Maths

Unit - 3

Application of PDE :-

Q Solve $2x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0$ by method of separation of variables.

Sol→ Given :- $2x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0$ ---------- (1)

Let $Z = XY$ ---------- (2)

where X and Y is function of x and y respectively.

$\Rightarrow \frac{\partial z}{\partial x} = X'Y \Rightarrow \frac{\partial z}{\partial y} = XY'$

put these values in eq. (1)

$$2x\,X'Y - 3y\,XY' = 0$$

$$2xX'Y = 3yXY'.$$

$$\frac{2xX'}{X} = \frac{3yY'}{Y} = k.$$

(i) (ii) (iii)

from (i) & (iii) & from (ii) & (iii)

$$\frac{2xX'}{X} = k \qquad \frac{3yY'}{Y} = k$$

$$\Rightarrow \frac{X'}{X} = \frac{k}{2x} \qquad \frac{Y'}{Y} = \frac{k}{3y}$$

On Integrating.

$$\log X = \frac{k}{2}\log x + \log c_1 \qquad \log Y = \frac{k}{3}\log y + \log c_2$$

$$\log X = \log c_1 x^{k/2} \qquad \log Y = \log c_2 y^{k/3}.$$

$$X = c_1 x^{k/2} \qquad Y = c_2 y^{k/3}.$$

$$Z = c_1 c_2 x^{k/2} y^{k/3}. \quad \underline{\underline{\text{Ans}}}$$

Q. Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ by method of seperation of variables.

Sol → Given :- $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ —— ①

Let $Z = XY$ —— ②

where X & Y is function of x & y respectively.

$$\frac{\partial z}{\partial x} = X'Y \qquad \frac{\partial z}{\partial y} = XY' \qquad \frac{\partial^2 z}{\partial x^2} = X''Y.$$

put these values in eq. ①

$$X''Y - 2X'Y + XY' = 0$$

$$(X'' - 2X')Y = -XY'$$

$$\Rightarrow \underset{(i)}{\frac{X'' - 2X'}{X}} = \underset{(ii)}{\frac{-Y'}{Y}} = \underset{(iii)}{k}.$$

from (i) & (iii) & from (ii) & (iii)

$$\frac{X'' - 2X'}{X} = k \qquad \frac{-Y'}{Y} = k$$

$X'' - 2X' - kX = 0$

$D^2 X - 2DX - kX = 0$

$(D^2 - 2D - k)X = 0$

A.E is $m^2 - 2m - k = 0$

$m = 1 \pm \sqrt{1+k}$.

$X = CF = C_1 e^{(1+\sqrt{1+k})x} + C_2 e^{(1-\sqrt{1+k})x}$.

$\frac{Y'}{Y} = -k$

$\log Y = -ky + c$

$Y = e^{-ky+c}$

$Y = e^{c} e^{-ky}$

$Y = C_3 e^{-ky}$

$\Rightarrow Z = \left[C_1 e^{(1+\sqrt{1+k})x} + C_2 e^{(1-\sqrt{1+k})x}\right] C_3 e^{-ky}$

Ans

Q By the method of seperation of variables, solve $4\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 3z$ subject to $z = e^{-5y}$ when $x = 0$.

Sol→ $4\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 3z$ --- (1)

Let $Z = XY$ --- (2)

where X and Y are function of x & y.

$\frac{\partial z}{\partial x} = X'Y$ & $\frac{\partial z}{\partial y} = XY'$.

Putting these values in eq. (1), we get

$4X'Y + XY' = 3XY$.

$4X'Y - 3XY = -XY'$

$\Rightarrow \underset{(i)}{\frac{4X' - 3X}{X}} = \underset{(ii)}{\frac{-Y'}{Y}} = \underset{(iii)}{k}$

from (i) & (iii)

$$\frac{4x' - 3x}{x} = k$$

$$4\frac{x'}{x} - 3 = k$$

$$\frac{x'}{x} = \frac{k+3}{4}$$

$$\log x = \frac{k+3}{4}x + C_1$$

$$x = e^{\frac{k+3}{4}x + C_1}$$

$$x = C_1 e^{\frac{k+3}{4}x}$$

from (ii) & (iii).

$$\frac{-y'}{y} = k$$

$$\frac{y'}{y} = -k$$

$$\log y = -ky + C$$

$$y = e^{-ky + C_2}$$

$$y = C_2 e^{-ky}$$

$$Z = C_3 e^{\left(\frac{k+3}{4}x\right)} e^{-ky}$$

$$Z = C_3 e^{\left(\frac{k+3}{4}\right)x - ky} \quad \text{--- --- (3)}$$

On putting $x = 0$.

$$e^{-5y} = C_3 e^{0 - ky}$$

$\Rightarrow$ On comparing $C_3 = 1$, $k = 5$

So, On putting these value in eq (3) we get

$$Z = e^{2x - 5y}$$ Ans

Classification of PDE of II^{nd} order :-

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + f(u) = 0$$

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + f(u) = 0.$$

(1) If $B^2 - 4AC < 0 \rightarrow$ Elliptic

(2) If $B^2 - 4AC = 0 \longrightarrow$ parabolic

(3) If $B^2 - 4AC > 0 \longrightarrow$ Hyperbolic.

Q $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow$ Laplace Equation

Sol→ On Comparing with

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + f(u) = 0$$

$$A = 1, \quad B = 0, \quad C = 1.$$

$$B^2 - 4AC = 0 - 4 \times 1 \times 1$$
$$= -4 < 0$$

So, $B^2 - 4AC < 0$

$\Rightarrow$ Elliptic.

Q. $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \longrightarrow$ 1-D Heat Equation

Sol→

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial u}{\partial t} = 0$$

$$A = 1, \quad B = 0, \quad C = 0$$

$B^2 - 4AC = 0 \Rightarrow$ parabolic

Q $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2}$ $\rightarrow$ 1-D Wave eq^n

Sol $\rightarrow$ $\frac{\partial^2 u}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2} = 0$

$A = 1,\ B = 0,\ C = -\frac{1}{\alpha^2}$

$B^2 - 4AC = 0 - 4\times 1\times\left(-\frac{1}{\alpha^2}\right)$

$= \frac{4}{\alpha^2} > 0$

So, $B^2 - 4AC > 0$

$\Rightarrow$ Hyperbolic.

