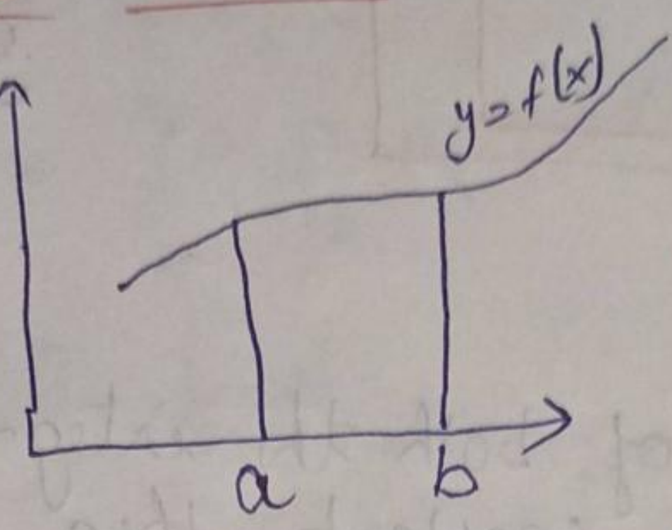
Maths

Unit-1 → Multiple Integral & Triple Integral

Recap of 12th class :-



→ Area $= \int_a^b y\,dx$

$= \int_a^b f(x)\,dx$

① $\int_0^{\pi/2} \sin\theta\, d\theta$

$= \left|-\cos\theta\right|_0^{\pi/2}$

$= -\left(\cos\frac{\pi}{2} - \cos 0\right)$

$= -(0-1)$

$= 1$ Ans

② $\int_0^{\pi/2} \sin^2\theta$

$= \int_0^{\pi/2} \frac{1-\cos 2\theta}{2}$

$= \frac{1}{2}\left|\frac{x}{2} - \frac{\sin 2\theta}{2}\right|_0^{\pi/2}$

$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\sin\pi}{2} - 0 + 0\right)$

$= \frac{\pi}{4}$ Ans

Overview of what we will study in this unit.

$\int_0^1 \int_0^2 f(x,y)\,dx\,dy$

* $f(x,y)$ → Constant, $f(x)$, $f(y)$, both

$\int_0^1 \int_0^x f(x,y)\,dx\,dy$

[→ We solve variable one first then solve the constant one]

Note :- ① When the limit of integral is variable, then there must be ~~one side~~ atleast one side curve.

② When the limits of integral is all constant, then all side is straight line.

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Multiple Integral :-

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy$$

x_1 & $x_2 \in x$
y_1 & $y_2 \in y$.

Properties :-

① If the limit of both the integral is constant and function is also constant, then we can integrate with respect to any variable.

Example → We have, $I = \int_0^1\int_0^2 dx\,dy$.

$$\Rightarrow \int_0^1\int_0^2 dx\,dy$$
$$= \int_0^1 |x|_0^2\,dy$$
$$= \int_0^1 2\,dy$$
$$= 2|y|_0^1$$
$$= 2(1-0)$$
$$= 2$$

$$\Rightarrow \int_0^1\int_0^2 dx\,dy$$
$$= \left(\int_0^1 dx\right)\left(\int_0^2 dy\right).$$
$$= |x|_0^1 \cdot |y|_0^2$$
$$= (1-0)(2-0)$$
$$= 2.$$

Both give same answer.

② If the limit of both the integral is constant and function is given then we can seperat the integral.

Example → We have, $I = \int_0^3\int_0^2 xy\,dy\,dx$.

$$= \left(\int_0^3 x\,dx\right)\left(\int_0^2 y\,dy\right). = \left|\frac{x^2}{2}\right|_0^3 \left|\frac{y^2}{2}\right|_0^2$$

$$= \frac{9}{2}\times\frac{4}{2} = 9 \text{ Ans}$$

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(3) If one limit is constant and one is variable, then we will solve variable limit integral first.

Example → $I = \int_0^2 \int_0^x dy\,dx$

$= \int_0^2 |y|_0^x \, dx$

$= \int_0^2 x\,dx$

$= \left|\frac{x^2}{2}\right|_0^2 = \left(\frac{4}{2} - 0\right) = 2$ Ans

(4) We have $\int_0^1 \int_0^y xy\,dy\,dx$

$= \int_0^1 y \left|\frac{x^2}{2}\right|_0^y dy$

$= \int_0^1 y\left(\frac{y^2}{2}\right) dy = \int_0^1 \frac{y^3}{2}\,dy = \frac{1}{2}\left|\frac{y^4}{4}\right|_0^1$

$= \frac{1}{2}\left(\frac{1}{4} - 0\right) = \frac{1}{8}$ Ans

Q Solve $\int_0^\infty \int_0^\infty \frac{dy\,dx}{(1+x^2)(1+y^2)}$

$= \left(\int_0^\infty \frac{dx}{(1+x^2)}\right)\left(\int_0^\infty \frac{dy}{(1+y^2)}\right) = \left(|\tan^{-1}x|_0^\infty\right) \cdot \left(|\tan^{-1}y|_0^\infty\right)$

$= (\tan^{-1}\infty - \tan^{-1}0)(\tan^{-1}\infty - \tan^{-1}0) = \frac{\pi}{2} \times \frac{\pi}{2} = \frac{\pi^2}{4}$ Ans

Q $\int_0^1 \int_0^1 \frac{dy\,dx}{\sqrt{1-x^2}\sqrt{1-y^2}} \Rightarrow \left(\int_0^1 \frac{dx}{\sqrt{1-x^2}}\right)\left(\int_0^1 \frac{dy}{\sqrt{1-y^2}}\right)$

$= |\sin^{-1}x|_0^1 \cdot |\sin^{-1}y|_0^1 = (\sin^{-1}1 - \sin^{-1}0)(\sin^{-1}1 - \sin^{-1}0)$

$= \left(\frac{\pi}{2} - 0\right)\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{4}$ Ans

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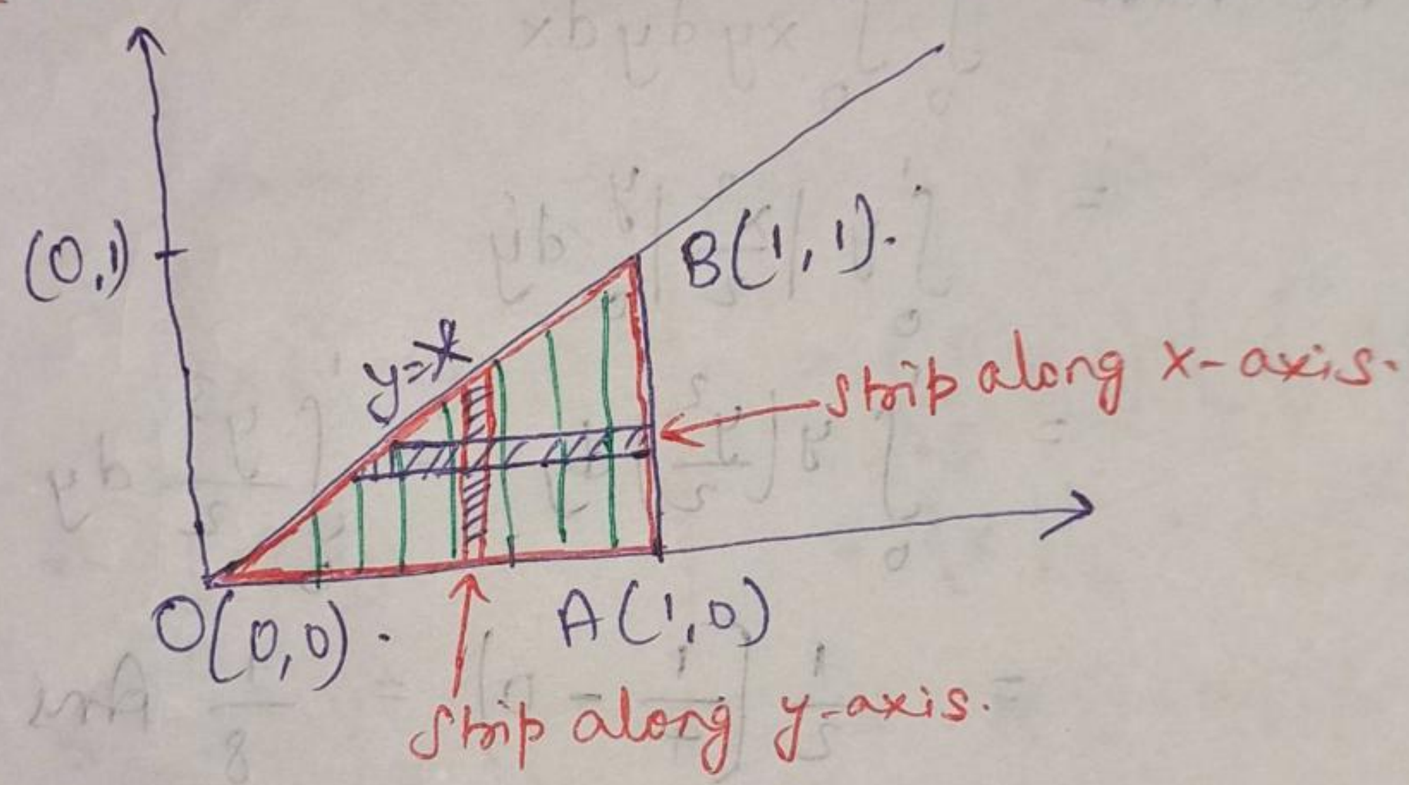
Q $\int_0^1 \int_0^x x^2 y \, dy \, dx$

Solution → $\int_0^1 \int_0^x x^2 y \, dy \, dx = \int_0^1 x^2 \left| \frac{y^2}{2} \right|_0^x dx.$

$= \int_0^1 x^2 \left(\frac{x^2}{2} \right) dx = \int_0^1 \frac{x^4}{2} = \frac{1}{2} \left| \frac{x^5}{5} \right|_0^1$

$= \frac{1}{2} \left(\frac{1}{5} - 0 \right) = \frac{1}{10}$ Ans

Question Type :- ① Draw the diagram for $y=x$, $y=0$, & $x=1$.



Along x-axis :-

$x = 0$, to 1

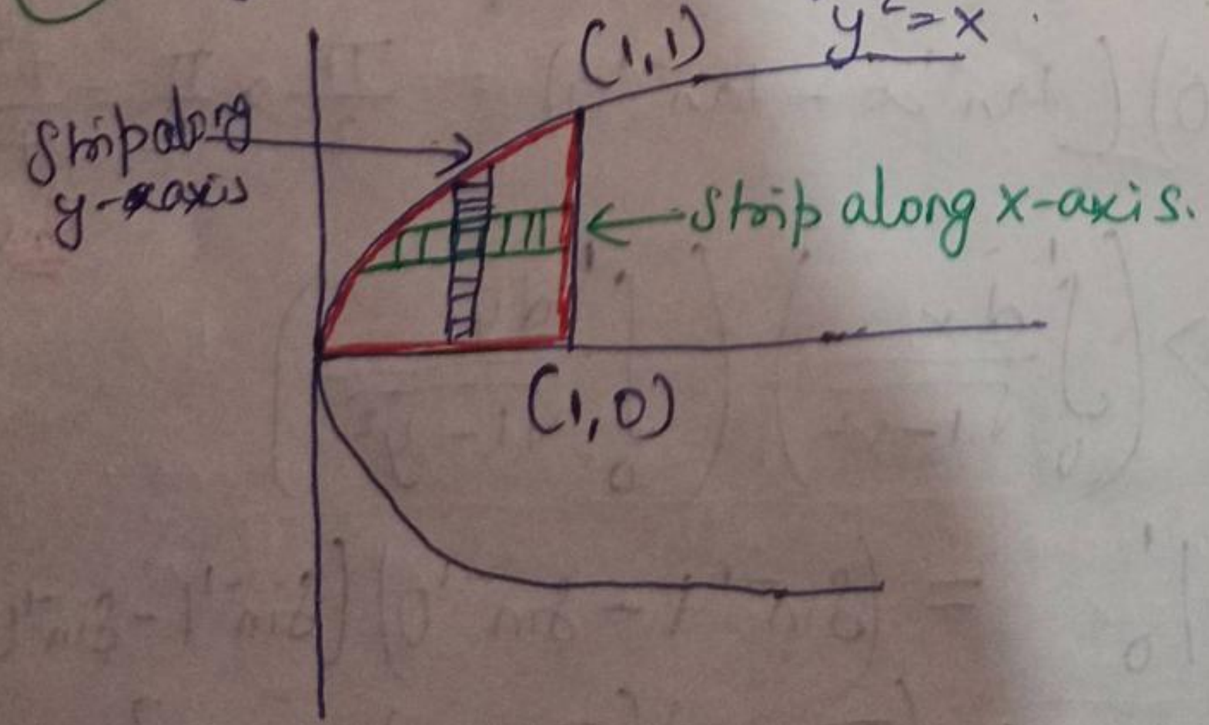
$y = 0$ to x.

Along Y-axis :-

$y = 0$ to 1

$x = y$ to 1

② Draw the diagram for $y^2 = x$, $y=0$ & $x=1$.



Along x-axis :-

$x = 0$ to 1

$y = 0$ to $\sqrt{x}$

Along Y-axis :-

$y = 0$ to 1

$x = y^2$ to 1

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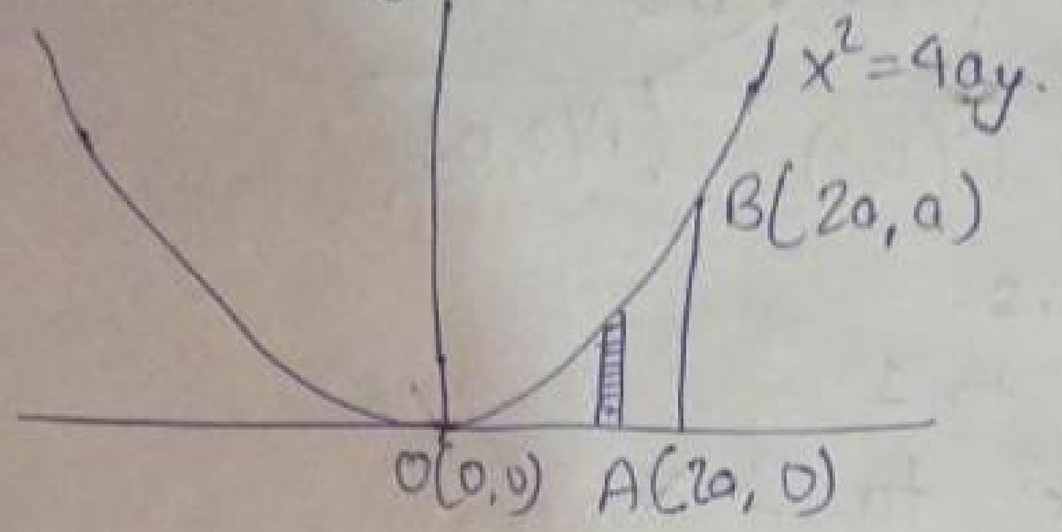
Q. Evaluat $\iint_R xy\,dxdy$ over the region bounded by the x-axis, ordinate at $x = 2a$ and parabola $x^2 = 4ay$.

Sol→ Let $I = \iint_R xy\,dxdy$ — — — (i)

Given. $x = 2a$, $x^2 = 4ay$.

According to data,



Along x-axis.

$x = 0$ to $2a$

$y = 0$ to $\frac{x^2}{4a}$.

Now, eq. (i) becomes $I = \int_0^{2a}\int_0^{x^2/4a} xy\,dxdy$.

$$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \int_0^{2a} x\left[\frac{x^4}{32a^2}\right] = \frac{1}{32a^2}\int x^5\,dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{32a^2}\times\frac{(2a)^6}{6}$$

$$= \frac{64a^6}{32\times 6\,a^2} = \frac{a^4}{3} \text{ Ans}$$

Q Evaluate $\iint_R y\,dydx$ over the region bounded by $x=0$, $y=x^2$ and $x+y=2$ in first quadrant.

Sol→ Let $I = \iint_R y\,dydx$.

Given. $x=0$, $y=x^2$, $x+y=2$

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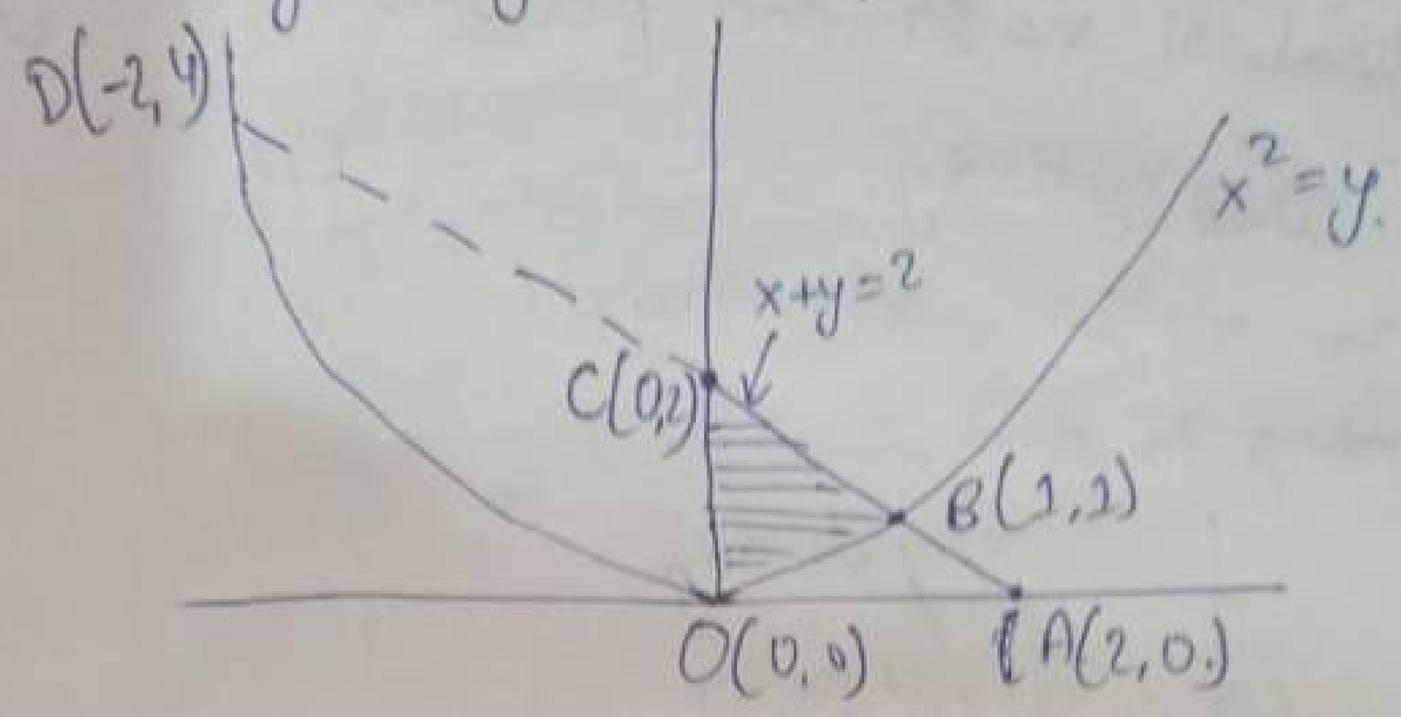
According to given data.



Along x-axis.

$x=0$ to 1.

$y=x^2$ to $2-x$

Now, eq. (i)

$$I=\int_0^1\int_{x^2}^{2-x} y\,dy\,dx.$$

$$=\int_0^1\left|\frac{y^2}{2}\right|_{x^2}^{2-x}dx$$

$$=\int_0^1\left[\frac{(2-x)^2}{2}-\frac{x^4}{2}\right]dx$$

$$=\frac{1}{2}\int_0^1(4+x^2-4x-x^4)\,dx$$

$$=\frac{1}{2}\left|4x+\frac{x^3}{3}-\frac{4x^2}{2}-\frac{x^5}{5}\right|_0^1$$

$$=\frac{1}{2}\left|4+\frac{1}{3}-2-\frac{1}{5}\right|$$

$$=\frac{1}{2}\left(\frac{30+5-3}{15}\right)$$

$$=\frac{\cancel{2}\,32}{30}=\frac{16}{15}$$ Ans

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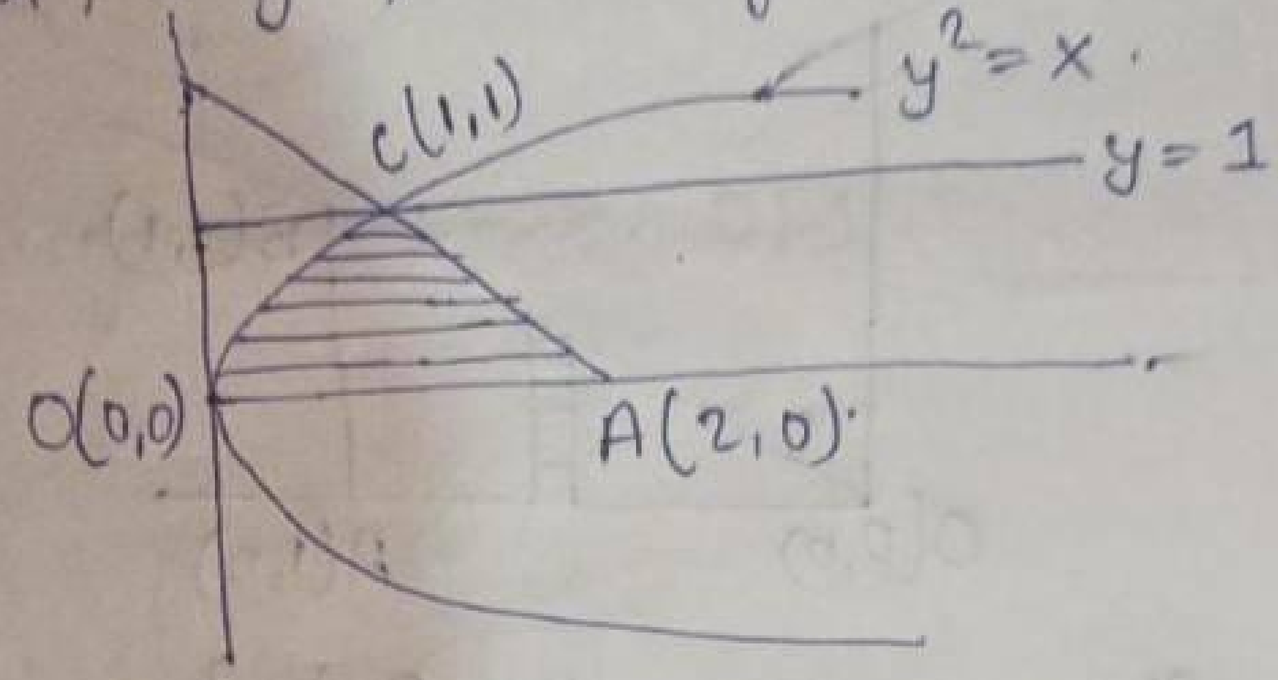
Q Evaluate $\iint_R xy\,dx\,dy$, over the region bounded by the curve $x=y^2$, $x=2-y$. $\underbrace{y=0 \;\&\; y=1}_{\text{may not be given}}$

Sol→ Let $I=\iint_R xy\,dx\,dy$.

Given, $x=y^2$, $x=2-y$. $x=0$, & $y=1$.



Along y-axis :- $y=0$ to 1
$x=y^2$ to $2-x$.

Now, eq. (i).

$$\int_0^1\int_{y^2}^{2-y} xy\,dx\,dy = \int_0^1 y\left|\frac{x^2}{2}\right|_{y^2}^{2-y} dy$$

$$=\frac{1}{2}\int_0^1 y\left[(2-y)^2-y^4\right]dy$$

$$=\frac{1}{2}\int_0^1 (4y+y^3-4y^2-y^5)\,dy$$

$$=\frac{1}{2}\left|\frac{4y^2}{2}+\frac{y^4}{4}-\frac{4y^3}{3}-\frac{y^6}{6}\right|_0^1$$

$$=\frac{1}{2}\left[\frac{4}{2}+\frac{1}{4}-\frac{4}{3}-\frac{1}{6}\right]$$

$$=\frac{1}{2}\left[\frac{24+3-16-2}{12}\right]=\frac{9}{24}$$ Ans

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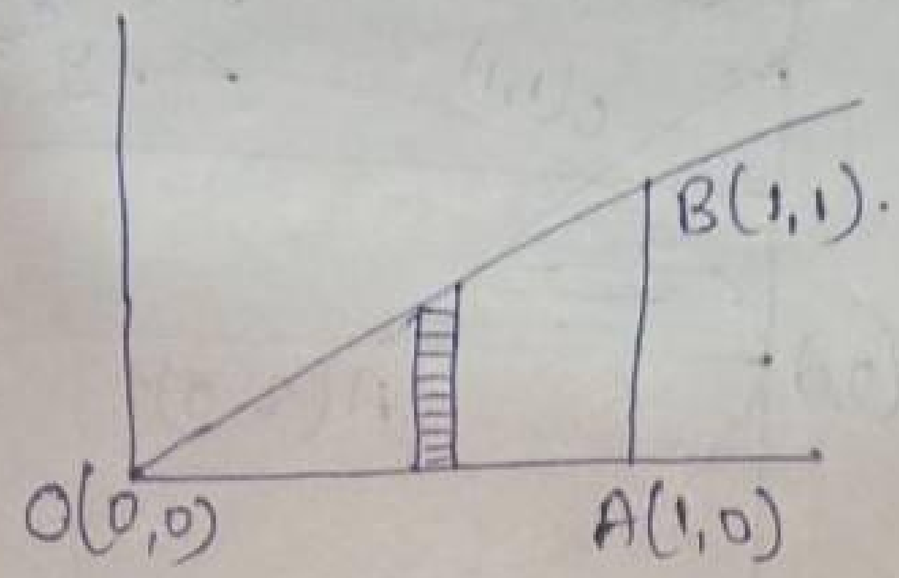
Q Evaluate $\iint_D e^{y/x}\,dx\,dy$ where D is the region bounded by straight line $y=x$, $y=0$ and $x=1$.

Sol→ Let $I = \iint_D e^{y/x}\,dx\,dy$.

Given $y=x$, $y=0$, $x=1$



Along x-axis, $x = 0$ to 1
$y = 0$ to x.

Now, eq. 1 becomes.

$$\int_0^1\int_0^x e^{y/x}\,dx\,dy = \int_0^1\left(\int_0^x e^{y/x}\,dy\right)dx.$$

$$= \int_0^1 \left| x e^{y/x} \right|_0^x dx.$$

$$= \int_0^1 x(e^1 - e^0).\,dx.$$

$$= (e-1)\int_0^1 x$$

$$= (e-1)\left|\frac{x^2}{2}\right|_0^1$$

$$= \frac{e-1}{2} \quad \text{Ans.}$$

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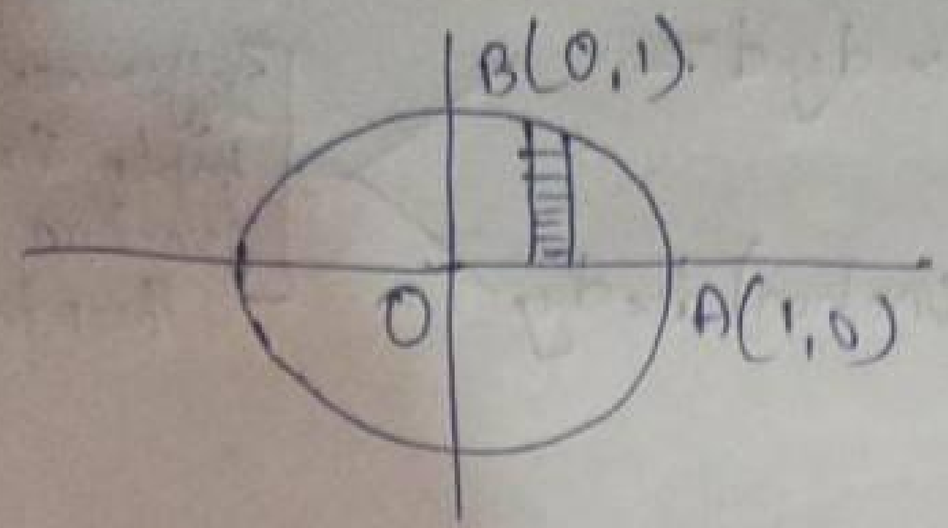
Q Evaluate $\iint_D x^2y^2\,dx\,dy$ where D is region in the circular disk $x^2+y^2 \le 1$.

