Unit-2

Fourier Series

Periodic function :- A function $f(x)$ is said to be periodic if for all x.

$$f(x+T) = f(x).$$

where T is a +ve constant.

The least value of $T > 0$ is called the period of $f(x)$.

Ex→ ① $f(x) = \sin x$

$f(x+2\pi) = \sin(x+2\pi) = \sin x.$

$\Rightarrow f(x) = f(x+2\pi)$

$\therefore \sin x$ is periodic function with period 2π.

② $f(x) = \tan x$

$f(x+\pi) = \tan(x+\pi) = \tan x.$

Here $f(x) = f(x+\pi)$

$\therefore \tan x$ is periodic function with period π.

③ $f(x) = \cos 2x$

$f(x+\pi) = \cos(2x+\pi) = \cos(2\pi+2x) = \cos 2x$

$\Rightarrow f(x) = f(x+\pi)$

$\therefore \cos 2x$ is periodic function with period π.

* Fourier Series :- A series whose each and every term is either of sine or cosine.

$$= a_0 \cos 0x + a_1 \cos 1x + a_2 \cos 2x + \text{- - - -}$$
$$+ b_0 \sin 0x + b_1 \sin 1x + b_2 \sin 2x + \text{- - -}$$

Or

It is a mathematical way to represent Non-trigonometric period function as an infinit sum of trigonometric function

Or

It is an expression used for period signals to expand into sum of set of simple oscillating functions.

Mathematically,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(n) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\, dx.$$

Standard form of fourier series :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

where,

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx.$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

NOTE:- ① Over the interval $(0, 2l)$:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$

where, $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

② Over the Interval $(-\pi, \pi)$:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

③ Over the Interval $(0, 2\pi)$:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

Fourier Series Interval

- Full Range
  - n limit
    - $(0, 2\pi)$
    - $(-\pi, \pi)$
  - Constant limit
    - $(0, 2c)$
    - $(-c, c)$
- Half Range
  - n limit
    - $(0, \pi)$
  - Constant limit
    - $(0, c)$

NOTE :- Among all these limits we can take additional benifits of being Even or odd function with $(-\pi, \pi)$ or $(-c, c)$ only.

Dirichelt's Boundary Conditions :-

A fourier series can be written for a function $f(x)$ if it will satisfy following properties :-

1. $F(x)$ must be periodic, finit & single valued.
2. $F(x)$ has discontinouties at finite no. of points.
3. $F(x)$ has at most a finit maxima and minima in any one period.

Then, fourier series of $f(x)$ converges to :-

(a) $f(x)$ if x is a point of continuity.

(b) $\dfrac{f(x+a) + f(x-a)}{2}$ if $x = a$ is a point of discontinuity.

Application of fourier Series :-

1. Any wave can be expressed in terms of sine or Cosine wave (fourier Series) which is ideal for transmission in Communication system.
2. used in compressing any wave form.
3. used in signal processing, modulation & demodulation of voice signals.

Even & Odd Functions :-

Even Function :- A function is said to be even if $f(-x) = f(x)$ for all x.

Ex → $f(x) = x^2$

$f(x) = \cos x$.

Odd function :- A function is said to be odd function if $\forall x$ $f(-x) = -f(x)$.

Ex → $f(x) = x^3$

$f(x) = \sin x$.

Properties of Even/odd function :-

(1) even function × even function = even function.

Ex → $x^2 \times x^4 = x^6$

(2) odd function × even function = odd function.

Ex → $x \times x^4 = x^5$

(3) even function × odd function = odd function.

Ex → $x^2 \times x = x^3$

(4) odd function × odd function = even function.

Ex → $x^3 \times x = x^4$

(5) If $f(x)$ is even function, then

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)$$

(6) If $f(x)$ is odd function, then

$$\int_{-l}^{l} f(x)\,dx = 0$$

Important Results :-

(1) If $f(x)$ is odd function, then.

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = 0$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x) \cdot \cos\frac{n\pi x}{l}}_{\text{odd} \times \text{even}}\,dx = 0$$

(2) If $f(x)$ is even function, then.

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)\,\sin\frac{n\pi x}{l}}_{\text{even} \times \text{odd}}\,dx = 0$$

Two Improper Integral :-

1. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

2. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$.

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3. Bernoulli's formula of Integration by parts -

$$\int uv\,du = uv_1 - u'v_2 + u''v_3 + \dots$$

where. $V_1 = \int v\,dx$

$V_2 = \int v_1\,dx$

$\vdots$

Ex $\rightarrow$ $\int (2x^2+x+2)\cos 2x\,dx$

$$= (2x^2+x+2)\left(\frac{\sin 2x}{2}\right) - 4(x+1)\left(-\frac{\cos 2x}{4}\right) + 4\left(-\frac{\sin 2x}{8}\right) + c.$$

Important formulae :-

1. $\sin n\pi = 0$, $\sin(n+1)\pi = 0$, $\sin(n-1)\pi = 0$
2. $\cos n\pi = (-1)^n$
3. $\sin -\theta = -\sin\theta$
4. $\cos(-\theta) = \cos\theta$
5. $\sin(2n+1)\frac{\pi}{2} = (-1)^n$
6. $\sin(2n+1)\frac{\pi}{2} = 0$

Q Find the fourier constants b_n for $x \sin x$ in $(-\pi, \pi)$.

Sol→ $f(x) = x \sin x$

$f(-x) = (-x)(\sin(-x))$

$= x \sin x = f(x)$.

$\Rightarrow f(x)$ is even function in $(-\pi, \pi)$.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{(x \sin x)}_{\text{even}} \cdot \underbrace{\sin nx}_{\text{odd}} \, dx = 0$$

$\Rightarrow b_n = 0$

Q find b_n in expansion of x^2 as fourier series in $(-\pi, \pi)$.

Sol→ $f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

$\Rightarrow f(x)$ is even function in $(-\pi, \pi)$.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cdot \sin nx \, dx = 0$$

$\Rightarrow b_n = 0$

Q. find a_0 & a_n in expansion of $x - x^3$ in $(-7, 7)$.

Sol→ $f(x) = x - x^3$

$f(-x) = -x - (-x)^3 = -[x - x^3] = -f(x)$

$\Rightarrow f(x)$ is odd function.

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$$a_0 = \frac{1}{7}\int_{-7}^{7} (x + x^3)\, dx.$$

$$= 0 \quad \underline{\underline{\text{Ans}}}$$

$$a_n = \frac{1}{7}\int_{-7}^{7} (x - x^3) \cos\frac{n\pi x}{7} \, dx$$

$$= 0 \quad \underline{\underline{\text{Ans}}}$$

<u>Q</u> Find the fourier series expansion of periodic function $f(x+2\pi) = f(x)$. $f(x) = x$ in range $-\pi < x < \pi$.

Sol→ $f(x) = x$

$\because f(-x) = -x = -f(x) \Rightarrow$ odd function.

