Unit-3

Laplace Transform :-

Let $f(t)$ be the function of t, then laplace ~~theorem~~ transform of $f(t)$ is denoted and defined by.

$$L(f(t)) = f(p) = \int_0^{\infty} e^{-pt} f(t)\, dt \quad ; p \geq 0.$$

where p is parameter.

Q Find $L(1)$.

Sol → We know $L(f(t)) = f(p) = \int_0^{\infty} e^{-pt} f(t) dt$

On putting $f(t) = 1$

$$L(1) = \int_0^{\infty} e^{-pt} dt.$$

$$= \left| \frac{e^{-pt}}{-p} \right|_0^{\infty}$$

$$= \frac{e^{-\infty} - e^0}{-p} = \frac{1}{p}.$$

Some formulae :-

① $L(1) = \frac{1}{p}$

② $L(e^{at}) = \frac{1}{p-a}$

③ $L(e^{-at}) = \frac{1}{p+a}$

④ $L(t^n) = \frac{n!}{p^{n+1}} = \frac{\Gamma(n+1)}{p^{n+1}}$

(5) $L(\sin at) = \dfrac{a}{p^2+a^2}$

(6) $L(\sinh at) = \dfrac{a}{p^2-a^2}$

(7) $L(\cos at) = \dfrac{p}{p^2+a^2}$

(8) $L(\cosh at) = \dfrac{p}{p^2-a^2}$

$$\cos 2t = 1 - 2\sin^2 t$$
$$= 2\cos^2 t - 1$$
$$= \cos^2 t - \sin^2 t$$

$$\cosh 2t = 1 + 2\sinh^2 t$$
$$= 2\cosh^2 t - 1$$
$$= \cosh^2 t + \sinh^2 t$$

Linear Property :-

If $L(f(t)) = f(p)$ & $L(g(t)) = g(p)$, Then,

$L[a f(t) + b g(t)] = a L(f(p)) + b L(g(p))$

Q. Find $L(3 + e^{-2t} + \sin 2t + \cosh 3t)$

Sol→ We have $L(3 + e^{-2t} + \sin 2t + \cosh 3t)$.

By linear property,

$3L(1) + L(e^{-2t}) + 3L(\sin 2t) + L(\cosh 3t)$.

$= 3 \cdot \dfrac{1}{p} + \dfrac{1}{p+2} + 3 \cdot \dfrac{2}{p^2+4} + \dfrac{p}{p^2-9}$ Ans

Q Find $L(t^3 + \cos^2 t + \sinh^2 t)$

Sol→ We have $L(t^3 + \cos^2 t + \sinh^2 t)$

By linear property.

$L(t^3) + L(\cos^2 t) + L(\sinh^2 t)$

$= \frac{3!}{p^4} + L\left(\frac{1+\cos 2t}{2}\right) + L\left(\frac{\cosh 2t - 1}{2}\right)$

$= \frac{3!}{p^4} + \cancel{\frac{1}{2}L(1)} + \frac{1}{2}L(\cos 2t) + \frac{1}{2}L(\cosh 2t) - \cancel{\frac{1}{2}L(1)}$.

$= \frac{3!}{p^4} + \frac{1}{2}\cdot\frac{p}{p^2+4} + \frac{1}{2}\frac{p}{p^2-4}$. Ans

First Shifting theorem :-

If $L(f(t)) = f(p)$, then $L[e^{at} f(t)] = f(p-a)$.

Q. Find $L(e^{2t}\sin t)$

Sol→ Here $f(t) = \sin t$

$L(f(t)) = L(\sin t)$

$L(f(t)) = \frac{1}{p^2+1} \Rightarrow L(p) = \frac{1}{p^2+1}$ — ①

By 1st shifting theorem,

$L(e^{at} f(t)) = f(p-a)$

$\Rightarrow L(e^{2t}\sin t) = f(p-2)$

$= \frac{1}{(p-2)^2+1}$ [using ①]

Q Find $L(e^t t^2)$.

Sol→ Here $f(t) = t^2$

$L(f(t)) = L(t^2) = \frac{2!}{p^3}$.

$\Rightarrow L(p) = \frac{2}{p^3}$ ---- (1)

By 1st Shifting theorem.

$L[e^{at} f(t)] = f(p-a)$.

$\Rightarrow L(e^t t^2) = f(p-1)$.

$= \frac{2}{(p-1)^3}$ [using (1)] Ans

Additional Q. for practice:-

(1) Q $L(e^{2t} \cos t)$

(2) Q $L(e^{2t} \cos^2 t)$.

(1) $L(e^{2t} \cos t)$

$\Rightarrow$ Here, $f(t) = \cos t$, $L(f(t)) = L(\cos t)$

$= \frac{p}{p^2+1} = L(p)$.

By 1st Shifting theorem,

$L(e^{at} f(t)) = f(p-a)$

$\Rightarrow L(e^{2t} \cos t) = f(p-2)$

$= \frac{p-2}{(p-2)^2+1}$ Ans

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(2) $L(e^{2t}\cos^2 t)$.

= Here $f(t) = \cos^2 t$

$L(f(t)) = L(\cos^2 t) = L\left(\frac{1+\cos t}{2}\right)$

$= \frac{1}{2}L(1) + \frac{1}{2}L(\cos t)$

$= \frac{1}{2p} + \frac{p}{2(p^2+1)} = f(p)$

By 1st Shifting theorem.

$L(e^{2t}\cos^2 t) = f(p-2)$

$= \frac{1}{2(p-2)} + \frac{(p-2)}{2[(p-2)^2+1]}$ Ans

2nd Shifting Property :-

If $L(f(t)) = f(p)$ and $\phi(t) = \begin{cases} f(t-a) & ; t>0 \\ 0 & ; t<0 \end{cases}$

Then,

$L(\phi(t)) = e^{-ap} f(p)$.

Q. Find $L(\phi(t))$ where $\phi(t) = \begin{cases} \cos\left(t-\frac{2\pi}{3}\right) & t>\frac{2\pi}{3} \\ 0 & t<\frac{2\pi}{3} \end{cases}$

Sol → We have $\phi(t) = \begin{cases} \cos\left(t-\frac{2\pi}{3}\right) & t>\frac{2\pi}{3} \\ 0 & t<\frac{2\pi}{3} \end{cases}$

--- ①

By 2^{nd} Shifting property,

If $L(f(t)) = f(p)$ and $\phi(t) = \begin{cases} f(t-a) & t>0 \\ 0 & t<0 \end{cases}$ ---- ②

Then, $L[\phi(t)] = e^{-ap} f(p)$ ------ ③

On Comparing ① and ②, we get,

$a = \frac{2\pi}{3}$ and $f(t) = \cos t$.

$L(f(t)) = L(\cos t) = \frac{p}{p^2+1} = f(p)$.

Now, putting the values of a and f(p) in eq. ③.

$L(\phi(t)) = e^{-2\pi/3 p} f(p)$.

