Computer oriented Numerical Analysis

Unit-1

Errors

Error can be defined as difference between true value and approximate value of a variable. We can define the following type of error as follows:-

(i) **Input Error** :-

Error created due to wrong input or the input data being measured with limited accuracy or real number represented in a computed with a fixed no. of object.

(ii) **Algorithm Error** :-

Error arise due to algorithm we use to solve a problem. Truncation errors are the examples of algorithm errors. These errors arise when an iterative method is terminated or a mathematical procedure is approximated, and the approximated solution differs from the exact solution. It arise when an infinite series replaced by a finite one.

Example:-

if $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\infty = x$

is replaced by:

$$x' = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

then the truncation error is $x - x'$

(iii) Computational Error :-

Are the errors that occur in the computational data, such as in your vectors and matrices, during a computation.

(iv) Round off Errors :-

A round off error, also called rounding error, is the difference between the result produced by a given algorithm using exact arithmetic and the result produced by the same algorithm using finite precision, rounded arithmetic.

(v) Inherent Error :-

An Inherent error is a program error that happens regardless of what the user does and is often unavoidable.

(vi) Truncate Error :-

Truncation Error is the difference between a truncated value and the actual value.

Ex:- Rounding off a number 20.776 is 20.78 while truncation gives 20.77

(vii) Absolute Error :-

The absolute Error Ea is defined as follows :-

e_a = absolute Error

$e_a = |\text{True value} - \text{Approx. value}|$

$e_a = |x - x'|$

(viii) Relative Error

The relative error e_r is defined as follows:-

e_r = relative error

$$e_r = \frac{e_a}{TV}$$

$$e_r = \frac{|\text{True Value} - \text{Approx value}|}{\text{True value}}$$

(ix) Percentage Error

The Percentage error e_p is defined as follows:-

e_p = percentage error

$$e_p = e_r \times 100$$

$$e_p = \frac{|\text{True value} - \text{Approx value}|}{\text{True value}} \times 100$$

Q. If $\pi = 3.14$ approx then find absolute Error, relative Error or percentage.

Ans. Absolute Error

$e_a = |\text{True value} - \text{Approx Value}|$

$e_a = |x - x'|$

True value = 3.14285

Approx value = 3.14

$$e_a = \frac{|\text{True value} - \text{Approx value}|}{\cancel{\text{True value}}} = \frac{|3.14285 - 3.14|}{\cancel{3.14285}}$$

$= 0.00285$

Relative Error

$$e_r = \frac{e_a}{TV} = \frac{|\text{True value} - \text{Approx Value}|}{\text{True Value}}$$

$$= \frac{0.00285}{3.14285} = 0.0009068$$

Percentage Error

$$e_p = e_r \times 100$$

$$= \frac{|\text{True Value} - \text{Approx value}|}{\text{True value}} \times 100$$

$$= 0.0009068 \times 100$$

$$= 0.09068$$

Q. True value $= 7.143928$
Absolute value $= 7.14$

Two decimal round off:-

find Absolute Error, relative Error and Percentage Error.

Ans. $e_a = |\text{True Value} - \text{Approx value}|$

$$= |7.143928 - 7.14|$$

$$= 0.003928$$

$$e_r = \frac{e_a}{\text{True value}} = \frac{|7.143928 - 7.14|}{7.143928} = 0.0005498$$

$$e_p = e_r \times 100 = \frac{e_a}{\text{True value}} \times 100 = 0.00054\text{98} \times 100$$

$$= 0.05498$$

Q. 7.31214 convert it in 4 significant figure.

True value = 7.3124

Approx value = 7.31200

Ans. $e_a = |\text{True value} - \text{Approx value}|$

$= 73124 - 731200$

$= 14$

$e_r = \frac{e_a}{T.V} = \frac{14}{7.3124} = 1.914624$

$e_p = e_r \times 100$

$= 1.914624 \times 100$

$= 1914624$ Ans.

Q. If 0.333 is the approximate value of 1/3 find:-

(i) absolute Error

(ii) Relative Error

(iii) Percentage Error

Ans.

