Maths

Unit-5

Graph Theory

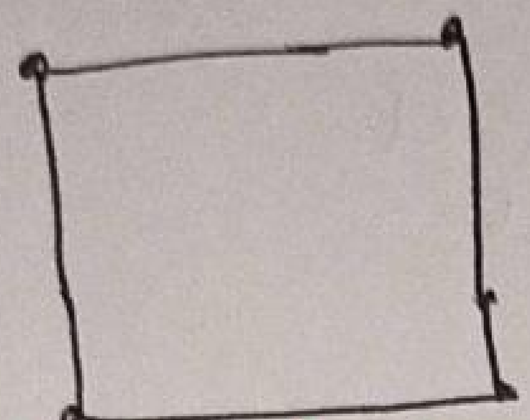
Graph → $G = (V, E)$

V → Vertices (nodes/points)

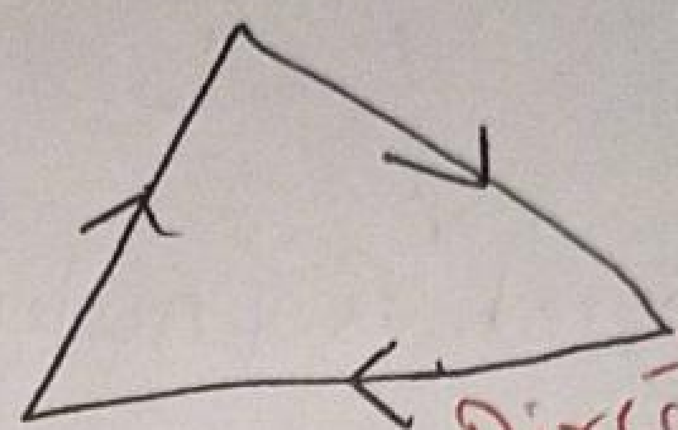
E → Edges

* The edge e that connects the nodes u and v is said to be incident on each node.
The pair of nodes that are connected by an edge is called Adjacent nodes.
* A node of a graph which is not adjacent to any other other is called an isolated node.
* A graph containing only isolated nodes is null graph.
* If in graph each e is associated with an ordered pair of vertices then G is called a directed graph or digraph. If each edge is associated with an unordered pair of vertices then G is called an Undirected graph.
* A graph in which there is only one edge between pair of vertices is called simple graph.
* A graph which contain some parallel edge is called mutigraph.
* Ege of a graph that joins a vertex to itself is called Loop.

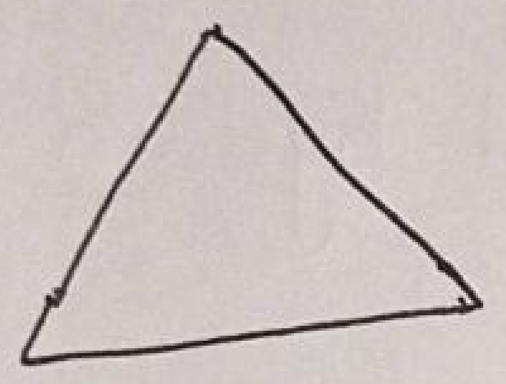
* A graph in which loop and ~~parra~~ parallel edge are allowed is called a pseudograph.
Graph in which a number (weight) is assigned is called weighted graph.



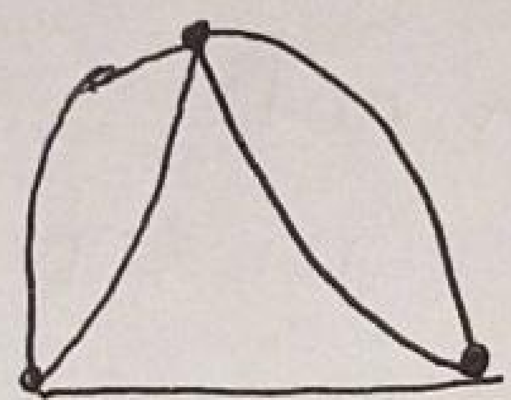
Undirected



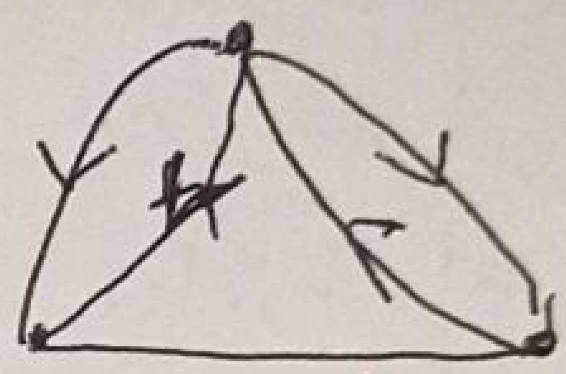
Directed



simple



Undirected multigraph



directed multigraph

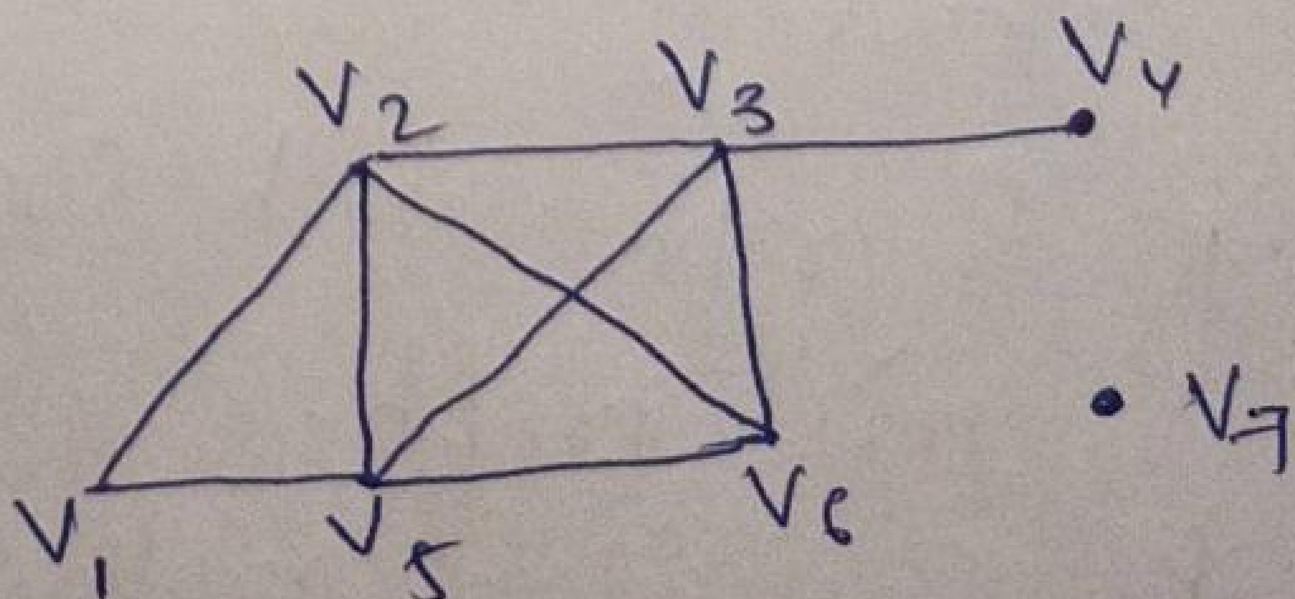
directed graph with distinct edge.

Degree of a vertex :- The degree of a vertex in an undirected graph is number of edge incident with it.

* Self loop will be counted 2 time.

Degree of vertex v is denoted by deg(v).
degree of isolated vertex = 0.
If degree of a vertex = 1 → It is called pendant vertex

Example :-



$deg(v_1) = 2$
$deg(v_2) = 4$
$deg(v_3) = 4$
$deg(v_4) = 1$
$deg(v_5) = 4$
$deg(v_6) = 3$
$deg(v_7) = 0$

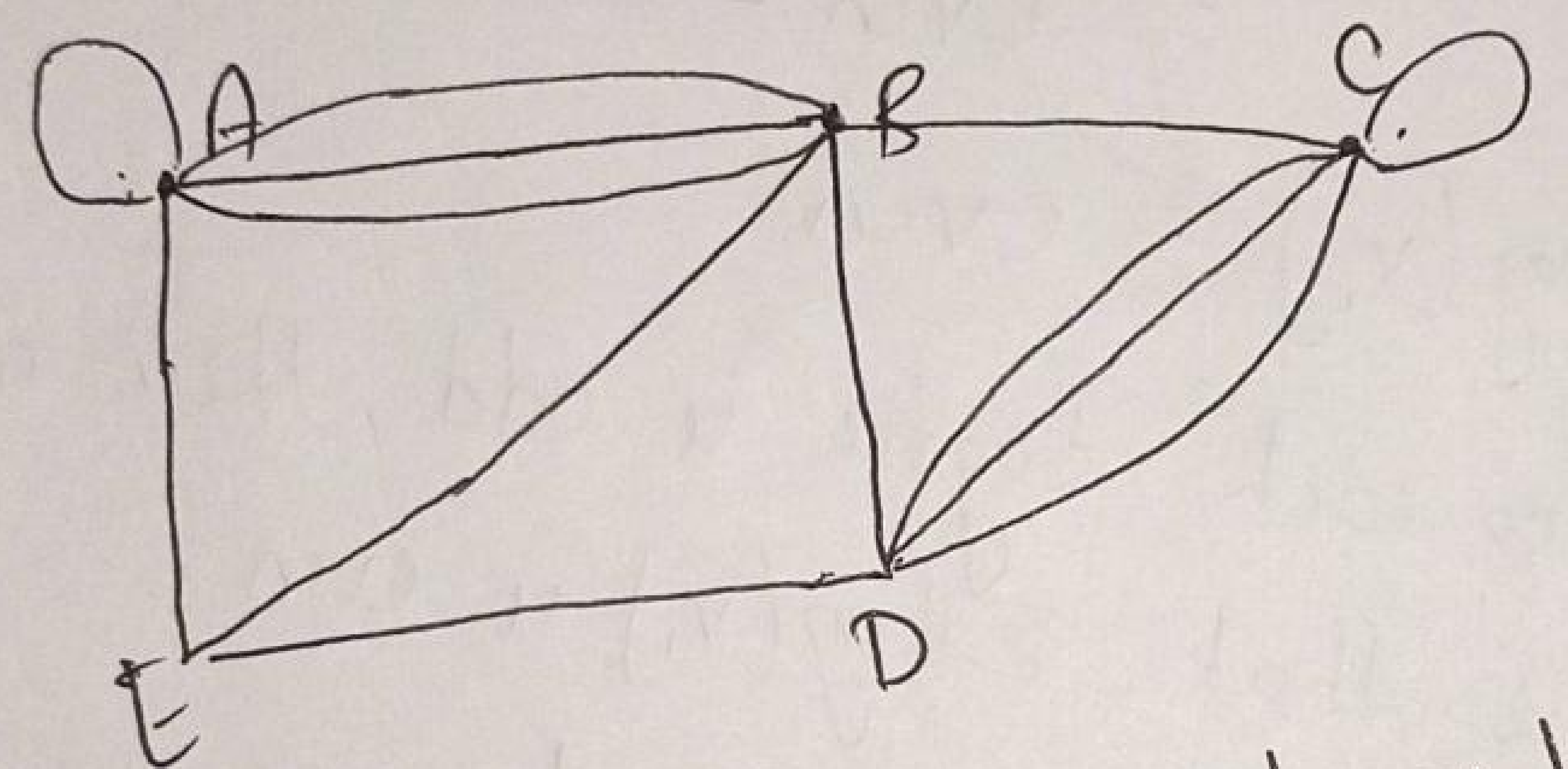
Theorem (Hand-Shaking Theorem) :-

If $G = (V, E)$ is an undirected graph with e edge. Then $\sum_i deg(v_i) = 2e$.

Proof :- Since every edge is incident to 2 vertex So, each edge contributes two times to the sum of ~~edge~~ degree of vertices.

$\therefore \sum_i deg(v_i) = 2e$.

Q.



Find No. of edges, No. of vertices, deg. of each vertex Also verify Hand shaking theorem.

Sol →

(i) 13

(ii) 5

(iii) $deg(A) = 6$ $deg(C) = 6$ $deg(E) = 3$

$deg(B) = 6$ $deg(D) = 5$

(iv) $\sum deg(i) = 2 \cdot e$

$6+6+6+5+3 = 2 \cdot 13$

$\Rightarrow 26 = 26$

LHS = RHS

Hence verified

Theorem :- The no. of vertices of odd degree in an undirected graph is even.

Proof :- $\sum_i \deg(v_i) = 2e$.

$\Rightarrow \underbrace{\sum_i \deg(v_i)}_{\text{odd}} + \underbrace{\sum_j \deg(v_j)}_{\text{even}}$.

$\sum \deg(v_i) = 2e - \sum_j \deg(v_j)$

$= \text{even} - \text{even}$

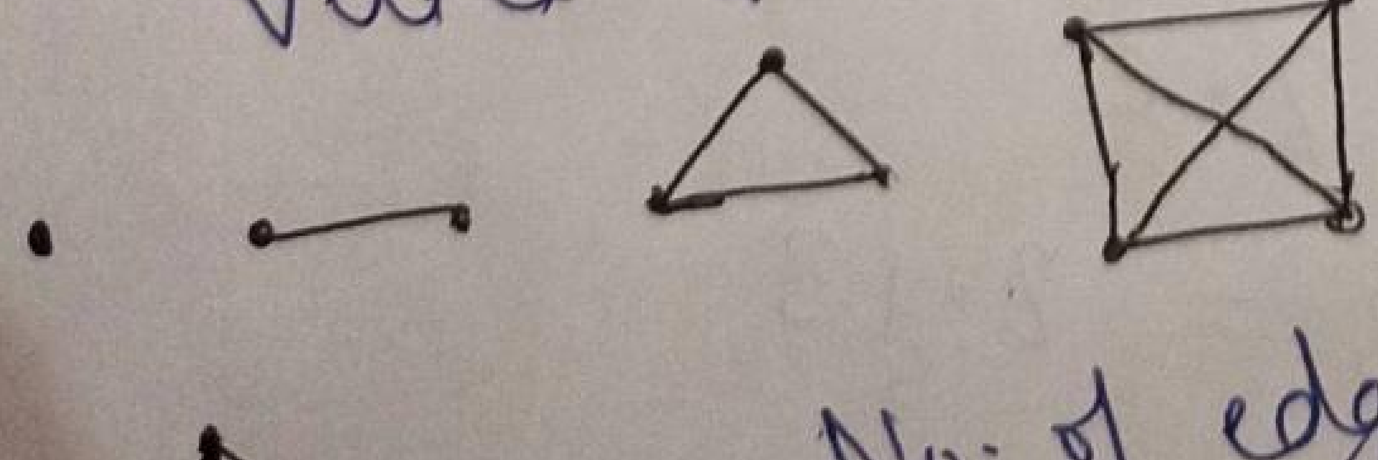
$\sum \deg(v_i) = \text{even}$

So, since each degree is odd, there must be even no. of terms so that $\sum \deg(v_i)$ is even.

