# Fourier Transform :-

Fourier Transform / Complex :-

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

Inversion Formula :-

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, dS$$

Some Important Formula :-

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

Q Find the Complex fourier transform of

$$f(x) = \begin{cases} x & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$$

Sol →

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{-a} 0 + \int_{-a}^{a} e^{isx} x\, dx + \int_{a}^{\infty} 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} x\, dx.$$

$$= \frac{1}{\sqrt{2\pi}}\left[x \cdot \frac{e^{isx}}{is} + \frac{e^{isx}}{s^2}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left\{\frac{ae^{ias}}{is} + \frac{e^{ias}}{s^2}\right\} - \left\{\frac{-ae^{-ias}}{s} + \frac{e^{-ias}}{s^2}\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{is}\left(e^{ias} + e^{-ias}\right) + \frac{1}{s^2}\left(e^{ias} - e^{-ias}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-ai}{s}(2\cos as) + \frac{1}{s^2}(2i\sin as)\right]$$

Ans

Q. Find the fourier transform of $f(x) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases}$

Also Evaluate $\int_{-\infty}^{\infty} \frac{\sin\lambda a \cos\lambda x}{\lambda} d\lambda$.

Sol→ $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0 + \int_{-a}^{a} 1 e^{isx} dx + \int_{a}^{0} 0\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left|\frac{e^{isx}}{is}\right|_{-a}^{a} = \frac{1}{\sqrt{2\pi}\, is}\left(e^{ias} + e^{-ias}\right)$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}\, is} 2i \sin as$$

$$\Rightarrow F(s) = \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}.$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s)\, ds.$$

$$\begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}\, ds.$$

$$\Rightarrow \begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases} = \frac{\sqrt{2}}{\sqrt{2\pi\pi}} \int_{-\infty}^{\infty} (\cos sx - i \sin sx) \frac{\sin as}{s}\, ds$$

On Comparing real part of LHS.

$$\begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cos sx\, ds.$$

on putting $s = \lambda$.

$$\Rightarrow \begin{cases} \pi & |x| \le a \\ 0 & |x| > a \end{cases} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \lambda a \cos \lambda x}{\lambda}\, d\lambda.$$

Q Find the fourier transform of $f(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

Also Evaluate $\int_{-\infty}^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2}\, dx.$

Sol→ $$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{-1} 0 + \int_{-1}^{1} (1 - x^2) e^{isx} dx + \int_{1}^{\infty} 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\frac{e^{isx}}{is} - (-2x)\frac{e^{isx}}{i^2s^2} + (-2)\frac{e^{isx}}{i^3s^3}\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left\{0 - \frac{2e^{isx}}{s^2} - \frac{2i}{s^3}e^{isx}\right\} - \left\{0 + \frac{2e^{-is}}{s^2} - \frac{2i}{s^3}e^{-is}\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2}{s^2}(e^{is}+e^{-is}) - \frac{2i}{s^3}(e^{is}-e^{-is})\right]$$

$$= \frac{-2}{\sqrt{2\pi}}\left[\frac{2\cos s}{s^2} - \frac{2\sin s}{s^3}\right]$$

$$= \frac{-4}{\sqrt{2\pi}}\left[\frac{s\cos s - \sin s}{s^3}\right]$$

Now, using inverse inversion formula :-

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-4}{\sqrt{2\pi}}\left(\frac{s\cos s - \sin s}{s^3}\right)e^{-isx}\,ds.$$

$$\begin{cases}1-x^2\\ 0\end{cases} = \frac{-2}{\pi}\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}e^{-isx}\,ds$$

$$\begin{cases}1-x^2 & |x|<1\\ 0 & |x|>1\end{cases} = \frac{-2}{\pi}\int_{-\infty}^{\infty}(\cos sx - i\sin sx)\frac{(s\cos s - \sin s)}{s^3}\,ds$$

On Comparing real parts.