(i) Absolute Error:

The absolute error ea is given by:-

$\varepsilon_a = |\text{True value} - \text{Approx value}|$

True value $= \frac{1}{3}$ Approx value $= 0.333$

$e_a = |\text{True value} - \text{Approx value}|$

$= \left| \frac{1}{3} - 0.333 \right|$

$= \left| \frac{1}{3} - \frac{0.333}{1000} \right|$

$= \left| \frac{1000 - 999}{3000} \right| = \frac{1}{3000} = 0.00033$

(ii) Relative Error

$e_r = \frac{e_a}{T.V} = \frac{0.00033}{\frac{1}{3}} = 0.00033 \times 3$

$= 0.00099$

(iii) Percentage Error

$e_p = e_r \times 100$

$= 0.00099 \times 100 = 0.099$

Q. Define Absolute Error, Relative Error. Now, we have to find the absolute and relative error in y and when $x = 0.5 \pm 0.1$ $y = \frac{(0.31x + 2.73)}{x + 0.35}$

Ans when $x = 0.5 + 0.1$

$= 0.6$

$y = \frac{(0.31 \times 0.6 + 2.73}{0.6 + 0.35} = \frac{2.916}{0.95} = 3.069$

when $x = 0.5 - 1$

$= 0.4$

$y = \frac{(0.31 \times 0.4 + 2.73)}{0.4 + 0.35}$

$$\therefore = \frac{2.854}{0.75} = 3.805$$

(a) Absolute Error at $x = 0.6$

$T.V = 3.069$ $\quad A.V = 3.000$

$$e_a = |T.V - A.V|$$
$$= |3.069 - 3000|$$
$$= 0.069$$

Relative Error at $x = 0.6$

$$e_r = \frac{e_a}{T.V} = \frac{0.069}{3.069} = 0.02248$$

(b) Absolute Error at $x = 0.4$

$$e_a = |T.V - A.V|$$

$TV = 3.805$

$A.V = 3.000$

$$= |3.805 - 3.000|$$
$$= 0.805 \text{ Ans.}$$

Relative error $= \frac{e_a}{TV}$

$$= \frac{0.805}{3.805} = 0.21156 \text{ Ans}$$

Q. Let we have 3 non-zero number a, b and c is 10, 15, 20 and its absolute error i.e. Δa, Δb and Δc be 2, 3, 4 respectively.

Ans: So, er of a $= \frac{\Delta a}{a} = \frac{2}{10} = \frac{1}{5}$

er of b $= \frac{\Delta b}{b} = \frac{3}{15} = \frac{1}{5}$

er of c $= \frac{\Delta c}{c} = \frac{4}{20} = \frac{1}{5}$

So, Sum of the relative Error:-

$= \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5} = 0.6$

Now, we find the product of these numbers:-

$d = a * b * c = 10 \times 15 \times 20 = 3000$

and product of these errors

$d = \Delta a * \Delta b * \Delta c = 2 \times 3 \times 4$

$\Delta d = 24$

So, the relative error of $d = \frac{\Delta d}{d} = \frac{24}{3000}$

$= 0.008$ Ans.

Bisection Method

Q: Describe Bisection Method.

Ans This Method is used to Solve an equation of the type f(x)=0 is based on the repeated application of The Intermediate value theorem.

In this method, we start the iterative cycle by picking two trial points. Say, x_0 and x_1 such That $F(x_0)F(x_1) < 0$ which ensures that the root of The equation $F(x)=0$ lies b/w x_0 and x_1.
Two points x_0 and x_1 enclose a root if $f(x_0)$ and $f(x_1)$ are of opposite signs.