$= e^{-\frac{2\pi}{3}p} \cdot \frac{p}{p^2+1}$ Ans

Q Find $L(\phi(t))$ when

$\phi(t) = \begin{cases} (t-2)^3 & ; t>2 \\ 0 & ; t<2 \end{cases}$

Sol → We have, $\phi(t) = \begin{cases} (t-2)^3 & ; t>2 \\ 0 & ; t<0 \end{cases}$ -- ①

from 2^{nd} Shifting property.

If $\phi(t) = \begin{cases} f(t-a) & t>0 \\ 0 & t<0 \end{cases}$ -- ②

$L[\phi(t)] = e^{-ap} f(p)$ ---- ③

On Comparing eq. 1 and 2.

$a = 2, \quad f(t) = t^3.$

$$L(f(t)) = L(t^3) = \frac{3!}{p^4} = f(p).$$

Putting value of $f(p)$ and a in eq. 3.

$$L[\phi(t)] = e^{-2p} f(p).$$

$$= e^{-2p} \cdot \frac{3!}{p^4} \quad \underline{\underline{\text{Ans}}}$$

<u>Q</u> find $L(\phi t)$ where $\phi(t) = \begin{cases} \sin^2(t - \pi/2) & t > \pi/2 \\ 0 & t < \pi/2. \end{cases}$

Sol → We have. $\phi(t) = \begin{cases} \sin^2(t - \pi/2) & ; t > \pi/2 \\ 0 & ; t < \pi/2 \end{cases}$ --- 1

from 2nd shifting property,

$$\phi(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases} \text{ --- } 2$$

$$L(\phi(t)) = e^{-ap} f(p) \text{ ——— } 3$$

On comparing eq. 1 and 2.

$a = \frac{\pi}{2}, \quad f(t) = \sin^2 t.$

$$L(f(t)) = L(\sin^2 t) = L\left(\frac{1 - \cos 2t}{2}\right)$$

$$= \frac{1}{2} L\left(\frac{1}{2}\right) - \frac{1}{2} L(\cos 2t).$$

$$= \frac{1}{2}\left[\frac{1}{p} - \frac{1}{p^2+4}\right] = f(p)$$

Now, putting value of a and $f(p)$ in eq. 3.

$$L[\phi(t)] = e^{-\pi/2 p} f(p)$$

$$= e^{-\frac{\pi}{2}p} \cdot \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right] \text{ Ans}$$

Laplace Transform of Integral

If $L(f(t)) = F(p)$, then, $L\left(\int_0^t f(u)\,du\right) = \frac{f(p)}{p}$.

* We have to get $f(t)$ from $f(u)$.

Q. Find $L\left(\int_0^t x^2\,dx\right)$

Sol→ Here, $f(t) = t^2$

$$L(f(t)) = L(t^2) = \frac{2!}{p^3} = f(p). \quad \text{--- (1)}$$

From, Laplace transform of integral,

$$L\left(\int_0^t x^2\,dx\right) = \frac{L(p)}{p}$$

$$= \frac{2!}{p^3 \cdot p} \quad [\text{from (1)}]$$

$$= \frac{2}{p^4} \text{ Ans}$$

Q Find $L\left(\int_0^t \cos u\,du\right)$.

Sol→ $f(t) = \cos t$

$$L[f(t)] = L(\cos t) = \frac{p}{p^2+1} = f(p) \quad \text{--- (1)}$$

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By laplace transform of Integral,

$$L\left(\int_0^t \cos u \, du\right) = \frac{f(p)}{p}.$$

$$= \frac{p}{(p^2+1)\cdot p} \quad [\text{using (i)}]$$

$$= \frac{1}{(p^2+1)} \quad \underline{\underline{\text{Ans}}}$$

Q $L\left(\int_0^t \cos^2 u \, du\right)$

Sol → Here, $f(t) = \cos^2 t$.

$$L[f(t)] = = f(\cos^2 t) = L\left(\frac{1+\cos 2t}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{p} + \frac{p}{p^2+4}\right] = f(p) \quad \text{--- (1)}$$

By laplace transform of Integral.

$$L\left(\int_0^t \cos^2 u \, du\right) = \frac{f(p)}{p}$$

$$= \frac{1}{2p}\left[\frac{1}{p} + \frac{p}{p^2+4}\right] \quad [\text{using (1)}]$$

$$= \frac{1}{2}\left[\frac{1}{p^2} + \frac{1}{p^2+4}\right] \quad \underline{\underline{\text{Ans}}}$$

Multiple by t^n :-

If $L(f(t)) = f(p)$ Then,

$$L(t^n f(t)) = (-1)^n \frac{d^n}{dp^n}[f(p)] \;.\; n \geq 1$$

Q find $L(t\sin t)$

Sol→ Here $f(t) = \sin t$.

$L(f(t)) = L(\sin t) = \dfrac{1}{p^2+1} = f(p)$ --- ①

By ~~Mult~~ Multiple theorem :-

$L(t\sin t) = (-1)^1 \dfrac{d}{dp}[f(p)]$

$= -\dfrac{d}{dp}\left(\dfrac{1}{p^2+1}\right)$ [from ①]

$= -\left[\dfrac{-2p}{(p^2+1)^2}\right]$

$= \dfrac{2p}{(p^2+1)^2}$ Ans

Q find $L(t\cos t)$.

Sol→ Here, $f(t) = \cos t$

$L[f(t)] = L(\cos t) = \dfrac{p}{p^2+1} = f(p)$ --- ①

From Multiple theorem.

$L(t\cos t) = (-1)^1 \dfrac{d}{dp}[f(p)]$

$= (-1)\dfrac{d}{dp}\left(\dfrac{p}{p^2+1}\right)$.

$= (-1)\dfrac{(p^2+1)(1) - p(2p)}{(p^2+1)^2} = -\left(\dfrac{-p^2+1}{(p^2+1)^2}\right)$

$= \dfrac{p^2-1}{(p^2+1)^2}$ Ans

Division by t property :-

If $L(f(t)) = f(p)$ then $L\left(\frac{f(t)}{t}\right) = \int_p^\infty f(x)\,dx$.
where $f(x) = f(p)$, Provided $\lim_{t \to 0} \frac{f(t)}{t}$ exists.

Q Find $L\left(\frac{\sin t}{t}\right)$.

Sol → We have $\lim_{t \to 0} \frac{f(t)}{t} = \lim_{t \to 0} \frac{\sin t}{t} = 1$

⇒ limit exist
So, Laplace transform will exist

$f(t) = \sin t$

$L(f(t)) = L(\sin t) = \frac{1}{p^2+1} = f(p)$.

By division by t property,

$$L\left(\frac{\sin t}{t}\right) = \int_p^\infty \frac{1}{x^2+1}\,dx.$$

$$= \left|\tan^{-1} x\right|_p^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} p.$$

$$= \frac{\pi}{2} - \tan^{-1} p$$

$$= \cot^{-1} p$$

Ans

* Not → Laplace transform exist when there is 'p' term in it.
If $L(f(t)) = \infty$ or $\pi/2$ ⇒ Do not exist

Q Find $L\left(\frac{\cos t}{t}\right)$.

Sol → We have $\lim_{t\to 0} \frac{\cos t}{t} = \frac{\cos 0}{0} = \frac{1}{0} = \infty$.

$\Rightarrow$ limit does not exist.

