December 2018

Objective: Find a closed form price-to-yield formula for a mortgage pool with a constant CPR rate

For any fixed rate mortgage loan, the current balance can be determined using the ordinary annuity formula (assuming no prior curtailments):

$$B = \frac{\pi}{r}\left[1 - \frac{1}{(1+r)^N}\right]$$

where π is the loan's monthly P&I, r is the note rate (divided by 12), and N is the remaining months to maturity. Therefore, we can write the loan's P&I in the i^{th} period as a function of its UPB:

$$\pi = \frac{rB}{1-\frac{1}{(1+r)^N}} = \frac{rB}{q_N}$$

where

$$q_N = 1 - \frac{1}{(1+r)^N}$$

Theorem 1: Consider a mortgage pool prepaying at a constant CPR rate. Let s be the corresponding single month mortality rate. Furthermore, let π_i and B_i be the pool's P&I and beginning-of-period UPB in the i^{th} period. Then the P&I of the pool in the i^{th} period can be written as

$$\pi_i = (1-s)^i \pi_0$$

Proof outline: As an annuity, the P&I of a pool would be constant if there are no prepayments/curtailments. Therefore, the P&I in the i^{th} period is

$$\pi_i = \pi_{i-1}\frac{B_i}{B_{i-1} - (\pi_{i-1} - rB_{i-1})}$$

However,

$$B_i = \big(B_{i-1} - (\pi_{i-1} - rB_{i-1})\big)(1-s)$$

Therefore,

$$\pi_i = \pi_{i-1}(1-s)$$

Finally, using a simple induction argument, we have our result. *QED*

Theorem 2: The balance of the pool in the i^{th} period can be written as

$$B_i = (1-s)^i B_i'$$

where B_i' is the balance of the of the pool in the i^{th} period assuming no prepayments/curtailments.

Proof outline: The balance in each period can be written as

$$
\begin{aligned}
B_i &= \big(B_{i-1} - (\pi_{i-1} - rB_{i-1})\big)(1-s) = (B_{i-1} - \pi_{i-1} + rB_{i-1})(1-s) = ((1+r)B_{i-1} - \pi_{i-1})(1-s) \\
&= (1+r)(1-s)B_{i-1} - (1-s)\pi_{i-1} = (1+r)(1-s)B_{i-1} - (1-s)(1-s)^{i-1}\pi_0 \\
&= (1+r)(1-s)B_{i-1} - (1-s)^i\pi_0
\end{aligned}
$$

Therefore,

$$B_1 = (1+r)(1-s)B_0 - (1-s)^1\pi_0$$

$$
\begin{aligned}
B_2 &= (1+r)(1-s)B_1 - (1-s)^2\pi_0 = (1+r)(1-s)\big((1+r)(1-s)B_0 - (1-s)\pi_0\big) - (1-s)^2\pi_0 \\
&= [(1+r)(1-s)]^2B_0 - (1+r)(1-s)^2\pi_0 - (1-s)^2\pi_0 \\
&= [(1+r)(1-s)]^2B_0 - (1-s)^2\pi_0[1+(1+r)]
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= (1+r)(1-s)B_2 - (1-s)^3\pi_0 \\
&= (1+r)(1-s)([(1+r)(1-s)]^2B_0 - (1-s)^2\pi_0[1+(1+r)]) - (1-s)^3\pi_0 \\
&= [(1+r)(1-s)]^3B_0 - (1-s)^3\pi_0(1+r)[1+(1+r)] - (1-s)^3\pi_0 \\
&= [(1+r)(1-s)]^3B_0 - (1-s)^3\pi_0\big[1+(1+r)\big(1+(1+r)\big)\big] \\
&= [(1+r)(1-s)]^3B_0 - (1-s)^3\pi_0[1+(1+r)+(1+r)^2] \\
&= [(1+r)(1-s)]^3B_0 - (1-s)^3\pi_0\sum_{k=0}^{2}(1+r)^k
\end{aligned}
$$

