Maths

Unit - 1 :-

Partial differential Equation :-

$y = f(x)$ → one independent variable
⇒ ordinary.

$y = f(x, y)$ → More than one independent variable
⇒ partial.

Formation of PDE :-

Elimination of arbitrary constant

Elimination of arbitrary function.

* $p = \frac{\partial z}{\partial x}$
* $q = \frac{\partial z}{\partial y}$
* $r = \frac{\partial^2 z}{\partial x^2}$
* $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$
* $t = \frac{\partial^2 z}{\partial y^2}$

Remark :- 1. If no. of constant are equal to the no. of independent variable. then the resultant P.D.E will be of 1^{st} order.

2. If no. of constant > no. of independent variable, then resulting PDE will be of higher order.

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Method of finding P.D.E :-

Q. Form the partial differential equation by eliminating the arbitrary constants a and b from

$$z = (x+a)(y+b).$$

Solution → Given $z = (x+a)(y+b)$ ---- ①

Partially differentiat w.r.t x and y.

$$\frac{\partial z}{\partial x} = (y+b)\frac{\partial}{\partial x}(x+a)$$

$$p = (y+b)(1) \quad \text{---- ②}$$

$$\frac{\partial z}{\partial y} = (x+a)\frac{\partial}{\partial y}(y+b)$$

$$= (x+a)\cdot 1 \quad \text{---- ③}$$

Putting value of ② and ③ in ①.

$$z = pq.$$

Ans

Q Form the partial differential equation by eliminating the arbitrary constant a and b from

$$z = (x^2+a)(y^2+b).$$

Sol → Given $z = (x^2+a)(y^2+b)$ ---- ①

On paritially differentiat w.r.t x and y-

$$\frac{\partial z}{\partial y} = (x^2+a)\frac{\partial}{\partial y}(y^2+b)$$

$$q = (x^2+a)\cdot 2y \quad \text{---- ②}$$

$$\Rightarrow (x^2+a) = q/2y.$$

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$$\frac{\partial z}{\partial x} = (y^2+b)\frac{\partial}{\partial x}(x^2+a)$$

$$p = (y^2+b)\cdot 2x. \quad \text{---} \; (3)$$

$$\Rightarrow (y^2+b) = p/2x$$

Put the value of (2) and (3) in (1).

$$z = \frac{pq}{4xy}$$ **Ans**

Q Form the partial differential equation by eliminating the arbitrary constant a and b from $\log(az-1) = x+ay+b$.

Sol → Given :- $\log(az-1) = x+ay+b$ --- (1)

On differentiating partially w.r.t x and y.

$$\frac{1}{(az-1)}\, a\frac{\partial z}{\partial x} = 1$$

$$\Rightarrow \frac{1}{(az-1)}\, ap = 1 \quad \text{---} \; (2)$$

and $$\frac{1}{(az-1)}\, a\frac{\partial z}{\partial y} = a$$

$$\Rightarrow \frac{1}{(az-1)}\, aq = a \quad \text{---} \; (3)$$

On dividing 2 and 3, we get

$$\frac{1}{(az-1)}\, aq \times \frac{(az-1)}{ap} = \frac{a}{1}$$

$$\Rightarrow \frac{q}{p} = a$$

$$\Rightarrow q = ap$$

Putting this value of $q = ap$ in ①

$$\frac{1}{(az-1)} q = 1$$

$$\Rightarrow q = (az-1)$$

$$\Rightarrow q = \left(\frac{q}{p} z - 1\right)$$

$$\Rightarrow q = \frac{qz - p}{p}$$

$$\Rightarrow pq + p = qz$$

$$\Rightarrow qz = p(q+1)$$ Ans

Q Form the partial differential equation by eliminating the 3 arbitrary constant a, b, c from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Sol→ Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — ①

On differentiating eq. ① w.r.t x and y partially

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad \text{— ②}$$

$$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \quad \text{— ③}$$

Paritially differentiating eq. ② w.r.t. y.

$$0 + \frac{2}{c^2}\frac{\partial}{\partial y}\left[z\frac{\partial z}{\partial x}\right] = 0$$

$$\Rightarrow \quad z\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad zs + pq = 0$$

Ans

Q Form the partial differential equation by eliminating the arbitrary constant a and b from $(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha$ where α is not an arbitrary constant.

$$(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha \quad \text{--- (1)}$$

Sol→ Given :- $(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha$

On differentiating w.r.t x and y partially.

$$2(x-a) = 2z\cot^2\alpha\,\frac{\partial z}{\partial x}$$

$$\Rightarrow \quad (x-a) = pz\cot^2\alpha \quad \text{--- (2)}$$

and $$2(y-b) = 2z\cot^2\alpha\,\frac{\partial z}{\partial y}$$

$$\Rightarrow \quad (y-b) = qz\cot^2\alpha \quad \text{--- (3)}$$

On squaring eq. (2) and (3) and putting the value in eq. (1), we get

$$p^2z^2\cot^4\alpha + q^2z^2\cot^4\alpha = z^2\cot^2\alpha.$$

$$\Rightarrow \quad z^2\cot^4\alpha\,(p^2+q^2) = z^2\cot^2\alpha$$

$$\Rightarrow \quad (p^2+q^2) = \frac{z^2\cot^2\alpha}{z^2\cot^4\alpha}$$

$$\Rightarrow \quad p^2+q^2 = \frac{1}{\cot^2\alpha}$$

$$\Rightarrow \quad p^2+q^2 = \tan^2\alpha$$

Ans

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Q Form the partial differential equation by eliminating the arbitrary constant from $z = ax^n + by^n$.

Sol→ Given $z = ax^n + by^n$ ---- (1)

On differentiating w.r.t x and y partially.

$$\frac{\partial z}{\partial x} = nax^{n-1}$$

$$\Rightarrow p = nax^{n-1}$$

$$\Rightarrow xp = nax^n$$

$$\Rightarrow ax^n = \frac{xp}{n} \quad \text{---} \quad (2)$$

and.

$$\frac{\partial z}{\partial y} = nby^{n-1}$$

$$\Rightarrow q = nby^{n-1}$$

$$\Rightarrow yq = nby^n$$

$$\Rightarrow by^n = \frac{yq}{n} \quad \text{------} \quad (3)$$

Putting the value of (2) and (3) in eq. (1)

$$\Rightarrow z = \frac{xp}{n} + \frac{yq}{n}$$

$$\Rightarrow nz = xp + yq.$$

Ans

By elimination of arbitrary functions :-

Let u and v be any two given functions of $x, y,$ and z. Let u and v be connected by an arbitrary function ϕ by the relation

$$\phi(u, v) = 0 \quad \text{---} \quad (1)$$

Now, we want to eliminate ϕ.

Differentiating partially w.r.t x and y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \cdot p\right) = 0 \quad \text{---} \quad (2)$$

and
$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \cdot q\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \cdot q\right) = 0 \quad \text{---} \quad (3)$$

Instead of eliminating ϕ, we eliminate $\frac{\partial \phi}{\partial u}$ & $\frac{\partial \phi}{\partial v}$, we get an equation of the form.

$$\boxed{Pp + Qq = R}$$

This is called lagrange's linear equation.

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (3) and (2).

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = 0$$

which simplified to

$$Pp + Qq = R$$

where,

$$P = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y} = \frac{\partial(u,v)}{\partial(y,z)}$$

$$Q = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} = \frac{\partial(u,v)}{\partial(z,x)}$$

$$R = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} = \frac{\partial(u,v)}{\partial(x,y)}$$

Q Form the P.D.E by eliminating arbitrary function from
$$z = f(x^2+y^2).$$

Sol→ $z = f(x^2+y^2)$ ---------- (1)

On differentiating w.r.t x and y partially.

$\frac{\partial z}{\partial x} = p = f'(x^2+y^2)\cdot 2x$ ------- (2)

$\frac{\partial z}{\partial y} = q = f'(x^2+y^2)\cdot 2y$ ------- (3)

Eq. (2) / (3)

$\frac{p}{q} = \frac{x}{y}$

$\Rightarrow py - xq = 0$ Ans

Q Form the P.D.E by eliminating arbitrary ~~constant~~ function from
$$z = x^2 + 2f\left(\frac{1}{y} + \log x\right).$$

Sol→ $z = x^2 + 2f(1/y + \log x)$ -------- (1)

$\frac{\partial z}{\partial x} = 2x + 2f'\left(\frac{1}{y} + \log x\right)\cdot \frac{1}{x}$

$\Rightarrow p = \frac{2x^2 + 2f'\left(\frac{1}{y} + \log x\right)}{x}$

$\Rightarrow px - 2x^2 = 2f'\left(\frac{1}{y} + \log x\right)$ ---- (2)

$\frac{\partial z}{\partial y} = 2f'\left(\frac{1}{y} + \log x\right)\cdot\left(\frac{-1}{y^2}\right)$

$\Rightarrow -qy^2 = 2f'\left(\frac{1}{y} + \log x\right)$. ------ (3)

$\Rightarrow px - 2x^2 = -qy^2$ [on putting (3) in (2)]

$\Rightarrow xp + y^2q = 2x^2$ Ans

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Q Form P.D.E by eliminating arbitrary function from.

