Unit - 4

Function of Complex Variable

$$f(z) = z^2 + 2z + c$$

$$= u(x,y) + i\,v(x,y).$$

f(z) is called function of complex variable.

$$w = f(z) = u(x,y) + i\,v(x,y)$$

$$\Rightarrow w = u + iv.$$

Let $z = x + iy$ be a complex variable where x and y are real variable. Then function of complex variable is denoted and defined by

$$w = f(z) = u(x,y) + i\,v(x,y)$$

$$w = f(z) = u + iv$$

$$w = v + iv$$

Not → w is function of complex variable
$w = f(z)$.

Analytic function

→ It is also known as Regular function / Holomorphic function.

→ A function defined at point z_0 is said to be analytic at z_0 if it has derivative at z_0 and every point in the neighbourhood of z_0.

Condition for analytic for function f(z) :-

We have $w = f(z) = u + iv$ ---- ①

Condition :- $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These equation are called Cauchy Reman eqn.

Note → If CR eq are satisfied then the function is analytic. These equation is known as cartesian form.

Q Test whether $w = \bar{z}$ is analytic.

Sol → $w = f(z) = u + iv$

Given $w = \bar{z}$

$\Rightarrow w = x - iy$

$\Rightarrow u + iv = x - iy$

On equating real and imaginary part.

$u = x$ ---- ①

$v = -y$ ---- ②

Differentiating ① and ② w.r.t x & y partially.

$\frac{\partial u}{\partial x} = 1$ $\quad \frac{\partial u}{\partial y} = 0$

$\frac{\partial v}{\partial x} = 0$ $\quad \frac{\partial v}{\partial y} = -1$

We know. CR equation.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow 1 \neq -1$ & $0 = 0$

Since cauchy Remann eqⁿ is not satisfied.

So, function is not analytic

Q Show that $f(z) = \bar{z}$ is nowhere differentiable

Sol → $w = f(z) = u + iv$

$f(z) = \bar{z}$

$u + iv = x - iy$

On equating,

$u = x$ ---- (1)

$v = -y$ ---- (2)

On differentiating

$\frac{\partial u}{\partial x} = 1$ $\quad \frac{\partial u}{\partial y} = 0$

$\frac{\partial v}{\partial x} = 0$ $\quad \frac{\partial v}{\partial y} = -1$

We know Cauchy Remann eqⁿ.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\Rightarrow \underbrace{1 = -1}_{\text{false}}$

So, cauchy's Reemann eqn is not satisfied.

So, function is not differentiable.

Q Investigat the function $f(z) = z^2$.

Sol → We have $f(z) = z^2$

$u + iv = z^2 \Rightarrow u + iv = (x+iy)^2$

$$u + iv = x^2 - y^2 + i(2xy).$$

On equating eq. ⓐ real & imaginary part

$$u = x^2 - y^2 \quad --- ①$$

$$v = 2xy \quad --- ②$$

On differentiating both w.r.t x & y partially.

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2x$$

We know Cauchy Remann eqn :-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x = 2x$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -2y = -2y$$

So, Cauchy Remann equation is satisfied

Hence function $f(z)$ is analytic.

Derivative of $f(z)$:-

$$f(z) = u + iv.$$

$$f(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Q Test for analyticity of function $e^x(\cos y + i\sin y)$

Sol → Let $u + iv = e^x(\cos y + i\sin y)$

On equating real and imaginary part

$u = e^x \cos y$ ---- ①

$v = e^x \sin y$ ---- ②

On differentiating w.r.t. x & y partially.

$\frac{\partial u}{\partial x} = e^x \cos y$ $\quad$ $\frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial v}{\partial x} = e^x \sin y$ $\quad$ $\frac{\partial v}{\partial y} = e^x \cos y$.

We know CR equation.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow e^x \cos y = e^x \cos y$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -e^x \sin y = -e^x \sin y$.

So, CR equation is satisfied.

So, function is differentiable

Now, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= e^x \cos y + i e^x \sin y$

$= e^x(\cos y + i \sin y)$

$= e^x e^{iy}$

$= e^{x+iy}$.

$= e^z$.

So, $f'(z) = e^z$ Ans

Q Show that $f(z) = \sin z$ is an analytic function.

Sol → Given $f(z) = \sin z$

$$u + iv = \sin(x+iy)$$

$$u + iv = \sin x \cos iy + \cos x \sin iy.$$

$$u + iv = \sin x \cosh y + \cos x \,(i \sinh y)$$

$$\boxed{\because \sin i\theta = i \sinh\theta \qquad \cos i\theta = \cosh\theta}$$

$$u + iv = \sin x \cosh y + i \cos x \sinh y$$

On equating,

$$u = \sin x \cosh y$$

$$v = \cos x \sinh y$$

On differentiating w.r.t x & y partially.

$$\frac{\partial u}{\partial x} = \cos x \cosh y \qquad \frac{\partial u}{\partial y} = \sin x \sinh y.$$

$$\frac{\partial v}{\partial x} = -\sin x \sinh y \qquad \frac{\partial v}{\partial y} = \cos x \cosh y$$

We know, CR equation.

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow +\sin x \sinh y = +\sin x \sinh y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \cos x \cosh y = \cos x \cosh y$$

So, CR equation is satisfied

So, function is analytic

Q $f(z) = e^z$ is analytic

Sol → $f(z) = e^z$

$u + iv = e^{x+iy}$

$= e^x[\cos y + i\sin y]$

$u + iv = e^x\cos y + i e^x \sin y.$

On equating,

$u = e^x \cos y$

$v = e^x \sin y$

· On differentiating w.r.t x & y partially,

$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial v}{\partial x} = e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y.$

We know CR equation

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow e^x\cos y = e^x \cos y$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -e^x \sin y = -e^x \sin y$

So, CR equation is satisfied

So, $f(z) = e^z$ is analytic.

CR equation in polar form :-

We have $f(z) = u + iv$.

Condition :- $\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}\right\} \rightarrow$ these eqn is called CR equation in polar form.

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Q Discuss the analytic of function $r^n(\cos n\theta + i\sin n\theta)$

Sol→ Let $w = r^n(\cos n\theta + i\sin n\theta)$

$u + iv = r^n\cos n\theta + i r^n \sin n\theta$

On Equating real & imaginary part.

$u = r^n \cos n\theta$

$v = r^n \sin n\theta$

On differentiating w.r.t ~~x~~ r & θ partially.

$\frac{\partial u}{\partial r} = n r^{n-1}\cos n\theta \qquad \frac{\partial u}{\partial \theta} = -n r^n \sin n\theta$

$\frac{\partial v}{\partial r} = n r^{n-1} \sin n\theta \qquad \frac{\partial v}{\partial \theta} = n r^n \cos n\theta$

Now, CR equation of polar form.

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \Rightarrow n r^{n-1}\cos n\theta = \frac{1}{r} n r^n \cos n\theta$

$\Rightarrow n r^{n-1}\cos n\theta = n r^{n-1}\cos n\theta$

$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \Rightarrow n r^{n-1}\sin n\theta = -\frac{1}{r}(-n r^n \sin n\theta)$

$\Rightarrow n r^{n-1}\sin n\theta = n r^{n-1}\sin n\theta$

So, CR equation is satisfied

So, it is analytic.

