Maths

Unit-3 (Mathematical Logic)

Preposition (or Statement) :- It is a declarative sentence which is either true or false but not both.

These 2 values True or false is denoted by T & F respectively. Sometimes it is also represented by 1 & 0.

Sentences which are exclamatory, interrogative in nature are not preposition.

Eg.

Preposition
- New Delhi is capital of india.
- ~~New Delh~~ $9 > 10$
- Blood is green
- It is raining
- The Sum will come out tomorrow.

Not preposition
- $x + y = z$.
- How beautiful is rose?
- Who are you?
- Close the door.

Truth value of a statement :- A preposition can have one and only one of 2 possible value namely True or False. These 2 values are called as truth value of a statement.

Connectives :- The words or symbols which are used to make a sentence by 2 sentences are called logical connectives.

Eg. Ram is boy.
Shayam is boy.
⇒ Ram and Shayam are boys.
↑
Connective

Connective	Word	Symbol
Negation	NOT	$\sim$ or $\neg$
Conjuaction	AND	$\wedge$
Disjuction	OR	$\vee$
Conditional (Implication)	IF .. THEN	$\longrightarrow$
BiConditional	If and only If	$\longleftrightarrow$

1. Negation (NOT) :- Negation of P is False if P is true and vice-versa. Denoted by $\sim$ & $\neg$.

e.g. Ram is boy.
Ram is not a boy.

p	$\sim p$
T	F
F	T

2. ~~Di~~ Conjunction (AND) :- If p and q are any two proposition then $p \wedge q$ are called Conjunction of p and q.

p	q	$p \wedge q$	
T	T	T	$1 \cdot 1 = 1$
T	F	F	$1 \cdot 0 = 0$
F	T	F	$0 \cdot 1 = 0$
F	F	F	$0 \cdot 0 = 0$

3. Disjunction (OR) :- If p and q are any two proposition then $p \vee q$ are ~~conjunc~~ disjunction of p & q.

p	q	$p \vee q$		
T	T	~~1~~	T	$1+1=1$
T	F	~~1~~	T	$1+0=1$
F	T	~~1~~	T	$0+1=1$
F	F	~~0~~	F	$0+0=0$

4. Conditional (IF -- THEN) :- If p & q are two proposition then "If p then q" is denoted by $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

5. Bi-Conditional (If and only If) :- If p and q are two proposition then the compound proposition "p If and only If q" is written as $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

* Truth Table :- A Table that shows the truth value of Compound proposition for all possible cases.

Q. Construct truth table of (i) $p \wedge \sim p$ (ii) $p \vee \sim q$ (iii) $(p \vee q) \vee \sim p$. (iv) $(p \vee q) \rightarrow (p \wedge q)$

Sol →

p	q	$\sim p$	$\sim q$	$p \wedge \sim p$	$p \vee \sim q$	$p \vee q$	$(p \vee q) \vee \sim p$
T	T	F	F	F	T	T	T
T	F	F	T	F	T	T	T
F	T	T	F	F	F	T	T
F	F	T	T	F	T	F	T

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Tautology :- A proposition which is True for all possible Truth value of its propositional variable is called a tautology.

Contradiction → If it contains only false value.

- Contigency → Neither Tautology Nor Contradiction

Q Show that $[p \wedge (p \to q)] \to q$ is a tautology.

Sol →

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

⇓

All true

⇓

tautology.

Question for Practice :-

1) Show that $(p \vee q) \vee \sim p$ is tautology.
2) Show that $p \wedge \sim p$ is Contradiction.
3) Construct truth table for $(p \vee q) \wedge r$.
4) Construct truth table for $(p \vee q) \to (p \wedge q)$.

Equivalence of preposition :-

$A \equiv B$ or $A \Leftrightarrow B$
if Both A & B have same truth value.

Q. Show that $\neg(p \rightarrow q) \equiv p \wedge \neg q$.

Sol →

~~p~~	~~q~~	~~$p \rightarrow q$~~	~~$\neg(p \rightarrow q)$~~	~~$\sim q$~~
~~T~~	~~T~~	~~T~~	~~0 F~~	~~1 T~~
~~T~~	~~F~~	~~T~~	~~0 F~~	~~0 F~~
~~F~~	~~T~~	~~F~~	~~1 T~~	~~T~~
~~F~~	~~F~~	~~T~~	~~0 F~~	~~F~~

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\sim q$	$p \wedge \sim q$
F	F	T	F	T	F
F	T	T	F	F	F
T	F	F	T	T	T
T	T	T	F	F	F

same

$\Rightarrow \neg(p \rightarrow q) \equiv p \wedge \neg q$.