Now, we know that,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \cos nx \, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx.$$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \, dx = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx \, dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(n)\cdot \sin nn \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} n \sin nn \, dx \qquad \left[\because \text{ for even } fn \;\; \int_{-n}^{n} f(n) = 2\int_0^n fn\right]$$

$$= \frac{2}{\pi}\left[n\left(-\frac{\cos nn}{n}\right) - 1\left(-\frac{\sin nn}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi\left(-\frac{\cos n\pi}{n}\right) + \frac{\sin n\pi}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \times \frac{\pi}{n}(-\cos n\pi).$$

$$= \frac{-2}{n}(-1)^n.$$

Now, put value of a_0, a_n, b_n in eq. 1

$$f(n) = 0 + \sum_{n=1}^{\infty}(0) + \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nn$$

$$\Rightarrow f(n) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nn.$$

Q If $f(n) = \begin{cases} -k & \text{when } -\pi < n < 0 \\ k & \text{when } 0 < n < \pi \end{cases}$ and $f(n+2\pi) = f(n)$

for all n, derive the fourier series for $f(n)$

Deduce that.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{——}.$$

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Solution $\rightarrow$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

$$a_0 = \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x) dx + \int_{0}^{\pi} f(x) dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -k \, dx + \int_{0}^{\pi} k \, dx \right]$$

$$= \frac{1}{\pi} \left[[-kx]_{-\pi}^{0} + [kx]_{0}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[0 - [-k \cdot (-\pi)] + [\pi k - 0] \right]$$

$$= \frac{1}{\pi} \left[0 - k\pi + k\pi - 0 \right]$$

$$= 0.$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x) \cos nx \, dx + \int_{0}^{\pi} f(x) \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -k \cos nx \, dx + \int_{0}^{\pi} k \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\left(\frac{-k \sin nx}{n} \right)_{-\pi}^{0} + \left(\frac{k \sin nx}{n} \right)_{0}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{k \sin 0}{n} + \frac{k \sin n\pi}{n} + \frac{k \sin n\pi}{n} - \frac{k \sin 0}{n} \right]$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k \sin nx \, dx + \int_{0}^{\pi} k \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{k \cos nx}{n}\right)_{-\pi}^{0} + \left[\frac{-k \cos nx}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{k \cos 0}{n} - \frac{k \cos n\pi}{n} - \frac{k \cos n\pi}{n} - \frac{k \cos 0}{n}\right]$$

$$= \frac{k}{n\pi}\left[1 - \cos n\pi - \cos n\pi + 1\right]$$

$$= \frac{k}{n\pi}\left[2 - 2\cos n\pi\right]$$

$$= \frac{2k}{n\pi}\left[1 - \cos n\pi\right]$$

Now, $f(x) = 0 + \sum_{n=1}^{\infty} 0 + \frac{2k}{\pi}\sum_{n=1}^{\infty} \frac{(1 - \cos n\pi)}{n} \sin nx$

$\Rightarrow f(x) = \frac{2k}{\pi}\sum_{n=1}^{\infty} \frac{(1 - \cos n\pi)}{n} \sin nx$. Ans

~~Now, On putting $x = \frac{\pi}{2}$.~~

~~$f\left(\frac{\pi}{2}\right) = \frac{2k}{\pi} \sum_{n=1} n(1 - \cos n\pi)$~~

$\Rightarrow f(x) = \frac{2k}{\pi}\sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n} \sin nx$ Ans

Now, On putting $x = \frac{\pi}{2}$

$f\left(\frac{\pi}{2}\right) = \frac{2k}{\pi}\sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n} \sin n\frac{\pi}{2}$.

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$$\Rightarrow \quad k = \frac{2k}{\pi}\left[\frac{2}{1}\sin\frac{\pi}{2} + \frac{0}{2}\sin\pi + \frac{2}{3}\sin\frac{3\pi}{2} + \frac{0}{4}\sin 2\pi + \frac{2}{5}\sin\frac{5\pi}{2} + \cdots\right]$$

$$\Rightarrow \quad \frac{k}{k} = \frac{2}{\pi}\times 2\left[\sin\frac{\pi}{2} + \frac{\sin\frac{3\pi}{2}}{3} + \frac{\sin\frac{5\pi}{2}}{5} + \cdots\right]$$

$$\Rightarrow \quad 1 = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

$$\Rightarrow \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Hence <u>proved</u>

<u>Q</u> Obtain the fourier series of periodicity 2π for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$. Hence deduce the value of $\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$. Further derive a series for $\operatorname{cosech}\pi$.

Sol $\rightarrow$
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

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$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{-1} - \frac{e^{0}}{-1}\right]$$

$$= \frac{1}{\pi}\left[1 - e^{-2\pi}\right].$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\, dx.$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left[-\cos nx + n\sin nx\right]\right]_0^{2\pi}$$

$$= \frac{1}{(1+n^2)\pi}\left[e^{-2\pi}\left(-\cos 2n\pi + n\sin 2n\pi\right) - e^{0}\left[-\cos 0 + n\sin 0\right]\right]$$

$$= \frac{1}{(1+n^2)\pi}\left[e^{-2\pi}\left(-1+0\right) - 1\left(-1+0\right)\right]$$

$$a_n = \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\, dx.$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left[(-1)\sin nx - n\cos nx\right]\right]_0^{2\pi}$$

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$$= \frac{1}{\pi(1+n^2)}\left[\left[(-1)\sin 2n\pi - n\cos 2n\pi\right] - \left[(-1)\sin 0 - n\cos 0\right]\right]$$

$$\Rightarrow \frac{1}{\pi(1+n^2)}\left[0 - n(1) - 0 + n\right.$$

$$= \frac{1}{\pi(1+n^2)}\left[\left\{\frac{e^{-2\pi}}{\cancel{1+n^2}}\left((-1)\sin 2n\pi - n\cos 2n\pi\right)\right\} - \left\{\frac{e^{0}}{\cancel{1+n^2}}\left((-1)\sin 0 - n\cos 0\right)\right\}\right]$$

$$\Rightarrow \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(0-n) - 1(0-n)\right]$$

$$b_n = \frac{n}{\pi(1+n^2)}\left[1 - e^{-2\pi}\right]$$

$$f(x) = \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty}\left(\frac{1-e^{-2\pi}}{\pi(1+n^2)}\cos nx + \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}\sin nx\right)$$

$$f(x) = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2+1}\left(\cos nx + n\sin nx\right)\right]$$

Now, on putting $x = \pi$, we get

$$f(\pi) = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} + 0\right]$$

$$\Rightarrow e^{-\pi} = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

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$$\Rightarrow \frac{\pi e^{-\pi}}{1-e^{-2\pi}} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$\Rightarrow \frac{\pi e^{-\pi}}{1-e^{-2\pi}} = \frac{1}{2} - \frac{1}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}.$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi e^{-\pi}}{1-e^{-2\pi}} \quad \underline{\underline{\text{Ans}}}$$

Now, $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$$\Rightarrow \operatorname{cosech} x = \frac{2}{e^x - e^{-x}} \Rightarrow \frac{1}{e^x - e^{-x}} = \frac{\operatorname{cosech} x}{2}$$

$$\Rightarrow \operatorname{cosech} \pi = \frac{2}{e^{\pi} - e^{-\pi}}.$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(1+n^2)} = \frac{\pi}{e^{\pi} - e^{-\pi}} \quad \left[\because \frac{\pi e^{-\pi}}{1-e^{-2\pi}} = \frac{\pi}{e^{\pi} - e^{-\pi}}\right]$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(1+n^2)} = \frac{\pi \operatorname{cosech} \pi}{2}.$$

$$\Rightarrow \operatorname{cosech} \pi = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} \quad \underline{\underline{\text{Ans}}}$$

Q Expand in Fourier series of periodicity 2π of $f(x) = x\sin x$, for $0 < x < 2\pi$.