$$\begin{cases}1-x^2 & |x|<1\\ 0 & |x|>1\end{cases} = \frac{-2}{\pi}\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\cos sx\,ds.$$

Now, put $x = \frac{1}{2}$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\cos\frac{s}{2}\,ds = \begin{cases}\left(1-\frac{1}{4}\right)\frac{\pi}{2} & |x|<1\\ 0 & |x|>1\end{cases}$$

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$$= 2\int_{-\infty}^{\infty} \frac{s\cos s - \sin s}{s^3} \cos\frac{s}{2}\, ds = -\frac{3\pi}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{s\cos s - \sin s}{s^3} \cos\frac{s}{2}\, ds = -\frac{3\pi}{8}$$

Ans

Q Show that the transform of $e^{-x^2/2}$ is $e^{-s^2/2}$ by finding fourier transform of $e^{-a^2x^2}$, $a>0$.

Sol→ $$F(e^{-a^2x^2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2 - \frac{s^2}{4a^2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-s^2/4a}\, dx$$

On putting $ax - \frac{is}{2a} = t \Rightarrow dx = \frac{dt}{a}$.

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \frac{e^{-s^2/4a}}{a} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \frac{e^{-s^2/4a}}{a} \cdot 2\int_{0}^{\infty} e^{-t^2}\, dt$$

Let $t^2 = u \Rightarrow 2t\,dt = du$

$\Rightarrow dt = \frac{du}{2t}$.

$$\Rightarrow \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \cdot 2 \times \int_0^\infty e^{-u} \times \frac{du}{2\sqrt{u}}$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \cdot \frac{2}{2} \int_0^\infty e^{-u} u^{-1/2} du$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \cdot \sqrt{\frac{1}{2}}$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \sqrt{\pi}$$

$$F(e^{-a^2x^2}) = \frac{e^{-s^2/4a^2}}{\sqrt{2}\, a}$$

Now, on putting $a = \frac{1}{\sqrt{2}}$

$$F\left(e^{-\frac{u^2}{2}}\right) = \frac{e^{s^2/4\times\frac{1}{2}}}{\sqrt{2}\times\frac{1}{\sqrt{2}}}$$

$$F\left(e^{-u^2/2}\right) = e^{s^2/2}.$$

## Properties of Fourier Transform :-

(1) Fourier Transform is linear.

$$F\left(af(u) + bg(u)\right] = aF[f(u)] + bF[g(u)]$$

$$\Rightarrow F[af(u) + bg(u)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} (af(u) + bg(u)) e^{isx} dx$$

$$= a \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx + b \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{isx} dx$$

$$= a F[f(x)] + b \{F[g(x)].$$

✓

② Shifting Theorem :-

If $F(f(x)) = F(s)$ then $F[f(x-a)] = e^{isa} F(s)$.

$$\Rightarrow F\{F(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx .$$

$$= \frac{1}{\sqrt{\pi 2}} \int_{-\infty}^{\infty} f(t) e^{i(a+i)s} dt \quad \begin{bmatrix} \text{On putting} \\ x-a=t \end{bmatrix}$$

$$= e^{ias} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{its} dt$$

$$= e^{ias} F(s)$$

③ $F(e^{iax} f(x)\} = F(s+a).$

$$\Rightarrow F\{e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix(s+a)} f(x)$$

$$= F(s+a)$$

4. Change of Scale ~~of~~ property :-

If $F(f(x)) = F(s)$.

then $F\{f(ax)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right)$ where $|a| \neq 0$.