Duality Law :- The two statement is said said to be dual of each other if either one can be obtained from the other by replacing.

$\wedge$ by $\vee$, $\vee$ by $\wedge$, T by F, F by T.

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Q Find dual of (i) $p \vee q \wedge r$ (ii) $p \wedge q \vee T$
(iii) $\neg(p \vee q) \wedge (p \vee \sim(p \wedge s))$.

Sol → i) $p \vee q \wedge r \Rightarrow p \wedge q \vee r$
ii) $p \wedge q \vee T \Rightarrow p \vee q \wedge F$
iii) $\neg(p \vee q) \wedge (p \vee \sim(p \wedge s)) \Rightarrow \neg(p \wedge q) \vee (p \wedge \sim(p \vee s))$

Algebra of proposition :-

S.No.	Name of Law	Primal form	Dual form
1.	Commutative Law	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
2.	Associative Law	$p \vee (q \vee r) \equiv (p \vee q) \vee r$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
3.	Distributive Law	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
4.	Idempotent Law	$p \vee p \equiv p$	$p \wedge p \equiv p$.
5.	Identity Law	$p \vee F \equiv p$.	$p \wedge T \equiv p$
6.	Dominant Law	$p \vee T \equiv T$	$p \wedge F \equiv F$
7.	Complement Law	$p \vee \sim p \equiv T$	$p \wedge \sim p \equiv F$
8.	Absorption Law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
9.	De Morgan's Law	$\sim(p \vee q) \equiv \sim p \wedge \sim q$	$\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Equivalence Using Conditionals :-

1) $p \to q \equiv \sim p \vee q$
2) $p \to q \equiv \sim q \to \sim p$.
3) $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
4) $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
5) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
6) $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$.

Equivalence using Bi Conditional :-

1) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
2) $p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$.

Q. Without truth table Prove that

$(p \vee q) \vee \sim p \equiv T$

Sol → $(p \vee q) \vee \sim p$

$\sim p \vee (p \vee q)$ [Using Commulative Law]

$(\sim p \vee p) \vee q$ [Using Associative Law]

$T \vee q$ [Using Complement Law]

T [Using Dominant Law]

Q. $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \equiv p \wedge q$

Sol → $(\neg p \vee q) \wedge (p \wedge (p \wedge q))$

$(\neg p \vee q) \wedge ((p \wedge p) \wedge q)$ [Associative Law]

$(\neg p \vee q) \wedge (p \wedge q)$ [Idempotent Law]

$((\neg p \vee q) \wedge p) \wedge q$ [Associative Law]

$(p \wedge (\neg p \vee q)) \wedge q$ [Commutative Law]

$((p \wedge \neg p) \vee (p \wedge q)) \wedge q$ [Distributive Law]

$(F \vee (p \wedge q)) \wedge q$ [Complement Law]

$(p \wedge q) \wedge q$ [Identity Law]

$p \wedge (q \wedge q)$ [Associative Law]

$p \wedge q$ [Idempotent Law]

$p \wedge q$ = RHS

Hence proved

Q. $p \rightarrow (q \rightarrow p) \equiv \neg p \rightarrow (p \rightarrow q)$.

Sol → LHS: $p \rightarrow (q \rightarrow p)$

$p \rightarrow (\sim q \vee p)$.

$\sim p \vee (\sim q \vee p)$

$= (\sim q \vee p) \vee \sim p$ [Commutative Law]

$= \sim q \vee (p \vee \sim p)$ [Associative Law]

$= \sim q \vee T$ [Complement Law]

$= T$ [Dominant Law]

RHS :- $7p \rightarrow (p \rightarrow q)$

$= \sim p \rightarrow (\sim p \vee q)$

$= \sim(\sim p) \vee (\sim p \vee q)$

$= p \vee (\sim p \vee q)$

$= (p \vee \sim p) \vee q$ [Associative Law]

$= T \vee q$ [Complement Law]

$= T$

LHS = RHS

Q $p \rightarrow (q \rightarrow r) \equiv p \rightarrow (7q \vee r) \equiv (p \wedge q) \rightarrow r$.