Sol→



Along x-axis, $x = 0$ to 1

$y = 0$ to $\sqrt{1-x^2}$

$$\int_0^1\int_0^{\sqrt{1-x^2}} x^2y^2\,dx\,dy$$

$$= \int_0^1 x^2 \left|\frac{y^3}{3}\right|_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{3}\int_0^1 x^2(1-x^2)^{3/2}\,dx.$$

Now, put $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

$$\Rightarrow \frac{1}{3}\int_0^{\pi/2} \sin^2\theta(1-\sin^2\theta)^{3/2}\cos\theta\,d\theta$$

$$\Rightarrow \int_0^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta$$

By Beta-Gama function, $\int_0^{\pi/2}\sin^m\theta\cos^n\theta = \dfrac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$

$$\Rightarrow \frac{1}{3}\,\frac{\Gamma\frac{2+1}{2}\,\Gamma\frac{4+1}{2}}{2\,\Gamma\frac{2+4+2}{2}} = \frac{1}{3}\times\frac{\Gamma\frac{3}{2}\times\Gamma\frac{5}{2}}{2\,\Gamma\frac{8}{2}} = \frac{\frac{1}{3}\times\frac{1}{2}\sqrt{\pi}\times\frac{3}{4}\sqrt{\pi}}{3\times2\times3\times2\times1}$$

$$\Rightarrow \frac{\pi}{96}$$

So, for full circle, Value is $4\times\frac{\pi}{96} = \frac{\pi}{24}$ Ans

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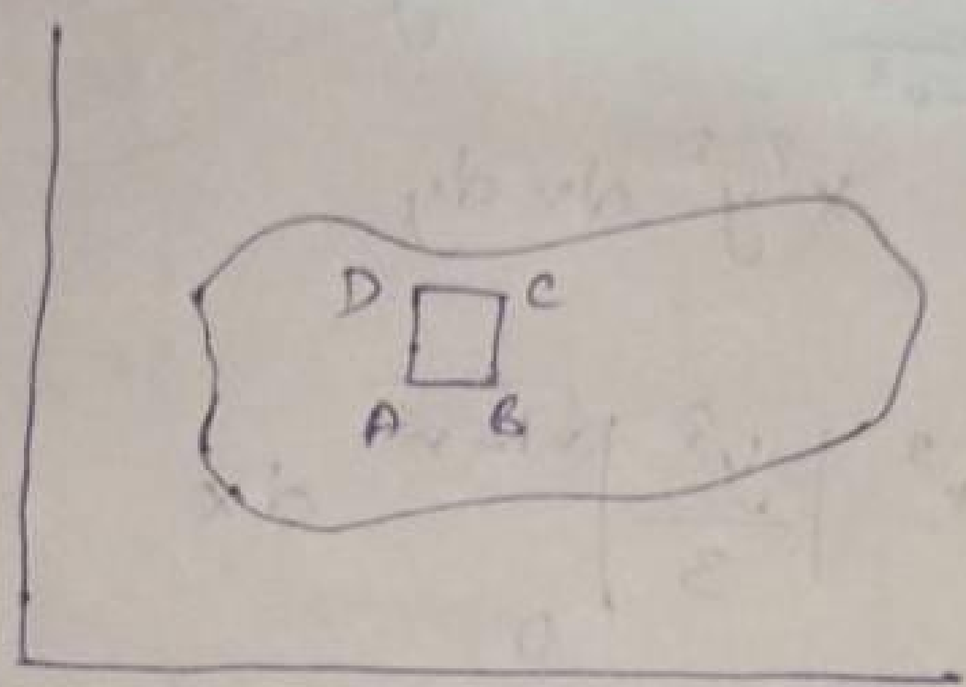
Note → * If function exist in the double integral then we find Volume not area.

Volume → $\iiint dx\,dy\,dz$

Volume → $\iint (\text{function})\,dx\,dy$

[Difference is that in triple there is no function which is there in double]

Determine the plain area by using double Integration



$AB \to \delta x \to 0$

$BC \to \delta y \to 0$

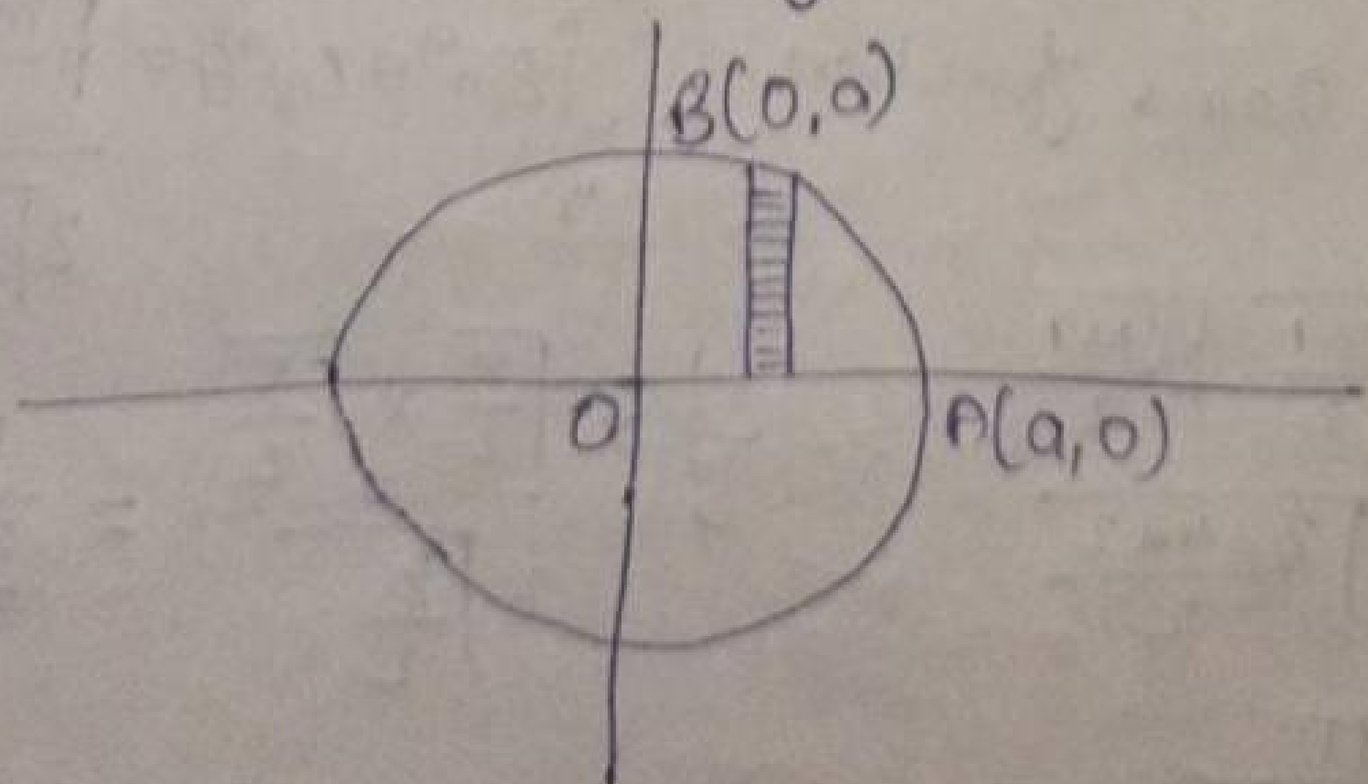
Area of ABCD = $\delta x \cdot \delta y$.

$\sum\sum \delta x \cdot \delta y$.

$$\boxed{\text{Area} = \iint_R dx\,dy.}$$

Q Find the area of circle $x^2+y^2=a^2$.

Sol → Given that $x^2+y^2=a^2$.



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Along x-axis.

$x = 0$ to a

$y = 0$ to $\sqrt{a^2 - x^2}$

We know, area $= \int_0^a \int_0^{\sqrt{a^2-x^2}} dx\,dy$

$$= \int_0^a |y|_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left(\sqrt{a^2-x^2}\right) dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \left(0 + \frac{a^2}{2}\sin^{-1}1 - 0 - 0\right)$$

$$= \frac{a^2}{2}\sin^{-1}1 = \frac{a^2}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi a^2}{4}$$

Now, Area of full Circle $= 4 \times \frac{\pi a^2}{4}$

$= \underline{\pi a^2}$ Ans

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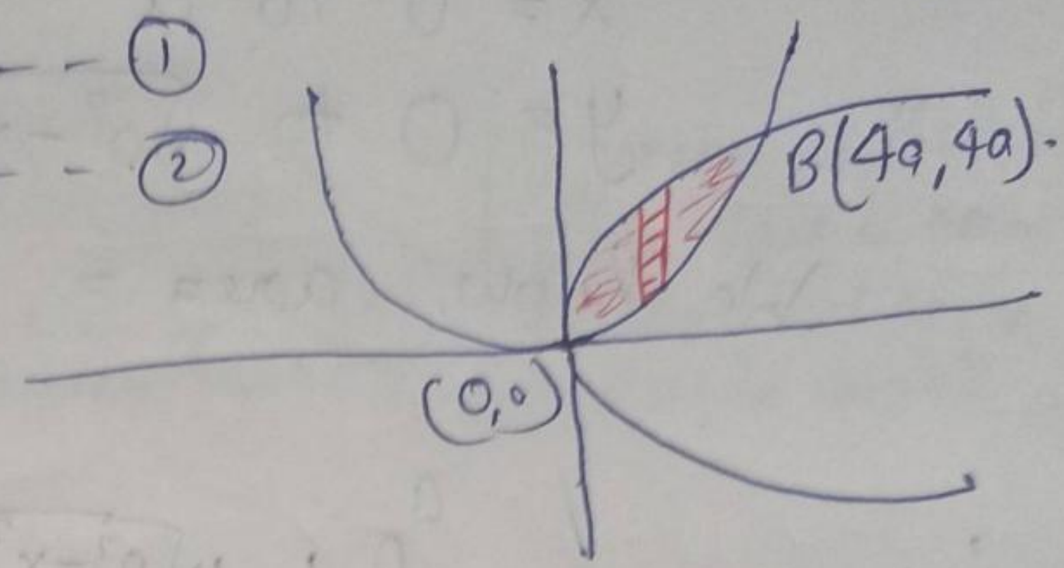
Q Find the area common to the parabola $y^2 = 4ax$ and $x^2 = 4ay$.

Sol→ Given. $y^2 = 4ax$ --- ①

$x^2 = 4ay$ --- ②



Along x-axis.

$x = 0$ to $4a$

$y = \frac{x^2}{4a}$ to $\sqrt{4ax}$

We know, Area $= \iint dy\,dx$

$$= \int_0^{4a}\int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dy\,dx = \int_0^{4a} |y|_{x^2/4a}^{\sqrt{4ax}}\,dx$$

$$= \int_0^{4a}\left(\sqrt{4ax} - \frac{x^2}{4a}\right)dx$$

$$= \sqrt{4a}\left|\frac{x^{3/2}}{3/2}\right|_0^{4a} - \frac{1}{4a}\left|\frac{x^3}{3}\right|_0^{4a}$$

$$= \sqrt{4a} \times \frac{2}{3} \times (4a)^{3/2} - \frac{1}{4a} \times \frac{(4a)^3}{3}$$

$$= \frac{2(4a)^2}{3} - \frac{(4a)^2}{3}$$

$$= \frac{(4a)^2}{3} = \frac{16a^2}{3}.$$

Q Find the area between curve $x = 0$, $x = y$, & $y = 1$.

Sol→ Given, that

$x = 0$, $x = y$.

$y = 1$.

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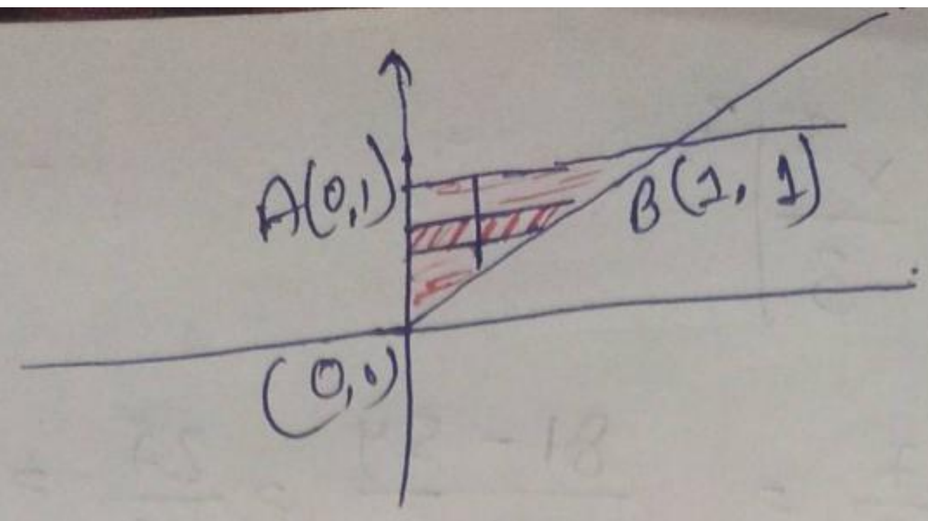


Along y-axis. $y = 0$ to 1

$x = 0$ to y

We know, Area $= \iint dy\,dx$

$$= \int_0^1 \int_0^y dx\,dy$$

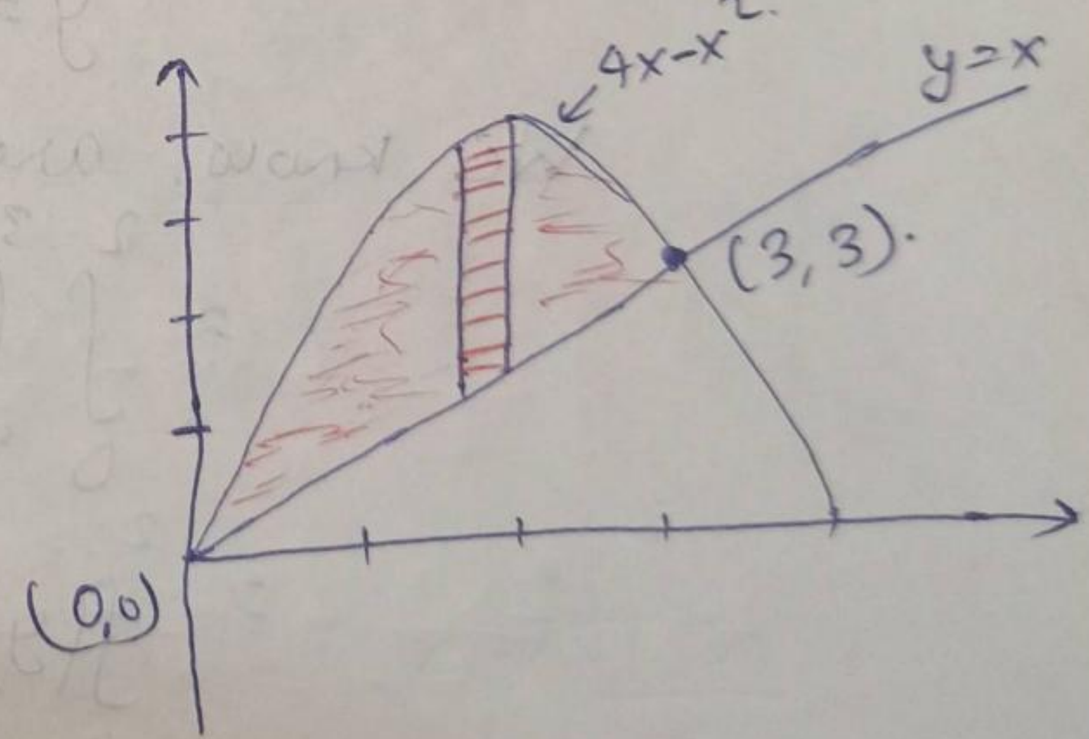
$$= \int_0^1 |x|_0^y\,dy = \int_0^1 y\,dy$$

$$= \left|\frac{y^2}{2}\right|_0^1 = \frac{1}{2}$$

Q. Find the area between the curve $y = 4x - x^2$ and line $y = x$.

Sol→ Given, $y = 4x - x^2$

$y = x$



Along x-axis

$x = 0$ to 3.

$y = x$ to $4x - x^2$

We know, Area $= \iint dy\,dx$.

$$= \int_0^3 \int_x^{4x-x^2} dy\,dx = \int_0^3 |y|_x^{4x-x^2}\,dx$$

$$= \int_0^3 (4x - x^2 - x)\,dx = \int_0^3 \left(\frac{3x^2}{2} - x^2\right)dx$$

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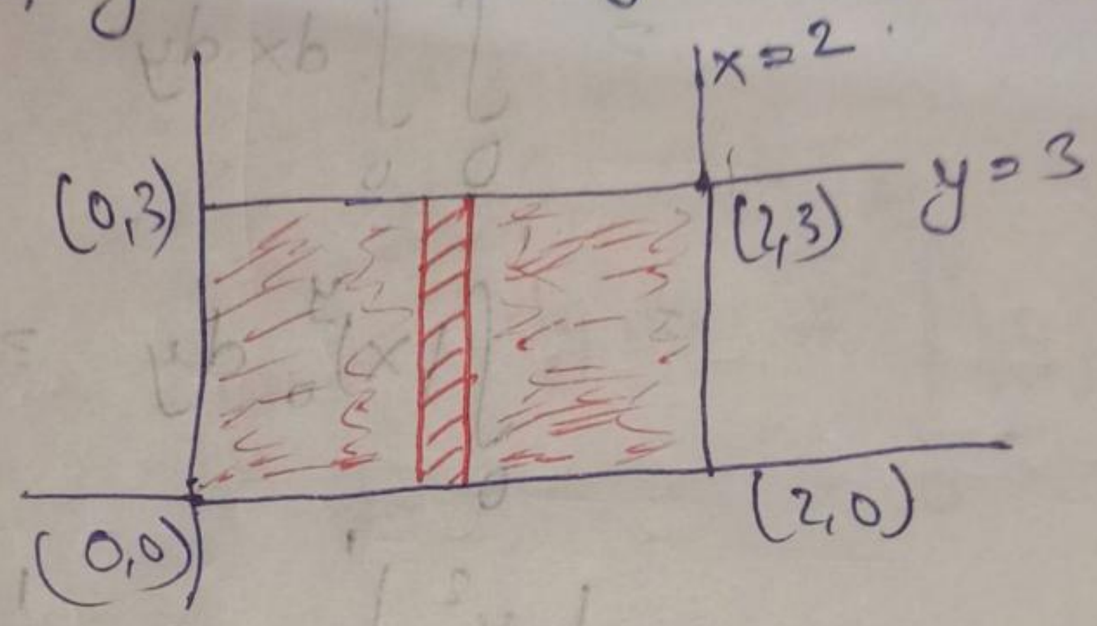
$$= \left| \frac{3x^2}{2} - \frac{x^3}{3} \right|^3$$

$$= \frac{27}{2} - \frac{27}{3} = \frac{81-54}{6} = \frac{27}{6} = \frac{9}{2} \text{ Ans}$$

Q. Find the area of the curve $x=0$, $y=0$, $x=2$, $y=3$.

Sol→ Given, $x=0$, $y=0$, $x=2$, $y=3$.



~~We~~ Along x-axis.

$x = 0$ to 2.

$y = 0$ to 3

We know, area $= \iint dy\,dx$

$$= \int_0^2 \int_0^3 dx\,dy$$

$$= \int_0^2 (y)_0^3\,dx = \int_0^2 3\,dx$$

$$= 3\int_0^2 dx = 3\left|x\right|_0^2$$

$$= 3 \times 2$$

$$= 6 \text{ Ans}$$

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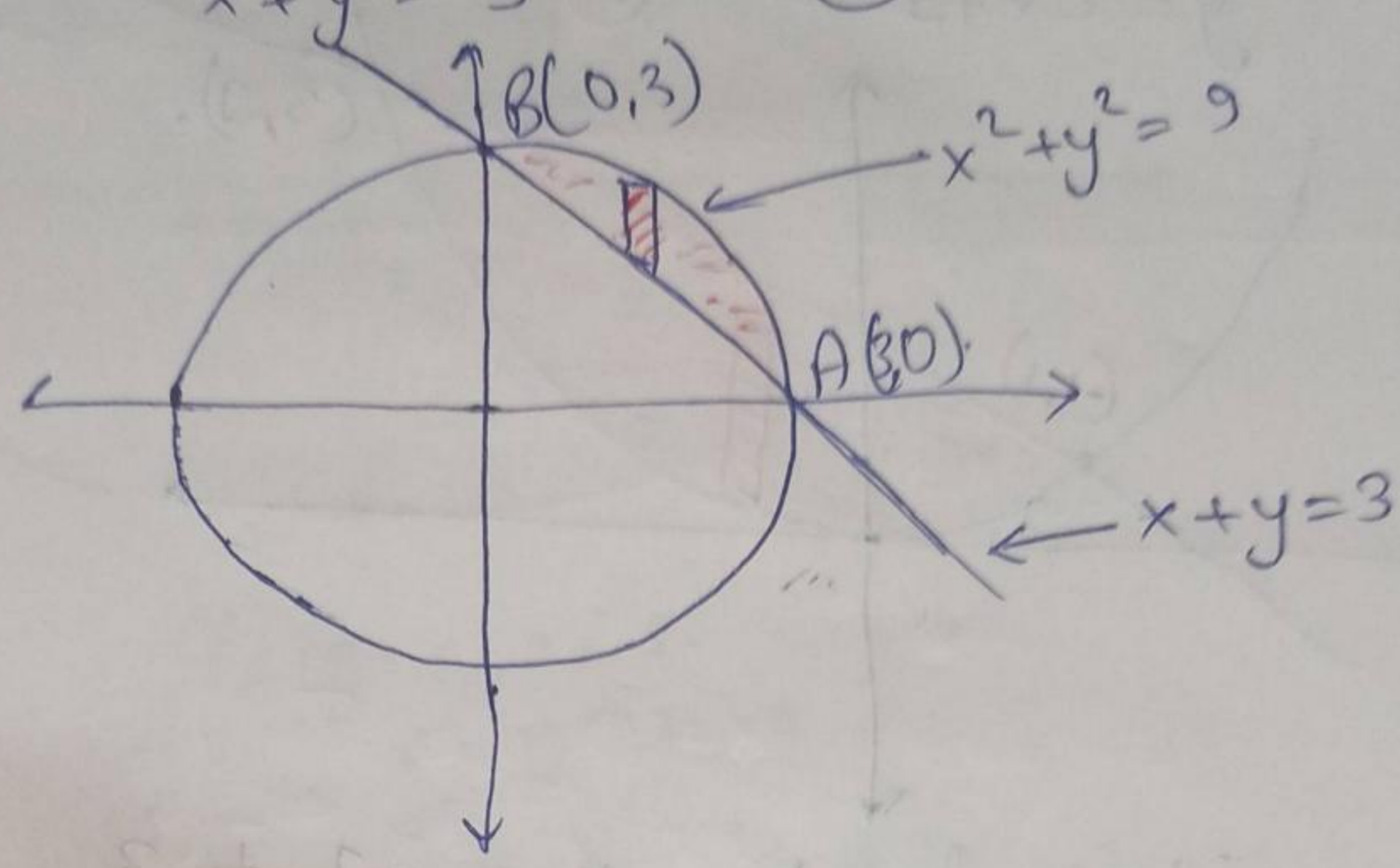
Q Find the area between the curve $x+y=3$ and $x^2+y^2=9$.

Sol→ Given ; $x^2+y^2=9$ --- ①

$x+y=3$ --- ②



$x=0$ to 3.

$y=3-x$ to $\sqrt{9-x^2}$

We know, Area $= \iint dy\,dx = \int_0^3 \int_{3-x}^{\sqrt{9-x^2}} dy\,dx$

$= \int_0^3 \left| y \right|_{3-x}^{\sqrt{9-x^2}} dx = \int_0^3 \left[\left(\sqrt{9-x^2}\right) - (3-x) \right] dx$

$= \left| \frac{1}{2} x\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} - 3x + \frac{x^2}{2} \right|_0^3$

$= \frac{1}{2}\times 0 + \frac{9}{2}\times \sin^{-1}1 - 3\times 3 - \frac{3\times 3}{2}$

$= 0 + \frac{9}{2}\cdot\frac{\pi}{2} - 9 - \frac{9}{2}$

$= \frac{9\pi}{4} - \frac{9}{2}$

$= \frac{9}{4}(\pi - 2)$ Ans

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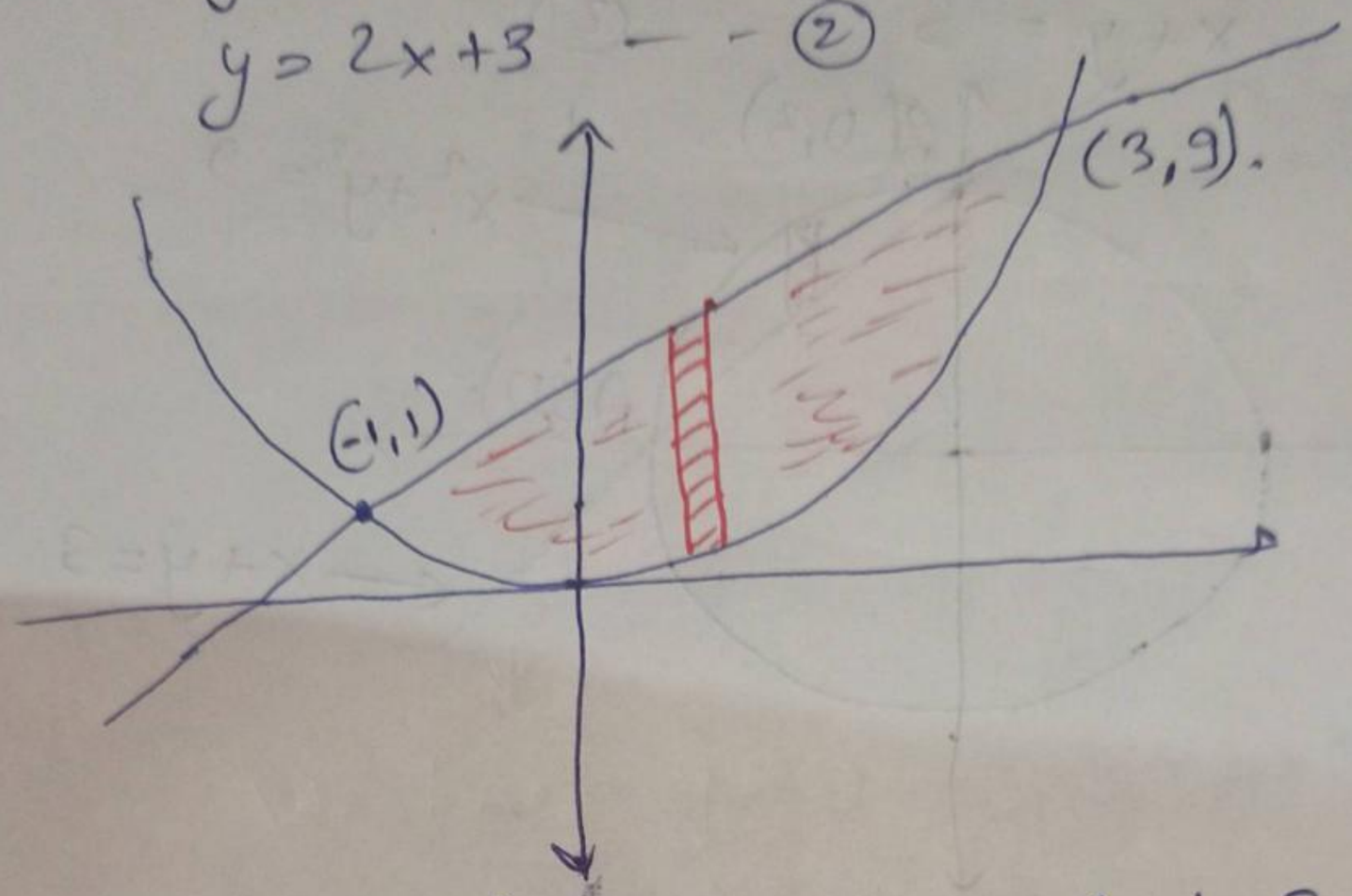
Q Find the area of the curve $y = x^2$ and $y = 2x+3$.

Sol → Given $y = x^2$ — (1)

$y = 2x+3$ — (2)



We have limit, $x = -1$ to 3.

