MATRIX FACTORIZATIONS FOR COMPLETE INTERSECTIONS AND MINIMAL FREE RESOLUTIONS

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ABSTRACT. We describe the asymptotic structure of minimal free resolutions over complete intersections of arbitrary codimension. To do this we define a higher matrix factorization of a regular sequence f_1, \ldots, f_c in a way that extends Eisenbud's definition of a matrix factorization of one element. Using this notion we can describe the minimal free resolutions, both over a regular local ring S and over the complete intersection ring S and over the complete intersection ring S and syzygies over S.

1. Introduction

Let S be a regular local ring and f_1, \ldots, f_c be a regular sequence. If N is a finitely generated module over the complete intersection $R := S/(f_1, \ldots, f_c)$, then it can be also considered as an S-module annihilated by f_1, \ldots, f_c . In this paper we will describe the minimal free resolutions of N as an S module and as an S-module when S is a high syzygy over S.

The case when N is the residue field of S is classical: its minimal free resolution over S is the Koszul complex. Perhaps motivated by questions of group cohomology, Tate [Ta], in 1957, gave an elegant description of its minimal free resolution over R.

The understanding of minimal resolutions of an arbitrary module N over R began with the 1974 paper [Gu] of Gulliksen, who showed that $\operatorname{Ext}_R(N,k)$ can be regarded as a finitely generated graded module over a polynomial ring $\mathcal{R} = k[\chi_1, \ldots, \chi_c]$, where c is the codimension of R. He used this to show that the Poincaré series $\sum_i b_i^R(N) x^i$, the generating function of the Betti numbers $b_i^R(N)$, is rational and that the denominator divides $(1-x^2)^c$.

Gulliksen's finite generation result implies that the even Betti numbers $b_{2i}^R(N)$ are eventually given by a polynomial in i, and similarly for the odd Betti numbers. In 1989 Avramov [Av] proved that the two polynomials have the same leading coefficient, and he also extended constructions from group cohomology to the general case. In 1997 Avramov, Gasharov and Peeva [AGP] gave further restrictions on the Betti numbers, establishing in particular that the Betti sequence $\{b_i^R(N)\}_{i\geq q}$

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is either strictly increasing or constant for $q \gg 0$. Examples in [Ei1] and [AGP] show that, as with the Betti numbers, minimal free resolutions over a complete intersection can have intricate structure, but the examples exhibit stable patterns when sufficiently truncated.

The theory of matrix factorizations entered the picture in the 1980 paper [Ei1] of Eisenbud, who introduced them to describe the minimal free resolutions of modules that are high syzygies over hypersurface rings—the case of codimension one. They have had many applications:

Starting with Kapustin and Li [KL], who followed an idea of Kontsevich, physicists discovered amazing connections with string theory — see [As] for a survey. A major advance was made by Orlov [Or1, Or3, Or4, Or5], who showed that matrix factorizations could be used to study Kontsevich's homological mirror symmetry by giving a new description of singularity categories. Matrix factorizations have also proven useful for the study of cluster tilting [DH], Cohen-Macaulay modules and singularity theory [BGS, BHU, CH, Kn], Hodge theory [BFK], Khovanov-Rozansky homology [KR1, KR2], moduli of curves [PV2], quiver and group representations [AM, Av, KST, Re], and other topics, for example, [BDFIK, CM, DM, Dy, Ho, HW, Is, PV1, Se, Sei, Sh].

Orlov [Or2] and subsequent authors, for example [Bu, BW, PV2], have studied modules over a complete intersection $S/(f_1, \ldots, f_c)$ by reducing to families of codimension 1 matrix factorizations over the hypersurace $\sum z_i f_i = 0$ in the projective space \mathbf{P}_S^{c-1} , where the z_i are the homogeneous coordinates of \mathbf{P}_S^{c-1} . By contrast, our theory is focused on understanding minimal free resolutions.

Minimal free resolutions of high syzygies over a codimension two complete intersection were constructed by Avramov and Buchweitz in [AB] in 2000 using the classification of modules over the exterior algebra on two variables. In higher codimension, non-minimal resolutions have been known for over forty years from the work of Shamash [Sh], but minimal free resolutions, which carry much more information and exhibit much more varied behavior, have remained mysterious. We introduce the concept of higher matrix factorization in order to describe the structure of minimal resolutions of high syzygies.

What is a Matrix Factorization?

We briefly review the codimension 1 case. If $0 \neq f \in S$ is an element in a commutative ring then a matrix factorization of f is a pair (d, h) of maps of finitely generated free modules

$$A_0 \xrightarrow{h} A_1 \xrightarrow{d} A_0$$

such that the diagram

$$A_1 \xrightarrow{f} A_0 \xrightarrow{h} A_1 \xrightarrow{d} A_0$$

commutes or, equivalently:

$$dh = f \cdot \mathrm{Id}_{A_0}$$
$$hd = f \cdot \mathrm{Id}_{A_1}.$$

If f is a non-zerodivisor and S is local, then the matrix factorization describes the minimal free resolutions of $M := \operatorname{Coker}(d)$ over the rings S and R := S/(f); if M has no direct summand then the free resolutions are:

$$(1.1) 0 \longrightarrow A_1 \stackrel{d}{\longrightarrow} A_0 \longrightarrow M \longrightarrow 0 \text{ over } S; \text{ and}$$

$$\cdots \stackrel{h}{\longrightarrow} R \otimes A_1 \stackrel{d}{\longrightarrow} R \otimes A_0 \stackrel{h}{\longrightarrow} R \otimes A_1 \stackrel{d}{\longrightarrow} R \otimes A_0 \longrightarrow M \longrightarrow 0 \text{ over } R.$$

Minimal free resolutions of all sufficiently high syzygies over a hypersurface ring are always of this form by [Ei1].

What is a Higher Matrix Factorization?

To extend the theory to higher codimensions, we make a new definition. After giving the definition and an example, we will outline the main results of this paper.

Definition 1.2. Sometimes we will abbreviate "higher matrix factorization" to "HMF". Let $f_1, \ldots, f_c \in S$ be elements of a commutative ring, and set $R = S/(f_1, \ldots, f_c)$. A higher matrix factorization (d, h) with respect to f_1, \ldots, f_c is:

(1) A pair of free finitely generated S-modules A_0 , A_1 with filtrations

$$0 \subseteq A_{\mathfrak{o}}(1) \subseteq \cdots \subseteq A_{\mathfrak{o}}(c) = A_{\mathfrak{o}}, \text{ for } s = 0, 1,$$

such that each $A_s(p-1)$ is a free summand of $A_s(p)$;

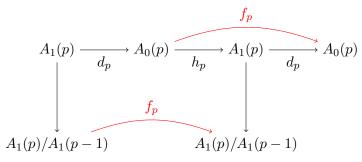
(2) A pair of maps d, h preserving filtrations,

$$\bigoplus_{q=1}^{c} A_0(q) \xrightarrow{h} A_1 \xrightarrow{d} A_0,$$

where we regard $\bigoplus_q A_0(q)$ as filtered by the submodules $\bigoplus_{q \leq p} A_0(q)$; such that, writing

$$A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{d_p} A_0(p)$$

for the induced maps, the diagrams



commute modulo (f_1, \ldots, f_{p-1}) for all p; or, equivalently,

- (a) $d_p h_p \equiv f_p \operatorname{Id}_{A_0(p)} \operatorname{mod}(f_1, \dots, f_{p-1}) A_0(p);$ (b) $\pi_p h_p d_p \equiv f_p \pi_p \operatorname{mod}(f_1, \dots, f_{p-1}) (A_1(p)/A_1(p-1)),$ where π_p denotes the projection $A_1(p) \longrightarrow A_1(p)/A_1(p-1).$

We define the module of the higher matrix factorization (d,h) to be

$$M := \operatorname{Coker}(R \otimes d)$$
.

We refer to modules of this form as higher matrix factorization modules or HMF modules.

In Section 13, we show that a homomorphism of HMF modules induces a morphism of the whole higher matrix factorization structure; see Definition 13.1 and Theorem 13.2 for details. In Section 12 we show that our constructions yield functors to stable categories of Cohen-Macaulay modules. In Section 14 we give a stronger version of Definition 1.2 requiring that the map h is part of a homotopy, and we prove in Theorem 14.2 that an HMF module always has such a strong matrix factorization.

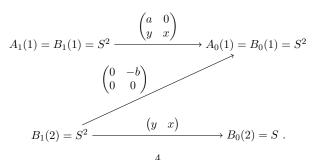
For each $1 \leq p \leq c$, we have a higher matrix factorization $(d_p, (h_1|\cdots|h_p))$ with respect to f_1, \ldots, f_p , where $(h_1|\cdots|h_p)$ denotes the concatenation of the matrices h_1, \ldots, h_p and thus an HMF module

$$M(p) = \operatorname{Coker}(S/(f_1, \dots, f_p) \otimes d_p).$$

This allows us to do induction on p.

If S is local, then we call the higher matrix factorization minimal if d and h are minimal (that is, the image of each map is contained in the maximal ideal times the target).

Example 1.3. Let S = k[a, b, x, y] over a field k, and consider the complete intersection R = S/(xa, yb). Let N = R/(x, y). The module N is a maximal Cohen-Macaulay R-module. The earliest syzygy of N that is an HMF module is the third syzygy M. We can describe the higher matrix factorization for M as follows. After choosing a splitting $A_s(2) = A_s(1) \oplus B_s(2)$, we can represent the map d as



The pair of maps

$$d_1: A_1(1) \xrightarrow{\begin{pmatrix} a & 0 \\ y & x \end{pmatrix}} A_0(1) \quad \text{and} \quad h_1: A_0(1) \xrightarrow{\begin{pmatrix} x & 0 \\ -y & a \end{pmatrix}} A_1(1)$$

is a matrix factorization for the element xa since $d_1h_1 = h_1d_1 = xa \operatorname{Id}$. The map $h_2: A_0 = A_0(2) \longrightarrow A_1 = A_1(2)$ is given by the matrix

$$h_2 = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ x & 0 & b \\ -y & a & 0 \end{pmatrix}, \quad \text{and} \quad d_2 = \begin{pmatrix} a & 0 & 0 & -b \\ y & x & 0 & 0 \\ 0 & 0 & y & x \end{pmatrix}.$$

Hence

Thus d_2h_2 is congruent, modulo (xa), to yb Id. Furthermore, condition (b) of Definition 1.2 is the statement that the two bottom rows in the latter matrix are congruent modulo (xa) to $yb\pi_2$. In the context of the diagram in the definition, with p=2, the fact that the lower left (2×2) -matrix is congruent to 0 modulo $f_1=xa$ is necessary for the map $d_2h_2:A_1(2)\longrightarrow A_1(2)$ to induce a map $A_1(2)/A_1(1)=B_1(2)\longrightarrow A_1(2)/A_1(1)=B_1(2)$.

In the rest of the introduction we focus on the case when S is a regular local ring and $R = S/(f_1, \ldots, f_c)$ is a complete intersection, although most of our results are proved in greater generality. We will keep the notation of Definition 1.2 throughout the introduction.