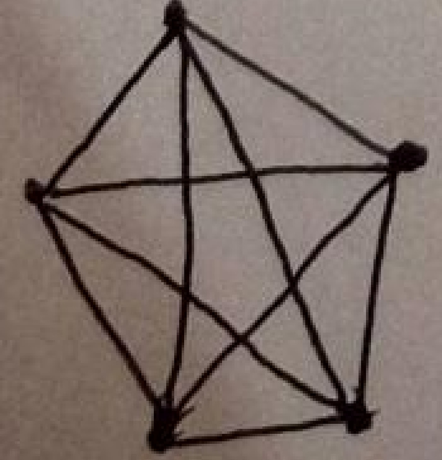
So, No. of odd degree vertex is even.

Some Special Graph :-

1) Complete Graph :- A graph in which there is exactly one edge between each pair of distinct vertex is called complete graph.
↳ denoted by K_n

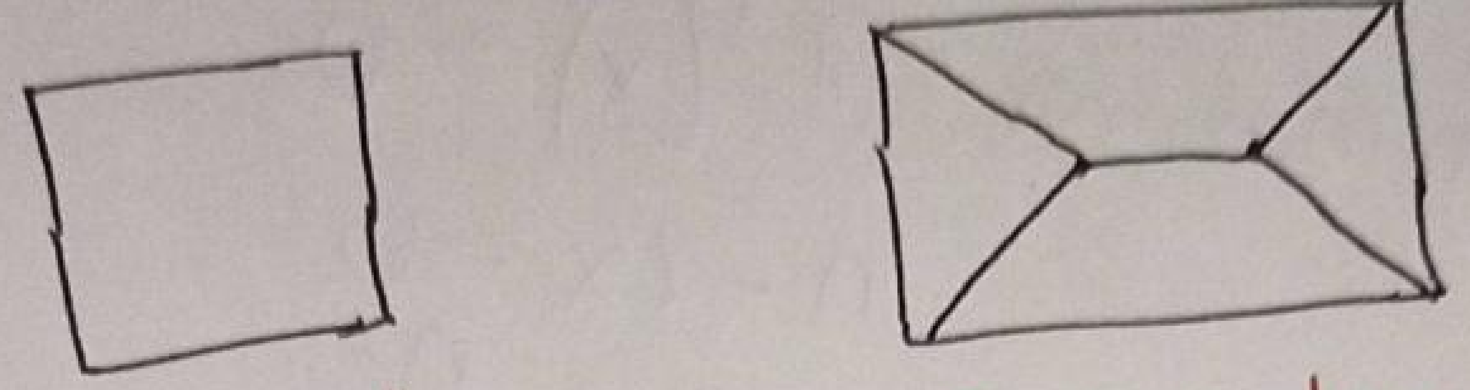


No. of edges in $K_n = {}^nC_2 = \frac{n(n-1)}{2}$



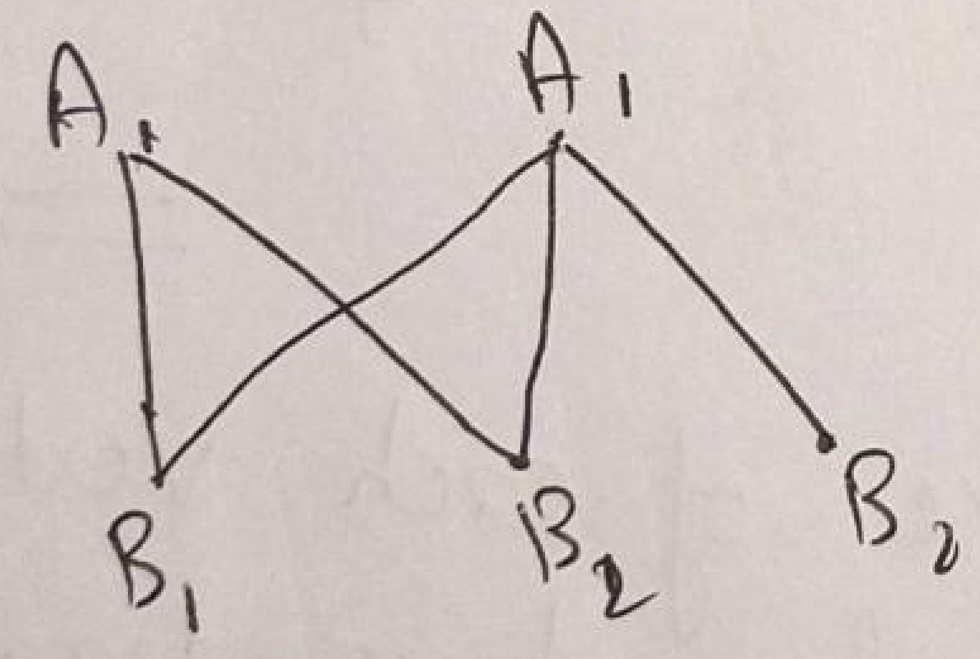
∴ Maximum no. of vertices in simple graph with n vertices $= \frac{n(n-1)}{2}$

2) Regular Graph :- If every vertex of a graph has the same degree then the graph is called Regular graph. If every ~~graph~~ vertices of a regular graph has degree n, then the graph is called n-regular.

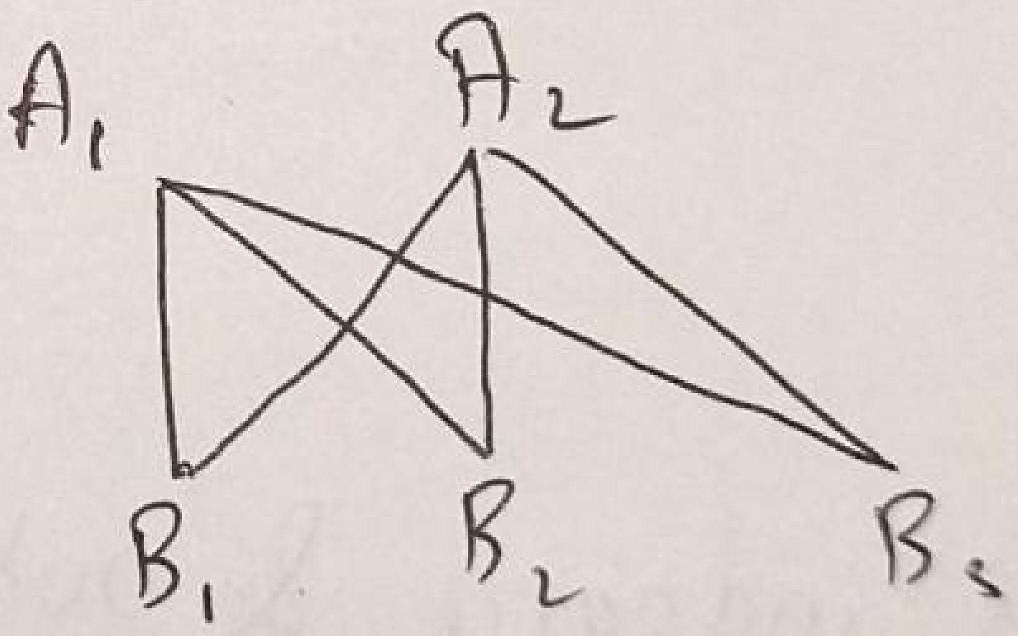


2-regular 3-regular.

3) Bipartite graph :- If the vertex set V of a simple graph can be partitioned into two subset V_1 and V_2 such that each edge of G connects a vertex in V_1 and a vertex in V_2 (so that no edge connect two vertices in V_1 or two vertices in V_2) then G is Called Bipartite graph.



Bipartit



Complet Bipartite.

↓

if every vertices of V_1 connect every vertices of V_2.

Q. Prove that no. of edge in a Biparted graph with n vertices is atmost $\left(\frac{n}{2}\right)^2$.

Sol → Let the vertex set be partition into the subset V_1 and V_2. Let V_1 contain x vertex & V_2 contain n−x vertices.

The largest no. of edges = $n(n-x)$

$$f(n) = n(n-x)$$
$$nx - x^2$$

$$f'(x) = n - 2x$$

for maxima. $f'(x) = 0$

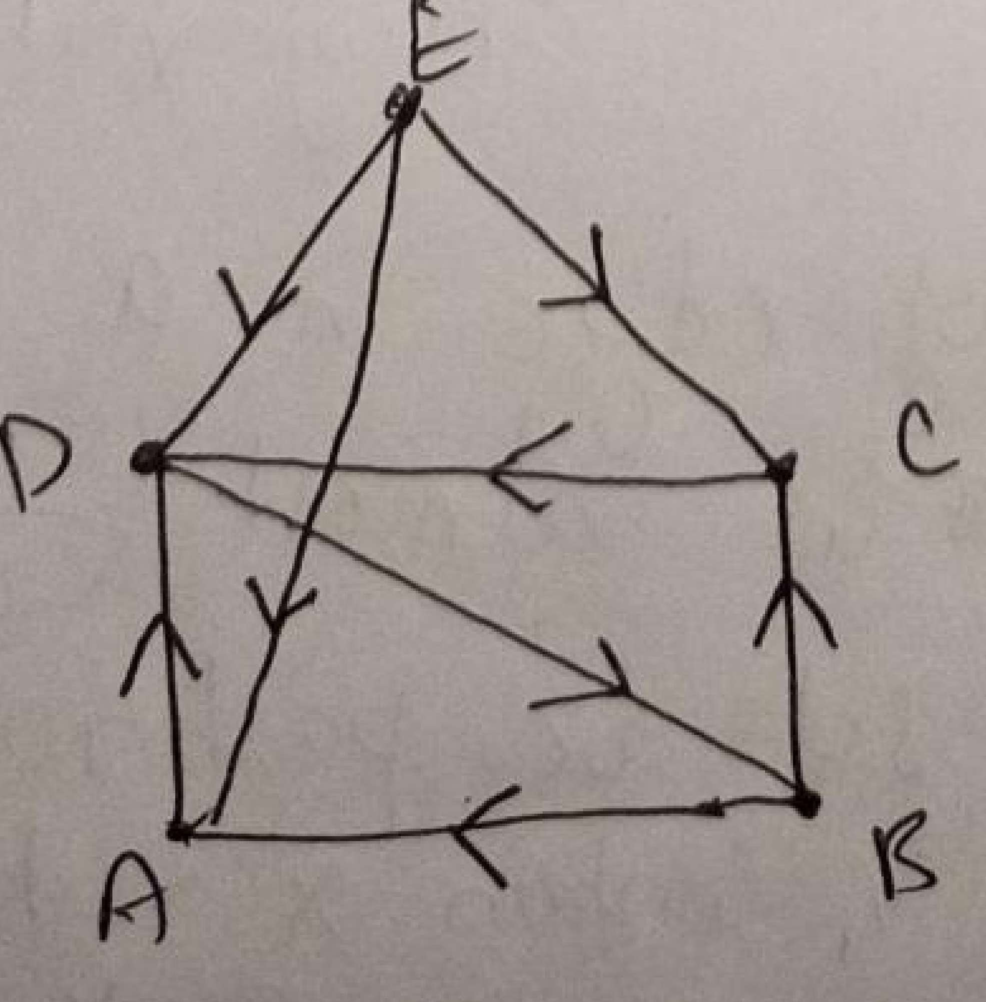
$$n - 2x = 0$$
$$x = \frac{n}{2}.$$

$$f''(n) = -2 < 0$$
$$\Rightarrow f'(n) = \frac{n}{2} \text{ is maxima}$$

So. Largest No. of edges $= n(n-n)$

$$= \frac{n}{2}\left(n - \frac{n}{2}\right)$$
$$= \frac{n^2}{4}. \text{ or } \left(\frac{n}{2}\right)^2.$$

Q Find indegree & out degree of each vertex of given graph. Also verify that sum of in-degree (or the out-degree) is equal to no. of edge.



Sol → ~~indegree (A)~~

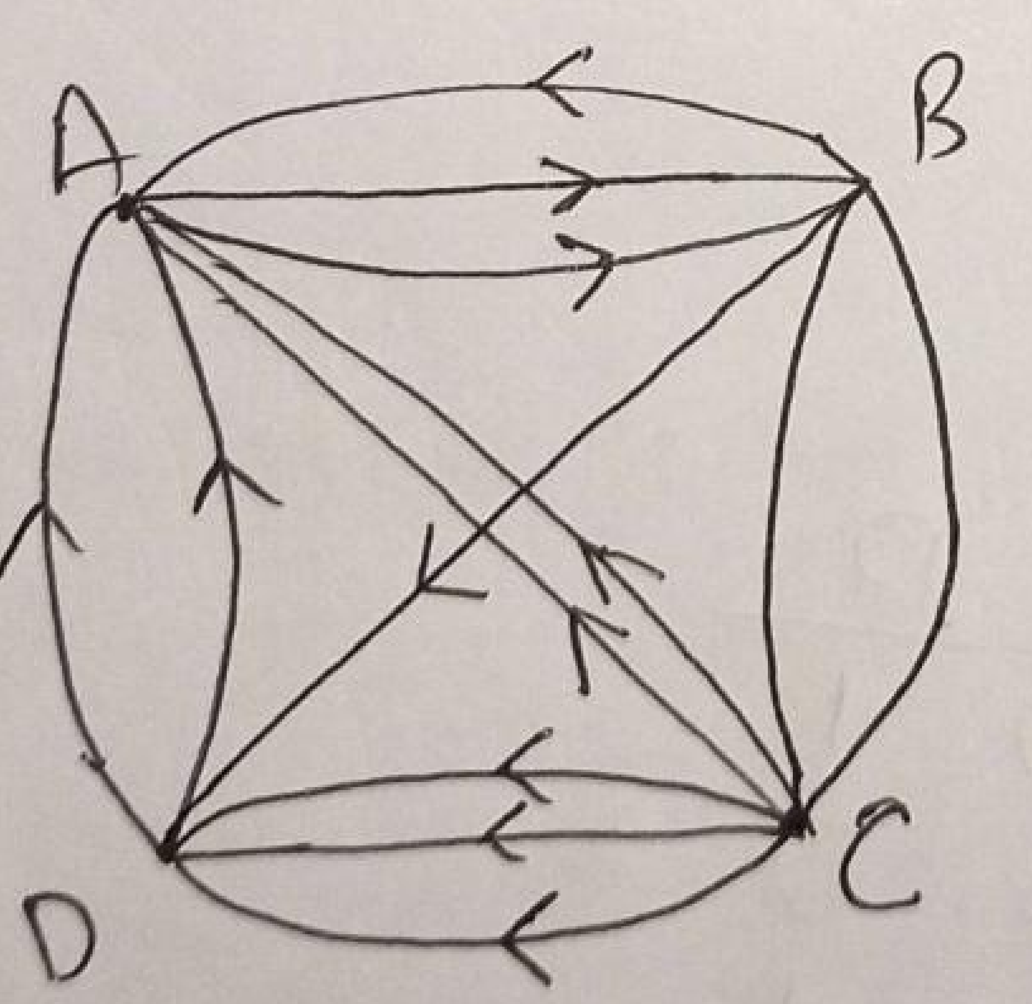
Vertex	indegree	Out-degree
A	2	1
B	1	2
C	2	2
D	3	1
E	0	3
	8	8

Sum

Sum of indegree = 8
Sum of outdegree = 8
~~Sum~~ No. of edge = 8.