Q $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

Sol $\rightarrow$ $A = 1,\ B = 2,\ C = 1.$

$B^2 - 4AC = 4 - 4 = 0$

$\Rightarrow$ parabolic

Q. $(x+1)U_{xx} - 2(x+2)U_{xy} + (x+3)U_{yy} = 0$

Sol $\rightarrow$ $A = (x+1),\ B = 2(x+2),\ C = (x+3)$

$B^2 - 4AC = 4(x^2+4+4x) - 4(x^2+4x+3)$

$= 16 - 12 = 4 > 0$

So, $B^2 - 4AC > 0$

$\Rightarrow$ Hyperbolic

Q $x f_{xx} + y f_{yy} = 0 \quad x > 0,\ y > 0$

Sol→ $A = x,\ B = 0,\ C = y$

$B^2 - 4AC = -4xy.$

$-4xy < 0 \quad [\because x > 0, y > 0]$

$\Rightarrow$ Eliptic.

Q $x^2 f_{xx} + (1-y^2) f_{yy} = 0$

Sol→ $A = x^2,\ B = 0,\ C = (1-y^2).$

$B^2 - 4AC = -4x^2(1-y^2)$

$= -4x^2 + 4y^2x^2 = 4x^2(y^2-1).$

for all value of x, x^2 is +ve

Case I :- If $-1 < y < 1$, (y^2-1) is −ve.

$\Rightarrow B^2 - 4AC < 0$

$\Rightarrow$ Elliptic except $x = 0$

Case II :- If $y > 1$ or $y < -1$, then (y^2-1) is +ve.

$\Rightarrow B^2 - 4AC$ is hyperbolic except at $x = 0$.

Case III :- If $y = \pm 1$ and $\forall x$ and $x = 0$ and $\forall y$ $\Big] \rightarrow$ $B^2 - 4AC$ is equal to 0

$\Rightarrow$ parabolic

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Solution of one dimensional wave Equation :-

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (1)}$$

Where $\alpha^2 = \frac{T}{m} \frac{(\text{tension})}{\text{mass/unit length}} = \text{Constant}$

Sol → Let $y = X(x)\,T(t)$ be the solution of (1)

where, X is function of x &

T is function of t.

$$\frac{\partial^2 y}{\partial x^2} = X'' T$$

$$\frac{\partial^2 y}{\partial t^2} = X T''$$

Putting these in eq. (1)

$$X T'' = \alpha^2 X'' T$$

$$\underset{\text{(i)}}{\frac{X''}{X}} = \underset{\text{(ii)}}{\frac{T''}{\alpha^2 T}} = \underset{\text{(iii)}}{k}.$$

from (i) & (iii) & from (ii) & (iii)

$$\frac{X''}{X} = k \qquad\qquad \frac{T''}{\alpha^2 T} = k.$$

$$X'' - kX = 0 \qquad\qquad T'' - \alpha^2 k T = 0$$

Case I :- When $k = \lambda^2$, a positive no.

$$X'' - \lambda^2 X = 0 \qquad T'' - \alpha^2\lambda^2 T = 0$$

$$(D^2 - \lambda^2)X = 0 \qquad (D^2 - \alpha^2\lambda^2)T = 0$$

$$m^2 - \lambda^2 = 0 \qquad (m^2 - \alpha^2\lambda^2) = 0$$

$$m = \pm\lambda \qquad m = \pm\alpha\lambda.$$

$$cf = X = A_1 e^{+\lambda x} + B_1 e^{-\lambda x} \qquad T = C_1 e^{\alpha\lambda t} + D_1 e^{-\alpha\lambda t}.$$

Case II :- $k = -\lambda^2$, a negative no.

$$X'' + \lambda^2 X = 0 \qquad T'' + \alpha^2\lambda^2 T = 0$$

$$(D^2 + \lambda^2)X = 0 \qquad (D^2 + \alpha^2\lambda^2)T = 0$$

$$m^2 + \lambda^2 = 0 \qquad m^2 + \alpha^2\lambda^2 = 0$$

$$m = \pm\lambda i \qquad m = \pm\alpha\lambda i.$$

$$T = A_2 \cos\lambda x + B_2 \sin\lambda x \qquad T = C_2 \cos\alpha\lambda t + D_2 \sin\alpha\lambda t$$

Case III :- $k = 0$

$$X'' = 0 \qquad T'' = 0$$

$$T' = C$$

$$X' = A$$

$$T = C_3 t + D_3$$

$$X = A_3 x + B$$

$$Y = (A_1 e^{\lambda x} + B_1 e^{-\lambda t}) \cdot (C_1 e^{\alpha\lambda t} + D_1 e^{-\alpha\lambda t}).$$

$$Y = (A_2 \cos\lambda x + B_2 \sin\lambda x)(C_2 \cos\alpha\lambda t + D_2 \sin\alpha\lambda t).$$

$$Y = (A_3 x + B)(C_3 t + D_3).$$

we get these 3 solution. 2^{nd} is most appropiat solution.

Q. A tightly stretched string with fixed end point $x=0$ and $x=l$ is initially in the position $y=f(x)$. It is set vibrating by giving to each of its point a velocity $\frac{\partial y}{\partial t} = g(x)$ at $t=0$.

Find $y(x,t)$ in form of Fourier series.

Sol → The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (1)}$$

The boundary condition under which (1) is to be solved are :-

(i) $y(0,t) = 0$ for $t \geq 0$ --- (A)

(ii) $y(l,t) = 0$ for $t \geq 0$ --- (B)

(iii) $y(x,0) = f(x)$ for $0<x<l$. --- (C)

(iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$ for $0<x<l$ --- (D)

Solving eq^n (1) by method of seperation of variable we get

$$y = (A_2 \cos \lambda x + B_2 \sin \lambda x)(C_2 \cos \lambda a t + D_2 \sin \lambda a t).$$

$$\therefore y(x,t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda a t + D \sin \lambda a t). \quad \text{--- (2)}$$

where A, B, C, D are arbitrary constant.

Now, using boundary condition (A) in eq. (2),

$$y(0,t) = (A \cos 0 + B \sin 0)(C \cos \lambda a t + D \sin \lambda a t).$$

$$0 = A [C \cos \lambda a t + D \sin \lambda a t]$$

$\therefore A = 0$.

Now, applying the Boundary condition (B) in eq. (2)

$$y(l,t) = (0 + B\sin\lambda l)(C\cos\lambda at + D\sin\lambda at).$$

$$0 = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at).$$

If $B = 0$ then solution become $y = 0$, which is not true.