$$x_2 = \frac{x_0+1}{2}$$

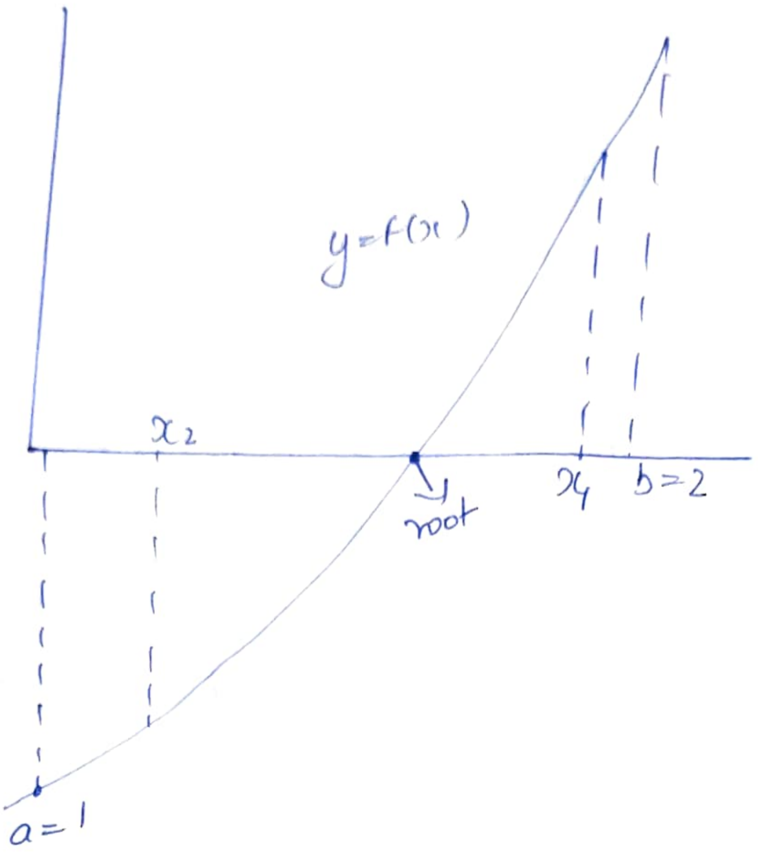
If $F(x_1)=0$ then x is The actual root. otherwise x is present either between the root of b or b/w The root of a.

Suppose in The example $F(x_1)>0$ is +ve. Hence, it lies b/w root and b. Hence, new points for further approximation are
(a and x_1)

$$x_2 = \frac{a+x_1}{2}$$

Let $f(x_2)$ is −ve in the example hence the new points for approximation are x_1 and x_2.

Similarly, do the Same step till we get The roots.



Q. Perform five iterations of Bisection method to obtain The Smallest positive root of equation:-

$$f(x) = x^3 - 5x + 1 = 0$$

The root lies b/w .2016 and .2017

Ans. $f(x) = x^3 - 5x + 1 = 0$

$f(.2016) = (.2016)^3 - 5 \times .2016 + 1$

$= .0001935$ (+ve)

$f(.2017) = (.2017)^3 - 5 \times .2017 + 1$

$= -.0002943$ i.e. (−ve)

First Approximation:-

$$X_1 = \frac{.2016 + .2017}{2} = \frac{0.4033}{2} = 0.20165$$

$f(x_1) = (.20165)^3 - 5 \times .20165 + 1 = 0$

$= -ve \quad -.00005036$

The root lies b/w .20165 and .2016

Second Approximation

$\frac{x_1 + x_2}{2}$

$$x_2 = \frac{.2016 \cancel{\text{and}} + .20165}{2} = .201625$$

$$F(x_2) = (.201625)^3 - 5 \times .201625 + 1$$

$$= -ve \simeq -.00007159$$

The root lies b/w = .201625 & .20165

Third Approximation:-

$$x_3 = \frac{.201625 + .20165}{2} = .2016375.$$

$$f(x_3) = (.2016375)^3 - 5 \times (.2016375) + 1 = 0$$

$$= -.00001061 \text{ i.e. } (-ve)$$

The root lies b/w .2016375 and .20165.

fourth Approximation

$$x_4 = \frac{.2016375 + .20165}{2} \qquad \frac{x_2 + x_3}{2}$$

$$= .20164375$$

$$f(x_4) = -.00001987 \; (-ve)$$

The root lies b/w .20164375 and .2016375 fifth Approximation:

$$x_5 = \frac{.2016375 + .20164375}{2} \qquad \frac{x_3 + x_4}{2}$$

$$= .201640625 \ (+ve)$$

Hence Smallest +ve root of the given equation is $\cdot 2064$ Correct to 5 decimal point.

of the equation:

Iteration Method

This Method is also known as Direct Substitution method or method of fixed iterations.