By an induction argument,

$$B_i = [(1+r)(1-s)]^iB_0 - (1-s)^i\pi_0\sum_{k=0}^{i-1}(1+r)^k$$

But $\sum_{k=0}^{i-1}(1+r)^k$ is just a geometric series. Therefore,

$$\sum_{k=0}^{i-1}(1+r)^k = \frac{1-(1+r)^{i-1}}{1-(1+r)} = \frac{(1+r)^{i-1}-1}{r}$$

So

$$
\begin{aligned}
B_i &= [(1+r)(1-s)]^i B_0 - (1-s)^i \pi_0 \frac{(1+r)^{i-1}-1}{r} \\
&= [(1+r)(1-s)]^i B_0 - (1-s)^i \frac{B_0}{q_N}((1+r)^{i-1}-1) \\
&= (1-s)^i B_0 \left[(1+r)^i - \frac{1}{q_N}((1+r)^{i-1}-1)\right] = (1-s)^i B_0 \left[\left(1-\tfrac{1}{q_N}\right)(1+r)^i + \tfrac{1}{q_N}\right] \\
&= (1-s)^i B_0 \left[\left(1 - \frac{1}{1-\frac{1}{(1+r)^N}}\right)(1+r)^i + \frac{1}{1-\frac{1}{(1+r)^N}}\right] \\
&= (1-s)^i B_0 \left[\frac{1-\frac{1}{(1+r)^N}-1}{1-\frac{1}{(1+r)^N}}(1+r)^i + \frac{1}{1-\frac{1}{(1+r)^N}}\right] = (1-s)^i B_0 \left[\frac{-\frac{(1+r)^i}{(1+r)^N}}{1-\frac{1}{(1+r)^N}} + \frac{1}{1-\frac{1}{(1+r)^N}}\right] \\
&= (1-s)^i B_0 \left[\frac{1-\frac{(1+r)^i}{(1+r)^N}}{1-\frac{1}{(1+r)^N}}\right] = (1-s)^i B_0 \left[\frac{(1+r)^N-(1+r)^i}{(1+r)^N-1}\right]
\end{aligned}
$$

But, since π is constant is the there are no prepayments/curtailments,

$$
\pi = \frac{rB_0}{q_N} = \frac{rB_i'}{q_{N-i}}
$$

Or

$$
B_i' = B_0 \frac{q_{N-i}}{q_N} = B_0 \frac{1-\frac{1}{(1+r)^{N-i}}}{1-\frac{1}{(1+r)^N}} = B_0 \frac{1-\frac{(1+r)^i}{(1+r)^N}}{1-\frac{1}{(1+r)^N}} = B_0 \frac{(1+r)^N-(1+r)^i}{(1+r)^N-1}
$$

Finally, by substitution,

$$
B_i = (1-s)^i B_0 \left[\frac{(1+r)^N-(1+r)^i}{(1+r)^N-1}\right] = (1-s)^i B_i'
$$

QED.

Theorem 3: The cash flows of a mortgage pool with a constant CPR rate can be decomposed into two components that can be evaluated using the annuity-with-growth formula. One grows at the rate -s and then second at a rate [(1+r)(1-s)-1].

Proof Outline: The cash flow in the i[th] period is:

$$
\begin{aligned}
C_i = \pi_i + s(B_i - (\pi_i - rB_i)) &= \pi_i + s(B_i - \pi_i + rB_i) = \pi_i + sB_i - s\pi_i + srB_i \\
&= \pi_0(1-s)^i + s(1-s)^i B_i' - s\pi_0(1-s)^i + sr(1-s)^i B_i' \\
&= (1-s)^i(\pi_0 + sB_i' - s\pi_0 + srB_i') = (1-s)^i\left(\tfrac{rB_0}{q_N} + sB_i' - \tfrac{srB_0}{q_N} + srB_i'\right) \\
&= (1-s)^i\left(\tfrac{rB_0}{q_N}(1-s) + s(1+r)B_i'\right)
\end{aligned}
$$

Now,

$$B_i' = \frac{q_{N-i}B_0}{q_N} = B_0\left(1-\tfrac{1}{q_N}\right)(1+r)^i + \tfrac{B_0}{q_N}$$