So. $\sum deg^+ = \sum deg^- =$ No. of edge $= \underline{8}$

Q



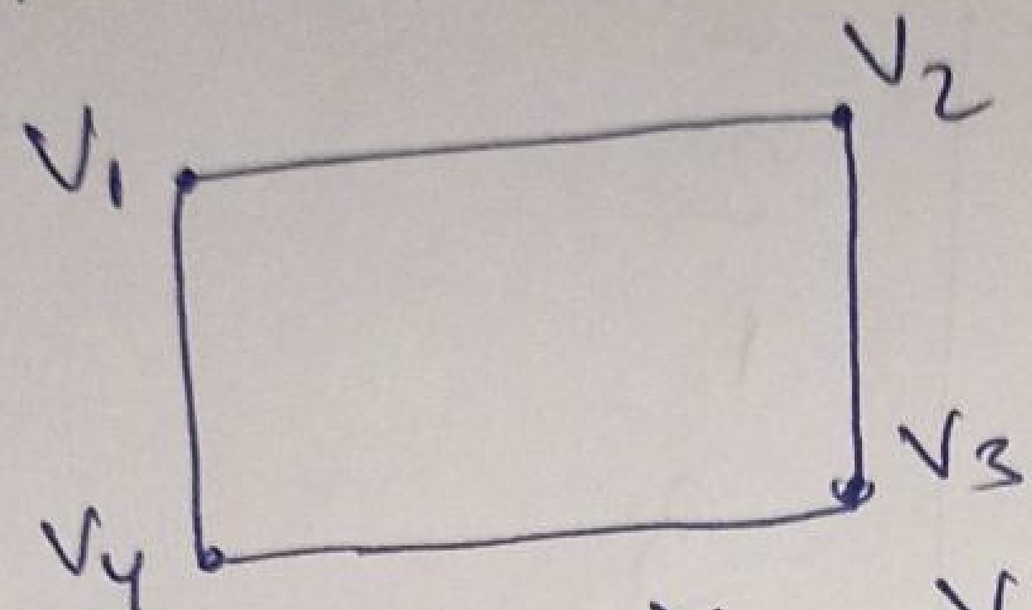
Sol →

Vertex	Indegree	outdegree
A	5	2
B	3	3
C	1	6
D	4	2
	13	13

Matrix Representation of Graph :-

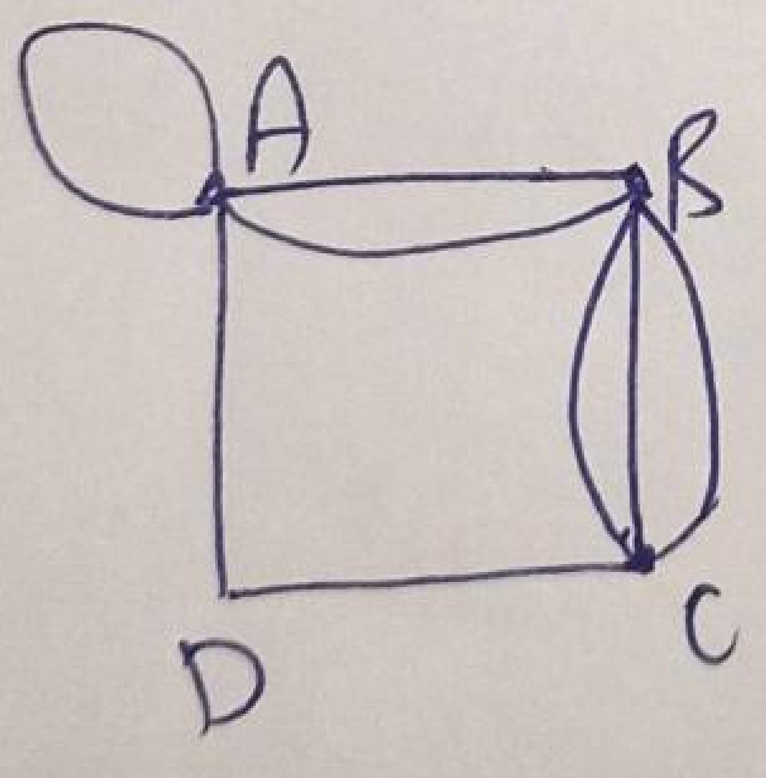
1) Adjacency Matrix :- Vertex to vertex.

(a) Simple graph :-



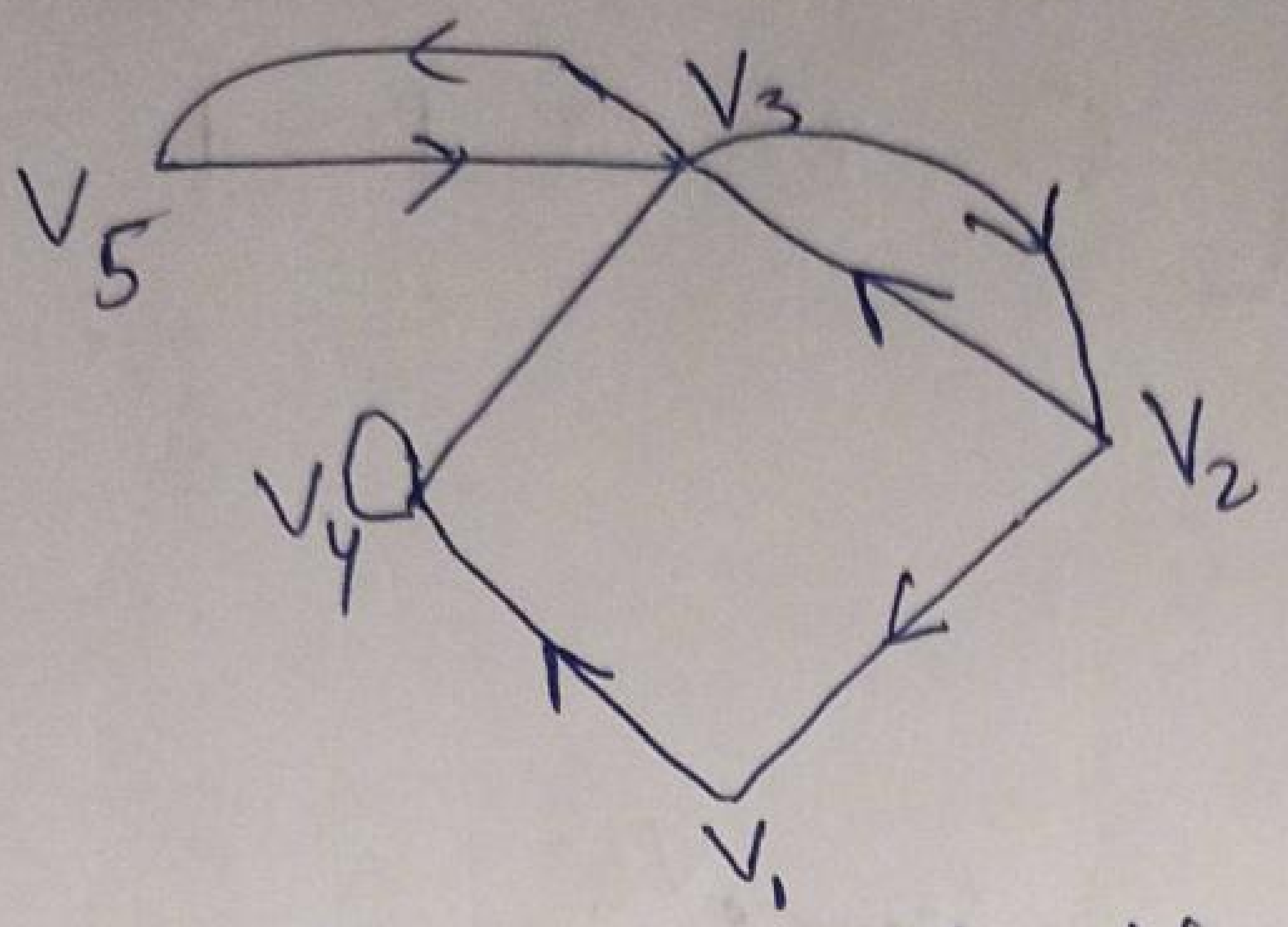
	V_1	V_2	V_3	V_4
V_1	0	1	0	1
V_2	1	0	1	0
V_3	0	1	0	1
V_4	1	0	1	0

(b) Multigraph :-



	A	B	C	D
A	1	2	0	1
B	2	0	3	0
C	0	3	1	1
D	1	0	1	0

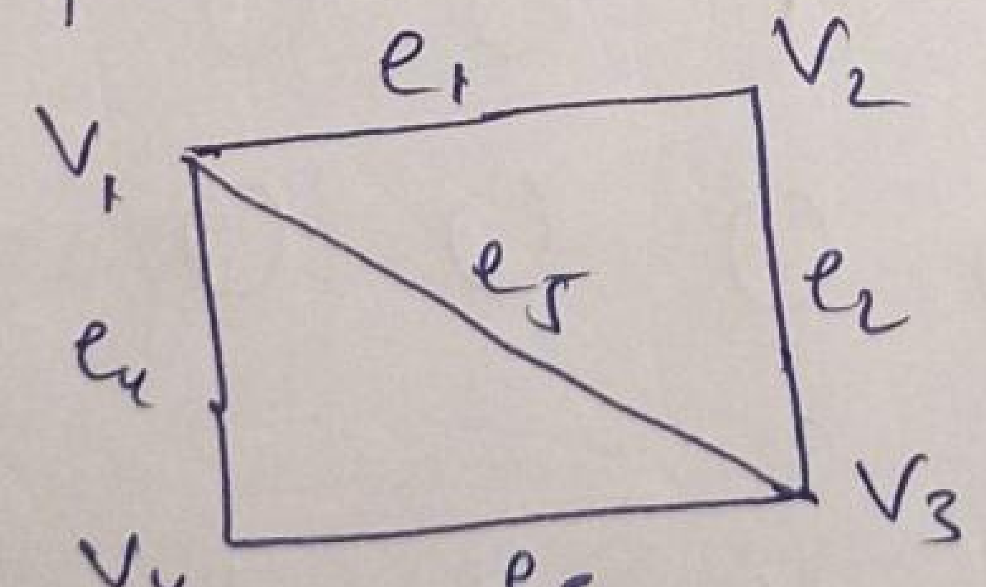
(c) Directed graph :-



	V1	V2	V3	V4	V5
V1	0	0	0	1	0
V2	1	0	1	0	0
V3	0	1	0	0	1
V4	0	0	1	1	0
V5	0	0	1	0	0

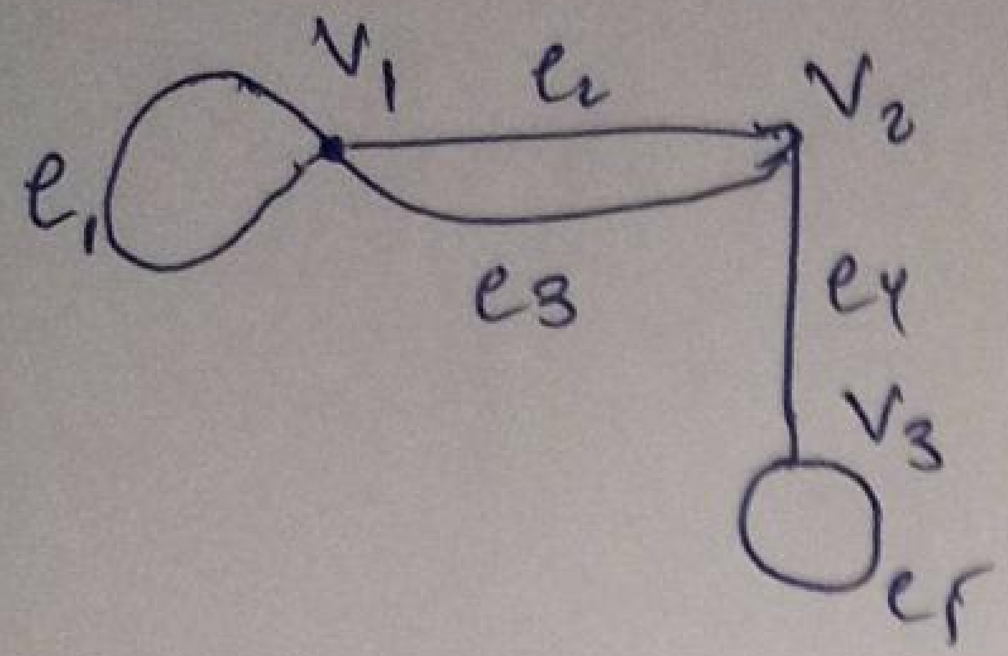
2) Incidence Matrix :- vertex & edge ka relation