Sol → $p \rightarrow (q \rightarrow r)$

$\Rightarrow p \rightarrow (\sim q \vee r)$

$\Rightarrow \sim p \vee (\sim q \vee r)$

$\Rightarrow \sim p \vee \sim q \vee r$

~~$p \to (\neg q \vee r)$~~

~~$= p \to (\sim(\sim q)$~~

$p \to (\neg q \vee r)$

$= p \to (\sim q \vee r)$

$= \sim p \vee (\sim q \vee r)$

$= \sim p \vee \sim q \vee r$

$(p \wedge q) \to r$

$\Rightarrow \sim(p \wedge q) \vee r$

$= \sim p \vee \sim q \vee r$

So, since all 3 resulted in same value. So,

$p \to (q \to r) \equiv p \to (\neg q \vee r) \equiv (p \wedge q) \to r$.

Q Prove the following equivalence by proving the Equivalence of the duals.

$\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) \equiv p$.

Sol $\to$ $\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) \equiv p$.

$\Rightarrow \sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$.

Dual of it :-

$\sim((\sim p \vee q) \wedge (\sim p \vee \sim q)) \wedge (p \vee q) \equiv p$

$\sim((\sim p \vee (q \wedge \sim q))) \wedge (p \vee q)$. [Distributive Law]

$\Rightarrow \sim((\sim p \vee F)) \wedge (p \vee q)$ [Complement Law]

$\Rightarrow \sim(\sim p) \wedge (p \vee q)$ [Idempotent Law]

$= p \wedge (p \vee q)$.

$= p$. [Absorption Law].

$=$ RHS

Hence proved

Tautological Implication :-

A compound proposition A is said to be tautologically imply the compound proposition if B is true whenever A is T.

or. $A \rightarrow B$ is tautology.

i.e $A \Rightarrow B$

Q. Prove that $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \Rightarrow r$

Sol $\rightarrow$ we need to prove

$(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \rightarrow r \equiv T$

$(p \vee q) \wedge (\sim p \vee r) \wedge (\sim q \vee r) \rightarrow r$

$= (p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \rightarrow r$

$= (p \vee q) \wedge [(p \vee q) \rightarrow r] \rightarrow r$

$= (p \vee q) \wedge [\sim(p \vee q) \vee r] \rightarrow r$

$\Rightarrow [((p \vee q) \wedge \sim(p \vee q)) \vee ((p \vee q) \wedge r)] \rightarrow r$ [Distributive Law]

$\Rightarrow [F \vee (p \vee q) \wedge r] \rightarrow r$ [Complement Law]

$\Rightarrow [(p \vee q) \wedge r] \rightarrow r$ [Identity Law]

$\Rightarrow \sim[(p \vee q) \wedge r] \vee r$

$\Rightarrow \sim(p \vee q) \vee (\sim r) \vee r$ [Demorgan's Law].

$\Rightarrow \sim(p \vee q) \vee (T)$. [Complement]

[dominant Law]

$= T$

Mathematical Induction

Q. Prove that $1^2 + 3^2 + 5^2 + \cdots (2n-1)^2 = \frac{1}{3}(n)(2n-1)(2n+1)$

Sol→ Let $S_n = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3} n (2n-1)(2n+1)$. --- (1)

Step 1 :- We shall prove it is true for $n = 1$

LHS $S_1 = (1)^2 = 1$

RHS $= \frac{1}{3}(1)(2 \cdot 1 - 1)(2 \cdot 1 + 1) = 1$

LHS = RHS

$\Rightarrow$ It is true for $n = 1$

Step 2 :- Now we assume that it is true for $n = k$.

$\Rightarrow S_k = 1^2 + 3^2 + 5^2 + \cdots (2k-1)^2 = \frac{k}{3}(2k-1)(2k+1)$ --- ②

Step 3 :- We shall prove that it is true for $k+1$.