So, laplace transform does not exist.

$f(t) = \cos t$

$L(f(t)) = L(\cos t) = \frac{p}{p^2+1} = f(p)$.

By division by t property.

$$L\left(\frac{\cos t}{t}\right) = \int_p^\infty \frac{x}{x^2+1}\, dx$$

$$= \frac{1}{2}\int_p^\infty \frac{2x}{x^2+1}\, dx$$

$$= \frac{1}{2}\left[\log(x^2+1)\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\infty - \log(p^2+1)\right]$$

$$= \infty - \frac{1}{2}\log(p^2+1)$$

$$= \infty$$

So, laplace transform does not exist.

Q find $L\left(\frac{e^{at}}{t}\right)$

Sol → We have $\lim_{t\to 0} \frac{e^{at}}{t} = \frac{e^0}{0} = \frac{1}{0} = \infty$

$\Rightarrow$ Limit does not exist.

So, laplace transform will not exist.

$f(t) = e^{at}$.

$L(f(t)) = L(e^{at}) = \frac{1}{p-a} = f(p)$

By division by t property.

$$L\left(\frac{e^{at}}{t}\right) = \int_p^\infty f(x)\,dx$$

$$= \int_p^\infty \frac{1}{x-a}\,dx$$

$$= \left[\log(x-a)\right]_p^\infty$$

$$= \log(\infty - a) - \log(p-a)$$

$$= \infty - \log(p-a)$$

$$= \infty$$

So, laplace transform does not exist.

Q Find $L\left(\frac{e^{at}-e^{bt}}{t}\right)$

Sol→ We have $\lim_{t\to 0} \frac{e^{at}-e^{bt}}{t}$

$$= \frac{e^0 - e^0}{0} = \frac{1-1}{0} = \frac{0}{0}.$$

By L' Hospital Rule

$$\lim_{t\to 0}\left(\frac{e^{at}-e^{bt}}{t}\right) = \lim_{t\to 0} \frac{ae^{at} - be^{bt}}{1}$$

$$= a - b$$

⇒ Limit exists.

So, laplace transform will exist.

$f(t) = e^{at} - e^{bt}$

$L(f(t)) = L(e^{at} - e^{bt}) = \frac{1}{p-a} - \frac{1}{p-b} = f(p)$

By division by t property.

$L\left(\frac{e^{at} - e^{bt}}{t}\right) = \int_p^\infty f(x)\,dx$

$= \int_p^\infty \left(\frac{1}{x-a} - \frac{1}{x-b}\right) dx$

$= \left| \log(x-a) - \log(x-b) \right|_p^\infty$

$= \log \left| \frac{x-a}{x-b} \right|_p^\infty$

$= \left| \log \frac{x(1-\frac{a}{x})}{x(1-\frac{b}{x})} \right|_p^\infty$

$= \log\left(\frac{x(1-\frac{a}{\infty})}{x(1-\frac{b}{\infty})}\right) - \log\left(\frac{\cancel{x}}{\cancel{x}}\left(\frac{p-a}{p-b}\right)\right)$

$= \log 1 - \log\left(\frac{p-a}{p-b}\right)$

$= 0 - \log\left(\frac{p-a}{p-b}\right)$

$= \log\left(\frac{p-b}{p-a}\right)$. Ans

Initial Value Theorem :-

If $L(f(t)) = f(p)$, then $\lim_{t \to 0} f(t) = \lim_{p \to \infty} p[f(p)]$.

Q Verify initial value Theorem for the function.
$3 - 2\cos 2t$.

Sol → let $f(t) = 3 - 2\cos 2t$

$$L(f(t)) = L(3) - 2L(\cos t).$$

$$= 3 \cdot \frac{1}{p} - 2\frac{p}{p^2+1} = f(p) \quad \text{---} (1)$$

Now, $\lim_{t \to 0} f(t) = \lim_{t \to 0} (3 - 2\cos t)$

$$= 3 - 2\cos 0$$

$$= 3 - 2$$

$$= 1 \quad \text{-------} (2)$$

Now, $\lim_{p \to \infty} p \cdot f(p) = \lim_{p \to \infty} p \cdot \left[\frac{3}{p} - \frac{2p}{p^2+1}\right]$

$$= \lim_{p \to \infty} \left[3 - \frac{2p^2}{p^2+1}\right]$$

$$= \lim_{p \to \infty} \left[3 - \frac{2p^2}{p^2\left[1 + \frac{1}{p^2}\right]}\right]$$

$$= 3 - \frac{2}{\left(1 + \frac{1}{\infty}\right)}$$

$$= 3 - 2$$

$$= 1 \quad \text{----} (3)$$

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from, equation ② and ③, we get

$$\lim_{t \to 0} f(t) = \lim_{p \to \infty} p \cdot f(p).$$

Hence initial value theorem is verified

Final Value Theorem

If $L(f(t)) = f(p)$, Then,

$$\lim_{t \to \infty} f(t) = \lim_{p \to 0} p f(p).$$

Q. Verify final value theorem for the function $1 + e^{-t} \sin t$.

Sol→ Let $f(t) = 1 + e^{-t} \sin t$

$$L(f(t)) = L(1 + e^{-t} \sin t)$$

$$= L(1) + L(e^{-t} \sin t) \quad \text{--- ①}$$

Let $\phi(t) = \sin t$

$$L(\phi(t)) = \frac{1}{p^2+1} = \phi(p) \quad \text{--- ②}$$

By first shifting :-

$$L(e^{-t} \sin t) = \phi(p+1)$$

$$= \frac{1}{(p+1)^2+1} \quad \text{-- ③}$$

Using ③ in ①

$$L(f(t)) = L(1) + L(e^{-t} \sin t)$$

$$= \frac{1}{p} + \frac{1}{(p+1)^2+1} = f(p).$$

Now, $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 1+e^{-t}\sin t$

$= 1+0$

$= 1$ ---- (4)

Now, $\lim_{p\to 0} p \cdot f(p)$

$$= \lim_{p\to 0} p \cdot \left[\frac{1}{p} + \frac{1}{(p+1)^2+1}\right]$$

$$= \lim_{p\to 0} \left[1 + \frac{p}{(p+1)^2+1}\right]$$

$$= 1 + \frac{0}{(0+1)^2+1} = 1+0$$

$= 1$ ---- (5)

By using (4) & (5), we see that

$\lim_{t\to\infty} f(t) = \lim_{p\to 0} p \cdot f(p)$.

So, final value theorem is verified

Home-Work

Q Verify final value theorem for function

$2+e^{-t}(\sin t + \cos t)$.