Proof :- As we know,

$$F(f(x)) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) dx.$$

$$F(f(ax)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(ax) dx$$

Let $ax = t$, $a > 0$.

$$dx = \frac{dt}{a}.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is\frac{t}{a}} f(t) \frac{dt}{a}.$$

$$= \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\frac{s}{a}t} f(t) dt. \qquad a > 0$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right).$$

Similarly for $a < 0$,

$$F(f(ax)) = \frac{-1}{a} F\left(\frac{s}{a}\right).$$

Hence, $F(f(ax)) = \frac{1}{|a|} F\left(\frac{s}{a}\right)$

5 Modulation Theorem :-

If $F[f(x)] = F(s)$, then

$$F\{f(x)\cos ax\} = \frac{1}{2}[F(s-a) + F(s+a)]$$

Proof :- $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$

$$F\{f(x)\cos ax\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) \cdot \cos ax\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) \cdot \frac{e^{iax} + e^{-iax}}{2}\, dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix(s+a)} f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix(s-a)} f(x)\, dx\right]$$

$$= \frac{1}{2}[F(s+a) + F(s-a)]$$

6. If $F\{f(x)\} = F(s)$.

Then $F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} F(s)$.

Proof :- $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$

Differentiating w.r.t s both side, n times.

$$\frac{d^n F(s)}{ds^n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix)^n e^{isx} f(x)\, dx.$$

$$= (i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{isx} f(x)\, dx$$

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$$\frac{d^n}{ds^n} F(s) = i^n F\{x^n f(x)\}$$

$$\Rightarrow F(x^n f(x)) = \frac{1}{i^n} \frac{d^n}{ds^n} F(s)$$

$$F(x^n f(x)) = (-1)^n \frac{d^n}{ds^n} F(s)$$

**7.** $F\{f'(x)\} = -is F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

Proof:- $$F\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f'(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, d\{f(x)\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \{e^{isx} f(x)\}_{-\infty}^{\infty} - is \int_{-\infty}^{\infty} f(x) e^{isx}\, dx \right]$$

$$= \cancel{\frac{1}{\sqrt{2\pi}} \Big[}$$ If $f(x) \to 0$

$\Rightarrow -is F(s)$.

$F\{f^n(x)\} = (-is)^n F(s)$ if $f, f', f'', \ldots f^{n-1} \to 0$ as $x \to \pm\infty$

8 $F\left\{\int_a^x f(x)\,dx\right\} = \frac{F(s)}{(-is)}$

Proof :- Let $\int_a^x f(x) = g(x)$ — — ①

then $f(x) = g'(x)$ — — ②

Using theorem ⑦.

$F\{g'(x)\} = -is\, G(s)$

$G(s) = \frac{F\{g'(x)\}}{(-is)}$

$= \frac{F\{f(x)\}}{-is}$

$F\{g(x)\} = \frac{F(s)}{-is}$

$\Rightarrow F\left\{\int_a^x f(x)\,dx\right\} = \frac{F(s)}{-is}$

## Convolution Theorem

The convolution of two function $f(x)$ and $g(x)$ is defined as

$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$

Theorem → The FT of the convolution of f(n) and g(n) is product of their FT.

i.e. $F\{f(n) * g(n)\} = F(s) \cdot G(s)$

$= F\{f(n)\} \cdot F\{g(n)\}$.

Proof :- $F\{f*g\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(n-t) dt \right) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(n-t) e^{isx} dx \right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \; F\{g(n-t)\} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \; e^{its} G(s) dt \quad \left[\text{Using Shifting theorem}\right]$$

$$= G(s) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$= G(s) \cdot F(s).$$

By <u>Inversion</u>

$$F^{-1}\{F(s) \cdot G(s)\} = f*g = F^{-1}\{f(s)\} * F^{-1}\{G(s)\}$$

Parseval's Identity :- If $F(s)$ is the Fourier transform of $f(x)$ then.

$$\int_{-\infty}^{\infty} |F(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(s)|^2 \, ds.$$

Q. If $F(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases}$

Then using parseval's Identity prove that

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

Sol → $F(s) = \sqrt{\frac{2}{\pi}} \frac{\sin a s}{s}.$

Using parseval's Identity.