High Syzygies are Higher Matrix Factorization Modules

The next result was the key motivation for our definition of a higher matrix factorization. A more precise version of this result is proved in Corollary 9.3.

Theorem 1.4. Let S be a regular local ring with infinite residue field, and let $I \subset S$ be an ideal generated by a regular sequence of length c. Set R = S/I, and suppose that N is a finitely generated R-module. Let f_1, \ldots, f_c be a generic choice of elements minimally generating I. If M is a sufficiently high syzygy of N over R, then M is the HMF module of a minimal higher matrix factorization (d,h) with respect to f_1, \ldots, f_c . Moreover $d \otimes R$ and $h \otimes R$ are the first two differentials in the minimal free resolution of M over R.

The meaning of "a sufficiently high syzygy" is explained in Section 7, where we introduce a class of R-modules that we call pre-stable syzygies and show that they have the property given in Theorem 1.4. Given an R-module N we give in

Corollary 9.3 a sufficient condition, in terms of $\operatorname{Ext}_R(N,k)$, for the r-th syzygy module of N to be pre-stable. We also explain more about the genericity condition. Over a local Gorenstein ring, we introduce the concept of a stable syzygy in Section 7 and discuss it in Section 10.

Minimal R-free and S-free Resolutions

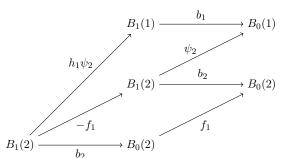
Theorem 1.4 shows that in order to understand the asymptotic behavior of minimal free resolutions over the complete intersection R it suffices to construct the resolutions of HMF modules. This is accomplished by Construction 5.1 and Theorem 5.2.

The finite minimal free resolution over S of an HMF module is given by Construction 3.3 and Theorem 3.4. Here is an outline of the codimension 2 case: Let (d,h) be a codimension 2 higher matrix factorization. We first choose splittings $A_s(2) = B_s(1) \oplus B_s(2)$. Since $d(B_1(1)) \subset B_0(1)$, we can represent the differential d as

$$\mathbf{B}(1): \qquad \qquad B_1(1) \xrightarrow{\qquad b_1 \qquad} B_0(1)$$

$$\mathbf{B}(2): \qquad \qquad B_1(2) \xrightarrow{\qquad b_2 \qquad} B_0(2) \,,$$

which may be thought of as a map of two-term complexes $\psi_2 : \mathbf{B}(2)[-1] \longrightarrow \mathbf{B}(1)$. This extends to a map of complexes $\mathbf{K}(f_1) \otimes \mathbf{B}(2)[-1] \longrightarrow \mathbf{B}(1)$, as in the following diagram:



Theorem 3.4 asserts that this is the minimal S-free resolution of the HMF module $M = \operatorname{Coker}(S/(f_1, f_2) \otimes d)$.

Strong restrictions on the finite minimal S-free resolution of a high syzygy M over the complete intersection $S/(f_1, \ldots, f_c)$ follow from our results: for example, by Corollary 3.13 the minimal presentation matrix of M must include c-1 columns of the form

$$\begin{pmatrix} f_1 & \cdots & f_{c-1} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

for a generic choice of f_1, \ldots, f_c . For instance, in Example 1.3, the presentation matrix of M is

$$\begin{pmatrix} a & 0 & 0 & -b & 0 \\ y & x & 0 & 0 & 0 \\ 0 & 0 & y & x & xa \end{pmatrix},$$

and the last column is of the desired type. There are numerical restrictions as well; see Corollary 9.15 and the remark following it.

Every maximal Cohen-Macaulay $S/(f_1)$ -module is a pre-stable syzygy, but this is not true in higher codimension — one must go further back in the syzygy chain. This is not surprising, since every S-module of finite length is a maximal Cohen-Macaulay module over an artinian complete intersection, and it seems hopeless to characterize the minimal free resolutions of all such modules.

In Corollary 3.10 and Corollary 5.7 we get formulas for the Betti numbers of an HMF module over S and over R respectively. Furthermore, the vector spaces $\operatorname{Ext}_S^i(M,k)$ and $\operatorname{Ext}_R^i(M,k)$ can be expressed as follows.

Corollary 1.5. Suppose that f_1, \ldots, f_c is a regular sequence in a regular local ring S with infinite residue field k, so that $R = S/(f_1, \ldots, f_c)$ is a local complete intersection. Let M be the HMF module of a minimal higher matrix factorization (d,h) with respect to f_1, \ldots, f_c . Using notation as in Definition 1.2, for s = 0,1, choose splittings $A_s(p) = A_s(p-1) \oplus B_s(p)$ for s = 0,1, so

$$A_s(p) = \bigoplus_{1 \le q \le p} B_s(q) .$$

Set $B(p) = B_1(p) \oplus B_0(p)$, where we think of $B_s(p)$ as placed in homological degree s. There are decompositions

$$\operatorname{Ext}_{S}(M,k) \cong \bigoplus_{p=1}^{c} k \langle e_{1}, \dots, e_{p-1} \rangle \otimes \operatorname{Hom}_{S}(B(p),k)$$
$$\operatorname{Ext}_{R}(M,k) \cong \bigoplus_{p=1}^{c} k[\chi_{p}, \dots, \chi_{c}] \otimes \operatorname{Hom}_{S}(B(p),k),$$

as vector spaces, where $k\langle e_1, \ldots, e_{p-1} \rangle$ denotes the exterior algebra on variables of degree 1 and $k[\chi_p, \ldots, \chi_c]$ denotes the polynomial ring on variables of degree 2.

The former formula in 1.5 follows from Remark 3.5 and the latter from Corollary 5.6. We explain in [EPS1] and Corollary 5.6 how the given decompositions reflect certain natural actions of the exterior and symmetric algebras on the graded modules $\operatorname{Ext}_S(M,k)$ and $\operatorname{Ext}_R(M,k)$.

Syzygies over intermediate quotient rings

For each $0 \le p \le c$ set $R(p) := S/(f_1, \ldots, f_p)$. In the case of a codimension 1 matrix factorization (d, h), one can use the data of the matrix factorization to describe two minimal free resolutions, as explained in (1.1). In the case of a codimension c higher matrix factorization we construct the minimal free resolutions

of its HMF module over all c+1 rings

$$S = R(0), S/(f_1) = R(1), \dots, S/(f_1, \dots, f_c) = R(c).$$

See Theorem 6.4 for the intermediate cases.

By Definition 1.2 an HMF module M with respect to the regular sequence f_1, \ldots, f_c determines, for each $p \leq c$, an HMF R(p)-module M(p) with respect to f_1, \ldots, f_p . In the notation and hypotheses as in Theorem 1.4, Corollary 10.5 shows that

$$M(p-1) = \operatorname{Syz}_{2}^{R(p-1)} \left(\operatorname{Cosyz}_{2}^{R(p)} \left(M(p) \right) \right),$$

where Syz(-) and Cosyz(-) denote syzygy and cosyzygy, respectively. Furthermore, Corollary 10.6 says that if we replace M by its first syzygy, then all the modules M(p) are replaced by their first syzygies:

$$(\operatorname{Syz}_1^{R(p)}(M(p)))(p-1) = \operatorname{Syz}_1^{R(p-1)}(M(p-1)).$$

Theorem 11.1 expresses the modules M(p) as syzygies of $Y := \operatorname{Cosyz}_{c+1}^R(M)$ over the intermediate rings R(p) as follows:

$$\operatorname{Syz}_{c+1}^{R(p)}(Y) \cong M(p) \text{ for } p \geq 0.$$

The package CompleteIntersectionResolutions, available from the first author, can compute in Macaulay2 examples of many of the constructions in this paper.

2. Notation and Conventions

Unless otherwise stated, in the rest of the paper all rings are assumed commutative and Noetherian, and all modules are assumed finitely generated.

If S is a local ring with maximal ideal \mathbf{m} then a map of S-modules is called *minimal* if its image is contained in \mathbf{m} times the target.

To distinguish a matrix factorization for one element from the general concept, sometimes we will refer to the former as a codimension 1 matrix factorization or a hypersurface matrix factorization.

We will frequently use the following notation.

Notation 2.1. A higher matrix factorization

$$(d: A_1 \longrightarrow A_0, h: \bigoplus_{p=1}^c A_0(p) \longrightarrow A_1)$$

with respect to f_1, \ldots, f_c as in Definition 1.2 involves the following data:

- a ring S over which A_0 and A_1 are free modules;
- for $1 \le p \le c$, the rings $R(p) := S/(f_1, \ldots, f_p)$, and in particular R = R(c);
- for s = 0, 1, the filtrations $0 = A_s(0) \subseteq \cdots \subseteq A_s(c) = A_s$, preserved by d;
- the induced maps

$$A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{d_p} A_0(p);$$

• the quotients $B_s(p) = A_s(p)/A_s(p-1)$ and the projections $\pi_p : A_1(p) \longrightarrow B_1(p)$;

• the two-term complexes induced by d:

$$\mathbf{A}(p): A_1(p) \xrightarrow{d_p} A_0(p)$$

$$\mathbf{B}(p): B_1(p) \xrightarrow{b_p} B_0(p)$$

• the modules

$$M(p) = \operatorname{Coker}(R(p) \otimes d_p : R(p) \otimes A_1(p) \longrightarrow R(p) \otimes A_0(p)),$$

and in particular, the HMF module M = M(c) of (d, h).

We sometimes write $h = (h_1 | \cdots | h_c)$. We say that the higher matrix factorization is *trivial* if $A_1 = A_0 = 0$.

If $1 \leq p \leq c$ then d_p together with the maps h_q for $q \leq p$, is a higher matrix factorization with respect to f_1, \ldots, f_p ; we write it as $(d_p, h(p))$, where $h(p) = (h_1|\cdots|h_p)$. We call (d_1, h_1) the codimension 1 part of the higher matrix factorization; (d_1, h_1) is a hypersurface matrix factorization for f_1 over S (it could be trivial). If $q \geq 1$ is the smallest number such that $A(q) \neq 0$ and $R' = S/(f_1, \ldots, f_{q-1})$, then writing -' for $R' \otimes -$, the maps

$$b'_q: B_1(q)' \longrightarrow B_0(q)'$$
 and $h'_q: B_0(q)' \longrightarrow B_1(q)'$

form a hypersurface matrix factorization for the element $f_q \in R'$. We call it the top nonzero part of the higher matrix factorization (d, h).

For each $0 \le p \le c$ set $R(p) := S/(f_1, \ldots, f_p)$. The HMF module

$$M(p) = \operatorname{Coker}(R(p) \otimes d_p)$$

is an R(p)-module.

Next, we make some conventions about complexes.

We write $\mathbf{U}[-a]$ for the shifted complex, with $\mathbf{U}[-a]_i = \mathbf{U}_{i+a}$ and differential $(-1)^a d$.