So, $B \neq 0$. $\Rightarrow \sin\lambda l = 0$

$$\Rightarrow \lambda l = n\pi$$

$$\Rightarrow \lambda = \frac{n\pi}{l}.$$

$$\therefore y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right]$$

$$\Rightarrow y(x,t) = \sin\frac{n\pi x}{l}\left[BC\cos\frac{n\pi at}{l} + BD\sin\frac{n\pi at}{l}\right]$$

$$\Rightarrow y(x,t) = \sin\frac{n\pi x}{l}\left[C_n\cos\frac{n\pi at}{l} + D_n\sin\frac{n\pi at}{l}\right] \quad --(3)$$

Since the wave eq is linear and homogenous the most general solution of it is

$$y(x,t) = \sum_{n=1}^{\infty}\left[C_n\cos\frac{n\pi at}{l} + D_n\sin\frac{n\pi at}{l}\right]\sin\frac{n\pi x}{l} \quad --(4)$$

Now, on applying boundary Condition C in eq. (4).

$$y(x,0) = \sum_{n=1}^{\infty}\left[C_n\cos 0 + D_n(0)\right]\sin\frac{n\pi x}{l}.$$

$$y(x,0) = \sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{l} = f(x). \quad --(5)$$

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and $\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left[C_n\left(-\sin\frac{n\pi a t}{l}\right)\frac{n\pi a}{l} + D_n\left(\cos\frac{n\pi a t}{l}\right)\cdot\frac{n\pi a}{l}\right]\cdot\left(\sin\frac{n\pi x}{l}\right)$

At $t=0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty}\frac{n\pi a}{l}D_N \sin\frac{n\pi x}{l} = g(x) \quad (6)$$

The left hand side of (6) & (5) are fourier sine series of right hand side function.

Hence. $C_n = \frac{2}{n}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

and $\frac{n\pi a}{l}D_n = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$

Substituting the values of C_n in D_n in eq. (4), we get the solution of wave equation satisfying the given boundary condition.

Q A tightly stretched string with fixed end point $x=0$ and $x=l$ is initially in a position given by $y(x,0) = y_0\sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement y at any time and at any distance from the end $x=0$.

Sol → The displacement y of particle at a distance x from end $x=0$ at time t is given by.

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (1)}$$

the boundary condition are :

$y(0,t) = 0 \quad , t \geq 0$ --- (A)

$y(l,t) = 0 \quad t \geq 0$ --- (B)

$y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right). \quad 0 \leq x \leq l$ --- (C)

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad$ for $0 \leq x \leq l$. --- (D)

Now, solving (1) and selecting proper solution to suit the physical nature of problem and making use of boundary condition (A) & (B) just like previous ques.

$$y(x,t) = B \sin\frac{n\pi x}{l}\left[C \cos\frac{n\pi at}{l} + D \sin\frac{n\pi at}{l}\right] \quad \text{--- (2)}$$

Using boundary condition (D) .

$$\frac{\partial y}{\partial t} = B \sin\frac{n\pi x}{l}\left[C\left(-\sin\frac{n\pi at}{l}\right)\frac{n\pi a}{l} + D \cos\frac{n\pi at}{l} \cdot \frac{n\pi a}{l}\right]$$

At $t = 0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = B \sin\frac{n\pi x}{l}\left[0 + D \cos\frac{n\pi a 0}{l} \cdot \frac{n\pi a}{l}\right]$$

$$0 = B \sin\frac{n\pi x}{l} \cdot D \cdot \frac{n\pi a}{l}$$

$\Rightarrow \because B \neq 0$, so, $D = 0$

$$\Rightarrow y(x,t) = B \sin\frac{n\pi x}{l}\left[C \cos\frac{n\pi at}{l}\right]$$

$$\Rightarrow y(x,t) = B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}.$$

The most general solution satisfying ① and boundary condition is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}. \quad \text{--- ③}$$

Now, using boundary condition © in eq. ③.

$$y_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cdot\cos 0$$

$$\Rightarrow \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = B_1\sin\frac{\pi x}{l} + B_2\sin\frac{2\pi x}{l} + B_3\sin\frac{3\pi x}{l} + \cdots$$

On comparing: $B_1 = \frac{3y_0}{4}$, $B_2 = 0$, $B_3 = \frac{-y_0}{4}$

& $B_n = 0$ [for all $n > 3$]

Put these value in eq ③ we get-

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l}$$

Ans

Q. If a string of lenght l is initially at rest in equilibrium position and each point of it is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = V_0 \sin\frac{3\pi x}{l}$, $0<x<l$, determine the transverse displacement $y(x,t)$.

Sol→ The displacement is governed by.

$$\frac{\partial^2 y}{\partial^2 t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2} \quad \text{---} \; (1)$$

Boundary Condition are.

$$y(0,t) = 0 \quad \text{for } t \geq 0 \quad \text{---} \; (i)$$

$$y(l,t) = 0 \quad \text{for } t \geq 0 \quad \text{---} \; (ii)$$

$$y(x,0) = 0 \quad \text{for } 0 \leq x \leq l \quad \text{---} \; (iii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = V_0 \sin^3 \frac{\pi x}{l} \quad \text{for } 0 \leq x \leq l \quad \text{---} \; (iv)$$

Solving eq. (1) and using boundary Condition (i) & (ii).

we get,

$$y(x,t) = B \sin \frac{n\pi x}{l} \left[C \cos \frac{n\pi a t}{l} + D \sin \frac{n\pi a t}{l} \right] \quad \text{---} \; (2)$$

Using boundary Condition (iii).

$$y(x,0) = B \sin \frac{n\pi x}{l} \left[C \cos 0 + 0 \right].$$

$$\Rightarrow C = 0 \quad [\because B \neq 0].$$

Therefore $y(x,t) = B \sin \frac{n\pi x}{l} \, D \sin \frac{n\pi a t}{l}$.

∴ The most common general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi a t}{l}.$$

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \cdot \frac{n\pi a}{l}.$$

At $t = 0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot \frac{n\pi a}{l}.$$

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$$V_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\left(\frac{\pi n a}{l}\right).$$

$$\frac{V_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = B_1 \sin\frac{\pi x}{l}\cdot\frac{\pi a}{l} + B_3 \sin\frac{3\pi x}{l}\cdot\frac{3\pi a}{l} + \ldots$$

On Comparing.

$$\frac{\pi a}{l}B_1 = \frac{3V_0}{4} \quad \& \quad B_3 \cdot \frac{3\pi a}{l} = \frac{-V_0}{4} \quad \& \quad B_n = 0.$$

$$B_1 = \frac{3V_0 l}{4\pi a}, \quad B_3 = \frac{-V_0 l}{12\pi a}, \quad B_n = 0.$$

On putting this in eq. 3, we get.

$$y(x,t) = \frac{3lV_0}{4\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi a t}{l} - \frac{V_0 l}{12\pi a}\sin\frac{3\pi x}{l}\sin\frac{3\pi a t}{l}$$

Ans

Homework :- Do yourself.