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Sol → Let $f(t) = 2 + e^{-t}(\sin t + \cos t)$

$$L(f(t)) = L[2 + e^{-t}(\sin t + \cos t)]$$

$$= L(f(t)) = L(2) + L(e^{-t}\sin t) + L(e^{-t}\cos t). \quad --- (1)$$

Let $\phi_1(t) = \sin t$

$$L(\phi_1(t)) = \frac{1}{p^2+1} = \phi_1(p) \quad --- (2)$$

By first shifting.

$$L(e^{-t}\sin t) = \phi_1(p+1)$$

$$= \frac{1}{(p+1)^2+1} \quad --- (3)$$

Let $\phi_2(t) = \cos t$

$$L(\phi_2(t)) = L(\cos t) = \frac{p}{p^2+1} \quad --- (4)$$

By first shifting.

$$L(e^{-t}\cos t) = \phi_2(p+1)$$

$$= \frac{(p+1)}{(p+1)^2+1} \quad --- (5)$$

using (3) and (5) in eq. (1).

$$L(f(t)) = 2L(t) + L(e^{-t}\sin t) + L(e^{-t}\cos t)$$

$$= \frac{2}{p} + \frac{1}{(p+1)^2+1} + \frac{(p+1)}{(p+1)^2+1} = f(p)$$

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Now, $\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \left[2 + e^{-t}(\sin t + \cos t)\right]$

$= 2 + 0$

$= 2$ ---------- (6)

Now, $\lim_{p \to 0} p \cdot f(p) = \lim_{p \to 0} p \cdot \left[\frac{2}{p} + \frac{1}{(p+1)^2+1} + \frac{(p+1)}{(p+1)^2+1}\right]$

$= \lim_{p \to 0} \left[2 + \frac{p}{(p+1)^2+1} + \frac{p \cdot (p+1)}{(p+1)^2+1}\right]$

$= 2 + \frac{0}{(0+1)^2+1} + \frac{0(0+1)}{(0+1)^2+1}$

$= 2 + 0$

$= 2$ ---------- (7)

Thus from (6) and (7) we see that

$\lim_{t \to \infty} f(t) = \lim_{p \to 0} p \cdot f(p)$

So, final value theorem is verified.

Change of Scale property :-

If $L(f(t)) = f(p)$ Then $L(f(at)) = \frac{1}{a} f\left(\frac{p}{a}\right)$.

Q. Find the laplace of $L(\sin 2t)$ by using change of scale property.

Sol → Let $f(t) = \sin t$

$$L(f(t)) = L(\sin t) = \frac{1}{p^2+1} = f(p) \quad \text{---} (1)$$

By change of scale property.

$$L(\sin 2t) = \frac{1}{2} f\left(\frac{p}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{\left(\frac{p}{2}\right)^2+1}\right)$$

$$= \frac{1}{2} \times \frac{1}{\frac{p^2+4}{4}}$$

$$= \frac{2}{p^2+4}. \quad \text{Ans}$$

Q. If $L(f(t)) = \frac{p^2+1}{p^2+2p}$. find $L(f(2t))$.

Sol → Given $L(f(t)) = \frac{p^2+1}{p^2+2p} = f(p)$

By change of scale property.

$$L(f(2t)) = \frac{1}{2} f(p/2)$$

$$= \frac{1}{2} \frac{\left[\left(\frac{p}{2}\right)^2+1\right]}{\left[\left(\frac{p}{2}\right)^2+2\left[\frac{p}{2}\right]\right]}$$

$$= \frac{1}{2} \cdot \frac{p^2+4}{4} \times \frac{4}{p^2+4p}.$$

$$= \frac{1}{2}\frac{p^2+4}{p^2+4p} \quad \underline{\underline{Ans}}$$

Some Important problem

Q Evaluate $\int_0^\infty e^{-t} t \sin t \, dt$

Sol → Let $I = \int_0^\infty e^{-t} t \sin t \, dt$ ------ (1)

$$f(t) = t \sin t$$

$$L(f(t)) = L(t \sin t)$$

$$= (-1)^1 \frac{d}{dp}\left(\frac{1}{p^2+1}\right)$$

$$= -\left[\frac{-2p}{(p^2+1)^2}\right]$$

$$= \frac{2p}{(p^2+1)^2}$$

$$\int_0^\infty e^{-pt} f(t) \, dt = \frac{2p}{(p^2+1)^2}$$

$$\Rightarrow \int_0^\infty e^{-pt} t \sin t \, dt = \frac{2p}{(p^2+1)^2}.$$

put $p = 1$

$$\int_0^\infty e^{-t} t \sin t = \frac{2 \times 1}{(1+1)^2} = \frac{1}{2} \quad \underline{Ans}$$

Q. Evaluate $\int_0^\infty t \sin t \, dt$

Sol → Let $I = \int_0^\infty t \sin t \, dt$ ------ (1)

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$f(t) = t\sin t$

$L(f(t)) = L(t\sin t)$

$= (-1)^1 \frac{d}{dp}\left(\frac{1}{p^2+1}\right) = \frac{2p}{(p^2+1)^2}$.

$\int_0^\infty e^{pt} f(t)\, dt = \frac{2p}{(p^2+1)^2}$

$\Rightarrow \int_0^\infty e^{pt}\, t\sin t\, dt = \frac{2p}{(p^2+1)^2}$

put $p = 0$

$\Rightarrow \int_0^\infty t\sin t\, dt = \frac{2\times 0}{(0+1)^2} = 0$ Ans

Q Evaluat $\int_0^\infty e^{-t} \frac{\sin\sqrt{3}t}{t}\, dt$.

Sol → Let $f(t) = \frac{\sin\sqrt{3}t}{t}$.

$L[f(t)] = L\left(\frac{\sin\sqrt{3}t}{t}\right)$

$= \int_p^\infty \frac{\sqrt{3}}{x^2+(\sqrt{3})^2}\, dx$.

$= \sqrt{3}\left[\frac{1}{\sqrt{3}} \tan^{-1}\frac{x}{\sqrt{3}}\right]_p^\infty$

$= \tan^{-1}\infty - \tan^{-1}\frac{p}{\sqrt{3}}$

$= \frac{\pi}{2} - \tan^{-1}\frac{p}{\sqrt{3}}$

$= \cot^{-1} p/\sqrt{3}$

$$\int_0^\infty e^{-pt} f(t)\, dt = \cot^{-1} \frac{p}{\sqrt{3}}$$

$$\int_0^\infty e^{-pt} \frac{\sin\sqrt{3}\,t}{t}\, dt = \cot^{-1} \frac{p}{\sqrt{3}}$$

putting $p = 1$.

$$\int_0^\infty e^{-t} \frac{\sin\sqrt{3}\,t}{t} = \cot^{-1} \frac{1}{\sqrt{3}}$$

$$= \pi/3 \quad \underline{\underline{\text{Ans}}}$$

Q. Evaluat $\int_0^\infty \frac{\sin t}{t}\, dt$

Sol → Let $f(t) = \frac{\sin t}{t}$

$$L(f(t)) = L\left(\frac{\sin t}{t}\right) = \int_p^\infty \frac{1}{x^2+1}\, dx$$

$$= \left|\tan^{-1} x\right|_p^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} p$$

$$= \pi/2 - \tan^{-1} p$$

$$= \cot^{-1} p.$$

$$\int_0^\infty e^{-pt} \frac{\sin t}{t}\, dt = \cot^{-1} p$$

put $p = 0$

$$\Rightarrow \int_0^\infty \frac{\sin t}{t}\, dt = \cot^{-1} 0$$

$$= \pi/2 \quad \underline{\text{Ans}}$$

Laplace Transform of periodic function :-

If $f(t)$ has period $T>0$ Then

$$L(f(t)) = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\, dt$$

Q Find the laplace Transform of periodic function of period 2π given by.