(a) Simple Graph :-



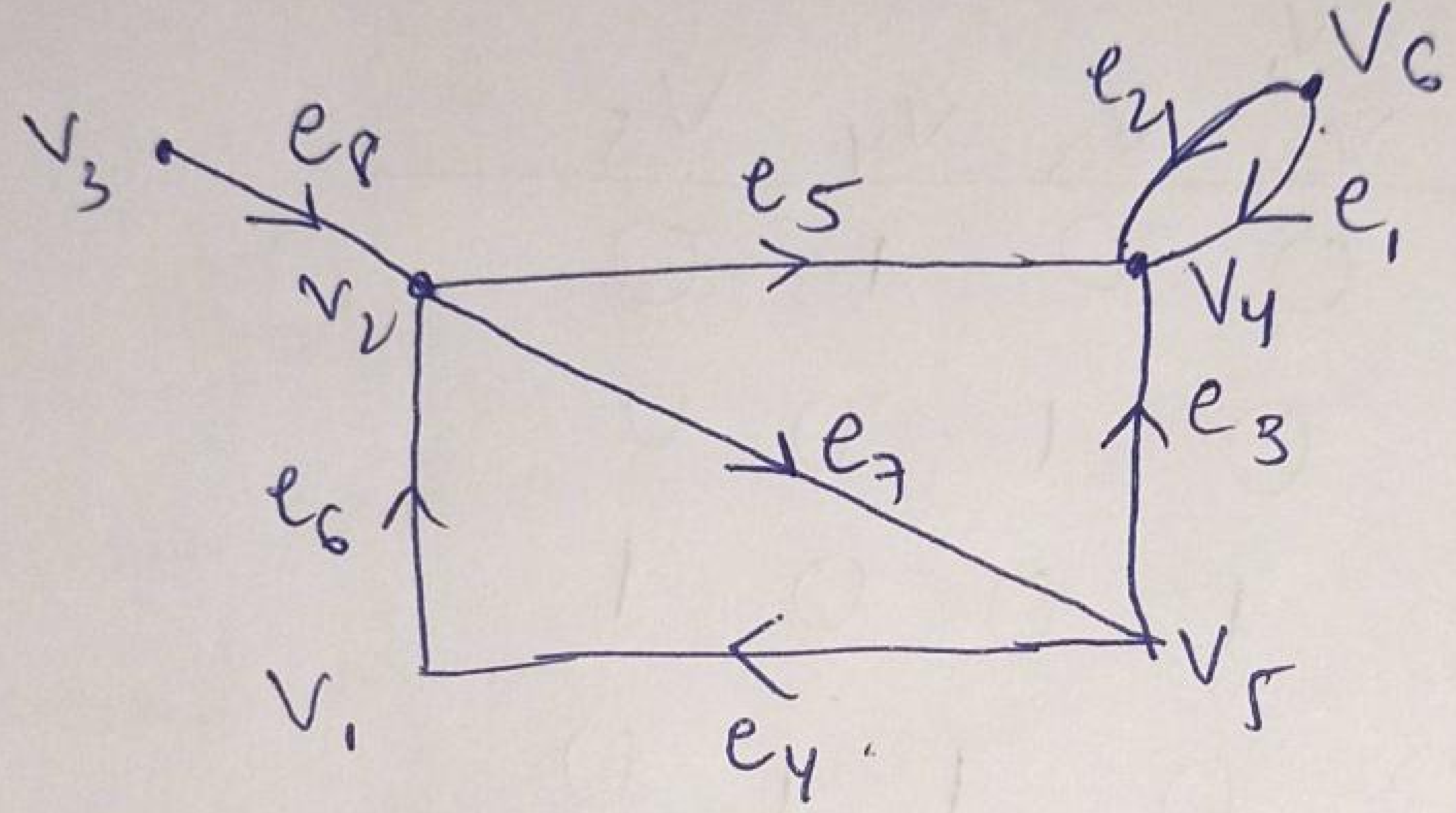
	e1	e2	e3	e4	e5
V1	1	0	0	1	1
V2	1	1	0	0	0
V3	0	1	1	0	1
V4	0	0	1	1	0

(b) Multigraph :-



$$
\begin{array}{c|ccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 \\
\hline
V_1 & 1 & 1 & 1 & 0 & 0 \\
V_2 & 0 & 1 & 1 & 1 & 0 \\
V_3 & 0 & 0 & 0 & 1 & 1
\end{array}
$$

(c) Directed :-



$$
\begin{array}{c|cccccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\
\hline
V_1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
V_2 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\
V_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
V_4 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\
V_5 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\
V_6 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$-1 \longrightarrow$ oppo. direction

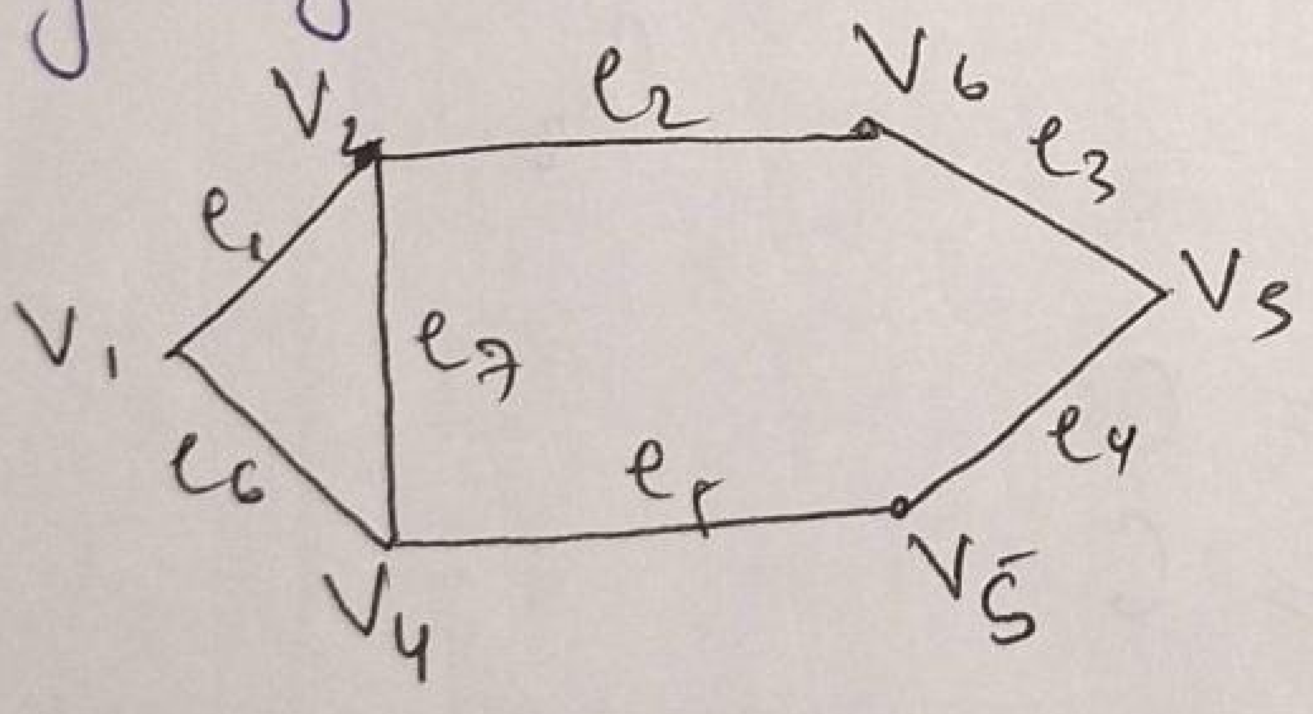
Subgraph :-

A graph $H=(V', E')$ is called subgraph of $G=(V,E)$ if $V' \subseteq V$ and $E' \subseteq E$.

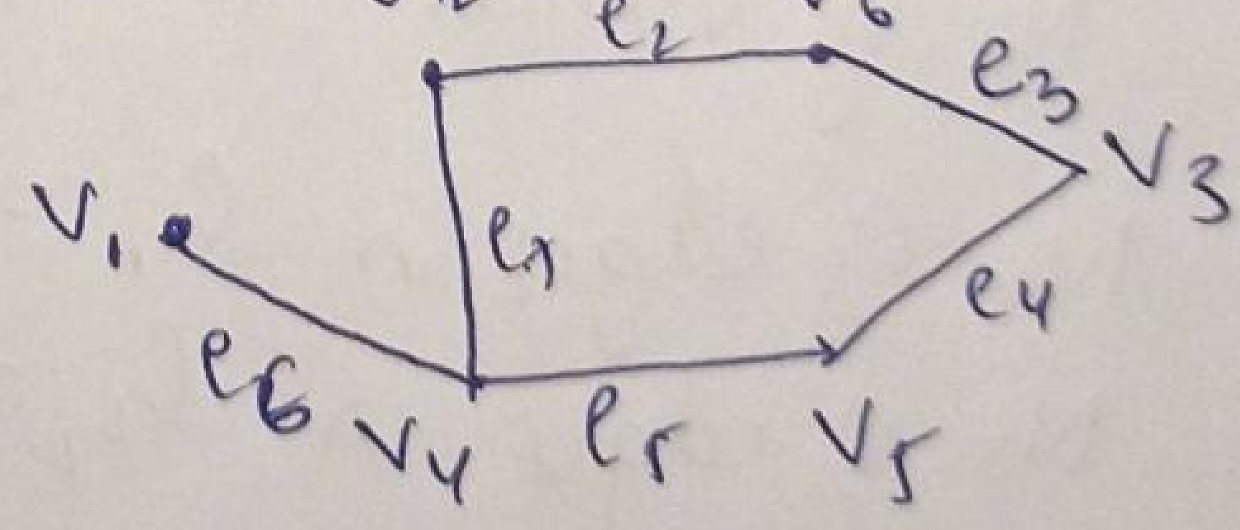
If $V' \subset V$ & $E' \subset E$ then H is called proper subgraph

If $V' = V$, then H is called spanning subgraph of G. A spanning subgraph need not contain all the edge.

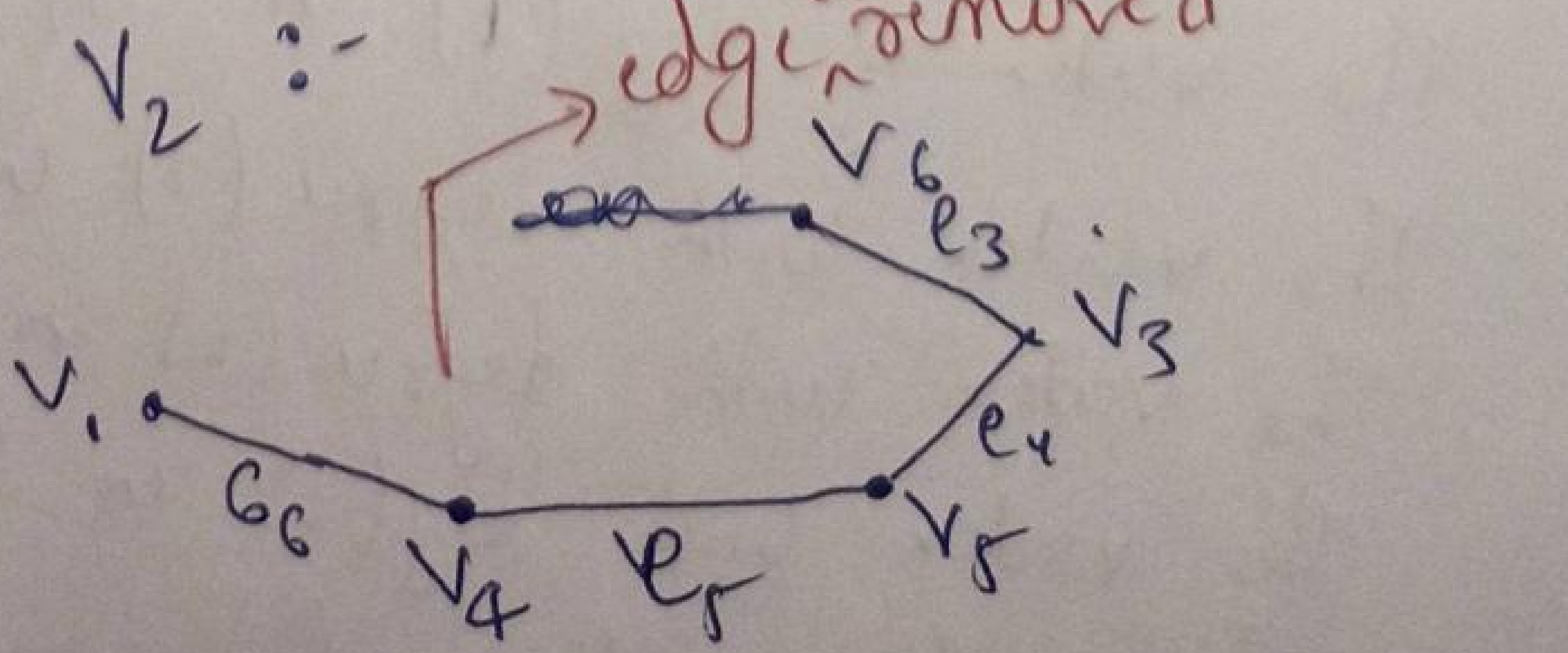
Any subgraph of a graph G can be obtained by removing certain vertices and edges from G. It is ~~so~~ to be noted that removal of an edge does not go with removal of it adjacent vertex whereas removal of vertex goes with removal of any edge incident on it.



On removing e_1 :-



On removing V_2 :-



Isomorphic Graph :-

Two graph G_1 & G_2 are said be to isomorphic to each other if $\exists$ (there exist) one-to-one corrauponderce between the vertex sets which preserve adjacency of the vertex.