Q Example No. 7 :-

A tightly stretched string with end points $x=0$ & $x=l$ is initially at rest at position $y(x,0) = y_0 \sin\frac{\pi x}{l}$. It is released from rest find the displacement.

Sol→ Same as 2nd previous till Eq. 3.

Now, using boundary condition (c) in eq. 3

$$y(x,0) = y_0 \sin\frac{\pi x}{l}.$$

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$\Rightarrow y_0 \sin\frac{\pi x}{l} = B_1 \sin\frac{\pi x}{l} + B_2 \sin\frac{2\pi x}{l} + \dots$

On comparing. $B_1 = y_0$. $B_n = 0$.

$\Rightarrow y(x,t) = y_0 \sin\frac{\pi x}{l} \sin\frac{\pi a t}{l}$. Ans.

Q. A tightly stretched string with fixed end point $x=0$ and $x=l$ is initially at rest in equilibrium position. If it is set vibrating giving each point a velocity $3x(l-x)$. find the displacement.

Sol→ The displacement is governed by.

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (1)}$$

Boundary condition are

$$y(0,t) = 0 \quad t \geq 0 \quad \text{--(i)}$$
$$y(l,t) = 0 \quad t \geq 0 \quad \text{--(ii)}$$
$$y(x,0) = 0 \quad 0 \leq x \leq l \quad \text{--(iii)}$$
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 3x(l-x) \quad 0 \leq x \leq l. \quad \text{--(iv)}$$

Now, on solving (1) and using boundary condition (i) & (ii)

$$y(x,t) = B \sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi a t}{l} + D\sin\frac{n\pi a t}{l}\right). \quad \text{--(2)}$$

On using (iii) in eq. (2)

$$0 = B \sin\frac{n\pi x}{l}\, C\cos\frac{n\pi a t}{l}$$
$$\Rightarrow C = 0 \quad [\because B \neq 0].$$

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$$\therefore\ y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}.$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}\left(\frac{n\pi a}{l}\right).$$

at $t=0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n\left(\sin\frac{n\pi x}{l}\right)\cdot\left(\frac{n\pi a}{l}\right).$$

$$3x(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l 3x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{6}{l}\left[\frac{x(l-x)}{\frac{n\pi}{l}}\left(-\cos\frac{n\pi x}{l}\right) - \frac{(l-2x)}{\frac{n^2\pi^2}{l^2}}\left(-\sin\frac{n\pi x}{l}\right) + \frac{(-2)}{\frac{n^3\pi^3}{l^3}}\left(\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{6}{l}\left[0 - 0 - 2\frac{l^3}{n^3\pi^3}\cos n\pi - 0 + 0 + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{12l^3}{l n^3\pi^3}\left[1 - \cos n\pi\right]$$

$$= \frac{12l^2}{n^3\pi^3}\left[1-(-1)^n\right].$$

$$b_n = \begin{cases} 0 & \text{when } n = \text{even} \\ \dfrac{24l^2}{n^3\pi^3} & \text{when } n = \text{odd} \end{cases}$$

$$B_n\left(\frac{n\pi a}{l}\right) = b_n$$

$$\Rightarrow B_n = \begin{cases} 0 & \text{when } n = \text{even} \\ \dfrac{24l^3}{an^4\pi^4} & \text{when } n = \text{odd} \end{cases}$$

$$\Rightarrow y(x,t) = \frac{24l^3}{a\pi^4} \sum_{n=1,3,5}^{\infty} \frac{1}{n^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}.$$

$\boxed{\text{Or}}$

$$y(x,t) = \frac{24l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi x}{l} \sin\frac{(2n-1)\pi a t}{l}.$$

Ans

Q. A taut string of length $2l$ is fastened at both ends. The mid point of string is taken to a height b and then released from the rest in that position. Find the displacement of the string.

Sol→ The displacement is governed by

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$

The Boundary conditions are :-

$$y(x,0) = 0 \quad t \geq 0$$

$$y(2l, t) = 0 \quad t \geq 0$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad 0 \leq x \leq l$$

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$$y(x,0) = \frac{b}{l}x \qquad 0 \le x \le l$$
$$= \frac{-b}{l}(x-2l) \qquad l \le x \le 2l.$$

$$\left[\because \text{ eq. of } OA \Rightarrow y = \frac{b}{l}x \right.$$
$$\left. \& \text{ eq. of } AB \Rightarrow \frac{y-0}{x-2l} = \frac{b-0}{l-2l}\right]$$

Now, on solving the eq. ① ~~and~~, we get

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$$

On using 1st boundary condition.

$$y(0,t) = A(C\cos\lambda at + D\sin\lambda at)$$
$$\Rightarrow A = 0$$

On using second boundary condition.

$$y(2l,t) = 0$$
$$\Rightarrow B\sin 2l\lambda(C\cos\lambda at + D\sin\lambda at) = 0$$
$$\Rightarrow \sin 2l\lambda = 0$$
$$\Rightarrow \lambda = \frac{n\pi}{2l}.$$

On using 3rd boundary condition.

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$
$$\Rightarrow D = 0$$

$$\therefore\ y(x,t) = B\sin\frac{n\pi x}{2l}\, C\cos\frac{n\pi a t}{2l}.$$

$$y(x,t) = \sum B_n \sin\frac{n\pi x}{2l}\cos\frac{n\pi a t}{2l}.$$

On using boundary condition (iv) in eq. 2

$$\sum B_n \sin\frac{n\pi x}{2l}\cos 0^{\circ} = \frac{b}{l}x \qquad 0\le x\le l$$

$$B_n \sin\frac{n\pi x}{2l} = \frac{b}{l}(x-2l) \qquad l\le x\le 2l.$$

$$B_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}\,dx$$

$$= \frac{1}{l}\left[\int_0^{l}\frac{b}{l}x\sin\frac{n\pi x}{2l} + \int_l^{2l}\frac{b}{l}(x-2l)\sin\frac{n\pi x}{2l}\right]$$

$$B_n = \frac{b}{l^2}\left[\left\{\frac{x\cdot 2l}{n\pi}\left(-\cos\frac{n\pi x}{2l}\right) - \left(-\sin\frac{n\pi x}{2l}\right)\cdot\frac{4l^2}{n^2\pi^2}\right\}_0^{l}\right.$$

$$\left. - \left\{\frac{(x-2l)\cdot 2l}{n\pi}\left(-\cos\frac{n\pi x}{2l}\right) - \left(-\sin\frac{n\pi x}{2l}\right)\cdot\frac{4l^2}{n^2\pi^2}\right\}_l^{2l}\right]$$

$$B_n = \frac{b}{l^2}\left[\left\{\cancel{\frac{2l^2}{n\pi}\left(-\cos\frac{n\pi}{2}\right)} + \frac{4l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right) - 0 - 0\right\}\right.$$

$$\left. - \left\{0 - 0 - \cancel{\left(\frac{2l^2}{n\pi}\cos\frac{n\pi}{2}\right)} - \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$B_n = \frac{b}{l^2}\left[\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$B_n = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