Q Show that $f(z) = z^n$ is analytic where n is positive integer.

Sol→ put $z = re^{i\theta}$.

$u + iv = (re^{i\theta})^n$

$$U + iV = r^n e^{in\theta}$$

$$\Rightarrow r^n[\cos n\theta + i \sin_n \theta]$$

$$U + iV = r^n \cos n\theta + i r^n \sin n\theta$$

Now same process as previous ques.

function is analytic

Q find the value of C_1 and C_2 such that the function $f(z) = C_1 xy + i(C_2 x^2 + y^2)$. is analytic

Sol → $f(z) = C_1 xy + i(C_2 x^2 + y^2)$

On equating real and imaginary.

$$U = C_1 xy \quad \text{--- (1)}$$

$$V = C_2 x^2 + y^2 \quad \text{--- (2)}$$

On differentiating w.r.t. x & y partially.

$$\frac{\partial u}{\partial x} = C_1 y \qquad \frac{\partial u}{\partial y} = C_1 x$$

$$\frac{\partial v}{\partial x} = 2C_2 x \qquad \frac{\partial v}{\partial y} = 2y$$

Since function is analytic. So,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow C_1 y = 2y$$

$$\Rightarrow C_1 = 2 \quad \text{Ans}$$

$$\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 2C_2 x = -C_1 x$$

$$\Rightarrow 2C_2 x = -2x$$

$$\Rightarrow C_2 = -1 \quad \text{Ans}$$

Harmonic function :-

If u & v are the function of x and y then.
satisfy Laplace equation such that u and v
is called harmonic function.

i.e. $$\boxed{\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0\end{aligned}}$$

Q Show that function $u = e^x \cos y$ is harmonic.

Sol→ $\frac{\partial u}{\partial x} = e^x \cos y$ $\frac{\partial^2 u}{\partial x^2} = e^x \cos y$ —— ①

$\frac{\partial u}{\partial y} = -e^x \sin y$ $\frac{\partial^2 u}{\partial y^2} = -e^x \cos y$ —— ②

Adding ① and ②

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$$

So, since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

So, u is harmonic.

Home Work

Q Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic

Sol→ $\frac{\partial u}{\partial x} = \frac{1\,(\cancel{2}x)}{\cancel{2}(x^2+y^2)}$, $\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cancel{2} - \cancel{2}x(2x)}{\cancel{2}(x^2+y^2)^2}$

$$= \frac{(y^2-x^2)}{(x^2+y^2)^2} \text{ —— ①}$$

$$\frac{\partial v}{\partial x} = \frac{1}{z} \cdot \frac{xy}{(x^2+y^2)}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2} \quad (2)$$

So. on adding (1) and (2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{(y^2-x^2)}{(x^2+y^2)^2} + \frac{(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}$$

$$= 0$$

Since

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

So, u is harmonic

Harmonic Conjugat

Let $f(z) = u + iv$ be an analytic function then u is called conjugat of v and v is called conjugat of u.

Case I $\rightarrow$ When u is given

$$\partial v = \frac{\partial v}{\partial x}\partial x + \frac{\partial v}{\partial y}\partial y$$

By CR equation

$$\boxed{\partial v = -\frac{\partial u}{\partial y}\partial x + \frac{\partial u}{\partial x}\partial y}$$

Case II $\rightarrow$ When v is given

$$\partial u = \frac{\partial u}{\partial x}\partial x + \frac{\partial u}{\partial y}\partial y$$

By CR equation

$$\boxed{\partial u = \frac{\partial v}{\partial y}\partial x - \frac{\partial v}{\partial x}\partial y}$$

Q find the harmonic conjugat of $u = e^x \cos y$.

Sol $\rightarrow$ $\frac{\partial u}{\partial x} = e^x \cos y$ $\quad \frac{\partial u}{\partial y} = -e^x \sin y$

we have $\partial v = -\frac{\partial u}{\partial y}\partial x - \frac{\partial u}{\partial x}\partial y$

$$= e^x \sin y\,\partial x + e^x \cos y\,\partial y$$

$$\partial v = \partial(e^x \sin y).$$

$$\int \partial v = \int \partial(e^x \sin y)$$

$$v = e^x \sin y + C \quad \text{Ans}$$

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Q find the Conjugat of $u = \frac{1}{2}\log(x^2+y^2)$

Sol → $\frac{\partial u}{\partial x} = \frac{1}{2}\frac{2x}{(x^2+y^2)}$, $\frac{\partial u}{\partial y} = \frac{1}{2}\frac{(2y)}{(x^2+y^2)}$

we have

$$\partial v = \frac{-\partial u}{\partial y}\partial x + \frac{\partial u}{\partial x}\partial y$$

$$\partial v = \frac{-y}{(x^2+y^2)}\partial x + \frac{x}{(x^2+y^2)}\partial y$$

$$\partial v = \frac{(x\partial y - y\partial x)}{x^2+y^2}$$

$$\partial v = \partial\left(\tan^{-1}\frac{y}{x}\right)$$

On Integrating

$$v = \tan^{-1}\frac{y}{x} + c \quad \text{Ans}$$

Q find harmonic Conjugate of $v = e^x \sin y$

Sol → $v = e^x \sin y$

$\frac{\partial v}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial y} = \cancel{e^x \cos y}\; e^x \sin y$

we have. $\partial u = \frac{\partial v}{\partial y}\partial x - \frac{\partial v}{\partial x}\partial y$

$$\partial u = e^x \sin y\, \partial x - e^x \cos y\, \partial y$$

$$\partial u = \partial(e^x \cos y)$$

On Integrating

$$u = e^x \cos y + c \quad \text{Ans}$$

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Operator formula

$$\boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}}$$

Q If $f(z)$ is analytic, show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Sol → Taking LHS.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2$$

$$= 4\left(\frac{\partial^2}{\partial z \partial \bar{z}}\right)|f(z)|^2$$

$$= 4\left(\frac{\partial^2}{\partial z \partial \bar{z}}\right)\left[f(z) \cdot f(\bar{z})\right]$$

$$= 4\frac{\partial}{\partial z}\left[f(z) \cdot f'(\bar{z})\right]$$

$$= 4 \cdot \left[f'(z) \cdot f'(\bar{z})\right]$$

$$= 4|f'(z)|^2$$

So, LHS = RHS

Hence proved

Q If $f(z)$ is analytic show that
$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)\left\{\log |f(z)|\right\} = 0$$

Sol → taking LHS.
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left\{\log\left(|f(z)|^2\right)^{1/2}\right\}.$$

$$\Rightarrow 4\frac{\partial^2}{\partial z\partial \bar{z}}\left\{\frac{1}{2}\log|f(z)|^2\right\}$$

$$= 4\frac{\partial^2}{\partial z\partial \bar{z}} \times \frac{1}{2} \times \log\left[f(z)\cdot f(\bar{z})\right].$$

$$= 2\frac{\partial^2}{\partial z\partial \bar{z}}\left[\log f(z) + \log f'(\bar{z})\right]$$

$$= 2\frac{\partial}{\partial z}\left[0 + \frac{1}{f(\bar{z})}\cdot f'(\bar{z})\right]$$

$$= 2\left[0+0\right]$$

$$= 0$$

So, LHS = RHS.