To find the root of Equation $f(x)=0$ we write it in $x=\phi(x)$. The roots of $f(x)=0$ are same as the point of Intersection of straight line $y=0$. Hence, the curve of $y=\phi(x)$.

Let $x=x_0$ is the initial approximation. Then first approximation

$$x_1=\phi(x_0)$$

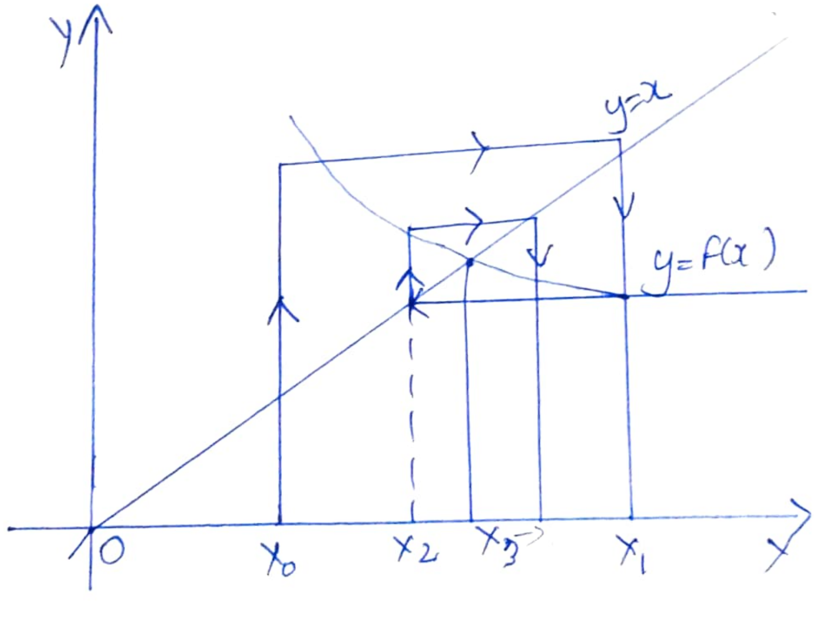
And other approximation

$$x_2=\phi(x_1)$$

$$x_3=\phi(x_2)$$

$$\vdots$$

$$x_{n+1}=\phi(x_n)$$



Solve by iteration method:

Q. (i) $x^3+x+1=0$

$f(x)=x^3+x+1=0$ —(1)

$F(0)=1$

$F(-1)=1$

If $f(0)$ is positive and $f(-1)$ is negative

Therefore, Root lies between 0 and -1.

∴ The negative root exist for given equation

$=x^3-x+1=0$ —(2)

$f(0)=-1$

$f(1)=1$

Root lies between 0 and 1

we can write the equation in the form:

$$x=\frac{1}{1+x^2}=\phi(x)$$

$$x_1=\frac{1}{1+1^2}=\frac{1}{2}=0.5$$

taking, $x_0=1$

$$x_2=\frac{1}{1+(0.5)^2}=0.8000$$

$$x_3=\frac{1}{1+(0.8000)^2}=0.610$$

$$x_4=\frac{1}{1+(0.610)^2}=0.729$$

$x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}$.

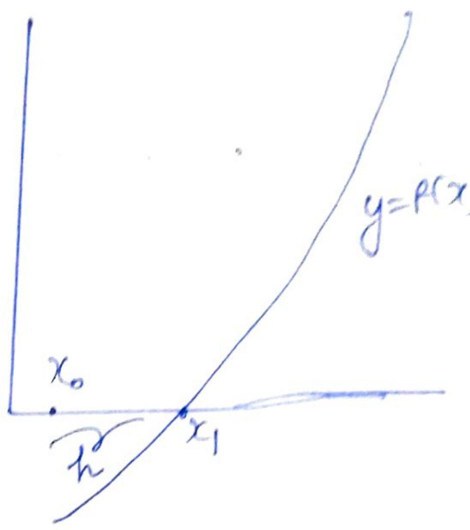
Root of equation $x^3+x-1=0$ is given by $a=0.6823$ Ans.

Newton Raphson Method

· Prove that the rate of convergence of NRM is Two.