Therefore,

$$\begin{aligned}
C_i &= (1-s)^i\left(\tfrac{rB_0}{q_N}(1-s) + s(1+r)B_i'\right) \\
&= (1-s)^i\left(\frac{rB_0}{q_N}(1-s) + s(1+r)\left(B_0\left(1-\tfrac{1}{q_N}\right)(1+r)^i + \tfrac{B_0}{q_N}\right)\right) \\
&= (1-s)^i\left(\tfrac{rB_0}{q_N}(1-s) + B_0\left(1-\tfrac{1}{q_N}\right)s(1+r)^{i+1} + \tfrac{B_0}{q_N}s(1+r)\right) \\
&= (1-s)^i\left(\tfrac{rB_0}{q_N}(1-s) + \tfrac{sB_0}{q_N}(1+r) + B_0\left(1-\tfrac{1}{q_N}\right)s(1+r)^{i+1}\right) \\
&= B_0(1-s)^i\left(\tfrac{r(1-s)+s(1+r)}{q_N} + \left(1-\tfrac{1}{q_N}\right)s(1+r)^{i+1}\right) \\
&= B_0\left(\tfrac{r(1-s)+s(1+r)}{q_N}(1-s)^i + \left(1-\tfrac{1}{q_N}\right)s(1+r)^{i+1}(1-s)^i\right) = \\
&= B_0\left(\tfrac{r(1-s)+s(1+r)}{q_N}(1-s)^i + \left(1-\tfrac{1}{q_N}\right)s(1+r)(1+r)^i(1-s)^i\right) \\
&= B_0\left(\tfrac{r(1-s)+s(1+r)}{q_N}(1-s)^i + \left(1-\tfrac{1}{q_N}\right)s(1+r)[(1+r)(1-s)]^i\right) \\
&= B_0\tfrac{r(1-s)+s(1+r)}{q_N}(1-s)^i + B_0\left(1-\tfrac{1}{q_N}\right)s(1+r)[(1+r)(1-s)]^i \\
&= B_0\lambda_1(1-s)^i + B_0\lambda_2[(1+r)(1-s)]^i
\end{aligned}$$

QED.

The present value of the cash flows can be computed directly using the annuity-with-growth formula:

$$P_0 = \tfrac{B_0\lambda_1}{y+s}\left[1-\left(\tfrac{1-s}{1+y}\right)^N\right] + \tfrac{B_0\lambda_2}{y-(1+r)(1-s)+1}\left[1-\left(\tfrac{(1+r)(1-s)}{1+y}\right)^N\right]$$

where N is the months remaining until maturity.

Now consider the case where there is a balloon cash flow. Let N' be the number of months until the balloon date. Then,

$$P_0 = \frac{B_0\lambda_1}{y+s}\left[1-\left(\frac{1-s}{1+y}\right)^{N'}\right] + \frac{B_0\lambda_2}{y-(1+r)(1-s)+1}\left[1-\left(\frac{(1+r)(1-s)}{1+y}\right)^{N'}\right] + \frac{B_i}{(1+y)^{N'}}$$

$$= \frac{B_0\lambda_1}{y+s}\left[1-\left(\frac{1-s}{1+y}\right)^{N'}\right] + \frac{B_0\lambda_2}{y-(1+r)(1-s)+1}\left[1-\left(\frac{(1+r)(1-s)}{1+y}\right)^{N'}\right] + \frac{(1-s)^{N'}B_i'}{(1+y)^{N'}}$$

$$= \frac{B_0\lambda_1}{y+s}\left[1-\left(\frac{1-s}{1+y}\right)^{N'}\right] + \frac{B_0\lambda_2}{y-(1+r)(1-s)+1}\left[1-\left(\frac{(1+r)(1-s)}{1+y}\right)^{N'}\right]$$

$$+ \frac{(1-s)^{N'}(B_0\left(1-\frac{1}{q_N}\right)(1+r)^{N'} + \frac{B_0}{q_N})}{(1+y)^{N'}}$$