Sol→ $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\Big[x(-\cos x) - (-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[-2\pi(\cos 2\pi) + \sin 2\pi + 0 - 0\Big]$$

$$a_0 = \frac{-2\pi}{2} = -2$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\big[\sin(n+1)x - \sin(n-1)x\big]dx.$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\sin(n+1)n - \int_0^{2\pi} x\sin(n-1)x\,dx\right].$$

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$$= \frac{1}{2\pi}\left[x\left[\frac{-\cos(n+1)x}{(n+1)}\right] - 1\left[\frac{-\sin(n+1)x}{(n+1)^2}\right]\right]_0^{2\pi}$$

$$- \frac{1}{2\pi}\left[x\left[\frac{-\cos(n-1)x}{(n-1)} - 1\left[\frac{-\sin(n-1)x}{(n-1)^2}\right]\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{\cos(n+1)2\pi}{(n+1)}\right) - 1\left(-\frac{\sin(n+1)2\pi}{(n+1)^2}\right) + 0 - \sin 0\right] - \frac{1}{2\pi}\left[2\pi \cdot \frac{-\cos(n-1)2\pi}{(n-1)} - 1\left(\frac{-\sin(n-1)2\pi}{(n-1)^2}\right) + 0 - 0\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi}{(n+1)}\right] + \frac{1}{2\pi}\left[\frac{2\pi}{(n-1)}\right]$$

$$= \frac{-n+1+n+1}{n^2-1}$$

$$a_n = \frac{2}{n^2-1}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(1-n)x - \cos(1+n)x\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\cos(1-n)x - \int_0^{2\pi} x\cos(1+n)x\right]$$

$$= \frac{1}{2\pi}\left[\frac{x\sin(1-n)x}{(1-n)} - 1\left(-\frac{\cos(1-n)x}{(1-n)^2}\right)\right]_0^{2\pi}$$

$$- \frac{1}{2\pi}\left[x\frac{\sin(1+n)x}{(1+n)} - 1\left(-\frac{\cos(1+n)x}{(1+n)}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(\frac{2\pi\sin 2(1-n)\pi}{(1-n)} + \frac{\cos 2(1-n)\pi}{(1-n)^2}\right] - 0 - \frac{1}{(1-n)^2}\Bigg]$$

$$- \frac{1}{2\pi}\left[\frac{2\pi\sin(1+n)2\pi}{(1+n)} - \frac{\cos(1+n)2\pi}{(1-n)^2} - 0 - \frac{1}{(1+n)^2}\right]$$

$$= \frac{1}{2\pi}\left[0 + \frac{1}{(1-n)^2} - \frac{1}{(1-n)^2}\right] - \frac{1}{2}\left[0 + \frac{1}{(1+n)^2} - \frac{1}{(1+n)^2}\right]$$

$b_n = 0$

$\because b_n = 0$, we can try find b_1.

$$\Rightarrow b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x\, dx$$

$$\Rightarrow b_1 = \frac{1}{\pi}\int_0^{2\pi} x\left[1 - \cos 2x\right] dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x\sin 2x}{2} + 1\cdot\frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$b_1 = \frac{1}{\pi}\left[\frac{4\pi^2}{2} - \frac{2\pi\sin 4\pi}{2} - \frac{\cos 4\pi}{4} - 0 + 0 + \frac{1}{4}\right]$$

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$$b_1 = \frac{1}{2\pi}\left[2\pi^2 - \frac{1}{4} + \frac{1}{4}\right]$$

$$b_1 = \pi$$

Now, putting a_0, a_n, b_1 in $f(x)$.

$$f(x) = \frac{-2}{2} + \left[\sum_{n=1}^{\infty} \frac{2}{n^2-1} \cos nx\right] + \pi \sin x$$

$$f(x) = -1 + \left[2\sum_{n=1}^{\infty} \frac{1}{n^2-1} \cos nx\right] + \pi \sin x.$$

Ans

Q Expand fourier series for $f(x) = x^2$ in interval $(-\pi, \pi)$.

Sol → Then, Deduce :-

① $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \text{---} + \infty = \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

② $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \text{---} \infty = \frac{\pi^2}{12}$.

③ $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \text{---} \infty = \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Sol → $f(x) = x^2$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

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$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2 \frac{\sin n\pi}{n} + 2\pi\left[\frac{\cos \pi nx}{n^2}\right] - 2\frac{\sin n\pi}{n^3} - (0 + 0 - 0)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{n} \times 0 + 2\pi \frac{(-1)^n}{n^2} - \frac{2}{n^3} \times 0\right]$$

$$a_n = 4\frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \, dx$$

$$b_n = 0$$

$$f(x) = \frac{2\pi^2}{2 \times 3} + 4\sum \frac{(-1)^n}{n^2} \cos nx.$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} \cos nx.$$

Now, putting $x = 0$, we get

$$f(x) = 0 = \frac{\pi^2}{3} + 4\left[-\frac{\cos 0}{1^2} + \frac{1}{2^2}\cos 0 + -\right]$$

$$\Rightarrow 0 = \frac{\pi^2}{3} + 4\left[-1 + \frac{1}{4} - \frac{1}{9} + \cdots\right]$$

$$\Rightarrow \frac{-\pi^2}{3\times 4} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \cdots$$

$$\Rightarrow \left[1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{9^2} \cdots \infty\right] = \frac{\pi^2}{12} \quad \text{--- (1)}$$

Now, putting $x = \pi$; we get

$$\pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\Rightarrow \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty\right] = \left(\pi^2 - \frac{\pi^2}{3}\right)\frac{1}{4}$$

$$\Rightarrow 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty = \frac{\pi^2}{6} \quad \text{--- (2)}$$

Now, on adding (1) & (2), we get

$$2\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$\Rightarrow \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty\right] = \frac{\pi^2}{8}$$