$y = x^2$ to $2x+3$

We know, Area $= \iint dx\,dy$.

$$= \int_{-1}^{3}\int_{x^2}^{2x+3} dx\,dy.$$

$$= \int_{-1}^{3} |y|_{x^2}^{2x+3} dx = \int_{-1}^{3} (2x+3-x^2)\,dx$$

$$= \left| \frac{2x^2}{2} + 3x - \frac{x^3}{3} \right|_{-1}^{3}$$

$$= \frac{2\times 9}{2} + 3\times 3 - \frac{27}{3} - \left(\frac{2\times 1}{2} + 3(-1) + \frac{1}{3} \right).$$

$$= 9 + \cancel{9} - \cancel{9} - 1 + 3 - \frac{1}{3}.$$

$$= 11 - \frac{1}{3} = \frac{33-1}{3}$$

$$= \frac{32}{3}$$ Ans

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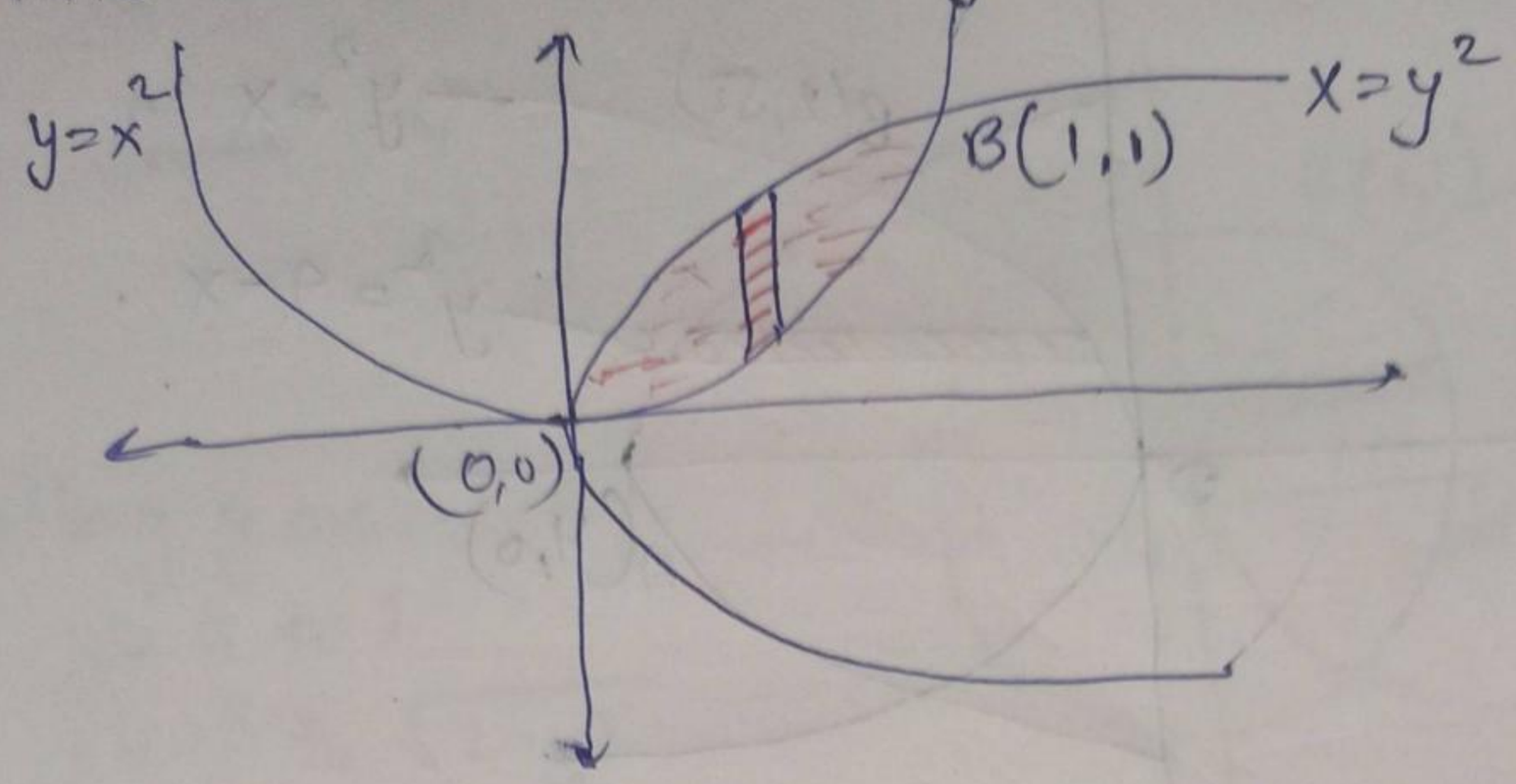
Q Find the area between $y^2 = x$, $x^2 = y$.



Along x-axis.

$x = 0$ to 1

$y = x^2$ to $\sqrt{x}$.

We know

$$\text{Area} = \iint dy\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} dy\,dx.$$

$$= \int_0^1 |y|_{x^2}^{\sqrt{x}} = \int_0^1 (\sqrt{x} - x^2)\,dx$$

$$= \left|\frac{x^{3/2}}{3/2}\right|_0^1 - \left|\frac{x^3}{3}\right|_0^1$$

$$= 1 \times \frac{2}{3} - \frac{1}{3}$$

$$= \frac{2-1}{3} = \frac{1}{3} \quad \text{Ans}$$

Q Find the area between the curve $y^2 = 4 - x$, $y^2 = x$

Sol → Given. $y^2 = 4 - x$. --- (1)

$y^2 = x$ --- (2)

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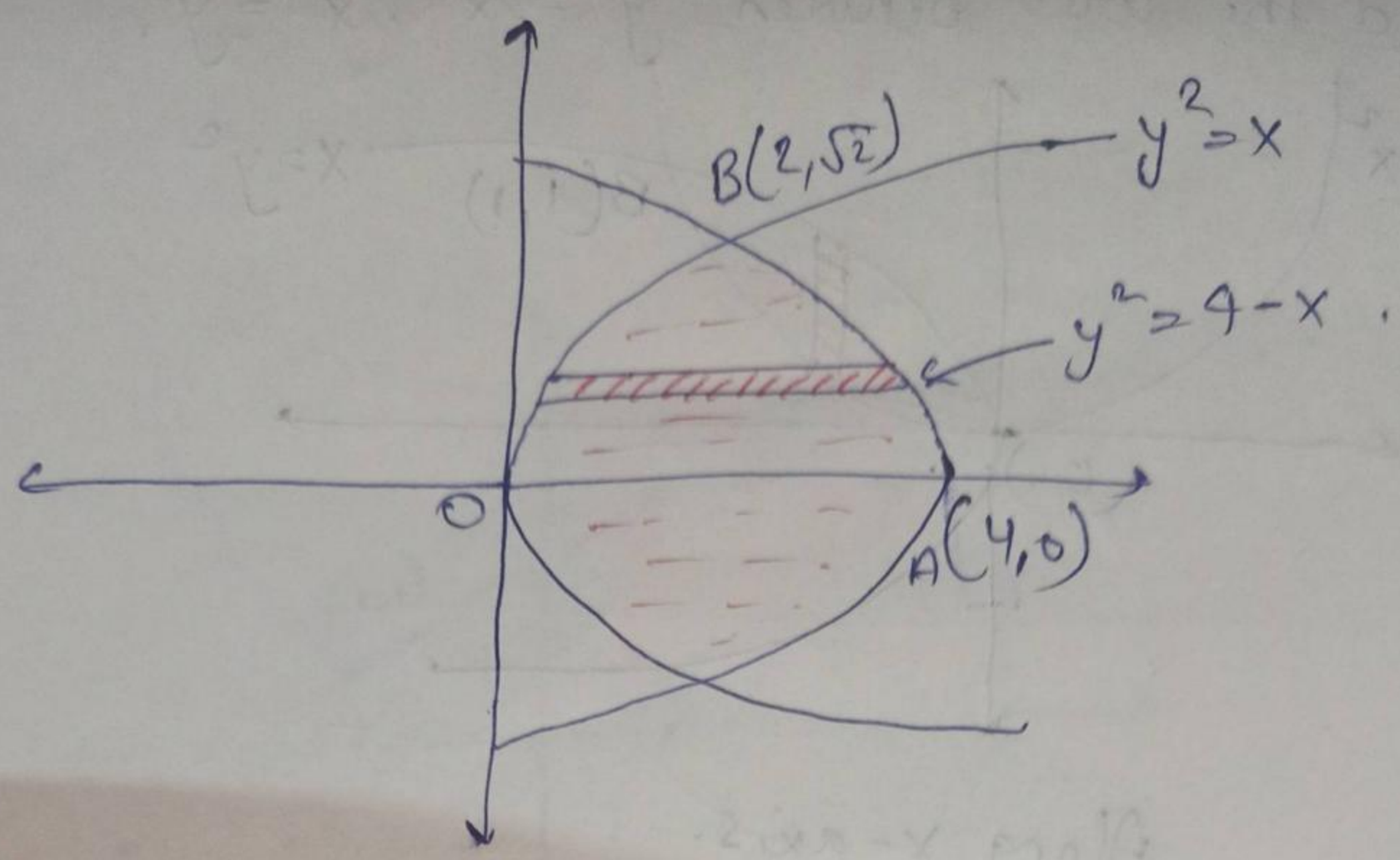

We have. $y=0$ to $\sqrt{2}$

$x=y^2$ to $4-y^2$

Whole area $= 2\times$ Area of OAB.

$$= 2\times\int_0^{\sqrt{2}}\int_{y^2}^{4-y^2} dy\,dx.$$

$$= 2\int_0^{\sqrt{2}} \left|\cancel{x}\right|_{y^2}^{4-y^2} = 2\int_0^{\sqrt{2}} (4-y^2-y^2)\,dy.$$

$$= 2\int_0^{\sqrt{2}} (4-2y^2)\,dy = 2\cdot\left|4y-\frac{2}{3}y^3\right|_0^{\sqrt{2}}$$

$$= 2\cdot\left[4\sqrt{2}-\frac{2}{3}\times 2\sqrt{2}\right]$$

$$= 2\times 4\sqrt{2}\left[\frac{3-1}{3}\right] = 8\sqrt{2}\times\frac{2}{3}$$

$$= \frac{16\sqrt{2}}{3} \quad \text{Ans}$$

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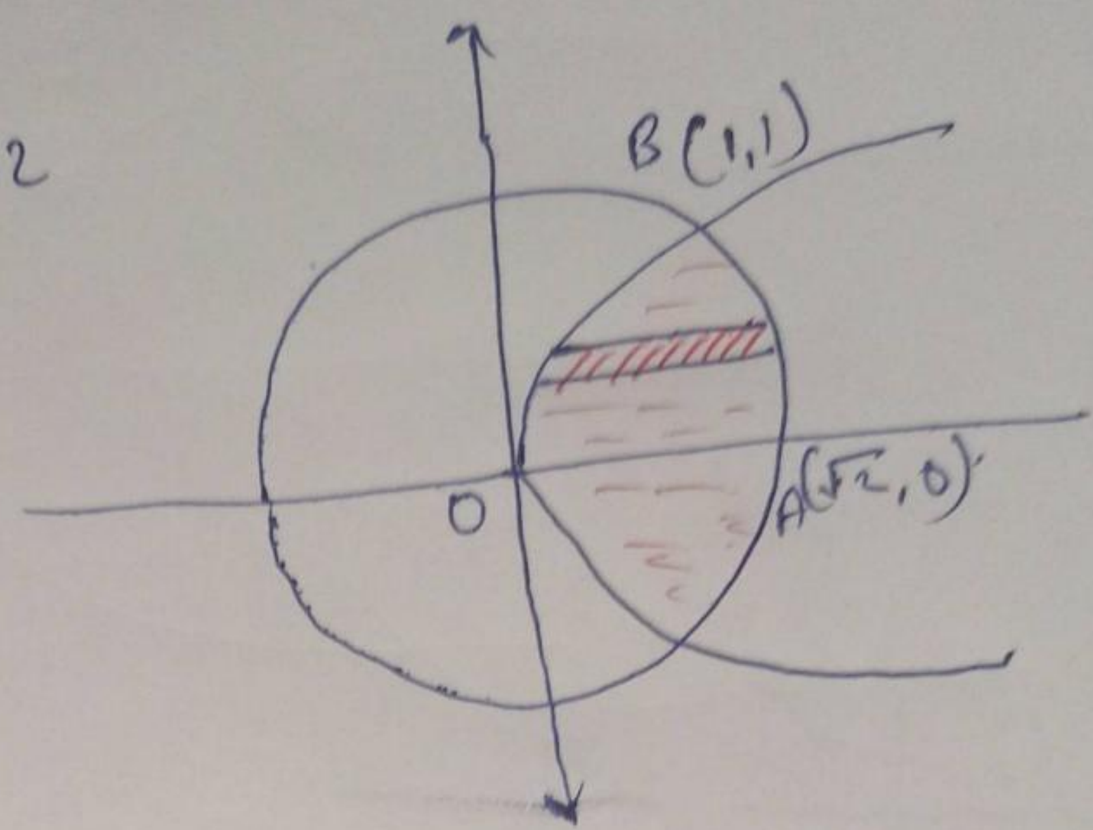
Q Find the area common to parabola. $y^2 = x$ & $x^2+y^2=2$.

Sol→ Given, $y^2 = x$,
$x^2+y^2 = 2$



Along y-axis
$y = 0$ to 1
$x = y^2$ to $\sqrt{2-y^2}$

Whole area = 2 × Area of OAB

$$= 2\iint dx\,dy$$

$$= 2\int_0^1\int_{y^2}^{\sqrt{2-y^2}} dx\,dy = 2\int_0^1 \left(\sqrt{2-y^2} - y^2\right) dy.$$

$$= 2\left[\frac{y}{2}\sqrt{2-y^2} + \frac{2}{2}\sin^{-1}\frac{y}{\sqrt{2}} - \frac{y^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2}\sqrt{2-1} + 1\times\sin^{-1}\frac{1}{\sqrt{2}} - \frac{1}{3}\right]$$

$$= 2\left[\frac{1}{2} + \sin^{-1}\frac{1}{\sqrt{2}} - \frac{1}{3}\right]$$

$$= 2\left[\frac{1}{6} + \frac{\pi}{4}\right] = \frac{\pi}{2} + \frac{1}{3}.$$ Ans

$$= \frac{3\pi + 2}{6}$$

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Change of Order of Integration

$$\underbrace{\iint dx\,dy}_{\text{given}} = \underbrace{\iint dy\,dx}_{\text{converted}}$$

* Use → We use this to integrate the integrals which cannot be solved directly.

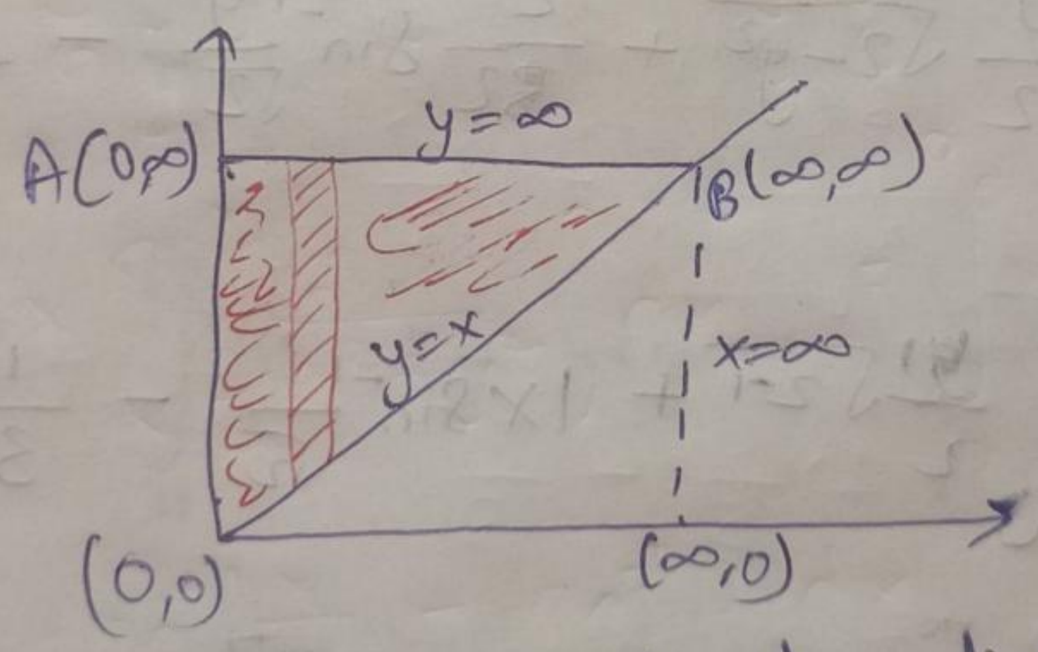
Q $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$, by change of order.

Sol → Let $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$. ——— (1).

On Comparing the limits, we get

$y = x$, $y = \infty$

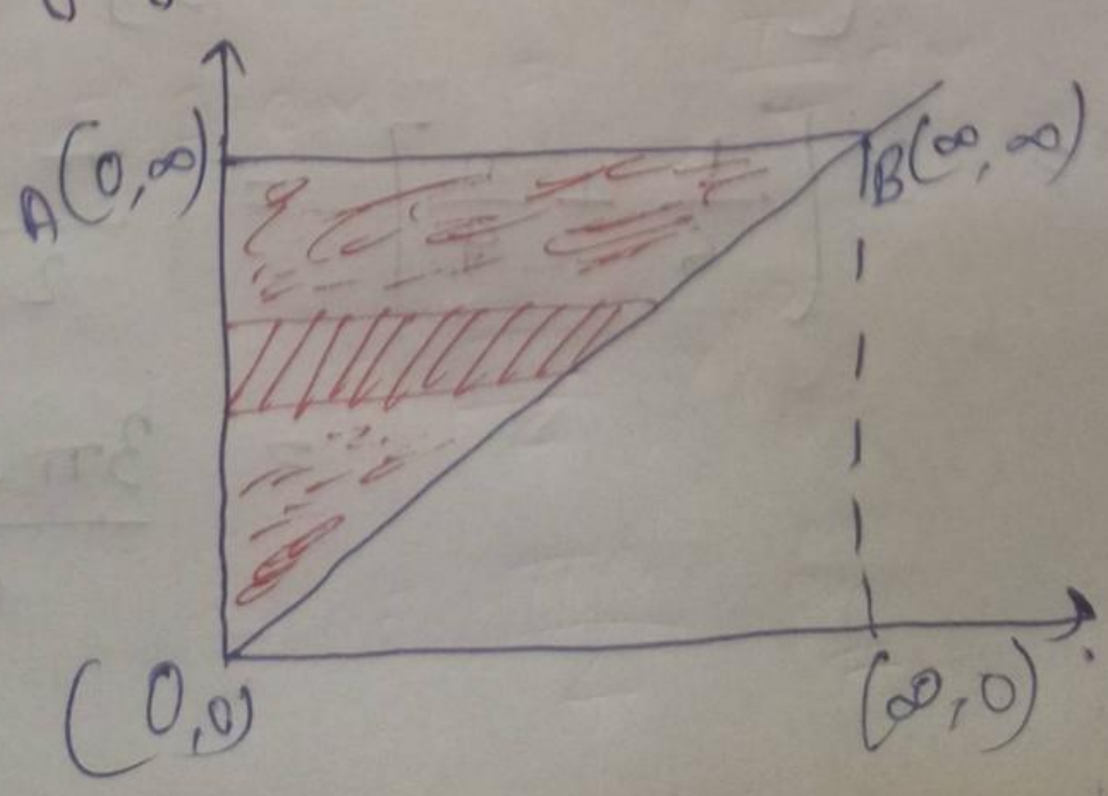
$x = 0$, $x = \infty$



On changing order of integration.

$y = 0$ to ∞

$x = 0$ to y



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Now, eq. (i)

$$I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy.$$

$$= \int_0^\infty \frac{e^{-y}}{y} \left| x \right|_0^y dy = \int_0^\infty \frac{e^{-y}}{y}\, y\, dy$$

$$= \int_0^\infty e^{-y}\, dy = \left| \frac{e^{-y}}{-1} \right|_0^\infty$$

$$= \frac{e^{-\infty} - e^{-0}}{-1} = \frac{0-1}{-1} = 1 \quad \underline{\underline{\text{Ans}}}$$

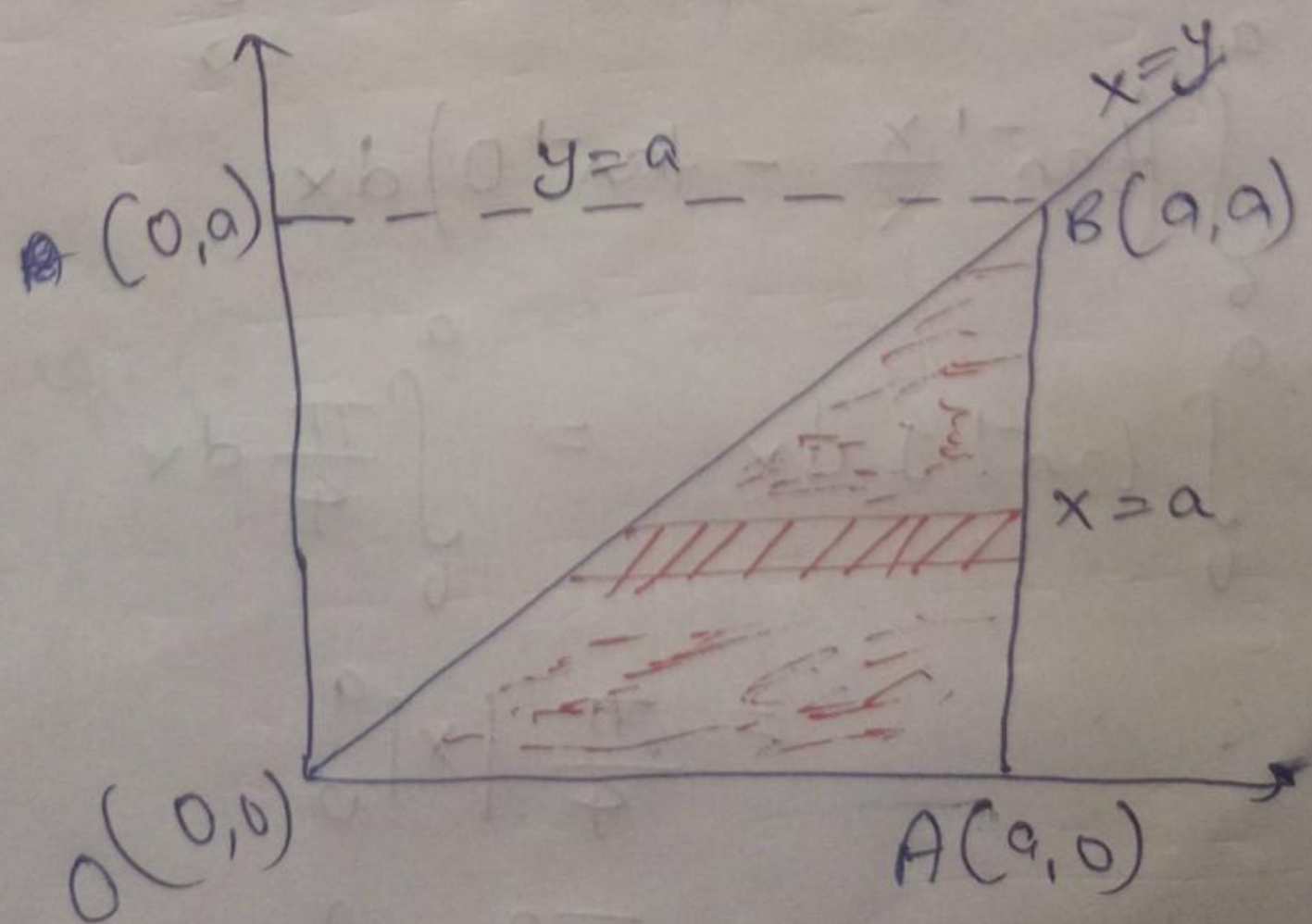
Q Evaluate $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ by using change of order.

Sol→ Let $I = \int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ ——— (1)

On comparing the limits we get

$y = 0 \quad y = a$

$x = y \quad x = a$



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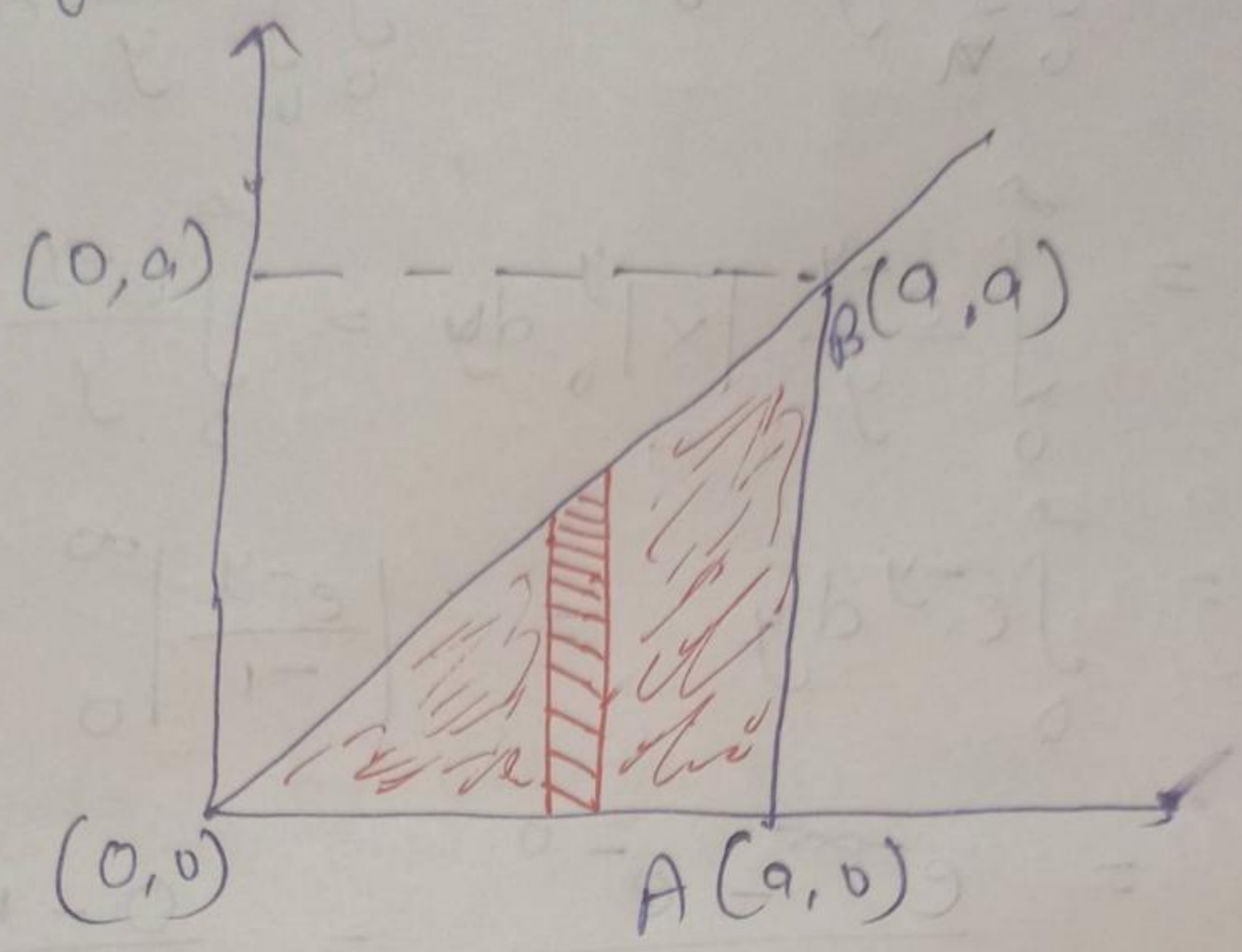
On changing the order of integration.



$x = 0$ to a

$y = 0$ to x

Now, eq. (i)

$$I = \int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \int_0^a \int_0^x \frac{x\,dy\,dx}{x^2+y^2}$$

$$= \int_0^a x \left\{ \int_0^x \frac{dy}{x^2+y^2} \right\} dx = \int_0^a x \left\{ \frac{1}{x} \left| \tan^{-1}\frac{y}{x} \right|_0^x \right\} dx$$

$$= \int_0^a \left(\tan^{-1}\frac{x}{x} - \tan^{-1} 0 \right) dx$$

$$= \int_0^a \tan^{-1} 1 \, dx = \int_0^a \frac{\pi}{4}\, dx$$

$$= \frac{\pi}{4} \left| x \right|_0^a$$

$$= \frac{\pi a}{4} \quad \text{Ans}$$

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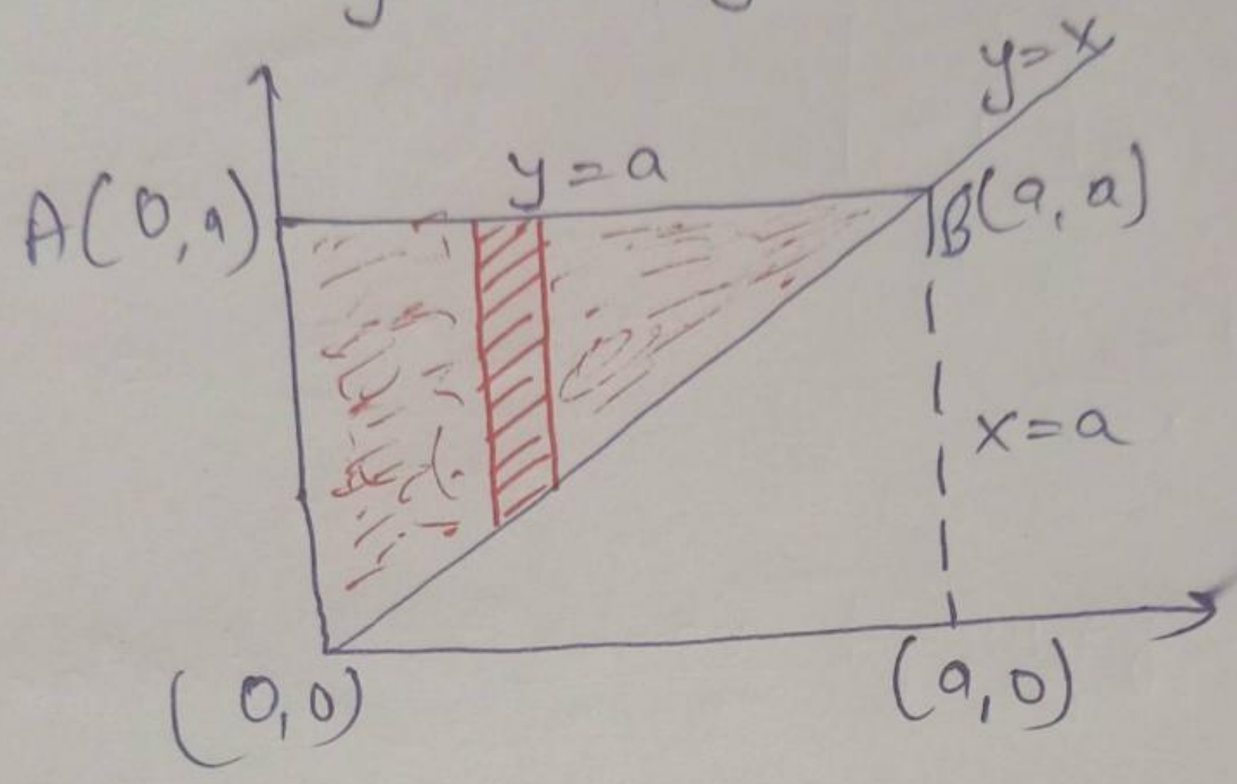
Q Evaluat $\int_0^a \int_x^a (x^2+y^2)\,dy\,dx$ by change of order.