Let (\mathbf{W}, ∂^W) and (\mathbf{Y}, ∂^Y) be complexes. The complex $\mathbf{W} \otimes \mathbf{Y}$ has differential

$$\partial_q^{W \otimes Y} = \sum_{i+j=q} \left((-1)^j \partial_i^W \otimes \operatorname{Id} + \operatorname{Id} \otimes \partial_j^Y \right).$$

A map of complexes $\gamma: \mathbf{W}[a] \longrightarrow \mathbf{Y}$ is homotopic to 0 if there exists a map $\alpha: \mathbf{W}[a+1] \longrightarrow \mathbf{Y}$ such that

$$\gamma = \partial^{\mathbf{Y}} \alpha - \alpha \partial^{\mathbf{W}[a+1]} = \partial^{\mathbf{Y}} \alpha - (-1)^{a+1} \alpha \partial^{\mathbf{W}}.$$

If $\varphi : \mathbf{W}[-1] \longrightarrow \mathbf{Y}$ is a map of complexes, so that $-\varphi \partial^W = \partial^Y \varphi$, then the mapping cone $\mathbf{Cone}(\varphi)$ is the complex $\mathbf{Cone}(\varphi) = \mathbf{Y} \oplus \mathbf{W}$ with modules $\mathbf{Cone}(\varphi)_i = Y_i \oplus W_i$ and differential

$$\begin{array}{ccc} Y_i & W_i \\ Y_{i-1} & \left(\begin{matrix} \partial_i^Y & \varphi_{i-1} \\ 0 & \partial_i^W \end{matrix} \right). \\ 9 \end{array}$$

If f is an element in a ring S then we write $\mathbf{K}(f)$ for the two-term Koszul complex $f: eS \longrightarrow S$, where we think of e as an exterior variable. If (\mathbf{W}, ∂) is any complex of S-modules we write $\mathbf{K}(f) \otimes \mathbf{W} = e\mathbf{W} \oplus \mathbf{W}$; it is the mapping cone of the map $\mathbf{W} \longrightarrow \mathbf{W}$ that is $(-1)^i f: W_i \longrightarrow W_i$.

3. The minimal S-free resolution of a higher matrix factorization module

We will use the notation in 2.1 throughout this section. Suppose that M is the HMF module of a higher matrix factorization (d, h) with respect to a regular sequence f_1, \ldots, f_c in a local ring S. Theorem 3.4 expresses the minimal S-free resolution of M as an iterated mapping cone of Koszul extensions, which we will now define in 3.1. We say that a complex (\mathbf{U}, d) is a left complex if $U_j = 0$ for j < 0; thus for example the free resolution of a module is a left complex.

Definition 3.1. Let S be a ring. Let \mathbf{B} and \mathbf{L} be S-free left complexes, and let $\psi : \mathbf{B}[-1] \longrightarrow \mathbf{L}$ be a map of complexes. Note that ψ is zero on B_0 . Denote $\mathbf{K} := \mathbf{K}(f_1, \dots, f_p)$ the Koszul complex on $f_1, \dots, f_p \in S$. An (f_1, \dots, f_p) -Koszul extension of ψ is a map of complexes

$$\Psi: \mathbf{K} \otimes \mathbf{B}[-1] \longrightarrow \mathbf{L}$$

extending

$$\mathbf{K}_0 \otimes \mathbf{B}[-1] = \mathbf{B}[-1] \xrightarrow{\psi} \mathbf{L}$$

whose restriction to $\mathbf{K} \otimes B_0$ is zero.

The next proposition shows that Koszul extensions exist in the case we will use.

Proposition 3.2. Let f_1, \ldots, f_p be elements of a ring S. Let \mathbf{L} be a free resolution of an S-module N annihilated by f_1, \ldots, f_p . Let $\psi : \mathbf{B}[-1] \longrightarrow \mathbf{L}$ be a map from an S-free left complex \mathbf{B} .

- (1) There exists an (f_1, \dots, f_p) -Koszul extension of ψ .
- (2) If S is local, the elements f_i are in the maximal ideal, **L** is minimal, and the map ψ is minimal, then every Koszul extension of ψ is minimal.

PROOF: Set $\mathbf{K} = \mathbf{K}(f_1, \dots, f_p)$, and let $\varphi : \mathbf{K} \otimes \mathbf{L} \longrightarrow \mathbf{L}$ be any map extending the identity map $S/(f_1, \dots f_p) \otimes N \longrightarrow N$. The map φ composed with the tensor product map $\mathrm{Id}_{\mathbf{K}} \otimes \psi$ is a Koszul extension, proving existence. For the second statement, note that if ψ is minimal, then so is the Koszul extension we have constructed. Since any two extensions of a map from a free complex to a resolution are homotopic, it follows that every Koszul extension is minimal.

We can now describe our construction of an S-free resolution of an HMF module.

Construction 3.3. Let (d, h) be a higher matrix factorization with respect to a regular sequence f_1, \ldots, f_c in a ring S. Using notation as in 2.1, we choose splittings $A_s(p) = A_s(p-1) \oplus B_s(p)$ for s = 0, 1, so

$$A_s(p) = \bigoplus_{1 < q < p} B_s(q)$$

and denote by ψ_p the component of d_p mapping $B_1(p)$ to $A_0(p-1)$.

- Set $\mathbf{L}(1) := \mathbf{B}(1)$, a free resolution of M(1) with zero-th term $B_0(1) = A_0(1)$.
- For $p \ge 2$, suppose that $\mathbf{L}(p-1)$ is an S-free resolution of M(p-1) with zero-th term $L_0(p-1) = A_0(p-1)$. Let

$$\psi_p': \mathbf{B}(p)[-1] \longrightarrow \mathbf{L}(p-1)$$

be the map of complexes induced by $\psi_p: B_1(p) \longrightarrow A_0(p-1)$, and let

$$\Psi_p: \mathbf{K}(f_1,\ldots,f_{p-1})\otimes \mathbf{B}(p)[-1] \longrightarrow \mathbf{L}(p-1)$$

be an (f_1, \ldots, f_{p-1}) -Koszul extension. Set $\mathbf{L}(p) = \mathbf{Cone}(\Psi_p)$.

The following theorem implies that $H_0(\mathbf{L}(p)) = M(p)$, so that the construction can be carried through to $\mathbf{L}(c)$. Note that $\mathbf{L}(c)$ has a filtration with successive quotients of the form $\mathbf{K}(f_1, \ldots, f_{p-1}) \otimes \mathbf{B}(p)$.

Theorem 3.4. With notation and hypotheses as in 3.3 the complex $\mathbf{L}(p)$ is an S-free resolution of M(p) for $p=1,\ldots,c$. Moreover, if S is local and (d,h) is minimal, then the resolution $\mathbf{L}(p)$ is minimal.

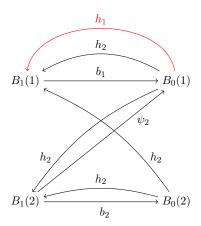
Remark 3.5. The underlying free module of the Koszul complex $\mathbf{K}(f_1, \ldots, f_{p-1})$ is the exterior algebra on generators e_i corresponding to the f_i . Set $B(p) = B_0(p) \oplus B_1(p)$, and thus we get that as an S-free module $\mathbf{L}(p)$ is

$$\mathbf{L}(p) = \mathbf{L}(p-1) \oplus S\langle e_1, \dots, e_{p-1} \rangle \otimes_S B(p)$$
.

The only non-zero components of the differential that land in $B_0(p)$ are those of the map d and

$$f_i: e_i B_0(p) \longrightarrow B_0(p)$$
 for $i < p$.

Example 3.6. Here is the case of codimension 2. After choosing splittings $A_s(2) = B_s(1) \oplus B_s(2)$, a higher matrix factorization (d, h) for a regular sequence $f_1, f_2 \in S$ is a diagram of free S-modules



where d has components b_1, b_2, ψ_2 , and for some C, D we have

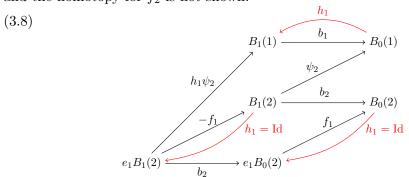
$$b_1 h_1 = f_1 \operatorname{Id}_{B_0(1)} \quad \text{on } B_0(1)$$

$$h_1 b_1 = f_1 \operatorname{Id}_{B_1(1)} \quad \text{on } B_1(1)$$

$$dh_2 = f_2 \operatorname{Id} + f_1 C \quad \text{on } B_0(1) \oplus B_0(2)$$

$$\pi_2 h_2 d_2 = f_2 \pi_2 + f_1 D \pi_2 \quad \text{on } B_1(1) \oplus B_1(2).$$

Applying Theorem 3.4, we may write the S-free resolution of the HMF module $M = \operatorname{Coker}(S/(f_1, f_2) \otimes d)$ in (3.8). The homotopy for f_1 is shown with red arrows, and the homotopy for f_2 is not shown.



Before giving the proof of Theorem 3.4 we exhibit some consequences for the structure of modules that can be expressed as HMF modules. We keep notation as in 2.1.

Corollary 3.9. With notation and hypotheses as in 3.3, if in addition S is local and the higher matrix factorization is minimal, then the minimal S-free resolution of M has a filtration by minimal S-free resolutions of the modules $M(p) := \operatorname{Coker}(S/(f_1, \ldots, f_p) \otimes d_p)$, whose successive quotients are the complexes

$$\mathbf{K}(f_1,\ldots,f_{p-1})\otimes_S\mathbf{B}(p).$$

Corollary 3.10. With notation and hypotheses as in 3.3, if in addition S is local and the higher matrix factorization (d,h) is minimal, then the Poincaré series of the HMF module M of the higher matrix factorization (d,h) is

$$\mathcal{P}_{M}^{S}(x) = \sum_{1 \le p \le c} (1+x)^{p-1} \left(x \operatorname{rank} \left(B_{1}(p) \right) + \operatorname{rank} \left(B_{0}(p) \right) \right).$$

Corollary 3.11. With notation and hypotheses as in 3.3, if $M(p) \neq 0$ then its projective dimension over S is p, and f_{p+1} is a non-zerodivisor on M(p). If S is a local Cohen-Macaulay ring then the module M(p) is a maximal Cohen-Macaulay R(p)-module.

PROOF: The resolution $\mathbf{L}(p)$ has length p, and no module annihilated by a regular sequence of length p can have projective dimension < p. The Cohen-Macaulay statement follows from this and the Auslander-Buchsbaum formula.

Suppose that f_{p+1} is a zerodivisor on M(p). Hence, f_{p+1} is contained in a minimal prime \mathbf{n} over $\operatorname{ann}_S(M(p))$. Since f_1, \ldots, f_p annihilate M(p), they are contained in \mathbf{n} as well. Therefore, the height of \mathbf{n} is $\geq p+1$. The projective dimension of $M(p)_{\mathbf{n}}$ over $S_{\mathbf{n}}$ is less or equal to p, so it is strictly less than $\dim(S_{\mathbf{n}})$. Thus the minimal $S_{\mathbf{n}}$ -free resolution of $M(p)_{\mathbf{n}}$ is a complex of length $< \dim(S_{\mathbf{n}})$ and its homology $M(p)_{\mathbf{n}}$ has finite length. This is a contradiction by the New Intersection Theorem, cf. [PW].

Corollary 3.12. With notation and hypotheses as in 3.3, if in addition S is local and the higher matrix factorization is minimal, then M(p) has no R(p)-free summands.