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Q Express $f(x) = (\pi - x)^2$ as a fourier series of periodicity 2π in $0 < x < 2\pi$ and hence deduce the the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Sol →

$$f(x) = (\pi - x)^2$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (\pi - x)^2\,dx$$

$$= \frac{1}{\pi} \left[\frac{-(\pi - x)^3}{3} \right]_0^{2\pi}$$

$$= \frac{-1}{3\pi} \left[-\pi^3 - \pi^3 \right]$$

$$a_0 = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{\pi} \left[(\pi - x)^2 \left(\frac{\sin nx}{n} \right) + 2(\pi - x) \left(-\frac{\cos nx}{n^2} \right) + 2 \left(-\frac{\sin nx}{n^3} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\pi^2 \frac{\sin \pi x}{n} + 2(-\pi) \left(-\frac{\cos \pi x}{n^2} \right) + 2 \left(-\frac{\sin n 2\pi}{n^3} \right) - \frac{\pi^2 \sin 0}{n} - \frac{2\pi(-\cos 0)}{n^2} - \frac{2 \sin 0}{n^3} \right]$$

$$= \frac{1}{\pi} \left[\frac{4\pi}{n^2} \right] = \frac{4}{n^2}$$

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$$b_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx \, dx$$

$$= \frac{1}{\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right) + 2(\pi-x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\pi^2\left(-\frac{\cos 2n\pi}{n}\right) + 2(-\pi)\left(-\frac{\sin 2n\pi}{n^2}\right) + \frac{2\cos 2n\pi}{n^3} + \pi^2\left(\frac{\cos 0}{n}\right) + 2\pi\frac{\sin 0}{n^2} - \frac{2\cos 0}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n} + 0 + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right]$$

$$b_n = 0$$

$$f(x) = \frac{2\pi^2}{2\times 3} + 4\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos nx + 0\right)$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx.$$

Now put $x = 0$,

$$(\pi - 0)^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \cdots \infty\right]$$

$$\Rightarrow \frac{2\pi^2}{3\times 4} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots \infty$$

$$\Rightarrow \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty\right] = \frac{\pi^2}{6}$$ Ans

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Q. Obtain the fourier series of periodic function defined by

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}.$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \infty = \frac{\pi^2}{8}$.

Sol → $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\,[x]_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi(0+\pi) + \frac{\pi^2}{2}\right]$$

$$= \frac{-\pi^2}{2\pi} = \frac{-\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{-\pi\left(\frac{\sin nx}{n}\right)\right\}_{-\pi}^{0} + \left\{x\left(\frac{\sin nx}{n}\right) - 1\cdot\left(\frac{-\cos nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\{0-0\} + \left\{0 + \frac{1}{n^2}\cos n\pi - 0 - \frac{1}{n^2}\right\}\right]$$

$$a_n = \frac{1}{n^2}(\cos n\pi - 1).$$

$$= \frac{1}{n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx + \int_{0}^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{(-\pi)\left(\frac{-\cos nx}{n}\right)\right\}_{-\pi}^{0} + \left\{x\left[\frac{-\cos nx}{n}\right] - (1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{n} - \frac{\pi\cos n\pi}{n}\right) + \left(-\frac{\pi}{n}\cos n\pi + 0 - 0 + 0\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - 2\cos n\pi)\right]$$

$$= \frac{1}{n}(1 - 2\cos n\pi).$$

$$\therefore f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}(\cos n\pi - 1)\cos nx + \frac{1}{n}(1 - 2\cos n\pi)\sin nx\right]$$

$$= \frac{-\pi}{4} + \frac{1}{\pi}\left[\left[\frac{1}{1}(-2)\cos x + \frac{1}{2^2}\times 0\times\cos 2x + \frac{1}{3^2}(-2)\cos 3x + \cdots\right] + \left[\frac{1}{1}3\sin x + \frac{1}{2}(-1)\sin 2x + \frac{1}{3}3\sin 3x + \cdots\right]\right.$$

$$f(x) = \frac{-\pi}{4} + \left(-\frac{2}{\pi}\right)\left[\left\{\cos x + \frac{1}{9}\cos 3x + \cdots\right\} + \left[3\sin x + \frac{1}{3}\sin 3x - \frac{\sin 2x}{2} - \frac{\sin 4x}{2} - \cdots\right\}\right]$$

Now, Zero is point of discontinuity of $f(x)$.

$$\therefore \quad f(x) = \frac{f(0^-) + f(0^+)}{2}$$

$$= \frac{-\pi + x}{2}$$

$$f(0) = \frac{-\pi}{2}$$

$$\Rightarrow \frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow -\frac{\pi}{2} + \frac{\pi}{4} = \frac{-2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{-\pi}{4} \times \frac{\pi}{-2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\Rightarrow \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty\right] = \frac{\pi^2}{8}$$

Q find fourier series of periodic function given by

$$f(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x < 2\pi \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$.

Sol → $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left[-\frac{(2\pi-x)^2}{2}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - 0\right) - \left(0 - \frac{\pi^2}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right\}_0^{\pi} + \left\{(2\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{0 + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right\} + \left\{0 - \frac{1}{n^2} + 0 + \frac{\cos n\pi}{n^2}\right\}\right]$$

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$$= \frac{1}{\pi}\left[\frac{1}{n^2}[(-1)^n - 1] - \frac{1}{n^2}[1-(-1)^n]\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1 - 1 + (-1)^n\right]$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\pi} + \left\{(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{-\pi}{n}\cos n\pi + 0 - 0 + 0\right\} + \left\{0 - 0 + \frac{\pi}{n}\cos n\pi + 0\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}\cos n\pi + \frac{\pi}{n}\cos n\pi\right]$$

$$b_n = 0$$

$$\therefore f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx.$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{1}{1^2}(-2)\cos x + \frac{1}{2^2}\times 0\times\cos 2x + \frac{1}{3^2}(-2)\cos 3x + \cdots\right]$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \cdots \infty\right]$$

$x = 0$ is the end point of range.

The value of fourier series at end ~~of~~ is equal to average of $f(0) + f(2\pi)$.

$$f(x) = \frac{f(0) + f(2\pi)}{2} = \frac{0+0}{2}$$

$$\Rightarrow f(x) = 0$$

$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$$

Half-Range Series

If a function is defined over half the range say 0 to l, instead of the full range from $-l$ to l, it may be expanded in a series of sine terms only or of cosine terms only. The series produced is then called a half-range fourier series.

1. Half-Range Cosine Series [Even function] :-

* An even function can be expanded using half its range from
(i) 0 to l or (ii) $-l$ to 0 or (iii) l to $2l$.

If $F(x) = f(x)$ is $(0, l)$, assign $f(x) = f(-x)$ in $(-l, 0)$, so that

$$F(x) = \begin{cases} f(-x) & -l < x < 0 \\ f(x) & 0 < x < L \end{cases}$$

the corresponding Fourier series is

$$\boxed{\begin{aligned} F(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \\ \text{where} \quad a_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$

2. Half-range sine series [odd function]

· An odd function can be expanded using half it range from 0 to L.

From the definition of $F(x)$, a $F(x)$ is an odd function of x and it is defined in the interval $-\pi$ to π.

$$F(x) = \begin{cases} f(x) & \text{when } 0 < x < L \\ -F(x) & \text{when } -L < x < 0 \end{cases}$$

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$\therefore$ $a_n = 0$ & $a_0 = 0$, we have

$$\boxed{\begin{aligned} F(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$

Q Expand the function, $F(x) = \sin x \quad 0 < x < \pi$ in fourier cosine series.