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

$$\int_{-a}^{a} 1 \cdot dt = \int_{-\infty}^{\infty} \frac{2}{\pi} \left(\frac{\sin a s}{s}\right)^2 ds.$$

$$|t|_{-a}^{a} = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin a s}{s}\right)^2 ds$$

$$as = t \Rightarrow ds = \frac{dt}{a}.$$

$$\Rightarrow 2a = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{\frac{t}{a}}\right)^2 \frac{dt}{a}.$$

$$\Rightarrow \pi = \frac{1}{a^2} \times a^2 \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi$$

$$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

Q $F(x) = \begin{cases} 1-|x| & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$

Hence find value of $\int_0^\infty \frac{\sin^4 t}{t^4}\, dt$

Sol→ $F\{f(x)\} = \int_{-\infty}^{\infty} (1-|x|)\, e^{isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \cdot \int_0^1 (1-x)\cos sx\, dx \quad \left[\because f(x) \text{ is even}\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1-x)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x)\frac{\sin sx}{s} - (-1)\left(-\frac{\cos sx}{s^2}\right)\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos s}{s^2}\right).$$

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Using Parseval's Identity.

$$\int_{-1}^{1} (1-|x|)^2 \, dx = \int_{-\infty}^{\infty} \frac{2}{\pi} \left(\frac{1-\cos s}{s^2}\right)^2 ds.$$

$$= 2\int_0^1 (1-x)^2 dx = \frac{4}{\pi}\int_0^\infty \frac{(1-\cos s)^2}{s^4} ds$$

$$\frac{2}{3} = \frac{4}{\pi}\int_0^\infty \frac{\left(2\sin^2\frac{s}{2}\right)^2}{s^4} ds$$

Let $\frac{s}{2} = t \Rightarrow ds = 2dt$.

$$\Rightarrow \frac{2}{3} = \frac{16}{\pi}\int_0^\infty \frac{\sin^4 t}{16t^4} \times 2dt$$

$$\Rightarrow \frac{2}{3} = \frac{2\times 16}{16\pi}\int_0^\infty \frac{\sin^4 t}{t^4} dt$$

$$\Rightarrow \int_0^\infty \frac{\sin^4 t}{t^4} dt = \frac{\pi}{3}$$

## Fourier Cosine Transform

$$F_c(s) = f_c(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx \, dx$$

Inverse, $\rightarrow$ $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx \, ds.$

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# Fourier Sine Transform

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

Inverse $\rightarrow$ $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds.$$

Q Find fourier cosine and sine transform of $e^{-ax}$, $a>0$ and hence deduce the inversion formula.

Sol $\rightarrow$ $$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-ax}}{a^2+s^2} (-a\cos sx + s\sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-\infty}}{e^2+s^2} (-a\cos\infty + s\sin\infty) - \frac{e^0}{a^2+s^2} (-a\cos 0 + 0) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{-e^0}{a^2+s^2} (-a\cos 0 + 0) \right] \quad [\because e^{-\infty} = 0]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2} \quad , a>0$$

Ans

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-ax}}{a^2+s^2} (-a\sin sx - s\cos sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left\{ \frac{e^{-\infty}}{a^2+s^2} (-a\sin\infty - s\cos\infty) \right\} - \left\{ \frac{e^0}{a^2+s^2} (0 - s\cos 0) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 + \frac{s}{s^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{s}{a^2+s^2}, \quad a>0$$ Ans.

Now, By Inversion formula of (i).

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2} \cos sx \, ds.$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{a^2+s^2} \, ds.$$

$$\Rightarrow \int_0^\infty \frac{\cos sx}{a^2+s^2} \, ds = \frac{\pi}{2a} \cdot e^{-ax}.$$

On putting $s \to x$ and $x \to \alpha$.

$$\int_0^\infty \frac{\cos \alpha x}{a^2+x^2} \, dx = \frac{\pi}{2a} \cdot e^{-a\alpha}.$$

Now, By Inversion formula of (ii)

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sin sx \, ds$$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{s^2+a^2}\,ds.$$

$$\Rightarrow \int_0^\infty \frac{s\sin sx}{a^2+s^2}\,ds = \frac{\pi}{2}e^{-ax}$$

On putting $s \to x$ and $x \to \alpha$

$$\int_0^\infty \frac{x\sin \alpha x}{a^2+x^2}\,dx = \frac{\pi}{2}e^{-a\alpha}$$

Q Find the fourier sine transform of $\frac{1}{x}$.