PROOF: If M(p) had an R(p)-free summand, then with respect to suitable bases the minimal presentation matrix $R(p) \otimes d_p$ of M(p) would have a row of zeros. Thus a matrix representing $R(p-1) \otimes d_p$ would have a row of elements divisible by f_p . Composing with h_p we see that a matrix representing $R(p-1) \otimes d_p h_p$ would have a row of elements in $\mathbf{m} f_p$. However $R(p-1) \otimes (d_p h_c) = f_p \mathrm{Id}$, a contradiction.

The following result shows that HMF modules are quite special. Looking ahead to Corollary 9.3, we see that it can be applied to any S module that is a sufficiently high syzygy over R.

Corollary 3.13. With notation and hypotheses as in 3.3, suppose in addition that S is local and that the higher matrix factorization (d, h) is minimal, and let $n = \sum_{p} \operatorname{rank} B_0(p)$, the rank of the target of d. In a suitable basis, the minimal presentation matrix of the HMF module M consists of the matrix d concatenated with an $(n \times \sum_{p} (p-1)\operatorname{rank} B_0(p))$ -matrix that is the direct sum of matrices of the form

We remark that a similar property holds for all matrices of the differential in the minimal free resolution of M.

PROOF: In the notation of Construction 3.3, the given direct sum is the part of the map $\mathbf{L}_1(c) \longrightarrow \mathbf{L}_0(c)$ that corresponds to

$$\bigoplus_{p} (\mathbf{K}(f_1,\ldots,f_{p-1}))_1 \otimes B_0(p) \longrightarrow \bigoplus_{p} B_0(p).$$

Theorem 3.4 and Corollary 3.12 allow us to express the Betti numbers of an HMF module in terms of the ranks of the modules $B_s(p)$. Recall that if S is a local ring with residue field k then the Betti numbers of a module N over S are $b_i^S(N) = \dim_k(\operatorname{Tor}_i^S(N, k))$. They are often studied via the Poincaré series:

$$\mathcal{P}_N^S(x) = \sum_{i>0} b_i^S(N) x^i.$$

Corollary 3.10 makes it worthwhile to ask whether there are interesting restrictions on the ranks of the $B_s(p)$. Here is a first result in this direction:

Corollary 3.14. With notation and hypotheses as in 3.3, suppose in addition that S is local and Cohen-Macaulay and that the higher matrix factorization (d,h) is minimal. If $B_1(p) = 0$ for some p, then $B_1(q) = B_0(q) = 0$ for all $q \leq p$.

PROOF: Suppose that $B_1(p) = 0$. If $B_0(p) \neq 0$ then M(p) would have a free summand, contradicting Corollary 3.12, so $B_0(p) = 0$ as well. It follows that h_p restricts to a map $A_0(p-1) \longrightarrow A_1(p-1)$, and thus M(p-1) is annihilated by f_p . However, if $M(p-1) \neq 0$ then by Corollary 3.11 it would be a maximal Cohen-Macaulay module over the ring R(p-1), and this is a contradiction. Thus M(p-1) = 0, so $B_s(q) = 0$ for $q \leq p$.

Example 3.15. Let S = k[x, y, z] and let f_1, f_2 be the regular sequence xz, y^2 . We give an example of a higher matrix factorization with respect to f_1, f_2 such that $B_1(2) \neq 0$, but $B_0(2) = 0$. If

$$B_1(1) = S^2 \xrightarrow{\begin{pmatrix} z & -y \\ 0 & x \end{pmatrix}} B_0(1) = S^2$$

$$B_1(2) = S \xrightarrow{\qquad \qquad } B_0(2) = 0,$$

and

$$h_1 = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$$
 and $h_2 = \begin{pmatrix} 0 & 0 \\ -y & 0 \\ x & y \end{pmatrix}$,

then (d, h) is a higher matrix factorization.

In the case of higher matrix factorizations that come from high syzygies (stable matrix factorizations) Corollary 3.14 can be strengthened further: $B_0(p) = 0$ implies $B_1(p) = 0$ as well; see Corollary 9.14. This is not the case in general, as the above example shows.

PROOF OF THEOREM 3.4: The minimality statement follows at once from the construction and Proposition 3.2(2). Thus it suffices to prove the first statement.

Note that $d_1 = b_1$. The equations in the definition of a higher matrix factorization imply in particular that $h_1b_1 = b_1h_1 = f_1\mathrm{Id}$, so b_1 is a monomorphism. Note that $\operatorname{Coker}(d_1)$ is annihilated by f_1 . Thus $\mathbf{L}(1) = \mathbf{B}(1)$ is an S-free resolution of

$$M(1) = \operatorname{Coker}(R(1) \otimes d_1) = \operatorname{Coker}(d_1).$$

To complete the proof we do induction on p. By induction hypothesis

$$\mathbf{L}(p-1): \cdots \longrightarrow L_1(p-1) \longrightarrow L_0(p-1)$$

is a free resolution of M(p-1). Since $L_0(p-1)=A_0(p-1)$, the map ψ_p defines a morphism of complexes $\psi_p': \mathbf{B}(p)[-1] \longrightarrow \mathbf{L}(p-1)$ and thus a mapping cone

$$\cdots \longrightarrow L_2(p-1) \longrightarrow L_1(p-1) \xrightarrow{\psi_p} L_0(p-1)$$

$$B_1(p) \xrightarrow{b_p} B_0(p).$$

To simplify the notation, denote by **K** the Koszul complex $\mathbf{K}(f_1,\ldots,f_{p-1})$ of f_1, \ldots, f_{p-1} , and write $\kappa_i : \wedge^i S^{p-1} \longrightarrow \wedge^{i-1} S^{p-1}$ for its differential. Also, set $B_s := B_s(p) \text{ and } \mathbf{B} : B_1 \xrightarrow{b_p} B_0.$

Since M(p-1) is annihilated by (f_1,\ldots,f_{p-1}) , Proposition 3.2 shows that there exists a Koszul extension $\Psi_p: \mathbf{K} \otimes \mathbf{B}[-1] \longrightarrow \mathbf{L}(p-1)$ of ψ_p' . Let $(\mathbf{L}(p), \epsilon)$ be the mapping cone of Ψ_p , and note that the zero-th terms of $\mathbf{L}(p)$ is $L_0 =$ $L_0(p-1) \oplus B_0 = A_0(p)$. We will show that $\mathbf{L}(p)$ is a resolution of M(p).

We first show that $H_0(\mathbf{L}(p)) = \operatorname{Coker}(\epsilon_1) = M(p)$. If we drop the columns corresponding to B_1 from a matrix for ϵ_1 we get a presentation of $M(p-1) \oplus$ $(R(p-1)\otimes B_0(p-1))$, so $Coker(\epsilon_1)$ is annihilated by (f_1,\ldots,f_{p-1}) . Moreover, the map $h_p: A_0(p) \longrightarrow A_1(p) \subset L_1(p)$ defines a homotopy for multiplication by f_p modulo (f_1, \ldots, f_{p-1}) , so $\operatorname{Coker}(\epsilon_1)$ is annihilated by f_p as well. Thus $\operatorname{Coker}(\epsilon_1) =$ $\operatorname{Coker}(R(p) \otimes \epsilon_1) = M(p)$ as required.

We next analyze the homology of the complex $\mathbf{K} \otimes \mathbf{B}$. It is isomorphic to $\mathbf{B} \otimes \mathbf{K}$, which is the mapping cone of the map

$$(-1)^i b_p \otimes \operatorname{Id}: B_1[-1] \otimes K_i \longrightarrow B_0 \otimes K_i,$$

so there is a long exact sequence

$$\cdots \longrightarrow H_i(\mathbf{K} \otimes B_1) \longrightarrow H_i(\mathbf{K} \otimes B_0) \longrightarrow H_i(\mathbf{K} \otimes \mathbf{B}) \longrightarrow H_{i-1}(\mathbf{K} \otimes B_1) \longrightarrow \cdots$$

Since $\mathbf{K} \otimes B_s$ is a resolution of $R(p-1) \otimes B_s$ we see that $H_i(\mathbf{K} \otimes \mathbf{B}) = 0$ for i > 1 and there is a four-term exact sequence

$$0 \longrightarrow H_1(\mathbf{K} \otimes \mathbf{B}) \longrightarrow R(p-1) \otimes B_1 \xrightarrow{R(p-1) \otimes b_p} R(p-1) \otimes B_0 \longrightarrow H_0(\mathbf{K} \otimes \mathbf{B}) \longrightarrow 0.$$

Since $\mathbf{L}(p)$ is the mapping cone of Ψ_p , we have a long exact sequence in homology of the form

$$\cdots \longrightarrow H_i(\mathbf{L}(p-1)) \longrightarrow H_i(\mathbf{L}(p)) \longrightarrow H_i(\mathbf{K} \otimes \mathbf{B}) \xrightarrow{\Psi_{p_*}} H_{i-1}(\mathbf{L}(p-1)) \longrightarrow \cdots,$$

so from the vanishing of the $H_i(\mathbf{K} \otimes \mathbf{B})$ for i > 1 we see that $H_i(\mathbf{L}(p)) = 0$ for i > 1.

It remains to prove only that $H_1(\mathbf{L}(p)) = 0$, or equivalently that the map

$$\Psi_{p_*}: H_1(\mathbf{K} \otimes \mathbf{B}) \longrightarrow H_0(\mathbf{L}(p-1)) = M(p-1)$$

is a monomorphism. From the four-term exact sequence above we see that

$$H_1(\mathbf{K} \otimes \mathbf{B}) = \operatorname{Ker}(R(p-1) \otimes b_p).$$

Also note that by construction the map

$$\Psi_{p_*}: \operatorname{Ker}(R(p-1) \otimes b_p) \longrightarrow H_0(\mathbf{L}(p-1)) = \operatorname{Coker}(R(p-1) \otimes d_{p-1})$$

is induced by

$$\psi_p: R(p-1) \otimes B_1(p) \longrightarrow R(p-1) \otimes A_0(p-1)$$
.

Since $\overline{L}_0(p-1) = \overline{A}_0(p-1)$, the proof is finished by the next Lemma 3.16, which we will use again in Section 5.

Lemma 3.16. With notation and hypotheses as in Construction 3.3, ψ_p induces a monomorphism from $\operatorname{Ker}(R(p-1)\otimes b_p)$ to $\operatorname{Coker}(R(p-1)\otimes d_{p-1})$.

PROOF: To simplify notation we write $\overline{-}$ for $R(p-1) \otimes -$. Consider the diagram:

$$u \in \overline{A}_1(p-1) \xrightarrow{\overline{d}_{p-1}} \overline{A}_0(p-1)$$

$$v \in \overline{B}_1(p) \xrightarrow{\overline{b}_p} \overline{B}_0(p).$$

We must show that if $v \in \text{Ker}(\overline{b}_p)$ and $\overline{\psi}_p(v) = \overline{d}_{p-1}(u)$ for some $u \in \overline{A}_1(p-1)$, then v = 0.

Write $\overline{\pi}_p$ for the projection of $\overline{A}_1(p) = \overline{A}_1(p-1) \oplus \overline{B}_1(p)$ to $\overline{B}_1(p)$, and note that \overline{d}_p is the sum of the three maps in the diagram above. Our equations say that $d_p(-u,v)=0$. By condition (b) in Definition 1.2,

$$f_p v = f_p \overline{\pi}_p(-u, v) = \overline{\pi}_p \overline{h}_p \overline{d}_p(-u, v) = 0.$$

Since f_p is a non-zero divisor in R(p-1), it follows that v=0.