$$f(t) = \begin{cases} t & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$$

Sol $\rightarrow$ Given $f(t) = \begin{cases} t & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$

We have

$$L(f(t)) = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi p}} \int_0^{2\pi} e^{-pt} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi p}} \left[\int_0^{\pi} e^{-pt}\, t\, dt + \int_{\pi}^{2\pi} e^{-pt} \times 0\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi p}} \left[\int_0^{\pi} e^{-pt}\, t\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi p}}\left[(t)\frac{e^{-pt}}{-p} - \frac{(-1)e^{-pt}}{p^2}\right]_0^{\pi}$$

$$= \frac{1}{(1-e^{-2\pi p})}\left[\frac{\pi e^{-\pi p}}{-p} + \frac{e^{-\pi p}}{-p} - \left(0 + \frac{1\times e^0}{p^2}\right)\right]$$

$$= \frac{1}{(1-e^{-2\pi p})}\left[\frac{p\pi e^{-\pi p} + e^{-\pi p} - 1}{(-1)\ p^2}\right]$$

$$= \frac{1 - e^{-\pi p} - \pi p e^{-\pi p}}{(1-e^{-2\pi p})\ p^2}$$ Ans

Home Work

Q Find the laplace transform of periodic function

$$f(t) = \begin{cases} 1 & 0<t<1 \\ -1 & 1<t<2 \end{cases}$$

Sol→ Given $f(t) = \begin{cases} 1 & 0<t<1 \\ -1 & 1<t<2. \end{cases}$

We have

$$L(f(t)) = \frac{1}{1-e^{-pT}}\int_0^T e^{-pt} f(t)\,dt$$

$$= \frac{1}{1-e^{-2p}}\left[\int_0^1 e^{-pt}(1)\,dt + \int_1^2 e^{-pt}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2p}}\left[\left|\frac{e^{-pt}}{-p}\right|_0^1 + \left|\frac{e^{-pt}}{(-p)}\right|_1^2\right]$$

$$= \frac{1}{1-e^{-2p}}\left[\left(\frac{e^{-p}}{-p} - \frac{e^{0}}{-p}\right) - \left\{\left(\frac{e^{-2p}}{-p}\right) - \left(\frac{e^{-p}}{(-p)}\right)\right\}\right]$$

$$= \frac{1}{1-e^{-2p}}\left[\frac{e^{-p} - 1 - e^{-2p} + e^{-p}}{-p}\right]$$

$$= \frac{1}{(1-e^{-2p})}\left(\frac{1 - 2e^{-p} + e^{-2p}}{p}\right]$$

$$= \frac{(1-e^{-p})^2}{(1-e^{-2p})\,p}$$

$$= \frac{(1-e^{-p})(1-e^{-p})}{(1-e^{-p})(1+e^{-p})} \times \frac{1}{p}$$

$$= \frac{(1-e^{-p})}{(1+e^{-p})} \times \frac{1}{p}$$

The Inverse Laplace Transform

If $L(f(t)) = f(p)$, then $F(t)$ is called inverse laplace transform of $f(p)$.

i.e. $L(f(t)) = f(p)$.

$$L^{-1}(f(p)) = f(t)$$

The symbol L^{-1} is called Inverse Laplace operator.

Linear property

If $L^{-1}(f(p)) = f(t)$ Then. $L^{-1}\left[af(p) + b\,g(p)\right]$

$$= aL^{-1}(f(p)) + bL^{-1}(g(p))$$

Q find the inverse Laplace $L^{-1}\left[\frac{2}{2p-4} - \frac{3p}{p^2-9} + \frac{1}{p^2}\right]$

Sol → We have.

$$L^{-1}\left[\frac{2}{2p-4} - \frac{3p}{p^2-9} + \frac{1}{p^2}\right]$$

By Inverse Laplace linear property.

$$= L^{-1}\left(\frac{2}{2p-4}\right) - 3L^{-1}\left(\frac{p}{p^2+9}\right) + L^{-1}\left(\frac{1}{p^2}\right)$$

$$= L^{-1}\left(\frac{1}{(p-2)}\right) - 3L^{-1}\frac{p}{p^2+9} + L\left(\frac{1}{p^2}\right)$$

$$= e^{2t} - 3L^{-1}\cos t + \frac{t^1}{1!}$$

$$= e^{2t} - 3\cos t + t \quad \text{Ans}$$

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Q $L^{-1}\left(\frac{(p-1)^2}{p^4}\right)$.

Sol → $L^{-1}\left(\frac{p^2+2p+1}{p^4}\right)$

$= L^{-1}\left(\frac{1}{p^2}\right) + L^{-1}\left(\frac{1}{p^4}\right) - 2L^{-1}\left[\frac{1}{p^3}\right]$

$= \frac{t}{1!} + \frac{t^3}{3!} - \frac{t^2}{2!} \times 2$

$= t + \frac{t^3}{6} - t^2$ Ans

First Shifting property :-

If $L^{-1}(f(p)) = f(t)$ then $L^{-1}(f(p-a)) = e^{at} f(t)$.

$= e^{at} L^{-1}(f(p))$

Q Evaluat $L^{-1}\left[\frac{1}{p^2+2p+3}\right]$

Sol- $L^{-1}\left[\frac{1}{p^2+2p+3}\right]$

$= L^{-1}\left[\frac{1}{(p+1)^2+2}\right]$

By first Shifting

$L^{-1}\left[\frac{1}{(p+1)^2+2}\right] = e^{-t} L^{-1}\left[\frac{1}{p^2+(\sqrt{2})^2}\right]$

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$$= e^{-t} L^{-1}\left[\frac{\sqrt{2}}{p^2+(\sqrt{2})^2} \times \frac{1}{\sqrt{2}}\right]$$

$$= e^{-t} \times \frac{1}{\sqrt{2}} L^{-1}\left[\frac{\sqrt{2}}{p^2+(\sqrt{2})^2}\right]$$

$$= \frac{e^{-t} \sin\sqrt{2}t}{\sqrt{2}}$$ Ans

Q find $L^{-1}\left[\frac{p}{p^2+p+1}\right]$

Sol→ $L^{-1}\left[\frac{p}{p^2+p+1}\right] = L^{-1}\left[\frac{p}{\left(p+\frac{1}{2}\right)^2+1-\frac{1}{4}}\right]$

$$= L^{-1}\left[\frac{p}{\left(p+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$= L^{-1}\left[\frac{\left(p+\frac{1}{2}\right)-\frac{1}{2}}{\left(p+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$= L^{-1}\left[\frac{\left(p+\frac{1}{2}\right)}{\left(p+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right] + \frac{1}{2} L^{-1}\left(\frac{1}{\left(p+\frac{1}{2}\right)^2+\frac{3}{4}}\right)$$

$$= e^{-\frac{1}{2}t} \cos\frac{\sqrt{3}}{2}t - \frac{1}{2} \times \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t$$

$$= e^{-1/2 t} \cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t$$ Ans

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2^{nd} Shifting Theorem :-

~~Let~~ If $L^{-1}(f(p)) = f(t)$, then,

$$L^{-1}(e^{-ap} f(p)) = \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$$