Let x_0 is the approximate root of $f(x)=0$

Let $x_1 = x_0+h$ is the correct root.



$X_1 = X_0+h$

$F(X_1)=0$

$F(X_0+h)=0$

Expand $F(X_0+h)$ by Taylor's Series

$F(X_0)+hF'(X_0)+\frac{h^2}{2!}F''(x_0)+\dots\dots=0$

Truncate the higher power term

$F(X_0)+hF'(X_0)=0$

$$h = \frac{-F(X_0)}{F'(X_0)}$$

$X_1 = X_0+h$

$$\boxed{X_1 = X_0 - \frac{F(X_0)}{F'(X_0)}}$$

$$\boxed{X_3 = X_2 - \frac{F(X_2)}{F'(X_2)}}$$

$$\boxed{X_2 = X_1 - \frac{F(X_1)}{F(X_1)}}$$

$$\boxed{X_{n+1} = X_n - \frac{F(X_n)}{F'(X_n)}}$$

We have $X_{i+1} = a + e_{i+1}$

where e_{i+1} is the error

X_{i+1} be the approximate root of $f(x)=0$ at $(i+1)$th stage of iteration.

$$e_{i+1} = \frac{e_i^2 F''(a)}{2F'(a)} + O(e_i^2)$$

$e_{i+1} = \frac{f''(a)}{2f'(a)}$ $\quad \frac{e_{i+1}}{e_i^2} = C$

$$C = \frac{f''(a)}{2'f(a)}$$

Q. Evaluate $\sqrt{12}$ upto 4 decimal by Newton Raphson Method.

Ans $x = \sqrt{12}$

$x^2 = 12$

OR

$F(x) = x^2 - 12 = 0$

$F(3) = (3)^2 - 12 = 9 - 12 = -3$ (-ve)

$F(3.5) = (3.5)^2 - 12 = .25$ (+ve)

The root lies between $x = 3$ and $x = 3.5$ and it is nearer to $x = 3.5$

Now, $F(x) = x^2 - 12$

$F'(x) = 2x$

$F'(x) = 2 \times 3.5 = 7.0$

$x_0 = 3.5$

$F(x_0) = 3.5$

$F'(x_0) = 7.0$

From Newton Raphson method we have

$$x_{i+1} = x_i - \frac{F(x_i)}{F'(x_i)}$$

$$x_i = x_i - \frac{x_i^{\,\circ} - 12}{2x_i}$$

$$x_{i+1} = \frac{x_i^2 + 12}{2x_i}$$

For $i=0$, first Approximation x_i is given by:

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

$$x_{i+1} = \frac{x_0^2 + 12}{2x_0}$$

$$= \frac{(3.5)^2 + 12}{2 \times 3.5} = 3.4643$$

Now for $i=1$; second approximation is given by

$$x_2 = \frac{x_1^2 + 12}{2x_1}$$

$$= \frac{(3.4643)^2 + 12}{2(3.4643)} = 3.4641$$

Similarly, repeat this process to have the third approximation x_3 as given by:

$$x_3 = \frac{x_i^2 + 12}{2x_2}$$

$$= \frac{(3.4641)^2 + 12}{2 \cdot (3.4641)} = 3.4641$$

Ans $\sqrt{12} = 3.4641$ Approximately

Rate of convergence of Iterative method ? Explain :-

Ans: Order (or Rate) of convergence of Iterative methods :-

The convergence of an iterative method is judged by the order (or rate) at which the error between successive approximations to the root (or between the true and calculated root) decreases. The order of convergence of an iterative method is defined in terms of the errors e_i and e_{i+1} in successive approximations.

An iterative method is said to be K^{th} order convergent if K is the largest positive real number such that:

$$\lim_{i \to \infty} \left(\frac{e_{i+1}}{e_i^K} \right) \leq A,$$

where A is a finite number, not zero, called the asymptotic error constant and usually depends on derivative of the function $f(x)$ at x = approximate root.

In other words, the error in any step is proportional to the K^{th} ~~order~~ power of the error in the previous step. Physically, the K^{th} order convergence means that each iteration, the no. of significant digits in each approximation increases K times.