4. Resolutions with a surjective CI operator

We begin by recalling the definition of CI operators. Suppose that $f_1, \ldots, f_c \in S$ is a regular sequence and (\mathbf{V}, ∂) is a complex of free modules over $R = S/(f_1, \ldots, f_c)$. Suppose that $\widetilde{\mathbf{V}}$ is a lifting of \mathbf{V} to S, that is, a sequence of free modules \widetilde{V}_i and maps $\widetilde{\partial}_{i+1} : \widetilde{V}_{i+1} \longrightarrow \widetilde{V}_i$ such that $\partial = R \otimes \widetilde{\partial}$. Since $\partial^2 = 0$ we can choose maps $\widetilde{t}_j : \widetilde{V}_{i+1} \longrightarrow \widetilde{V}_{i-1}$, where $1 \leq j \leq c$, such that

$$\widetilde{\partial}^2 = \sum_{j=1}^c f_j \widetilde{t}_j.$$

We set

$$t_j := R \otimes \widetilde{t_j}.$$

Since

$$\sum_{j=1}^{c} f_{j} \widetilde{t}_{j} \, \widetilde{\partial} = \widetilde{\partial}^{3} = \sum_{j=1}^{c} f_{j} \widetilde{\partial} \, \widetilde{t}_{j},$$

and the f_i form a regular sequence, we see that each t_j commutes with ∂ , and thus the t_j define a map of complexes $\mathbf{V}[-2] \longrightarrow \mathbf{V}$, [Ei1, 1.1]. In the case c = 1, we have $\widetilde{\partial}^2 = f_1 \widetilde{t}_1$ and we sometimes write $\widetilde{t}_1 = \frac{1}{f_1} \widetilde{\partial}^2$ and call it the *lifted CI operator*.

[Ei1, 1.2 and 1.5] shows that the operators t_j are, up to homotopy, independent of the choice of liftings. They are called the *CI operators* (sometimes called Eisenbud operators) associated to the sequence f_1, \ldots, f_c .

We next recall the definition of higher homotopies and the Shamash construction. The version for a single element is due to Shamash [Sh]; [Ei2] treats the more general case of a collection of elements.

Definition 4.1. Let $f_1, \ldots, f_c \in S$, and **G** be a free complex of S-modules. We denote $\mathbf{a} = (a_1, \ldots, a_c)$, where each $a_i \geq 0$ is an integer, and set $|\mathbf{a}| = \sum_i a_i$. A system of higher homotopies σ for f_1, \ldots, f_c on **G** is a collection of maps

$$\sigma_{\mathbf{a}}: \mathbf{G} \longrightarrow \mathbf{G}[-2|\mathbf{a}|+1]$$

of the underlying modules such that the following three conditions are satisfied:

- (1) σ_0 is the differential on **G**.
- (2) For each $1 \leq i \leq c$, the map $\sigma_0 \sigma_{\mathbf{e}_i} + \sigma_{\mathbf{e}_i} \sigma_0$ is multiplication by f_i on \mathbf{G} , where \mathbf{e}_i is the *i*-th standard vector.
- (3) If **a** is a multi-index with $|\mathbf{a}| \ge 2$, then $\sum_{\mathbf{b}+\mathbf{s}=\mathbf{a}} \sigma_{\mathbf{b}} \sigma_{\mathbf{s}} = 0$.

A system of higher homotopies σ for one element $f \in S$ on \mathbf{G} consists of maps $\sigma_j : \mathbf{G} \longrightarrow \mathbf{G}[-2j+1]$ for $j = 0, 1, \ldots$, and will be denoted $\{\sigma_j\}$.

Proposition 4.2. [Ei2, Sh] If **G** is a free resolution of an S-module annihilated by elements $f_1, \ldots, f_c \in S$, then there exists a system of higher homotopies on **G** for f_1, \ldots, f_c .

For the reader's convenience we present a short proof following [Sh]:

PROOF: It is well-known that homotopies $\sigma_{\mathbf{e}_i}$ satisfying (2) in Definition 4.1 exist. Equation (3) in 4.1 can be written as

$$d\sigma_{\mathbf{a}} = -\sum_{\substack{\mathbf{b}+\mathbf{s}=\mathbf{a}\\\mathbf{b}\neq\mathbf{0}}} \sigma_{\mathbf{b}}\sigma_{\mathbf{s}}.$$

As G is a free resolution, in order to show by induction on a and on the homological degree that the desired $\sigma_{\mathbf{a}}$ exists, it suffices to show that the right-hand side is annihilated by d. Indeed,

$$\begin{split} &-\sum_{\substack{\mathbf{b}+\mathbf{s}=\mathbf{a}\\\mathbf{b}\neq\mathbf{0}}}(d\sigma_{\mathbf{b}})\sigma_{\mathbf{s}} = \sum_{\substack{\mathbf{b}+\mathbf{s}=\mathbf{a}\\\mathbf{b}\neq\mathbf{0}}}\sum_{\substack{\mathbf{m}+\mathbf{r}=\mathbf{b}\\\mathbf{r}\neq\mathbf{0}}}\sigma_{\mathbf{r}}\sigma_{\mathbf{m}}\sigma_{\mathbf{s}} - \sum_{\{i:\,\mathbf{e_i}<\mathbf{a}\}}f_i\sigma_{\mathbf{a}-\mathbf{e_i}} \\ &= \sum_{\substack{\mathbf{m}+\mathbf{r}+\mathbf{s}=\mathbf{a}\\\mathbf{r}\neq\mathbf{0}}}\sigma_{\mathbf{r}}\sigma_{\mathbf{m}}\sigma_{\mathbf{s}} - \sum_{\{i:\,\mathbf{e_i}<\mathbf{a}\}}f_i\sigma_{\mathbf{a}-\mathbf{e_i}} \\ &= -\sum_{\{i:\,\mathbf{e_i}<\mathbf{a}\}}f_i\sigma_{\mathbf{a}-\mathbf{e_i}} + \sum_{\mathbf{r}\neq\mathbf{0}}\sigma_{\mathbf{r}}\left(\sum_{\mathbf{m}+\mathbf{s}=\mathbf{a}-\mathbf{r}}\sigma_{\mathbf{m}}\sigma_{\mathbf{s}}\right) \\ &= \sum_{\substack{\mathbf{r}\neq\mathbf{0}\\\mathbf{r}\neq\mathbf{a}-\mathbf{e}:}}\sigma_{\mathbf{r}}\left(\sum_{\mathbf{m}+\mathbf{s}=\mathbf{a}-\mathbf{r}}\sigma_{\mathbf{m}}\sigma_{\mathbf{s}}\right) + \sum_{\{i:\,\mathbf{e_i}<\mathbf{a}\}}\sigma_{\mathbf{a}-\mathbf{e_i}}(\sigma_{\mathbf{e_i}}\sigma_{\mathbf{0}} + \sigma_{\mathbf{0}}\sigma_{\mathbf{e_i}} - f_i) = 0\,, \end{split}$$

where the first and the last equalities hold by induction hypothesis.

Construction 4.3. (cf. [Ei1, Section 7]) Suppose that f_1, \ldots, f_c are elements in a ring S, and that G is a free complex over S with a system σ of higher homotopies. This gives rise to a new complex $Sh(\mathbf{G}, \sigma)$. To define it, we will write $S\{y_1, \ldots, y_G\}$ for the divided power algebra over S on variables y_1, \ldots, y_c ; thus,

$$S\{y_1,\ldots,y_c\} \cong \operatorname{Hom}_{\operatorname{graded}} S\operatorname{-modules}(S[t_1,\ldots,t_c],S) = \oplus Sy_1^{(i_1)}\cdots y_c^{(i_c)}$$

where the $y_1^{(i_1)} \cdots y_c^{(i_c)}$ form the dual basis to the monomial basis of the polynomial ring $S[t_1,\ldots,t_c]$. We will use the fact that $S\{y_1,\ldots,y_c\}$ is an $S[t_1,\ldots,t_c]$ -module with action $t_j y_j^{(i)} = y_j^{(i-1)}$ (see [Ei3, Appendix 2]). Set $R = S/(f_1, \ldots, f_c)$. The graded module

$$S\{y_1,\ldots,y_c\}\otimes \mathbf{G}\otimes R,$$

where each y_i has degree 2, becomes a free complex over R when equipped with the differential

$$\delta := \sum t^{\mathbf{a}} \otimes \sigma_{\mathbf{a}} \otimes R.$$

This complex is called the *Shamash complex* and denoted $Sh(\mathbf{G}, \sigma)$.

In the case when we consider only one element $f \in S$, we denote the divided power algebra by $S\{y\}$, where the $y^{(i)}$ form the dual basis to the basis t^i of the polynomial ring S[t].

Proposition 4.4. [Ei1, Sh] Let f_1, \ldots, f_c be a regular sequence in a ring S, and let N be a module over $R := S/(f_1, \ldots, f_c)$. If G is an S-free resolution of N and σ is a system of higher homotopies for f_1, \ldots, f_c on \mathbf{G} , then $\mathrm{Sh}(\mathbf{G}, \sigma)$ is an R-free resolution of N.

Construction 4.5. In [Ei1, 1.2 and 1.5] Eisenbud shows that the CI operators are, up to homotopy, independent of the choice of liftings, and also that they commute up to homotopy. If S is local with maximal ideal \mathbf{m} and residue field k, and \mathbf{V} is an R-free resolution of an R-module N, then the CI operators t_i induce well-defined, commutative maps χ_j on $\operatorname{Ext}_R(N,k)$, and thus make $\operatorname{Ext}_R(N,k)$ into a module over the polynomial ring $\mathcal{R} := k[\chi_1, \cdots, \chi_c]$, where the variables χ_i have degree 2. The χ_i are also called CI operators. By [Ei1, Proposition 1.2], the action of χ_j can be defined using any CI operators on any R-free resolution of N. Because the χ_j have degree 2, we may split any \mathcal{R} -module into even degree and odd degree parts; in particular, we write

$$\operatorname{Ext}_R(N,k) = \operatorname{Ext}_R^{even}(N,k) \oplus \operatorname{Ext}_R^{odd}(N,k).$$

A version of the following result was first proved in [Gu] by Gulliksen, who used a different construction of operators on Ext. Other constructions of operators were introduced and used by Avramov [Av], Avramov-Sun [AS], Eisenbud [Ei1], and Mehta [Me]. The relations between these constructions were explained by Avramov and Sun [AS]. We will use only the construction from [Ei1] outlined at the beginning of this section. Using that construction, we provide a new and short proof of the following result.

Theorem 4.6. [AS, Ei1, Gu] Let f_1, \ldots, f_c be a regular sequence in a local ring S with residue field k, and set $R = S/(f_1, \ldots, f_c)$. If N is an R-module with finite projective dimension over S, then the action of the CI operators makes $\operatorname{Ext}_R(N,k)$ into a finitely generated $\mathcal{R} := k[\chi_1, \dots, \chi_c]$ -module.