Sol → $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$= \frac{2}{\pi}-\left[(-1) - 1\right]$$

$$a_0 = \frac{4}{\pi}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx.$$

$$= \frac{2}{2\pi}\int_0^{\pi} [\sin(1+n)x + \sin(1-n)x] \, dx$$

$$= \frac{2}{2\pi}\left[-\frac{\cos(1+n)x}{(1+n)} - \frac{\cos(1-n)x}{(1-n)}\right]_0^{\pi}$$

$$= \frac{2}{2\pi}\left[-\frac{\cos(1+n)\pi}{(1+n)} - \frac{\cos(1-n)\pi}{(1-n)} + \frac{\cos 0}{1+n} + \frac{\cos 0}{1-n}\right]$$

$$= \frac{2}{2\pi}\left[+\frac{\cos n\pi}{(1+n)} - \frac{\cos n\pi}{(1-n)} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{2}{2\pi}\left[\frac{\cos n\pi + 1}{n+1} + \frac{\cos n\pi + 1}{1-n}\right]$$

$$= \frac{2(1+\cos n\pi)}{2\pi(1-n^2)}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx.$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi} - \left[\cos 2x\right]_0^{\pi}$$

$$= \frac{1}{\pi} - (1-1)$$

$$a_1 = \underline{\underline{0}}$$

$$f(x) = \frac{a_0}{2} + a_1 + \sum_{n=2}^{\infty} a_n \cos nx.$$

$$= \frac{2}{\pi} + \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{1+\cos n\pi}{-(n+1)(n-1)} \cos nx.$$

$$= \frac{2}{\pi} - \frac{2}{\pi}\left[\frac{2\cos 2x}{1\cdot 3} + \frac{2\cos 4x}{3\cdot 5} + \frac{2\cos 6x}{5\cdot 7} + \cdots\infty\right]$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{1\cdot 3} + \frac{\cos 4x}{3\cdot 5} + \frac{\cos 6x}{5\cdot 7} + \cdots\infty\right]$$

Ans

Q find half-range sine series for the function

$$f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$$

Sol→ $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) + 1\cdot\left(+\frac{\sin nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left\{-\frac{\pi}{2n}\cos\frac{\pi n}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + 0 - 0\right\} + \left\{0 - 0 + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

Case I :- Let $n = 2r$.

$$b_{2r} = \frac{4}{\pi 4 r^2}\sin\frac{2r\pi}{2}$$

$$= \frac{4}{4\pi r^2}\sin \pi r$$

$$b_{2r} = 0$$

On putting $r = 1, 2, 3 \ldots$

$$b_2 = b_4 = b_6 = b_8 = 0$$

Case II :- Let $n = (2r+1)$.

$$b_{2r+1} = \frac{4}{\pi(2r+1)^2}\sin\frac{(2r+1)\pi}{2}$$

$$= \frac{4}{\pi(2r+1)^2}\sin\left[\frac{\pi}{2}+\pi r\right]$$

$$= \frac{4}{\pi(2r+1)^2}\cos\pi r$$

$$= \frac{4}{\pi(2r+1)^2}(-1)^r.$$

$b_1 = \frac{4}{\pi}$

$b_3 = \frac{-4}{\pi 3^2}$

$b_5 = \frac{4}{\pi 5^2}$

$f(x) = b_1 \sin nx + b_2 \sin 2x + b_3 \sin 3x + \cdots$

$= \frac{4}{\pi}\sin x + 0 - \frac{4}{\pi 3^2}\sin 3x + 0 + \frac{4}{\pi 5^2}\sin 5x + \cdots \infty$

$\Rightarrow f(x) = \frac{4}{\pi}\left[\sin x - \sin 3x + \sin 5x + \cdots \infty\right]$

Q Express $f(x) = x(\pi - x)$ $0 < x < \pi$

(i) Sine terms only.

(ii) Cosine terms only.

Hence deduce $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots \infty = \frac{\pi^2}{3^2}$

Sol $\rightarrow$ (i) Sine series :-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \, dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx \, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{+\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + 0 - \frac{2}{n^3}\cos n\pi - 0 - 0 + \frac{2}{n^3}\cos 0\right]$$

$$= \frac{2}{\pi}\left[\frac{-2}{n^3}\left((-1)^n - 1\right)\right]$$

$$= \frac{4}{\pi n^3}\left[1 - (-1)^n\right]$$

$$b_n = \begin{cases} 0 & \text{when } n \text{ is even} \\ +\frac{8}{\pi n^3} & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \cdots \infty$$

$$f(x) = \frac{8}{\pi}\sin x + \frac{8}{27\pi}\sin 3x + \frac{8}{125\pi}\sin 5x + \cdots \infty$$

Now, for $x = \pi/2$

$$f(\pi/2) = \frac{8}{\pi}\sin \pi/2 + \frac{8}{3^3\pi}\sin\frac{3\pi}{2} + \frac{8}{5^3\pi}\sin\frac{5\pi}{2} + \cdots \infty$$

$$\frac{\pi^2}{4} = \frac{8}{\pi}\left[\frac{\sin \pi/2}{1} + \frac{\sin\frac{3\pi}{2}}{3^3} + \frac{\sin\frac{5\pi}{2}}{5^2} + \cdots \infty\right]$$

$$\Rightarrow \left[1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots \infty\right] = \frac{\pi}{32}$$

Now, solve (ii) by yourself.

Q. Expand $f(x) = x\sin x$ as cosine series in $0<x<\pi$ and show that $1+\frac{2}{1\cdot3}-\frac{2}{3\cdot5}+\frac{2}{5\cdot7}-\cdots\infty=\frac{\pi}{2}$.

Sol $\rightarrow$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

$$= \frac{2}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi(-\cos\pi) + \sin\pi + 0 - 0\right]$$

$$= \frac{2}{\pi}[\pi]$$

$$a_0 = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin(n+1)x\,dx - \int_0^{\pi} x\sin(n-1)x\,dx\right]$$

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$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{(n+1)}\right\} - 1\left(-\frac{\sin(n+1)x}{(n+1)^2}\right\} - x\left[-\frac{\cos(n-1)x}{(n-1)}\right\} + 1\left\{-\frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{-(-1)^{n+1}}{n+1}\right) - 0 + 0 - 0 + \pi\left[\frac{(-1)^{n-1}}{n-1}\right] - 0 + 0 - 0\right].$$

$$= \frac{\pi}{\pi}\left[\frac{-(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}\right]$$

$$= (-1)^n\left[\frac{+1}{n+1} - \frac{1}{n-1}\right]$$

$$= (-1)^n\left[\frac{n-1-n-1}{n^2-1}\right]$$

$$= \frac{-2(-1)^n}{n^2-1}$$

$$a_n = \frac{2(-1)^{n+1}}{n^2-1}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x$$

$$= \frac{1}{\pi}\left[(x)\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(-\frac{\cos 2\pi}{2}\right) - 0 + 0 - 0\right]$$

$$= \frac{1}{\pi}\left(\frac{-\pi}{2}\right) = \frac{-1}{2}$$

So, $f(x) = a_0 + a_1 \cos x + \sum_{n=2}^{\infty} b_n \cos nx.$

$$f(x) = \frac{2}{2} - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \cos nx$$

~~f(x) =~~ Now, set $x = \pi/2$.