$$z = f(x+ct) + \phi(x-ct).$$

Sol $\rightarrow$ $z = f(x+ct) + \phi(x-ct)$ — — (1)

$$\frac{\partial z}{\partial x} = f'(x+ct)\cdot(1) + \phi'(x-ct)\cdot(1) \quad \text{— — (2)}$$

$$\frac{\partial z}{\partial t} = f'(x+ct)\cdot(c) + \phi'(x-ct)\cdot(-c) \quad \text{— (3)}$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct)\cdot(1) + \phi''(x-ct)\cdot(1) \quad \text{— (4)}$$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+ct)\cdot(c^2) + \phi'(x-ct)\cdot(c^2) \quad \text{— (5)}$$

$$\frac{\partial^2 z}{\partial t^2} = c^2\left[f''(x+ct) + \phi''(x-ct)\right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Ans

Q Form P.D.E by eliminating arbitrary function from.

$$\phi(x^2+y^2+z^2,\ lx+my+nz) = 0$$

Sol $\rightarrow$ $\phi(x^2+y^2+z^2,\ lx+my+nz) = 0$

Let $u = x^2+y^2+z^2$

$v = lx+my+nz$

Now, the P.D.E is given by

$$Pp + Qq = R$$

where,

$$P = \frac{\partial(u,v)}{\partial(y,z)} = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y} = (2y)(n) - (2z)(m)$$

$$Q = \frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z} = (2z)(l) - (2x)(n)$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} = (2x)(m) - (2y)\cdot(l)$$

Putting value of P, Q, R in $Pp + Qq = R$.

$$\Rightarrow (2ny - 2mz)p - (2zl - 2xn)q = (2xm - 2yl).$$

$$\Rightarrow (ny - mz)p - (zl - xn)q = (xm - yl)$$ Ans

2nd Method

Sol → Let $x^2 + y^2 + z^2 = f(lx + my + nz)$

Partially differentiate w.r.t x and y.

$$2x + 2z\frac{\partial z}{\partial x} = f'(lx + my + nz)\cdot\left(l + n\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow 2x + 2zp = f'(lx + my + nz)\cdot(l + np) \quad \text{--- (1)}$$

$$2y + 2z\frac{\partial z}{\partial y} = f'(lx + my + nz)\cdot\left(m + n\frac{\partial z}{\partial y}\right)$$

$$\Rightarrow 2y + 2zq = f'(lx + my + nz)\cdot(m + nq) \quad \text{--- (2)}$$

On dividing (2) and (3)

$$\frac{2x + 2zp}{2y + 2zq} = \frac{f'(lx + my + nz)\cdot(l + np)}{f'(lx + my + nz)\cdot(m + nq)}$$

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$$\Rightarrow \frac{2x+2zp}{2y+2zq} = \frac{l+np}{m+nq}$$

$$\Rightarrow (2x+2zp)(m+nq) = (l+np)(2y+2zq).$$

$$\Rightarrow \cancel{2}[mx + xnq + mzp + zpnq] = \cancel{2}[ly + lzq + ynp + npzq]$$

$$\Rightarrow [mx + xnq + mzp + \cancel{zpnq} - ly - lzq - ynp - \cancel{npzq}$$

$$\Rightarrow (mx - ly) + (xn - lz)q + (mz - yn)p.$$

$$\Rightarrow (zm - ny)p + (xn - lz)q = (ly - mx)$$

Ans

Q Form P.D.E by eliminating arbitrary function from

$$\phi\left(z^2 - xy, \frac{x}{z}\right) = 0.$$

Sol →

$$\phi\left(z^2 - xy, \frac{x}{z}\right) = 0$$

Let $u = z^2 - xy$. $v = \frac{x}{z}$.

Now, the P.D.E is given by.

$$Pp + Qq = R \quad - - - - (1)$$

where,

$$P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial v}{\partial z} \cdot \frac{\partial u}{\partial y} = (-x) \cdot \left(\frac{-x}{z^2}\right) - (2z) \cdot (0)$$

$$\Rightarrow P = \frac{x^2}{z^2} \quad - - (2)$$

$$Q = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial v}{\partial z}\cdot\frac{\partial u}{\partial x} = (2z)\cdot\left(\frac{1}{z}\right) - (-y)\cdot\left(\frac{-x}{z^2}\right).$$

$$\Rightarrow Q = \left(2 - \frac{xz}{z^2}\right) \quad \text{--- (3)}$$

$$R = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} = (-x)\cdot 0 - (-x)\cdot\left(\frac{1}{z}\right)$$

$$\Rightarrow R = \frac{x}{z}. \quad \text{--- (4)}$$

Putting (2), (3) (4) in eq. (1)

$$\frac{x^2}{z^2}p + \left(2 - \frac{xz}{z^2}\right)q = \frac{x}{z}.$$

$$= \frac{x^2p + (2z^2 - xy)q}{z^2} = \frac{x}{z}.$$

$$\Rightarrow x^2p + (2z^2 - xy)q = \frac{x}{z}\cdot z^2$$

$$\Rightarrow x^2p + (2z^2 - xy)q = xz \quad \text{Ans}$$

Q. Form the partial differential equation by eliminating arbitrary function from $z = f(ax+by) + g(\alpha x + \beta y)$.

Sol → $z = f(ax+by) + g(\alpha x + \beta y)$ --- (1)

Partially differentiating w.r.t x and y.

$$\frac{\partial z}{\partial x} = f'(ax+by)\cdot a + g'(\alpha x + \beta y)\cdot \alpha. \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = f'(ax+by)\cdot b + g'(\alpha x + \beta y)\cdot \beta. \quad \text{--- (3)}$$

On Partially differentiating ② and ③ with respect to x and y respectively.

$$\frac{\partial^2 z}{\partial x^2} = f''(ax+by)a^2 + g''(\alpha x+\beta y)\cdot\alpha^2 \quad \text{--- ④}$$

$$\frac{\partial^2 z}{\partial y^2} = f''(ax+by)b^2 + g''(\alpha x+\beta y)\cdot\beta^2 \quad \text{--- ⑤}$$

On differentiating eq. ② w.r.t. y.

$$\frac{\partial^2 x}{\partial x\partial y} = f''(ax+by)ab + g''(\alpha x+\beta y)\alpha\beta \quad \text{--- ⑥}$$

Now, by $(b\beta)\times④ + (\alpha a)⑥ - (a\beta+b\alpha)⑤$ gives

$$\Rightarrow b\beta\frac{\partial^2 z}{\partial x^2} + \alpha a\frac{\partial^2 z}{\partial y^2} - (a\beta+b\alpha)\frac{\partial^2 z}{\partial x\partial y}$$

$$= b\beta\left[f''(ax+by)a^2 + g''(\alpha x+\beta y)\cdot\alpha^2\right]$$

$$-(\alpha a+\beta b)\left[f''(ax+by)ab + g''(\alpha x+\beta y)\alpha\beta\right]$$

$$+\alpha a\left[f''(ax+by)b^2 + g''(\alpha x+\beta y)\beta^2\right]$$

$$\Rightarrow f''(ax+by)a^2b\beta + g''(\alpha x+\beta y)\cdot\alpha^2 b\beta -$$

$$f''(ax+by)a^2b\alpha - f''(ax+by)b^2a\beta -$$

$$g''(\alpha x+\beta y)\alpha^2 a\beta - g''(\alpha x+\beta y)\beta^2\alpha b +$$

$$f''(ax+by)b^2\alpha a + g''(\alpha x+\beta y)\beta^2\alpha a$$

$$\Rightarrow b\beta r + \alpha a t - (\alpha\beta+b\alpha)s = 0$$

Ans

Solution of P.D.E

1. Complet Solution / Integral
   ↳ Same no. of variable & Constant

2. Particular Integral
   ↳ when arbitrary const. can be calculated

3. Singular Integral :-

$$f(x, y, z, p, q) = 0$$

$$f(x, y, z, a, b) = 0$$

$$\frac{\partial f}{\partial a} = 0, \quad \frac{\partial f}{\partial b} = 0$$

4. General Solution :-
   ↳ a, b

$$b = \phi(a).$$ [all in one variable]

(least Imp. Topic) **Solution of PDE by Direct Integration**

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Q Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x$

Sol → Integrating w.r.t x.

$$\frac{\partial z}{\partial y} = -\cos x + f(y).$$

When we Integrate w.r.t x then const will be of y. and when with y, then will be function of x

Integrate w.r.t y.

$$Z = -y\cos x + f'(y) + \phi(x).$$

Ans

Q. Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$ given that $u=0$ when $t=0$ and $\frac{\partial u}{\partial t} = 0$ when $x=0$.

Sol→ $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$. — ①

Integrat partially w.r.t x.

$\frac{\partial u}{\partial t} = e^{-t}(\sin x) + \phi(t)$. — ②

At $x=0$, $\frac{\partial u}{\partial t} = 0$

$\Rightarrow 0 = 0 + \phi(t)$.

$\Rightarrow \phi(t) = 0$.

Putting it in ② and Integrat w.r.t. t.

$u = -e^{-t}\sin x + f(x)$. — ③

At $t=0$, $u=0$.

$\Rightarrow 0 = -e^{0}\sin x + f(x)$

$\Rightarrow f(x) = \sin x$.

Putting it in eq. ③. we get

$u = -e^{-t}\sin x + \sin x$

$u = \sin x(1-e^{-t})$ Ans.

Method to Solve 1st Order partial Diff. eq^n.

Type 1 :- $F(p,q) = 0$.

Then the solution will be $z = ax + by + c$.

Q. Solve $p^2 + q^2 = 1$. --- (1)

Sol→ Let the solution is $z = ax + by + c$. --- (2)

Now, $p = \frac{\partial z}{\partial x} = a$, $q = \frac{\partial z}{\partial y} = b$.

putting values in eq. (1)

$$a^2 + b^2 = 1$$
$$b = \sqrt{1 - a^2}$$

Putting the values of b in eq. (2)

$$z = ax + \sqrt{1-a^2}\, y + c \quad \text{--- (3)}$$

This is required solution.