$$= \frac{B_0\lambda_1}{y+s}\left[1-\left(\frac{1-s}{1+y}\right)^{N'}\right] + \frac{B_0\lambda_2}{y-(1+r)(1-s)+1}\left[1-\left(\frac{(1+r)(1-s)}{1+y}\right)^{N'}\right]$$

$$+ B_0\frac{\left(1-\frac{1}{q_N}\right)(1+r)^{N'}(1-s)^{N'} + \frac{1}{q_N}(1-s)^{N'}}{(1+y)^{N'}}$$

$$= \frac{B_0\lambda_1}{y+s}\left[1-\left(\frac{1-s}{1+y}\right)^{N'}\right] + \frac{B_0\lambda_2}{y-(1+r)(1-s)+1}\left[1-\left(\frac{(1+r)(1-s)}{1+y}\right)^{N'}\right]$$

$$+ B_0\frac{\left(1-\frac{1}{q_N}\right)[(1+r)(1-s)]^{N'} + \frac{1}{q_N}(1-s)^{N'}}{(1+y)^{N'}}$$

Finally, the dollar price of the security can be written as

$$\varphi_0 \equiv \frac{P_0}{B_0} = \frac{\lambda_1}{y+s}\left[1-\left(\frac{1-s}{1+y}\right)^{N'}\right] + \frac{\lambda_2}{y-(1+r)(1-s)+1}\left[1-\left(\frac{(1+r)(1-s)}{1+y}\right)^{N'}\right] + \frac{\lambda_3}{(1+y)^{N'}}$$

where

$$\lambda_1 = \frac{r(1-s)+s(1+r)}{q_N}$$

$$\lambda_2 = \left(1-\frac{1}{q_N}\right)s(1+r)$$

$$\lambda_3 = \left(1-\frac{1}{q_N}\right)[(1+r)(1-s)]^{N'} + \frac{1}{q_N}(1-s)^{N'}$$

$$q_N = 1-\frac{1}{(1+r)^N}$$

where N in the latter formula is months to *maturity*, not months to the balloon date.

Modified Duration:

$$Mod.Duration \equiv -\frac{1}{\varphi_0}\frac{d\varphi_0}{dw}$$

where w is the bond equivalent yield. Using the chain rule,

$$\frac{d\varphi_0}{dw} = \frac{d\varphi_0}{dy}\frac{dy}{dw}$$

Now,

$$y = \left(1 + \frac{w}{2}\right)^{\frac{1}{6}}$$

$$\frac{dy}{dw} = \frac{1}{12\left(1 + \frac{w}{2}\right)^{\frac{5}{6}}}$$

Furthermore,

$$\begin{aligned}
\frac{d\varphi_0}{dy} = \frac{d}{dy}&\left[\frac{\lambda_1}{y+s}\left[1 - \left(\frac{1-s}{1+y}\right)^{N'}\right] + \frac{\lambda_2}{y-(1+r)(1-s)+1}\left[1 - \left(\frac{(1+r)(1-s)}{1+y}\right)^{N'}\right] + \frac{\lambda_3}{(1+y)^{N'}}\right] \\
&= \frac{\lambda_1}{y+s}\left[N'\frac{(1-s)^{N'}}{(1+y)^{N'+1}}\right] - \frac{\lambda_1}{(y+s)^2}\left[1 - \left(\frac{1-s}{1+y}\right)^{N'}\right] + \frac{\lambda_2}{y-(1+r)(1-s)+1}\left[N'\frac{\left((1+r)(1-s)\right)^{N'}}{(1+y)^{N'+1}}\right] \\
&- \frac{\lambda_2}{(y-(1+r)(1-s)+1)^2}\left[1 - \left(\frac{(1+r)(1-s)}{1+y}\right)^{N'}\right] - \frac{N'\lambda_3}{(1+y)^{N'+1}} \\
&= \left(\frac{\lambda_1}{y+s}\right)\left(\frac{N'}{1+y}\right)\left(\frac{1-s}{1+y}\right)^{N'} - \frac{\lambda_1}{(y+s)^2}\left[1 - \left(\frac{1-s}{1+y}\right)^{N'}\right] + \left(\frac{\lambda_2}{y-(1+r)(1-s)+1}\right)\left(\frac{N'}{1+y}\right)\left(\frac{(1+r)(1-s)}{1+y}\right)^{N'} \\
&- \frac{\lambda_2}{(y-(1+r)(1-s)+1)^2}\left[1 - \left(\frac{(1+r)(1-s)}{1+y}\right)^{N'}\right] - \frac{N'\lambda_3}{(1+y)^{N'+1}}
\end{aligned}$$