Hence proved

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Q If $f(z)$ is analytic function show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2 \{f(z)\}^{p-2} |f(z)|^2$$

Sol→ Taking LHS.

$$4 \frac{\partial^2}{\partial z \partial \bar{z}} |f(z)|^p$$

$$\Rightarrow 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left\{|f(z)|^2\right\}^{p/2}$$

$$\Rightarrow 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left[f(z)\, f(\bar{z})\right]^{p/2}$$

$$\Rightarrow 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left([f(z)]^{p/2} \cdot [f(\bar{z})]^{p/2}\right)$$

$$\Rightarrow 4 \frac{\partial}{\partial z} \left[f(z)^{p/2} \cdot \frac{p}{2} \cdot f(\bar{z})^{\frac{p}{2}-1} \cdot f'(\bar{z})\right]$$

$$\Rightarrow 4 \left[\frac{p}{2} f(z)^{p/2-1} f'(z) \cdot \frac{p}{2} f(\bar{z})^{p/2-1} \cdot f'(\bar{z})\right]$$

$$= 4 \times \frac{p^2}{4} \times [f(z) \cdot f(\bar{z})]^{\frac{p-2}{2}} \cdot [f'(z) \cdot f'(\bar{z})]$$

$$= p^2 |f(z)|^{p-2} |f'(z)|^2 = RHS$$

Hence proved

Q If $f(z)$ is analytic function of z. find the value of $4\frac{\partial^2}{\partial z \partial \bar{z}}\left[Re\left(f(z)\right)\right]^2$.

Sol→ We have

$$4\frac{\partial^2}{\partial z \partial \bar{z}}\left|Re f(z)\right|^2 \quad \text{--- } (1)$$

$$\therefore f(z) = u + iv \quad \text{--- } (2)$$

$$f(\bar{z}) = u - iv \quad \text{--- } (3)$$

Adding (2) & (3)

$$u = \left[\frac{f(z) + f(\bar{z})}{2}\right] \quad \text{--- } (4)$$

Using (4) in (1)

$$4\frac{\partial^2}{\partial \bar{z} \partial z}\left[\frac{f(z) + f(\bar{z})}{2}\right]^2$$

$$\Rightarrow \frac{4}{4}\frac{\partial^2}{\partial \bar{z} \partial z}\left(\{f(z) + f(\bar{z})\}\{\overline{f(z) + f(\bar{z})}\}\right)$$

$$\Rightarrow \frac{4}{4}\frac{\partial^2}{\partial \bar{z} \partial z}\left[\{f(z) + f(\bar{z})\}\{f(\bar{z}) + f(z)\}\right]$$

$$\left[\because \overline{A+B} = \bar{A} + \bar{B}, \quad \overline{\bar{A}} = A\right]$$

$$\Rightarrow \frac{\partial^2}{\partial \bar{z} \partial z}\left[f(z) + f(\bar{z})\right]^2$$

$$\Rightarrow \frac{\partial}{\partial z}\left[2\{f(z) + f(\bar{z})\} \cdot f'(\bar{z})\right]$$

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$$= 2\left[f'(z) + 0\right] \cdot f'(\bar{z})$$

$$= 2 f'(z) \cdot f'(\bar{z})$$

$$= 2|f(z)|^2$$ Ans

Construction of f(z) :-

[u & v → given
u or v → given
find f(z)]

Milne Thomson Method :-

By using milne Thomson Method, f(z) is find when u or v is given.

Case I :- When u is given :-

$$f(z) = u + iv$$

Differentiating w.r.t x.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

By CR equation.

$$f'(z) = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Taking Integral.

$$f(z) = \int\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)dz + C$$

where $\phi_1(z,0) = \frac{\partial u}{\partial x}$

$$\phi_2(z,0) = \frac{\partial u}{\partial y}$$

$$\Rightarrow f(z) = \int [\phi_1(z,0) - \phi_2(z,0)]\, \partial z + C.$$

Case II :- When v is given.

$$f(z) = u + iv$$

Differentiating w.r.t to x.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

By CR equation,

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

On Integrating.

$$f(z) = \int \left(\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}\right) dz + C$$

where $\psi_1(z,0) = \frac{\partial v}{\partial y}$

$$\psi_2(z,0) = \frac{\partial v}{\partial x}$$

$$\Rightarrow f(z) = \int [\psi_1(z,0) + \psi_2(z,0)]\, dz + C$$

Q If $u = e^x \cos y$ find $f(z)$

Sol →

$u = e^x \cos y$

$\frac{\partial u}{\partial x} = e^x \cos y$.

$\phi_1(z,0) = e^z \cos 0 = e^z$

$\frac{\partial u}{\partial y} = -e^x \sin y$

$\phi_2(z,0) = e^z \sin 0$

$= 0$

By Milne Method.

$f(z) = \int [\phi_1(z,0) + i\,\phi_2(z,0)]\,dz + c$

$= \int (e^z - i \cdot 0)\,dz + c$

$f(z) = e^z + c$ Ans

Q If ① $u+v = \frac{x}{x^2+y^2}$ and $f(1) = 1$

② $u-v = e^x(\cos y - \sin y)$

③ $2u+v = e^x(\cos y - \sin y$

④ $u-v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

find $f(z) = u+iv$.

Solution → (1) $u + v = \dfrac{x}{x^2+y^2}$ ---- (1)

we have $f(z) = u + iv$ ---- (2)

$i \cdot f(z) = iu - v$ ---- (3)

Adding (2) & (3)

$$(1+i)f(z) = (u-v) + i(u+v)$$

$$\Rightarrow F(z) = U + iV \quad \text{---- (4)}$$

where $F(z) = (1+i)f(z)$

$$U = (u-v)$$

$$V = (u+v)$$

$$V = \frac{x}{x^2+y^2} \quad [\text{from (1)}]$$

$$\frac{\partial V}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\psi_2(z,0) = \frac{0 - z^2}{(z^2+0)^2} = \frac{-1}{z^2}$$

$$\frac{\partial V}{\partial y} = x\left[\frac{-2y}{(x^2+y^2)^2}\right]$$

$$\psi_1(z,0) = 0$$

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$$F(z) = \int \varphi_1(z,0) + i\varphi_2(z,0).$$

$$= \int \left[0 + i\left(-\frac{1}{z^2}\right)\right] dz + C$$

$$= -i \int \frac{-1}{z^2} dz + C$$

$$= -i\left(\frac{-1}{z}\right) + C$$

$$F(z) = \frac{i}{z} + C$$

$$\Rightarrow \quad (1+i) f(z) = \frac{i}{z} + C$$

$$f(z) = \frac{i}{z(1+i)} + \frac{C}{(i+1)}$$

$$= \frac{i(1-i)}{z(i+1)(1-i)} + C_1$$

$$= \frac{i - i^2}{z(1+1)} + C_1$$

$$= \frac{i+1}{z \times 2} + C_1$$

$$f(z) = \left(\frac{i+1}{2}\right)\frac{1}{z} + C_1 \quad -- (5)$$

Now, Given $f(1) = 1$

Put $z = 1$ in (5)

$$f(z) = \frac{i+1}{1 \cdot 2} + C_1$$

$$1 = \frac{i+1}{2} + C_1$$

$$C_1 = 1 - \left(\frac{1+i}{2}\right)$$

$$C_1 = \frac{1-i}{2}$$

Putting value of C_1 in eq. ⑤, we get

$$f(z) = \frac{(1+i)}{2} \cdot \frac{1}{z} + \frac{(1-i)}{2} \quad \text{Ans}$$

② $u - v = e^x(\cos y - \sin y)$ —— ①

We have $f(z) = u + iv$ —— ②

$if(z) = iu - v$ —— ③

Adding ② and ③.

$$(1+i)f(z) = (u-v) + i(u+v)$$

$$\Rightarrow F(z) = U + iV \quad \text{—— ④}$$

where $F(z) = (1+i)f(z)$

$U = u - v$

$V = u + v$.