Proof: Let G be a finite S-free resolution of N. By Proposition 4.2, there exists a system of higher homotopies on G. Proposition 4.4 shows that $Sh(G, \sigma)$ is an R-free resolution of N. Consider its dual. By [Ei1, Theorem 7.2] (also see Construction 4.7), the CI operators can be chosen to act on $Sh(\mathbf{G}, \sigma)$ as multiplication by the variables, and thus they commute. By the construction of the Shamash resolution, it is clear that $\operatorname{Hom}_R(\operatorname{Sh}(\mathbf{G},\sigma),k)$ is a finitely generated module over \mathcal{R} . As the CI operators commute with the differential, it follows that both the kernel and the image of the differential are submodules, so they are finitely generated as well. Thus, so is the quotient module $\operatorname{Ext}_{R}(N,k)$.

In this paper we will use higher homotopies and the Shamash construction for one element $f \in S$. We focus on that case in the rest of the section.

Construction 4.7. Suppose that $f \in S$, and that (G, ∂) is a free complex over S with a system σ of higher homotopies. We use the notation in Construction 4.3. The standard lifting $Sh(\mathbf{G}, \sigma)$ of the Shamash complex to S is $S\{y\} \otimes \mathbf{G}$ with the maps $\widetilde{\delta} = \sum_{j} t^{j} \otimes \sigma_{j}$. In particular, $\widetilde{\delta}|_{\mathbf{G}} = \partial$, so of course $\widetilde{\delta}^{2}|_{\mathbf{G}} = \partial^{2} = 0$. Moreover, the equations of Definition 4.1 say precisely that, $\tilde{\delta}^2$ acts on the complementary summand $\mathbf{G}' = \bigoplus_{i>0} y^{(i)}\mathbf{G}$ by ft; that is, it sends each $y^{(i)}\mathbf{G}$ isomorphically to $fy^{(i-1)}\mathbf{G}$. Thus

$$\widetilde{\delta}^2 = ft \otimes 1$$
.

The standard CI operator for f on $Sh(\mathbf{G}, \sigma)$ is $t \otimes 1$. Note that $t : Sh(\mathbf{G}, \sigma) \longrightarrow Sh(\mathbf{G}, \sigma)[2]$ is surjective, and is split by the map sending $y^{(i)}u \in S\{y\} \otimes \mathbf{G} \otimes S/(f)$ to $y^{(i+1)}u$. Also, the standard lifted CI operator

$$\widetilde{t} := t \otimes 1 : \widetilde{\operatorname{Sh}}(\mathbf{G}, \sigma) \longrightarrow \widetilde{\operatorname{Sh}}(\mathbf{G}, \sigma)$$

commutes with the lifting $\tilde{\delta} = \sum_{i} t^{j} \otimes \sigma_{i}$ of the differential δ .

We will use the following modified version of Proposition 4.4:

Proposition 4.8. Let $\widetilde{\mathbf{G}}$ be a complex of S-free modules with a system of higher homotopies σ for a non-zerodivisor f in a ring S. If $\mathbf{F} = \operatorname{Sh}(\widetilde{\mathbf{G}}, \sigma)$, then $H_j(\mathbf{F}) = 0$ for all $0 < j \le i$ if and only if $H_j(\widetilde{\mathbf{G}}) = 0$ for all $j \le i$. In particular, $\operatorname{Sh}(\widetilde{\mathbf{G}}, \sigma)$ is an S/(f)-free resolution of a module N if and only if $\widetilde{\mathbf{G}}$ is an S-free resolution of N.

PROOF: We first show that (without any exactness hypothesis) $H_0(\widetilde{\mathbf{G}}) = H_0(\mathbf{F})$. Since the standard lifted CI operator $\widetilde{t}: \widetilde{F}_i \longrightarrow \widetilde{F}_{i-2}$ is surjective, f annihilates $N := \operatorname{Coker}(\widetilde{\delta}: \widetilde{F}_1 \longrightarrow \widetilde{F}_0)$, and thus $N = \operatorname{Coker}(\delta: F_1 \longrightarrow F_0) = H_0(\mathbf{F})$. But for $i \leq 1$ we have $\widetilde{G}_i = \widetilde{F}_i$, so $H_0(\widetilde{\mathbf{G}}) = H_0(\mathbf{F})$ as required.

Set $\overline{\mathbf{G}} = R \otimes \widetilde{\mathbf{G}}$. We now use the short exact sequences of complexes

$$0 \longrightarrow \overline{\mathbf{G}} \longrightarrow \mathbf{F} \stackrel{t}{\longrightarrow} \mathbf{F}[2] \longrightarrow 0$$
$$0 \longrightarrow \widetilde{\mathbf{G}} \stackrel{f}{\longrightarrow} \widetilde{\mathbf{G}} \longrightarrow \overline{\mathbf{G}} \longrightarrow 0.$$

which yield long exact sequences

$$(4.9) \cdots \longrightarrow H_{j-1}(\mathbf{F}) \longrightarrow H_{j}(\overline{\mathbf{G}}) \longrightarrow H_{j}(\mathbf{F}) \longrightarrow H_{j-2}(\mathbf{F}) \longrightarrow H_{j-1}(\overline{\mathbf{G}}) \longrightarrow \cdots$$

$$(4.10)$$

$$\cdots \longrightarrow H_{j+1}(\overline{\mathbf{G}}) \longrightarrow H_j(\widetilde{\mathbf{G}}) \stackrel{f}{\longrightarrow} H_j(\widetilde{\mathbf{G}}) \longrightarrow H_j(\overline{\mathbf{G}}) \longrightarrow H_{j-1}(\widetilde{\mathbf{G}}) \longrightarrow \cdots$$

respectively. Since σ_1 is a homotopy for f on $\widetilde{\mathbf{G}}$, the latter sequence breaks up into short exact sequences

$$(4.11) 0 \longrightarrow H_j(\widetilde{\mathbf{G}}) \longrightarrow H_j(\overline{\mathbf{G}}) \longrightarrow H_{j-1}(\widetilde{\mathbf{G}}) \longrightarrow 0.$$

First, assume that $H_j(\mathbf{F}) = 0$ for $1 \le j \le i$. From the long exact sequence (4.9) we conclude that $H_j(\widetilde{\mathbf{G}}) = 0$ for $2 \le j \le i$, and then (4.11) implies that $H_j(\widetilde{\mathbf{G}}) = 0$ for $1 \le j \le i$.

Conversely, suppose that $H_j(\widetilde{\mathbf{G}}) = 0$ for $1 \leq j \leq i$. It is well known that if we apply the Shamash construction to a resolution then we get a resolution, but since the bound i is not usually present we give an argument:

Assume that $H_j(\widetilde{\mathbf{G}}) = 0$ for $1 \leq j \leq i$. By (4.11) it follows that $H_j(\overline{\mathbf{G}}) = 0$ for $2 \leq j \leq i$. Applying (4.9), we conclude that $H_j(\mathbf{F}) \cong H_{j-2}(\mathbf{F})$ for $3 \leq j \leq s$. Hence, it suffices to prove that $H_1(\mathbf{F}) = H_2(\mathbf{F}) = 0$.

We will prove that $H_1(\mathbf{F}) = 0$. Let $g_1 \in \widetilde{G}_1$ be an element that reduces modulo f to \overline{g}_1 . We have

$$\widetilde{\partial}(g_1) = fg_0 = \widetilde{\partial}\sigma_1(g_0)$$

for some $g_0 \in G_0$. Thus $g_1 - \sigma_1(g_0) \in \text{Ker}(\widetilde{\partial})$ is a cycle in $\widetilde{\mathbf{G}}$. Since $H_1(\widetilde{\mathbf{G}}) = 0$, we must have $g_1 - \sigma_1(g_0) = \widetilde{\partial}(g_2)$ for some $g_2 \in \widetilde{G}_2$. Using the isomorphism $\widetilde{F_2} = \widetilde{G}_2 \oplus \widetilde{G}_0$ we see that

$$g_1 = \sigma_1(g_0) + \widetilde{\partial}(g_2) = \widetilde{\delta}(g_0 + g_2).$$

It follows that $\overline{g}_1 = \delta(\overline{g}_0 + \overline{g}_2)$ is a boundary in **F**, as required.

Finally, we show that $H_2(\mathbf{F}) = 0$. Part of (4.9) is the exact sequence

$$H_2(\overline{\mathbf{G}}) \longrightarrow H_2(\mathbf{F}) \longrightarrow H_0(\mathbf{F}) \xrightarrow{\beta} H_1(\overline{\mathbf{G}}) \longrightarrow H_1(\mathbf{F}).$$

Since $H_2(\overline{\mathbf{G}}) = 0$, it suffices to show that the map marked β is a monomorphism. But we already showed that $H_1(\mathbf{F}) = 0$, so β is an epimorphism. Since its source and target are isomorphic finitely generated modules over the ring S, this implies that it is an isomorphism, whence $H_2(\mathbf{F}) = 0$.

It follows from Theorem 4.6 that CI operators on the resolutions of high syzygies over complete intersections are often surjective, in a sense we will make precise. To prepare for the study of this situation, we consider what can be said when a CI operator is surjective.

Proposition 4.12. Let $f \in S$ be a non-zerodivisor in a ring S, and let

$$(\mathbf{F}, \delta): \cdots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\delta_1} F_0$$

be a complex of free R := S/(f)-modules. Let $(\widetilde{\mathbf{F}}, \widetilde{\delta})$ be a lifting of (\mathbf{F}, δ) to S. Set

$$\widetilde{t} := (1/f)\widetilde{\delta}^2 : \ \widetilde{\mathbf{F}} \longrightarrow \widetilde{\mathbf{F}}[2],$$

 $\widetilde{\mathbf{G}} = \operatorname{Ker}(\widetilde{t}).$

Suppose that \tilde{t} is surjective. Then:

(1) [Ei1, Theorem 8.1] The maps $\widetilde{\delta}: \widetilde{F}_i \longrightarrow \widetilde{F}_{i-1}$ induce maps $\widetilde{\partial}: \widetilde{G}_i \longrightarrow \widetilde{G}_{i-1}$, and

$$\widetilde{\mathbf{G}}: \cdots \longrightarrow \widetilde{G}_{i+1} \xrightarrow{\widetilde{\partial}_{i+1}} \widetilde{G}_i \longrightarrow \cdots \longrightarrow \widetilde{G}_1 \xrightarrow{\widetilde{\partial}_1} \widetilde{G}_0$$

is an S-free complex. If S is local and \mathbf{F} is minimal, then so is $\widetilde{\mathbf{G}}$.

(2) We may write $\widetilde{F}_i = \bigoplus_{j \geq 0} \widetilde{G}_{i-2j}$ in such a way that the lifted CI operator \widetilde{t} consists of the projections

$$\widetilde{F}_i = \bigoplus_{0 \le j \le i/2} \widetilde{G}_{i-2j} \xrightarrow{\widetilde{t}} \bigoplus_{\substack{0 \le j \le (i-2)/2 \\ 21}} \widetilde{G}_{i-2-2j} = \widetilde{F}_{i-2}.$$

If $\sigma_j: \widetilde{G}_{i-2j} \longrightarrow \widetilde{G}_{i-1}$ denotes the appropriate component of the map $\widetilde{\delta}: \widetilde{F}_i \longrightarrow \widetilde{F}_{i-1}$, then $\sigma = \{\sigma_j\}$ is a system of higher homotopies on $\widetilde{\mathbf{G}}$, and $\mathbf{F} \cong \operatorname{Sh}(\widetilde{\mathbf{G}}, \sigma)$.