Weighted Average Life:

$$WAL \equiv \frac{\sum_{i=1}^{N'}(C_i - rB_i)i}{\sum_{i=1}^{N'}(C_i - rB_i)} = \frac{\sum_{i=1}^{N'}(C_i - rB_i)i + N'B_{N'}}{B_1}$$
$$= \frac{\sum_{i=1}^{N'}\left(B_0\lambda_1(1-s)^i + B_0\lambda_2[(1+r)(1-s)]^i - rB_i'(1-s)^i\right)i + N'B_{N'}}{B_0}$$
$$= \frac{\sum_{i=1}^{N'}\left(B_0\lambda_1(1-s)^i + B_0\lambda_2[(1+r)(1-s)]^i - rB_0\frac{q_{N-i}}{q_N}(1-s)^i\right)i + N'B_0\frac{q_{N-N'}}{q_N}(1-s)^{N'}}{B_0}$$
$$= \sum_{i=1}^{N'}\left(\lambda_1(1-s)^i + \lambda_2[(1+r)(1-s)]^i - r\frac{q_{N-i}}{q_N}(1-s)^i\right)i + N'\frac{q_{N-N'}}{q_N}(1-s)^{N'}$$
$$= \sum_{i=1}^{N'}\left[\lambda_1 i(1-s)^i + \lambda_2 i[(1+r)(1-s)]^i - \frac{r}{q_N}iq_{N-i}(1-s)^i\right] + N'\frac{q_{N-N'}}{q_N}(1-s)^{N'}$$
$$= \sum_{i=1}^{N'}\left[\lambda_1 i(1-s)^i + \lambda_2 i[(1+r)(1-s)]^i - \frac{r}{q_N}i\left(1-\frac{1}{(1+r)^{N-i}}\right)(1-s)^i\right] + N'\frac{q_{N-N'}}{q_N}(1-s)^{N'}$$
$$= \sum_{i=1}^{N'}\left[\lambda_1 i(1-s)^i + \lambda_2 i[(1+r)(1-s)]^i - \frac{r}{q_N}i\left((1-s)^i-\frac{(1-s)^i}{(1+r)^{N-i}}\right)\right] + N'\frac{q_{N-N'}}{q_N}(1-s)^{N'}$$
$$= \sum_{i=1}^{N'}\left[\lambda_1 i(1-s)^i + \lambda_2 i[(1+r)(1-s)]^i - \frac{r}{q_N}i\left((1-s)^i-\frac{((1+r)(1-s))^i}{(1+r)^{N}}\right)\right] + N'\frac{q_{N-N'}}{q_N}(1-s)^{N'}$$
$$= \sum_{i=1}^{N'}\left[(\lambda_1 - \frac{r}{q_N})i(1-s)^i + (\lambda_2 + \frac{r}{q_N(1+r)^N})i[(1+r)(1-s)]^i\right] + N'\frac{q_{N-N'}}{q_N}(1-s)^{N'}$$
$$= (\lambda_1 - \frac{r}{q_N})\sum_{i=1}^{N'} i(1-s)^i + (\lambda_2 + \frac{r}{q_N(1+r)^N})\sum_{i=1}^{N'} i[(1+r)(1-s)]^i + N'\frac{q_{N-N'}}{q_N}(1-s)^{N'}$$