$$f(\pi/2) = \frac{2}{2} - \frac{1}{2}\cos\frac{\pi}{2} - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos\frac{n\pi}{2}.$$

$$\frac{\pi}{2} = 1 - 0 - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cdot\cos\frac{n\pi}{2}.$$

$$\pi/2 = 1 - 2\left[\frac{1}{1\cdot 3}\cos 2\pi + \frac{(-1)}{2\cdot 4}\cos\frac{3\pi}{2} + \frac{1}{3\cdot 5}\cos 2\pi + \frac{(-1)}{4\cdot 6}\cos\frac{5\pi}{2} + \frac{1}{5\cdot 7}\cos 3\pi + \cdots \infty\right]$$

$$\frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} + \cdots$$

So, $1 + 2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots \infty\right] = \frac{\pi}{2}$

Change of order

* For full-range :- $2L$ = Period
* For half-range :- L = Period

Q. Obtain the fourier series expansion for function

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 & 0 < x < 2 \end{cases} \quad \text{and } f(x+3) = f(x)$$

Sol →

$$2L = 3$$

$$L = \frac{3}{2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$$

$$= \frac{2}{3}\int_0^{3} f(x)\,dx$$

$$= \frac{2}{3}\left[\int_0^1 1\,dx + \int_1^3 2\,dx\right]$$

$$= \frac{2}{3}\left[\{1-0\} + \{6-2\}\right]$$

$$= \frac{2}{3}[5]$$

$$a_0 = \frac{10}{3}$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x) \cos\frac{n\pi x}{L}$$

$$= \frac{2}{3}\int_0^3 f(x) \cos\frac{2n\pi x}{3}\, dx.$$

$$= \frac{2}{3}\left[\int_0^1 1\cdot \cos\frac{2n\pi x}{3} + \int_1^3 2\cdot\cos\frac{2n\pi x}{3}\right]$$

$$= \frac{2}{3}\left[\left\{\frac{3}{2n\pi}\sin\frac{2n\pi x}{3}\right\}_0^1 + 2\left\{\frac{3}{2n\pi}\sin\frac{2n\pi x}{3}\right\}_1^3\right]$$

$$= \frac{2}{3}\left[\frac{3}{2n\pi}\left\{\sin\frac{2n\pi}{3} - 0\right\} + 2\left(\frac{3}{2n\pi}\right)\left\{0 - \sin\frac{2n\pi x}{3}\right\}\right]$$

$$= \frac{2}{3}\left[-\frac{3}{2n\pi}\sin\frac{2n\pi}{3}\right]$$

$$a_n = \frac{-1}{n\pi}\sin\frac{2n\pi}{3}$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{3}\left[\int_0^1 1\cdot\sin\frac{2\pi n x}{3} + 2\int_1^3 \sin\frac{2n\pi x}{3}\right]$$

$$= \frac{2}{3}\left[\left\{\frac{-3}{2n\pi}\cos\frac{2n\pi x}{3}\right\}_0^1 + 2\left\{\frac{-3}{2n\pi x}\cos\frac{2n\pi x}{3}\right\}_1^3\right]$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}\right]\left[\cos\frac{2n\pi}{3} - \cos 0 + 2\cos 2\pi - 2\cos\frac{2n\pi}{3}\right]$$

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$$b_n = \frac{-1}{n\pi}\left[1 - \cos\frac{2n\pi}{3}\right].$$

Now, $a_1 = \frac{-\sqrt{3}}{2\pi}$, $a_2 = \frac{\sqrt{3}}{4\pi}$, $a_3 = 0$.

$$a_4 = \frac{-\sqrt{3}}{8\pi}.$$

Also, $b_1 = \frac{-3}{2\pi}$, $b_2 = \frac{-3}{4\pi}$, $b_3 = 0$

$$b_4 = \frac{-3}{8\pi}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{3} + b_n \sin\frac{2n\pi x}{3}\right).$$

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{2\pi x}{3} + a_2 \frac{\cos 4\pi x}{3} + a_3 \cos\frac{6\pi x}{3}$$

$$+ a_4 \cos\frac{8\pi x}{3} + \cdots + b_1 \sin\frac{2\pi x}{3} +$$

$$b_2 \sin\frac{4\pi x}{3} + b_3 \sin\frac{6\pi x}{3} + b_4 \sin\frac{8\pi x}{3} + \cdots$$

$$f(x) = \frac{5}{3} - \frac{\sqrt{3}}{2\pi}\cos\frac{2\pi x}{3} + \frac{\sqrt{3}}{4\pi}\cos\frac{4\pi x}{3} - \frac{\sqrt{3}}{8\pi}\cos\frac{8\pi x}{3}$$

$$+ -\frac{3}{2\pi}\sin\frac{2\pi x}{3} - \frac{3}{4\pi}\sin\frac{4\pi x}{3} - \frac{3}{8\pi}\sin\frac{8\pi x}{3}$$

$$+ - \cdot \infty.$$

Ans

Q Find the fourier series of function :-

$$f(t) = \begin{cases} 0 & \text{when } -2<t<-1 \\ k & \text{when } -1<t<1 \\ 0 & \text{when } 1<t<2 \end{cases} \quad \& \ f(t+4) = f(t).$$

Sol →

$$2L = 4$$
$$L = 2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + b_n \cos\frac{n\pi t}{L}.$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\, dt$$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0 \cdot dt + \int_{-1}^{1} k\, dt + \int_{1}^{2} 0 \cdot dt\right]$$

$$= \frac{1}{2}\left[\left[kt\right]_{-1}^{1}\right]$$

$$a_0 = \frac{1}{2}\left[k+k\right] = k.$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x) \cos\frac{n\pi t}{2}\, dt$$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0 \cdot dt + \int_{-1}^{1} k \cdot \cos\frac{n\pi t}{2}\, dt + \int_{1}^{2} 0 \cdot dt\right]$$

$$= \frac{1}{2}\left[\int_{-1}^{1} k \cdot \cos\frac{n\pi t}{2}\, dt\right]$$

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$$a_n = \frac{1}{2}\left[\frac{2k}{n\pi}\sin\frac{n\pi t}{2}\right]_{-1}^{1}$$

$$= \frac{k}{n\pi}\left[\sin\frac{\pi n}{2} + \sin\frac{\pi n}{2}\right]$$

$$a_n = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi t}{2}\,dt$$

$$= \frac{1}{2}\int_{-2}^{2} k\cdot\sin\frac{n\pi t}{2}\,dt$$

$$b_n = 0$$

So, $f(t) = \frac{k}{2} + \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n}\cos\frac{n\pi t}{2}$

$$f(t) = \frac{k}{2} + \frac{2k}{\pi}\left[\frac{1}{1}\cos\frac{\pi t}{2} + \frac{0}{2}\cos\frac{2\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{0}{4}\cos\frac{4\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2}\right]$$

$$\Rightarrow f(t) = \frac{k}{2} + \frac{2k}{\pi}\left[\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} + - \infty\right]$$

Ans

Q If $f(x) = \begin{cases} \sin x & \text{for } 0 \le x \le \pi/4 \\ \cos x & \text{for } \pi/4 \le x \le \pi/2 . \end{cases}$

express $f(x)$ in a series of sines.