Putting value of $(u-v)$ in eq. ①

$$U = e^x(\cos y - \sin y)$$

$$U = e^x(\cancel{\sin}\cos y - \sin y).$$

$$\frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$$

$$\phi_1(z,0) = e^z(1-0)$$

$$= e^z$$

$$\frac{\partial U}{\partial y} = e^x[-\sin y - \cos y]$$

$$\phi_2(z,0) = e^z[-\sin 0 - \cos 0]$$

$$= -e^z.$$

Now, $F(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]\,dz + C$

$$(1+i)f(z) = \int[e^z - i(-e^z)]\,dz + C$$

$$(1+i)f(z) = (1+i)\cancel{2}\int e^z\,dz + C$$

$$(1+i)f(z) = (1+i)\cancel{2}e^z + C$$

$$f(z) = \frac{(1+i)\cancel{2}e^z}{(1+i)} + \frac{C}{\cancel{6}(i+1)}$$

$$= \frac{\cancel{2}(1+i)e^z}{\cancel{2}(1+i)} + C_1$$

$$f(z) = \cancel{(\ldots)}e^z + C_1 \quad \text{Ans}$$

③ $2u + v = e^x (\cos y - \sin y)$ ---- ①

$f(z) = u + iv$ --- ②

$if(z) = iu - v$ --- ③

$2if(z) = 2iu - v$ --- ④

Adding ② and ④

$(1+2i) f(z) = (u-2v) + i(2u+v)$.

$F(z) = U + iV$ --- ⑤

where $F(z) = (1+2i) f(z)$

$U = u - 2v$

$V = 2u + v$

Now, eq. ① can be written as.

$V = e^x (\cos y - \sin y)$

$\frac{\partial V}{\partial x} = e^x (\cos y - \sin y)$

$\psi_2 (z, 0) = e^z (\cos 0 - \sin 0)$

$= e^z$

$\frac{\partial V}{\partial y} = e^x (-\sin y - \cos y)$

$\psi_1 (z, 0) = e^z [-\sin 0 - \cos 0]$

$= -e^z$

By Milne's Thomson method

$F(z) = \int \psi_1 (z, 0) + i \psi_2 (z, 0)\, dz + c$

$$= \int \left(-e^{z} + i e^{z}\right) dz + C$$

$$= \int (i-1)\, e^{z} dz + C$$

$$(1+2i)\, f(z) = (i+1)\, e^{z} + C_1$$

$$f(z) = \frac{(i-1)}{(1+2i)} e^{z} + \frac{C}{(1+2i)}$$

$$f(z) = \frac{1+3i}{5} e^{z} + C_2$$

Ans

④ $U - V = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ —— ①

$f(z) = u + iv$ —— ②

$if(z) = iu - v$ —— ③

Adding ② and ③

$(1+i) f(z) = (u-v) + i(u+v)$

$F(z) = U + iV$

where $F(z) = (1+i) f(z)$

$U = u - v$

$V = u + v$

Now, eq ①

$$U = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

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$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)\cdot(2\cos 2x) - \sin 2x \cdot (2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{(\cos 0 - \cos 2z)\cdot 2\cos 2z - \sin 2z\cdot(2\sin 2z)}{(\cosh 2\cdot 0 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2(\cos 2z - 1)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{(1-\cos 2z)} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

$$\frac{\partial v}{\partial y} = \frac{(\cosh 2y - \cos 2x)\cdot(2\sinh 2y) - \sin 2x\,(\cdot 2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = \frac{(1-\cos 2z)\times 0 - \sin 2z\,(2\sin 0)}{(\cosh 2y - \cos 2x)^2}$$

$$= 0$$

By Milne Thomson Method

$$F(z) = \int[\phi_1(z,0) - \phi_2(z,0)]\,dz + c$$

$$= \int -\operatorname{cosec}^2 z \; dz$$

$$F(z) = \cot z + c.$$

$\Rightarrow (1+i)f(z) = \cot z + C$

$$f(z) = \frac{\cot z}{(1+i)} + \frac{C}{(1+i)}$$

$$f(z) = \frac{\cot z}{(1+i)} + C_1 \quad \text{Ans}$$

Q Find the analytic function if $U + 2V = (x-y)(x^2+4xy+y^2)$.

Sol → $U + 2V = (x+y)(x^2+4xy+y^2)$

$U + 2V = x^3 + 3x^2y - 3xy^2 - y^3$ ——— (1)

$f(z) = U + iV$ —— (2)

$if(z) = iU - V$ —— (3)

$2f(z) = 2U + i(2V)$ —— (4)

Adding (3) and (4)

$(2+i)f(z) = (2U - V) + i(2V + U)$

$F(x) = U + iV$

where $F(x) = (2+i)f(z)$

$U = 2U - V$

$V = U + 2V$

Now, eq. (1)

$V = x^3 + 3x^2y - 3xy^2 - y^3$

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$$\frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2$$

$$\psi_2(z,0) = 3z^2$$

$$\frac{\partial v}{\partial y} = 3x^2 - 6xy - 3y^2$$

$$\psi_1(z,0) = 3z^2$$

By Milne Thomson Method.

$$f(z) = \int \psi_2(z,0) + i\,\psi_1(z,0)\,dz + C$$

$$(2+i)\,f(z) = \int (3z^2 + i\,3z^2)\,dz + C$$

$$(2+i)\,f(z) = (1+i)\int 3z^2\,dz + C$$

$$(2+i)\,f(z) = (1+i)\,z^3 + C$$

$$f(z) = \frac{1+i}{2+i}\,z^3 + \frac{C}{2+i}$$

$$= \frac{(1+i)(2-i)}{(4+1)}\,z^3 + C_1$$

$$= \frac{3+i}{5}\,z^3 + C_1$$

So, $f(z) = \frac{3+i}{5}\,z^3 + C_1$ Ans

Q. Find the analytic function $f(z)$ if.

$$U - V = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}.$$

Sol → $$U - V = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}.$$

$$= \frac{\cos x + \sin x - (\cosh y - \sinh y)}{2\cos x - 2\left(\frac{e^{y} + e^{-y}}{2}\right)}$$

$$\left[\because e^{-\theta} = \cosh\theta - \sinh\theta \quad \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}\right]$$

$$U - V = \frac{\cos x + \sin x - \cosh y + \sinh y}{(2\cos x - 2\cosh y)} \quad \text{--- (1)}$$