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$$B_n = \begin{cases} 0 & n = \text{even} \\ \dfrac{8b}{n^2\pi^2} \sin\dfrac{n\pi}{2} & n = \text{odd} \end{cases}$$

$$\Rightarrow y(x,t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin\frac{(2n-1)\pi x}{2l} \cdot \sin\frac{(2n-1)\pi a t}{2l} \cdot \sin(2n-1)\frac{\pi}{2}$$

$$\Rightarrow y(x,t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \sin\frac{(2n-1)\pi x}{2l} \cos\frac{(2n-1)\pi a t}{2l}$$

Ans

Q. Solve the problem of vibrating string for the following boundary condition :-

(i) $y(0,t) = 0$. (ii) $y(l,t) = 0$

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = x(x-l) \quad 0 \le x \le l$

(iv) $y(x,0) = x$ in $0 \le x \le l/2$

$\qquad = l - x$ in $l/2 \le x \le l$.

Sol→ This is solved same as Question 1. but in that $f(x)$ and $g(x)$ were given & in this question it is specified

$$g(x) = x(x-l)$$

$$f(x) = x \qquad 0 \le x \le \frac{l}{2}.$$

$$(l-x) \qquad \frac{l}{2} \le x \le l.$$

Hence. $C_n = \dfrac{2}{l}\displaystyle\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$

$$C_n = \frac{2}{l}\left[\int_0^{l/2} x \sin\frac{n\pi x}{2} + \int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{2}\right]$$

$$C_n = \frac{2}{l}\left[\left\{\frac{xl}{n\pi}\left(-\cos\frac{n\pi x}{l}\right) - \frac{\left(-\sin\frac{n\pi x}{l}\right)}{n^2\pi^2/l^2}\right\}_0^{l/2} + \left\{(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$C_n = \frac{2}{l}\left[\left\{\frac{l^2}{2n\pi}\left(-\cos\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - 0 + 0\right\} + \left\{0 - 0 - \left(-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$C_n = \frac{2}{l}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$C_n = \begin{cases} 0 & \text{when } n = \text{even} \\ \frac{4l}{n^2\pi^2} & \text{when } n = \text{odd} \end{cases}$$

Now, $$D_n = \frac{2}{n\pi a}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

$$D_n = \frac{2}{n\pi a}\int_0^l x(x-l)\sin\frac{n\pi x}{l}\,dx$$

$$D_n = \frac{2}{n\pi a}\left[(x^2-lx)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2x-l)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2}{2\pi n a}\left[\cancel{0+0 + 2\cos\frac{n\pi}{2}\cdot\frac{l^2}{n^3\pi^3}} - \right.$$

$$= \frac{2}{2 n\pi a}\left[0 + 0 + \frac{2l^3}{n^3\pi^3}\cos n\pi - 0 - 0 - \frac{2l^3}{n^3\pi^3}\cos 0\right]$$

$$= \frac{2}{n\pi a}\left[\frac{2l^3}{n^3\pi^3}\left((-1)^n - 1\right)\right]$$

$$D_n = \begin{cases} 0 & \text{for } n = \text{even} \\ \dfrac{-8l^3}{n^4\pi^4 a} & \text{for } n = \text{odd} \end{cases}$$

So,

$$y(x,t) = \sum_{n=1,3,5}^{\infty} \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi a t}{l}$$

$$- \sum_{n=1,3,5}^{\infty} \frac{-8l^3}{n^4\pi^4 a}\sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}$$

Ans

One dimensional Heat flow :-

We shall consider the heat of flow of heat and the accompanying variation of temperature with position and with time in conducting solids.

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$\alpha^2 = \frac{k}{\rho c} \quad \text{diffusivity} \left(\frac{cm^2}{sec}\right)$$

Solution of One dimensional heat eq^n :-

we have to solve the eq.

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

Assume a solution of form.

$$u(x,t) = X(x) \cdot T(t).$$

where X and T is function of x & t respectively.

$$\frac{\partial u}{\partial t} = XT' \quad , \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$\Rightarrow XT' = \alpha^2 X'' T.$$

$$\Rightarrow \frac{T'}{T} = \alpha^2 \frac{X''}{X} = k.$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{\alpha^2 T} = k$$

Hence $X'' - kX = 0$ & $T' - \alpha^2 kT = 0.$

Case I :- Let $k = \lambda^2$ a positive no.

$$X'' - \lambda^2 X = 0$$

$$(D^2 - \lambda^2) X = 0$$

$$m^2 - \lambda^2 = 0$$

$$\Rightarrow m = \pm \lambda$$

$$\Rightarrow X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}$$

$$T' - \alpha^2 \lambda^2 T = 0$$

$$(D - \alpha^2 \lambda^2) T = 0$$

$$m - \alpha^2 \lambda^2 = 0$$

$$m = \alpha^2 \lambda^2$$

$$\Rightarrow T = C_1 e^{\alpha^2 \lambda^2 t}$$

Case 2 :- $k = -\lambda^2$ (a negative no.)

$$X'' + \lambda^2 X = 0 \qquad T' + \alpha^2\lambda^2 T = 0$$

$$\Rightarrow m = \pm \lambda i; \qquad m = -\alpha^2\lambda^2.$$

$$\Rightarrow X = A_2 \cos\lambda x + B_2 \sin\lambda x \qquad T = C_2 e^{-\alpha^2\lambda^2 t}$$

Case 3 :- $k = 0$

$$X'' = 0 \qquad T' = 0$$

$$X = A_3 x + B_3 \qquad T = C_3$$

Thus, the various possible solution of heat eq. are :-

$$U(x,t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})\, C_1 e^{a^2\lambda^2 t} \quad \text{—— (I)}$$

$$U(x,t) = (A_2 \cos\lambda x + B_2 \sin\lambda x)\, C_2 e^{-\alpha^2\lambda^2 t} \quad \text{—— (II)}$$