$S_{k+1} = 1^2 + 3^2 + 5^2 + \cdots (2k-1)^2 + (2k+1)^2 = \frac{k+1}{3}(2k+3)(2k+1)$

LHS :- $1^2 + 3^2 + 5^2 + \cdots (2k-1)^2 + (2k+1)^2$

$= \frac{k}{3}(2k-1)(2k+1) + (2k+1)^2$

$= (2k+1)\left[\frac{k(2k-1)}{3} + (2k+1)\right]$

$= (2k+1)\frac{(2k^2 - k + 6k + 3)}{3}$

$= \frac{(2k+1)(2k^2+5k+3)}{3}$

$= \frac{(2k+1)(k+1)(2k+3)}{3} = \text{RHS}$

$\Rightarrow S_{k+1}$ is true.

So, By Mathematical Induction S_n is true for all n.

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Q

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\frac{1}{n(n+1)}=\frac{n}{n+1}$$

$\Rightarrow$ Try yourself. Same method as above.

Q. Prove that n^3+2n is divisible by 3.

Sol $\rightarrow$ for $n=1$:-

$1^3+2\cdot 1 = 3$

It is divisible by 3

$\Rightarrow S_n$ is true for $n=1$.

for $n=k$:

we assume k^3+2k is divisible by 3.

for $n=k+1$:-

$(k+1)^3+2(k+1)$

$= k^3+1+3k^2+3k+2k+2$

$= k^3+2k+3k^2+3k+3$

$= k^3+2k+3(k^2+k+1)$

$=$ k^3+2k is divisible by 3 and $3(k^2+k+1)$ is divisible by 3.

So, $k^3+2k+3(k^2+k+1)$ is divisible by 3

$\Rightarrow$ ~~k+1~~ True for $k+1$.

$\Rightarrow S_n$ is true for all n.

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Q Prove that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Sol → for $n = 1$.

$1^3 + (1+1)^3 + (1+2)^3 = 36 \Rightarrow$ divisible by 9.

$\Rightarrow S_n$ is true for $n = 1$.

for $n = k$:- We assume it is true for $n = k$

$\Rightarrow k^3 + (k+1)^3 + (k+2)^3$ is divisible by 9.

for $n = k+1$:-

$(k+1)^3 + (k+2)^3 + (k+3)^3$.

$\Rightarrow (k+1)^3 + (k+2)^3 + k^3 + 3^3 + 27k + 9k^2$

$\Rightarrow \underbrace{k^3 + (k+1)^3 + (k+2)^3}_{I} + \underbrace{9(3 + 3k + k^2)}_{II}$

$\Rightarrow$ I is divisible by 9. II is factor of 9 so divisible by 9.

$\Rightarrow (k+1)^3 + (k+2)^3 + (k+3)^3$ is divisible by

$\Rightarrow S_n$ is true for $n = k+1$.

$\Rightarrow S_n$ is true for all n.

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Theory of Inference :-

p_1
p_2
p_3
$\vdots$
p_n } → premise

$\overline{\quad C \quad}$ ← Conclusion

An argument of sequence of statement is called premises.

An argument is said to be logically valid argument iff the conjunction of the premises implies the conclusion. i.e If all the premises are true, the conclusion must also be true.

$(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \rightarrow C$ is a Tautology.

Inference Theory is concerned with the inferring of a conclusion from certain hypothesis or premises by applying certain principle of reasoning; called rules of inference.

Rules of Inference :-

Rule P :- A premises may be introduced at any step in the derivation.

Rule T :- A formula S may be introduced in the derivation.

Rules of inference	Tautological form	Name of the Rule..
1.(a) $\frac{p}{\therefore p \vee q}$ (b) $\frac{q}{p \vee q}$	1.(a) $p \to p \vee q$ (b) $q \to p \vee q$	Addition
2.(a) $\frac{p \wedge q}{\therefore p}$ (b) $\frac{p \wedge q}{\therefore q}$	2. (a) $p \wedge q \to p$ (b) $p \wedge q \to q$	Simplification
3. $\frac{\begin{matrix} p \\ q \end{matrix}}{p \wedge q}$	3. $(p) \wedge (q) \to p \wedge q$	Conjunction
4. $\frac{\begin{matrix} p \to q \\ p \end{matrix}}{\therefore q}$	4. $[(p \to q) \wedge p] \to q$	modus ponens
5. $\frac{\begin{matrix} p \to q \\ \neg q \end{matrix}}{\neg p}$	5. $[(p \to q) \wedge \neg q] \to \neg p$	modus tollens
6. $\frac{\begin{matrix} p \to q \\ q \to r \end{matrix}}{\therefore p \to r}$	6. $[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical Syllogism
7. $\frac{\begin{matrix} p \vee q \\ \neg p \end{matrix}}{q}$	7. $(p \vee q) \wedge \neg p \to q$	Disjunctive Syllogism
8. $\frac{\begin{matrix} (p \to q) \wedge (r \to s) \\ p \vee r \end{matrix}}{\therefore q \vee s}$	8. $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to q \vee s$	Constructive dilemma
9. $\frac{\begin{matrix} (p \to q) \wedge (r \to s) \\ \neg q \vee \neg s \end{matrix}}{\neg p \vee \neg r}$	9. $[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$	Destructive dilemma