$f(z) = U + iV$ --- (2)

$if(z) = iU - V$ --- (3)

Adding (2) and (3)

$(1+i)f(z) = (U-V) + i(U+V) \Rightarrow F(z) = U + iV$

where $F(z) = (1+i)f(z)$

$U = (U - V)$

$V = U + V.$

Now, Eq. (1)

$$U = \frac{\cos x + \sin x - \cosh y + \sinh y}{2\cos x - 2\cosh y}.$$

$$\frac{\partial u}{\partial x} = \frac{(\cos x - \cosh y)\cdot(-\sin x + \cos x) - (\cos x + \sin x - \cosh y + \sinh y)\cdot(-\sin x)}{2(\cos x - \cosh y)^2}$$

$$\phi_1(z,0) = \frac{(\cos z - 1)\cdot(-\sin z + \cos z) - (\cos z - \sin z)\cdot(-\sin z)}{2(\cos z - \cosh 0)^2}$$

$$\phi_1(z,0) = \frac{\cancel{-\cos z \sin z} + \cos^2 z + \cancel{\sin z} - \cos z + \cancel{\cos z \sin z} + \sin^2 z - \cancel{\sin z}}{2(\cos z - 1)^2}$$

$$= \frac{(1-\cos z)}{2(\cos z - 1)^2} = \frac{1}{2(1-\cos z)}$$

$$= \frac{1}{2(\cancel{1} - (\cancel{1} - 2\sin^2\frac{z}{2}))} = \frac{1}{4\sin^2\frac{z}{2}}$$

$$= \frac{1}{4}\operatorname{cosec}^2\frac{z}{2}.$$

$$\frac{\partial u}{\partial y} = \frac{(\cos x - \cosh y)\cdot(-\sinh y + \cosh y) - (\cos x + \sin x - \cosh y + \sinh y)\cdot(-\sinh y)}{2(\cos x - \cosh y)^2}$$

$$\phi_2(z,0) = \frac{(\cos z - 1)\cdot(0+1) - (\cos z + \sin z - 1 + 0)\cdot 0}{2(\cos z - \cosh y)^2}$$

$$= \frac{1}{2(\cos z - 1)} = \frac{1}{2(1 - 2\sin^2\frac{z}{2} - 1)}$$

$$\phi_2(z,0) = \frac{-1}{4\sin^2\frac{z}{2}} = -\frac{1}{4}\operatorname{cosec}^2\frac{z}{2}.$$

By Milne Thomson Method,

$$F(z) = \int \phi_1(z,0) - i\phi_2(z,0)\, dz + C.$$

$$= \int \frac{1}{4}\operatorname{cosec}^2\frac{z}{2} + i\left(\frac{1}{4}\operatorname{cosec}^2\frac{z}{2}\right) dz + C$$

$$= \int \left(\frac{1+i}{4}\right)\operatorname{cosec}^2\frac{z}{2}\, dz + C$$

$$= \frac{1+i}{4}\left(-\cot\frac{z}{2}\times 2\right) + C.$$

$$(1+i)f(z) = -\frac{(1+i)}{2}\cot\frac{z}{2} + C$$

$$f(z) = \frac{-(1+i)}{(1+i)2}\cot\frac{z}{2} + \frac{C}{(1+i)}$$

$$f(z) = \frac{-1}{2}\cot\frac{z}{2} + C_1$$ Ans

Now, if asked to find for $f(\pi/2) = 0$

$$f(\pi/2) = \frac{-1}{2}\cot\frac{\pi}{4} + C_1$$

$$0 = \frac{-1}{2}\times 1 + C_1 \Rightarrow C_1 = \frac{1}{2}$$

So, $$f(z) = \frac{-1}{2}\cot\frac{z}{2} + \frac{1}{2}$$ Ans

Conformal mapping

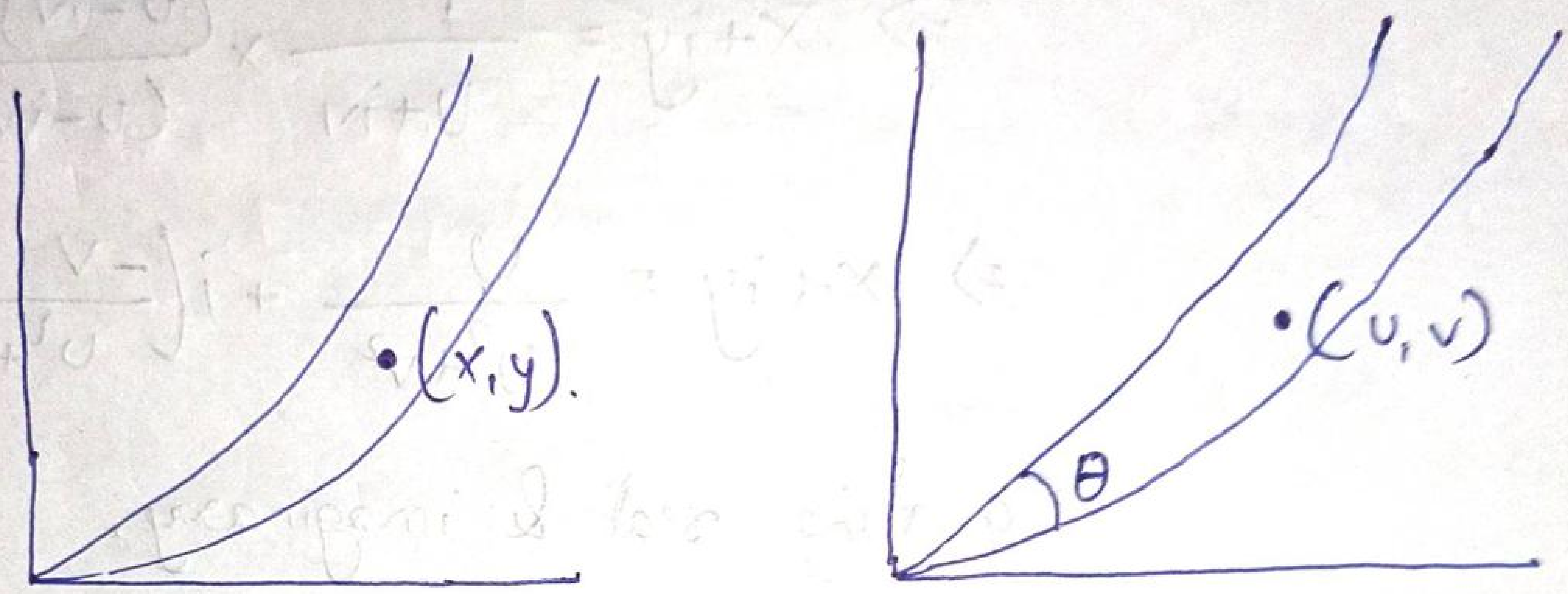
Z - plane

$Z = x+iy$

w - plane

$w = u+iv$

Given → (x, y) $\qquad w = f(z) \qquad (u, v)$ ← find.