Sol → Let $I = \int_0^a \int_x^a (x^2+y^2)\,dy\,dx$.

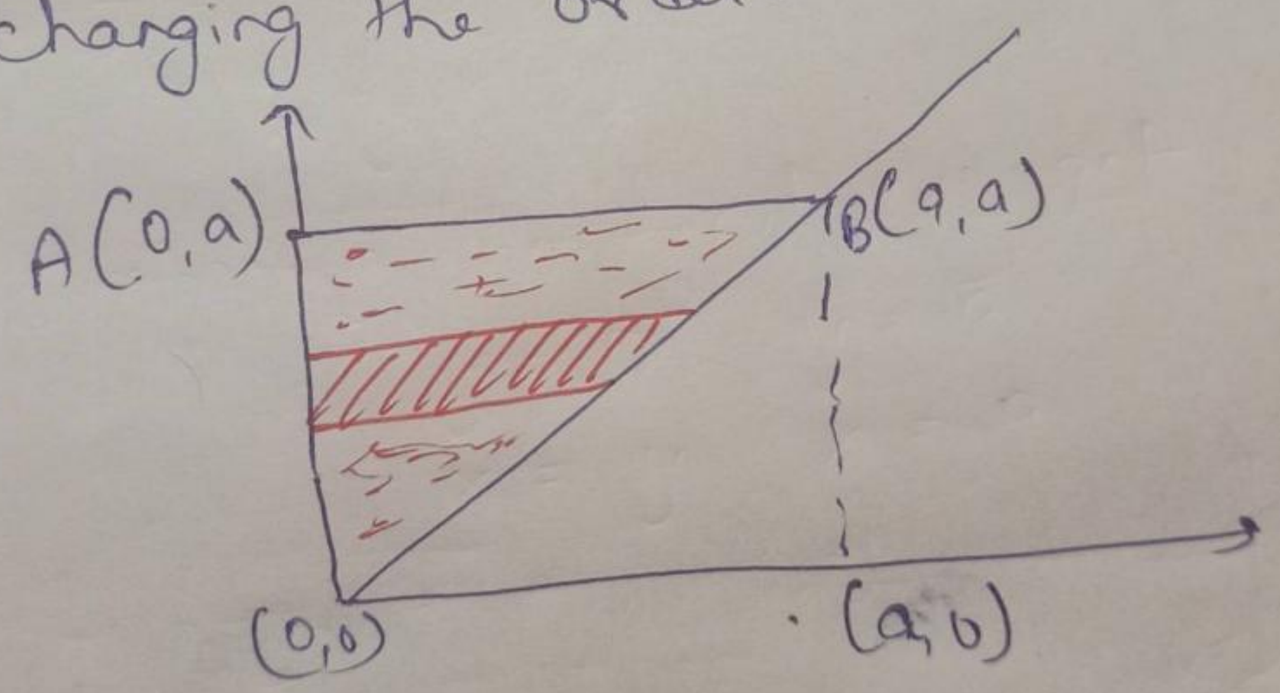
On Comparing limits,

$x = 0, \quad x = a$

$y = x \quad y = a.$



On changing the order.



$x = 0$ to y

$y = 0$ to a.

Now, eq. (1) become.

$$I = \int_0^a \int_x^a (x^2+y^2)\,dy\,dx = \int_0^a \int_0^y (x^2+y^2)\,dx\,dy.$$

$$= \int_0^a \left| \frac{x^3}{3} + y^2 x \right|_0^y dy = \int_0^a \left(\frac{y^3}{3} + y^3\right) dy = \int_0^a \frac{4y^3}{3}\,dy$$

$$= \frac{4}{3}\left|\frac{y^4}{4}\right|_0^a = \frac{4}{3} \times \frac{a^4}{4} = \frac{a^4}{3} \text{ Ans}$$

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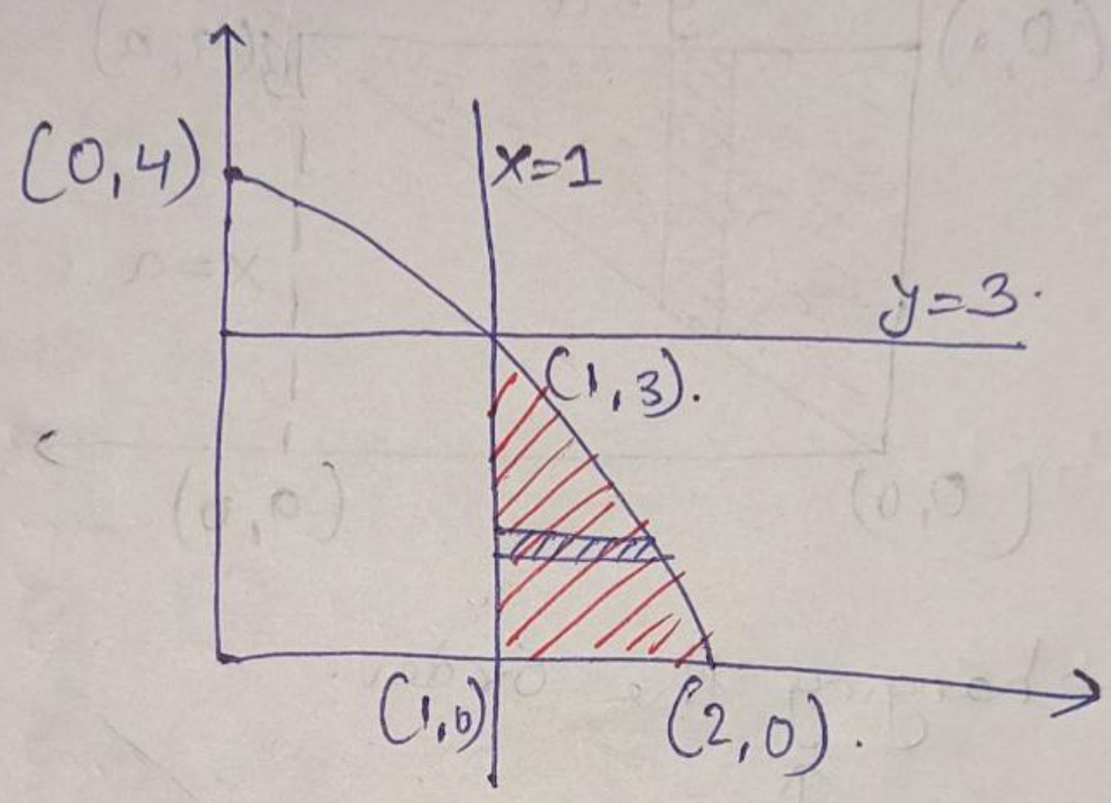
Q Solve $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ by using change of order of integration.

Sol→ Let $I = \int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$. ——— (1)

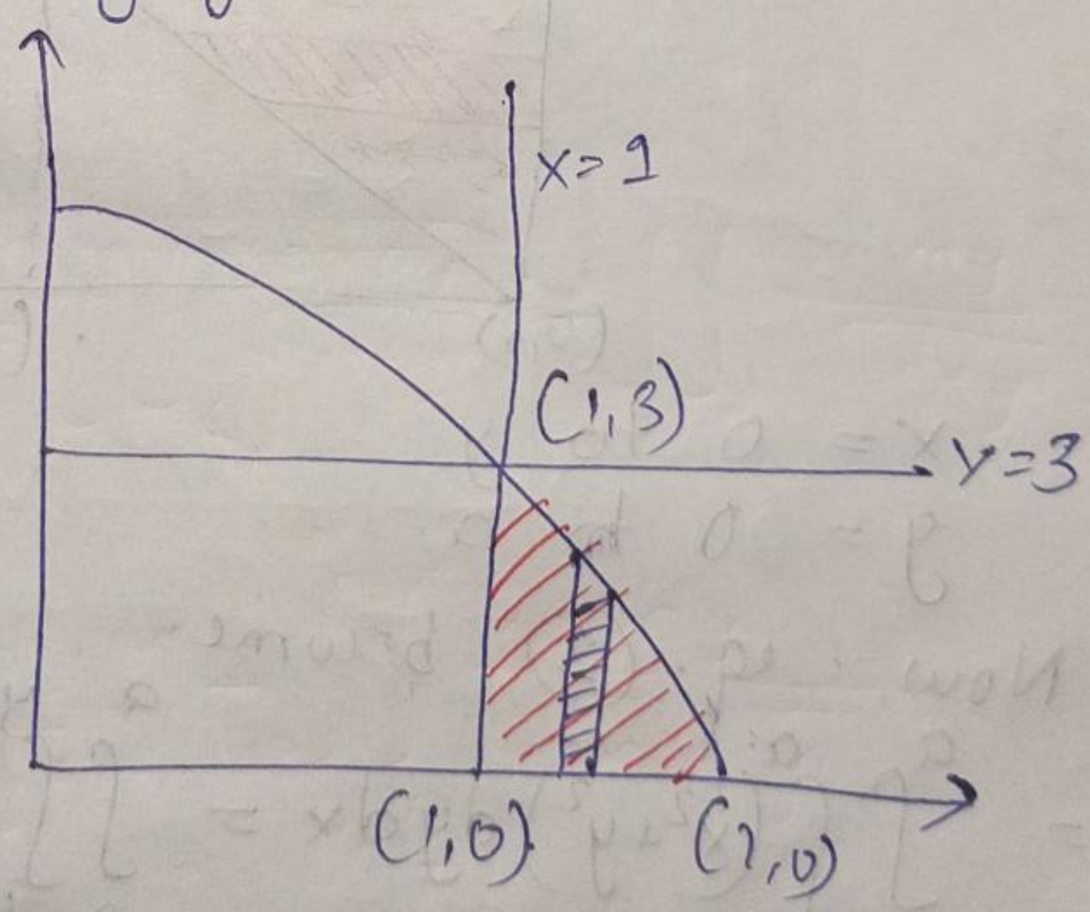
On Comparing limits

$y = 0$; $y = 3$.

$x = 1$; $x = \sqrt{4-y} \Rightarrow y = 4 - x^2$.



On changing the order.



$x = 1$ to 2

$y = 0$ to $4 - x^2$

Now, eq. (i)

$$I = \int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy = \int_1^2 \int_0^{4-x^2} (x+y)\,dy\,dx$$

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$$\Rightarrow I = \int_1^2 \left| xy + \frac{y^2}{2} \right|_0^{4-x^2} dx$$

$$= \int_1^2 \left[4x - x^3 + \frac{(4-x^2)^2}{2} \right] dx$$

$$= \int_1^2 \left(4x - x^3 + \frac{16}{2} + \frac{x^4}{2} - \frac{8x^2}{2} \right) dx$$

$$= \left| \frac{4x^2}{2} - \frac{x^4}{4} + 8x + \frac{x^5}{10} - \frac{4x^3}{3} \right|_1^2$$

$$= \frac{4\times 4}{2} - \frac{16}{4} + 16 + \frac{32}{10} - \frac{32}{3} - \frac{4}{2} + \frac{1}{4} - 8 - \frac{1}{10} + \frac{4}{3}$$

$$= \frac{480 - 240 + 960 + 192 - 640 - 120 + 15 - 480 - 6 + 80}{60}$$

$$= \frac{241}{60} \quad \text{Ans}$$

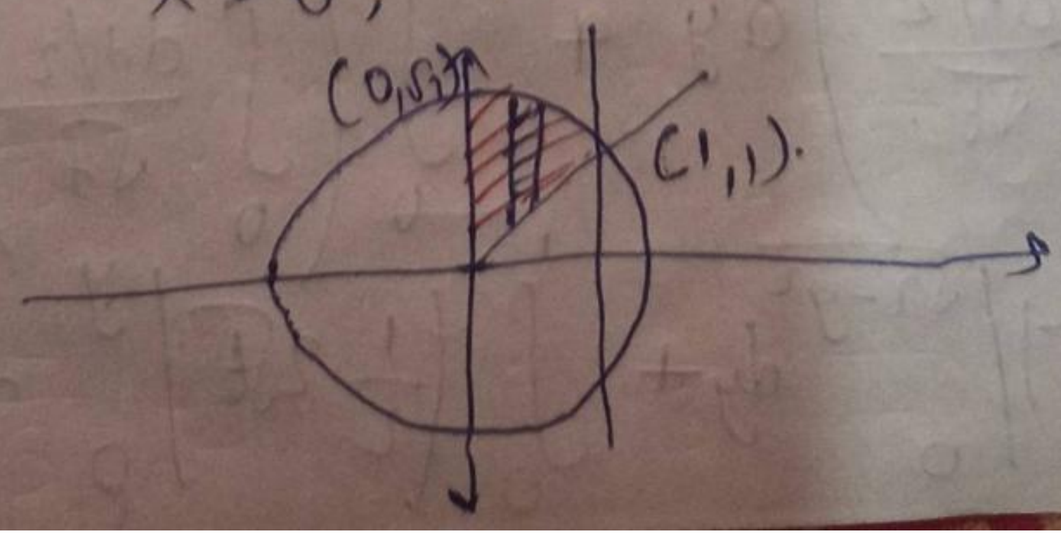
Q $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$

Sol→ Let $\displaystyle I = \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ — — (1)

Comparing the limits

$y = x$; $y = \sqrt{2-x^2} \Rightarrow x^2 + y^2 = 2$

$x = 0$; $x = 1$



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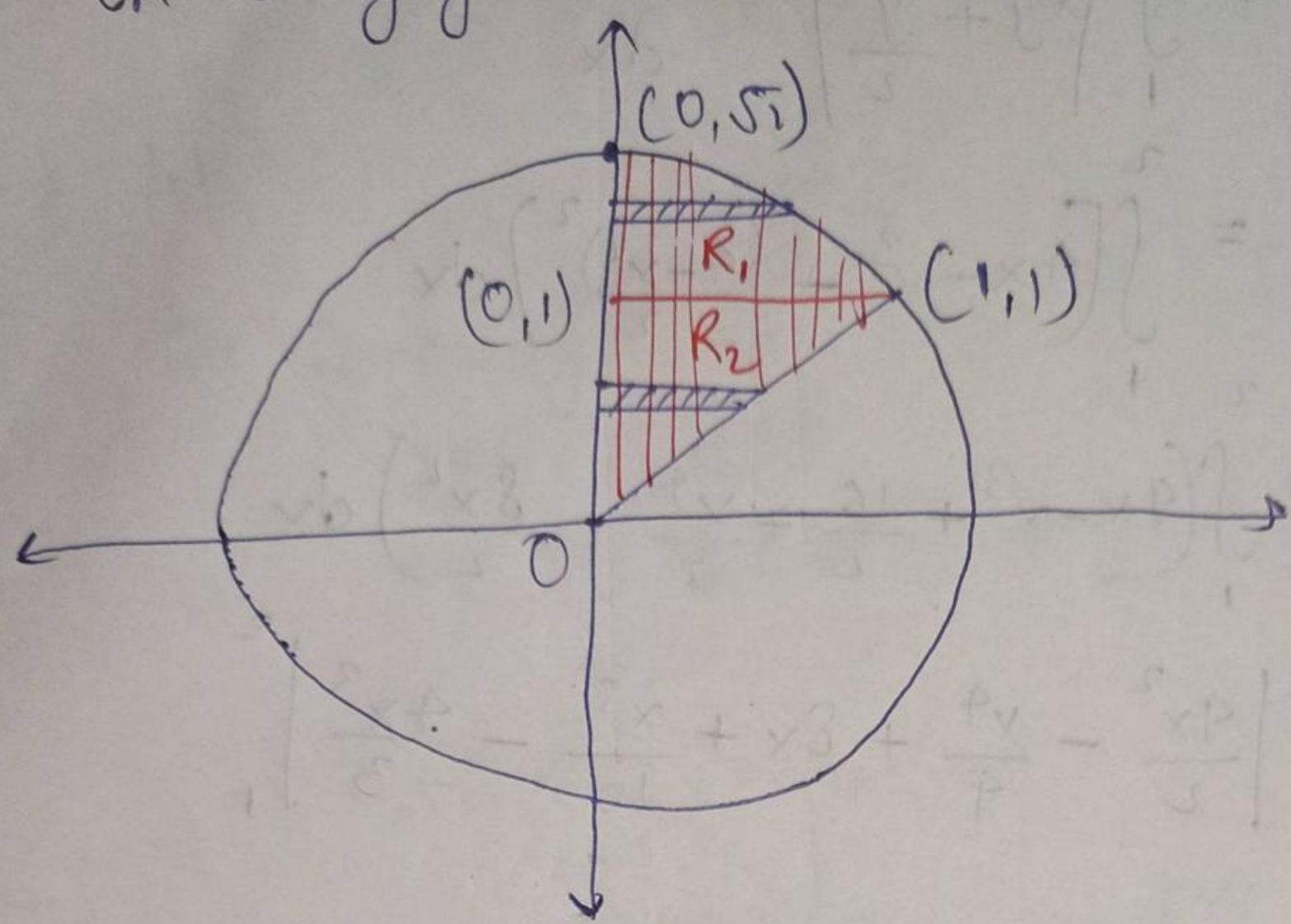
On changing the order.



We have limit in R_1

$y = 1$ to $\sqrt{2}$

$x = 0$ to $\sqrt{2-y^2}$

We have limit in R_2

$y = 0$ to 1

$x = 0$ to y.

Now, eq.(i)

$$I = \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \iint_{R_1} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \iint_{R_2} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$= \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \int_0^1 \int_0^y \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$= \int_1^{\sqrt{2}} \left(\int_0^{\sqrt{2-y^2}} \frac{dt/2}{\sqrt{t}} \right) dy + \int_0^1 \left(\int_0^y \frac{dt/2}{\sqrt{t}} \right) dy$$

$$= \int_1^{\sqrt{2}} \left| \frac{1}{2}(2\sqrt{t}) \right|_0^{\sqrt{2-y^2}} dy + \int_0^1 \left| \frac{1}{2} 2\sqrt{t} \right|_0^y dy$$

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$$= \int_1^{\sqrt{2}} \left|\sqrt{x^2+y^2}\right|_0^{\sqrt{2-y^2}} dy + \int_0^1 \left|\sqrt{x^2+y^2}\right|_0^{y} dy.$$

$$= \int_1^{\sqrt{2}} \left(\sqrt{2-y^2+y^2} - \sqrt{0+y^2}\right) dy + \int_0^1 \left(\sqrt{y^2+y^2} - \sqrt{0+y^2}\right) dy$$

$$= \int_1^{\sqrt{2}} \left(\sqrt{2} - y\right) dy + \int_0^1 \left(\sqrt{2}\, y - y\right) dy.$$

$$= \left|\sqrt{2}y - \frac{y^2}{2}\right|_1^{\sqrt{2}} + \int_0^1 (\sqrt{2}-1) \left|\frac{y^2}{2}\right|_0^1.$$

$$= \left(\sqrt{2} \times \sqrt{2} - \frac{2}{2} - \sqrt{2} + \frac{1}{2}\right) + \left(\frac{\sqrt{2}-1}{2}\right).$$

$$= \left(\frac{3}{2} - \sqrt{2} + \frac{\sqrt{2}}{2} - \frac{1}{2}\right)$$

$$= 1 - \frac{\sqrt{2}}{2}. \quad \underline{\underline{\text{Ans}}}$$

Q. Solve $\int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx$.

Sol→ Let $I = \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy \, dy \, dx$ ---- (1)

On Comparing limit

$y = \frac{x^2}{a} \Rightarrow x^2 = ay$; $y = 2a - x \Rightarrow x + y = 2a$

$x = 0$; $x = a$

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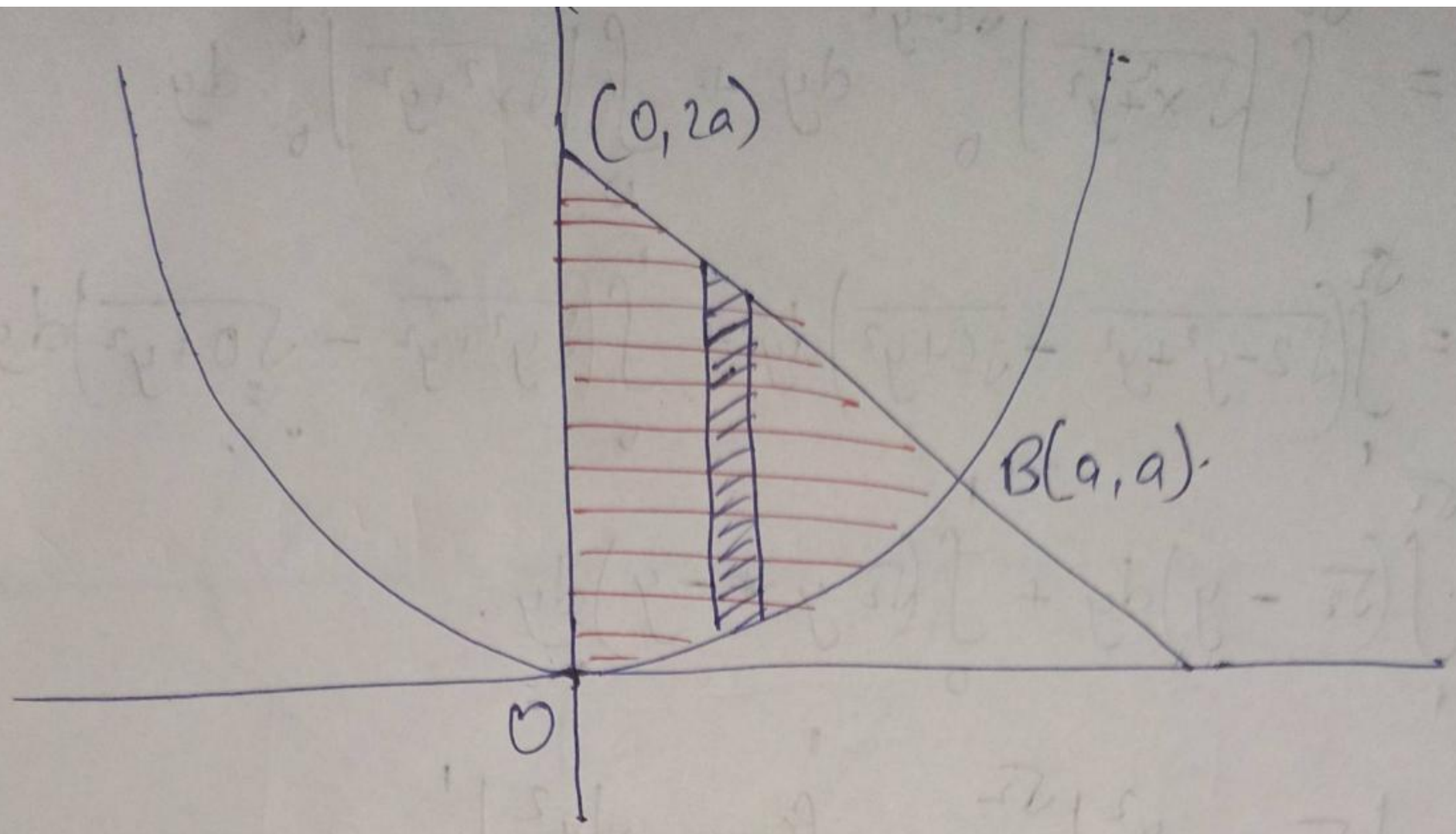
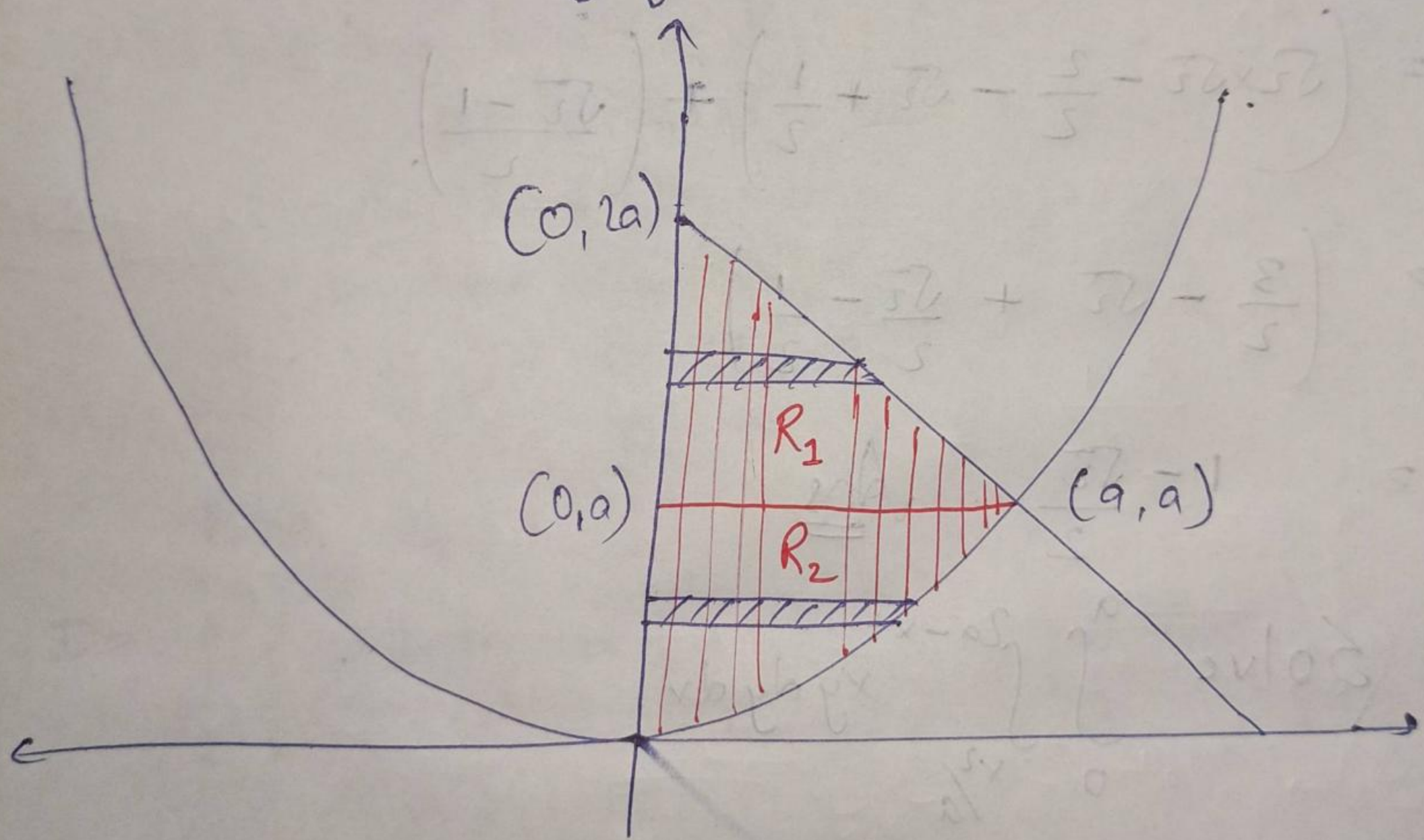


Changing the order.