Sol →

$$L = \frac{\pi}{2}$$

$$b_n = \frac{2}{\pi/2} \int_0^{\pi/2} f(x) \sin \frac{n\pi x}{\pi/2} \, dx$$

$$= \frac{4}{\pi} \left[\int_0^{\pi/4} \sin x \sin 2nx \, dx + \int_{\pi/4}^{\pi/2} \cos x \sin 2nx \, dx \right]$$

$$= \frac{2}{\pi} \left(\int_0^{\pi/4} [\cos(2n-1)x - \cos(2n+1)x] \, dx + \int_{\pi/4}^{\pi/2} [\sin(2n+1)x + \sin(2n-1)x] \, dx \right.$$

$$= \frac{2}{\pi} \left[\left\{ \frac{\sin(2n-1)x}{2n-1} - \frac{\sin(2n+1)x}{(2n+1)} \right\}_0^{\pi/4} + \left\{ -\frac{\cos(2n+1)x}{(2n+1)} - \frac{\cos(2n-1)x}{(2n-1)} \right\}_{\pi/4}^{\pi/2} \right]$$

$$= \frac{2}{\pi} \left[\left\{ \frac{\sin\left(\frac{n\pi}{2} - \frac{\pi}{4}\right)}{2n-1} - \frac{\sin\left(\frac{\pi n}{2} + \frac{\pi}{4}\right)}{2n+1} - 0 + 0 \right\} - \left\{ \frac{\cos\left(n\pi + \frac{\pi}{2}\right)}{2n+1} + \frac{\cos\left(\pi n - \frac{\pi}{2}\right)}{2n-1} - \frac{\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)}{2n+1} - \frac{\cos\left(\frac{n\pi}{2} - \frac{\pi}{4}\right)}{2n-1} \right\} \right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\cos\frac{\pi}{4}\sin\frac{n\pi}{2} - \cancel{\sin\frac{\pi}{4}\cos\frac{n\pi}{2}} + \cancel{\cos\frac{\pi}{4}\cos\frac{n\pi}{2}} + \sin\frac{\pi}{4}\sin\frac{n\pi}{4}}{(2n-1)}\right\}\right.$$

$$\left. - \left\{\frac{\cos\frac{\pi}{4}\sin\frac{n\pi}{2} + \cancel{\sin\frac{\pi}{4}\cos\frac{n\pi}{2}} - \cancel{\cos\frac{\pi}{4}\cos\frac{n\pi}{2}} + \sin\frac{\pi}{4}\sin\frac{n\pi}{2}}{(2n+1)}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\sin\frac{\pi}{4}\sin\frac{n\pi}{2}}{(2n-1)} - \frac{2\cos\frac{\pi}{4}\sin\frac{n\pi}{2}}{2n+1}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\cdot\frac{1}{\sqrt{2}}\cdot\sin\frac{n\pi}{2}}{(2n-1)} - \frac{2\cdot\frac{1}{\sqrt{2}}\cdot\sin\frac{n\pi}{2}}{2n+1}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\sin\frac{n\pi}{2}\left[\frac{2n+1-2n+1}{(2n-1)(2n+1)}\right]$$

$$b_n = \frac{4\sqrt{2}\sin\frac{n\pi}{2}}{\pi(2n-1)(2n+1)}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\pi/2} = \sum_{n=1}^{\infty} b_n \sin 2nx$$

$$= \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{(2n-1)(2n+1)}\cdot\sin 2nx.$$

$$= \frac{4\sqrt{2}}{\pi}\left[\frac{\sin 2x}{1\cdot 3} - \frac{\sin 6x}{5\cdot 7} + \frac{\sin 10x}{9\cdot 11} - \frac{\sin 14x}{13\cdot 15} + \cdots\right]$$

Ans

Root-Mean square value of a function and PARSEVAL's Theorem

Root-mean-square value (r.m.s.) or effective value of a function $y = f(x)$ over a given interval (a, b) is defined as

$$\boxed{\bar{y} = \sqrt{\frac{\int_a^b y^2\,dx}{b-a}}}$$

PARSEVAL's Theorem :-

$$\int_c^{c+2\pi} [f(x)]^2\,dx = 2\pi\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right.$$

Q. find the Fourier series of periodicity 2π for $f(x)=x^2$, in $-\pi < x < \pi$. Hence show that

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\text{to } \infty = \frac{\pi^4}{90}$$

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}, \quad b_n = 0$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Hence using the parseval Identity

$$2\pi\left[\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)\right] = \int_{-\pi}^{\pi}\left[f(x)\right]^2 dx$$

$$2\pi\left[\frac{\pi^4}{9\times 4} + \frac{1}{2}\sum\frac{16}{n^4}\right] = \frac{2}{5}\pi^5$$

$$\frac{\pi^4}{9} - 8\sum\frac{1}{n^4} = \frac{\pi^4}{5}$$

$$\frac{\pi^4}{9} - \frac{\pi^4}{5} = 8\sum\frac{1}{n^4}$$

$$\frac{-4\pi^4}{9\times 5} = 8\sum\frac{1}{n^4}$$

$$\sum\frac{1}{n^4} = \frac{\pi^4}{90} \quad \underline{\text{Ans}}$$

Q. Expand $f(x) = x - x^2$ as a fourier series in $-1 < x < 1$ and using this series find the R.M.S. value of $f(x)$ in the interval.

Sol. Here $2l = 2$, $l = 1$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\pi x + b_n \sin n\pi x]$

where $a_0 = \frac{1}{1} \int_{-1}^{1} f(x)\, dx$

$$= \int_{-1}^{1} (x - x^2)\, dx$$

$$= -\frac{2}{3}$$

$$a_n = \frac{1}{1} \int_{-1}^{1} (x - x^2) \cos n\pi x \, dx$$

$$= \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1} (x - x^2) \sin n\pi x \, dx$$

$$= -\frac{2}{n\pi} (-1)^n$$

$$\therefore (x - x^2) = -\frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{4}{n^2\pi^2} (-1)^{n+1} \cos n\pi x + \frac{2}{n\pi} (-1)^{n+1} \sin n\pi x\right]$$

R.M.S. Value of $f(x) = \sqrt{\dfrac{\int_{-1}^{1} (x-x^2)^2 \, dx}{1+1}}$

$$= \sqrt{\frac{\int_{-1}^{1} x^2 + x^4 - 2x^3 \, dx}{2}}$$

$$= \sqrt{\frac{2}{2} \int_0^1 x^2 + x^4 \, dx}$$

$$= \sqrt{\left(\frac{x^3}{3} + \frac{x^5}{5}\right)_0^1}$$

$$= \sqrt{\frac{1}{3} + \frac{1}{5}}$$

$$= \sqrt{\frac{8}{15}}$$ Ans

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Harmonic Analysis

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots$$

where $a_0 = 2$ (mean value of $f(x)$ in $(0, 2\pi)$)

$a_n = 2$ [mean value of $f(x) \cos nx$ in $(0, 2\pi)$]

$b_n = 2$ [mean value of $f(x) \sin nx$ in $(0, 2\pi)$]

The term $(a_1 \cos x + b_1 \sin x)$ in Fourier series is called the fundamental or first harmonic. The term $(a_2 \cos 2x + b_2 \sin 2x)$ is called second Harmonic and so on.