Q $L^{-1}\left[\frac{e^{-2p}}{p^2+1}\right]$

Sol→ Let $f(p) = \frac{1}{p^2+1}$

Taking inverse both side.

$$L^{-1}(f(p)) = L^{-1}\left(\frac{1}{p^2+1}\right)$$

$$= \sin t = f(t) \quad \text{--- (1)}$$

By 2^{nd} shifting theorem.

$$L^{-1}\left[\frac{e^{-2p}}{p^2+1}\right] = \begin{cases} f(t-2) & t>2 \\ 0 & t<0 \end{cases}$$

$$= \begin{cases} \sin(t-2) & t>2 \\ 0 & t<0 \end{cases}$$ Ans

Q $L^{-1}\left[\frac{e^{-3p}\, p}{p^2+9}\right]$

Sol→ Let $f(p) = \frac{p}{p^2+9}$

$$L^{-1}[f(p)] = L^{-1}\left[\frac{p}{p^2+p}\right]$$

$$= \cos 3t = f(t) \quad \text{--- (1)}$$

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By 2^{nd} shifting theorem.

$$L^{-1}\left(\frac{e^{-3p}p}{p^2+9}\right) = \begin{cases} f(t-3) & t>3 \\ 0 & t<0 \end{cases}$$

$$= \begin{cases} \cos 3(t-3) & t>3 \\ 0 & t<0 \end{cases} \quad \underline{\underline{\text{Ans}}}$$

Inverse Laplace transform of derivative :-

If $L^{-1}(f(p)) = f(t)$ then

$$L^{-1}(\underbrace{f^n(p)}_{n^{th}\ \text{derivative}}) = (-1)^n t^n L^{-1}(f(p)).$$

Q $L^{-1}[\tan^{-1}p]$

Sol → Let $f(p) = \tan^{-1}p$.

On differentiating w.r.t p.

$$f'(p) = \frac{d}{dp}\tan^{-1}p$$

$$f'(p) = \frac{1}{1+p^2}$$

$$L^{-1}(f'(p)) = L^{-1}\left(\frac{1}{1+p^2}\right) = \sin t \quad \text{--- (1)}$$

By Inverse Laplace transform of derivative

$$L^{-1}(f^n(p)) = (-1)^n t^n L^{-1}(f(p))$$

$$\Rightarrow L^{-1}f(p) = \frac{L^{-1}(f'(p))}{(-1)^1 t^1}$$

$\Rightarrow L^{-1}(\tan^{-1}p) = \dfrac{\sin t}{-1 \times t}$ [using ①]

$\Rightarrow L^{-1}(\tan^{-1}p) = -\dfrac{\sin t}{t}$ Ans

Q $L^{-1}\left[\cot^{-1}(p^2+1)\right]$

Sol→ Let $f(p) = \cot^{-1}(p^2+1)$

~~f'(p)~~ On differentiating w.r.t p.

$$f'(p) = \frac{-1}{(p^2+1)} \times (2p)$$

$$L^{-1}\left[f'(p)\right] = -2L^{-1}\left[\frac{p}{p^2+1}\right]$$

$$L^{-1}\left[f'(p)\right] = -2\cos t \quad \text{---} ①$$

By Inverse laplace transform of derivative

$$L^{-1}(f^n(p)) = (-1)^n t^n L^{-1}(f(p))$$

$$\Rightarrow L^{-1}[f'p] = (-1)^1 \times t^1 \times L^{-1}(f(p))$$

$$\Rightarrow -2\cos t = (-1)\, t\, L^{-1}(f(p))$$

$$\Rightarrow L^{-1}[f(p)] = \frac{2\cos t}{t}.$$

$$\Rightarrow L^{-1}\left[\cot^{-1}(p^2+1)\right] = \frac{2\cos t}{t} \quad \text{Ans}$$

Q $L^{-1}\left[\log(p-3)\right]$

Sol→ Let $f(p) = \log(p-3)$

On differentiating w.r.t p.

$$f'(p) = \frac{1}{(p-3)}$$

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$$L^{-1}[f'(p)] = L^{-1}\left[\frac{1}{p-3}\right]$$

$$= e^{3t} \quad \text{--- --- ---} \quad (i)$$

By Inverse laplace transform of derivative

$$L^{-1}[f^n(p)] = (-1)^n t^n L^{-1}(f(p))$$

$$\Rightarrow L^{-1}[f'(p)] = (-1)^1 (t)^1 [L^{-1}(f(p))]$$

$$\Rightarrow L^{-1}[f(p)] = \frac{e^{3t}}{-t}$$

$$\Rightarrow L^{-1}[\log(p-3)] = -\frac{e^{3t}}{t} \quad \text{Ans}$$

Q $L^{-1}\left[\frac{p}{(p^2+4)^2}\right]$

Sol → Let $f(p) = \frac{1}{(p^2+4)}$

On Differentiating w.r.t p.

$$f'(p) = \frac{-2p}{(p^2+4)^2}$$

$$L^{-1}[f'(p)] = 2L^{-1}\left[\frac{p}{(p^2+4)^2}\right]$$

By Inverse laplace transform of derivative.

$$L^{-1}(f^n(p)) = (-1)^n t^n (L^{-1}f(p))$$

$$\Rightarrow 2L^{-1}\left[\frac{p}{(p^2+4)^2}\right] = (-1)^1 t^1 L^{-1}[f(p)]$$

$$\Rightarrow 2L^{-1}\left[\frac{p}{(p^2+4)^2}\right] = -t\, L^{-1}\left[\frac{1}{p^2+4}\right]$$

$$\Rightarrow L^{-1}\left[\frac{p}{(p^2+4)^2}\right] = \frac{1}{2}\times\frac{1}{2}\sin 2t$$

$$= \frac{1}{4}\sin 2t \quad \underline{Ans}$$

Inverse Laplace transform of Integral :-

If $L^{-1}[f(p)] = f(t)$ then. $L^{-1}\left(\int_p^\infty f(x)dx\right) = \frac{f(t)}{t}$.

where, $f(x) = f(p)$.

Q $L^{-1}\left(\int_p^\infty \frac{x\,dx}{x^2+1}\right)$.

Sol → Let $f(p) = \frac{p}{p^2+1}$

$$L^{-1}(f(p)) = \cos t = f(t). \quad ---\text{①}$$

By Inverse laplace transform of Integral ;

$$L^{-1}\left(\int_p^\infty \frac{x\,dx}{x^2+1}\right) = \frac{f(t)}{t}$$

$$= \frac{\cos t}{t} \quad \underline{Ans}$$

Multiple by p :-

If $L^{-1}(f(p)) = f(t)$, then $L^{-1}\left[p\cdot f(p)\right] = \frac{d}{dt} f(t)$.