For example, if the value of the root is good to n significant digits in the i^{th} iteration

$$(|e_i| \leq 10^{-n}), \text{ then}$$

$$|e_{i+1}| \leq A|e_i^K| \leq A \times 10^{-nK}$$

Explain Algebraic equation, Polynomial equation and Transcendental equation.

(a) Algebraic equations:-

An equation of type $y = f(x)$ is said to be algebraic if it can be expressed in the form

$$f_n y_n + f_{n-1} y_{n-1} + f_{n-2} y_{n-2} + \dots + f_1 y_1 + f_0 = 0$$

where f_i is an i^{th} order Polynomial in x. we can write the above equation in general form

$$f(x, y) = 0$$

This implies that equation is dependent on x and y variable. Some examples are:

1. $2x + 7y + 20 = 0$ (linear)
2. $5xy + 7y + 3 = 0$ (non-linear)
3. $x^2 + y^3 - xy = 0$ (non-linear)

(b) Polynomial Equations:-

Polynomial equations are a simple class of algebraic equations that are represented as follows:-

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x_1 + a_0 = 0$$

This is called n^{th} degree polynomial which has n roots. The roots may be:

1. real and different
2. real and repeated
3. complex number

Since complex roots appear in pairs. If n is odd, then the polynomial has at least one real root. For example, a cubic equation of the type

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

will have at least one real root and the remaining two may be real or complex root. Some specific examples are as follows:-

1. $3x^5 - x^3 + 5x = 0$
2. $x^3 - 2x^2 + x + 6 = 0$
3. $x^2 - 4x + 4 = 0$

(c) Transcendental Equations:-

A non-algebraic equation is called a transcendental equation. In these equations some functions included like- trigonometric, exponential and logarithmic functions. Some example of transcendental equations are:-

1. $\cos x - xe^x$
2. $e^x \sin x - \frac{1}{2}x = 0$
3. $\log x + x\sin x = 0$

Explain with example the method of false position.

False Position Method :-

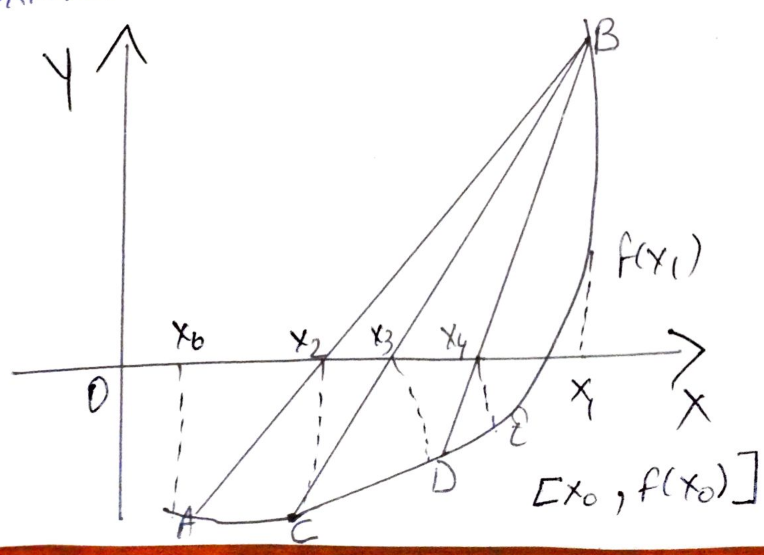
In this Method, we choose two point x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Since the graph $y = f(x)$ crosses the x-axis between these two points, a root must lie in between these points.

Consequently $f(x_0) \cdot f(x_1) < 0$

Equation of the chord joining points $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

The method consist in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with x-axis as an approximation to the root.



So, The abscissa of the point where chord cuts $y=0$ is given by

$$x_2 = x_0 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right] \cdot f(x_0) \quad \text{-----} \quad (1)$$

which is an approximation of the roots.

~~$x_2 \longrightarrow$~~ If now $f(x_0)$ and $f(x_2)$ are of opposite sign, then the root lie between x_0 and x_2.

So, replacing x_1 by x_2 in (1), we obtain the next approximation x_3. However the root could as well lie between x_1 and x_2 then we find x_3 accordingly.

This process is repeated till the root is found to the desired accuracy.