Is it singular solution?

$$\frac{\partial}{\partial c}(z) = 0 + 0 + 1$$
$$\Rightarrow 0 = 1 \leftarrow \text{Not possible.}$$

So, No singular solution

General solution :-

Let $c = \phi(a)$.

put it in eq. (3)

$$z = ax + \sqrt{1-a^2}\, y + \phi(a). \quad \text{--- (4)}$$

Now, differentiat w.r.t a.

$$\frac{\partial z}{\partial a} 0 = x + \frac{(-2a)}{2\sqrt{1-a^2}} y + \phi'(a). \quad \text{--- (5)}$$

Eliminating a between (4) and (5) will give general solution.

Q $p^2 + q^2 = npq.$

Sol→ $p^2 + q^2 = npq$ — — — (1)

Let solution will be $z = ax + by + c$ — — (2)

Then ~~p = a~~ on putting $p = a$, $q = b$ in eq (1)

$$a^2 + b^2 = nab.$$

$$a^2 - nab + b^2 = 0.$$

$$\Rightarrow b = \frac{na \pm \sqrt{n^2a^2 - 4 \cdot 1 a^2}}{2}$$

$$\Rightarrow b = \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right]$$

Putting this value in eq. (2)

$$z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2-4}\right] y + c \quad \text{--- (3)}$$

This is required solution.

It is not singular solution

General solution :-

Let $c = \phi(a)$.

Put it in eq. (3)

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$$Z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2-4}\right]y + \phi(a) \quad \text{---} \quad (4)$$

On differentiating w.r.t a partially.

$$\frac{\partial z}{\partial a}\ 0 = x + \frac{n \pm \sqrt{n^2-4}}{2}\,y + \phi'(a). \quad \text{---} \quad (5)$$

Eliminating a from (4) & (5) will give General Solution

Q. Solve $\sqrt{p} + \sqrt{q} = 1$.

Sol $\rightarrow$ $\sqrt{p} + \sqrt{q} = 1$ --- (1)

Solution will be $z = ax + by + c$ --- (2)

Put $p = a$, $q = b$ in eq. (1).

$$\sqrt{a} + \sqrt{b} = 1.$$

$$\Rightarrow \sqrt{b} = 1 - \sqrt{a}$$

$$b = (1-\sqrt{a})^2$$

Putting this in eq. (2), we get

$$Z = ax + (1-\sqrt{a})^2 y + c \quad \text{---} \quad (3)$$

This is required solution.

It is not singular solution.

General Solution :-

Let $c = \phi(a)$.

Putting it in eq. (3).

$$Z = ax + (1+\sqrt{a})^2 y + \phi(a) \quad \text{---} \quad (4)$$

Differentiating partially w.r.t a.

$$\frac{\partial z}{\partial a} = x + 2y(1+\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right) + \phi'(a).$$

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$$0 = \frac{\partial z}{\partial a} = x + \frac{2(1-\sqrt{a})}{2\sqrt{a}} y + \phi'(a) \quad \text{--- (5)}$$

Eliminating a from (4) & (5) gives the General solution.

Type II :- Clairaut's form

If eq. is of form

$$z = px + qy + f(p, q) \quad \text{--- (1)}$$

then the solution is

$$z = ax + by + f(a, b) \quad \text{--- (2)}$$

To find singular solution,

Differentiat partially w.r.t a & b.

$$\Rightarrow 0 = x + \frac{\partial f}{\partial a} \quad \text{--- (3)}$$

$$\Rightarrow 0 = y + \frac{\partial f}{\partial a} \quad \text{--- (4)}$$

To find general solution.

$$b = \phi(a).$$

$$z = ax + \phi(a) + f[a, \phi(a)]$$

Differentiat partially w.r.t a

and then by both eq. on Eliminating a we get general sol.

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Q Solve $z = px + qy + p^2 - q^2$.

Sol→ $z = px + qy + p^2 - q^2$ ——— (1)

Then the solution :-

$z = ax + by + a^2 - b^2$ ——— (2)

This is the complete solution.

To find singular solution,

Differentiate eq. (2) w.r.t 'a' & 'b' partially.

$0 = x + 2a$

$0 = y - 2b$

$x = -2a \Rightarrow a = -x/2$

$y = 2a \Rightarrow b = \frac{y}{2}$

Put the value of a and b in eq. (2)

$$z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

$$z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4}$$

$$4z = y^2 - x^2 \quad \text{Ans}$$

General solution :-

Let $b = \phi(a)$

Put it in eq. (2)

$z = ax + \phi(a)y + a^2 - [\phi(a)]^2$ ——— (3)

Differentiat eq. (3) w.r.t a partially.

$0 = x + \phi'(a)y + 2a - 2\phi(a)[\phi'(a)]$ ——— (4)

Eliminating (3) and (4), we get general solution.

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Q Solve $z = px + qy + p^2q^2$

Sol→ $z = px + qy + p^2q^2$ ---- ①

Then the solution is

$z = ax + by + a^2b^2$ ---- ②

This is complete solution.

To find singular solution,
differentiate eq. ② w.r.t 'a' & 'b' partially.

$\Rightarrow 0 = x + 2ab^2$

$\Rightarrow 0 = y + 2ba^2$

$\Rightarrow x = -2ab^2 \quad \Rightarrow \frac{x}{b} = -2ab$

$\Rightarrow y = -2a^2b \quad \Rightarrow \frac{x}{a} = -2ab.$

So, $\frac{x}{b} = \frac{y}{a} = -2ab = \frac{1}{K}$

I II ~~III~~ IV

from I and III.

$b = xk.$

from II & III,

$a = ky.$

also, on putting these values in eq.

$x = -2ab^2$; we get

$x = -2(ky)(kx)^2$

$\Rightarrow k^3 = \frac{-1}{2xy}$

On putting value of a and b in eq. (2), we get

$$Z = kyx + kxy + k^2y^2 \cdot k^2x^2$$

$$= 2kyx + k^4x^2y^2$$

$$= 2kxy + kx^2y^2(k^3)$$

$$= 2kxy + kx^2y^2\left(\frac{-1}{2xy}\right)$$

$$= 2kxy - \frac{kxy}{2}$$

$$Z = \frac{3kxy}{2}$$

On cube of this eq.

$$Z^3 = \frac{27k^3x^3y^3}{8}$$

$$Z^3 = \frac{27\left(-\frac{1}{2xy}\right)x^3y^3}{8}$$

$$16Z^3 = -27x^2y^2.$$

$$16z^3 + 27x^2y^2 = 0$$ Ans

General solution :-

Let $b = \phi(a)$

Putting it in eq. (2)

$$Z = ax + \phi(a)y + a^2[\phi(a)]^2.$$

On differentiating w.r.t a partially.

$$0 = x + \phi'(a)y + a^2 2\phi(a)\phi'(a) + [\phi(a)]^2 2a.$$

Ans

Q Solve ~~px~~ $z = px + qy + \sqrt{1+p^2+q^2}$

Sol→ $z = px + qy + \sqrt{1+p^2+q^2}$ ——— (1)

Then the solution is,

$z = ax + by + \sqrt{1+a^2+b^2}$ ——— (2)

This is the complet solution.

To find singular solution :-

Differntiat eq. (2) w.r.t 'a' & 'b' partially.

$$0 = x + \frac{2a}{2\sqrt{1+a^2+b^2}}$$

$$0 = y + \frac{2b}{2\sqrt{1+a^2+b^2}}$$

$$x = \frac{-\cancel{2}a}{\cancel{2}\sqrt{1+a^2+b^2}} \quad ; \quad y = \frac{-\cancel{2}b}{\cancel{2}\sqrt{1+a^2+b^2}}$$

——— (3) ——— (4)

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-(x^2+y^2) = 1 - \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-x^2-y^2 = \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2}$$

$$\Rightarrow 1-x^2-y^2 = \frac{1}{(1+a^2+b^2)}$$

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{1+a^2+b^2}} \quad \text{——— (6)}$$

or

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

from (3) & (6).

$$x = -a\sqrt{1-x^2-y^2}$$

$$a = \frac{-x}{\sqrt{1-x^2-y^2}}$$

from (4) & (6).

$$y = -b\sqrt{1-x^2-y^2}$$

$$b = \frac{-y}{\sqrt{1-x^2-y^2}}.$$

Putting values of 'a' and 'b' in eq. (2).

$$Z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \sqrt{1+a^2+b^2}$$

$$= \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$Z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} = \sqrt{1-x^2-y^2}$$

$$Z^2 = 1-x^2-y^2 \quad \text{Ans}$$

General Solution :- Let $b = \phi(a)$

Put it in eq (2).

$$Z = ax + \phi(a)y + \sqrt{1+a^2+[\phi(a)]^2} \quad \text{--- (7)}$$

Differentiating partially w.r.t a.

$$0 = x + \phi'(a)y + \frac{2a + 2\phi(a)\phi'(a)}{2\sqrt{1+a^2+[\phi(a)]^2}} \quad \text{--- (8)}$$

On Eliminating (7) & (8), we get General Solution.

Type III :- Case (a) :- If the eq. is of form.

$$f(z, p, q) = 0 :-$$

Let z is the function of u and $u = ax + y$.

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} \cdot 1$$

$$\Rightarrow p = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du} \cdot a$$

$$q = a \cdot \frac{dz}{du}$$

Q Solve $p(1+q) = qz$.

Sol→ $p(1+q) = qz$

Let $u = a + xy$, z is a function of u.