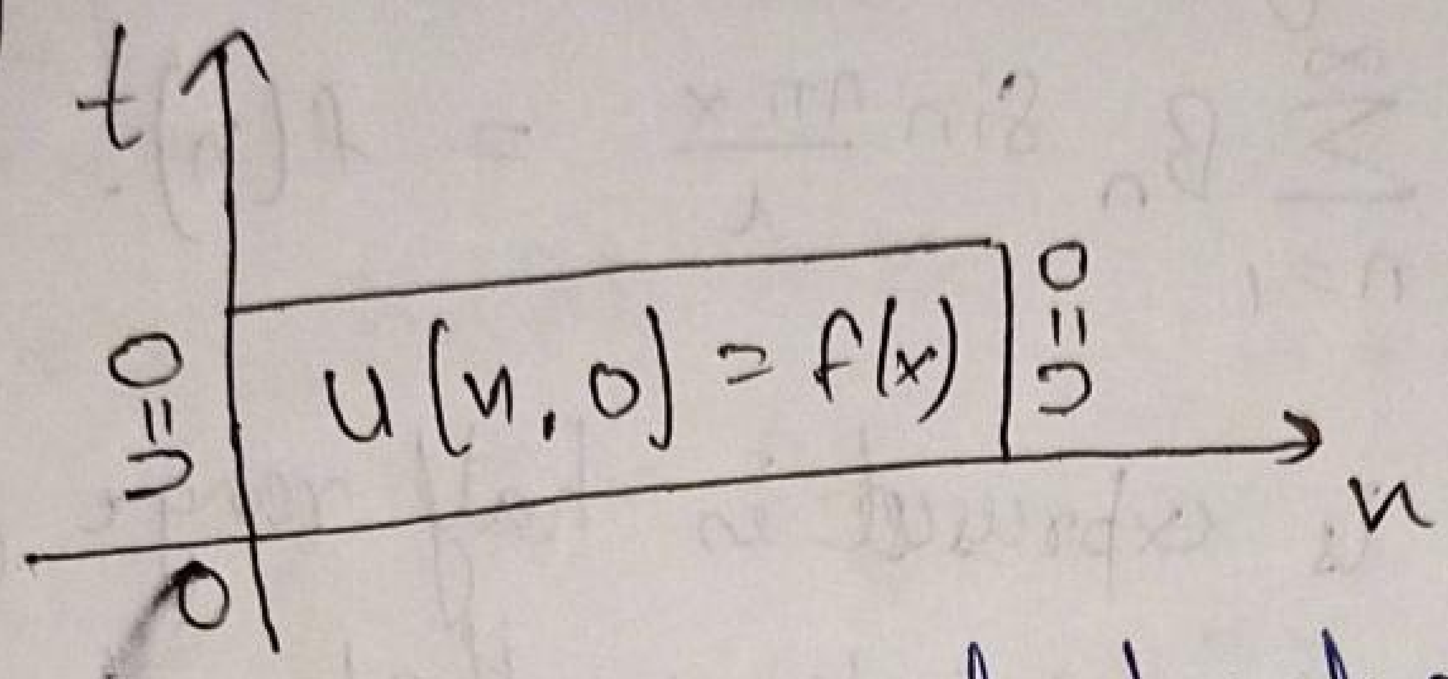
$$U(x,t) = (A_3 x + B_3)\, C_3 \quad \text{—— (III)}$$

Out of these 3, we have to choose the most appropiate one which suit the physical nature of the problem. As we are concerned that heat conduction must decrease with time increase.

Hence we select the solution (II).

In steady stat condition, when temp. no longer varies with time, the solution of diffusion eq. (1) will be the last solution (III).

Q. A rod l cm with insulated lateral surface is initially at temperature $f(x)$ at an inner point distant x cm from one end. If both ends are kept at zero temperature, find the temperature at any point of the rod at any subsequent time.



Sol→ Let $u(x,t)$ be the temperature at any point distant x from one end at any time t seconds.
Then u satisfies the P.D.E.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \quad \text{--- (1)}$$

The Boundary conditions are :-

$u(0,t) = u(l,t) = 0$ for all $t \geq 0$.

and $u(x,0) = f(x)$ for $0 < x < l$.

Solving eq. (1) by method of separation of variable and selecting the suitable solution, we get

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, Ce^{-\alpha^2\lambda^2 t} \quad \text{--- (2)}$$

Using Boundary Condition (i) in eq.

$$u(0,t) = ACe^{-\lambda^2\alpha^2 t} = 0$$

$$\Rightarrow A = 0$$

$$u(l,t) = B\sin\lambda l \cdot Ce^{-\lambda^2 a^2 t} = 0 \quad [\because B \neq 0]$$

$$\Rightarrow \sin\lambda l = 0$$

$$\Rightarrow \lambda = \frac{n\pi}{l}.$$

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$$\therefore \quad u(x,t) = B \sin\frac{n\pi x}{l} \cdot C e^{-\frac{\alpha^2 n^2 \pi^2}{l^2} t}$$

$$u(x,t) = \sum B_n \sin\frac{n\pi x}{l} \, e^{-\frac{\alpha^2 n^2 \pi^2}{l^2} t} \quad \text{--- (3)}$$

Using Boundary Condition (iii).

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x).$$

If $u(x,0)$ is expressed in half range fourier series in $0<x<l$. we know that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} \, dx$$

$$\therefore \quad u(x,t) = \sum \left[\frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} \, dx\right] \sin\frac{n\pi x}{l} \, C e^{-\frac{\alpha^2 n^2 \pi^2}{l^2}}$$

Ans

Q Solve $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to condition.

(i) u is not infinit as $t \to \infty$.

(ii) $\frac{\partial u}{\partial x} = 0$ for $x=0$ & $x=l$.

(iii) $u = lx - x^2$ for $t=0$, $0<x<l$.