Prove That :-

(i) $\nabla = \Delta E^{-1}$

we know that,

$\Delta y_x = y_{x+h} - y_x = E y_x - y_x = (E-1) y_x$

$\Rightarrow \Delta = E - 1$

or, $E = 1 + \Delta$

and $\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x$

$\therefore \nabla = 1 - E^{-1}$

$= (E-1) E^{-1}$

$= \Delta E^{-1}$ Proved

(ii) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$

$(E^{1/2} + E^{-\frac{1}{2}})(1+\Delta)^{1/2} = (E^{1/2} + E^{-1/2}) E^{1/2}$ $[\because 1 + \Delta = E]$

$= E + 1$

$= 1 + \Delta + 1$

$= 2 + \Delta$ Proved

(iii) $\nabla - \Delta = -\Delta\nabla$

L.H.S $\nabla - \Delta = (1-E^{-1}) - (E-1)$

$$= \frac{(E-1)}{E} - (E-1)$$

$$= (E-1)(E^{-1}-1)$$

$$= -(E-1)(E^{-1}-1)$$

$$= -(E-1)(1-E^{-1})$$

$$= -\nabla\Delta = \text{R.H.S}$$

Proved

(iv) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

R.H.S $= \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}$

$$= \frac{E-1}{\left(\frac{E-1}{E}\right)} - \frac{E}{E-1}$$

$$= E - \frac{1}{E}$$

$$= E - E^{-1}$$

$$= (E-1) + (1-E^{-1})$$

$$= \Delta + \nabla = \text{L.H.S}$$

Proved

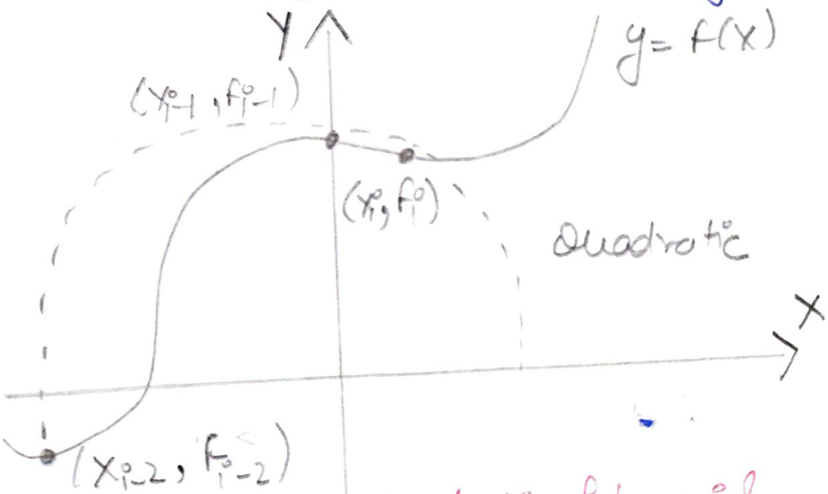
Muller's Method :- State and Prove Muller's Method.

In Muller's Method, the equation $f(x)=0$ is approximated by a second degree polynomial, that is by a quadratic equation that fits through three points in the vicinity of a root. The root of this quadratic equation are then assumed to be approximated to the roots of the equation $f(x)=0$. This method is iterative in nature and does not require the evaluation of derivatives as in the Newton-Raphson Method. This method can also used to determine both real and complex roots of $f(x)=0$.

Suppose x_{i-2}, x_{i-1}, x_i be any three distinct approximations to a root of $f(x)=0$.

Let $f(x_{i-2})=f_{i-2}$, $f(x_{i-1})=f_{i-1}$ and $f(x_i)=f_i$.

Noting that any three distinct points in the XY-Plane uniquely determine a polynomial of second degree.