(A) Direct Method :-

Q. Show that $t \wedge s$ can be derived from the premises $p \to q$, $q \to \neg r$, r, $p \vee (t \wedge s)$.

Sol →

S.No	Statement	Reason
1.	$p \to q$	Rule P.
2.	$q \to \neg r$	Rule P.
3.	$p \to \neg r$	Rule T 1, 2. Hypothetical Syllogism
4.	r	Rule P.
5.	$\neg p$	Rule T 3, 4, modus Tollens
6.	$p \vee (t \wedge s)$	Rule P
7.	$t \wedge s$	T, 5, 6, Disjunction Syllogism

Q. Show that $(p \to q) \wedge (r \to s)$, $(q \to t) \wedge (s \to u)$, $\neg(t \wedge u)$ and $(p \to r) \Rightarrow \neg p$.

Sol →

S.No	Statement	Reason
1.	$(p \to q) \wedge (r \to s)$	Rule P
2.	$(p \to q)$	T, 1, simplification
3.	$(r \to s)$	T, 1 simplification

S.No	Statement	Reason
4	$(q \to t) \wedge (s \to u)$	P
5.	$(q \to t)$	T, 4, Simplification
6.	$(s \to u)$	T, 4 Simplification
7.	$p \to t$	T, 2, 5, Hy. Syllogism
8.	$r \to u$	T, 3, 6, Hy. Syllogism
9.	$p \to r$	P
10.	$p \to u$	T, 8, 9, Hy. Syll.
11.	$(p \to t) \wedge (p \to u)$	T, 7, 10, Conjunction.
12.	$p \to (t \wedge u)$	T, 11, $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
13.	$\neg (t \wedge u)$	P
14.	$\neg p$	T, 12, 13. modus Tollens

B. Indirect Method :-

Q. Show that b can be derived from the ~~premises~~ premises $a \to b$, $c \to b$, $d \to (a \vee c)$, d, by indirect method.

Sol → Let us include $\neg b$ as additional premises.

S.No.	Statement	Reason
1.	$a \to b$	P
2.	$c \to b$	P

S.No	Statement	Reason
3.	$(a \to b) \wedge (c \to b)$	T, 1, 2, Conjunction
4.	$(a \vee c) \to b$	T, 3, Equivalence
5.	$d \to (a \vee c)$	P.
6.	$d \to b$	T, 4, 5. Hy. Syll.
7.	d	P
8.	b	T, 6, 7 Modus Po
9.	$\neg b$	P (Additional
10.	$b \wedge \neg b$	T, 8, 9 Conjunction
11.	F	T, 10, Complement Law.

<u>InConsistent</u> :-

Same set of prises me output False aane pe Inconsistent bolte h.

Q. Prove that the premises $a \to (b \to c)$, $d \to (b \wedge \neg c)$ and $(a \wedge d)$ are inconsistent.

Sol →

S.No	Statement	Reason
1.	$(a \wedge d)$	P
2.	a	T, 1, Simplification
3.	d	T, 1, Simplification
4.	$a \to (b \to c)$	P
5.	$b \to c$	T, 2, 4, Hy. Syllo.

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6.	$d \to (b \wedge \neg c)$	P.
7.	$(b \wedge \neg c)$	T, 3, 6, Hy. Syllo
8.	$\neg(\neg b \vee c)$	T, 7, demorgan's Law,
9.	$\neg b \vee c$	T, 5, Equivalence
10.	$\neg(\neg b \vee c) \wedge (\neg b \vee c)$	T, 8, 9, Conjunction
11	F	T, 10, Complement

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