PROOF: (2): Since the maps \tilde{t} are surjective, it follows inductively that we may write \tilde{F}_i and \tilde{t} in the given form. The component corresponding to $\tilde{G}_{i-2j} \longrightarrow \tilde{G}_{i-1}$ in $\tilde{\delta}: \tilde{F}_m \longrightarrow \tilde{F}_{m-1}$ is the same for any m with $m \geq i-2j$ and $m \equiv i \mod(2)$ because $\tilde{\delta}$ commutes with \tilde{t} . The condition that σ is a sequence of higher homotopies is equivalent to the condition that $\tilde{\delta}^2 = f\tilde{t}$, as one sees by direct computation. It is now immediate that $\mathbf{F} \cong \mathrm{Sh}(\tilde{\mathbf{G}}, \sigma)$.

Corollary 4.13. With hypotheses and notation as in Proposition 4.12, suppose in addition that S is a local ring and that (\mathbf{F}, δ) is a minimal R-free resolution of N. The minimal S-free resolution of N is $(\widetilde{\mathbf{G}}, \widetilde{\partial}) = \mathrm{Ker}(\widetilde{t})$. If we split the epimorphisms $t: F_i \longrightarrow F_{i-2}$ and correspondingly write $F_i = \overline{G}_i \oplus F_{i-2}$ then the differential $\delta: F_i \longrightarrow F_{i-1}$ has the form

$$\delta_i = \frac{\overline{G}_i}{F_{i-3}} \begin{pmatrix} \overline{\partial}_i & F_{i-2} \\ \overline{\partial}_i & \varphi_i \\ O & \delta_i \end{pmatrix}.$$

As an immediate consequence of Propositions 4.8 and 4.12 we obtain a result of Avramov-Gasharov-Peeva; their proof relies on the spectral sequence proof of [AGP, Theorem 4.3].

Corollary 4.14. [AGP, Proposition 6.2] Let $f \in S$ be a non-zerodivisor in a local ring. If N is a module over S/(f) then the CI operator χ corresponding to f is a non-zerodivisor on $\operatorname{Ext}_S(N,k)$ if and only if the minimal S/(f)-free resolution of N is obtained by a Shamash construction applied to the minimal free resolution of N over S.

PROOF: Nakayama's Lemma shows that the CI operator $t: \mathbf{F}[-2] \longrightarrow \mathbf{F}$ is surjective if and only if the operator $\chi: \operatorname{Ext}_R(N,k) \longrightarrow \operatorname{Ext}_R(N,k)$ is injective.

5. The minimal R-free resolution of a higher matrix factorization module

Let (d, h) be a higher matrix factorization with respect to a regular sequence f_1, \ldots, f_c in a ring S, and $R = S/(f_1, \ldots, f_c)$. We will describe an R-free resolution of the HMF module M that is minimal when S is local and (d, h) is minimal.

Construction 5.1. Let (d, h) be a higher matrix factorization with respect to a regular sequence f_1, \ldots, f_c in a ring S. Using notation as in 2.1, choose splittings $A_s(p) = A_s(p-1) \oplus B_s(p)$ for s = 0, 1, so

$$A_s(p) = \bigoplus_{\substack{1 \le q \le p \\ 22}} B_s(q) \,,$$

and write ψ_p for the component of d_p mapping $B_1(p)$ to $A_0(p-1)$. Set

$$\mathbf{A}(p): A_1(p) \xrightarrow{d_p} A_0(p) \quad \text{and} \quad \mathbf{B}(p): B_1(p) \xrightarrow{b_p} B_0(p).$$

• Set U(1) = B(1), and note that h_1 is a homotopy for f_1 . Set

$$\mathbf{T}(1) := \mathrm{Sh}(\mathbf{U}(1), h_1).$$

Its beginning is the complex $R(1) \otimes \mathbf{A}(1)$.

• Given an R(p-1)-free resolution $\mathbf{T}(p-1)$ of M(p-1) with beginning $R(p-1)\otimes \mathbf{A}(p-1)$, let

$$\Psi_p: R(p-1) \otimes \mathbf{B}(p)[-1] \longrightarrow \mathbf{T}(p-1)$$

be the map of complexes induced by $\psi_p: B_1(p) \longrightarrow A_0(p-1)$. Set

$$\mathbf{U}(p) := \mathbf{Cone}(\Psi_p)$$
.

We will show that $\mathbf{U}(p)$ is an R(p-1)-free resolution of M(p). Thus we can choose a system of higher homotopies $\sigma(p)$ for f_p on $\mathbf{U}(p)$ that begins with d_p (that is, $\sigma(p)_0 = d_p$) and

$$R(p-1)\otimes h_p: R(p-1)\otimes A_0(p)\longrightarrow R(p-1)\otimes A_1(p).$$

Set

$$\mathbf{T}(p) := \mathrm{Sh}(\mathbf{U}(p), \sigma(p)).$$

The underlying graded module of $\mathbf{T}(p)$ is $\mathbf{U}(p) = \mathbf{Cone}(\Psi_p)$ tensored with a divided power algebra on a variable y_p of degree 2. Its first differential is

$$R(p) \otimes \mathbf{A}(p) : R(p) \otimes A_1(p) \xrightarrow{R(p) \otimes d_p} R(p) \otimes A_0(p),$$

which is the presentation of M(p). We see by induction on p that the term $T_j(p)$ of homological degree j in $\mathbf{T}(p)$ is a direct sum of the form

(5.1)
$$T_j(p) = \bigoplus y_{q_1}^{(a_1)} \cdots y_{q_i}^{(a_i)} B_s(q) \otimes R(p)$$

where the sum is over all terms with

$$p \ge q_1 > q_2 > \dots > q_i \ge q \ge 1,$$

 $a_m > 0 \text{ for } 1 \le m \le i,$
 $j = s + \sum_{1 \le m \le i} 2a_m.$

We say that an element $y_{q_1}^{(a_1)} \cdots y_{q_i}^{(a_i)} v$ with $v \in B_s(q)$ and $a_1 > 0$ is admissible of weight q_1 , and we make the convention that the admissible elements in $B_s(q)$ have weight 0.

The complex $\mathbf{T}(c)$ is thus filtered by:

$$\mathbf{T}(0) := 0 \subseteq R \otimes \mathbf{T}(1) \subseteq \cdots \subseteq R \otimes \mathbf{T}(p-1) \subseteq \mathbf{T}(c)$$
,

where $R \otimes \mathbf{T}(p)$ is the subcomplex spanned by elements of weight $\leq p$ with with $v \in B_s(q)$ for $q \leq p$.

Theorem 5.2. With notation and hypotheses as in 5.1:

(1) The complex $\mathbf{T}(p)$ is an R(p)-free resolution of M(p) whose first differential is $R(p) \otimes d_p$ and whose second differential is

$$R(p) \otimes \left(\left(\bigoplus_{q \leq p} A_0(q) \right) \xrightarrow{h} A_1(p) \right),$$

where the q-th component of h is $h_q: A_0(q) \longrightarrow A_1(q) \hookrightarrow A_1(p)$.

(2) If S is local then $\mathbf{T}(p)$ is the minimal free resolution of M(p) if and only if the higher matrix factorization $(d_p, h(p) = (h_1|\cdots|h_p))$ (see 2.1 for notation) is minimal.

PROOF OF THEOREM 5.2(1): We do induction on p. To start the induction, note that $\mathbf{U}(1)$ is the two-term complex $\mathbf{A}(1) = \mathbf{B}(1)$. By hypothesis, its differential d_1 and homotopy h_1 form a hypersurface matrix factorization for f_1 , and $\mathbf{T}(1)$ has the form

$$\mathbf{T}(1): R(1) \otimes \left(\cdots \xrightarrow{h_1} A_1(1) \xrightarrow{d_1} A_0(1) \xrightarrow{h_1} A_1(1) \xrightarrow{d_1} A_0(1) \right).$$

Inductively, suppose that $p \geq 2$, and that

$$\mathbf{T}(p-1): \cdots \longrightarrow T_2 \longrightarrow T_1 \longrightarrow T_0$$

is an R(p-1)-free resolution of M(p-1) whose first two maps are as claimed. We write $\overline{-}$ for $R(p-1) \otimes -$. It follows that the first map of $\mathbf{U}(p)$ is

$$\overline{d}(p): \overline{A}_1(p) = T_1 \oplus \overline{B}_1(p) \longrightarrow \overline{A}_0(p) = T_0 \oplus \overline{B}_0(p)$$

Since $R(p-1) \otimes (d_p h_p) = f_p \operatorname{Id}_{A_0(p)}$ we may take $R(p-1) \otimes h_p$ to be the start of a system of higher homotopies $\sigma(p)$ for f_p on $R(p-1) \otimes \mathbf{U}(p)$. It follows from the definition that the first two maps in $\mathbf{T}(p) = \operatorname{Sh}(\mathbf{U}(p), \sigma(p))$ are as asserted.

By Proposition 4.8, the Shamash construction takes an R(p-1)-free resolution to an R(p)-free resolution of the same module. Thus for the induction it suffices to show that $\mathbf{U}(p)$ is an R(p-1)-free resolution of M(p). Since the first map of $\mathbf{U}(p)$ is $\overline{d}(p)$, and since $\overline{h}(p)$ is a homotopy for f_p , we see at once that

$$H_0(\mathbf{U}(p)) = \operatorname{Coker}(\overline{d}(p)) = \operatorname{Coker}(R(p) \otimes d_p) = M(p).$$

To prove that $\mathbf{U}(p)$ is a resolution, note first that $\mathbf{U}(p)_{\geq 2} = \mathbf{T}(p-1)_{\geq 2}$, and the image of $U(p)_2 = T(p-1)_2$ is contained in the summand $T(p-1)_1 \subseteq U(p)_1$, so $H_i(\mathbf{U}(p)) = H_i(\mathbf{T}(p-1)) = 0$ for $i \geq 2$. Thus it suffices to prove that $H_1(\mathbf{U}(p)) = 0$

Let
$$(y,v) \in U(p)_1 = T(p-1)_1 \oplus \overline{B}(p)_1$$
 be a cycle in $\mathbf{U}(p)$. Thus, $\overline{b}_p(v) = 0$ and $\overline{\psi}_p(v) = -\overline{d}_{p-1}(y)$. By Lemma 3.16, we conclude that $v = 0$.

For the proof of part (2) of Theorem 5.2 we will use the form of the resolutions $\mathbf{T}(p)$ to make a special lifting of the differentials to S, and thus to produce especially "nice" CI operators. We pause in the proof of Theorem 5.2 to describe this construction and deduce some consequences.

