**December 2018**

*Objective: Find a closed form price-to-yield formula for a mortgage pool with a constant CPR rate*

For any fixed rate mortgage loan, the current balance can be determined using the ordinary annuity formula (assuming no prior curtailments):

where π is the loan’s monthly P&I, r is the note rate (divided by 12), and N is the remaining months to maturity. Therefore, we can write the loan’s P&I in the ith period as a function of its UPB:

where

*Theorem 1:* Consider a mortgage pool prepaying at a constant CPR rate. Let s be the corresponding single month mortality rate. Furthermore, let πi and Bi be the pool’s P&I and beginning-of-period UPB in the ith period. Then the P&I of the pool in the ith period can be written as

*Proof outline:* As an annuity, the P&I of a pool would be constant if there are no prepayments/curtailments. Therefore, the P&I in the ith period is

However,

Therefore,

Finally, using a simple induction argument, we have our result. *QED*

*Theorem 2:* The balance of the pool in the ith period can be written as

where is the balance of the of the pool in the ith period assuming no prepayments/curtailments.

*Proof outline:* The balance in each period can be written as

Therefore,

By an induction argument,

But is just a geometric series. Therefore,

So

But, since π is constant is the there are no prepayments/curtailments,

Or

Finally, by substitution,

*QED.*

*Theorem 3:* The cash flows of a mortgage pool with a constant CPR rate can be decomposed into two components that can be evaluated using the annuity-with-growth formula. One grows at the rate -s and then second at a rate [(1+r)(1-s)-1].

*Proof Outline:* The cash flow in the ith period is:

Now,

Therefore,

*QED.*

The present value of the cash flows can be computed directly using the annuity-with-growth formula:

where N is the months remaining until maturity.

Now consider the case where there is a balloon cash flow. Let N’ be the number of months until the balloon date. Then,

Finally, the dollar price of the security can be written as

where

where N in the latter formula is months to *maturity*, not months to the balloon date.

*Modified Duration:*

where w is the bond equivalent yield. Using the chain rule,

Now,

Furthermore,

*Weighted Average Life:*

Now the first two terms are of the form

To get this series, note that

Therefore,

Now the first two terms are of the form

To get this series, note that

Therefore,

Let

Therefore,

**Settlement Date Adjustment (May 2017):**

*Modified Duration:*

As before,

where w is the bond equivalent yield. Where is the forward to account forward settlement date.

Therefore,

**Servicing Retained (Added December 2018):**

From theorems 1 and 2 above,

Let ε be the servicing fee percentage per month. Then the cash flow in the ith period is:

Now,

Therefore,

*QED.*

Every things above still applies except for different versions of the 2 lambdas.