We have limit in R_1

$y = a$ to $2a$

$x = 0$ to $2a - y$

We have limit in R_2

$y = 0$ to a

$x = 0$ to $\sqrt{ay}$

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Now, eq. (i)

$$I = \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx = \int_a^{2a}\int_0^{2a-y} xy\,dx\,dy + \int_0^a\int_0^{\sqrt{ay}} xy\,dx\,dy.$$

Let $I_1 = \int_a^{2a}\int_0^{2a-y} xy\,dx\,dy.$

$$= \int_a^{2a} \left(y \left| \frac{x^2}{2} \right|_0^{2a-y} \right) dy.$$

$$= \int_a^{2a} y \frac{(2a-y)^2}{2}\,dy$$

$$= \frac{1}{2}\int_a^{2a} y(4a^2+y^2-4ay)\,dy = \frac{1}{2}\int (4a^2y+y^3-4ay^2)\,dy$$

$$= \frac{1}{2}\left| \frac{4a^2y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3} \right|_a^{2a}$$

$$= \frac{1}{2}\left[(8a^4-2a^4) + \left(\frac{16a^4}{4} - \frac{a^4}{4}\right) - \left(\frac{32a^4}{3} - \frac{4a^4}{3}\right) \right]$$

$$= \frac{1}{2}\left[6a^4 + \frac{15a^4}{4} - \frac{28a^4}{3} \right]$$

$$= \frac{1}{2}\left[\frac{72a^4+45a^4-112a^4}{12} \right] = \frac{5a^4}{24}.$$

Let $I_2 = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy.$

$$= \int_0^a y \left| \frac{x^2}{2} \right|_0^{\sqrt{ay}} dy = \int_0^a \frac{ay^2}{2}\,dy = \left| \frac{ay^3}{6} \right|_0^a$$

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$$= a\left(\frac{a^3}{6}\right) - 0$$

$$= \frac{a^4}{6}$$

Now, total area $= I_1 + I_2$.

$$= \frac{5a^4}{24} + \frac{a^4}{6}$$

$$= \frac{5a^4 + 4a^4}{24}$$

$$= \frac{9a^4}{24} \quad \underline{\text{Ans}}.$$

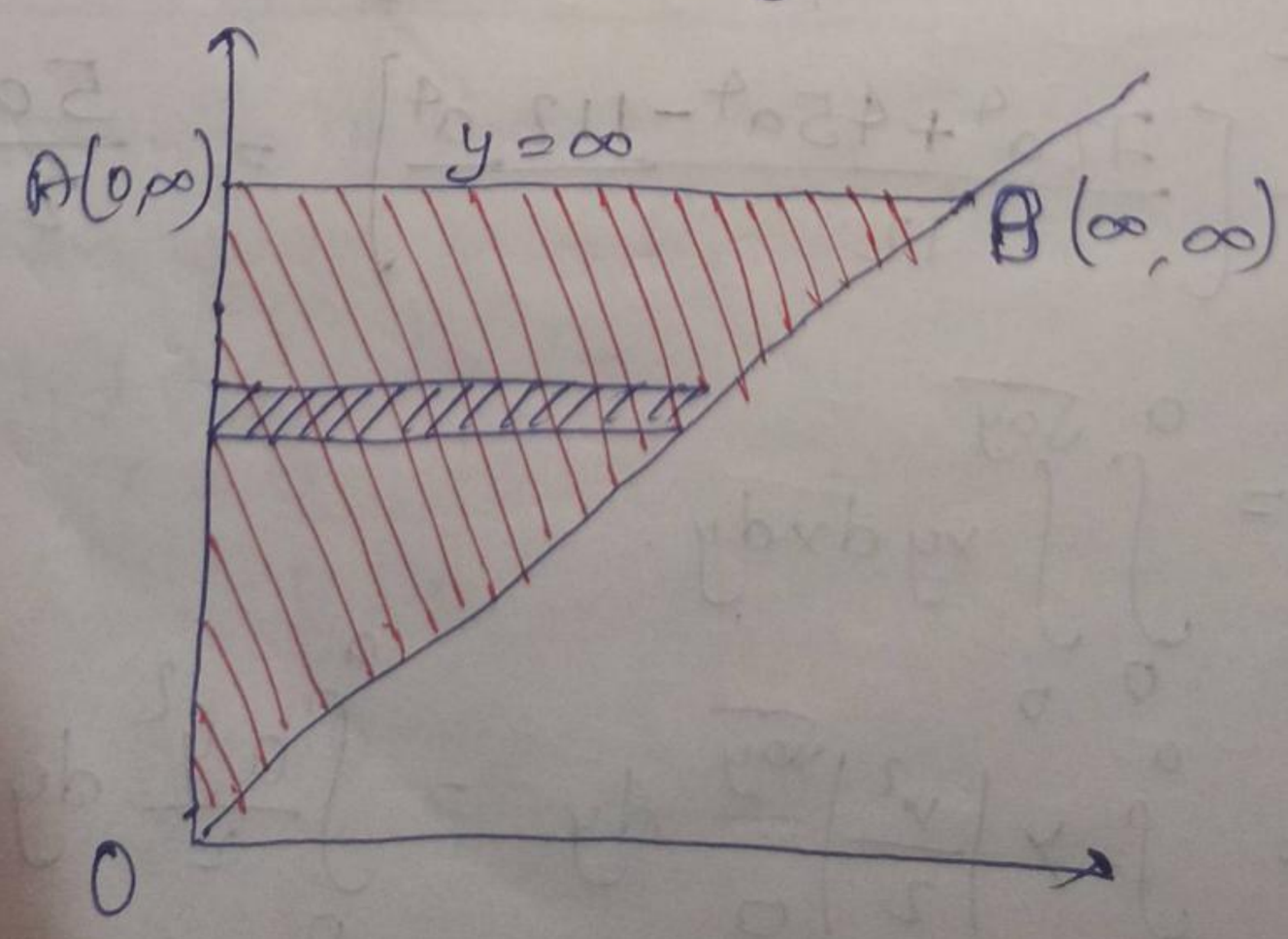
Q Solve $\int_0^\infty \int_0^y y e^{-y^2/x} \, dx \, dy$. by change of order

Sol→ Let $I = \int_0^\infty \int_0^y y e^{-\frac{y^2}{x}} \, dx \, dy$. ——— (1)

On comparision of limits

$y = 0 \qquad y = \infty$

$x = 0 \qquad x = y$



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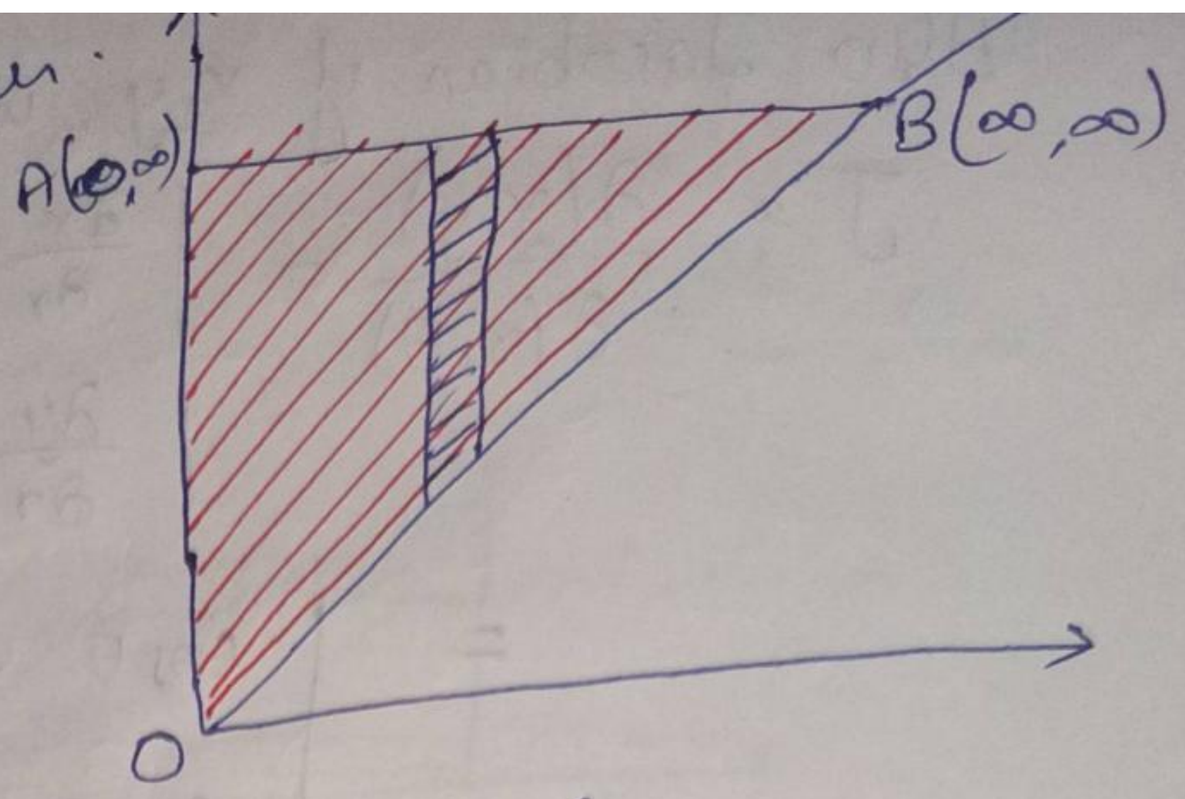
On changing Order:

$x = 0$ to ∞

$y = x$ to ∞



$$I = \int_0^\infty \int_0^y y e^{-\frac{y^2}{x}} dx\, dy = \int_0^\infty \int_x^\infty y e^{-\frac{y^2}{x}} dy\, dx$$

$$= \int_0^\infty \left| \frac{x e^{+}}{2} \right|_x^\infty dx = \int_0^\infty \frac{-x}{2} \left| \frac{e^{-\frac{y^2}{x}}}{\cancel{2}} \right|_{\cancel{x}}^\infty dx = \int_0^\infty \frac{-x}{2} \left(e^{-\infty} - e^{-x} \right) dx$$

$$= -\frac{1}{2} \int_0^\infty -x e^{-x} dx = \frac{1}{2} \int_0^\infty x e^{-x} dx.$$

$$= \frac{1}{2} \left[x e^{-x} - \int 1 \cdot \frac{e^{-x}}{-1} dx \right]_0^\infty$$

$$= \frac{1}{2} \left[-x e^{-x} - e^{-x} \right]_0^\infty$$

$$= \frac{1}{2} \left[0 - 0 - (0 - 1) \right]$$

$$= \frac{1}{2} [0 + 1] = \frac{1}{2} \quad \underline{\underline{\text{Ans}}}$$

Change of Order from Cartesian to polar coordinates.

We have $x = r\cos\theta$ --- (i); $y = r\sin\theta$ --- (ii)

On squaring and adding both.

$$x^2 + y^2 = r^2$$

also, on dividing we get $\tan\theta = \frac{y}{x}$.

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Also Jacobian of x, y. with respect to r, θ.

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$dx\,dy = |J|\,dr\,d\theta$$

$$dx\,dy = r\,dr\,d\theta$$

We know, Area $= \iint_R dx\,dy$ (in Cartesian)

Area $= \iint_R r\,dr\,d\theta$ (in Polar)

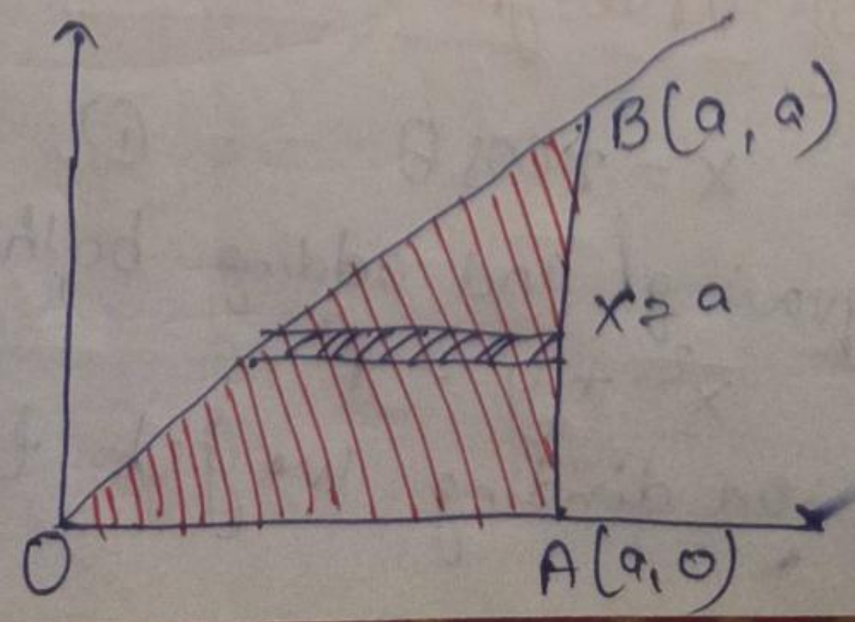
<u>Q</u> Change to polar coordinate and solve $\int_0^a \int_y^a x\,dx\,dy$.

Sol→ Let $I = \int_0^a \int_y^a x\,dx\,dy$ ——— (1)

On Comparing the limit we get

$x = y$; $x = a$

$y = 0$; $y = a$



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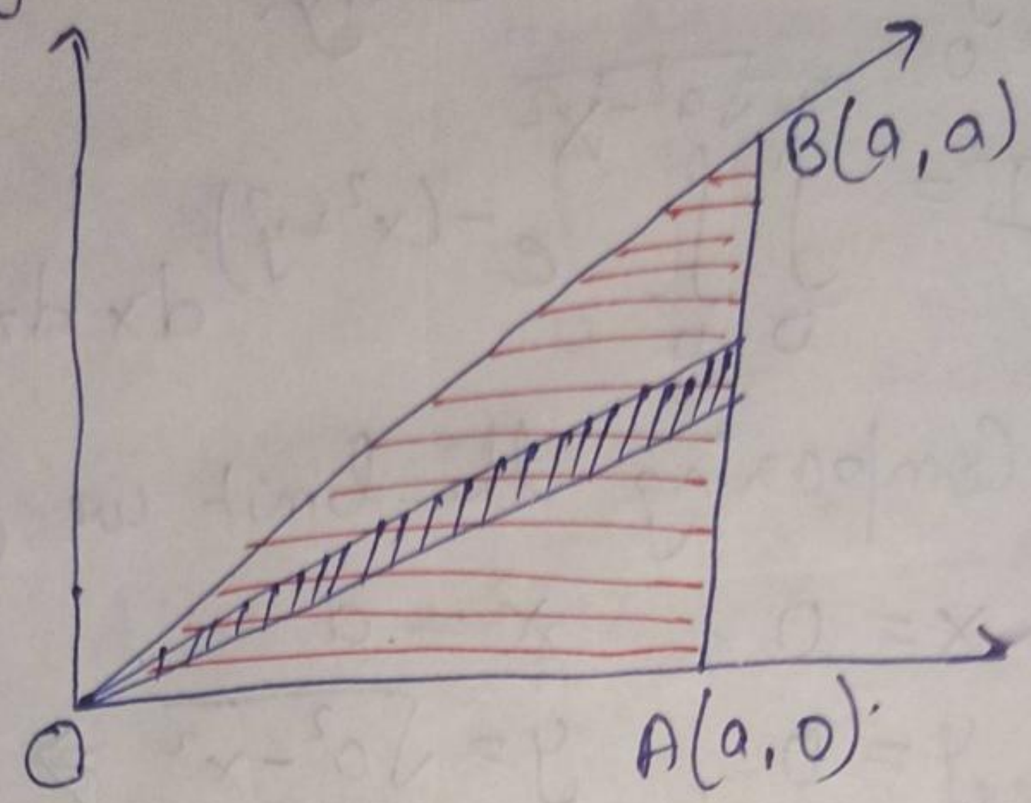
On Changing ~~into~~ in polar:



$\theta = 0$ to $\frac{\pi}{4}$.

$x = 0$ to $\frac{a}{\cos\theta}$ $\left[x = a, \quad r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta} \right]$

Now, eq. (i)

$$I = \int_0^a \int_y^a x\,dx\,dy = \int_0^{\pi/4} \int_0^{\frac{a}{\cos\theta}} (r\cos\theta)(r\,dr\,d\theta)$$

$$= \int_0^{\pi/4} \int_0^{a/\cos\theta} r^2 \cos\theta\,dr\,d\theta = \int_0^{\pi/4} \cos\theta \left| \frac{r^3}{3} \right|_0^{a/\cos\theta} d\theta$$

$$= \int_0^{\pi/4} \frac{a^3}{\cos^3\theta} \frac{\cos\theta}{3}\,d\theta = \int_0^{\pi/4} \frac{a^3}{3} \frac{1}{\cos^2\theta}\,d\theta = \int_0^{\pi/4} \frac{a^3}{3} \sec^2\theta\,d\theta$$

$$= \frac{a^3}{3} \left| \tan\theta \right|_0^{\pi/4} = \frac{a^3}{3} \left(\tan\frac{\pi}{4} - \tan\theta \right)$$

$$= \frac{a^3}{3}(1 - 0)$$

$$= \frac{a^3}{3} \quad \text{Ans}$$

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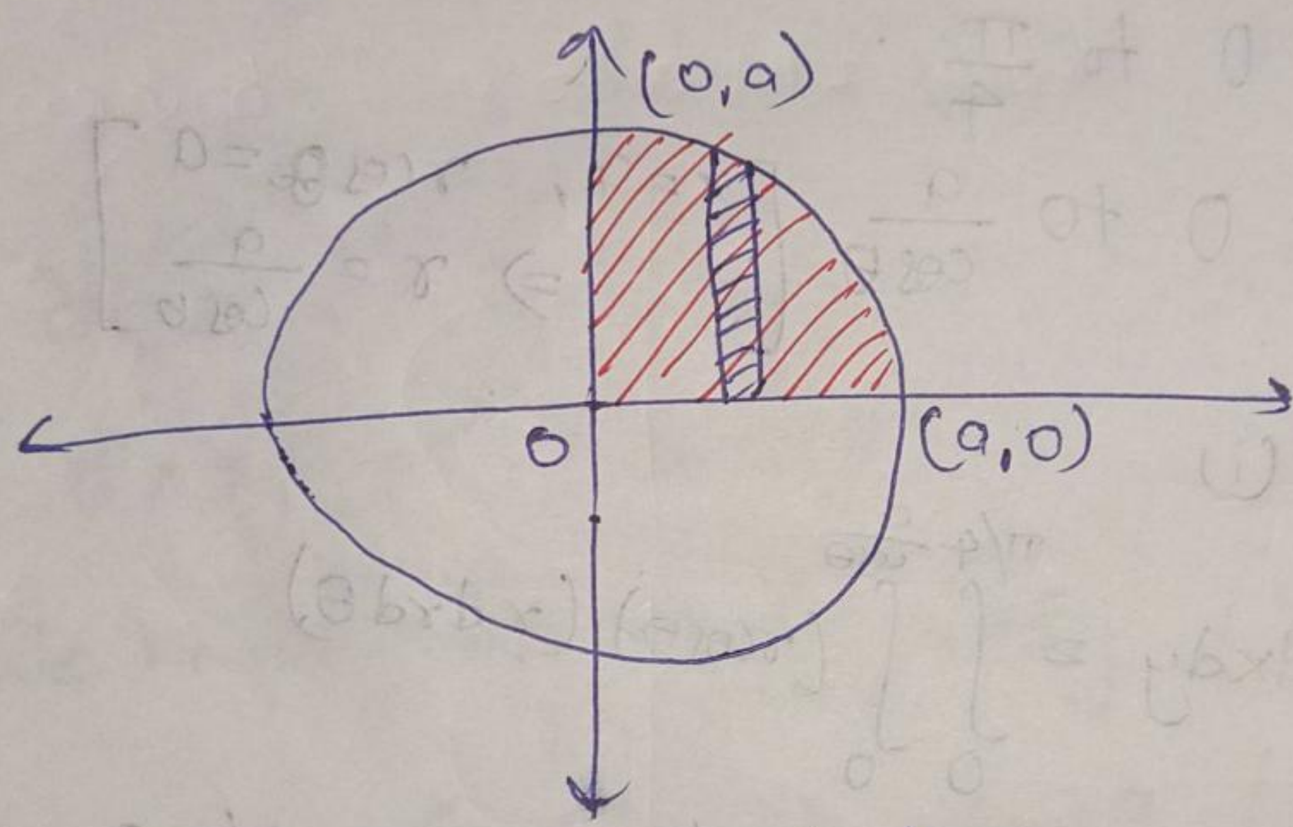
Q Solve $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dx\, dy$.

Sol→ Let $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dx\, dy$.

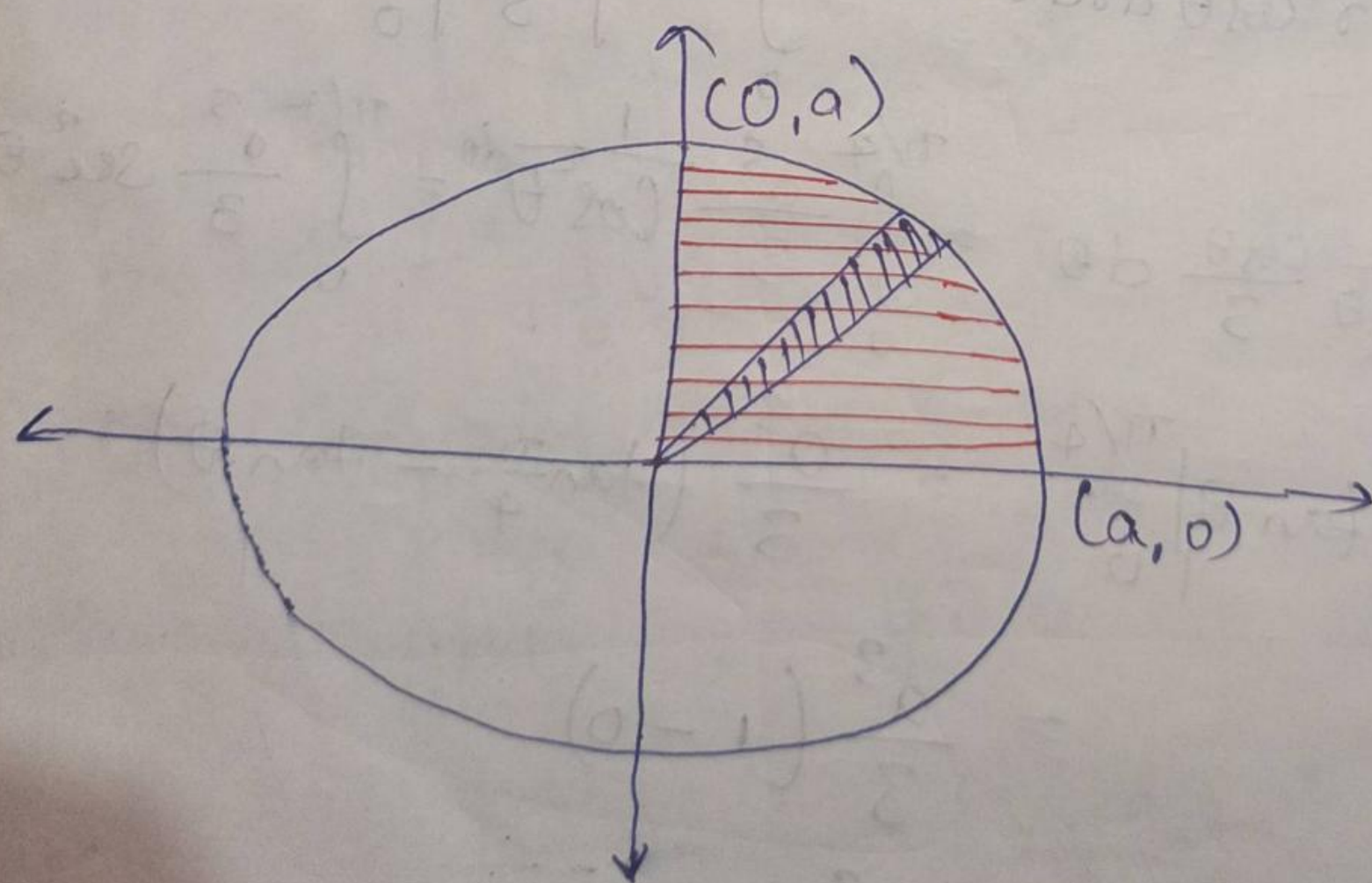
On Comparing the limit we get.

$x = 0, \quad x = a$

$y = 0, \quad y = \sqrt{a^2-x^2} \Rightarrow x^2+y^2 = a^2$



On changing to polar.



$\theta = 0$ to $\frac{\pi}{2}$

$r = 0$ to a.

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Now, eq. (i).

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^a e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left| \frac{e^t}{-2} \right|_0^a d\theta = \int_0^{\pi/2} \frac{-1}{2}\left(e^{-a^2} - e^0\right) d\theta$$

$$= -\frac{1}{2}\left(e^{-a^2} - 1\right)\int_0^{\pi/2} d\theta = \frac{(1-e^{-a^2})}{2} \left| \theta \right|_0^{\pi/2}$$

$$= \left(\frac{\pi}{2} - 0\right)\left(\frac{1-e^{-a^2}}{2}\right).$$

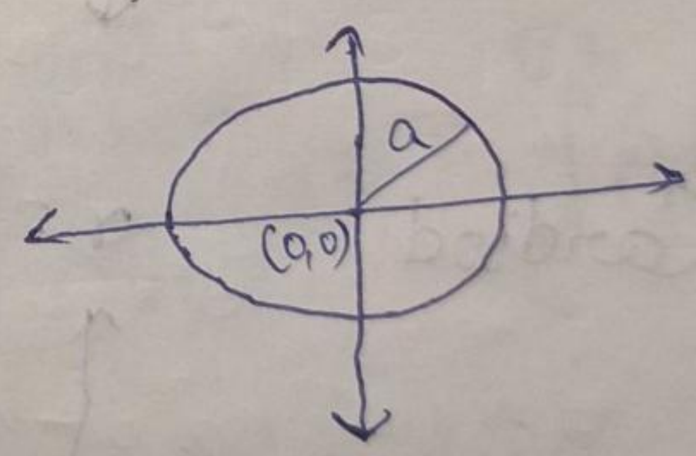
$$= \frac{\pi}{4}\left(1 - e^{-a^2}\right) \quad \underline{\underline{\text{Ans}}}$$

* Equation of Polar

① $r = a$

$\Rightarrow r^2 = a^2$.