Now the first two terms are of the form

$$\sum_{i=0}^{N} ir^i$$

To get this series, note that

$$\sum_{i=0}^{N} r^i = \frac{1-r^{N+1}}{1-r}$$

Therefore,

$$\sum_{i=1}^{N} ia^i = \frac{a(1-a^{N+1}) - (1-a)(N+1)a^{N+1}}{(1-a)^2}$$

$$WAL = (\lambda_1 - \tfrac{r}{q_N})\sum_{i=1}^{N'} i(1-s)^{i-1} + (\lambda_2 + \tfrac{r}{q_N(1+r)^N})\sum_{i=1}^{N'} i[(1+r)(1-s)]^{i-1} + N'\tfrac{q_{N-N'}}{q_N}(1-s)^{N'}$$

Now the first two terms are of the form

$$\sum_{i=0}^{N} ia^{i-1}$$

To get this series, note that

$$\sum_{i=0}^{N} a^i = \frac{1-a^{N+1}}{1-a}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{N} ia^{i-1} &= \frac{d}{da}\sum_{i=0}^{N} a^i = \frac{d}{da}\left[\frac{1-a^{N+1}}{1-a}\right] = \frac{(1-a)(-(N+1)a^N) + (1-a^{N+1})}{(1-a)^2}\\ &= \frac{(1-a^{N+1}) - (1-a)(N+1)a^N}{(1-a)^2} = \frac{(1-a^{N+1})}{(1-a)^2} - \frac{(N+1)a^N}{1-a}\\ &= \left[\frac{1}{1-a}\right]\left[\frac{(1-a^{N+1})}{1-a} - (N+1)a^N\right]\end{aligned}$$

Let

$$g(a,N) = \left[\frac{1}{1-a}\right]\left[\frac{(1-a^{N+1})}{1-a} - (N+1)a^N\right]$$

Therefore,

$$\begin{aligned}WAL = (\lambda_1 - \tfrac{r}{q_N})\sum_{i=1}^{N'} i(1-s)^{i-1} &+ (\lambda_2 + \tfrac{r}{q_N(1+r)^N})\sum_{i=1}^{N'} i[(1+r)(1-s)]^{i-1} + N'\tfrac{q_{N-N'}}{q_N}(1-s)^{N'} =\\ &= \left[\frac{\lambda_1 - \frac{r}{q_N}}{s}\right]\left[\frac{\left(1-(1-s)^{N'+1}\right)}{s} - (N'+1)(1-s)^{N'}\right]\\ &+ \left[\frac{\lambda_2 + \frac{r}{q_N(1+r)^N}}{1-(1+r)(1-s)}\right]\left[\frac{\left(1-[(1+r)(1-s)]^{N'+1}\right)}{1-(1+r)(1-s)} - (N'+1)[(1+r)(1-s)]^{N'}\right]\\ &+ N'\tfrac{q_{N-N'}}{q_N}(1-s)^{N'}\end{aligned}$$

Settlement Date Adjustment (May 2017):

Modified Duration:

As before,

$$Adj. Mod. Duration \equiv -\frac{1}{\vartheta_0}\frac{d\vartheta_0}{dw}$$

where w is the bond equivalent yield. Where ϑ is the forward to account forward settlement date.

$$\vartheta_0 = y^{\frac{d}{30}}\varphi_0 = \left[\left(1+\frac{w}{2}\right)^{\frac{1}{6}}\right]^{\frac{d}{30}}\varphi_0 = \left(1+\frac{w}{2}\right)^{\frac{d}{180}}\varphi_0$$

$$-\frac{1}{\vartheta_0}\frac{d\vartheta_0}{dw} = -\frac{1}{\left(1+\frac{w}{2}\right)^{\frac{d}{180}}\varphi_0}\frac{d}{dw}\left[\left(1+\frac{w}{2}\right)^{\frac{d}{180}}\varphi_0\right] = -\frac{\frac{d}{dw}\left[\left(1+\frac{w}{2}\right)^{\frac{d}{180}}\varphi_0\right]}{\left(1+\frac{w}{2}\right)^{\frac{d}{180}}\varphi_0}$$
$$= \frac{-\left(1+\frac{w}{2}\right)^{\frac{d}{180}}\frac{d\varphi_0}{dw} - \left(\frac{d}{360}\right)\left(1+\frac{w}{2}\right)^{\frac{d}{180}-1}\varphi_0}{\left(1+\frac{w}{2}\right)^{\frac{d}{180}}\varphi_0} = -\frac{1}{\varphi_0}\frac{d\varphi_0}{dw} - \frac{\left(\frac{d}{360}\right)}{\left(1+\frac{w}{2}\right)}$$