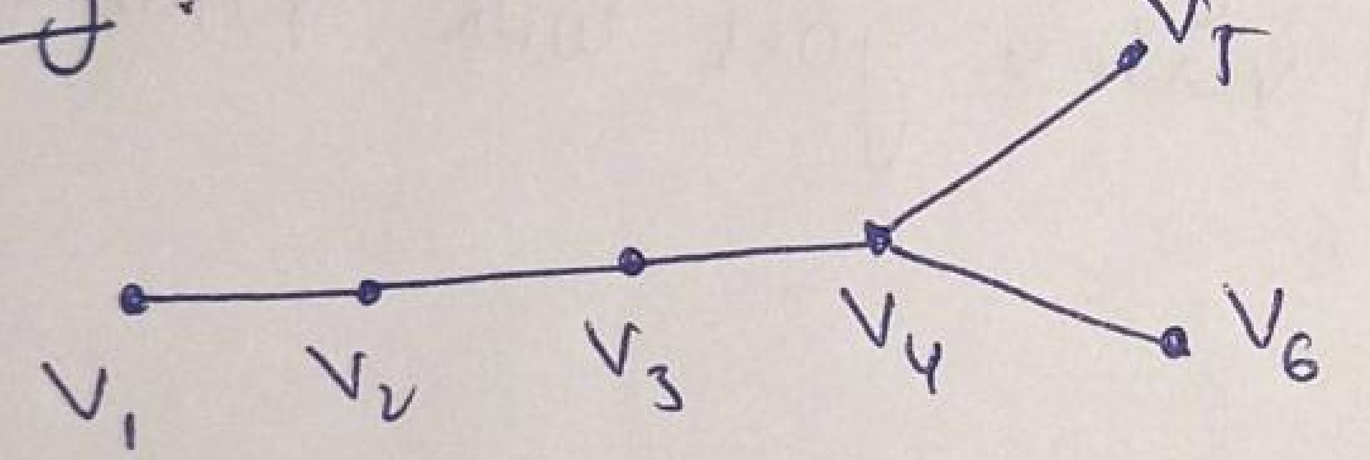
From the definition, it is clear that 2 graph are isomorphic if they

(i) have same no. of edge

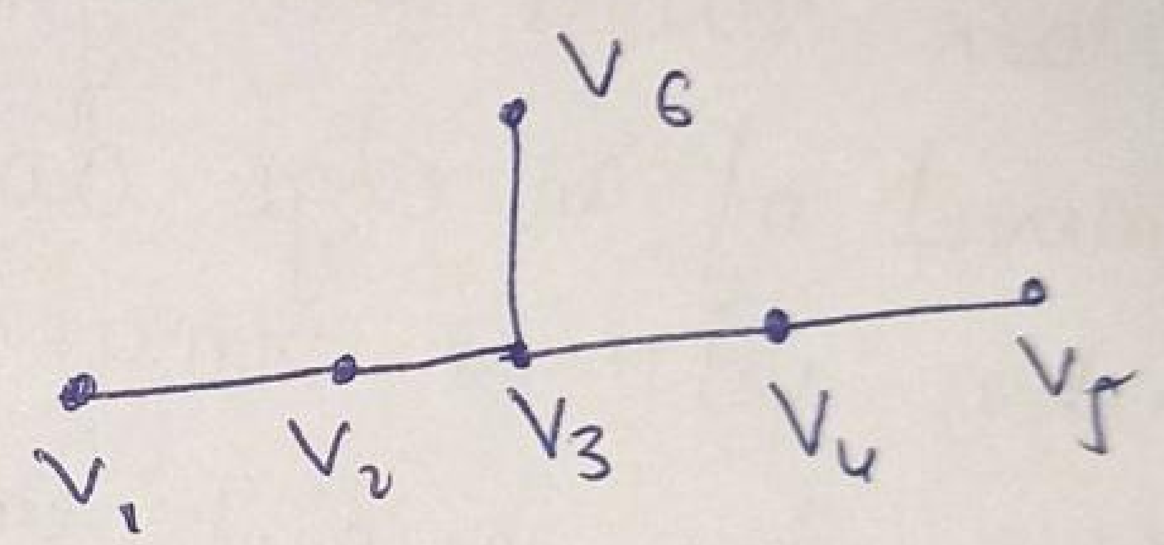
(ii) same no. of vertices

(iii) corresponding vertex with the same degree.

e.g :-



G_1



G_2.

No. of vertices in $G_1 = 6$

No of vertices in $G_2 = 6$

No. of edges in $G_1 = 5$

No. of edges in $G_2 = 5$

The vertex of deg (1) in G_1 are V_1, V_5, V_6.

The vertex of deg (1) in G_2 are V_1', V_5', V_6'

The vertex with degree (2) in G_1 are V_2, V_3.

The vertex with degree (2) in G_2 are V_2', V_4'

The vertex with deg(3) in $G_1 = V_4$

The vertex with deg(3) in $G_2 = V_3'$

$f(v_1) = v_1'$
$f(v_2) = v_2'$
$f(v_3) = v_4'$
$f(v_4) = v_3'$
$f(v_5) = v_5'$
$f(v_6) = v_6$

V_2 & V_3 are connected but V_2' & V_4' are not connected. So, Not Isomorphism.

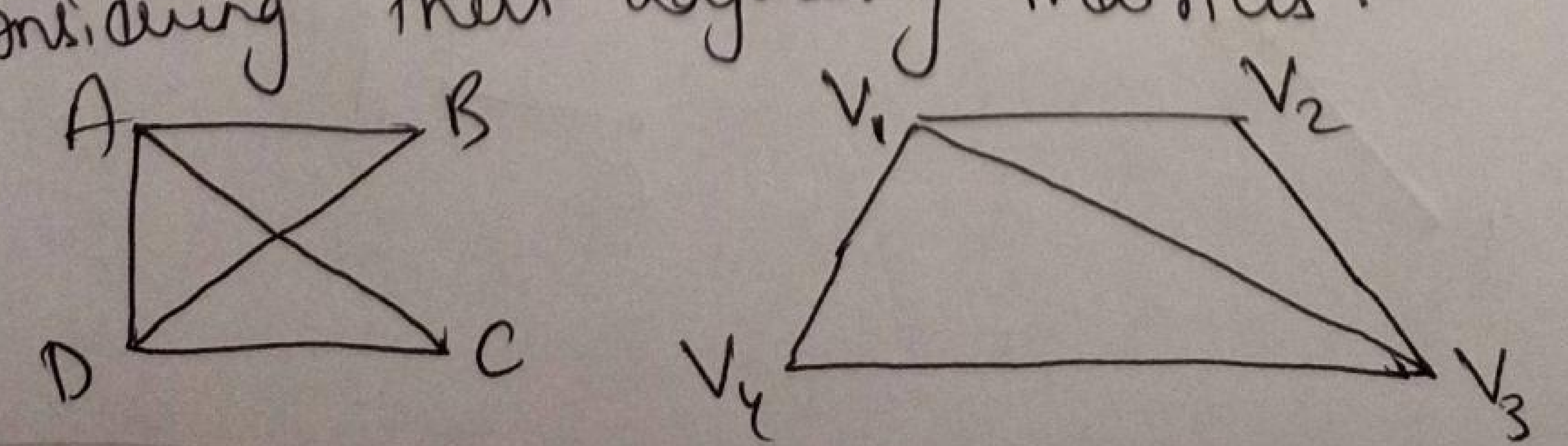
Isomorphism with Adjacency Matrix :-

→ Two graph ~~are~~ are isomorphic if and only if their vertices can be labeled in such a way that the corresponding adjacency matrices are equal.

→ 2 graph G_1 & G_2 with adjacency matrix A_1 & A_2 are isomorphic iff $\exists$ a permutation matrix such that $PA_1P^T = A_2$.

* A matrix whose rows are the rows of unit matrix, but not necessarily in their natural order is called permutation matrix.

Q. Establish the isomorphism of 2 graph by considering their adjacency matrices.



Sol → A_1 :-

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & & & & \\ B & & & & \\ C & & & & \\ D & & & & \end{array} \quad A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

A_2 :-

$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{array}$$

Since A_1 & A_2 are not same. So, graph G_1 & G_2 are not isomorphic.

$\deg(A) = 3 \xrightarrow{1,1} \deg(v_1) = 3$

$\deg(B) = 2 \xrightarrow{2,2} \deg(v_2) = 2$

$\deg(C) = 2 \xrightarrow{3,4} \deg(v_4) = 2$

$\deg(D) = 3 \xrightarrow{4,3} \deg(v_3) = 3$

Identity Matrix $\quad\quad$ Permutation Matrix

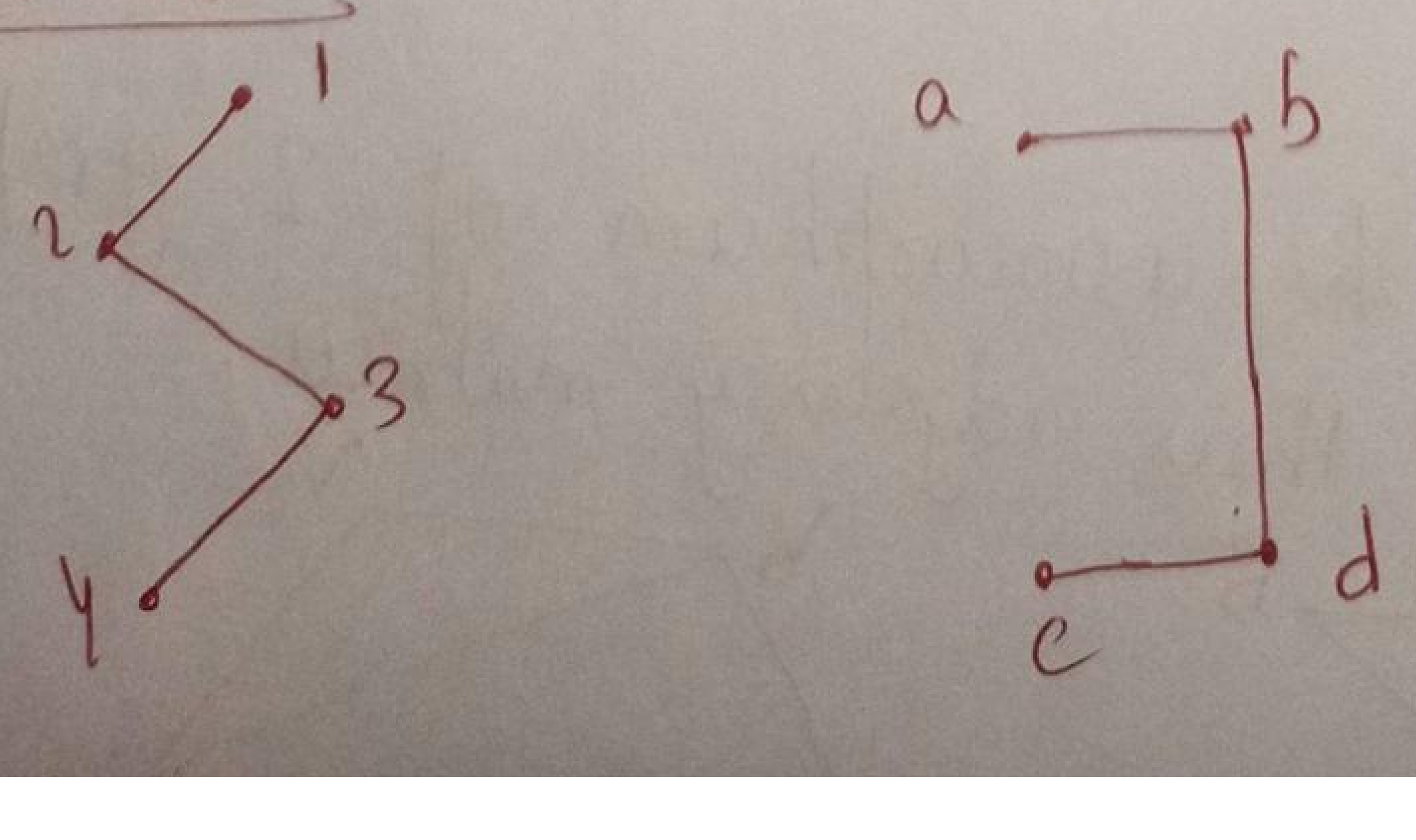
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{1,1;\ 2,2;\ 3,4;\ 4,3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

For this P, it can be easily verified $PA_1P^T = A_2$

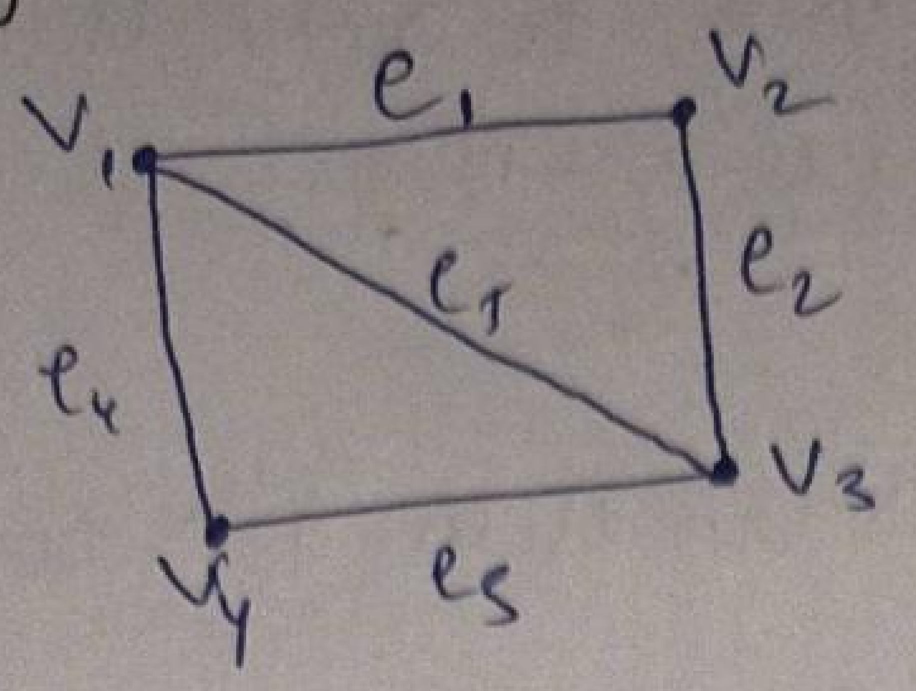
So, two graph G_1 & G_2 are isomorphic.

For pratise :-

Q



Paths, Cycles & Connectivity :-



A path in a graph is a finite alternating sequence of vertices and edges begining and ending with vertices, such that each edge is incident on the vertices preceding and following it

→ $v_1 e_1 v_2 e_2 v_3 e_3 v_4$ is a path.

