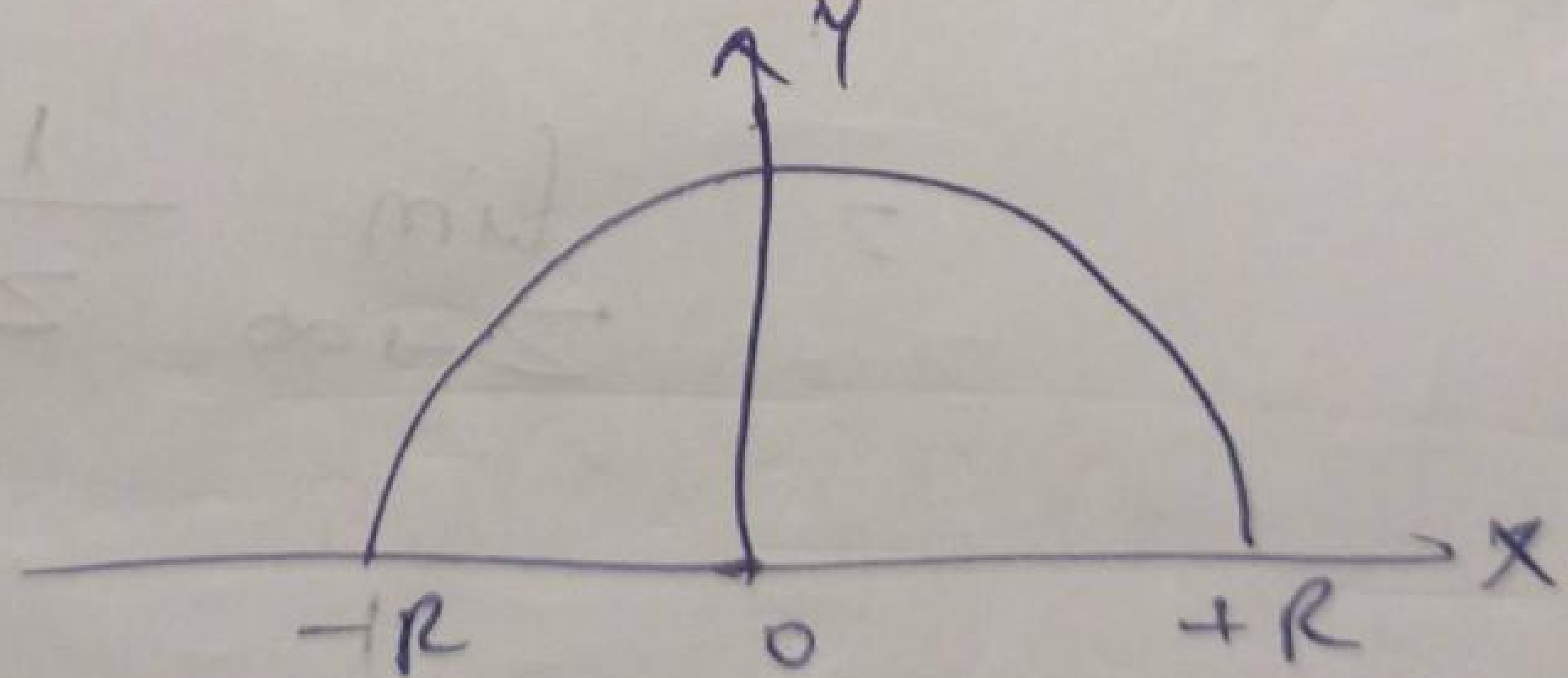
$$\Rightarrow 2\pi i R = 0 + \int_{-\infty}^{\infty} f(x)\,dx .$$

$$\Rightarrow 2\pi i \times \frac{1}{6i} = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{3} \quad \underline{\underline{\text{Ans}}}$$

**Q** Evalute $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$

Sol → Let $\displaystyle I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$




Let $\displaystyle f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$

For pole, $(z^2+a^2)(z^2+b^2) = 0$

$$\Rightarrow z = \pm ai, \pm bi$$

$z = +ai, +bi$ lie inside and $z = -ai, -bi$ lie outside C.

Now,

$$\oint_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx \quad \text{--- (1)}$$

For Curve C, we have

$$\oint_C f(z)\,dz = 2\pi i R \quad \text{--- (2)}$$

where R is sum of residue.