Sol→ $F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\cdot\sin sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}\,dx$$

Let $sx \to \theta \Rightarrow x = \frac{\theta}{s}$ & $dx = \frac{d\theta}{s}$.

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin\theta}{\frac{\theta}{\cancel{s}}}\cdot\frac{d\theta}{\cancel{s}}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin\theta}{\theta}\,d\theta$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{2}.$$

$$= \sqrt{\frac{\pi}{2}}$$ Ans

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Q Evaluate $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$ using transform method.

Sol $\rightarrow$ Let $f(x) = e^{-ax}$

$g(x) = e^{-bx}$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$ [See previous question for this]

Similarly, $G_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \cos sx$

$$= \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2}$$

Now, on using Identity

$$\int_0^\infty F_c(s) G_c(s) \, ds = \int_0^\infty f(x) g(x) \, dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{ab \, ds}{(a^2+s^2)\cdot(b^2+s^2)} = \int_0^\infty e^{-ax} \cdot e^{-bx} \, dx$$

$$= \frac{2}{\pi} \int_0^\infty \frac{ab \, ds}{(a^2+s^2)(b^2+s^2)} = \int_0^\infty e^{-(a+b)x} \, dx$$

On changing variable,

$$\frac{2}{\pi} \int_0^\infty \frac{ab \, dx}{(a^2+x^2)(b^2+x^2)} = \int_0^\infty e^{-(a+b)t} \, dt$$

$$= \left[ \frac{e^{-(a+b)t}}{-(a+b)} \right]_0^\infty$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+x^2)(b^2+x^2)}\,dx = \frac{e^{-\infty}}{-(a+b)} + \frac{e^0}{a+b}$$

$$\Rightarrow \int_0^\infty \frac{\cancel{ab}\ 1}{(a^2+x^2)(b^2+x^2)}\,dx = \frac{\pi}{2ab\,(a+b)}$$

Ans

## Identities :-

If $F_c(s)$, $G_c(s)$ are the fourier cosine ~~and~~ transform and $F_s(s)$ and $G_s(s)$ are fourier sine transform of $f(x)$ and $g(x)$ respectively, then.

1. $\int_0^\infty f(x)g(x)\,dx = \int_0^\infty F_c(s)\,G_c(s)\,ds$

2. $\int_0^\infty f(x)g(x)\,dx = \int_0^\infty F_s(s)\,G_s(s)\,ds$

3. $\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(s)|^2\,ds = \int_0^\infty |F_s(s)|^2\,ds$

## Parseval's Identity for fourier sine & cosine series :-

1. $\int_0^\infty [F_c(s)]^2\,ds = \int_0^\infty [f(x)]^2\,dx$

2. $\int_0^\infty [F_s(s)]^2\,ds = \int_0^\infty [F(x)]^2\,dx$

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Q Find fourier sine transform of $\frac{x}{a^2+x^2}$ and fourier cosine transform of $\frac{1}{a^2+x^2}$.

Sol→ $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$F_s\left(\frac{x}{a^2+x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{a^2+x^2} \sin sx \, dx \quad \text{--- (1)}$$

Let $f(x) = e^{-ax}$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-ax}}{a^2+s^2} (-a \sin sx - s \cos sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{s}{s^2+a^2} \right]$$

By Inversion formula,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sin sx \, ds$$

$$\Rightarrow \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds = \frac{\pi e^{-ax}}{2}$$

Putting this value in eq. (1)

$$F_s\left(\frac{x}{a^2+x^2}\right) = \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2} e^{-ax}\right) = \sqrt{\frac{\pi}{2}} e^{-ax}$$ Ans

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Similarly. $F_c\left(\frac{1}{a^2+x^2}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{a^2+x^2}\cos sx\,dx$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}\left[-a\cos sx + s\sin sx\right]\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+s^2}$$

By Inversion formula.