If there is no common edge then it is called simple path

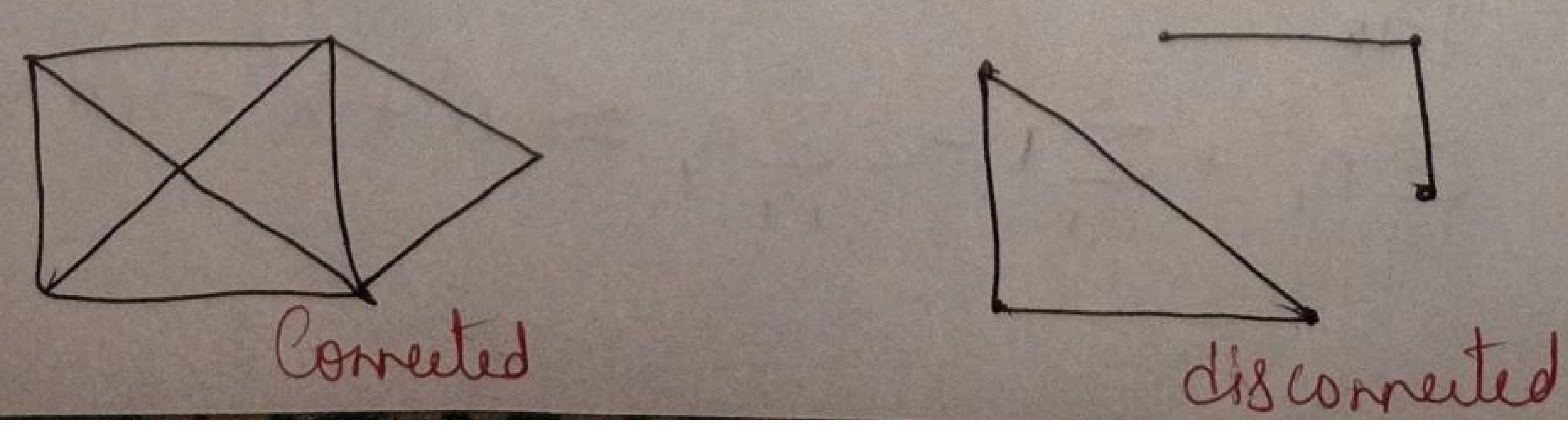
Circuit → If the initial & final vertices of a simple path of non-zero length is same then simple path is called a cycle or circuit.

$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$ is a cycle.

Max. Length of simple path ⇒ No. of edges.

Connected Graph :- An undirected graph is said to be connected if there is a path between every pair of distinct vertices of the graph.

Disconnected Graph :- A graph that is not connected.



Theorem :- The maximum no. of edge in a simple disconnected graph G with n vertices and k components is $\frac{(n-k)(n-k+1)}{2}$.

Proof :- Let the no. of vertices in i^{th} component be n_i

Then, $n_1 + n_2 + n_3 + \cdots n_k = n$

or $\sum_{i=1}^{k} n_i = n$.

$$\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$$

$$\sum_{i=1}^{k}(n_i - 1) = n - k$$

On squaring both side

$$\left[\sum_{i=1}^{k}(n_i - 1)\right]^2 = (n-k)^2$$

$$\Rightarrow \left[(n_1 - 1) + (n_2 - 1) + \cdots (n_k - 1)\right]^2 = n^2 + k^2 - 2nk$$

$$\Rightarrow (n_1-1)^2 + (n_2-1)^2 + \cdots (n_k-1)^2 + 2(n_1-1)(n_2-1) + \cdots 2(n_{k-1}-1)(n_k-1) = n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 + 2\sum_{i \neq j}^{k}(n_i-1)(n_j-1) = n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 \leq n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2\sum_{i=1}^{k} n_i \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - 2n \le n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \le n^2 + k^2 - 2nk - 2n - k \quad \text{--- (2)}$$

The maximum no. of edge in i^{th} component of G.

$$= \frac{n_i(n_i-1)}{2}$$

Total No. of edge in all component

$$= \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{k}(n_i^2 - n_i)$$

$$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum n_i$$

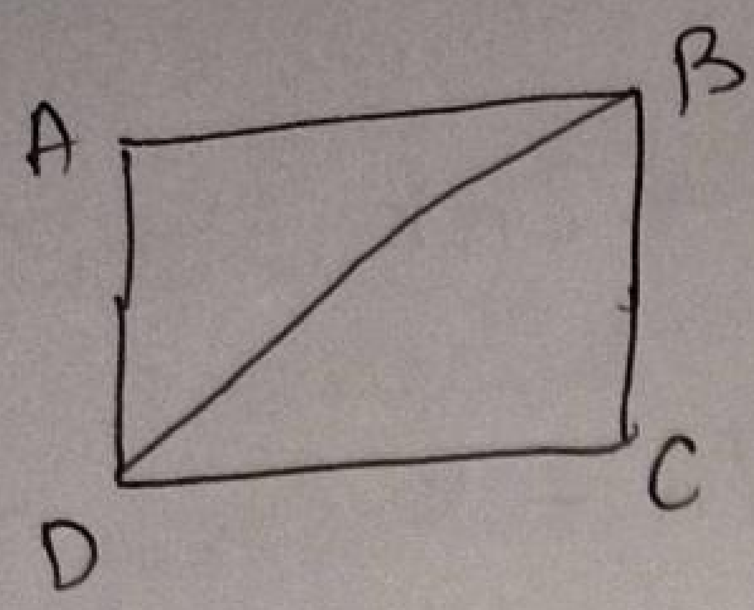
$$\le \frac{1}{2}\left[n^2 + k^2 - 2nk + 2n - k - n\right]$$

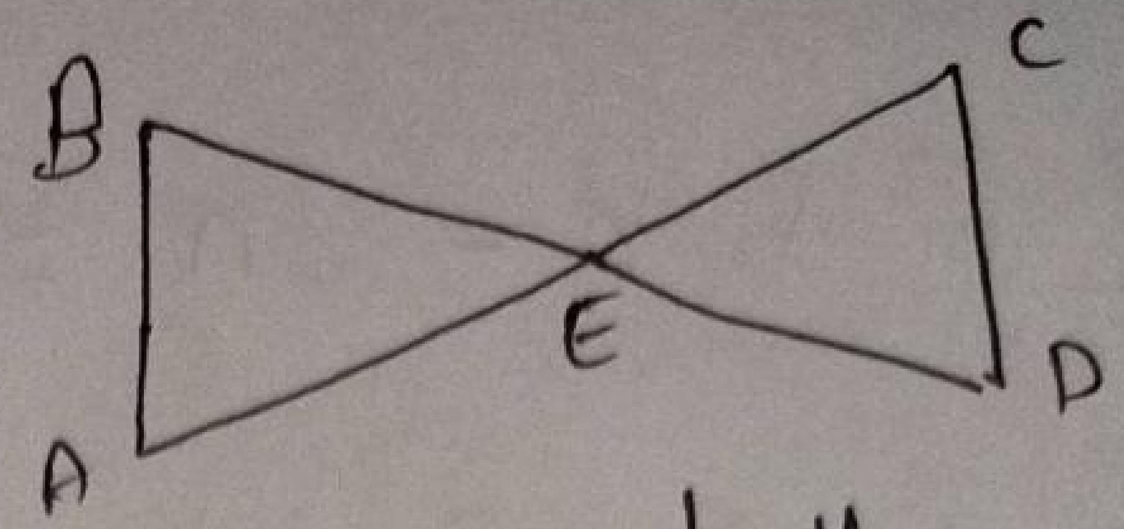
$$\le \frac{1}{2}\left[\frac{(n-k)^2 + (n-k)}{2}\right]$$

$$\le \frac{(n-k)(n-k+1)}{2}$$

Eulerian Graph :-

* A path of graph G is called an Eulerian path if it includes each edge of G exactly once.
* A circuit of graph G is called Eulerian circuit if it includes each edge of G exactly once.
* A graph containg eulerian circuit is called Eulerian graph





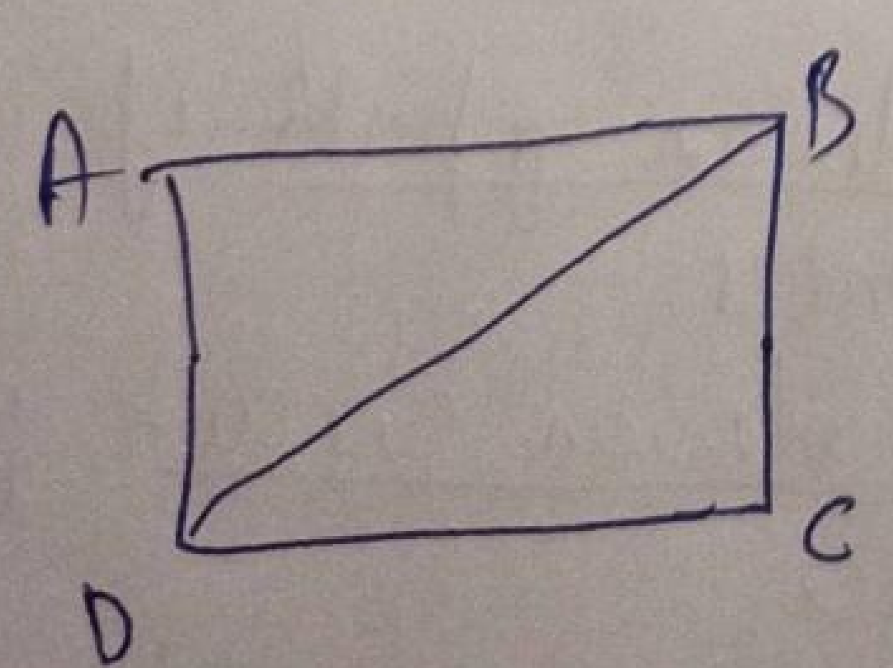
G_1 :- B - D - C - B - A - D → Eulerian path
G_2 :- ~~B~~ A - E - C - D - E - B - A → Eulerian circuit.
G_2 :- Eulerian Graph.

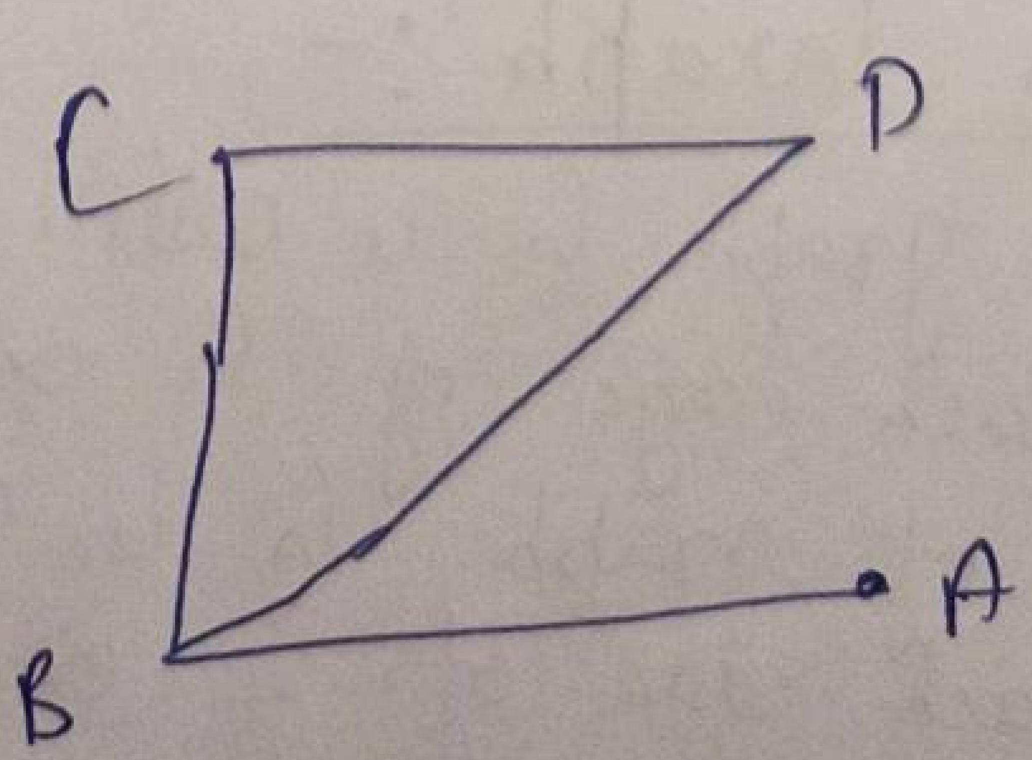
Necessary & Sufficient condition :-
1) A connected graph contain an Euler circuit if & only if each of it vertices is of even degree.
2) A connected graph contain Euler path iff ~~& on~~ it has exactly two vertices of odd degree.

Not → Euler path has odd degree vertices at end point.

Hamiltonian Graph :-

* A path of graph G is Hamiltonian if it includes each vertex of G exactly once.
* A circuit of graph G is Hamiltonian if it includes each vertex of G exactly once except the starting & end vertex.
* A graph containing Hamiltonian circuit is Hamiltonian graph.



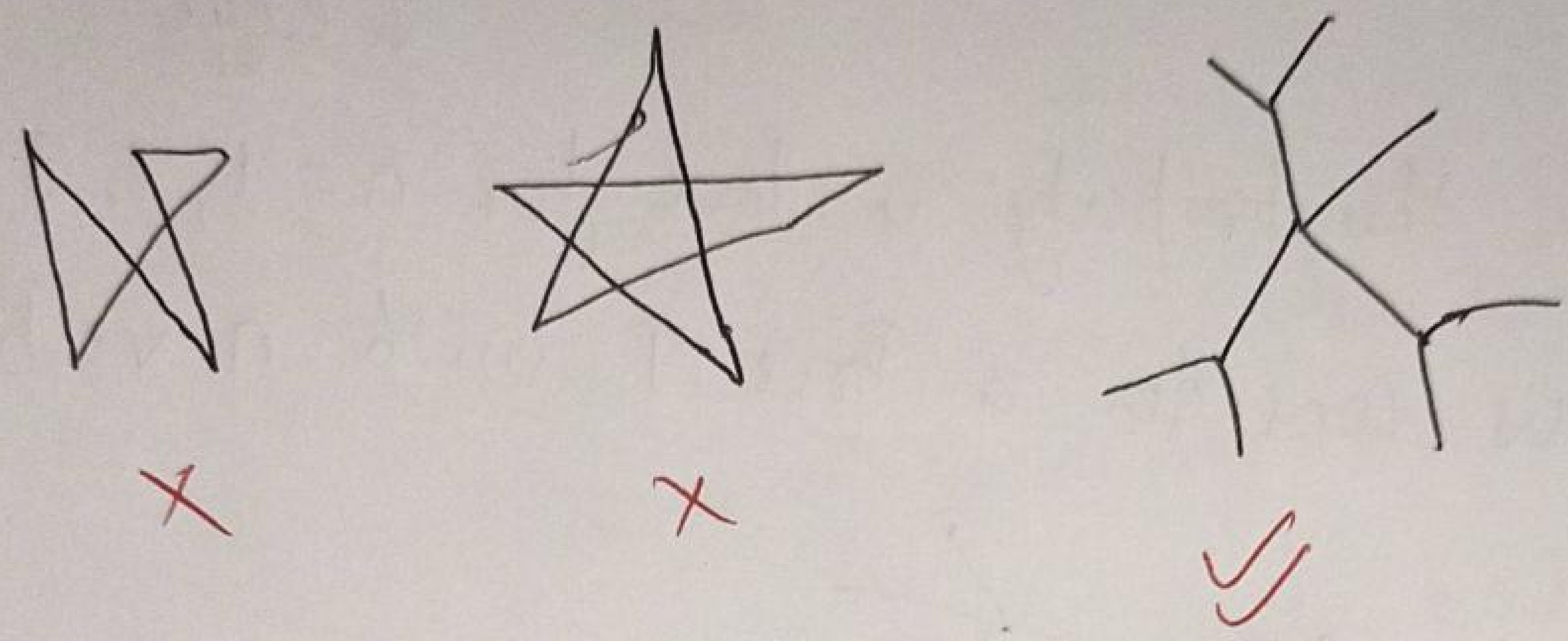


A - B - C - D - A
↳ circuit

A - B - C - D
↳ path but not cycle

Trees :-

A connected graph without any circuit is called a tree. So, obviously tree has to be a simple graph since loops and parallel edges are forming circuit.