Let each point (x, y) in the z-plane, the function $w = f(z)$. Determine a point (u, v) in the w-plane.

If the point z move along the curve C in z-plane and corresponding point w will move along the curve C'. The corresponding design is called mapping or transformation of the function.

A transform that preserve angle between every pair of curve through a point both in magnitude and sense is called to be conformal mapping at that point.

Magnification Transformation :- The transform $w = az$ is called magnification transformation where a is real value/ constant

i.e. $w = az$

$u+iv = a(x+iy)$.

On Equating real & imaginary.

$u = ax$ & $v = ay$.

Inverse Transformation → The transformation $w = \frac{1}{z}$ is called Inverse transformation.

i.e. $w = \frac{1}{z}$

$\Rightarrow z = \frac{1}{w}$

$\Rightarrow x+iy = \frac{1}{u+iv} \times \frac{(u-iv)}{(u-iv)}$

$\Rightarrow x+iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$

Equating real & imaginary.

$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

Q Find the image of the rectangular region in the z-plane bounded by line $x=0, y=0, x=2, y=1$. Under the transformation $w = 2z$.

Sol → We have,

$w = 2z$ —— (1)

$u+iv = 2(x+iy)$

Equating real & imaginary,

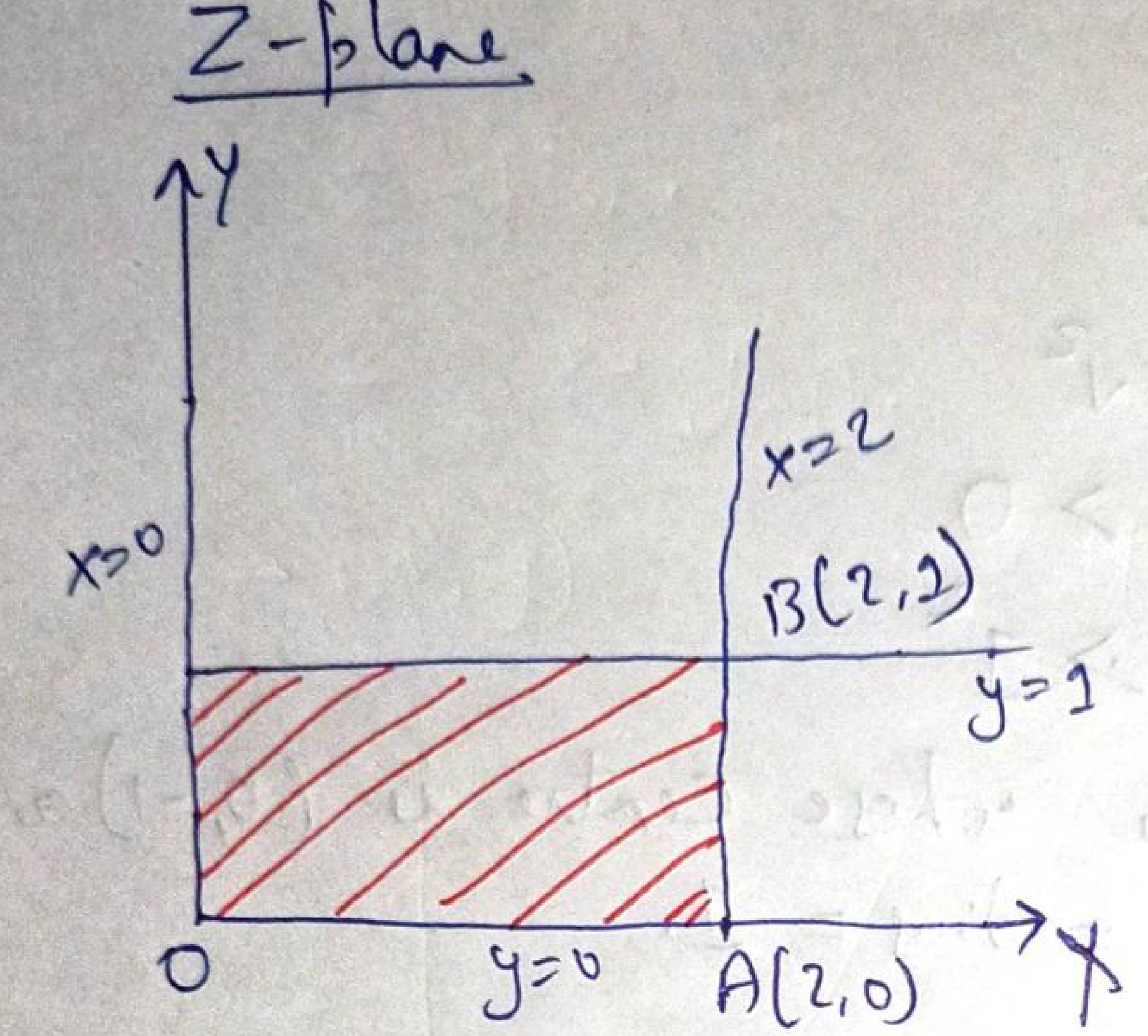
$u = 2x$ —— (2)

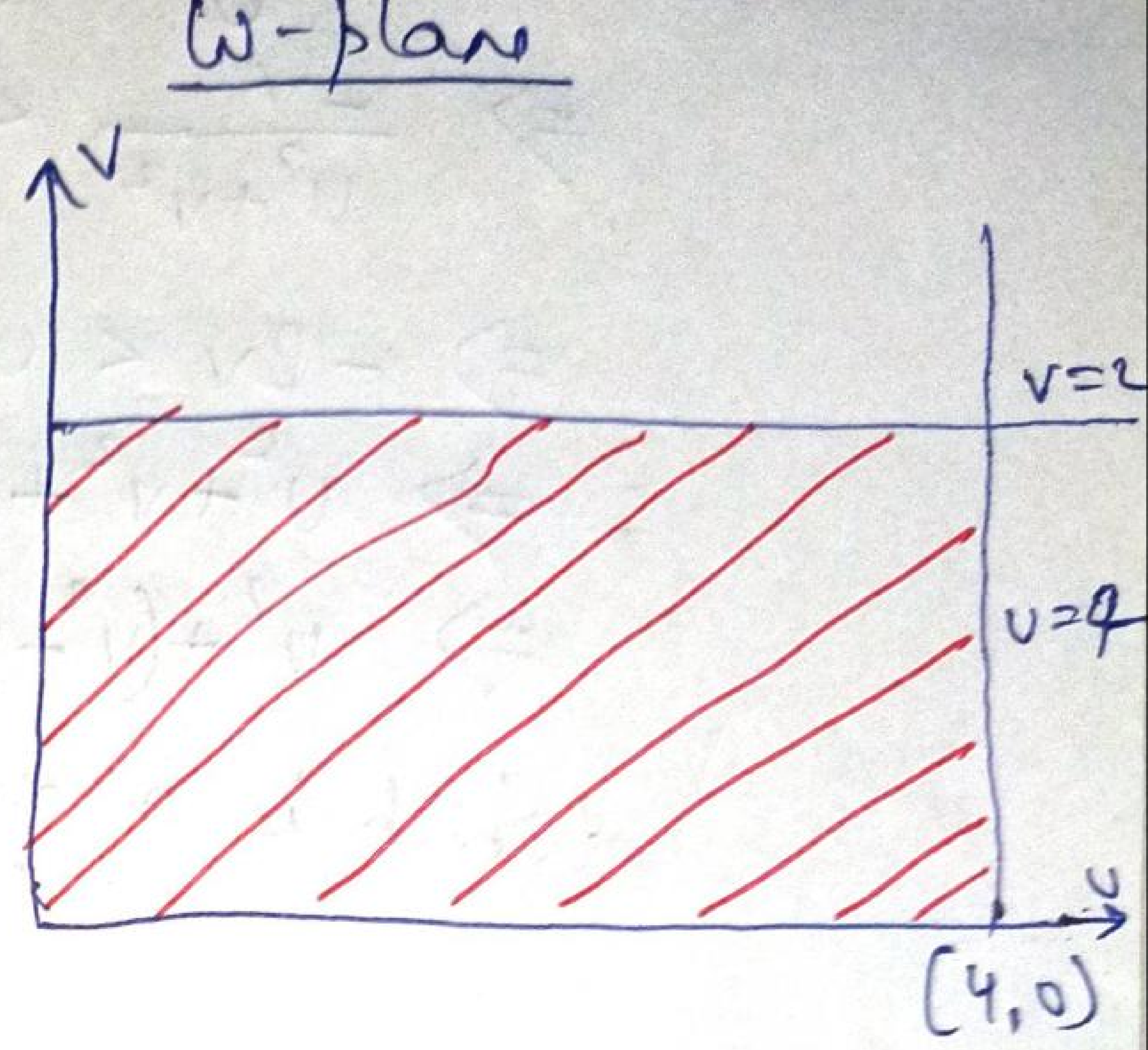
$v = 2y$ —— (3)