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\,dx$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+s^2}\cos sx\,dx$$

$$\Rightarrow \int_0^\infty \frac{1}{a^2+s^2}\cos sx\,dx = \frac{\pi}{2a}e^{-ax}$$

Putting this in eq. 2, we get

$$F_c\left(\frac{1}{a^2+x^2}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2a}e^{-ax}\right)$$

$$= \sqrt{\frac{\pi}{2}}\,\frac{e^{-ax}}{a}$$ Ans

Q Show that

(i) $F_s[x f(x)] = \frac{-d}{ds} F_c(s)$

(ii) $F_c[x f(x)] = \frac{d}{ds} F_s(s)$.

and hence find fourier sine and cosine transform of

$x \cdot e^{-ax}$.

Sol→ we know,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$\frac{d}{ds} F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty -x f(x) \sin sx \, dx$$

$$\frac{d}{ds} F_c(s) = -\sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \sin sx \, dx$$

$$\Rightarrow \frac{d}{ds} F_c(s) = -F_s[x \cdot f(x)]$$

$$\Rightarrow F_s[x \cdot f(x)] = \frac{-d}{ds} F_c(s)$$

Similarly, $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \cos sx \, dx$$

$$\frac{d}{ds} F_s(s) = F_c[x f(x)]$$

$$\Rightarrow F_c[x \cdot f(x)] = \frac{d}{ds} F_s(s).$$

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$$F_c\{x e^{-ax}\} = \frac{d}{ds} F_s\{e^{-ax}\}.$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx\right]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\cdot\frac{s}{s^2+a^2}\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\ \frac{(s^2+a^2)\cdot 1 - s(2s)}{(s^2+a^2)^2}$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{a^2-s^2}{(s^2+a^2)^2}.\quad \underline{\underline{\text{Ans}}}$$

$$F_s\{x\cdot e^{-ax}\} = \frac{-d}{ds} F_c\{e^{-ax}\}$$

$$= \frac{-d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx\right]$$

$$= \frac{-d}{ds}\left[\sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{-d}{ds}\left(\frac{a}{a^2+s^2}\right)$$

$$= \sqrt{\frac{2}{\pi}} - \left[\frac{(a^2+s^2)\cdot 0 - a(2s)}{(a^2+s^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{2as}{(a^2+s^2)^2}\quad \underline{\underline{\text{Ans}}}$$

Q Using Parseval's Identity. Evaluate.

(i) $\int_0^\infty \frac{dx}{(a^2+x^2)^2}$ . (ii) $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}dx$ if $a>0$.

Sol → $F(x) = e^{-ax}$.

$F_c(s) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$

$F_s(s) = \sqrt{\frac{2}{\pi}} \frac{s}{a^2+s^2}$

Using Parseval's Identity.

$$\int_0^\infty [F(x)]^2 dx = \int_0^\infty [F_c(s)]^2 ds$$

$$= \int_0^\infty e^{-2ax} dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}}\right)^2 \frac{a^2}{(a^2+s^2)^2} ds$$

$$= \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{a^2}{(a^2+s^2)^2} ds.$$

$$\Rightarrow \frac{1}{\infty} + \frac{1}{2a} = \frac{2a^2}{\pi}\int_0^\infty \frac{dx}{(a^2+x^2)^2}$$

$$\Rightarrow \frac{\pi}{4a^3} = \int_0^\infty \frac{dx}{(a^2+x^2)^2}$$

So, $\int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3}$ Ans

On using Parseval's Identity.