Q(1) Find the Fourier series as far as the second harmonic to represent the function given by the following table:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
$f(x)$	2.34	3.01	3.69	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

Soln

x	$\sin x$	$\sin 2x$	$\cos x$	$\cos 2x$	$f(x)$	$f(x)\sin x$	$f(x)\sin 2x$	$f(x)\cos x$	$f(x)\cos 2x$
0	0	0	1	1	2.34	0	0	2.340	2.340
30	0.50	0.87	0.87	0.5	3.01	1.505	2.619	2.619	1.505
60	0.87	0.87	0.50	−0.5	3.69	3.210	3.210	1.845	−1.845
90	1	0	0	−1	4.15	4.150	0	0	−4.150
120	0.87	−0.87	−0.50	−0.5	3.69	+3.210	−3.210	−1.845	−1.845
150	0.50	−0.87	−0.87	0.5	2.20	+1.100	−1.914	−1.914	1.100
180	0	0	−1	1	0.83	0	0	−0.830	0.830
210	−0.50	0.87	−0.87	0.5	0.51	0.255	0.444	−0.444	+0.255
240	−0.87	0.87	−0.50	−0.5	0.88	−0.766	0.766	−0.440	−0.440
270	−1	0	0	−1	1.09	−1.090	0	0	−1.090
300	−0.87	−0.87	0.50	−0.5	1.19	−1.035	−1.035	0.595	−0.595
330	−0.50	−0.87	0.87	0.5	1.64	−0.820	−1.427	1.427	0.820
			Total	(Σ)	25.22	9.209	−0.547	3.353	−3.115

$a_0 = 2\,[\text{mean of } f(x)] = 2\left(\frac{25.22}{12}\right) = 4.203$

$a_1 = 2\,[\text{Mean of } f(x)\cos x] = 2\left(\frac{3.353}{12}\right) = 0.559$

$a_2 = 2\,[\text{Mean of } f(x)\cos 2x] = 2\left(\frac{-3.115}{12}\right) = -0.519$

$b_1 = 2\,[\text{Mean of } f(x)\sin x] = 2\left(\frac{9.209}{12}\right) = 1.535$

$b_2 = 2\,[\text{Mean of } f(x)\sin 2x] = 2\left(-\frac{0.547}{12}\right) = -0.091$

Fourier Series is

$$f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) + \cdots$$

$$= \frac{4.203}{2} + (0.559\cos x + 1.535\sin x) + (-0.519\cos 2x - 0.091\sin 2x) + \cdots$$

Ans

Q. Compute the first three harmonics of the Fourier series of $f(x)$ given by the following table.

x :	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Soln We exclude the last pt. $x = 2\pi$ as $f(x)$ gives the same value as at $x = 0$ & $x = 2\pi$.

x	$\cos x$	$\cos 2x$	$\cos 3x$	$\sin x$	$\sin 2x$	$\sin 3x$	$f(x)$	$f(x)\cos x$	$f(x)\cos 2x$	$f(x)\cos 3x$
0	1.0	1	1	0	0	0	1.0	1	1	1
$\frac{\pi}{3}$	0.5	−0.5	−1	0.866	0.866	0	1.4	0.7	−0.7	−1.4
$2\pi/3$	−0.5	−0.5	1	0.866	−0.866	0	1.9	−0.95	−0.95	1.9
π	−1	1	−1	0	0	0	1.7	−1.7	+1.7	−1.7
$4\pi/3$	−0.5	−0.5	1	−0.866	0.866	0	1.5	−0.75	−0.75	1.5
$5\pi/3$	0.5	−0.5	−1	−0.866	−0.866	0	1.2	0.60	−0.60	−1.2
Total						$(\Sigma) =$	8.7	−1.1	−0.3	0.1

$a_0 = 2[\text{Mean of } f(x)] = 2\left(\frac{8.7}{6}\right) \Rightarrow 2.9$

$a_1 = 2[\text{Mean of } f(x)\cos x] = 2\left(\frac{-1.1}{6}\right) \Rightarrow -0.37$

$a_2 = 2[\text{Mean of } f(x)\cos 2x] = 2\left(\frac{-0.3}{6}\right) \Rightarrow -0.1$

$a_3 = 2[\text{Mean of } f(x)\cos 3x] = 2\left(\frac{0.1}{6}\right) \Rightarrow 0.03$

In similiar way; we can find;

$b_1 = 2(\text{Mean of } f(x)\sin x) = 0.17$

$b_2 = 2[\text{Mean of } f(x)\sin 2x] = -0.06$

$b_3 = 2[\text{Mean of } f(x)\sin 3x] = 0$

$\therefore f(x) = \frac{2.9}{2} - 0.37\cos x + 0.17\sin x - 0.1\cos 2x - 0.06\sin 2x + 0.03\cos 3x + 0.\sin 3x + \cdots$ Ans.

Q. Compute the first three Harmonics of the fourier series for $f(x)$ from the following data:

x:	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
$f(x)$:	6.824	7.976	8.026	7.204	5.676	3.674	1.764	0.552	0.262	0.904	2.492	4.736

Q. Find the first three Harmonics of fourier series of $y = f(x)$ from the following data:

x:	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
y:	298	356	373	337	254	155	80	51	60	93	147	221

The values of x and the corresponding values of $f(x)$ over a period T are given below. Show that $f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta$, where $\theta = \frac{2\pi x}{T}$

Soln

x:	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$y = f(x)$:	1.98	1.30	1.05	1.30	−0.88	−0.25	1.98

Soln We omit the last value as $f(x)$ at $x=0$ is same as $f(x)$ at $x=T$. $\because \theta = \frac{2\pi x}{T}$, $\therefore x=0 \Rightarrow \theta=0$

$x = \frac{T}{6} \Rightarrow \theta = \frac{2\pi \cdot T}{T \cdot 6} \Rightarrow \frac{2\pi}{6}$

θ	y	$\cos\theta$	$\sin\theta$	$y\cos\theta$	$y\sin\theta$
0	1.98	1.0	0	1.98	0
$\pi/3$	1.30	0.5	0.866	0.65	1.1258
$2\pi/3$	1.05	−0.5	0.866	−0.525	0.9093
π	1.30	−1.0	0	−1.3	0
$4\pi/3$	−0.88	−0.5	−0.866	0.44	0.762
$5\pi/3$	−0.25	0.5	−0.866	−0.125	0.2165
				1.12	3.013
Σ	4.6				

$$a_0 = 2 \times \frac{4.6}{6} \Rightarrow 1.5, \quad a_1 = 2 \times \frac{1.12}{6} = 0.37$$

$$b_1 = \frac{2}{6} \times 3.013 = 1.005$$

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta$$

$$\boxed{f(x) = 0.75 + 0.37\cos\theta + 1.005\sin\theta}$$ Ans.

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