We get $p = \frac{dz}{du}$, $q = \frac{dz}{du} \cdot a$.

Substituting this in eq. (1) we get

$$\frac{dz}{du}\left(1 + \frac{dz}{du} \cdot a\right) = \frac{dz}{du} az.$$

$$\Rightarrow \left(1 + \frac{dz}{du} a\right) = az$$

$$\Rightarrow \frac{a\,dz}{du} = az - 1$$

$$\Rightarrow \frac{a\,dz}{az-1} = du.$$

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On integrating.

$$\log(az-1) = u + c$$

$$\Rightarrow \log(az-1) = \cancel{x} + ay + c \quad \text{--- (2)}$$

This is required complete solution.

Singular solution does not exist

General solution :- Let $c = \phi(a)$

Put in eq. (2) we get

$$\log(az-1) = x + ay + \phi(a) \quad \text{--- (3)}$$

On differentiating w.r.t 'a' partially.

$$\frac{a}{(ax-1)} = y + \phi'(a) \quad \text{--- (4)}$$

By Eliminating a from (3) and (4), we will get general solution.

Q Solve $z = p^2 + q^2$.

Sol→ $z = p^2 + q^2$ ---- (1)

Let $u = ax + ay$ and z be a function of u.

Put $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in eq. (1)

$$z = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$z = (1+a^2)\left(\frac{dz}{du}\right)^2$$

$$\left(\frac{dz}{du}\right) = \frac{\sqrt{z}}{\sqrt{1+a^2}}$$

$$\Rightarrow \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}}\, du$$

$$\Rightarrow 2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}\, u + c$$

$$\Rightarrow 2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}(x+ay) + c$$

Singular solution will not exist.

General solution :- Let $c = \phi(a)$.

Put it in eq. (2) we get

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}(x+ay) + \phi(a). \quad \text{--- (3)}$$

On differentiating w.r.t 'a' partially.

$$0 = \frac{-2a}{2\sqrt{1+a}\,(1+a^2)}\, x + \frac{a\frac{2a}{\sqrt{1+a^2}(1+a^2)} - \frac{1}{(1+a^2)}}{(1+a^2)} + \phi'(a) \quad \text{--- (4)}$$

On Eliminating a from (3) and (4), we will get General solution.

Case (6) :- If the equation is of form.

$$F(x, p, q) = 0.$$

z is function of x and y.

$$\partial z = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

$$\partial z = p\,dx + q\,dy.$$

Let $q = a$.

Q Solve $p = 2qx$. ——— (1)

Sol→ $p = 2qx$.

$dz = p\,dx + q\,dy$

put $q = a$.

$dz = 2ax\,dx + a\,dy$.

On Integrating.

$z = 2a\frac{x^2}{2} + ay + c$ ——— (2)

This is complete solution.

Singular solution does not exist

General Solution :- Let $c = \phi(a)$

Put it in eq. (2).

$z = 2a\frac{x^2}{2} + ay + \phi(a)$ ——— (3)

On differentiating w.r.t 'a'.

$0 = \frac{2x^2}{2} + y + \phi'(a)$ ——— (4)

By Eliminating a from (3) & (4).

we will get general solution.

Case III :- If $F(y, p, q) = 0$

z is function of x and y.

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

$$dz = p\,dx + q\,dy.$$

Let $p = a$.

Q Solve $pq = y$.

Sol → $pq = y$.

$$dz = p\,dx + q\,dy.$$

Let $p = a$

$$\Rightarrow dz = a\,dx + \frac{y}{a} dy.$$

On Integrating.

$$z = ax + \frac{y^2}{2a} + c \quad \text{---} \quad (2)$$

Ans

Singular solution does not exist

General solution :- Let $c = \phi(a)$.

Put it in eq. (2).

$$z = ax + \frac{y^2}{2a} + \phi(a). \quad \text{---} \quad (3)$$

On differentiating w.r.t a partially

$$0 = x + \frac{y^2}{2} \log a + \phi'(a) \quad \text{---} \quad (4)$$

On solving (3) and (4) and Eliminating a, we get general solution.

Type 4 :- Separable Equations :-

If Equation is of form

$$F(x,p) = \phi(y,q)$$

Then put $F(x,p) = \phi(y,q) = a$

(i) (ii) (iii)

from (i) and (iii) & from (ii) and (iii)

$p = f(x,a)$ $\quad q = \phi(y,a)$

Put these in $dz = p\,dx + q\,dy$.

Q Solve $p^2 y(1+x^2) = qx^2$.

Sol → $p^2 y(1+x^2) = qx^2$

$$\Rightarrow p^2\left(\frac{1+x^2}{x^2}\right) = \frac{q}{y} = a$$

(i) (ii) (iii)

from (i) & (iii) and from (ii) & (iii)

$$p = \frac{x\sqrt{a}}{\sqrt{1+x^2}}, \quad q = ay$$

$dz = p\,dx + q\,dy$.

$$dz = \frac{x\sqrt{a}}{\sqrt{1+x^2}}dx + ay\,dy$$

On Integrating,

$$Z = \sqrt{a}\int \frac{x}{\sqrt{1+x^2}}\,dx + a\int dy$$

$$\Rightarrow Z = \sqrt{a}\sqrt{1+x^2} + \frac{ay^2}{2} + C \quad \underline{\text{Ans}}.$$

This Complete solution. Singular solution will not exist.

General solution :- Let $C = \phi(a)$

Put it in above eq.

$$Z = \sqrt{a}\sqrt{1+x^2} + \frac{ay^2}{2} + \phi(a)$$

On differentiating w.r.t a partially.

$$0 = \frac{\sqrt{1+x^2}}{2\sqrt{a}} + \frac{y^2}{2} + \phi'(a)$$

On Eliminating a from both of these eq. we get general solution.

Q. $p^2+q^2 = x^2+y^2$

Sol→ $p^2+q^2 = x^2+y^2$

$$\Rightarrow \underset{(i)}{p^2-x^2} = \underset{(ii)}{y^2-q^2} = \underset{(iii)}{a^2}$$

from (i) and (iii) & from (ii) and (iii)

$p^2-x^2 = a^2$ $\quad\quad$ $y^2-q^2 = a^2$

$p = \sqrt{a^2+x^2}$ $\quad\quad$ $q = \sqrt{y^2-a^2}$

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$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \sqrt{a^2+x^2}\,dx + \sqrt{y^2-a^2}\,dy$$

On ~~In~~ Integrating,

$$Z = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + C.$$

This is the complet solution.

There is no singular solution.

General solution can be find like all previous ques

Q $9(p^2 z + q^2) = 4$.

Sol → $p^2 z + q^2 = \frac{4}{9}$.

This is of form $f(z, p, q)$.

put $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$, we get

$$\left(\frac{dz}{du}\right)^2 z + a^2\left(\frac{dz}{du}\right)^2 = \frac{4}{9}.$$

$$\left(\frac{dz}{du}\right)^2 (z + a^2) = \frac{4}{9}.$$

$$\left(\frac{dz}{du}\right) = \frac{2}{3\sqrt{z+a^2}}$$

$$\Rightarrow \sqrt{z+a^2}\,dz = \frac{2}{3}\,du.$$

On Integrating.

$$\Rightarrow \frac{2}{3}(z+a^2)^{3/2} = \frac{2}{3}u + \frac{2}{3}c$$

$$\Rightarrow (z+a^2)^{3/2} = u + c.$$

$$\Rightarrow (z+a^2)^{3/2} = x + ay + c. \quad \text{--- (2)}$$

$$\Rightarrow (z+a^2)^{3} = (x+ay+c)^2 \quad \text{Ans}$$

There is no singular solution.

General solution can be find like previous ones

$\downarrow$

Let $c = \phi(a)$

Put it in above equation

$$(z+a^2)^3 = [x+ay+\phi(a)]^2 \quad \text{--- (3)}$$

On differentiating w.r.t a partially.

$$3(z+a^2)^2 2a = 2[x+ay+\phi(a)](y+\phi'(a)) \quad \text{--- (4)}$$

Eliminating a from (3) & (4),

we will get General solution.

Lagrange's Linear Equation

A linear partial differential equation of the first order is known Lagrange's linear equation is of form.

$$Pp + Qq = R \quad \text{--- (1)}$$

where P, Q, R are function of x, y, z.

Working Rule :-

1. Find auxiliary or subsidiary eqⁿ of eq. (1)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{--- (2)}$$

2. Solve auxiliary eqⁿ by (i) Grouping Method
   (ii) Method of Multiplier
   (iii) Combination of (i) & (ii)

3. Let $u = a$, $v = b$ are two solution of eq. (1) which is obtained by eq. (2).

4. Complete solution of (1). $\phi(u, v) = 0$.

Method to solve Auxiliary Eq^n

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(i) Grouping Method :-

$$\frac{dx}{P} = \frac{dy}{Q} \quad \text{or} \quad \frac{dy}{Q} = \frac{dz}{R} \quad \text{or} \quad \frac{dx}{P} = \frac{dz}{R}$$

$$u(x,y) = 0 \quad \text{or} \quad u(y,z) = 0 \quad \text{or} \quad u(x,z) = 0$$

(ii) Method of Multipliers :- Choose multipliers l, m, n such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{Pl + Qm + Rn}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

$$\Rightarrow l\,dx + m\,dy + n\,dz = 0$$

On Integrating

$$u = a$$

Q. Find the general integral of $px + qy = z$.

Sol → $px + qy = z$

On Comparing with $Pp + Qq = R$, we get

$P = x, \quad Q = y, \quad R = z.$

The subsidiary equation are :

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

(i) (ii) (iii)

from (i) & (ii).

$$\frac{dx}{x} = \frac{dy}{y}.$$

On Integrating.

$$\log x = \log y + \log c$$

$$\Rightarrow \log x - \log y = \log c$$

$$\Rightarrow \log\left(\frac{x}{y}\right) = \log c$$

$$\Rightarrow \frac{x}{y} = c$$

from (ii) & (iii).

$$\frac{dy}{y} = \frac{dz}{z}$$

On Integrating.

$$\log y = \log z + \log c$$

$$\log y - \log z = \log c$$

$$\log\left(\frac{y}{z}\right) = \log c$$

$\Rightarrow \quad \cancel{x}\ \frac{y}{z} = c$

Hence the general integral is $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$ Ans

Q Find the general integral of $yzp + zxq = xy$.