Q $L^{-1}\left[\frac{p^2}{p^2+1}\right]$

Sol → We have $L^{-1}\left[\frac{p^2}{p^2+1}\right]$

$= L^{-1}\left[p \cdot \frac{p}{p^2+1}\right]$

Let $f(p) = \frac{p}{p^2+1}$

$L^{-1}[f(p)] = L^{-1}\left[\frac{p}{p^2+1}\right]$

$= \cos t = f(t)$ --- (1)

By Multipl by p. property

$L^{-1}\left[p \cdot \frac{p}{p^2+1}\right] = \frac{d}{dt}[f(t)]$

$= \frac{d}{dt}\cos t$

$= -\sin t$ Ans

Q $L^{-1}\left[\frac{p}{p-1}\right]$

Sol → We have $L^{-1}\left[p \cdot \frac{1}{p-1}\right]$

Let $f(p) = \frac{1}{p-1}$

$L^{-1}(f(p)) = L^{-1}\left(\frac{1}{p-1}\right)$

$= e^t = f(t)$

By Multipl by p.

$L^{-1}\left(p \cdot \frac{1}{p-1}\right) = \frac{d}{dt} f(t). \Rightarrow \frac{d}{dt}(e^t)$

$= e^t$ Ans

Division by p :-

If $L^{-1}[f(p)] = f(t)$, then $L^{-1}\left(\frac{f(p)}{p}\right) = \int_0^t f(x)\,dx$

Q $L^{-1}\left[\frac{1}{p(p^2+1)}\right]$

Sol → $L^{-1}\left[\frac{1}{p} \cdot \frac{1}{p^2+1}\right]$

Let $f(p) = \frac{1}{p^2+1}$

$L^{-1}[f(p)] = L^{-1}\left[\frac{1}{p^2+1}\right] = \sin t = f(t)$ --- ①

By Division by p property :-

$$L^{-1}\left[\frac{\left(\frac{1}{p^2+1}\right)}{p}\right] = \int_0^t f(x)\,dx$$

$$= \int_0^t \sin x\,dt$$

$$= \left(-\cos x\right)\Big|_0^t$$

$$= -\cos t + 1$$

$$= 1 - \cos t \quad \text{Ans}$$

Q $L^{-1}\left[\frac{1}{p^2(p-1)}\right]$

Sol → We have $L^{-1}\left[\frac{1}{p^2} \cdot \frac{1}{(p-1)}\right]$

Let $f(p) = \frac{1}{p-1}$

$L^{-1}[f(p)] = L^{-1}\left(\frac{1}{p-1}\right) = e^t = f(t)$

By Division by p property.

$$L^{-1}\left(\frac{1/(p-1)}{p}\right) = \int_0^t f(x)\,dx$$

$$L^{-1}\left(\frac{1}{p(p-1)}\right) = \int_0^t e^x\,dx$$

$$= \left[e^x\right]_0^t$$

$$= e^t - 1 = \phi(t)$$

On again applying division by p property.

$$L^{-1}\left(\frac{\frac{1}{p(p-1)}}{p}\right) = \int_0^t f(x)\,dx$$

$$= \int_0^t (e^x - 1)\,dx$$

$$= \left[e^x - x\right]_0^t$$

$$= e^t - t - e^0 + 0$$

$$= e^t - t - 1 \quad \underline{\underline{Ans}}$$

Convolution Theorem :-

If $L^{-1}[f(p)] = f(t)$ & $L^{-1}[g(p)] = g(t)$.

Then. $L^{-1}[f(p)\cdot g(p)] = f*g = \int_0^t f(x)\,g(t-x)\,dx$

Q $L^{-1}\left[\frac{p}{(p^2+a^2)^2}\right]$ by using Convolution theorem.

Sol → $L^{-1}\left[\frac{1}{(p^2+a^2)^2}\cdot\frac{p}{(p^2+a^2)}\right]$

Let $f(p) = \frac{1}{(p^2+a^2)}$

$$L^{-1}(f(p)) = L^{-1}\left(\frac{1}{p^2+a^2}\right)$$

$$= \frac{1}{a}\sin at = f(t) \quad \text{--- (1)}$$

$$g(p) = \frac{p}{p^2+a^2}$$

$$L^{-1}(g(p)) = \cos at = g(t) \quad \text{--- (2)}$$

By Convolution theorem.

$$L^{-1}[f(p)\cdot g(p)] = f*g = \int_0^t f(x)\, g(t-x)\, dx$$

$$= \int_0^t \frac{1}{a}\sin ax \cdot \cos a(t-x)\, dx$$

$$= \frac{1}{2a}\int_0^t 2\sin ax \cdot \cos(at-ax)\, dx$$

$$= \frac{1}{2a}\int_0^t [\sin(ax+at-ax) + \sin(ax-at+ax)]\, dx$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin(2ax-at)]\, dx$$

$$= \frac{1}{2a}\left[\sin at\,|x|_0^t + \left|\frac{\cos(2ax-at)}{-2a}\right|_0^t\right]$$

$$= \frac{1}{2a}\left[\sin at\cdot t + \left(\frac{\cos at - \cos at}{-2a}\right)\right]$$

$$= \frac{1}{2a}(\sin at\cdot t + 0)$$

$$= \frac{t\cdot\sin at}{2a} \quad \text{Ans}$$

Home-Work

Q $L^{-1}\left(\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right)$

Sol → $L^{-1}\left[\frac{p}{(p^2+a^2)} \cdot \frac{p}{(p^2+b^2)}\right]$

$$f(p) = \frac{p}{p^2+a^2}$$

$$L^{-1}(f(p)) = L^{-1}\left(\frac{p}{p^2+a^2}\right)$$

$$= \cos at = f(t) \quad \text{--- (1)}$$

$$g(p) = \frac{p}{p^2+b^2}$$

$$L^{-1}(g(p)) = L^{-1}\left(\frac{p}{p^2+b^2}\right)$$

$$= \cos bt = g(t) \quad \text{--- (2)}$$

By Convolution Theorem.

$$L^{-1}[f(p) \cdot g(p)] = f * g = \int_0^t f(x)\, g(t-x)\, dx$$

$$= \int_0^t \cos ax \cdot \cos b(t-x)\, dx$$

$$= \frac{1}{2}\int_0^t 2\cos ax \cdot \cos(bt-bx)\, dx$$

$$= \frac{1}{2}\int_0^t \left[\cos(ax+bt-bx) + \cos(ax-bt+bx)\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin(ax+bt-bx)}{(a-b)} + \frac{\sin(ax-bt+bx)}{(a+b)}\right]_0^t$$

$$= \frac{1}{2} \times \left[\frac{\sin(at + bt - bt)}{(a-b)} + \frac{\sin(at - bt + bt)}{(a+b)} - \left(\frac{\sin bt}{(a-b)} + \frac{\sin(-bt)}{(a+b)} \right) \right]$$

$$= \frac{1}{2} \left[\left\{ \frac{(a+b)\sin at + (a-b)\sin at}{(a^2-b^2)} \right\} - \left\{ \frac{(a+b)\sin bt - (a-b)\sin bt}{(a^2-b^2)} \right\} \right]$$

$$= \frac{1}{2} \left[\frac{(a\sin at + \cancel{b\sin at} + a\sin at - \cancel{b\sin at}) - (\cancel{a\sin bt} + b\sin bt - \cancel{a\sin bt} + b\sin bt)}{a^2-b^2} \right]$$