Sol→ $u(x,t) = [A\cos\lambda x + B\sin\lambda x] \, C e^{-\lambda \alpha^2 t}$

$$\frac{\partial u}{\partial x} = [-A\lambda \sin\lambda x + B\lambda\cos\lambda x] \, C e^{-\alpha^2 \lambda^2 t}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = (\cancel{0} + B\lambda) C e^{-\lambda^2\alpha^2 t} = 0$$

$0 + B\lambda$

$$\Rightarrow B = 0.$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=l} = -A\lambda \sin\lambda l \, C e^{-\lambda^2\alpha^2 t} = 0.$$

$$\Rightarrow \sin\lambda l = 0$$

$$\Rightarrow \lambda = \frac{n\pi}{l}$$

$$\therefore u(x,t) = A_n \cos\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}} \quad \text{--- (2)}$$

$$u(x,t) = \sum A_n \cos\frac{n\pi x}{l} e^{-\frac{\alpha^2\pi^2 n^2 t}{l^2}}$$

using $u(x,0) = lx - x^2$ for $0 < x < l$ in eq (2)

$$u(x,0) = \sum A_n \cos\frac{n\pi x}{l} = lx - x^2$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} = lx - x^2$$

$$A_0 = \frac{a_0}{2} = \frac{1}{l}\int_0^l (lx - x^2)\,dx$$

$$= \frac{1}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l$$

$$= \frac{l^2}{6}$$

$$A_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[0 - \frac{l^3}{n^2\pi^2}\cos n\pi + 0 - 0 - \frac{l^3}{n^3\pi^3} + 0\right]$$

$$= \frac{2}{l}\left[\frac{-l^3}{n^2\pi^2}\left(\cos n\pi + 1\right)\right.$$

$$A_n = \begin{cases} 0 & \text{for } n = 0 \\ \dfrac{-4l^2}{n^2\pi^2} & \text{for } n = \text{even.} \end{cases}$$

$$\therefore\ u(x,t) = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\sum_{n=2,4,6}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}\cdot e^{-\frac{\alpha^2\pi^2 n^2 t}{l^2}}$$

Or

$$u(x,t) = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{l}\, e^{-\frac{4\alpha^2\pi^2 n^2 t}{l^2}}$$

Ans

Q. A rod 30 cm long has its end A and B kept at 20°C and 80°C respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature function $u(x,t)$ taking $x=0$ at A.

Sol→

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \qquad \text{--- (1)}$$

In steady state, the temperature at any particular point does not vary with time. Hence, P.D.E (1) in steady state becomes.

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{--- (2)}$$

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$\Rightarrow u = ax + b$.

In steady stat condition.

$u = 20$ at $x = 0$ & $u = 80$ at $x = 30$

$\Rightarrow 20 = b$ & $80 = 30a + 20$

$\Rightarrow a = 2$

$\Rightarrow u(x) = 2x + 20$ ——— (3)

When the temperature at A and B are reduced to 0, the temp distribution changes and the stat is no more steady stat.

Boundary condition at this condition :-

$u(0, t) = 0 \quad \forall\ t \geq 0$ — (i)

$u(30, t) = 0 \quad \forall\ t \geq 0$ — (ii)

$u(x, 0) = 2x + 20$ [$\because$ Initial temp of this stat is temp in previous stat]

— (iii)

The suitable solution of eq-(1) is

$u(x, t) = (A\cos\lambda t + B\sin\lambda x) \cdot e^{-\lambda^2 a^2 t}$ ——— (4)

On using Boundary condition (i) in eq (4)

$A = 0$

On using (ii) in (4)

$\lambda = \frac{n\pi}{30}$

So, eq(4) is reduced to

$u(x, t) = B_n \sin\frac{n\pi x}{30} \cdot e^{-\frac{n^2 a^2 \pi^2 t}{900}}$

$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30} \cdot e^{-\frac{n^2 \alpha^2 \pi^2 t}{900}}$ ——— (5)

On using (iii) on (5)

$$u(x,0) = 2x+20 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30}$$

$$\Rightarrow B_n = \frac{2}{30}\int_0^{30} (2x+20)\sin\frac{n\pi x}{30}$$

$$= \frac{1}{15}\left[(2x+20)\left(\frac{-\cos\frac{n\pi x}{30}}{\frac{n\pi}{30}}\right) - (2)\left(\frac{-\sin\frac{n\pi x}{30}}{\frac{n^2\pi^2}{30^2}}\right)\right]_0^{30}$$

$$= \frac{1}{15}\left[\frac{80\times 30}{n\pi}(-\cos n\pi) + 0 + \frac{20\cos 0\times 30}{n\pi} - 0\right]$$

$$= \frac{20\times 30}{15 n\pi}\left[1-4(-1)^n\right]$$

$$= \frac{40}{\cancel{15}\, n\pi}\left[1-4(-1)^n\right]$$

Substituting it in eq. (5)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{15 n\pi}\left[1-4(-1)^n\right]\sin\frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{900}}$$

Ans

Do Yourself :-

Example - 16

Example - 17

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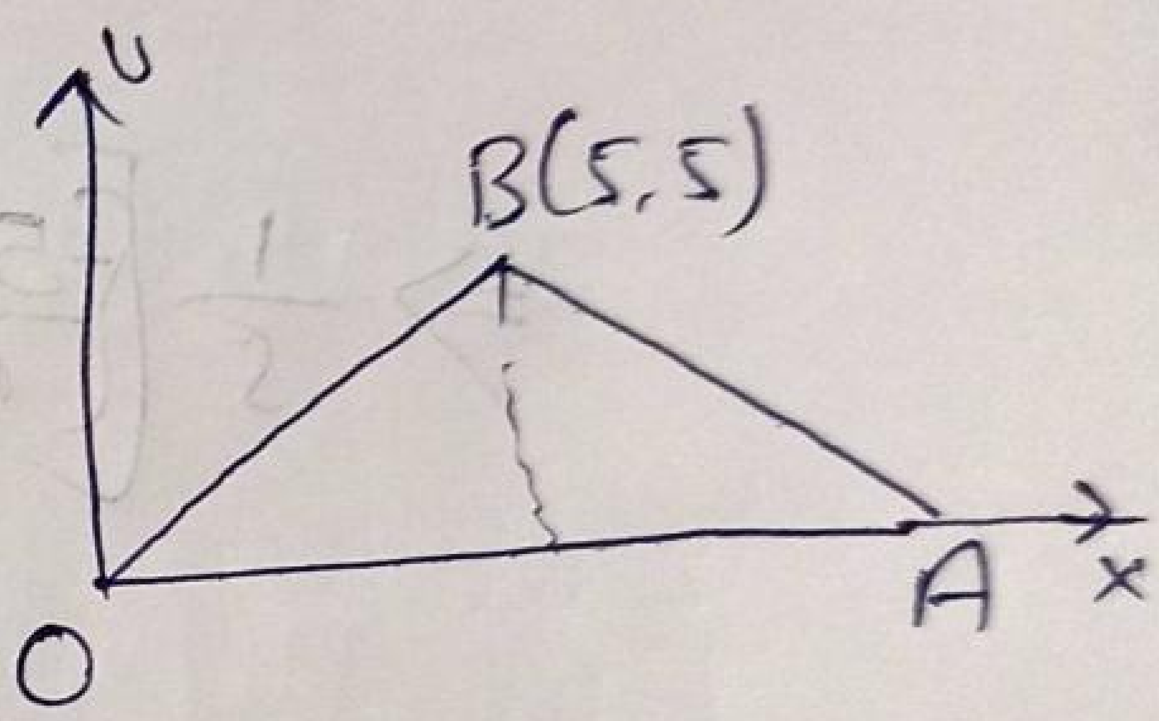
Q. Find the temperature $u(x,t)$ in a silver bar of lenght 10 cm. constant cross section of 1 cm^3 area. density 10.6 g/cm^3. thermal conductivity 1.04 calori/cm deg-sec. Specific heat (0.056 cal/gm.deg.) which is perfectly insultated laterally. if the end at italic 0°C and if initially the temp is italic 5°C at the centre of the bar. and falls uniformally to zero at ends.