Sol→ $yzp + zxq = xy$.

On comparing with $Pp + Qq = R$, we get

$P = yz$, $Q = zx$, $R = xy$.

The subsidiary equation are :-

$$\underset{(i)}{\frac{dx}{yz}} = \underset{(ii)}{\frac{dy}{zx}} = \underset{(iii)}{\frac{dz}{xy}}.$$

From (i) & (ii).

$$\frac{dx}{yz} = \frac{dy}{\cancel{z}x}$$

$\Rightarrow \quad x\,dx = y\,dy$

On integrating.

$$\frac{x^2}{2} = \frac{y^2}{2} + \frac{c}{2}.$$

$$\Rightarrow \frac{x^2 - y^2}{2} = \frac{c}{2}$$

$$\Rightarrow x^2 - y^2 = c$$

From (ii) & (iii)

$$\frac{dy}{\cancel{y}zx} = \frac{dz}{xy}.$$

$$\Rightarrow y\,dy = z\,dz$$

On Integrating

$$\frac{y^2}{2} = \frac{z^2}{2} + \frac{c}{2}$$

$$\Rightarrow y^2 - z^2 = c$$

Hence the general integral is $\phi(x^2-y^2,\, y^2-z^2) = 0$ Ans

Q Find the general integral of $p + 3q = 5z + \tan(y-3x)$.

Sol → $p + 3q = \tan(y-3x)$.

On comparing with $Pp + Qq = R$,

$P = 1$, $Q = 3$, $R = 5z + \tan(y-3x)$

The subsidiary equations are :-

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$$

(i) (ii) (iii)

from (i) & (ii)

$$\frac{dx}{1} = \frac{dy}{3}$$

$$3\,dx = dy$$

On Integrating, $3x = y + C$

$$\Rightarrow 3x - y = C.$$

or, $y - 3x = C_1$.

from (i) & (iii).

$$\frac{dx}{1} = \frac{dz}{5z+\tan(y-3x)}$$

$$\Rightarrow dx = \frac{dz}{5z + \tan C_1}$$

On Integrating

$$x = \frac{1}{5}\left[\int \frac{5\,dz}{5z+\tan C_1}\right]$$

$$\Rightarrow x = \frac{1}{5}\log(5z + \tan C_1) + \frac{C_2}{5}$$

$$\Rightarrow x - \frac{1}{5}\log(5z + \tan C_1) = \frac{C_2}{5}$$

$$\Rightarrow 5x - \log(5z + \tan C_1) = C_2$$

$$\Rightarrow 5x - \log[5z + \tan(y-3x)] = C_2$$

Hence general integral is:

$$\phi\left((y-3x),\ (5x - \log(5z+\tan(y-3x)))\right) = 0$$

Ans

Q Solve $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$

Sol→ $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$

On comparing with $Pp + Qq = R$

$P = x(z^2-y^2)$, $Q = y(x^2-z^2)$ $R = z(y^2-x^2)$

Auxialary eq. are:

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$$

By taking multiplier x, y, z ~~as~~

$$\frac{x\,dx + y\,dy + z\,dz}{x^2(z^2-y^2)+y^2(x^2-z^2)+z^2(y^2-x^2)}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{0}$$

Since denominator $= 0$

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0$$

On Integrating.

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{c^2}{2} = \frac{c}{2}$$

$$\Rightarrow x^2 + y^2 + z^2 = c$$

Now, on again taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multiplier,

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{(z^2-y^2)+(x^2-z^2)+(y^2-x^2)}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \quad [\because \text{denominator} = 0]$$

On Integrating,

$$\log x + \log y + \log z = \log c$$

$$\Rightarrow \log(xyz) = \log c$$

$$\Rightarrow xyc = c$$

So, general Integral is $\phi(x^2+y^2+z^2,\ xyz) = 0$

Q Find general Integral of $(mz-ny)p + (nx-lz)q = ly - mx$.

Sol→ $(mx-ny)p + (nx-lz)q = ly - mx$.

$(mx-ny)p + (nx-lz)q$

On Comparing with $Pp + Qq = R$

$P = mx + ny$, $Q = nx - lz$, $R = ly - mx$

Auxiliary equation are :-

$$\frac{dx}{(mx-ny)} = \frac{dy}{(nx-lz)} = \frac{dz}{(ly-mx)}$$

Taking x, y, z as multiplier.

$$\frac{xdx + ydy + zdz}{x(mx-ny) + y(nx-lz) + z(ly-mx)}$$

$$\Rightarrow xdx + ydy + zdz = 0$$

On Integrating.

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{c^2}{2} = \frac{c}{2}$$

$$\Rightarrow x^2 + y^2 + z^2 = c$$

Now, Taking l, m, n as multiplier.

$$\frac{ldx + mdy + ndz}{l(mx-ny) + m(nx-lz) + n(ly-mx)}$$

$$\Rightarrow ldx + mdy + ndz = 0$$

On Integrating $lx + my + nz = c$

So, general Integral is

$$\phi(x^2+y^2+z^2,\ lx+my+nz) = 0$$

Q Solve $(3z-4y)p + (4x-2z)q = (2y-3x)$

Sol→ $(3z-4y)p + (4x-2z)q = (2y-3x)$

On comparing with $pP + Qq = R$.

$P = 3z-4y$, $Q = 4x-2z$, $R = 2y-3x$.

Auxiliary equation are :-

$$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x}.$$

Taking x, y, z as multiplier.

$$\frac{xdx + ydy + zdz}{x(3z-4y) + y(4x-2z) + z(2y-3x)}$$

$\Rightarrow xdx + ydy + zdz = 0$

On Integrating. $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c}{2}$

$\Rightarrow x^2 + y^2 + z^2 = C.$

Again on Taking $2, 3, 4$ as multiplier.

$$\frac{2dx + 3dy + 4dz}{2(3z-4y) + 3(4x-2z) + 4(2y-3x)}$$

$\Rightarrow 2dx + 3dy + 4dz = 0$

On Integrating,

$2x + 3y + 4z = C.$

$\Rightarrow$ General Integral is

$\phi(x^2+y^2+z^2, 2x+3y+4z) = 0$

Ans

Q Solve $(x^2-y^2-z^2)p + 2xyq = 2xz$.

Sol→ $(x^2+y^2-z^2)p + 2xyq = 2xz$

On Comparing with $Pp + Qq = R$.

& $P = x^2-y^2-z^2$, $Q = 2xy$, $R = 2xz$.

Auxiliary Eq. are :-

$$\underset{(i)}{\frac{dx}{x^2-y^2-z^2}} = \underset{(ii)}{\frac{dy}{2xy}} = \underset{(iii)}{\frac{dz}{2xz}}$$

from (ii) & (iii)

$$\frac{dx}{2xy} = \frac{dz}{2xz} \Rightarrow \frac{dx}{y} = \frac{dz}{z}$$

On Integrating

$$\Rightarrow \log y = \log z + \log c$$

$$\Rightarrow \log\left(\frac{y}{z}\right) = \log c$$

$$\Rightarrow \frac{y}{z} = c.$$

Now, On taking x, y, z as multiplier.

$$\frac{dy}{2xy} = \frac{xdx + ydy + zdz}{x(x^2-y^2-z^2) + y(2xy) + z(2xz)}$$

$$\Rightarrow \frac{dy}{2xy} = \frac{xdx + ydy + zdz}{x(x^2+y^2+z^2)}$$

$$\Rightarrow \frac{dy}{y} = \frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2}$$

$$\Rightarrow \log y \cancel{+c} = \log(x^2+y^2+z^2) + \log c$$

$$\Rightarrow \log\left(\frac{y}{x^2+y^2+z^2}\right) = \log c$$

$$\Rightarrow \frac{y}{x^2+y^2+z^2} = c$$

So, general Integral is $\phi\left(\frac{y}{z}, \frac{y}{x^2+y^2+z^2}\right) = 0$ **Ans**

Q Solve $x^2\frac{dz}{dx} + y^2\frac{dz}{dy} = (x+y)z.$

Sol → $x^2 p + y^2 q = (x+y)z$

On Comparing with $Pp + Qq = R.$

$P = x^2$, $Q = y^2$, $R = (x+y)z$

Auxiliary equation are.

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x+y)}$$

(i) (ii) (iii)

From (i) & (ii)

$$\frac{dx}{x^2} = \frac{dy}{y^2}.$$

On Integrating.

$$-\frac{1}{x} = -\frac{1}{y} + c$$

$$c = \frac{1}{y} - \frac{1}{x} = \left(\frac{x-y}{yx}\right).$$

Now, on taking 1, −1, 0 as multiplier.