$$\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_s(s)]^2 ds$$

$$\Rightarrow \int_0^\infty e^{-2ax} dx = \int_0^\infty \frac{2}{\pi} \frac{s^2}{(a^2+s^2)^2} ds$$

$$\Rightarrow \frac{1}{2a} = \frac{2}{\pi}\int_0^\infty \frac{x^2}{(a^2+x^2)^2} dx$$

$$\Rightarrow \int_0^\infty \frac{x^2}{(a^2+x^2)^2} dx = \frac{\pi}{4a} \quad \text{Ans}$$

Q. Find fourier cosine transform of $e^{-a^2x^2}$ and hence evaluate, Fourier sine transform of $xe^{-a^2x^2}$.

Sol→ $F_c(e^{-a^2x^2}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos sx \, dx$

$$= \text{Real part of } \sqrt{\frac{2}{\pi}} \int e^{-a^2x^2} \cdot e^{isx} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\left(ax - \frac{is}{2a}\right)^2} \cdot e^{-\frac{s^2}{4a^2}} dx.$$

$$= e^{-\frac{x^2}{4a^2}} \cdot \frac{\sqrt{2}}{a\sqrt{\pi}} \int_0^\infty e^{-t^2} dt$$

$$= e^{-\frac{x^2}{4a^2}} \cdot \sqrt{\frac{2}{\pi}} \times \frac{1}{2a} \sqrt{\pi}$$

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$$= \frac{1}{\sqrt{2\pi}\, a} \sqrt{\pi} \cdot e^{-s^2/4a}$$

$$\Rightarrow \frac{1}{\sqrt{2}\, a} e^{-\frac{s^2}{4a^2}}$$

$$F_s\left(x e^{-a^2x^2}\right) = \frac{-d}{ds} F_c\left(e^{-a^2x^2}\right)$$

$$= -\frac{d}{ds}\left(\frac{1}{\sqrt{2}\, a} \cdot e^{-s^2/4a^2}\right)$$

$$= \frac{1}{\sqrt{2}\, a} \cdot e^{-s^2/4a} \cdot \frac{s}{2a^2}$$

$$= \frac{s}{2\sqrt{2}\, a^3} e^{-s/4a^2} \quad \underline{\underline{\text{Ans}}}$$

Q Solve for $f(x)$ from integral equation:

$$\int_0^\infty f(x) \sin sx\, dx = \begin{cases} 1 & \text{for } 0<s<1 \\ 2 & \text{for } 1<s<2 \\ 0 & \text{for } s>2. \end{cases}$$

Sol → On Multiplying $\sqrt{\frac{2}{\pi}}$ both the side

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin x\, dx = \begin{cases} \sqrt{\frac{2}{\pi}} & \text{for } 0<s<1 \\ 2\sqrt{\frac{2}{\pi}} & \text{for } 1<s<2 \\ 0 & \text{for } s>2 \end{cases}$$

$$F_s(s) = \begin{cases} \sqrt{\frac{2}{\pi}} \\ 2\sqrt{\frac{2}{\pi}} \\ 0 \end{cases}$$

$$f(x) = F^{-1}_{s}(s)$$

$$\Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{2}{\pi}} \sin sx \, ds + \sqrt{\frac{2}{\pi}} \int_1^2 2\sqrt{\frac{2}{\pi}} \sin sx \, ds + \int_2^\infty 0$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^1 \sin sx \, ds + \frac{4}{\pi} \int_1^2 \sin sx \, ds$$

$$= \frac{2}{\pi} \left[ \frac{-\cos sx}{x} \right]_0^1 + \frac{4}{\pi} \left[ \frac{-\cos sx}{\cancel{x}} \right]_1^2$$

$$= \frac{2}{\pi} \left[ \frac{-\cos x + 1}{x} \right] + \frac{4}{\pi} \left[ \frac{-\cos 2x + \cos x}{x} \right]$$

$$= \frac{2}{\pi x} \left[ 1 - 2\cos 2x \right].$$ Ans

Try Yourself :- Q Find fourier cosine transform of $e^{-x^2}$