Quadratic Polynomial

A general polynomial of second degree is given by :-

$$f(x) = ax^2 + bx + c \quad -(1)$$

Suppose it passes through the points (x_{i-2}, f_{i-2}), (x_{i-1}, f_{i-1}) and (x_i, f_i) as shown in figure, then the following equations will be satisfied:-

$$a x_{i-2}^2 + b x_{i-2} + c = f_{i-2} \quad -(2)$$

$$a x_{i-1}^2 + b x_{i-1} + c = f_{i-1} \quad -(3)$$

$$a x_i^2 + b x_i + c = f_i \quad -(4)$$

Eliminating a, b, c from equations (1) to (4)

$$\begin{vmatrix} x^2 & x & 1 & f \\ x_{i-2}^2 & x_{i-2} & 1 & f_{i-2} \\ x_{i-1}^2 & x_{i-1} & 1 & f_{i-1} \\ x_i^2 & x_i & 1 & f_i \end{vmatrix} = 0$$

Which can be conveniently written as:-

$$f = \frac{(x - x_{i-1})(x - x_i)}{(x_{i-2} - x_{i-1})(x_{i-2} - x_i)} f_{i-2} + \frac{(x - x_{i-2})(x - x_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)} f_{i-1}$$

$$+ \frac{(x - x_{i-2})(x - x_{i-1})}{(x_i - x_{i-2})(x_i - x_{i-1})} f_i \quad -(5)$$

This equation, obviously is a second degree polynomial.

Now, introducing the notations:

$$h = x - x_i, \quad h_i = x_i - x_{i-1}, \quad h_{i-1} = x_{i-1} - x_{i-2} \quad -(6)$$

The above equation (5) can be written as:-

$$f = \frac{(h+h_i)h}{-h_{i-1}(-h_{i-1}-h_i)} f_{i-2} + \frac{(h+h_i-h_{i-1})h}{(h_{i-1})(-h_i)} f_{i-1}$$

$$+ \frac{(h+h_i+h_{i-1})(h+h_i)}{(h_i+h_{i-1})h_i} f_i \quad -(7)$$

We further define

$$\lambda = \frac{h}{h_i} = \frac{x-x_i}{x_i-x_{i-1}}, \quad \lambda_i = \frac{h_i}{h_{i-1}}, \quad \delta_i = 1+\lambda_i \quad -(8)$$

The equation (7) can be further simplified to:-

$$f = \frac{1}{\delta_i}[\lambda(\lambda+1)\lambda_i^2 f_{i-2} - \lambda(\lambda+1+\lambda_i^{-1})\lambda_i\delta_i f_{i-1} +$$

$$(\lambda+1)(\lambda+1+\lambda_i^{-1})\lambda_i f_i]$$

or $f = \lambda^2 (f_i - 2\lambda_i^2 - f_{i-1} - \lambda_i\delta_i + f_i\lambda_i)\delta_i^{-1} + \lambda$

$$[f_{i-2}\lambda_i^2 - f_{i-1}\delta_i^2 + f_i(\lambda_i+\delta_i)\delta_i^{-1} + f_i \quad -(9)$$

Now, to compute λ, set $f = 0$ in equation (9) and obtain the quadratic equation as:-

$$\lambda_i (f_{i-2}\lambda_i - f_{i-1}\delta_i + f_i)\lambda^2 + g_i\lambda + \delta_i f_i = 0 \quad -(10)$$

where

$$g_i = f_i - 2\lambda_i^2 - f_{i-1}\delta_i^2 + f_i(\lambda_i + \delta_i)$$

A direct solution of equation (10) leads to loss of accuracy and therefore, to obtain maximum accuracy, we rewrite equation (10) as follows:-

$$\frac{f_i\delta_i}{\lambda^2} + \frac{g_i}{\lambda} + \lambda_i(f_{i-2}\lambda_i - f_{i-1}\delta_i + f_i) = 0 \quad -(12)$$

So that

$$\frac{1}{\lambda} = \frac{-g_i \pm [g_i^2 + 4f_i\delta_i\lambda_i(f_{i-2}\lambda_i - f_{i-1}\delta_i + f_i)]^{1/2}}{2f_i\delta_i}$$

or

$$\lambda = \frac{-2f_i\delta_i}{g_i \pm [g_i^2 - 4f_i\delta_i\lambda_i(f_i - 2\lambda_i - f_{i-1}\delta_i + f_i)]^{1/2}} \quad -(13)$$

Here, the positive sign must be so chosen that the denominator becomes largest in magnitude.

using $x_{i+1} = x_i + h_i\lambda$

we can get a better approximation to the root.