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$\Rightarrow \frac{dx-dy}{(x-y)(x+y)} = \frac{dz}{(x+y)z}$$

$$\Rightarrow \frac{dx-dy}{(x-y)} = \frac{dz}{z}$$

On Integrating.

$$\log(x-y) = \log z + \log c$$

$$\Rightarrow \log\left(\frac{x-y}{z}\right) = \log c$$

$$\Rightarrow \frac{x-y}{z} = c$$

So. general Integral is $\phi\left(\frac{x-y}{xy}, \frac{x-y}{z}\right) = 0$

Ans

Q. Solve $(y+z)p + (z+x)q = x+y$.

Sol → $(y+z)p + (z+x)q = (x+y)$.

On Comparing $Pp + Qq = R$.

$P = (y+z)$. $Q = (z+x)$. $R = (x+y)$

Auxiliary eq. are.

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}.$$

On taking $(1,-1,0)$, $(0,1,-1)$ & $(1,0,-1)$ as multiplier

$$\underset{(i)}{\frac{dx-dy}{y+z-z-x}} = \underset{(ii)}{\frac{dy-dz}{z+x-x-y}} = \underset{(iii)}{\frac{dx-dz}{y+z-x-y}}$$

From (i) and (iii)

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$$

$$\Rightarrow \frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)}$$

On Integrating, $\log(x-y) = \log(y-z) + \log c$

$$\Rightarrow \log\left(\frac{x-y}{y-z}\right) = \log c.$$

$$\Rightarrow \frac{x-y}{y-z} = c$$

Now, From (ii) & (iii)

$$\frac{dy-dz}{z-y} = \frac{dx-dz}{z-x}.$$

$$\Rightarrow \log(y-z) = \log(x-z) + \log c$$

$$\Rightarrow \log\left(\frac{y-z}{x-z}\right) = \log c.$$

$$\Rightarrow \left(\frac{y-z}{x-z}\right) = c$$

So, General Integral is

$$\phi\left(\frac{x-y}{y-z}, \frac{y-z}{x-z}\right) = 0$$

Ans

Q Solve $z(x-y) = px^2 - qy^2$.

Sol → $x^2p - y^2q = z(x-y)$.

On Comparing $Pp + Qq = R$.

$P = x^2$, $Q = y^2$, $R = z(x-y)$

Auxiliary eq. are:

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x-y)}$$

(i) (ii) (iii)

From (i) & (ii)

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

$\Rightarrow$ On Integrating $\frac{-1}{x} = \frac{+1}{y} + c$

$$\Rightarrow C = \frac{1}{x} + \frac{1}{y}.$$

Now, on taking 1, 1, 0 as multiplier

$$\frac{dx + dy}{x^2 - y^2} = \frac{dz}{z(x-y)}$$

$$\Rightarrow \frac{dx + dy}{(x-y)(x+y)} = \frac{dz}{z(x-y)}$$

On Integrating $\log(x+y) = \log z + \log c$

$c = \frac{x+y}{z}$

So, General Integral $\Rightarrow \phi\left(\frac{1}{x} + \frac{1}{y}, \frac{x+y}{z}\right) = 0$ Ans

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Linear Partial Differential Equation of higher Order with Constant Coefficient :-

A linear partial differential equation with constant coefficient in which all partial derivatives are of same order is called homogenous partial differential equation. otherwise it is called non-homogenous partial differiential equation.

Example :-

(i) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = xy$ i.e. $(D^2 + D'^2) z = xy$.

(ii) $\frac{\partial^3 z}{\partial x^3} + \frac{3(\partial^3 z)}{\partial y \partial x^2} - 4\frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{2x+y}$

i.e. $(D^3 + 3D^2 D' - 4DD'^2 + 2D'^3) z = e^{2x+y}$.

(iii) $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} + z = 2\sin y + x^3$

i.e. $(D^2 - 3D + 2D' + 1) z = 2\sin y + x^3$.

Where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

The (i) & (ii) of above equation is homogenous. whereas (iii) is non-homogenous.

Homogenous Linear Equation

A homogeneous linear P.D equation of n^{th} order with constant coefficient is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots a_n \frac{\partial^n z}{\partial y^n} = f(x,y) \quad \text{---} (1)$$

where $a_1, a_2, a_3 \cdots$ are constants and f is the known function of x and y.

Eq. (1) can also be written as

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots - a_n D'^n) z = f(x,y) \quad \text{---} (ii)$$

or

$$F(D, D') z = f(x,y) \quad \text{---} (iii)$$

The complete solution of eq.(iii) consists of two parts namely (i) Complementary function and (ii) Particular Integral.

Complementary Function of (iii) is the solution of $F(D, D') z = 0$ — (iv).

and the particular integral of (iii) is the solution of

$$\frac{1}{f(D, D')} F(x, y).$$

Hence the complete solution $= CF + PI$ of (iii)

Method of finding Complementary Function

In $F(D, D') = 0$ put $D = m$ & $D' = 1$ to get auxiliary equation.

$$a_0 m^n + a_1 m^{n-1} + \cdots a_n = 0$$

Let $m_1, m_2, \cdots m_n$ be root of above equation.

Case I :- If all root are distinct, then.

$$Cf = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots \phi_n(y + m_n x)$$

where ϕ_i are arbitrary.

Case II :- If r root are repeated, i.e. if root are m (r times), $m_{r+1}, m_{r+2} \cdots m_n$ then.

$$Cf = \phi_1(y + mx) + x\phi_2(y + mx) + \cdots x^r \phi_r(y + mx) + \phi_{r+1}(y + m_{r+1} x) + \phi_{r+2}(y + m_{r+2} x) + \cdots + \phi_n(y + m_n x).$$

Method to find Particular Integral :-

1. $\frac{1}{F(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by}$ if $f(a, b) \neq 0$.

2. $\frac{1}{F(D, D')} x^r y^s = [F(D, D')]^{-1} x^r y^s$.

where $[F(D, D')]^{-1}$ is to be expanded in power of D, D'.

3. $\frac{1}{F(D^2, DD', D'^2)} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax+by) = \frac{1}{f(-a^2, -ab, -b^2)} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax+by)$

4. $\frac{1}{F(D,D')} e^{ax+by} \phi(x,y) = e^{ax+by} \cdot \frac{1}{F(D+a, D'+b)} \phi(x,y)$

5. $\frac{\sin ax \sin by}{f(D^2, D'^2)} = \frac{\sin ax \sin by}{f(-a^2, -b^2)}$ if denominator $\neq 0$.

6. $\frac{\cos ax \cos by}{F(D^2, D'^2)} = \frac{\cos ax \cos by}{F(-a^2, -b^2)}$ if denominator $\neq 0$.

General Rule for PI :-

If $F(D,D') = D - mD'$

Then $P.I = \frac{1}{D-mD'} \cdot F(x,y)$

$= \int F(x, c-mx)\, dx$.

[Replace $y \rightarrow c - mx$]

(after Integration put $c = y + mx$).

Q Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0$.

Sol→ $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0$ --- (i)

Put $D = \frac{\partial}{\partial x}$ $D' = \frac{\partial}{\partial y}$.

then $(D^2 + DD' - 2D'^2)z = 0$. --- (ii)

Replacing D by m & D' by 1 in eq. (ii), we get

$m^2 + m - 2 = 0$

Solving for m; we get $(m-1)(m+2) = 0$

~~Here~~ $\Rightarrow m = 1, -2$.

$\because$ root are distinct.

$\therefore$ CF $= \phi_1(y+x) + \phi_2(y-2x)$.

$\because$ RHS of (1) is 0.

the general solution of given P.D.E is

$z = CF + PI$

$\Rightarrow z = \phi_1(y+x) + \phi_2(y-2x) + 0$

$\Rightarrow z = \phi_1(y+x) + \phi_2(y-2x)$ Ans

Q Solve $(D^2 - 2DD' + D'^2)z = 0$

Sol→ $(D^2 - 2DD' + D'^2)z = 0$ --- (1)

Replacing D by m & D' by 1, we get

$(m^2 - 2m + 1) \cancel{z} = 0$

Solving for m, we get $(m-1)^2 = 0$

$\Rightarrow m = 1, 1$

$\because$ roots are repeated

So, $CF = \phi_1(y+x) + x\phi_2(y+x)$

$\because$ RHS of Eq. (1) is 0.

$\therefore PI = 0$

Then the general solution of P.D.E

$z = CF + PI$

$\Rightarrow z = \phi_1(y+x) + x\phi_2(y+x) + 0$

$\Rightarrow z = \phi_1(y+x) + x\phi_2(y+x)$ Ans

Q Solve $(D^3 - 3D^2D' + 2DD'^2)z = 0$.

Sol→ $(D^3 - 3D^2D' + 2DD'^2)z = 0$ ——— (1)

Replace D by m & D' by 1, we get

$m^3 - 3m^2 + 2m = 0$

$m(m^2 - 3m + 2) = 0$

$m(m-1)(m-2) = 0$

$m = 0, 1, 2.$

then the general solution will be

$z = CF + PI$

$\Rightarrow z = \phi_1(y) + \phi_2(y+x) + \phi_3(y+2x) + 0$

Ans

Q Solve $(D^3 + DD'^2 - D^2D' - D'^3)z = 0$.

Sol→ $(D^3 + DD'^2 + D^2D' - D'^3)z = 0$ ---- (1)

Replace D by m & D' by 1, we get

$m^3 - m^2 + m - 1 = 0$

$\Rightarrow (m-1)(m^2+1) = 0$

$\Rightarrow m = 1, -i, +i$

∴ The general solution will be

$z = CF + PI$

$\Rightarrow z = \phi_1(y+x) + \phi_2(y+ix) + \phi_3(y-ix)$

Ans

Q Solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$.