Given $x=0, y=0, x=2, y=1$

Then putting these values in eq. (2) & (3).

$u=0, v=0, u=4, v=2.$





In this transformation, rectangle in z is mapped into the w-plane.

<u>Q</u> Find the image of the infinit loop $\frac{1}{4} < y < \frac{1}{2}$ under transformation $w = \frac{1}{z}$.

Sol→ We have. $w = \frac{1}{z}$.

$$z = \frac{1}{w}.$$

$$x + iy = \frac{1}{u+iv} \cdot \frac{u-iv}{u-iv}$$

$$x + iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$$

On Equating real & imaginary

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

Since $\frac{1}{4} < y < \frac{1}{2}$

or $y < \frac{1}{2}$ && $y > \frac{1}{4}$.

we have $y < \frac{1}{2}$.

$\Rightarrow \frac{-v}{u^2+v^2} < \frac{1}{2}$.

$\Rightarrow -2v < u^2+v^2$

$\Rightarrow u^2+v^2+2v > 0$

$\Rightarrow u^2+(v+1)^2 > 1^2$

Which is a circle whose centre is $(0,-1)$ and have radius = 1.

Also $y > \frac{1}{4}$.

$\frac{-v}{u^2+v^2} > \frac{1}{4}$

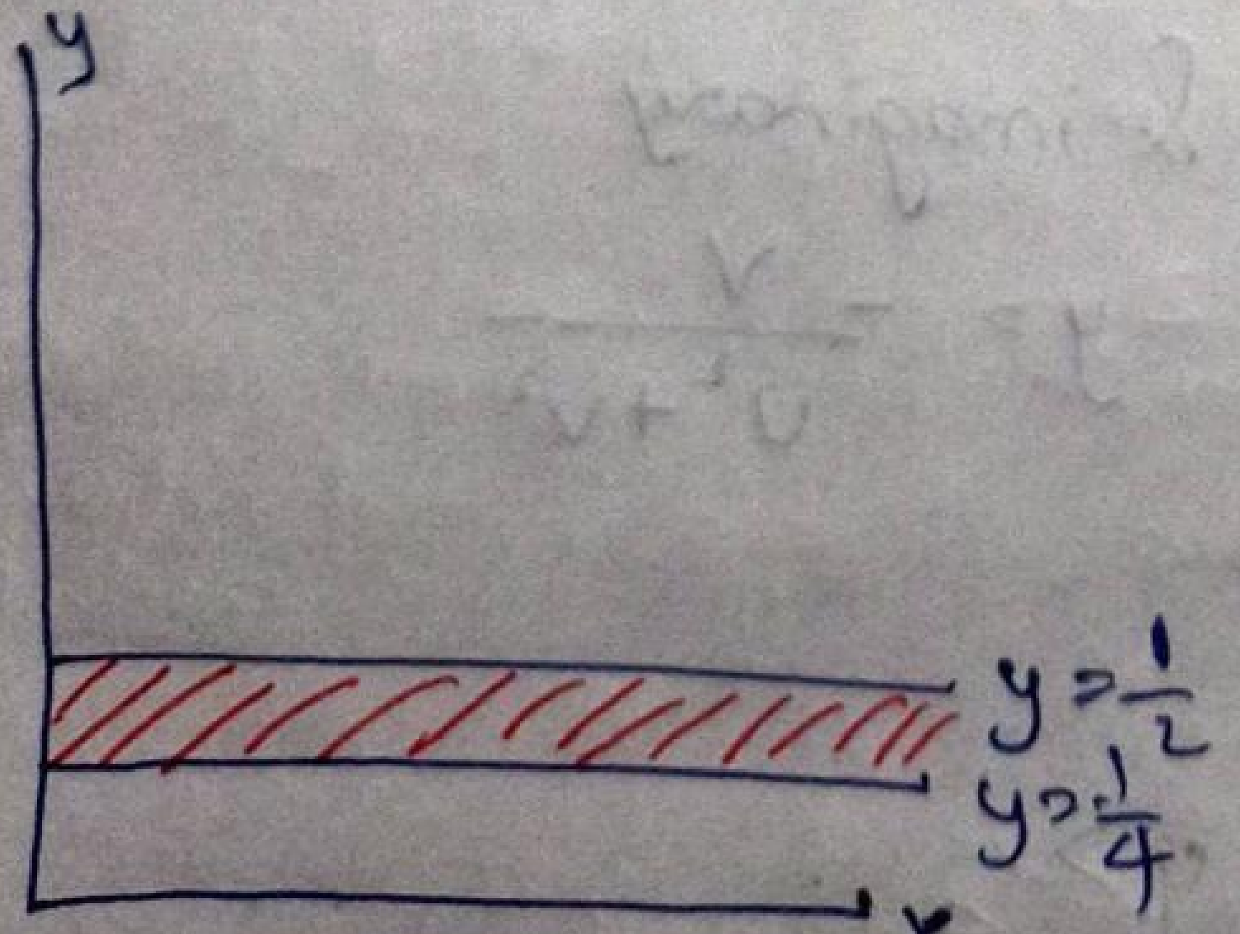
$\Rightarrow -4v > u^2+v^2$

$\Rightarrow u^2+v^2+4v < 0$

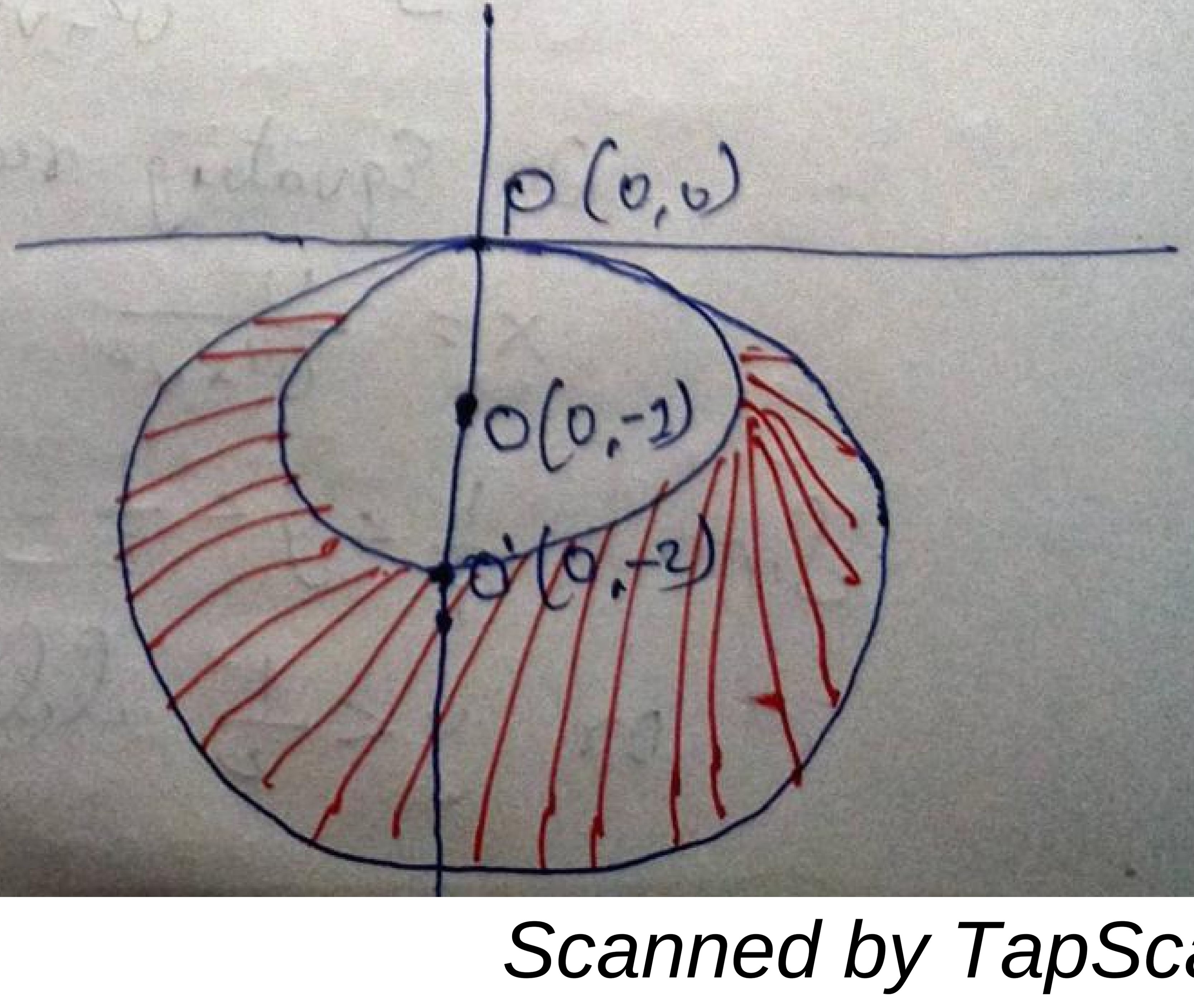
$\Rightarrow u^2+(v+2)^2 < 2^2$

Which is a circle of centre $(0,-2)$ and radius = 2.

Z-plane



W-plane



Fixed Point or Invariant point

If the image of a point z under a transformation $w = f(z)$ is itself, then the point is called fixed point or an invariant point of transformation.

i.e. $w = z$.

Q Find the invariant point of the transformation

$$w = \frac{2z+6}{z+7}$$

Sol → We have $w = \frac{2z+6}{z+7}$

Put $w = z$

$$z = \frac{2z+6}{z+7}$$

$$\Rightarrow z^2 + 7z = 2z + 6$$

$$\Rightarrow z^2 + 5z - 6 = 0$$

$$\Rightarrow z^2 + 6z - z - 6 = 0$$

$$\Rightarrow z(z+6) - 1(z+6) = 0$$