Sol→ $\alpha^2 = \frac{K}{\rho c} = \frac{1.04}{(10.6)(0.056)} \approx 1.75$



Equation to OB ⇒ $u = x$

Equation to BA $= u = -(x-10)$.

$\Rightarrow u = 10 - x$.

$$\Rightarrow f(x) = \begin{cases} x & 0<x<5 \\ 10-x & 5<x<10 \end{cases}$$

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{---} \quad (1)$$

Boundary Condition :-

$u(0,t) = 0$ ——— (i)

$u(l,t) = 0$ ——— (ii)

$u(x,0) = f(x)$. ——— (iii)

On solving (1) and using boundary condition (i) & (ii)

$A = 0$

$\lambda = \frac{n\pi}{10}$.

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} \cdot e^{-\alpha^2 \pi^2 n^2 t/(1.7)^2}$$

$$\Rightarrow B_n = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{1}{5}\left[\int_0^5 x\sin\frac{n\pi x}{10}\,dx + \int_5^{10}(10-x)\sin\frac{n\pi x}{10}\,dx\right]$$

$$= \frac{1}{5}\left[\left\{x\left(-\cos\frac{n\pi x}{10}\right)\cdot\frac{10}{n\pi} - 1\left(-\sin\frac{n\pi x}{10}\right)\cdot\frac{100}{n^2\pi^2}\right\}_0^5 + \left\{(10-x)\left(-\cos\frac{n\pi x}{10}\right)\cdot\frac{10}{n\pi} - (-1)\left(-\frac{\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right\}_5^{10}\right]$$

$$\Rightarrow \frac{1}{5}\left[\left\{-\frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + 0 - 0\right\} + \left\{0 - 0 + \frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$\Rightarrow \frac{1}{5}\left[\frac{200}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$\Rightarrow B_n = \begin{cases} 0 & \text{for } n = \text{even} \\ +\frac{40}{n^2\pi^2} & \text{for } n = 1, 5, 9 \ldots \\ -\frac{40}{n^2\pi^2} & \text{for } n = 3, 7, \ldots \end{cases}$$

$$\Rightarrow u(x,t) = \frac{-40}{\pi^2}\left[\frac{1}{1^2}\sin\frac{n\pi}{10}e^{-0.0175\pi^2 t} - \frac{1}{3^2}\sin\frac{3n\pi}{10}e^{-0.0125(3\pi)^2 t} + \ldots\right]$$

Ans

Q. A bar 10 cm long with insulated sides, has its end A and B kept at 20°C and 40°C respectively. until steady state conditions prevail that is until ~~sta~~ the temp. at any interior point no longer changes with time. The temp. at A is then suddenly raised to 50°C and at same instant that at B is lowered to 10°C. find the subsequently temp. function $u(x,t)$ at any time.

Sol→ $u(x,t) = \bar{U}_s(x) + U_t(x,t)$.

In steady condition.

$$\frac{\partial^2 u}{\partial x^2} = 0.$$

$\Rightarrow u = ax + b.$

$u = 20$ at $x = 0$ $\qquad$ $u = 40$ at $x = 10$.

$\Rightarrow b = 20$ $\qquad$ $\Rightarrow 40 = 10a + b.$

$\Rightarrow a = 2$

$\Rightarrow u(x) = 2x + 20$

Now, Boundary condition in 2nd stage.

$u(0,t) = 50.$

$u(10,t) = 10$

$u(x,0) = 2x + 50.$

We know, $u(x,t) = u_s(x) + U_t(x,t).$

$u_s(x)$ satisfy. $\frac{\partial^2 u}{\partial x} = 0.$

where $U_s(0) = 50$

$u_s(10) = 10$

$U_s(x) = ax + b$

$b = 50$, [from $U_s(0) = 50$]

$a = -4$ [by putting b & $U_s(10) = 10$]

$\Rightarrow U_s(x) = -4x + 50.$

Consequently.

$U_t(0,t) = u(0,t) - U_s(0)$

$= 50 - 50 = 0$ — (i)

$U_t(10,t) = u(10,t) - U_s(10) = 10 - 10$

$= 0.$ — (ii)

$U_t(x,0) = u(x,0) - U_s(x)$

$= 2x + 20 - (50 - 4x)$

$U_t(x,0) = 6x - 30$ — (iii)

Now, we know, $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

On solving eq. and using boundary condition (i)

$A = 0.$ (ii)

On using (ii)

$\lambda = \frac{n\pi}{10}$

$\Rightarrow U_t(x,t) = B_n \sin\frac{n\pi x}{10} e^{-\frac{\alpha^2\pi^2 n^2 t}{100}}$

On using (iii):

$u(x,0) = 6x - 30 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}.$

$$B_n = \frac{2}{10}\int_0^{10}(6x-30)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{1}{5}\left[(6x-30)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) + 6\left[\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right]\right]_0^{10}$$

$$\Rightarrow \frac{1}{5}\left[\frac{-300}{n\pi}\cos n\pi + 0 + \left(\frac{-300}{n\pi}\cos n\pi\right) + 0\right]$$

$$= \frac{-60}{n\pi}\left[1 + (-1)^n\right].$$

$$\Rightarrow B_n = \begin{cases} 0 & \text{for } n = \text{odd} \\ \frac{-120}{n\pi} & \text{for } n = \text{even} \end{cases}$$

$$u(x,t) = \sum_{n=2,4,6}^{\infty}\left(\frac{-120}{n\pi}\right)\sin\frac{n\pi x}{10}\,e^{-\frac{\alpha^2 n^2\pi^2 t}{100}}$$

or

$$u(x,t) = \sum_{n=1}^{\infty}\frac{-60}{n\pi}\sin\frac{n\pi x}{5}\,e^{-\frac{\alpha^2 n^2\pi^2 t}{25}}$$

Ans

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