Now, Residue at $z = +ai$

$$R_1(z = +ai) = \lim_{z \to ai} (z - ai) \left\{ \frac{z^2}{(z-ai)(z+ai)(z^2+b^2)} \right\}$$

$$= \frac{(ai)^2}{(ai+ai)((ai)^2+b^2)}$$

$$= \frac{-a^2}{2ai(-a^2+b^2)} \Rightarrow \frac{a}{2i(a^2-b^2)}$$

Residue at $z = +bi$

$$R_2(z = +bi) = \lim_{z \to bi} (z - bi) \frac{z^2}{(z+bi)(z-bi)(z^2+a^2)}$$

$$= \frac{(bi)^2}{(bi+bi)((bi)^2+a^2)} = \frac{-b}{2i(a^2-b^2)}$$

$R = R_1 + R_2$

$$= \frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}$$

$$= \frac{(a-b)}{2i(a-b)(a+b)}$$

$$= \frac{1}{2i(a+b)} \quad \text{--- (3)}$$

For Curve $C_1$, we have.

$$\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z \cdot z^2}{(z^2+a^2)(z^2+b^2)}$$

$$= \lim_{z\to\infty} \frac{z^3}{z^4\left(1+\frac{a^2}{z^2}\right)\left(1+\frac{b^2}{z^2}\right)}$$

$$= \frac{1}{\infty} = 0$$

So, $\int_{C_1} f(z)\, dz = 0$ --- (4)

Putting (2) (3) (4) in eq. (1), we get

$$2\pi i R = 0 + \int_{-\infty}^{\infty} f(x)\,dx.$$

$$2\pi i\left[\frac{1}{2i(a+b)}\right] = \int_{-\infty}^{\infty} f(x)\,dx.$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{(a+b)}$$

So, $$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{(a+b)}.$$

Q. Evaluat $\int_{0}^{\infty} \frac{dx}{x^4+1}$.

Sol $\rightarrow$ Let $f(z) = \frac{1}{z^4+1}$

for pole, $z^4+1=0$

$$\Rightarrow z^4 = -1$$

$$\Rightarrow z^4 = e^{i\pi}$$

$$\Rightarrow z^4 = e^{i(2n\pi+\pi)}$$

$$\Rightarrow z = e^{i\left(\frac{2n\pi+\pi}{4}\right)}, \quad n = 0, 1, 2$$

When, $n=0$, $z = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$n=1, \quad z = e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$n=2, \quad z = e^{i\frac{5\pi}{4}} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

$= \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$n=3, \quad z = e^{i\frac{7\pi}{4}} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$.

$z = e^{i\frac{\pi}{4}}$ and $e^{i\frac{3\pi}{4}}$ lie above the half plase $C_1$.

Now, $\oint_C f(z)dz = \int_{C_1} f(z)dz + \int_{-\infty}^{\infty} f(x)dx$ --- ①

For Curve C.

$\oint_C f(z)dz = 2\pi i R$. --- ②

R → sum of Residue.

Residue at $z = e^{i\frac{\pi}{4}}$.

$R_1(z = e^{i\frac{\pi}{4}}) = \lim_{z\to e^{i\frac{\pi}{4}}} (z - e^{i\pi/4}) f(z)$

$= \lim_{z\to e^{i\pi/4}} \frac{(z - e^{i\pi/4})}{z^4+1}$

$= \lim_{z\to e^{i\pi/4}} \frac{1}{4z^3}$ [By L-Hospital rule]

$= \frac{1}{4e^{i3\pi/4}} = \frac{1}{4}e^{-i\frac{3\pi}{4}}$

Residue at $z = e^{i3\pi/4}$.

$$R_2\left(z = e^{i3\pi/4}\right) = \lim_{z \to e^{i3\pi/4}} \left(z - e^{i3\pi/4}\right) f(z)$$

$$= \lim_{z \to e^{i3\pi/4}} \frac{\left(z - e^{i\frac{3\pi}{4}}\right)}{z^4 + 1}$$

$$= \lim_{z \to e^{i\frac{3\pi}{4}}} \frac{1}{4z^3} \quad \text{[By L-Hospital rule]}$$

$$= \frac{1}{4e^{i\frac{9\pi}{4}}}$$

$$= \frac{1}{4} e^{-i\frac{9\pi}{4}}$$

$$R = R_1 + R_2$$

$$= \frac{1}{4} e^{-i3\pi/4} + \frac{1}{4} e^{-i9\pi/4}$$

$$= \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right]$$

$$= \frac{1}{4}\left[\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{4} \times \left(\frac{-2i}{\sqrt{2}}\right)$$

$$= \frac{-2i}{4\sqrt{2}}$$

$$R = \frac{-2i}{4\sqrt{2}} \quad \text{--- (3)}$$

For Curve, $C_1$

$$\lim_{z\to\infty} z\, f(z) = \lim_{z\to\infty} z \cdot \frac{1}{z^4+1}$$

$$= \lim_{z\to\infty} \frac{z}{z\left(z^3+\frac{1}{z}\right)}.$$

$$= \frac{1}{\infty} = 0.$$

Hence, $\int_{C_1} f(z)\,dz = 0$ ——— (4)

Putting (2), (3), (4) in (1)

$$2\pi i\left(\frac{-2i}{4\sqrt{2}}\right). = 0 + \int_{-\infty}^{\infty} f(x)\,dx.$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{-\pi i^2}{\sqrt{2}}$$

$$\Rightarrow 2\int_{0}^{\infty} f(x)\,dx = \frac{\pi}{\sqrt{2}}$$

$$\int_{0}^{\infty} f(x)\,dx = \frac{\pi}{2\sqrt{2}}$$

So, $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$ Ans