$$\Rightarrow (z+6)(z-1) = 0$$

$$\Rightarrow z = 1, -6 \quad \text{Ans}$$

fixed point

Q Find the invariant point of the transformation. $w = \frac{1}{z}$

Sol → We have $w = \frac{1}{z}$, Put $w = z$

$$\Rightarrow z = \frac{1}{z} \Rightarrow z^2 = 1$$

$$\Rightarrow z = \pm 1 \quad \text{Ans}$$

Q Find the fixed point of transformation $w = \frac{z-2}{z+3}$.

Sol→ We have $w = \frac{z-2}{z+3}$

Put $w = z$

$\Rightarrow \quad z = \frac{z-2}{z+3}$

$\Rightarrow \quad z^2 + 3z = z - 2$

$\Rightarrow \quad z^2 + 2z + 2 = 0$

$\Rightarrow \quad z = \frac{-2 \pm \sqrt{4-8}}{2}$

$\Rightarrow \quad z = \frac{-2 \pm i2}{2}$

$\Rightarrow \quad z = -1 \pm i$ Ans

Q Find the fixed point of transformation $w = \frac{5z+4}{z+5}$

Sol→ We have $w = \frac{5z+4}{z+5}$

Put $w = z$.

$z = \frac{5z+4}{z+5}$

$z^2 + 5z = 5z + 4$

$z^2 = 4$

$z = \pm 2$ Ans

Q Find the fixed point of transformation $w = \frac{1}{z-2i}$

Sol→ We have $w = \frac{1}{z-2i}$

Put $w = z$

$\Rightarrow z = \frac{1}{z-2i}$

$\Rightarrow z^2 - 2iz = 1$

$\Rightarrow z^2 - 2iz - 1 = 0$

$\Rightarrow z = \frac{2i \pm \sqrt{(2i)^2 + 4}}{2}$

$\Rightarrow z = \frac{2i \pm \sqrt{-4+4}}{2}$

$\Rightarrow z = i$ Ans

Q Find the fixed point of transformation. $w = \frac{2z+9}{z+2}$

Sol→ We have $w\cancel{z} = \frac{2z+9}{z+2}$.

Put $w = z$

$\Rightarrow z = \frac{2z+9}{z+2}$

$\Rightarrow z^2 + 2z = 2z + 9$

$\Rightarrow z^2 = 9$

$\Rightarrow z = \pm 3$ Ans

Q Find the fixed point of transformation $w = \frac{z}{z+1}$

Sol→ We have $w\cancel{z} = \frac{z}{z+1}$

Put $w = z$

$\Rightarrow z = \frac{z}{z+1}$

$z^2 + z = z$

$\Rightarrow z^2 = 0$

$\Rightarrow z = 0$ Ans