Property 1 :- An undirected graph is a tree iff there is a unique simple path between every pair of vertices.

Proof :- We need to prove

Tree $\Longleftrightarrow$ Unique path b/w vertices.

(i) We assume graph T is a tree, then by definition of tree T is connected. T can't have circuit, so parallel edge or loop is not possible

$\Rightarrow$ Unique path between any two vertices.

(ii) Let a unique path exist between each pair of vertices in T,

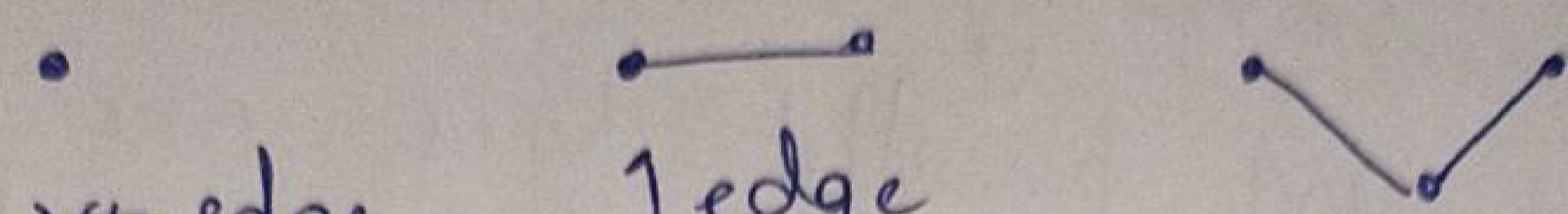
If possible Let ~~there~~ T ~~be~~ contain a circuit. This means that there is a pair of vertices v_1 & v_2 between which two distinct path exist which is against the data.

So, T cannot have circuit So, T is a tree

Imp

Property 2 :- A tree with n vertices has $(n-1)$ edges.

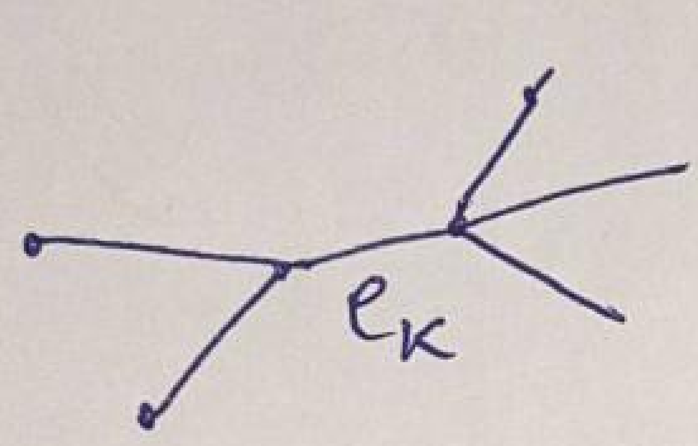
Proof :- $n=1$ $n=2$ $n=3$



0 ~~ve~~ edge 1 edge 2 edge

So, this property is true for $n=1, 2, 3$.

Let us consider a Tree T with n vertices.



Let e_k be the edge connecting the vertices v_i & v_j of T.
Then by property (1), e_k is only path between v_i & v_j.
If we delete e_k then T becomes disconnected and
contain 2 components namely T_1 & T_2.

Let T_1 contain k vertices $\Rightarrow (k-1)$ edge

Let T_2 contain $(n-k)$ vertex. $\Rightarrow (n-k)-1$ edge

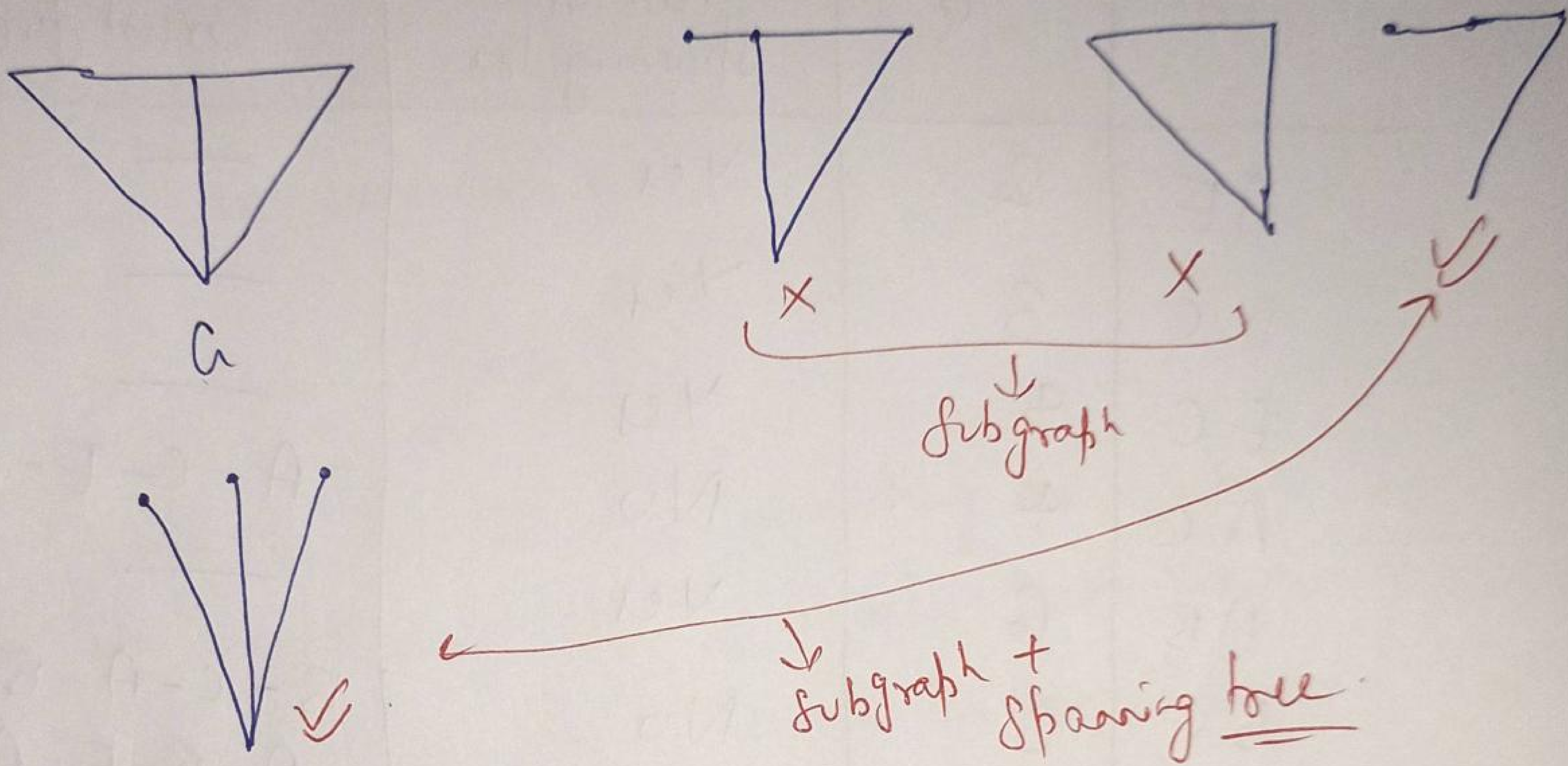
$$T_1 + T_2 = (k-1) + (n-k) - 1$$
$$\Rightarrow n-2 \text{ edge.}$$

Now, by adding that e_k edge which we remove

$$\text{No. of edges} = (n-2)+1$$
$$= (n-1) \text{ edge}$$

Spanning Tree :-

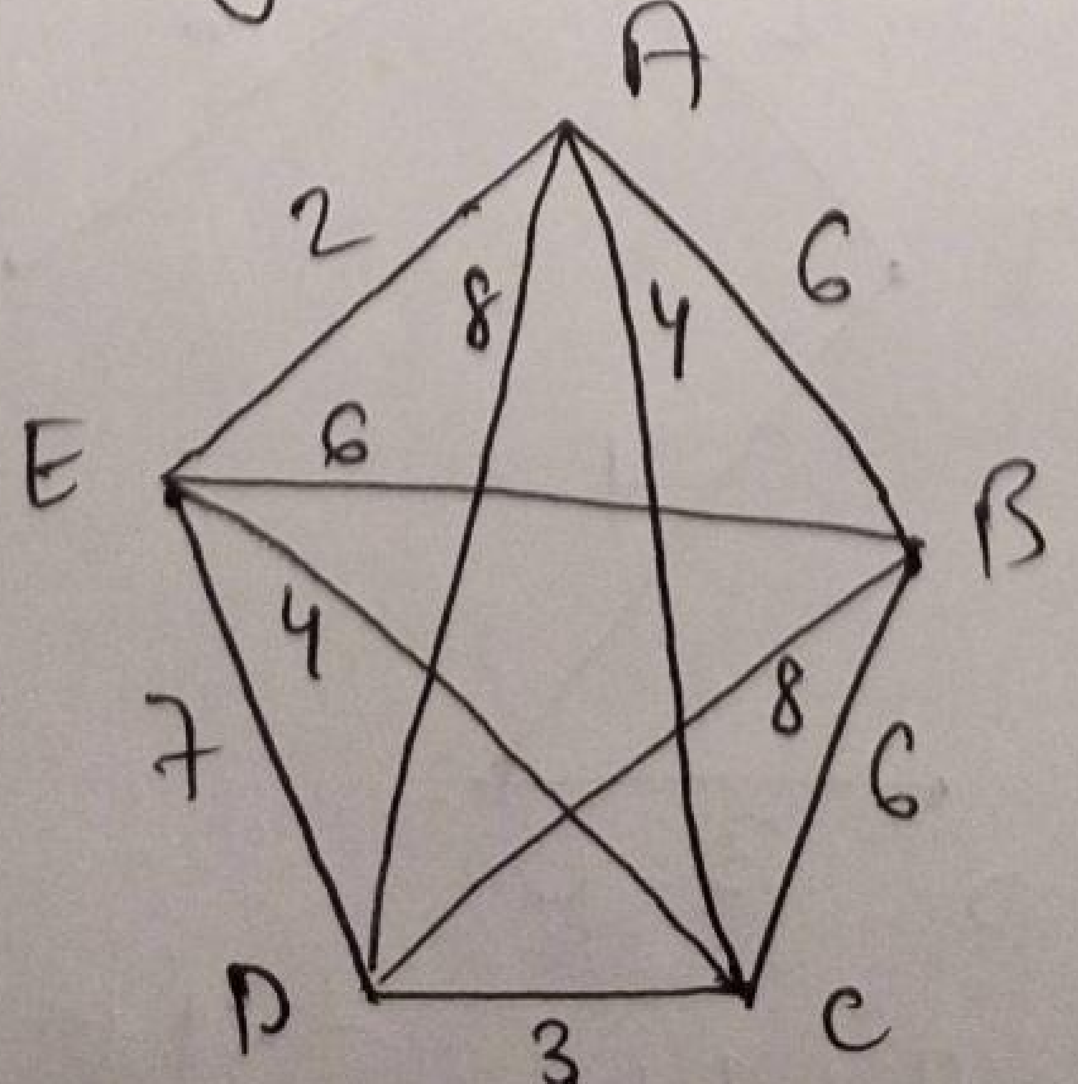
If the subgraph T of a connected graph G is a tree containing all the vertex of G, then T is called a spanning tree of G.



Minimum Spanning tree :- If G is a Connected weighted graph, the spanning tree of G with smallest total weight is called minimum spanning tree G.

Kruskal's Algorithm for MST :-

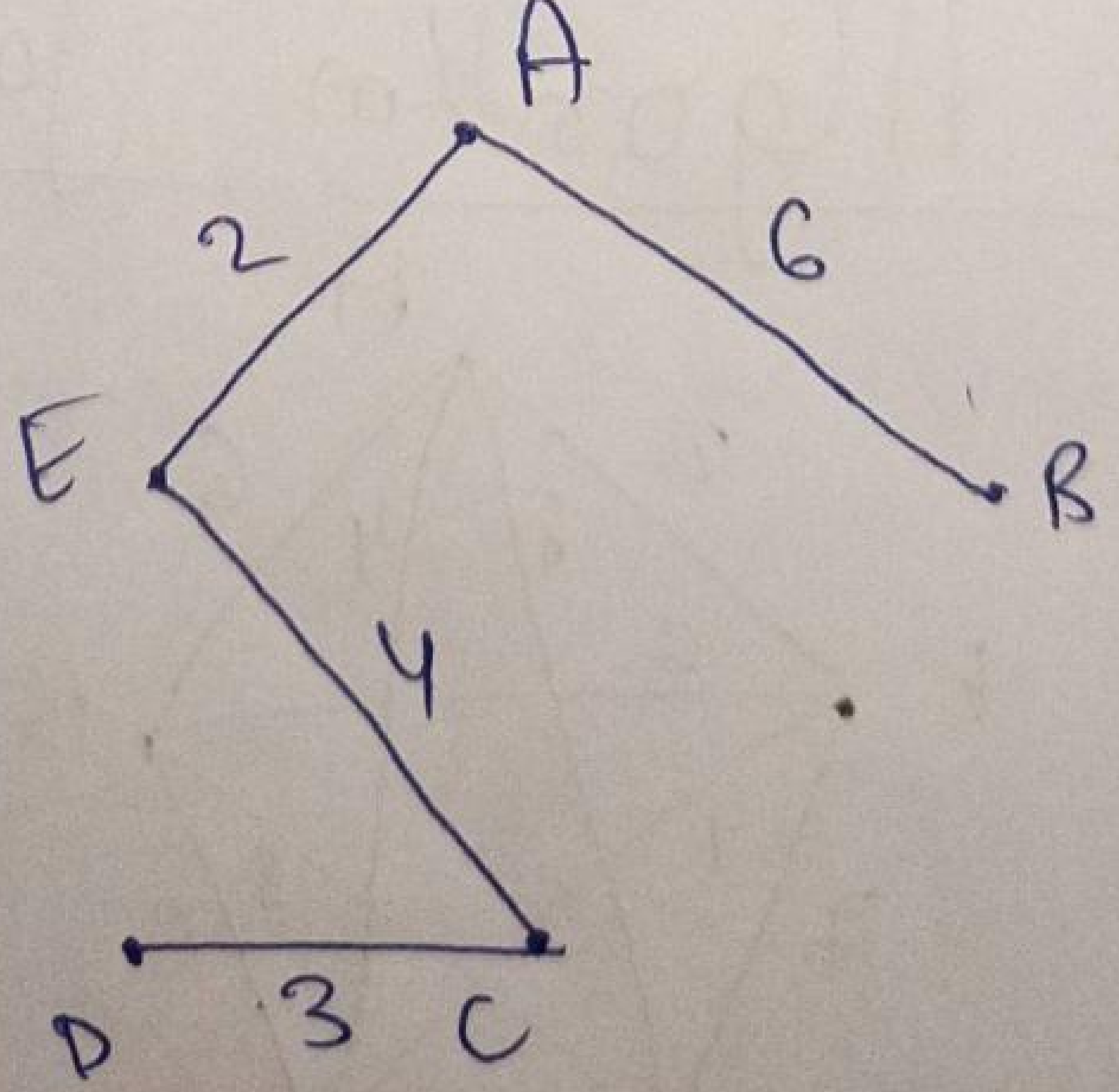
Q.