$$= \frac{1}{2} \left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2} \right]$$

$$= \frac{a\sin at - b\sin bt}{(a^2-b^2)} \quad \text{Ans}$$

Application of Laplace Transform to Solve Differential Equation

Derivatives of Laplace :-

$$y = f(x) \Rightarrow y' = f'(x)$$

$$y = f(t) \Rightarrow y' = f'(t)$$

$$L(y') = pL(y) - y(0)$$

$$L(y'') = p^2 L(y) - py(0) - y'(0)$$

$$L(y''') = p^3 L(y) - p^2 y(0) - py'(0) - y''(0)$$

Q Solve $\frac{d^2y}{dt^2} + y = 0$, given $y(0)$, $y'(0) = 1$

Sol $\rightarrow$ $y'' + y = 0$

$$L(y'') + L(y) = 0$$

$$\Rightarrow p^2 L(y) + p\,y(0) - y'(0) + L(y) = 0$$

$$\Rightarrow p^2 \cdot L(y) + 0 - 1 + L(y) = 0$$

$$\Rightarrow (p^2 + 1) L(y) = 1$$

$$L(y) = \frac{1}{p^2+1}$$

$$y = L^{-1}\left(\frac{1}{p^2+1}\right)$$

$$y = \sin t$$

Q Solve $y'' + 4y = \cos 2t$ given $y(0) = 3$ & $y'(0) = 4$

Sol $\rightarrow$ Given $y'' + 4y = \cos 2t$

$$L(y'') + 4L(y) = L(\cos 2t)$$

$$\Rightarrow \left[p^2 L(y) - p\,y(0) - y'(0)\right] + 4L(y) = \frac{p}{p^2+4}$$

$$\Rightarrow p^2 L(y) - 3p - 4 + 4L(y) = \frac{p}{p^2+4}$$

$$\Rightarrow (p^2+4) L(y) = \frac{p}{p^2+4} + 3p + 4$$

$$= \frac{\frac{p}{p^2+4} + 3p + 4}{p^2+4}$$

$$L(y) = \frac{3p}{p^2+4} + \frac{4}{p^2+4} + \frac{p}{(p^2+4)^2}$$

$$y = 3L^{-1}\left(\frac{p}{p^2+4}\right) + 4L^{-1}\left(\frac{1}{p^2+4}\right) + L^{-1}\left(\frac{p}{(p^2+4)^2}\right)$$

$$y = 3\cos 2t + \frac{4}{2}\sin 2t + L^{-1}\left(\frac{p}{(p^2+4)^2}\right) \quad \text{---} \ (1)$$

Now, $L^{-1}\left(\frac{p}{(p^2+4)^2}\right)$

Let $f(p) = \frac{1}{(p^2+4)}$

$$f'(p) = \frac{-2p}{(p^2+4)^2}$$

$$L^{-1}(f'(p)) = -2L^{-1}\left[\frac{p}{(p^2+4)^2}\right]$$

$$\Rightarrow (-1)^1 t^1 L^{-1}(f(p)) = -2L^{-1}\left[\frac{p}{(p^2+4)^2}\right]$$

$$\Rightarrow L^{-1}\left[\frac{p}{(p^2+4)^2}\right] = \frac{t}{2}L^{-1}(f(p))$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{p^2+4}\right)$$

$$= \frac{t}{2} \times \frac{1}{2} \times \sin 2t$$

$$= \frac{t}{4}\sin 2t \quad \text{---} \ (2)$$

using eq. (2) in (1)

$$y = 3\cos 2t + 2\sin 2t + \frac{t}{4}\sin 2t \quad \underline{\underline{\text{Ans}}}$$

Q Solve $y'' + 2y' - 3y = \sin t$, $y(0) = y'(0) = 0$.

Sol→ Given $y'' + 2y' - 3y = \sin t$

$$L(y'') + 2L(y') - 3L(y) = L(\sin t)$$

$$\Rightarrow \left[p^2 L(y) - p\,y(0) - y'(0)\right] + 2\left[p L(y) - y(0)\right] - 3L(y) = \frac{1}{1+p^2}$$

$$\Rightarrow p^2 L(y) + 2p L(y) - 3L(y) = \frac{1}{1+p^2}$$

$$\Rightarrow L(y) = \frac{1}{(1+p^2)(p^2+2p-3)}$$

$$\Rightarrow L(y) = \frac{1}{(1+p^2)(p-1)(p+3)}$$

$$y = L^{-1}\left[\frac{1}{(p^2+1)(p-1)(p+3)}\right] \longleftarrow (1)$$

Now, $L^{-1}\left[\frac{1}{(p^2+1)(p-1)(p+3)}\right]$

$$= \frac{A}{p-1} + \frac{B}{p+3} + \frac{Cp+D}{p^2+1} \quad - (2)$$

$$\Rightarrow \frac{1}{(p^2+1)(p-1)(p+3)} = \frac{A(p+3)(p^2+1) + B(p-1)(p^2+1) + (Cp+D)(p^2+2p-3)}{(p^2+1)(p-1)(p+3)}$$

$$\Rightarrow 1 = A\left[p^3+p+3p^2+3\right] + B\left[p^3-p^2+p-1\right] + C\left[p^3+2p^2-3p\right] + D\left[p^2+2p-3\right]$$

On Equating the like power of p.

$$A + B + C = 0$$
$$3A - B + 2C + D = 0$$
$$A + B + 3C + 2D = 0$$
$$3A - B - 3D = 1$$

For A :- $p - 1 = 0$

$$p = 1$$

$$A = \frac{1}{(1+1)(1+3)}$$

$$= \frac{1}{8}$$

For B :- $p + 3 = 0$

$$p = -3$$

$$B = \frac{1}{(-3-1)(3^2+1)}$$

$$= \frac{1}{-4 \times 10} = \frac{-1}{40}$$

For C :- $A + B + C = 0$

$$\Rightarrow C = \frac{1}{40} - \frac{1}{8} = \frac{1-5}{40} = \frac{-1}{10}$$

For D :- $3A - B + 2C + D = 0$

$$\Rightarrow \frac{3}{8} + \frac{1}{40} - \frac{2}{10} = -D.$$

$$\Rightarrow \frac{15+1-8}{40} = -D$$

$$\Rightarrow D = \frac{-8}{40} \Rightarrow \frac{-1}{5}.$$

Putting the value of A, B, C and D and taking Laplace ~~[illegible]~~ inverse on both side.

$$L^{-1}\left[\frac{1}{(p-1)(p+3)(p^2+1)}\right] = \frac{1}{8}L^{-1}\left(\frac{1}{p-1}\right) - \frac{1}{40}\left[\frac{1}{p+3}\right]$$

$$-\frac{1}{10}L^{-1}\left(\frac{p}{p^2+1}\right) - \frac{1}{5}L^{-1}\left(\frac{1}{p^2+1}\right)$$

$$\Rightarrow L^{-1}\left[\frac{1}{(p-1)(p+3)(p^2+1)}\right] = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t \quad \text{--- (3)}$$

Putting value of ~~[illegible]~~ eq (3) in (1).

$$y = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$$

Ans