Therefore,

$$Adj. Mod. Duration = Mod. Duration - \frac{\left(\frac{d}{360}\right)}{\left(1+\frac{w}{2}\right)}$$

Servicing Retained (Added December 2018):

From theorems 1 and 2 above,

$$\pi_i = (1-s)^i\pi_0$$

$$B_i = (1-s)^i B_i'$$

Let ε be the servicing fee percentage per month. Then the cash flow in the ith period is:

$$\begin{aligned}C_i &= \pi_i - \varepsilon B_i + s\big(B_i - (\pi_i - rB_i)\big) = \pi_i - \varepsilon B_i + s(B_i - \pi_i + rB_i) = \pi_i - \varepsilon B_i + sB_i - s\pi_i + srB_i \\ &= \pi_0(1-s)^i - \varepsilon(1-s)^i B_i' + s(1-s)^i B_i' - s\pi_0(1-s)^i + sr(1-s)^i B_i' \\ &= (1-s)^i(\pi_0 - \varepsilon B_i' + sB_i' - s\pi_0 + srB_i') = (1-s)^i\left(\frac{rB_0}{q_N} - \varepsilon B_i' + sB_i' - \frac{srB_0}{q_N} + srB_i'\right) \\ &= (1-s)^i\left(\frac{rB_0}{q_N}(1-s) + (s(1+r)-\varepsilon)B_i'\right)\end{aligned}$$

Now,

$$B_i' = \frac{q_{N-i}B_0}{q_N} = B_0\left(1-\frac{1}{q_N}\right)(1+r)^i + \frac{B_0}{q_N}$$

Therefore,

$$
\begin{aligned}
C_i = (1-s)^i \left(\frac{rB_0}{q_N}(1-s) + (s(1+r)-\varepsilon)B_i'\right) \\
&= (1-s)^i \left(\frac{rB_0}{q_N}(1-s) + (s(1+r)-\varepsilon)\left(B_0\left(1-\frac{1}{q_N}\right)(1+r)^i + \frac{B_0}{q_N}\right)\right) \\
&= (1-s)^i \left(\frac{rB_0}{q_N}(1-s) + B_0\left(1-\frac{1}{q_N}\right)s(1+r)^{i+1} + \frac{B_0}{q_N}s(1+r)\right. \\
&\left. - \varepsilon B_0\left(1-\frac{1}{q_N}\right)(1+r)^i - \varepsilon\frac{B_0}{q_N}\right) \\
&= (1-s)^i \left(\frac{rB_0}{q_N}(1-s) - \varepsilon\frac{B_0}{q_N} + \frac{B_0}{q_N}s(1+r) + B_0\left(1-\frac{1}{q_N}\right)s(1+r)(1+r)^i\right. \\
&\left. - \varepsilon B_0\left(1-\frac{1}{q_N}\right)(1+r)^i\right) \\
&= (1-s)^i \left(\frac{rB_0}{q_N}(1-s) - \varepsilon\frac{B_0}{q_N} + \frac{B_0}{q_N}s(1+r)\right. \\
&\left. + \left(B_0\left(1-\frac{1}{q_N}\right)s(1+r) - \varepsilon B_0\left(1-\frac{1}{q_N}\right)\right)(1+r)^i\right) \\
&= B_0\left(\frac{r(1-s)-\varepsilon+s(1+r)}{q_N}\right)(1-s)^i + B_0\left(1-\frac{1}{q_N}\right)(s(1+r)-\varepsilon)(1+r)^i(1-s)^i \\
&= B_0\lambda_1(1-s)^i + B_0\lambda_2[(1+r)(1-s)]^i
\end{aligned}
$$

QED.

Every things above still applies except for different versions of the 2 lambdas.