Sol → Step 1 → Arrange weight in ascending order :-

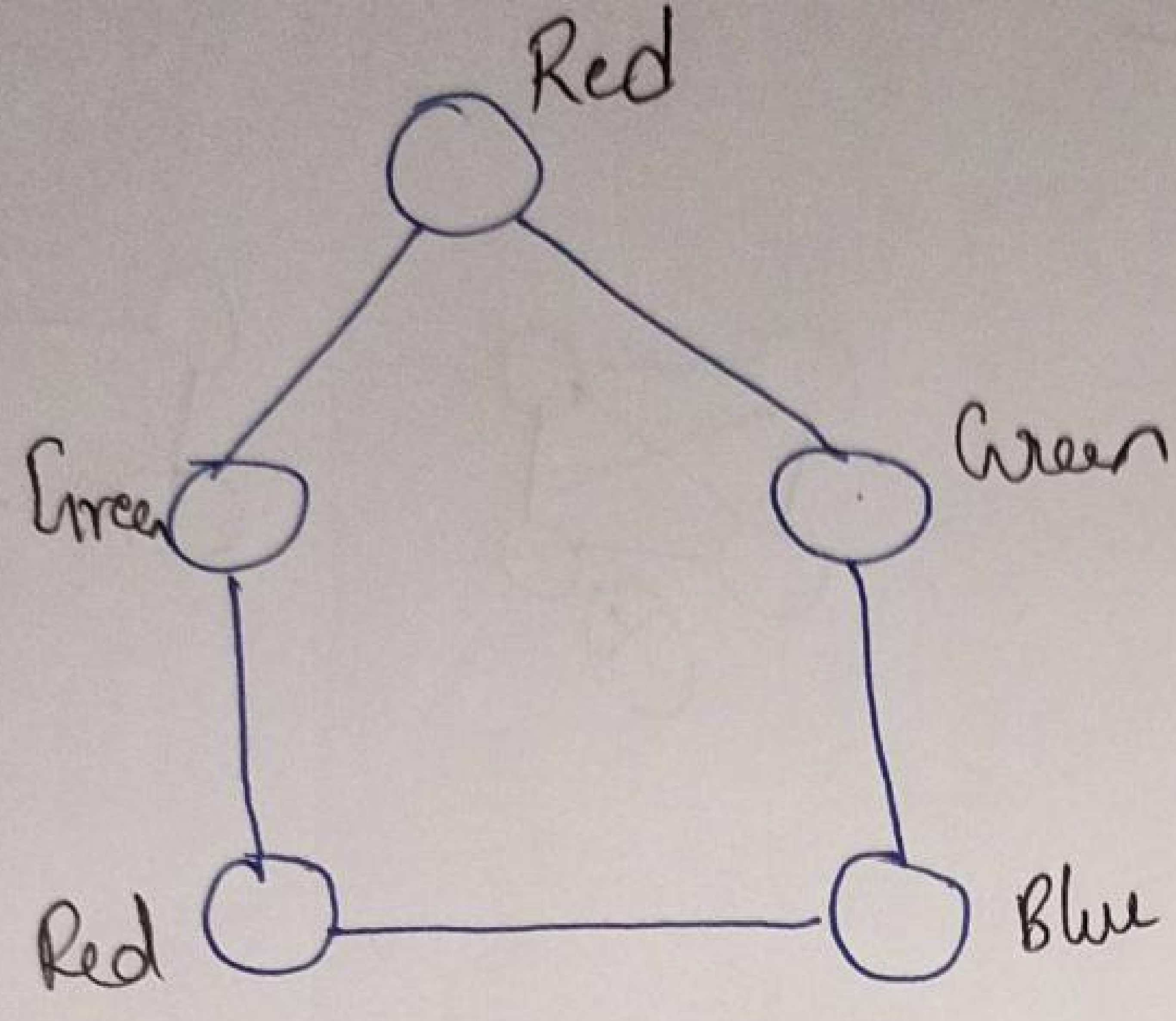
AE → 2 , DC → 3, EC → 4, AC → 4, AB → 6,
BC → 6, EB → 6 ED → 7, AD → 8 , BD → 8

edge	weight	Included or not in spanning tree	If not, then circuit formed
AE	2	Yes	—
DC	3	Yes	—
EC	4	Yes	—
AC	4	No	A-C-E-A
AB	6	Yes	—
BC	6	No	B-C-A-B
EB	6	No	A-B-E-A
ED	7	No	E-C-D-E
AD	8	No	A-D-C-E-A
BD	8	No	B-D-C-E-A-B



Weight = 2+3+4+6 = 15 Ans

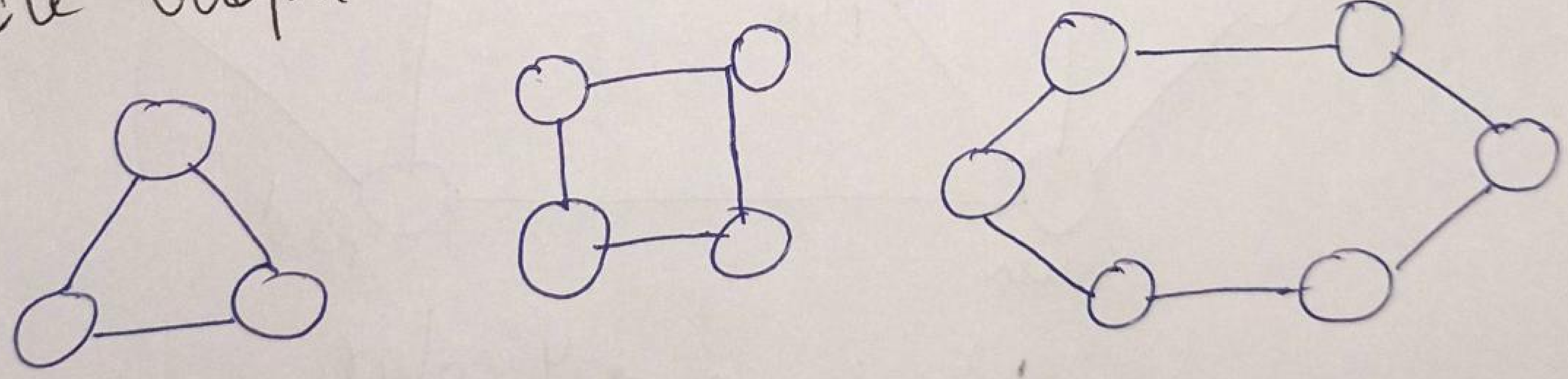
Graph Coloring :-



Chromatic No → Minimum No. of colors used to color graph.

$\chi(G) = K$ ↑ chromatic No.

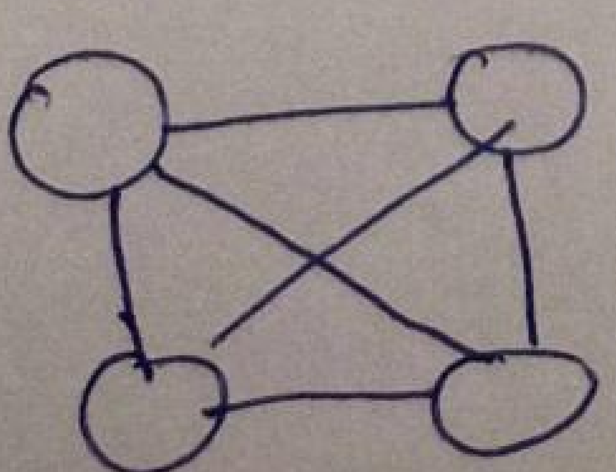
① Cycle Graph :-



No. of vertices is odd → $\chi(G) = 3$

No. of vertices is Even → $\chi(G) = 2$

② Complete Graph :-



No. of vertices → n

$\chi(G)$ = no. of vertices of graph

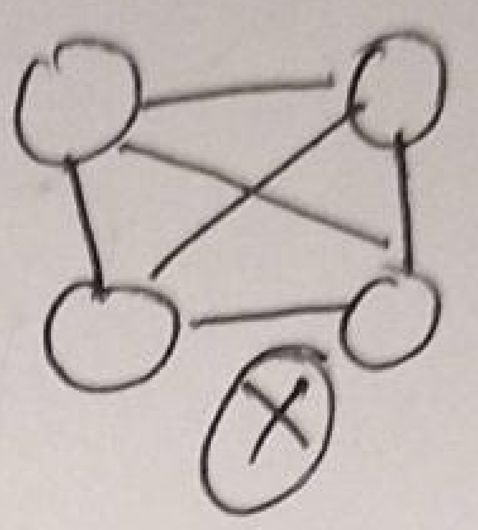
~~3 Bipolar :~~

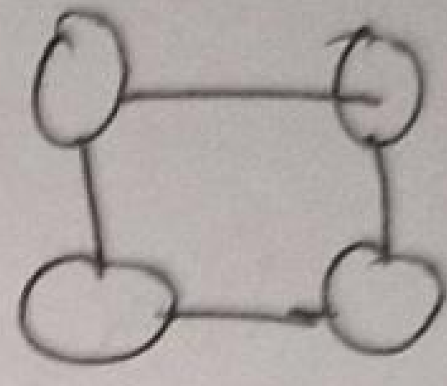
3 Biparted Graph :-

$$\chi(G) = 2$$

4 Planner Graph :-

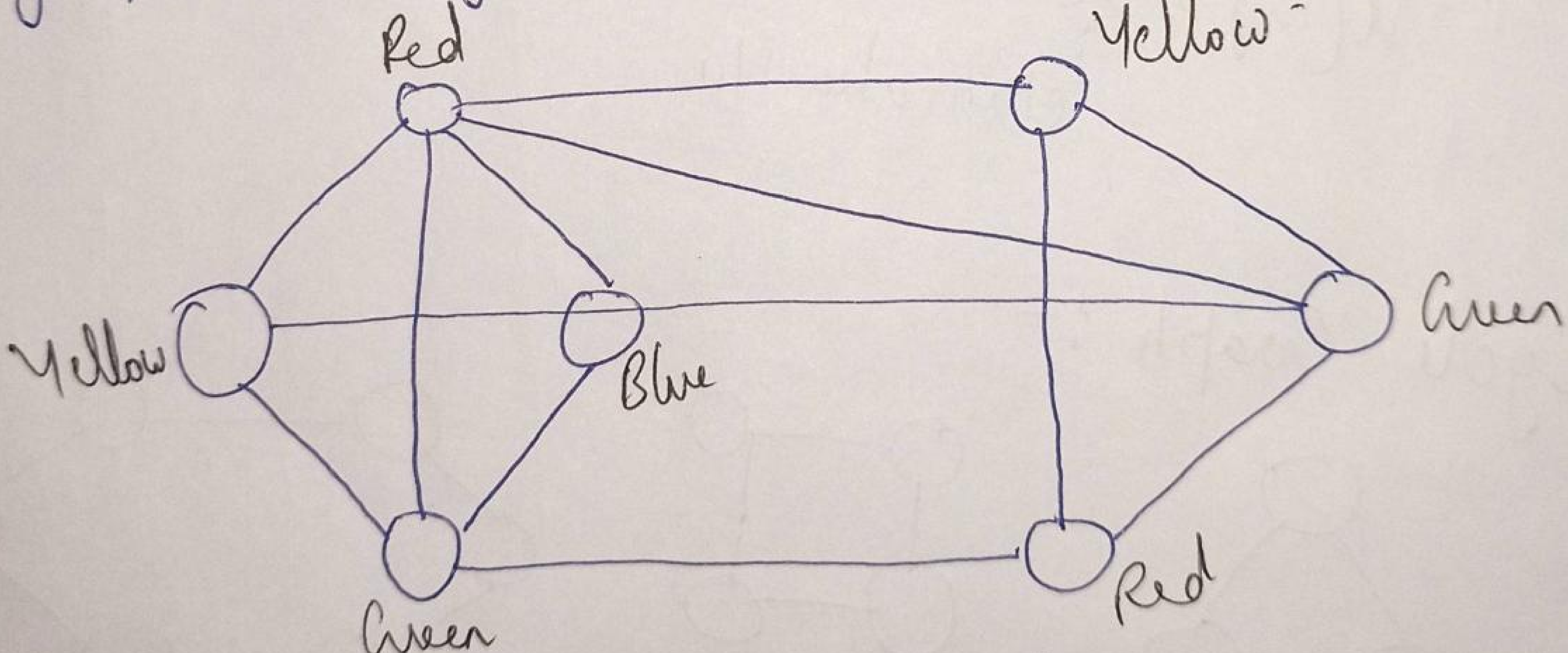
$$\chi(G) \le 4$$





Four Color Theorem :-

Any planner graph is atmost 4 colorable.



Maximum → 4 color