$x^2 + y^2 = a^2$



② $r = 2a\cos\theta$

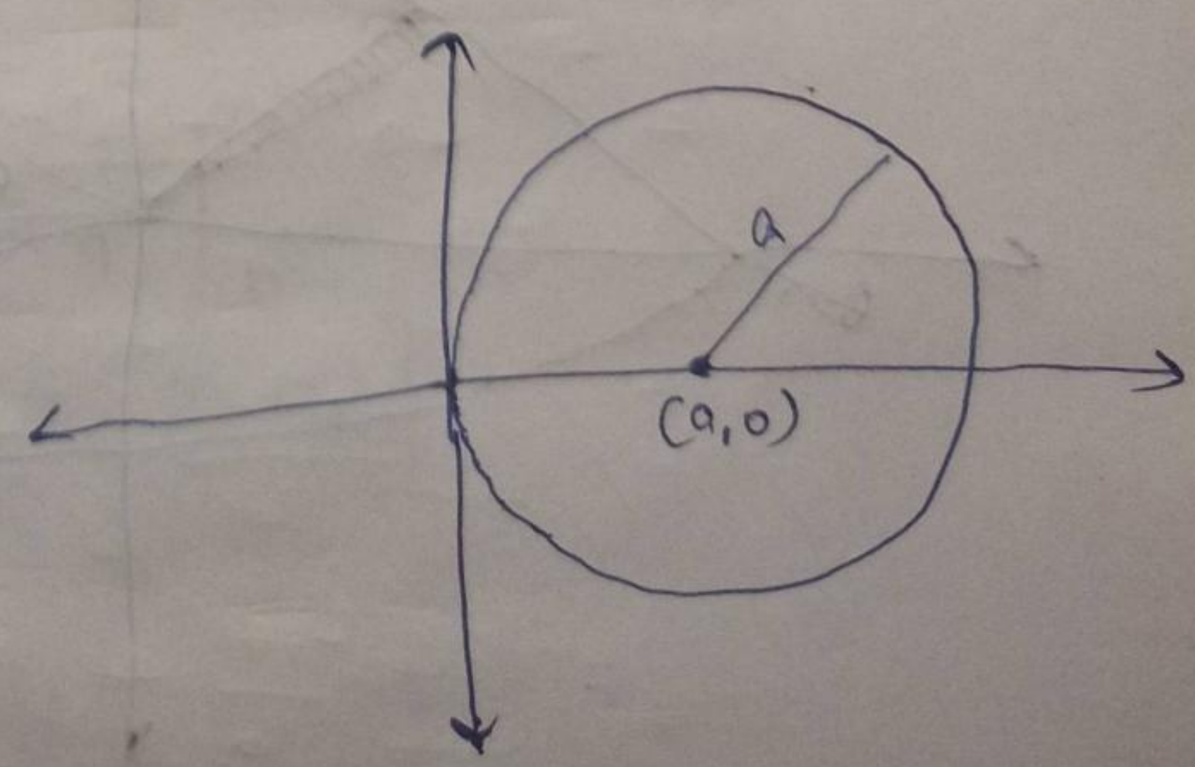
$r = 2a\left(\frac{x}{r}\right)$

$r^2 = 2ax$

$x^2 + y^2 = 2ax$

$x^2 - 2ax + y^2 = 0$

$(x-a)^2 + y^2 = a^2$



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3. $r = 2a \sin\theta$

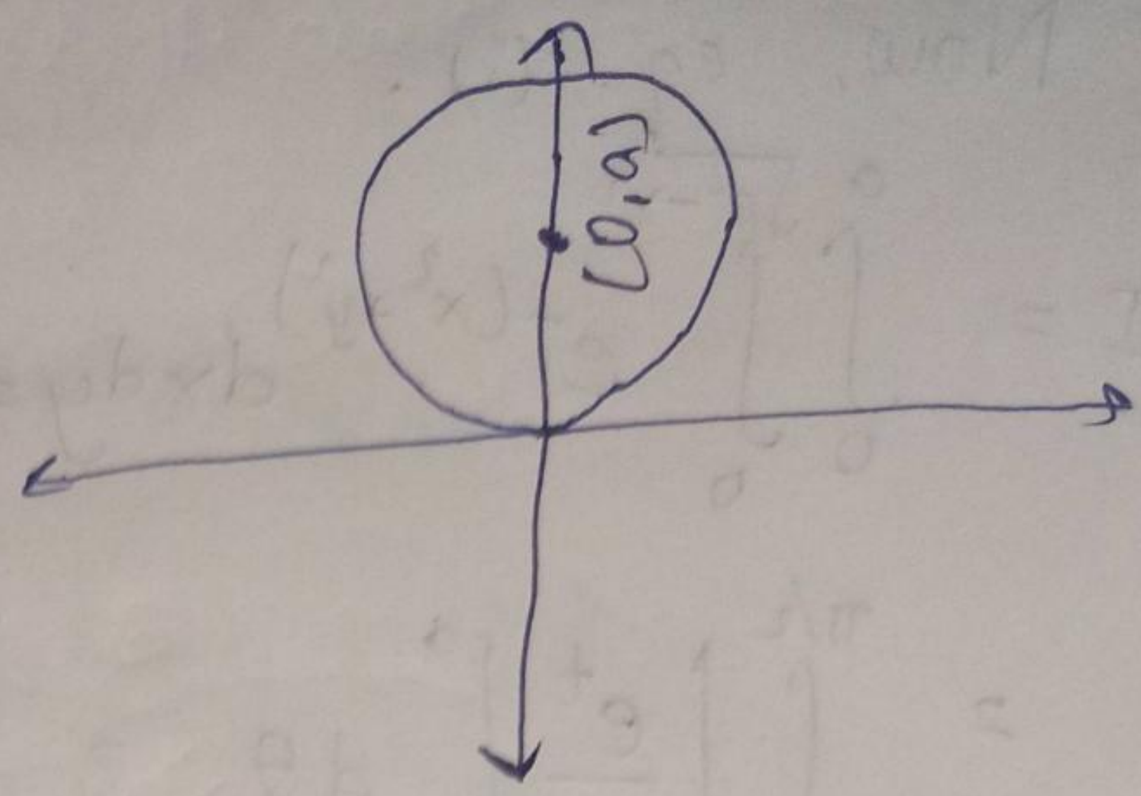
$\Rightarrow r = 2a\frac{y}{r}$

$\Rightarrow r^2 = 2ay$

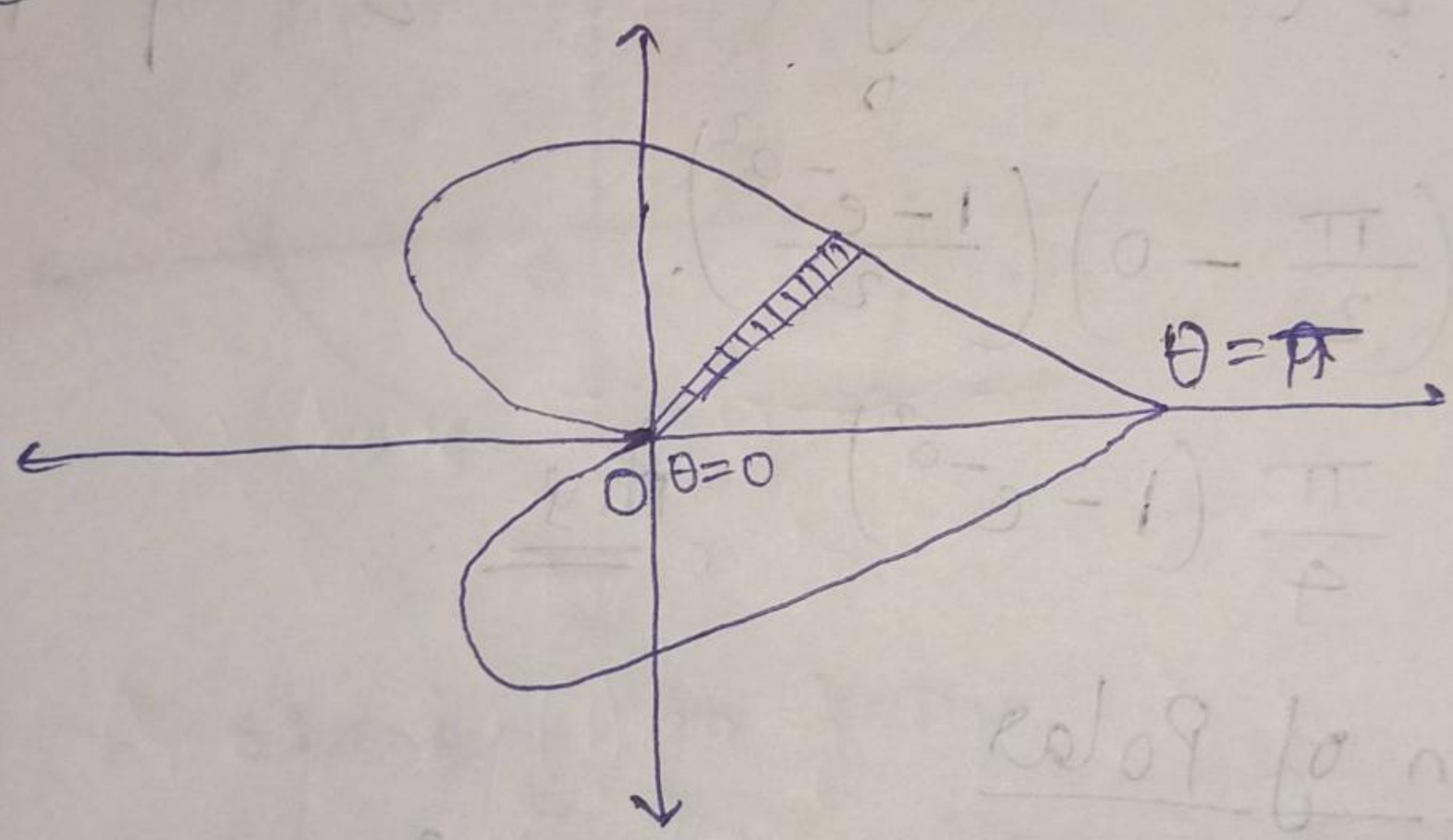
$\Rightarrow x^2 + y^2 = 2ay$

$\Rightarrow x^2 + y^2 - 2ay = 0$

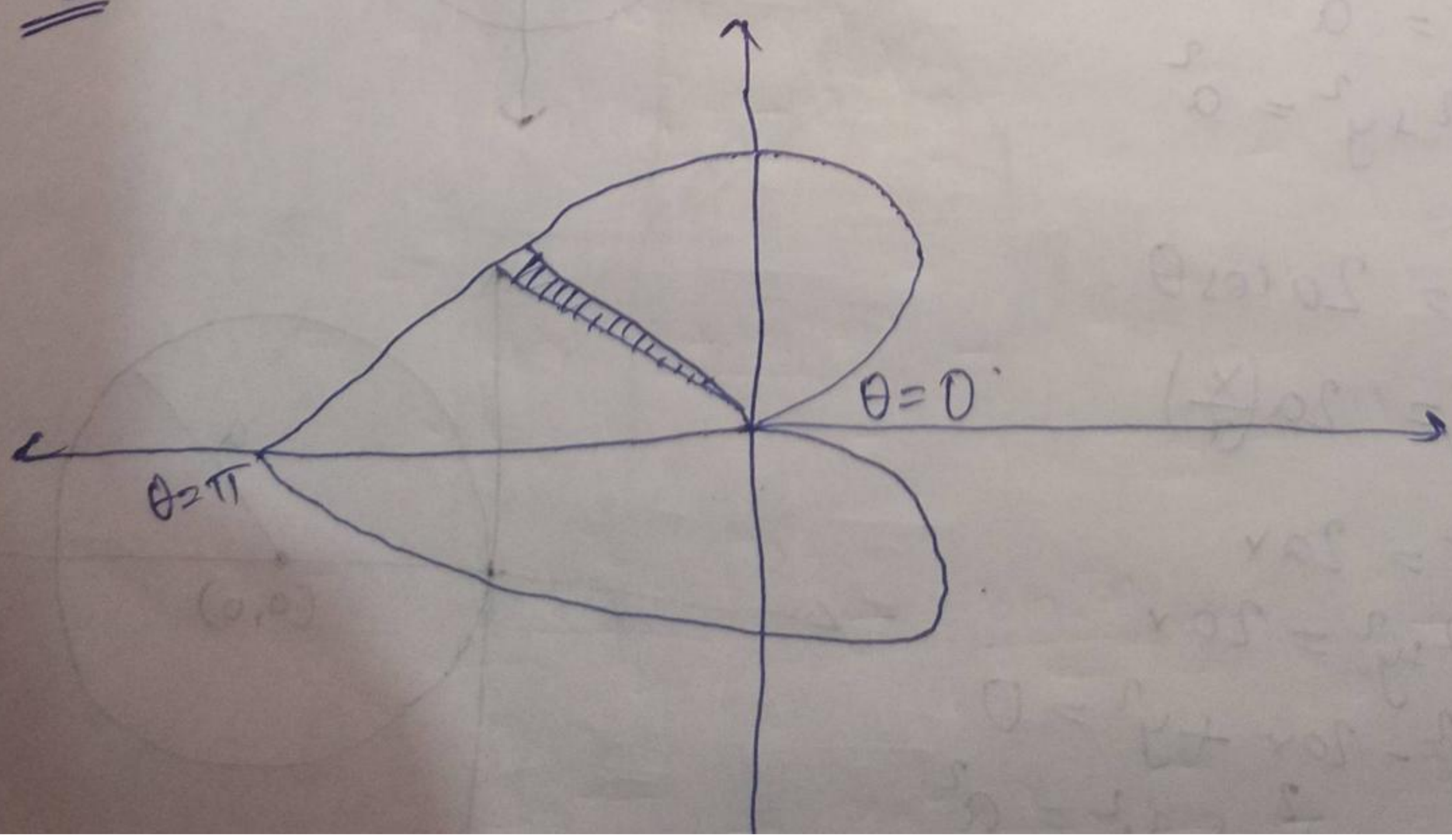
$\Rightarrow x^2 + (y-a)^2 = a^2$



4. Cardiod $r = a(1 + \cos\theta)$



5 Cardiod $r = a(1 - \cos\theta)$



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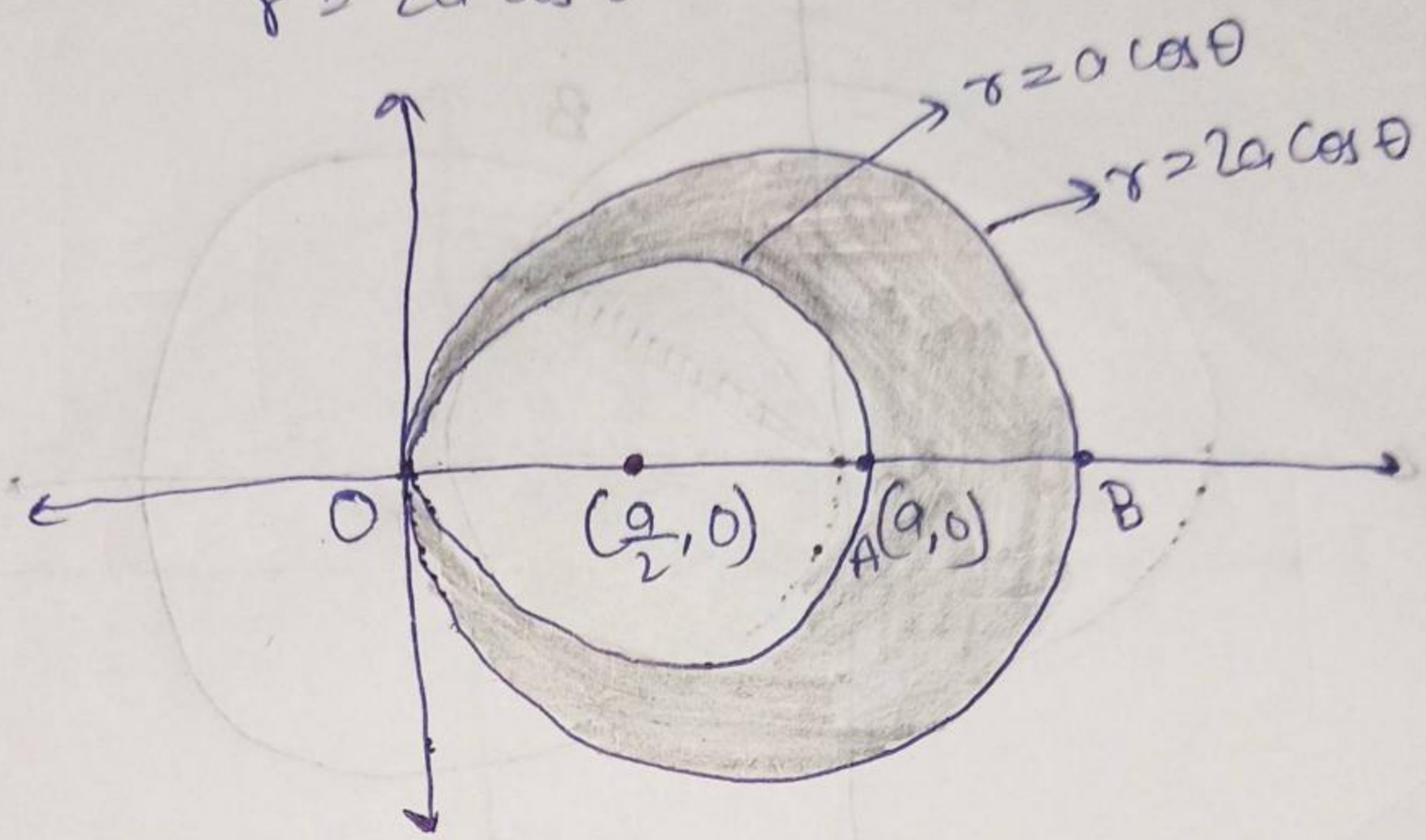
Q Find the area that lies outside the circle $r = a\cos\theta$ and inside $r = 2a\cos\theta$.

Sol→ Given $r = a\cos\theta$ ——— (1)

$r = 2a\cos\theta$ ——— (2)



We have $\theta = 0$ to $\pi/2$.

$r = a\cos\theta$ to $2a\cos\theta$

$$\text{Area} = 2\cdot(\text{Area of } OABO)$$

$$= 2\cdot\int_0^{\pi/2}\int_{a\cos\theta}^{2a\cos\theta} r\,dr\,d\theta$$

$$= 2\cdot\int_0^{\pi/2}\left|\frac{r^2}{2}\right|_{a\cos\theta}^{2a\cos\theta} d\theta = \int_0^{\pi/2}(4a^2\cos^2\theta - a^2\cos^2\theta)\,d\theta$$

$$= \int_0^{\pi/2} 3a^2\cos^2\theta\,d\theta = 3a^2\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 3a^2\left|\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right|_0^{\pi/2}$$

$$= 3a^2\left[\frac{\pi}{4} + 0 - (0+0)\right]$$

$$= \frac{3a^2\pi}{4} \text{ Ans}$$

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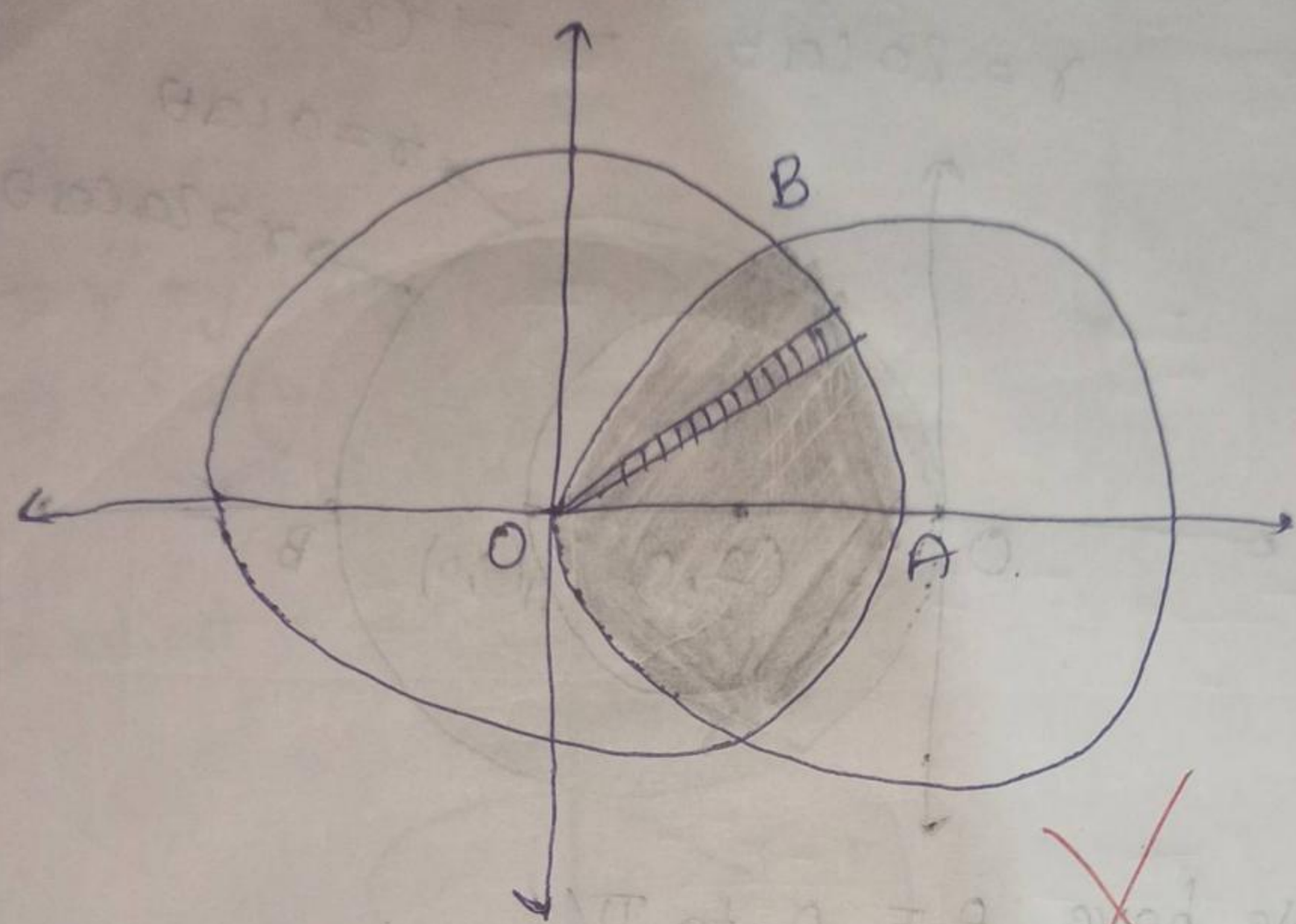
Q Find the common area to the curve $r=a$ & $r=2a\cos\theta$

Sol→ Given, $r=a$ ---- ①

$r=2a\cos\theta$ ---- ②



We have $\theta = 0$ to $\pi/3$.

$r = a$ to $2a\cos\theta$.

Correct Ans is on page 48

Area $= 2\cdot$(Area of OABO).

$$= 2\int_0^{\pi/3}\int_a^{2a\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3}\left|\frac{r^2}{2}\right|_a^{2a\cos\theta}$$

$$= \int_0^{\pi/3}\left(4a^2\cos^2\theta - a^2\right)d\theta$$

$$= \int_0^{\pi/3}\left[4a^2\left(\frac{1+\cos 2\theta}{2}\right) - a^2\right]d\theta$$

$$= \left|\frac{4a^2\theta}{2}\right|_0^{\pi/3} + \left|\frac{\sin 2\theta}{4}\times 4a^2\right|_0^{\pi/3} - \left|a^2\theta\right|_0^{\pi/3}$$

$$= 4a^2\frac{\pi}{6} + 4a^2\times\frac{1}{4}\times\frac{\sqrt{3}}{2} - a^2\frac{\pi}{3}$$

$$= a^2\left(\frac{\sqrt{3}}{2} - \frac{4\pi - 2\pi}{6}\right) = a^2\left(\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right)$$ Ans

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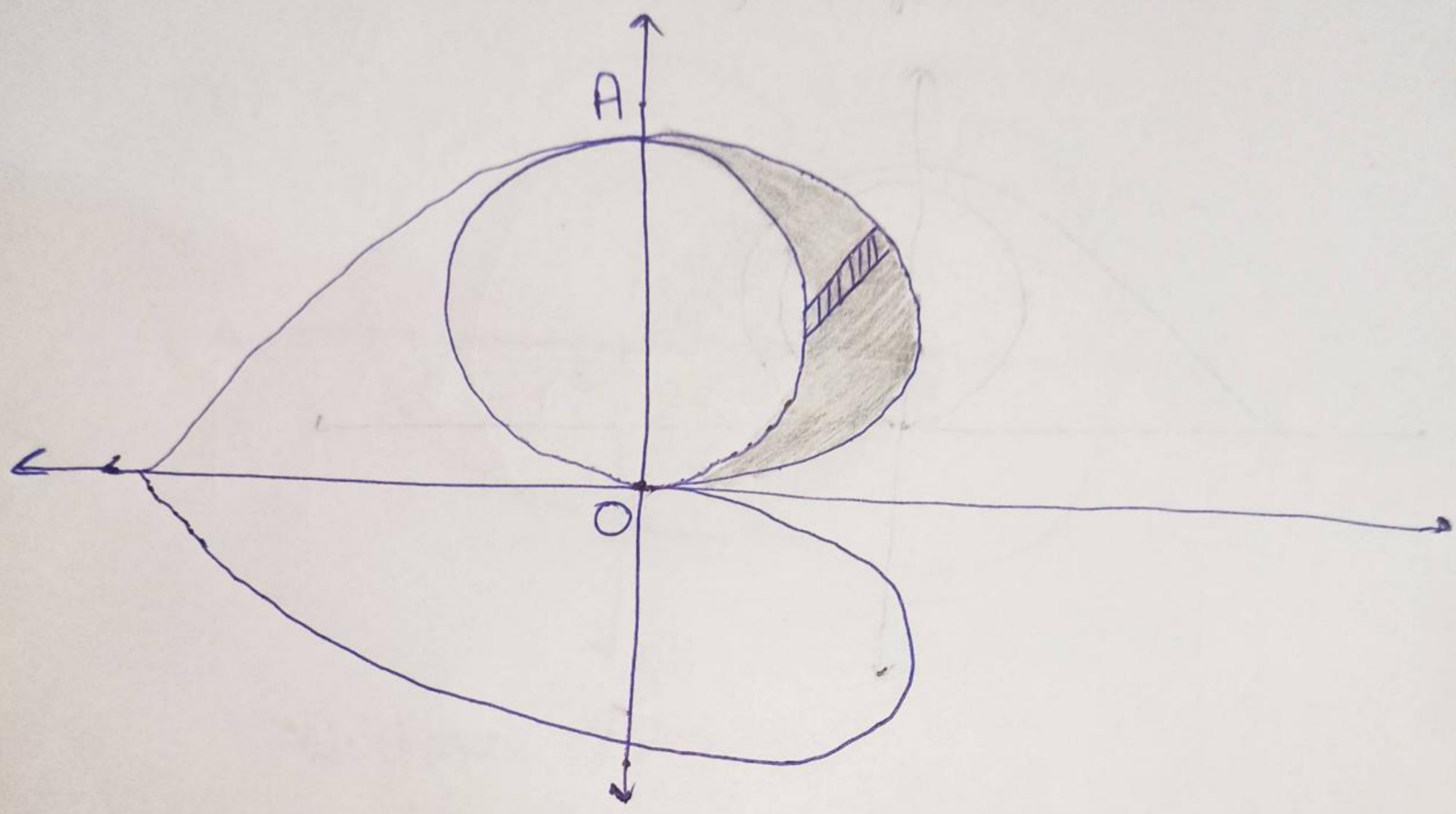
Q Find the area inside $r = a(1-\cos\theta)$ and outside $r = a\sin\theta$.

Sol→ Given, $r = a\sin\theta$ — — (1)

$r = a(1-\cos\theta)$ — — (2)



We have $\theta = 0$ to $\pi/2$

$r = a(1-\cos\theta)$ to $a\sin\theta$.

$$\text{Area} = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left|\frac{r^2}{2}\right|_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{a^2\sin^2\theta - a^2 - a^2\cos^2\theta + 2a^2\cos\theta}{2}\, d\theta$$

$$\Rightarrow \frac{a^2}{2}\int_0^{\pi/2}\left(2\cos\theta + \sin^2\theta - 1 - \cos^2\theta\right) d\theta$$

$$\Rightarrow \frac{a^2}{2}\int_0^{\pi/2}\left(2\cos\theta - \cos 2\theta - 1\right) d\theta$$

$$\Rightarrow \frac{a^2}{2}\left|2\sin\theta - \frac{\sin 2\theta}{2} - \theta\right|_0^{\pi/2}$$

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$$\Rightarrow \frac{a^2}{2}\left[2\times \sin\frac{\pi}{2} - 0 - \frac{\pi}{2} - (0-0-0)\right]$$

$$= \frac{a^2}{2}\left(2\times 1 - \frac{\pi}{2}\right)$$

$$= a^2\left(2-\frac{\pi}{2}\right)$$

$$= a^2\left(1-\frac{\pi}{4}\right) \quad \underline{\underline{\text{Ans}}}$$

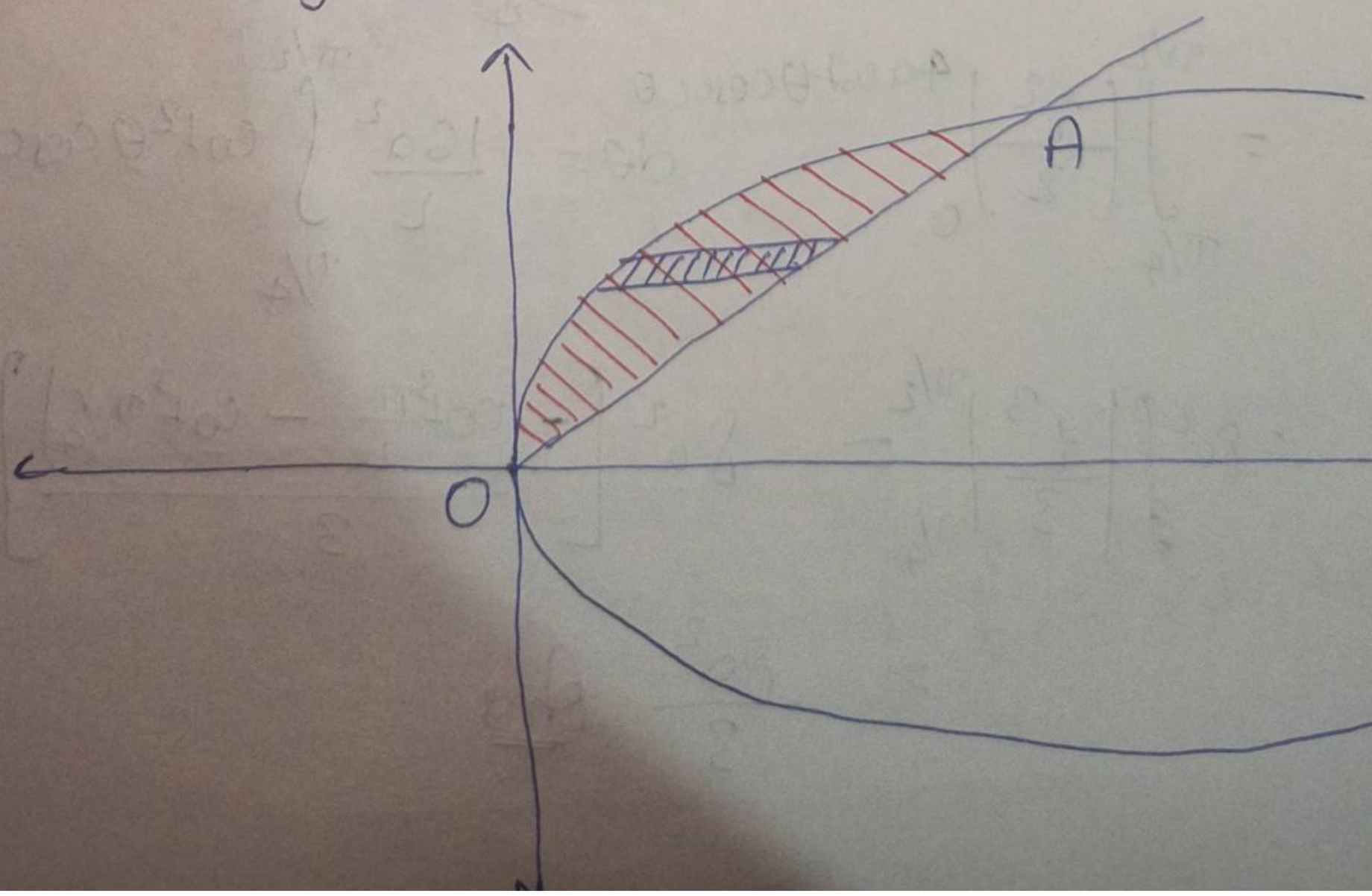
Q Evaluate $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} dx\,dy$ by changing to polar.

Sol→ let $I = \int_0^{4a}\int_{\frac{y^2}{4a}}^{y} dx\,dy$ — — (1)

Compare the limits.