Sol→ Putting $\frac{\partial}{\partial x} = D$ & $\frac{\partial}{\partial y} = D'$, we get

$(D^3 - 3D^2D' + 4D'^3) = e^{x+2y}$.

Replace D by m & D' by 1, we get

$m^3 - 3m^2 + 4 = 0$

On solving for m, we get

$m = 2, 2, -1$.

Hence the $CF = \phi_1(y+2x) + \phi_2(y+2x) + \phi_3(y-x)$

$$PI = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

On Comparing with

$$PI = \frac{1}{f(D, D')} e^{ax+by}$$

we get $a = 1$, $b = 2$

So, $$PI = \frac{1}{(1)^3 - 3(1)\cdot 2 + 4(2)^3} \cdot e^{x+2y}$$

$$\Rightarrow PI = \frac{1}{27} e^{x+2y}.$$

So, General Solution is $Z = CF + PI$

$$\Rightarrow z = \phi_1(x+2x) + \phi_2(x+2x) + \phi_3(x-y) + \frac{e^{x+2y}}{27}$$

Ans

On Comparing with
$$PI = \frac{1}{f(D,D')} e^{ax+by}$$

we get $a=1$, $b=2$

So, $PI = \frac{1}{(1)^3 - 3(1)\cdot 2 + 4(2)^3} \cdot e^{x+2y}$

$\Rightarrow PI = \frac{1}{27} e^{x+2y}$.

So, General Solution is $Z = CF + PI$

$\Rightarrow z = \phi_1(x+2x) + \phi_2(x+2x) + \phi_3(x-y) + \frac{e^{x+2y}}{27}$

Ans

Q $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x-y}$.

Sol→ $D^2 - 4DD' + 4D'^2 = e^{2x-y}$

Replace D by m & D' by 1.

$m^2 - 4m + 4 = 0$

Solving for m, we get $m = 2, 2$.

So, $CF = \phi_1(y+2x) + x\phi_2(y+2x)$

Now, for PI:

$$PI = \frac{1}{D^2 - 4D'D + 4D'^2} \times e^{2x-y}$$

On Comparing with $PI = \frac{1}{f(D,D')} e^{ax+by}$

we get $a=2, \; b=-1$.

So, $PI = \frac{1}{(2)^2 - 4(-1)(2) + 4(-1)^2} e^{2x-y}$

$\Rightarrow PI = \frac{e^{2x-y}}{16}$

So, General solution is $Z = Cf + PI$

$Z = \phi_1(x+2x) + n\phi_2(x+2x) + \frac{e^{2x-y}}{16}$ Ans

Alter Method by General Rule [DO NOT TOUCH THIS]

lenghty

$$\frac{1}{D^2 - 4DD' + 4D'^2} e^{2x-y}$$

$$\Rightarrow \frac{1}{(D-2D')(D-2D')} e^{2x-y}$$

$$\Rightarrow \frac{1}{(D-2D')} \int e^{2x-c+2x} dx$$

$$\Rightarrow \frac{1}{(D-2D')} \frac{e^{4x+c}}{4}$$

$$\Rightarrow \frac{1}{4} \frac{1}{(D+2D')} e^{4x-c}$$

$$\Rightarrow \frac{1}{4} \int e^{4x-y-2x}$$

$$\Rightarrow \frac{1}{4} \int e^{2x-y}$$

$$\Rightarrow \frac{1}{16} e^{2x-y}$$ Ans

Just for knowledge

Do not try this in Question

Type :- $\dfrac{1}{f(D^2, DD', D'^2)}$ Sin/Cos $(ax+by)$

Put $D^2 = -a^2$
$DD' = -ab$
$D'^2 = -b^2$

Q Solve $(D^2 - 2DD' + 2D'^2) z = \sin(x-y)$

Sol → Replace D by m & D' by 1

$m^2 - 2m + 2 = 0$

$\Rightarrow m = \dfrac{+2 \pm \sqrt{4-8}}{2}$

$\Rightarrow m = +1 \pm i$

So, $CF = \phi_1(y + (1+i)x) + \phi_2[y + (1-i)x]$.

For PI :- $\dfrac{1}{D^2 - 2DD' + 2D'^2} \sin(x-y)$

$\Rightarrow$ On comparing with $\dfrac{1}{f(D^2, DD', D'^2)} \sin(ax+by)$

$a = 1, \; b = -1$.

Then, $D^2 = -1$,
$DD' = 1$
$D'^2 = -1$

Putting it in PI

$\dfrac{1}{(-1) - 2(1) + 2(-1)} \sin(x-y)$

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$\Rightarrow PI = \frac{-1}{5} \sin(x-y)$.

So, General solution is $z = CF + PI$

$\Rightarrow \phi_1(y+(1+i)x) + \phi_2(y+(1-i)x) - \frac{1}{5}\sin(x-y)$

Ans

Q Solve $[D^3 - 7DD'^2 - 6D'^3]z = \sin(x+2y)$.

Sol→ Replace D by m & D' by 1.

$$m^3 - 7m - 6 = 0$$

$$\Rightarrow (m+1)(m^2 - m - 6) = 0$$

$$\Rightarrow m = -1, -2, 3.$$

$\therefore CF = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$.

Now, for PI :-

$$PI = \frac{1}{D^3 - 7DD'^2 - 6D'^2}\sin(x+2y)$$

$$= \frac{1}{D(D^2) - 7DD'^2 - 6D'(D')^2}\sin(x+2y)$$

$$= \frac{1}{D(-1) - 7D(-4) - 6D'(-4)} \cdot \sin(x+2y)$$

$$= \frac{1}{+27D + 24D'} \times \frac{24D' + 27D}{24D' + 27D} \cdot \sin(x+2y)$$

$$= \frac{24D' + 27D}{576D'^2 - 729D^2}\sin(x+2y)$$

$$\Rightarrow \frac{24D' + 27D}{576(-4) - 729(-1)}\sin(x+2y)$$

$$\Rightarrow \frac{24D' + 27D}{1575} \sin(x+2y)$$

$$\Rightarrow \frac{24 \times 2}{1575} \cos(x+2y) + \frac{27}{1575} \cos(x+2y)$$

$$\Rightarrow \cos(x+2y) \frac{\cancel{21}^{1}}{\cancel{1575}_{75}}$$

$$PI = \frac{\cos(x+2y)}{75}$$

So, General solution is $Z = CF + PI$

$$\Rightarrow Z = \phi_1(\cancel{y}-\cancel{x}) + \phi_2(\cancel{y}-2\cancel{y}) + \phi_3(\cancel{y}+3\cancel{x}) + \frac{\cos(x+2y)}{75}$$

$$\Rightarrow Z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) + \frac{\cos(x+2y)}{75}$$ Ans

Type 3:- $\frac{1}{f(D,D')} x^r y^s = [f(D,D')]^{-1} x^r y^s$.

Q Solve $(D^2 - DD' + D'^2)Z = 2x + 3y$.

Sol→ Replace D by m & D' by 1. we get

$$m^2 - m + 1 = 0$$

$$\Rightarrow m = \frac{1 \pm \sqrt{3}i}{2}$$

$$\therefore CF = \phi_1\left[y + \left(\frac{1+\sqrt{3}i}{2}\right)x\right] + \phi_2\left[y + \left(\frac{1-\sqrt{3}i}{2}\right)x\right]$$

for. P.I, $\frac{1}{D^2 - DD' + D'^2}(2x+3y)$.

$$\Rightarrow \frac{1}{D^2\left[1-\left(\frac{DD' - D'^2}{D^2}\right)\right]}(2x+3y)$$

$$\Rightarrow \frac{1}{D^2}\left[1-\left(\frac{D'}{D} - \frac{D'^2}{D^2}\right)\right]^{-1}(2x+3y)$$

$$\Rightarrow \frac{1}{D^2}\left[1-\left(\frac{D'}{D} - \frac{D'^2}{D^2}\right) + \left(\frac{D'}{D} - \frac{D'^2}{D^2}\right)^2 + \cdots\right](2x+3y)$$

$$\Rightarrow \frac{1}{D^2}\left[1 + \frac{D'}{D}\right](2x+3y)$$

$$\Rightarrow \frac{1}{D^2}\left(2x+3y + \frac{D'}{D}(2x+3y)\right]$$

$$\Rightarrow \frac{1}{D^2}\left[2x+3y + \frac{1}{D}(3)\right]$$

$$\Rightarrow \frac{1}{D^2}\left[2x+3y+3x\right]$$

$$\Rightarrow \frac{1}{D^2}\left[5x+3y\right]$$

$$\Rightarrow \frac{1}{D}\left[\frac{5x^2}{2} + 3xy\right]$$

$$\Rightarrow \left[\frac{5x^3}{6} + \frac{3x^2y}{2}\right]$$

So, General solution is $Z = CF + PI$

$$\Rightarrow Z = \phi_1\left[y + \left(\frac{1+\sqrt{3}i}{2}\right)x\right] + \phi_2\left[y + \left(\frac{1-\sqrt{3}i}{2}\right)x\right] + \frac{5x^3}{6} + \frac{3x^2y}{2}$$

Ans

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Type 4:- If $\frac{1}{f(D,D')} e^{ax+by} \phi(x,y)$ form is given

Then $= e^{ax+by} \frac{1}{f[(D+a),(D'+b)]} \phi(x,y)$.

Q Solve $[D^3 + D^2 D' - DD'^2 - D'^3] z = e^x \cos 2y$.