$x = \frac{y^2}{4a}$, $x = y$

$y = 0$, $y = 4a$



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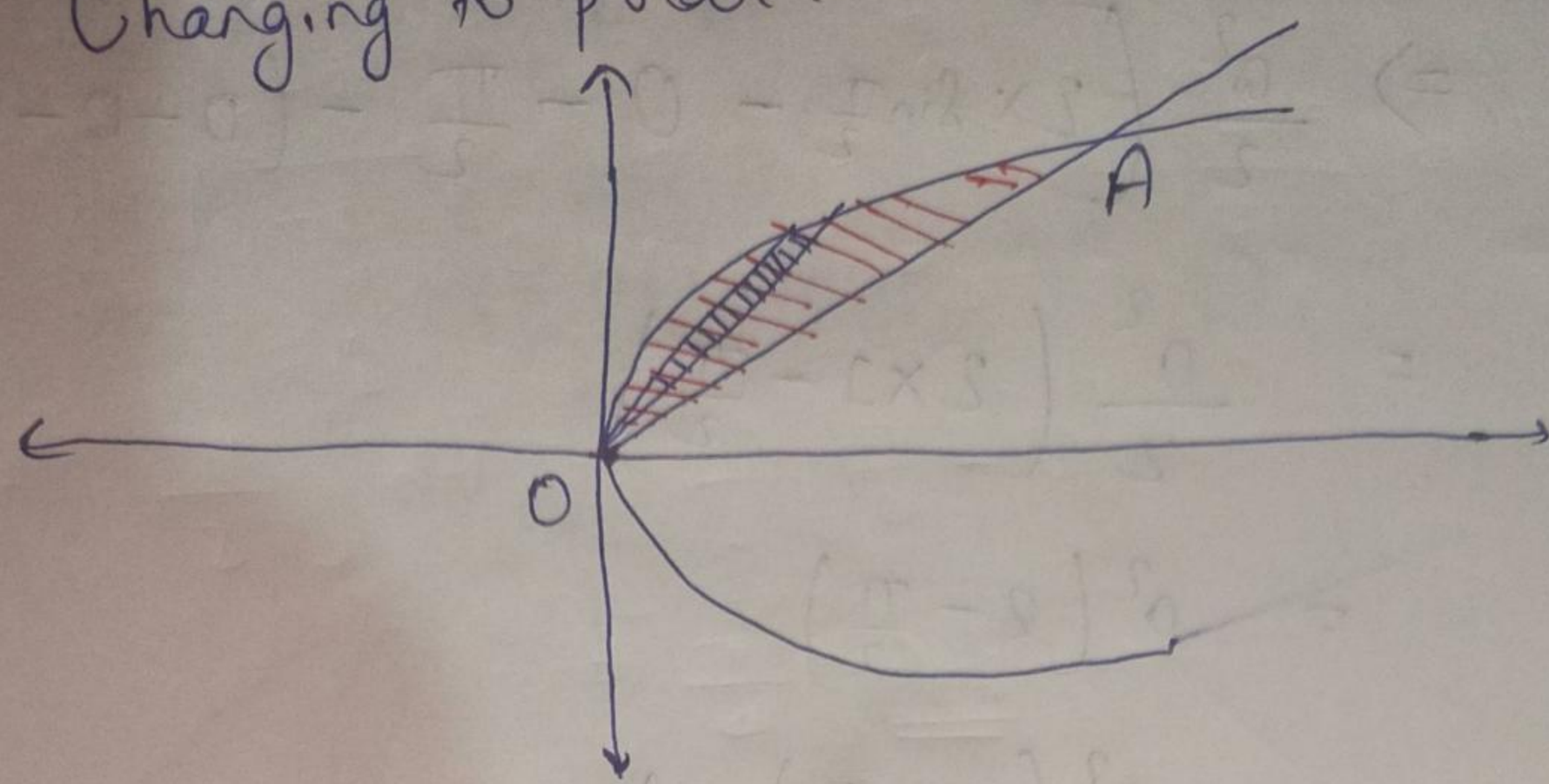
Changing to polar.



We have

$$y^2 = 4ax$$

$$r^2 \sin^2\theta = 4ar\cos\theta$$

$$r = \frac{4a\cos\theta}{\sin^2\theta} = 4a\cot\theta\operatorname{cosec}\theta$$

Here, $r = 0$ to $4a\cot\theta\operatorname{cosec}\theta$

$\theta = \frac{\pi}{4}$ to $\frac{\pi}{2}$.

Now, eq-(i)

$$I = \int_0^{4a}\int_{\frac{y^2}{4a}}^{y} dx\,dy = \int_{\pi/4}^{\pi/2}\int_0^{4a\cot\theta\operatorname{cosec}\theta} r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{4a\cot\theta\operatorname{cosec}\theta} d\theta = \frac{16a^2}{2}\int_{\pi/4}^{\pi/2}\cot^2\theta\operatorname{cosec}^2\theta\,d\theta$$

$$= 8a^2\left[\frac{t^3}{3}\right]_{\pi/4}^{\pi/2} = 8a^2\left[\frac{-\left(\cot^3\frac{\pi}{2} - \cot^3\pi/4\right)}{3}\right]$$

$$= \frac{8a^2}{3} \text{ Ans}$$

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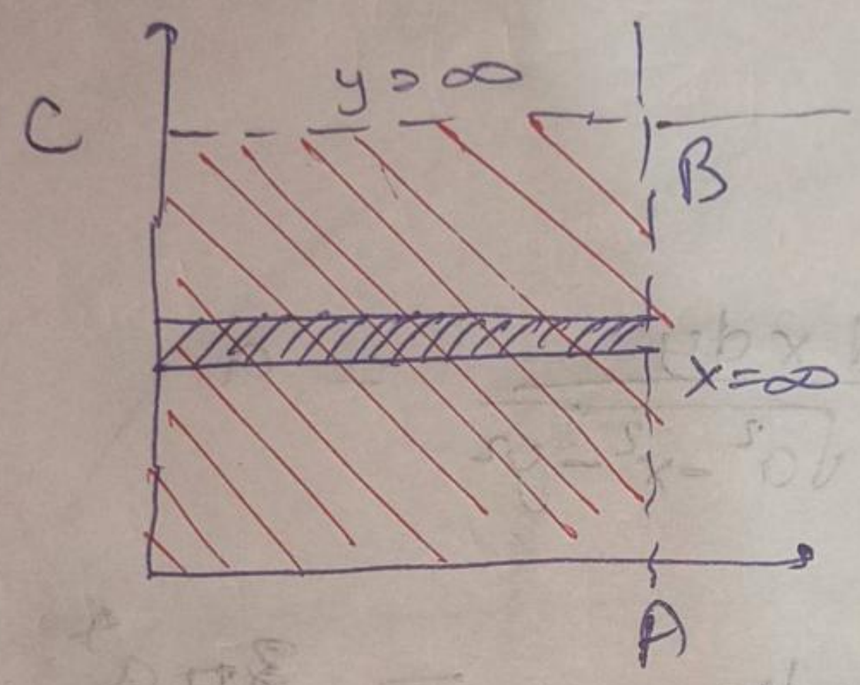
Q Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$.

Sol→ Let $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$. ——— (1)

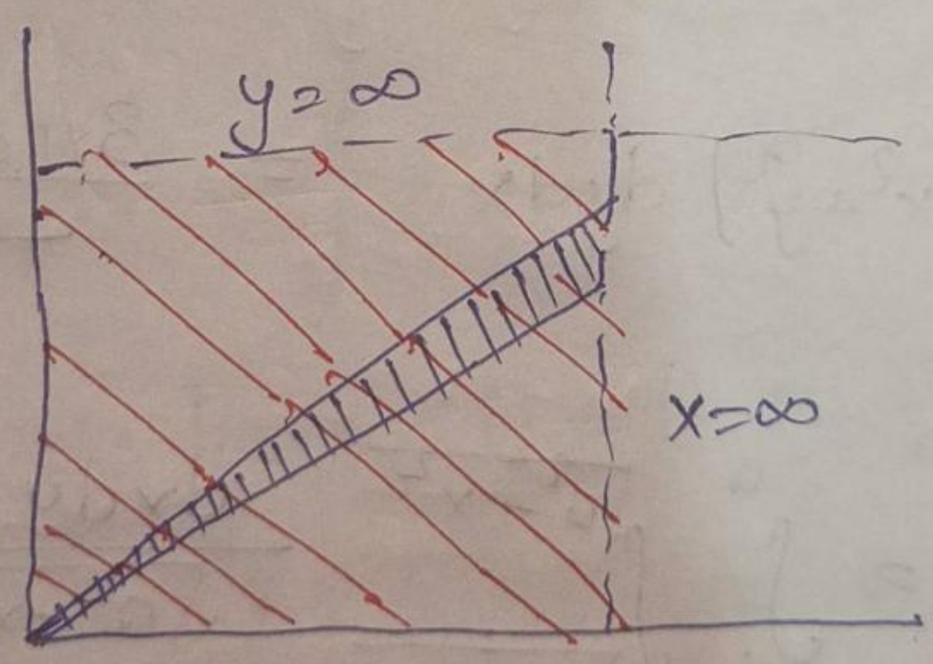
Compare the limits,

$x = 0$, $x = \infty$

$y = 0$, $y = \infty$



On changing to polar



We have, $\theta = 0$ to $\pi/2$

$r = 0$ to ∞.

Now, eq. (i).

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

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$$= \int_0^{\pi/2} \left| \frac{e^{-r^2}}{-2} \right|_0^{\infty} d\theta = \int_0^{\pi/2} \left(\frac{e^{-\infty} - e^0}{-2} \right) d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4} \quad \underline{\underline{\text{Ans}}}$$

Home Work

① Solve $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}} = a$

② $\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx = \frac{3\pi a^4}{4}$

③ $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx = \frac{3\pi a^4}{4}$

Sol → ① Let $I = \int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$

On comparing limits.

$x = 0$, $x = a$

$y = \sqrt{ax-x^2} \Rightarrow y^2 + x^2 = ax \Rightarrow y^2 + (x-a)^2 = \frac{a^2}{4}$

$y = \sqrt{a^2-x^2} \Rightarrow y^2 + x^2 = a^2$

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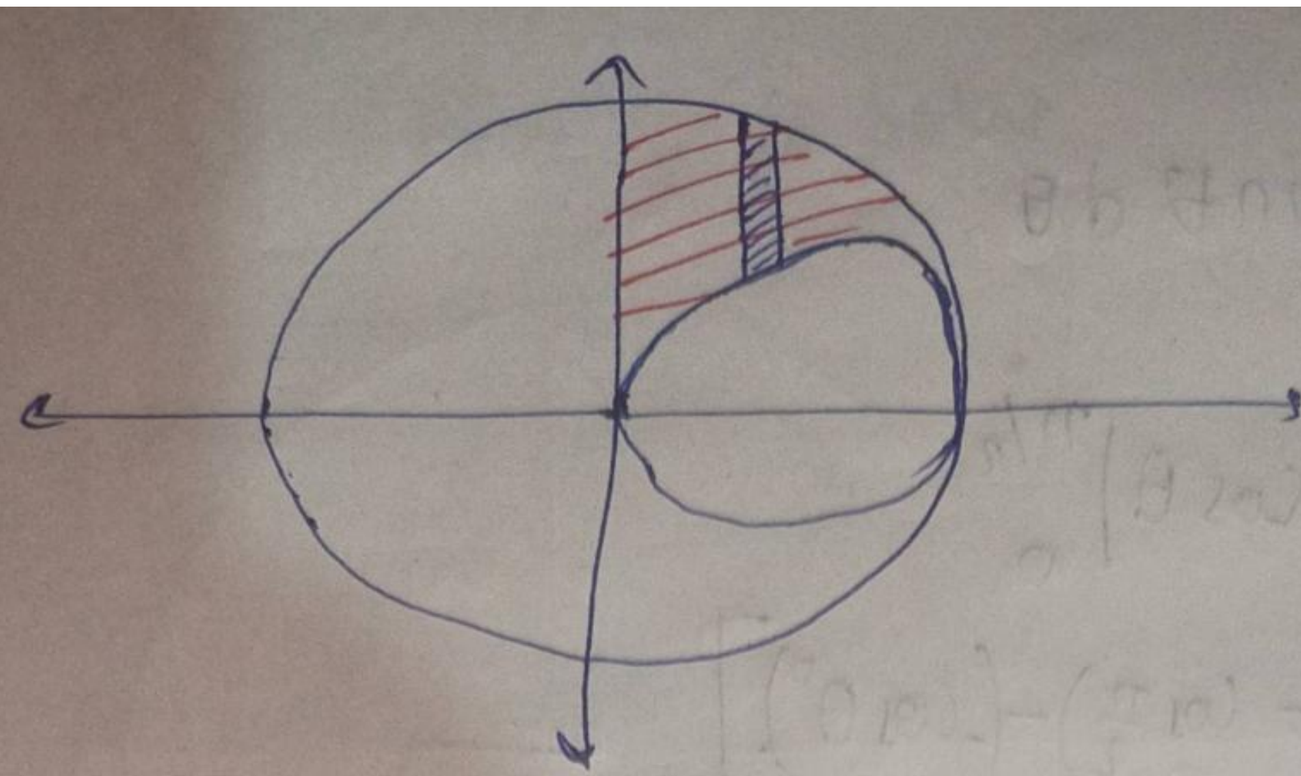
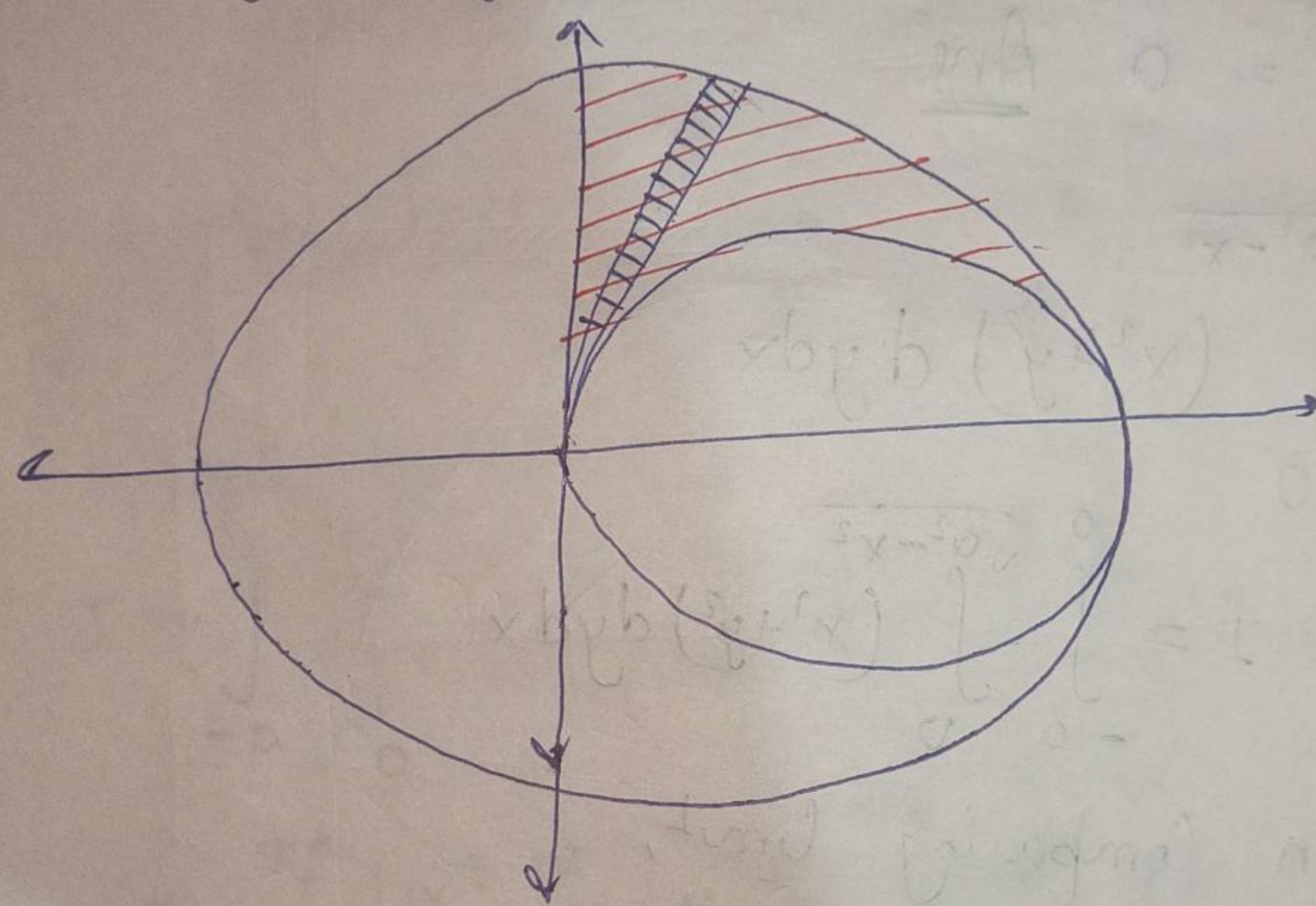


On changing to polar.



We have $\theta = 0$ to $\frac{\pi}{2}$.

$r = a\cos\theta$ to a

Now, eq. (i)

$$I = \int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}} = \int_0^{\pi/2}\int_{a\cos\theta}^{a} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$$

$$= \int_0^{\pi/2}\left(\frac{1}{-2}\int_{a\cos\theta}^{a} \frac{2r\,dr\,d\theta}{\sqrt{a^2-r^2}}\right) \Rightarrow \int_0^{\pi/2}\left(\frac{1}{2}\int_{a\cos\theta}^{a}\left(\sqrt{a^2-r^2}\right)^{-1} 2r\,dr\,\theta\right).$$

$$= -\int_0^{\pi/2} \frac{1}{2}\left(\frac{(a^2-r^2)^{1/2}}{\left(\frac{1}{2}\right)}\right)_{a\cos\theta}^{a} d\theta = -\int_0^{\pi/2} -a\sin\theta\, d\theta$$

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$$\int_0^{\pi/2} a \sin\theta \, d\theta$$

$$= a \left[-\cos\theta\right]_0^{\pi/2}$$

$$= a\left[\left(-\cos\frac{\pi}{2}\right)-\left(-\cos 0^\circ\right)\right]$$

$$= a(0+1)$$

$$= a \quad \underline{\text{Ans}}$$

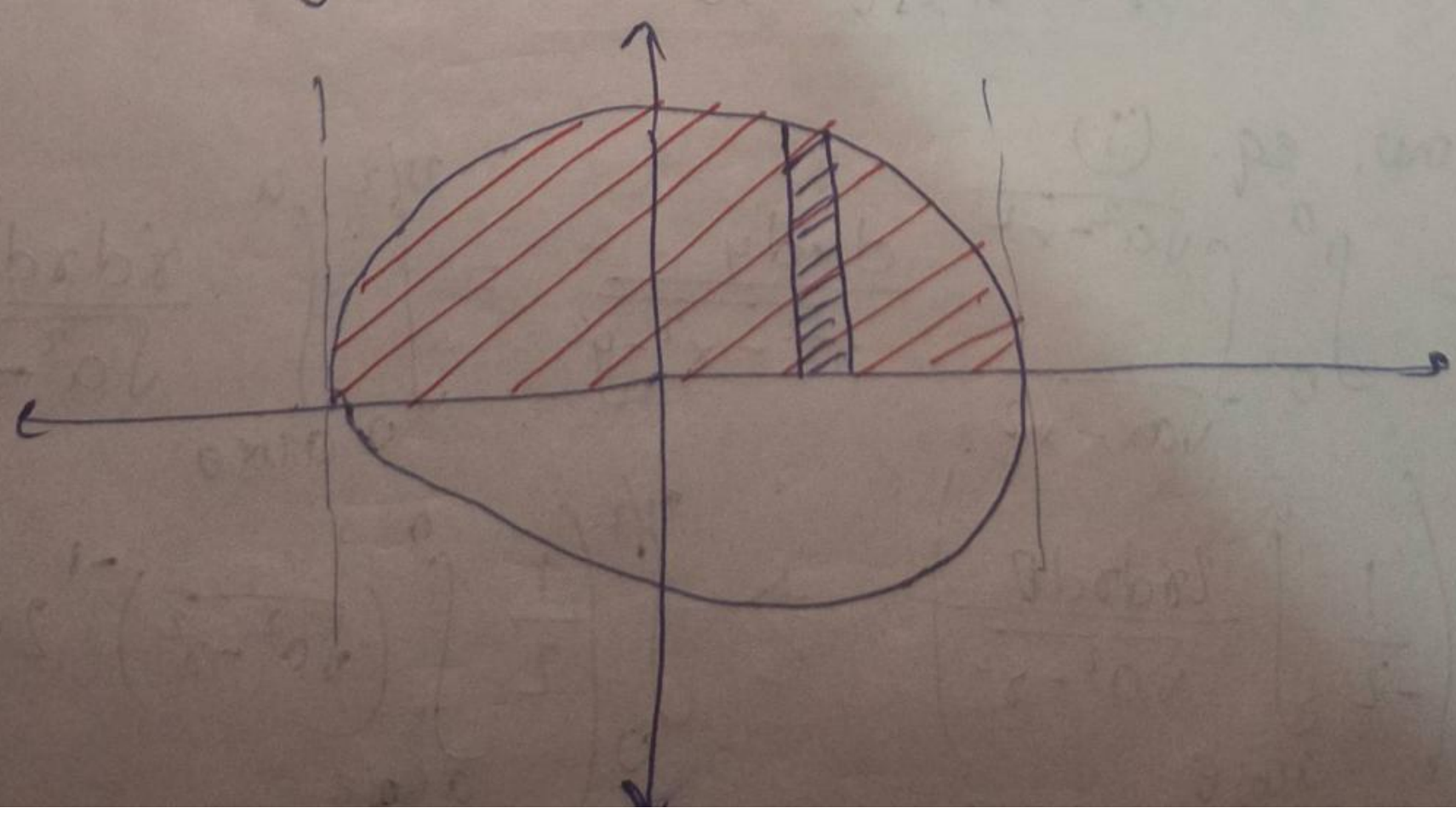
(2) $$\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\, dy\, dx$$

Sol → Let $I = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\, dy\, dx$.

On Comparing limit,

$x = -a, \quad x = a$

$y = 0, \quad y = \sqrt{a^2-x^2} \Rightarrow x^2+y^2 = a^2$



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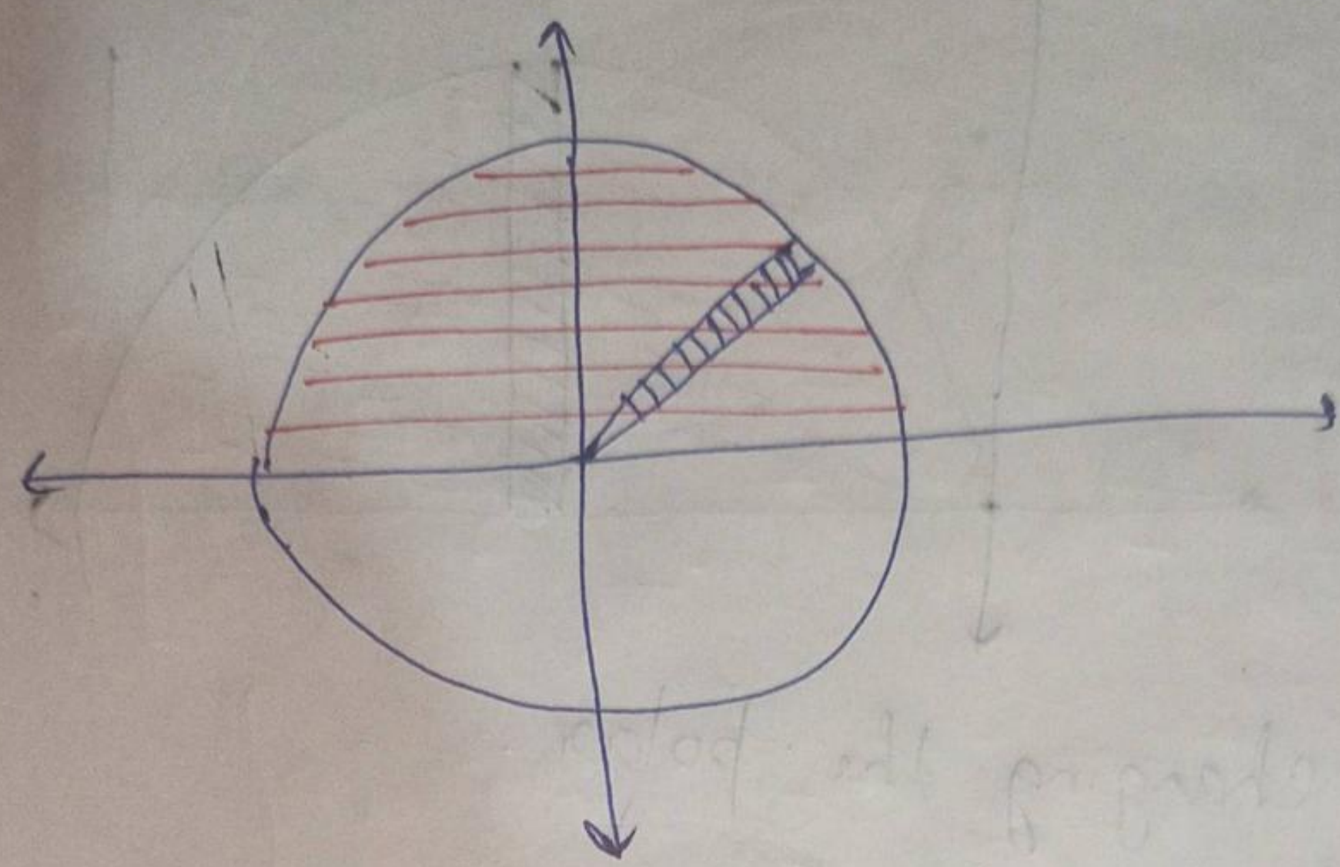
On changing to polar



We have, $\theta = 0$ to $\frac{\pi}{2}$

$r = 0$ to a

Now, eq. (i)

$$I = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)\, dy\, dx = \int_{0}^{\pi/2} \int_{0}^{a} r^2\, r\, dr\, d\theta$$

$$= \int_{0}^{\pi/2} \left[\frac{r^4}{4}\right]_0^a d\theta = \int_{0}^{\pi/2} \frac{a^4}{4}\, d\theta$$

$$= \frac{a^4}{4} \left|\theta\right|_0^{\pi/2}$$

$$= \frac{a^4}{4}\left(\frac{\pi}{2} - 0\right) = \frac{a^4 \pi}{8} \text{ Ans}$$

(3) Let $I = \int_{0}^{2a} \int_{0}^{\sqrt{2ax-x^2}} (x^2+y^2)\, dy\, dx.$

On comparing limits.