Sol→ Replacing D by m & D' by 1, we get

$$m^3 + m^2 - m - 1 = 0$$

$$\Rightarrow m = -1, -1, 1$$

$$\therefore CF = \phi_1(y-x) + x\phi_2(y-x) + \phi_3(y+x)$$

For PI :-

$$PI = \frac{1}{D^3 + D^2 D' - DD'^2 - D'^3} e^x \cos 2y.$$

$$\Rightarrow PI = e^x \frac{1}{(D+1)^3 + (D+1)^2 D' - D'^2(D+1) - D'^3} \cos 2y$$

$$= e^x \frac{\text{Real part of}}{(D+1)^3 + (D+1)^2 D' - D'^2(D+1) - D'^3} e^{2iy}$$

$$= e^x \frac{\text{Real part of}}{(0+1)^3 + (0+1)^2 \cdot (2i) - (2i)^2(0+1) - (2i)^3} e^{2iy}$$

$$= e^x \frac{\text{Real part of}}{1 + 2i + 4 + 8i} e^{2iy}$$

$$= e^x \frac{\text{Real part of}}{5 + 10i} e^{2iy}$$

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$$= \frac{e^x}{5} \text{ Real part of } \frac{e^{2iy}}{(1+2i)}$$

$$= \frac{e^x}{5} \text{ R.P of } \frac{(1-2i)}{(1+4)} e^{2iy}$$

$$= \frac{e^x}{5 \times 5} (1-2i)(\cos 2y + i\sin 2y)$$

$$= \frac{e^x}{25} [\cos 2y + i\sin 2y - 2i\cos 2y + 2\sin 2y]$$

$$= \frac{e^x}{25} [\cos 2y + 2\sin 2y]$$

So, General solution is: $Z = CF + PI$

$$\Rightarrow Z = \phi_1(y-x) + x\phi_2(y-x) + \phi_3(y+1) + \frac{e^x}{25}[\cos 2y + 2\sin 2y]$$

Ans

Problem for Practise

1) $[D^4 - D'^4] Z = e^{x+y}$

2) $r + s - 2t = e^{2x+y}$

3) $[D^2 + 3DD' + 2D'^2] = x + y$

Solution $\rightarrow$ ① Replacing D by m & D' by 1.

$$m^4 - 1 = 0$$

$$\Rightarrow m = 1, 1, -i, +i$$

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$\therefore CF = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+ix) + \phi_4(y-ix)$

For PI :- $\frac{1}{D^4 - D'^4} e^{x+y}$

$\Rightarrow \frac{1}{(D+D')(D-D')(D^2+D'^2)} e^{x+y}$

$\Rightarrow \frac{1}{(1+1)(D-D')(1+1)} e^{x+y}$

$\Rightarrow \frac{1}{4(D-D')} e^{x+y}$

Now, for $\frac{1}{D-D'} e^{x+y}$, we can use general form.

$\Rightarrow \frac{1}{4(D-D')} e^{x+y} = \frac{1}{4} \int e^{x+(y-x)}$

$= \frac{1}{4} \int e^{y}$

$\Rightarrow \frac{1}{4} x e^{y}$

So, $PI = \frac{xe^{y}}{4} \Rightarrow \frac{xe^{x+y}}{4}$

So, General Integral is $Z = CF + PI$

$\Rightarrow \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+ix) + \phi_4(y-ix) + \frac{xe^{x+y}}{4}$

Ans

3) Replacing D by m & D' by 1.

$m^2 + 3m + 2 = x+y$

$\Rightarrow m = -1, -2.$

$\therefore Cf = \phi_1(y-x) + \phi_2(y-2x)$

For PI :- $\frac{1}{D^2+3DD'+2D'^2}(x+y)$

$$= \frac{1}{D^2\left[1+\left(\frac{3DD'+2D'^2}{D^2}\right)\right]}(x+y)$$

$$= \cancel{\frac{d}{D^2}} \frac{1}{D^2}\left[1+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)\right]^{-1}(x+y)$$

$$\Rightarrow \frac{1}{D^2}\left[1-\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)+\left(\frac{3D'}{D}+\frac{2D'}{D^2}\right)-\ldots\right](x+y)$$

$$\Rightarrow \frac{1}{D^2}\left[1-\frac{3D'}{D}\right](x+y)$$

$$\Rightarrow \frac{1}{D^2}\left[x+y-\frac{3D'}{D}(x+y)\right]$$

$$\Rightarrow \frac{1}{D^2}\left[x+y-\frac{3}{D}\right]$$

$$= \frac{1}{D^2}\left[x+y-3x\right]$$

$$= \frac{1}{D^2}\left[y-2x\right]$$

$$= \frac{1}{D}\left[yx-\frac{2x^2}{2}\right]$$

$$PI = \frac{yx^2}{2}-\frac{x^3}{3}$$

So, General Integral is $Z = CF + PI$.

$$Z = \phi_1(y-x)+\phi_2(y-2x)+\frac{x^2y}{2}-\frac{x^3}{3}$$ Ans

2) $r + s - 2t = e^{2x+y}$.

Sol → $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 2\dfrac{\partial^2 z}{\partial y^2} = e^{2x+y}$

$\Rightarrow D^2 + DD' - 2D'^2 = e^{2x+y}$.

Replacing D by m & D' by 1

$m^2 + m - 2 = 0$

$\Rightarrow m^2 + 2m - m - 2 = 0$

$\Rightarrow m(m+2) - 1(m+2) = 0$

$\Rightarrow m = 1, -2$

$\therefore CF = \phi_1(y+x) + \phi_2(y-2x)$.

$PI = \dfrac{1}{D^2 + DD' - 2D'^2} e^{2x+y}$.

$= \dfrac{1}{(2)^2 + (2\times1) - 2\times(1)^2} e^{2x+y}$

$= \dfrac{1}{4} e^{2x+y}$

So, General Integral is $z = CF + PI$

$\Rightarrow z = \phi_1(y+x) + \phi_2(y-2x) + \dfrac{e^{2x+y}}{4}$ Ans

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Q $(D^2+2DD'+D'^2)z = \sinh(x+y) + e^{x+2y}$.

Sol $\rightarrow$ Replace D by m & D' by 1, we get

$$m^2+2m+1=0$$

$$\Rightarrow m=-1,-1.$$

$$\therefore CF = \phi_1(y-x) + x\phi_2(y-x).$$

For PI $= \dfrac{1}{D^2+2DD'+D'^2}\sinh(x+y)+e^{x+2y}$

$$= \frac{1}{(D+D')^2}\left[\sinh(x+y)+e^{x+2y}\right]$$

$$= \frac{1}{(D+D')^2}\left[\frac{e^{x+y}-e^{-(x+y)}}{2}\right] + \frac{1}{(D+D')^2}e^{x+2y}$$

$$= \frac{1}{(D+D')^2}\frac{e^{x+y}}{2} - \frac{1}{(D+D')^2}e^{-(x+y)} + \frac{1}{(D+D')^2}e^{x+2y}$$

$$= \frac{1}{4}\frac{e^{x+y}}{2} - \frac{1}{4}\frac{e^{-(x+y)}}{2} + \frac{1}{9}e^{x+2y}$$

$$\Rightarrow \frac{1}{4}\left[\frac{e^{x+y}-e^{-(x+y)}}{2}\right] + \frac{1}{9}e^{x+2y}$$

$$\Rightarrow \frac{1}{4}\sinh(x+y) + \frac{1}{9}e^{x+2y}$$

So, General Integral is $z = CF + PI$.

$$z = \phi_1(y-x) + x\phi_2(y-x) + \frac{1}{4}\sin(x+y) + \frac{1}{9}e^{x+2y}.$$

Q $(D^2 + DD' - 6D'^2)z = y\cos x$

Sol → Replacing D by m & D' by 1.

$$m^2 + m - 6 = 0$$

$$\Rightarrow m = 2, -3.$$

$$\therefore CF = \phi_1(y+2x) + \phi_2(y-3x).$$

For PI :-

$$\frac{1}{(D^2+DD'-6D'^2)} y\cos x$$

$$= \frac{1}{(D+3D')(D-2D')} y\cos x$$

$$= \frac{1}{(D-2D')}\left[\frac{1}{(D+3D')} y\cos x\right]$$

$$= \frac{1}{(D-2D')}\int (c+3x)\cos x\,dx.$$

$$= \frac{1}{(D-2D')}\int c\cos x + \int 3x\cos x$$

$$= \frac{1}{D-2D'}\left[c\sin x + 3\left[x\sin x - \int \sin x\right]\right]$$

$$= \frac{1}{D-2D'}\left[c\sin x - 3x\sin x + 3\cos x\right]$$

$$= \frac{1}{D-2D'}\left[(c-3x)\sin x + 3\cos x\right]$$

$$= \frac{1}{D-2D'}(y\sin x + 3\cos x)$$

$$= \int (c-2x)\sin x + \frac{1}{D-2D'} 3\cos x$$

$$= \int (c\sin x - 2x\sin x) + 3\frac{D+2D'}{D^2-4D'^2}\cos x.$$

$$= \int c\sin x - 2\int x\sin x + \left[3\frac{D+2D'}{-1}\cos x\right]$$

$$= -c\cos x - 2\left[x(-\cos x) - \int -\cos x\right] - 3D\cos x$$

$$= -c\cos x + 2x\cos x - 2\sin x + 3\sin x.$$

$$\Rightarrow \sin x - (c-2x)\cos x.$$

$$\Rightarrow \sin x - y\cos x$$

So, General Integral is $z = CF + PI$

$$z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x$$ Ans