$y = 0, \quad y = \sqrt{2ax-x^2} \Rightarrow (x-a)^2 + y^2 = a^2$

$x = 0, \quad x = 2a$

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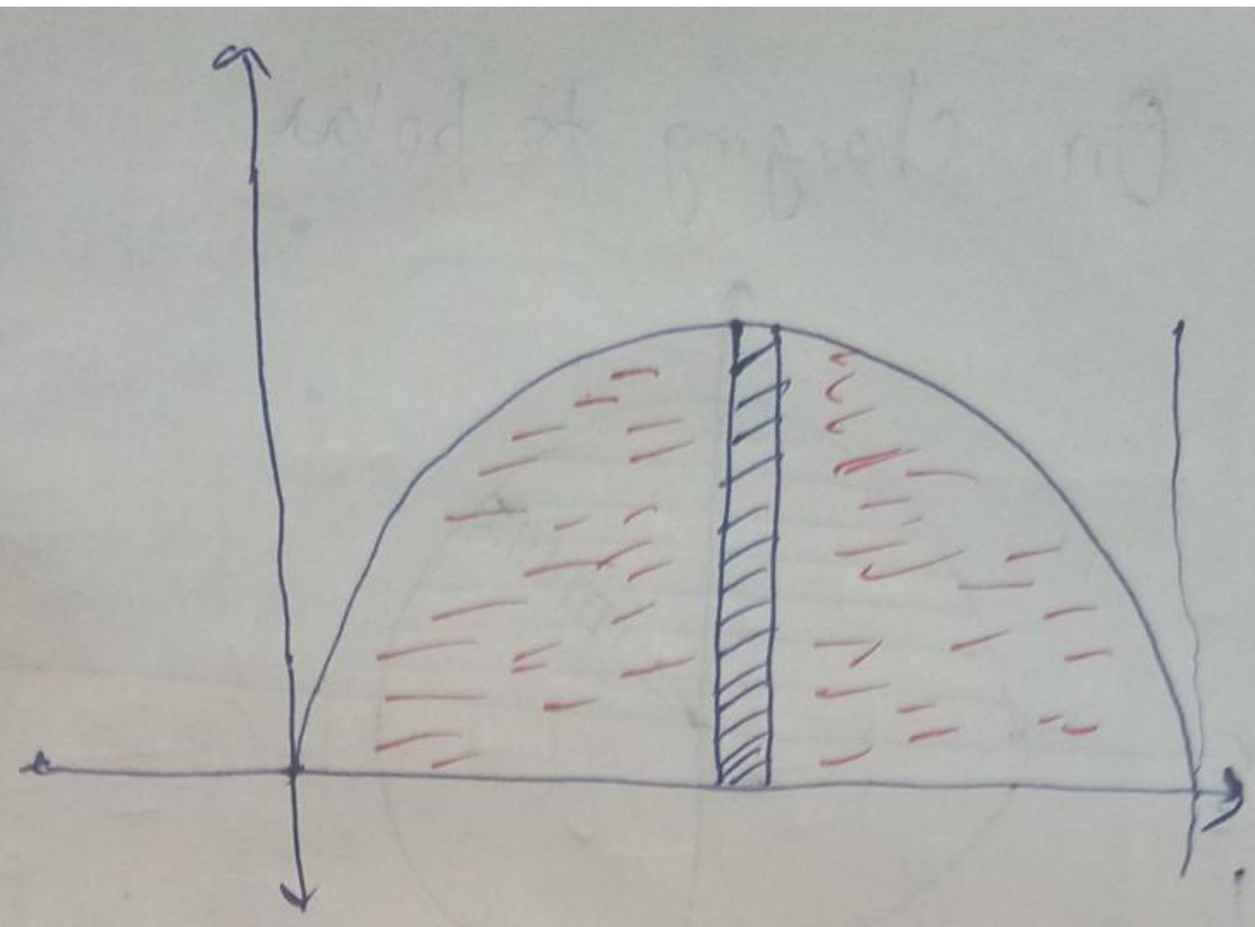
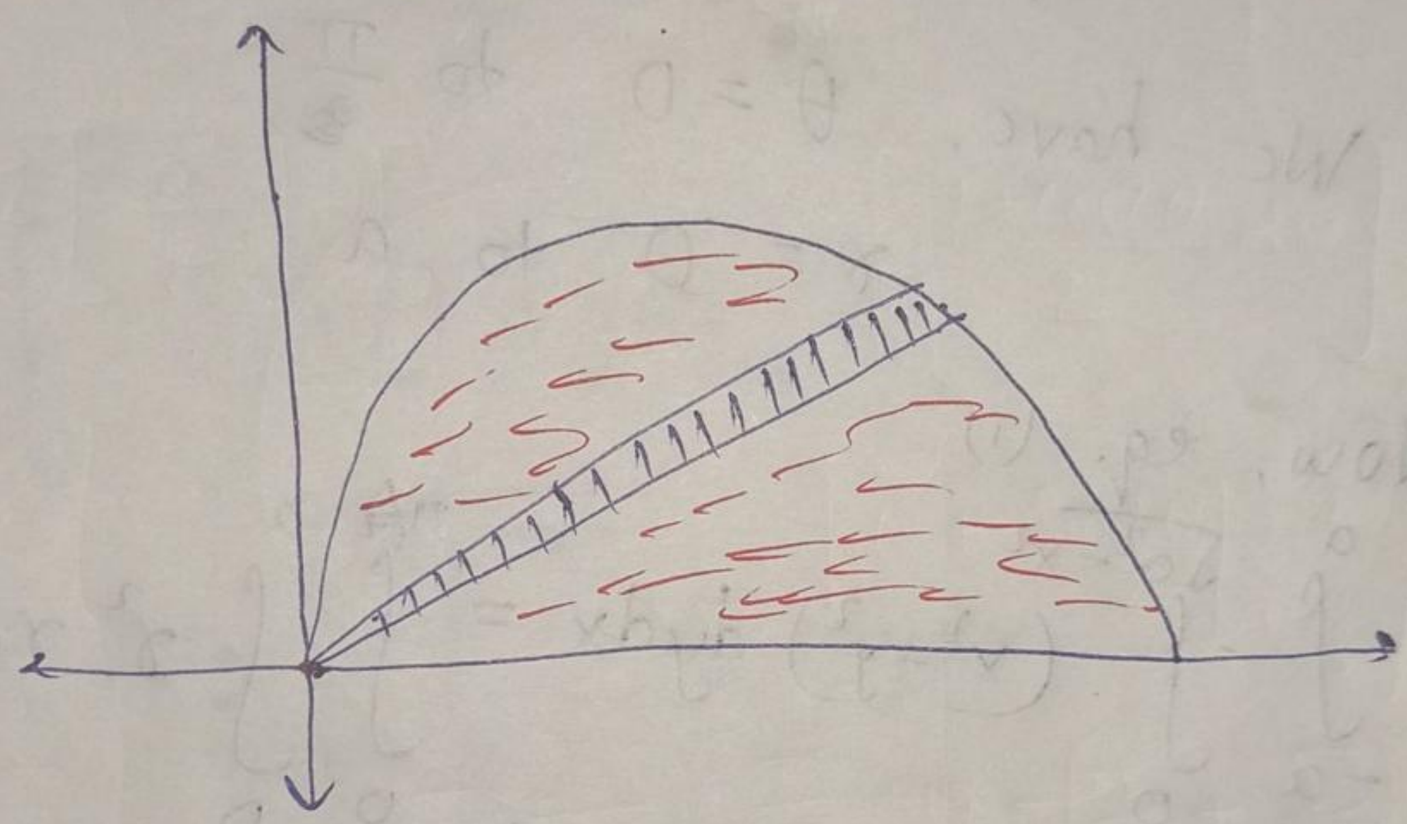


On changing the polar.



$r = 0$ to $2a\cos\theta$

$\theta = 0$ to $\pi/2$

Now, eq. (i)

$$I = \int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\, r\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} = \int_0^{\pi/2} (2a)^4 \frac{\cos^4\theta}{4}\,d\theta$$

$$= \frac{16a^4}{4}\int_0^{\pi/2} \cos^4\theta\,d\theta$$

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$$= \frac{16a^4}{4}\int_0^{\pi/2} \cos^4\theta \, d\theta$$

By Beta-Gamma function. $\int_0^{\pi/2} \cos^m\theta \sin^n\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$

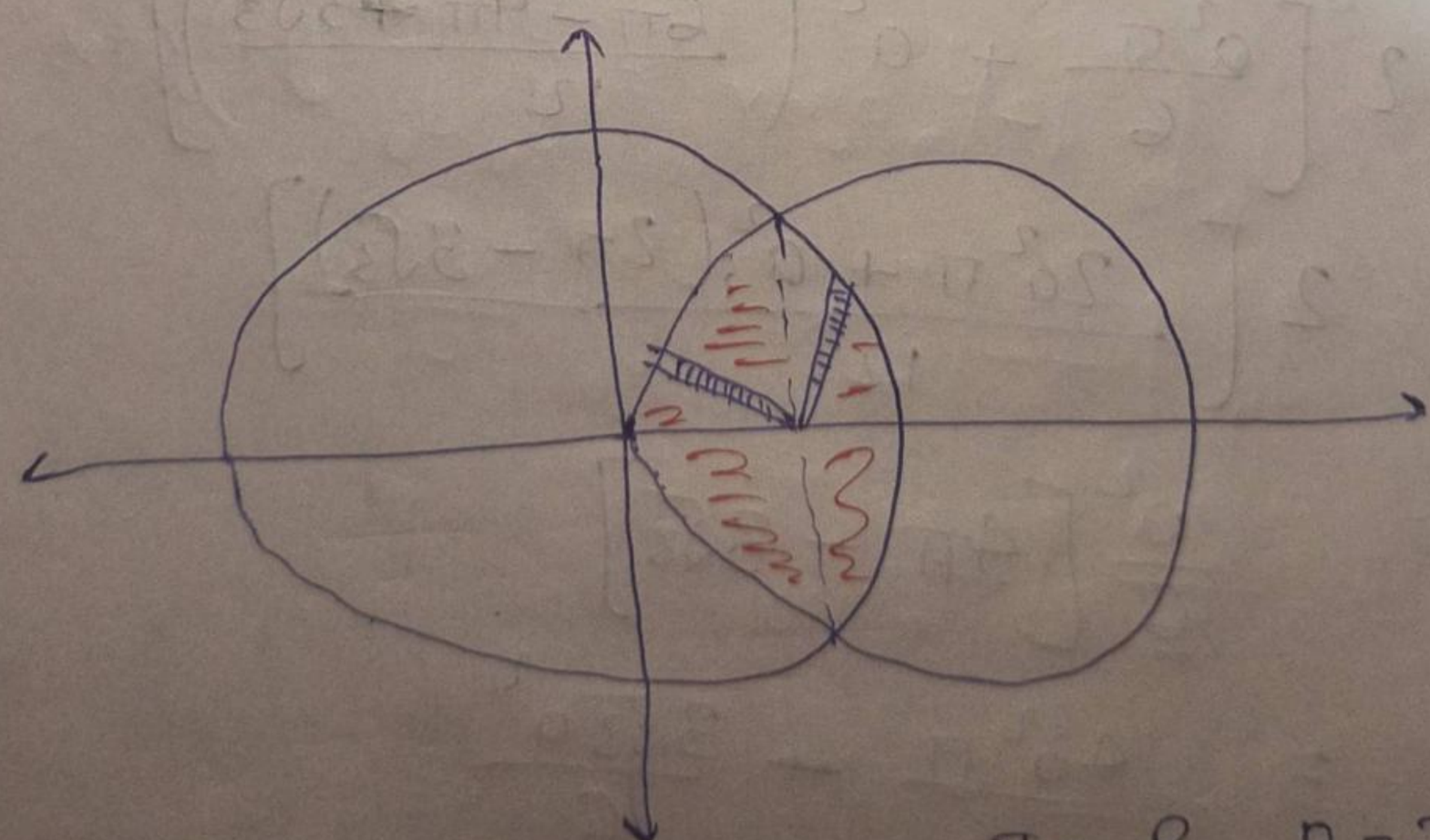
$$\Rightarrow 4a^4\int_0^{\pi/2} \cos^4\theta \, d\theta = 4a^4 \cdot \frac{\Gamma\frac{5}{2}\cdot\Gamma\frac{1}{2}}{2\,\Gamma\frac{6}{2}}$$

$$= 4a^4 \cdot \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\sqrt{\pi}}{2\,\Gamma 3}$$

$$= 4a^4 \cdot \frac{3\pi}{2\cdot 4\,\Gamma 3}$$

$$= \frac{3a^4\pi}{4}$$

Q $r = a$ & $r = 2a\cos\theta$



In R_1 $\theta = 0$ to $\pi/3$.
$r = 0$ to a

In R_2, $\theta = \pi/3$ to $\pi/2$
$r = 0$ to $2a\cos\theta$

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$$\text{Area} = 2\left[\iint_{R_2} r\,dr\,d\theta + \iint_{R_2} r\,dr\,d\theta\right]$$

$$= 2\left[\int_0^{\pi/3}\int_0^{a} r\,dr\,d\theta + \int_{\pi/3}^{\pi/2}\int_0^{2a\cos\theta} r\,dr\,d\theta\right]$$

$$= 2\left[\int_0^{\pi/3}\left|\frac{r^2}{2}\right|_0^{a} d\theta + \int_{\pi/3}^{\pi/2}\left|\frac{r^2}{2}\right|_0^{2a\cos\theta}\right]$$

$$= 2\left[\frac{1}{2}\int_0^{\pi/3} a^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} 4a^2\cos^2\theta\right]$$

$$= 2\left[\frac{a^2}{2}\left|\theta\right|_0^{\pi/3} + \frac{4a^2}{2}\int_{\pi/3}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta\right]$$

$$= 2\left[\frac{a^2}{2}\left(\frac{\pi}{3}-0\right) + \frac{2a^2}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_{\pi/3}^{\pi/2}\right]$$

$$= 2\left[\frac{a^2\pi}{6} + \frac{2a^2}{2}\left\{\frac{\pi}{2}+0-\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)\right\}\right]$$

$$= 2\left[\frac{a^2\pi}{6} + a^2\left(\frac{6\pi-4\pi-3\sqrt{3}}{12}\right)\right]$$

$$= 2\left[\frac{2a^2\pi + a^2(2\pi-3\sqrt{3})}{12}\right]$$

$$= \frac{a^2}{6}\left[4\pi - 3\sqrt{3}\right]$$

$$= \frac{4a^2\pi}{6} - \frac{3\sqrt{3}a^2}{6}$$

$$= \frac{2a^2\pi}{3} - \frac{\sqrt{3}}{2}a^2 \quad \text{Ans}$$

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Volume $= \iiint dx\,dy\,dz$.

Mass = density × volume

$= \iiint \rho \cdot dx\,dy\,dz$,

when $\rho = \underbrace{Kxy}_{\text{constant}}$

Mass $= \iiint Kxy\,dx\,dy\,dz$.

Q Solve $\int_0^1 \int_0^2 \int_0^3 dx\,dy\,dz$

Sol→ $\int_0^1 \int_0^2 \int_0^3 dx\,dy\,dz$

$= \int_0^1 \int_0^2 |x|_0^3\,dy\,dz$

$= \int_0^1 \int_0^2 (3-0)\,dy\,dz$

$= 3\int_0^1 |y|_0^2\,dz$

$= 3\int_0^1 (2-0)\,dz$

$= 6\int_0^1 dz = 6|z|_0^1 = 6(1-0)$

$= 6$ Ans

Q. Solve $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx$

Sol→ $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dz$

$= \int_0^1 \int_0^{1-x} |z|_0^{1-x-y}\,dy\,dz = \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$

$= \int_0^1 \left|y - xy - \frac{y^2}{2}\right|_0^{1-x} dx = \int_0^1 (1-x) - x(1-x) - \frac{(1-x)^2}{2}\,dx$

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$$\int_0^1 \left[(1-x)(1-x) - \frac{(1-x)^2}{2}\right] dx$$

$$= \int_0^1 \frac{(1-x)^2}{2}\, dx$$

$$= \frac{1}{2}\int_0^1 (1+x^2-2x)\, dx$$

$$= \frac{1}{2}\left|x + \frac{x^3}{3} - \frac{2x^2}{2}\right|_0^1$$

$$= \frac{1}{2}\left[1+\frac{1}{3}-1\right]$$

$$= \frac{1}{2}\times\frac{1}{3} = \frac{1}{6} \text{ Ans}$$

Q $\int_0^a \int_0^x \int_0^y dz\, dy\, dx$

Sol→ $\int_0^a \int_0^x \int_0^y dz\, dy\, dx$

$$= \int_0^a \int_0^x |z|_0^y\, dy\, dx = \int_0^a \int_0^x y\, dy\, dx$$

$$= \int_0^a \left|\frac{y^2}{2}\right|_0^x dx = \int_0^a \frac{x^2}{2}\, dx$$

$$= \frac{1}{2}\left|\frac{x^3}{6}\right|_0^a$$

$$= \left(\frac{a^3}{6} - 0\right)$$

$$= \frac{a^3}{6} \text{ Ans}$$

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Q Solve $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$.

Sol→ $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2)-z^2}}$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left| \sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}} \right|_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\sin^{-1} 1 - \sin^{-1} 0 \right) dy\,dx$$

$$= \frac{\pi}{2} \int_0^1 \left| y \right|_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2} \left| \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right|_0^1$$

$$= \frac{\pi}{2} \left[0 + \frac{1}{2} \sin^{-1} 1 \right]$$

$$= \frac{\pi}{2} \left[\frac{1}{2} \cdot \frac{\pi}{2} \right]$$

$$= \frac{\pi^2}{8}$$ Ans

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Q Find the volume of the tetrahedron by the coordinate plane and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Sol → We have.

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{--- (1)}$$

$$\frac{z}{c}=1-\frac{x}{a}-\frac{y}{b}$$

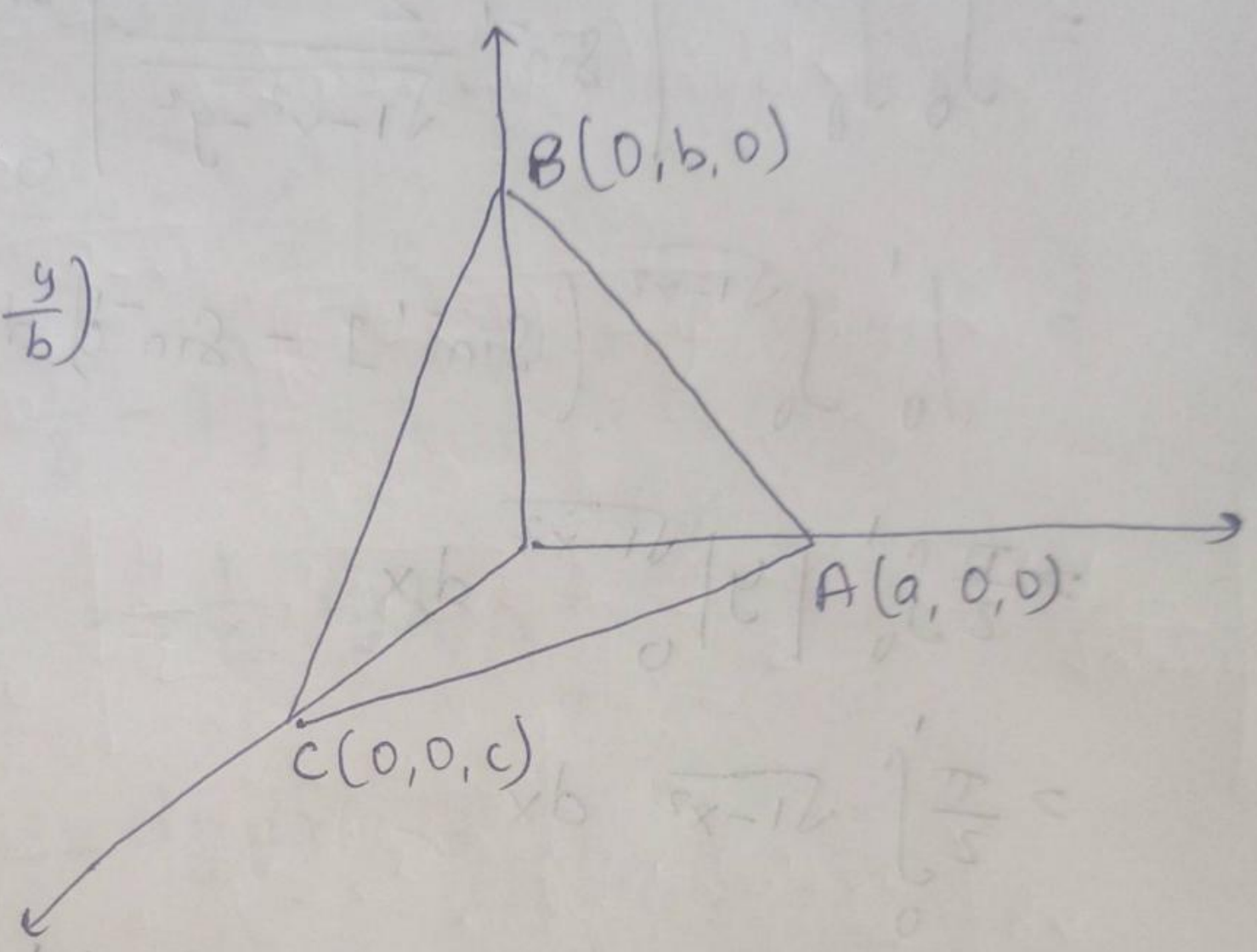
$$\Rightarrow z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

$$\frac{y}{b}=1-\frac{x}{a}$$

$$\Rightarrow y=b\left(1-\frac{x}{a}\right)$$

$$\frac{x}{a}=1$$

$$\Rightarrow x=a$$



Hence $x=0$ to a.
$y=0$ to $b\left(1-\frac{x}{a}\right)$
$z=0$ to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$$\text{Volume}=\iiint dz\,dy\,dx$$

$$=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx$$

$$=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$=c\int_0^a \left|\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right|_0^{b\left(1-\frac{x}{a}\right)} dx$$

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$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)\cdot b\left(1-\frac{x}{a}\right) - \frac{b^2\left(1-\frac{x}{a}\right)^2}{2b}\right] dx$$

$$= bc\int_0^a \left(1-\frac{x}{a}\right)^2\left(1-\frac{1}{2}\right) dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3\cdot\frac{-1}{a}}\right]_0^a$$

$$= \frac{abc}{6}\left[\frac{0-1}{-1}\right]$$

$$= \frac{abc}{6} \quad \text{Ans}$$

MCQ QUES. Q find the volume of for $x+y+z=1$.

Q find the volume for $\frac{x}{2}+\frac{y}{1}+\frac{z}{3}=1$

Q find the volume for $2x+3y+6z=6$.

Q find the volume of $x^2+y^2+z^2=a^2$.

Sol → We have $x^2+y^2+z^2=a^2$. --- (i)

$$z^2 = a^2-x^2-y^2$$

$$z = \pm\sqrt{a^2-x^2-y^2}$$

$$y^2 = a^2-x^2$$

$$y = \pm\sqrt{a^2-x^2}$$

$$z^2 = a^2$$

$$z = \pm a$$

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$$\text{Volume} = \int_{-a}^{+a}\int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{+\sqrt{a^2-x^2-y^2}} dz\,dy\,dx.$$

$$\Rightarrow \text{Volume} = 8\cdot\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8\cdot\int_0^a\int_0^{\sqrt{a^2-x^2}}\left(\sqrt{a^2-x^2-y^2}\right)dy\,dx.$$

$$= 8\cdot\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{\left(\sqrt{a^2-x^2}\right)^2-y^2}\;dy\,dx.$$

$$= 8\cdot\int_0^a\left|\frac{y}{2}\sqrt{a^2-y^2-x^2}+\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right|_0^{\sqrt{a^2-x^2}}dx$$

$$= 8\cdot\int_0^a\left(0+\frac{a^2-x^2}{2}\cdot\sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right)dx$$

$$= 8\cdot\int_0^a\frac{a^2-x^2}{2}\cdot\frac{\pi}{2}\,dx$$

$$= 8\cdot\frac{\pi}{4}\int_0^a\left(a^2-x^2\right)dx$$

$$= 2\pi\left|a^2x-\frac{x^3}{3}\right|_0^a$$

$$= 2\pi\left(a^3-\frac{a^3}{3}\right)$$

$$= \frac{4\pi a^3}{3}\ \text{Ans}$$

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Q find the volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol→ We have $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

$$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$\Rightarrow z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$\Rightarrow y=\pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$\frac{x^2}{a^2}=1 \Rightarrow x=\pm a$$

Hence, $x=-a$ to a

$y=-b\sqrt{1-\frac{x^2}{a^2}}$ to $b\sqrt{1-\frac{x^2}{a^2}}$

$z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ to $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$$\text{Volume}=\int_{-}^{+}\int_{-}^{+}\int_{-}^{+}dz\,dy\,dx$$

$$=8\cdot\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dz\,dy\,dx.$$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$=8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\frac{1}{b}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\,dy\,dx$$

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$$\Rightarrow \frac{8c}{b}\int_0^a \left| \frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2} + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}} \right|_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{8c}{2b}\int_0^a \left[0 + b^2\left(1-\frac{x^2}{a^2}\right)\left(\sin^{-1}1 - \sin^{-1}0\right)\right]dx$$

$$= \frac{4c}{b}\cdot\frac{\pi}{2}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$= \frac{2\pi b^2 c}{b}\int_0^a \left(1-\frac{x^2}{a^2}\right)dx$$

$$= \frac{2\pi b^2 c}{b}\left| x - \frac{x^3}{3a^2}\right|_0^a$$

$$= 2\pi bc\left(a - \frac{a^3}{3a^2}\right)$$

$$= \frac{2\pi bc \cdot 2a}{3}$$

$$= \frac{4\pi abc}{3} \quad \text{Ans}$$

MCQ QUES :- 1. find the volume of $\frac{x^2}{2}+\frac{y^2}{3}+\frac{z^2}{4}=1$

2. find the volume of $\frac{x^2}{4}+\frac{y^2}{1}+\frac{z^2}{9}=1$

3. find the volume of $2x^2+3y^2+6z^2=6$.

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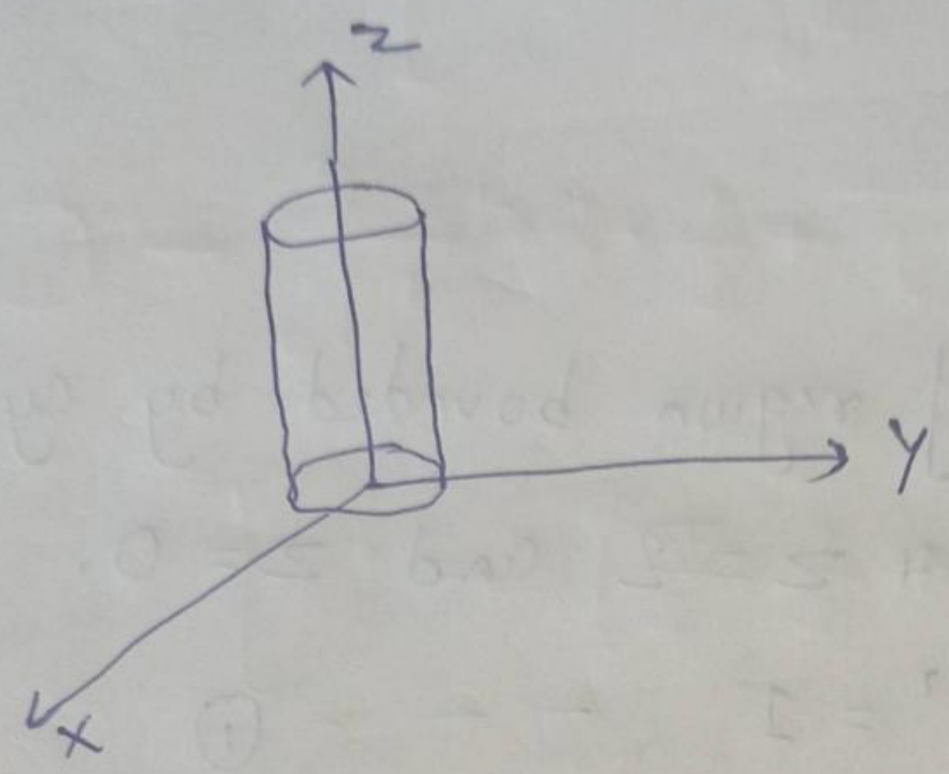
Q find the volume bonded by the cylinder $x^2+y^2=4$ and the plane $y+z=4$ and $z=0$.

Sol→ We have $x^2+y^2=4$ ---- ①

$y+z=4$, $z=0$ -- ②



from eq. ②, $z = 0$ to $4-y$.

from eq. ①, $y = -\sqrt{4-x^2}$ to $\sqrt{4-x^2}$

$x = -2$ to 2.

$$\text{Volume} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-y} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx$$

$$= \int_{-2}^{2}\left(4\times 2\int_{0}^{\sqrt{4-x^2}} dy\right)dx \quad \left[\text{By using property of Integration}\right]$$

$$= 8\int_{-2}^{2}\sqrt{4-x^2}\,dx.$$

$$= 8\times 2\int_{0}^{2}\sqrt{4-x^2}\,dx.$$

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$$= 16\left|\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right|_0^2$$

$$= 16\left[0 + \frac{4}{2}\cdot\frac{\pi}{2}\right]$$

$$= 16\pi \text{ Ans}$$

Q find the volume of region bounded by cylinder $x^2+y^2=1$. and $x+z=2$ and $z=0$.

Sol→ We have $x^2+y^2=1$ --- (1)

$x+z=2,\ z=0$ --- (2)

from eq. (2) $z=0$ to $2-x$

from eq. (1) $x=-\sqrt{1-y^2}$ to $\sqrt{1-y^2}$

$y=-1$ to 1.

$$\text{Volume} = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{2-x} dz\,dx\,dy.$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (2-x)\,dx\,dy$$

$$= \int_{-1}^{1}\left(2\times 2\int_0^{\sqrt{1-y^2}} dx\right) dy.$$

$$= 4\int_{-1}^{1}\sqrt{1-y^2}\,dy = 4\times 2\int_0^1\sqrt{1-y^2}\,dy$$

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$$= 8\left|\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1}\frac{y}{1}\right|_0^1$$

$$= 8\left(0 + \frac{1}{2}\times\frac{\pi}{2}\right)$$

$$= 8\times\frac{\pi}{4}$$

$$= 2\pi \text{ Ans}$$

Q find the volume bounded by x-y plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$.

Sol→ we have $x^2+y^2=1$. —— ①

$x+y+z=3$ —— ②

from eq. ② $z=0$ to $3-x-y$.

from eq. ① $y=-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

$x=-1$ to 1

$$\text{Volume} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{3-x-y} dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3-x-y)\,dy\,dx$$

$$= \int_{-1}^{1}\left(2\cdot(3-x)\int_{0}^{\sqrt{1-x^2}} dy\right)dx.$$

$$= 2\int_{-1}^{1}(3-x)\left(\sqrt{1-x^2}\right)dx.$$

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$$= 2 \times 2 \int_0^1 (3-x)(\sqrt{1-x^2})\, dx.$$

$$= 4\left[3\int_0^1 \sqrt{1-x^2} - \int_0^1 x\sqrt{1-x^2}\right]$$

$$= 4\left[3 \cdot \left|\frac{x}{2}\sqrt{1-y^2} + \frac{1}{2} \cdot \sin^{-1}\frac{x}{1}\right|_0^1 - 0\right]$$

$$= 4\left[3 \cdot \left(0 + \frac{1}{2} \cdot \frac{\pi}{2}\right) - 0\right]$$

$$= 4\left[3 \cdot \frac{\pi}{4} - 0\right]$$

$$= \frac{12\pi}{4} = 3\pi \text{ Ans}$$

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