Proposition 5.3. With notation and hypotheses as in 5.1, there exists a lifting of the filtration $\mathbf{T}(1) \subseteq \cdots \subseteq \mathbf{T}(c)$ to a filtration $\widetilde{\mathbf{T}}(1) \subseteq \cdots \subseteq \widetilde{\mathbf{T}}(c)$ over S, and a lifting $\widetilde{\delta}$ of the differential δ in $\mathbf{T}(c)$ to S with lifted CI operators $\widetilde{t}_1, \ldots, \widetilde{t}_c$ on $\widetilde{\mathbf{T}}(c)$ such that for every $1 \leq p \leq c$:

- (1) Both $\widetilde{\delta}$ and \widetilde{t}_p preserve $\widetilde{\mathbf{T}}(p)$, and $\widetilde{t}_p\big|_{\widetilde{\mathbf{T}}(p)}$ commutes with $\widetilde{\delta}\big|_{\widetilde{\mathbf{T}}(p)}$ on $\widetilde{\mathbf{T}}(p)$.
- (2) The CI operator t_p vanishes on the subcomplex $R \otimes \mathbf{U}(p)$ and induces an isomorphism from $R \otimes T(p)_j / U(p)_j$ to $R \otimes T(p)_{j-2}$ that sends an admissible element $y_{q_1}^{(a_1)} \cdots y_{q_i}^{(a_i)} v$ with $q_1 = p$ to $y_{q_1}^{(a_1-1)} \cdots y_{q_i}^{(a_i)} v$.

PROOF: If p = 1 the result is obvious. Thus we may assume by induction that liftings

$$0 \subset \widetilde{\mathbf{T}}(1) \subseteq \cdots \subseteq \widetilde{\mathbf{T}}(p-1),$$

 $\widetilde{\delta}(p-1)$ and $\widetilde{t}_1, \ldots, \widetilde{t}_{p-1}$ on $\widetilde{\mathbf{T}}(p-1)$ satisfying the Proposition have been constructed. We use the maps ψ_p and b_p from the definition of the higher matrix factorization to construct a lifting of $\mathbf{U}(p)$ from the given lifting of $\mathbf{T}(p-1)$. In addition, we choose liftings $\widetilde{\sigma}$ of the maps (other than the differential) in the system of higher homotopies $\sigma(p)$ for f_p on $\mathbf{U}(p)$.

By construction, $\mathbf{T}(p) = \operatorname{Sh}(\mathbf{U}(p), \sigma(p))$, so we take the standard lifting to S from 4.7, that is, take $\widetilde{\mathbf{T}}(p) = \bigoplus_{i \geq 0} y_p^{(i)} \widetilde{\mathbf{U}}(p)$ with lifting of the differential $\widetilde{\delta} = \sum_{i \geq 0} t^i \otimes \widetilde{\sigma}_i$, where t is the dual variable to y_p .

By Construction 4.7 it follows that, modulo (f_1, \ldots, f_{p-1}) , the map $\widetilde{\delta}^2$ vanishes on $\widetilde{\mathbf{U}}(p)$ and induces f_p times the projection $\widetilde{T}_j(p)/\widetilde{U}_j(p) \longrightarrow \widetilde{T}_{j-2}(p)$.

We choose \widetilde{t}_p to be the standard lifted CI operator, which vanishes on $\widetilde{\mathbf{U}}(p)$ and is the projection $\widetilde{T}_j(p)/\widetilde{U}_j(p) \longrightarrow \widetilde{T}_{j-2}(p)$. Then $\widetilde{\delta}_{i-2}\widetilde{t}_p = \widetilde{t}_p\widetilde{\delta}_i$ by construction; see 4.7.

Recall that $\widetilde{\delta}|_{\widetilde{\mathbf{T}}(p-1)}$ is the lifting $\widetilde{\delta}(p-1)$ given by induction. Therefore, from $\widetilde{\delta}$ we can choose maps $\widetilde{t}_1, \ldots, \widetilde{t}_{p-1}$ on $\widetilde{\mathbf{T}}(p)$ that extend the maps $\widetilde{t}_1, \ldots, \widetilde{t}_{p-1}$ given by induction on $\widetilde{\mathbf{T}}(p-1) \subseteq \widetilde{\mathbf{U}}(p)$.

The CI operators commute up to homotopy, and it is an open conjecture from [Ei1] (see also [AGP, Section 9]) that they can be chosen to commute when restricted to the minimal free resolution of a high syzygy in the local case. Proposition 5.3 allows us to give a partial answer, based on the following general criterion.

Proposition 5.4. Let f_1, \ldots, f_c be a regular sequence in a local ring S, and let $R = S/(f_1, \ldots, f_c)$. Suppose that (\mathbf{F}, δ) is a complex over R with lifting $(\widetilde{\mathbf{F}}, \widetilde{\delta})$ to S, and let $\widetilde{t}_1, \ldots, \widetilde{t}_c$ on $\widetilde{\mathbf{F}}$ define CI operators corresponding to f_1, \ldots, f_c . If, for some j, \widetilde{t}_j commutes with $\widetilde{\delta}^2$, then t_j commutes with each t_i .

PROOF: Since $\widetilde{\delta}^2 = \sum f_i \widetilde{t}_i$ by definition, we have $\sum f_i \widetilde{t}_j \widetilde{t}_i = \sum f_i \widetilde{t}_i \widetilde{t}_j$, or equivalently $\sum f_i (\widetilde{t}_j \widetilde{t}_i - \widetilde{t}_i \widetilde{t}_j) = 0$. Since f_1, \dots, f_c is a regular sequence it follows that $\widetilde{t}_j \widetilde{t}_i - \widetilde{t}_i \widetilde{t}_j$ is zero modulo (f_1, \dots, f_c) for each i.

As an immediate consequence, we have:

Corollary 5.5. Suppose that S is local. With CI operators on $\mathbf{T}(p)$ chosen as in Proposition 5.3 the operator t_p commutes on $\mathbf{T}(p)$ with each t_i for i < p.

Corollary 5.6. Let $k[\chi_1, ..., \chi_c]$ act on $\operatorname{Ext}_R(M, k)$ as in Construction 4.5. There is an isomorphism

$$\operatorname{Ext}_R(M,k) \cong \bigoplus_{p=1}^c k[\chi_p,\ldots,\chi_c] \otimes_k \operatorname{Hom}_S(\mathbf{B}(p),k)$$

of vector spaces such that, for $i \geq p$, χ_i preserves the summand

$$k[\chi_p,\ldots,\chi_c]\otimes \operatorname{Hom}_S(\mathbf{B}(p),k)$$

and acts on it via the action on the first factor.

PROOF: Since $\mathbf{T}(c)$ is a minimal free resolution of M, the $k[\chi_1, \ldots, \chi_c]$ -module $\operatorname{Ext}_R(M, k)$ is isomorphic to $\operatorname{Hom}_R(\mathbf{T}(c), k)$. Using the decomposition in (5.1) we see that the underlying graded free module of $\operatorname{Hom}_R(\mathbf{T}(c), k)$ is

$$\bigoplus_{p} k[\chi_p, \dots, \chi_c] \otimes_k \operatorname{Hom}_{S}(\mathbf{B}(p), k).$$

From part (2) of Proposition 5.3 we see that, for $i \geq p$, the action of χ_i on the summand $k[\chi_p, \ldots, \chi_c] \otimes_k \operatorname{Hom}_S(\mathbf{B}(p), k)$ is via the natural action on the first factor.

Corollary 5.6 provides a standard decomposition of $\operatorname{Ext}_R(M,k)$ in the sense of [EP].

We will complete the proof of Theorem 5.2:

PROOF OF THEOREM 5.2(2): We suppose that S is local with maximal ideal \mathbf{m} . If the resolution $\mathbf{T}(p)$ is minimal then it follows at once from the description of the first two maps that (d,h) is minimal. We will prove the converse by induction on p.

If p = 1 then $\mathbf{T}(1)$ is the periodic resolution

$$\mathbf{T}(1): \qquad \cdots \xrightarrow{h_1} A_1 \xrightarrow{d_1} A_0 \xrightarrow{h_1} A_1 \xrightarrow{d_1} A_0$$

and only involves the maps (d_1, h_1) ; this is obviously minimal if and only if d_1 and h_1 are minimal.

Now suppose that p > 1 and that $\mathbf{T}(q)$ is minimal for q < p. Let $\delta_i : T_i(p) \longrightarrow T_{i-1}(p)$ be the differential of $\mathbf{T}(p)$. We will prove minimality of δ_i by a second induction, on i, starting with i = 1, 2.

Recall that the underlying graded module of $\mathbf{T}(p) = \mathrm{Sh}(\mathbf{U}(p), \sigma)$ is the divided power algebra $S\{y_p\} = \sum_i Sy_p^{(i)}$ tensored with the underlying module of $R(p) \otimes \mathbf{U}(p)$. Thus the beginning of the resolution $\mathbf{T}(p)$ has the form

$$\cdots \longrightarrow R(p) \otimes y_p A_0(p) \oplus R(p) \otimes T_2(p-1) \xrightarrow{\delta_2} R(p) \otimes A_1(p) \xrightarrow{\delta_1} R(p) \otimes A_0(p).$$

The map δ_1 is induced by d_p , which is minimal by hypothesis. Further, $\delta_2 = (h_p, \partial_2)$ where the map ∂_2 is the differential of $\mathbf{T}(p-1)$ tensored with R(p). The map h_p

is minimal by hypothesis, and ∂ is minimal by induction on p, so δ_2 is minimal as well.

Now suppose that $j \geq 2$ and that δ_i is minimal for $i \leq j$. We must show that δ_{j+1} is minimal, that is, $\delta_{j+1}(w) \in \mathbf{m}T_j(p)$ for any $w \in T_{j+1}(p)$. By Construction 5.1, $\delta_{j+1}(w)$ can be written uniquely as a sum of admissible elements of the form

$$y_{q_1}^{(a_1)}\cdots y_{q_i}^{(a_i)}v$$

with $0 \neq v \in B_s(q)$ and

$$p \ge q_1 > q_2 > \dots > q_i \ge q \ge 1,$$

$$a_m > 0 \text{ for } 1 \le m \le i,$$

$$j = s + \sum_{1 \le m \le i} 2a_m.$$

If $\delta_{j+1}(w) \notin \mathbf{m}T_j(p)$ then there exists a summand $y_{q_1}^{(a_1)} \cdots y_{q_i}^{(a_i)}v$ in this expression that is not in $\mathbf{m}T_j(p)$. Since $\delta_{j+1}(w)$ has homological degree $j \geq 2$, the weight of this summand must be > 0, that is, a factor $y_{q_1}^{(a_1)}$ must be present. Choose such a summand with weight q_1' as large as possible. We choose $t_{q_1'}$

Choose such a summand with weight q'_1 as large as possible. We choose $t_{q'_1}$ as in Proposition 5.3. Then $t_{q'_1}$ sends every admissible element of weight $< q'_1$ to zero. The admissible summands of $\delta_{j+1}(w)$ with weight $> q'_1$ can be ignored since they are in $\mathbf{m}T_{j-2}(p)$. By Proposition 5.3 it follows that $t_{q'_1}\delta_{j+1}(w) \notin \mathbf{m}T_{j-2}(p)$. Since

$$t_{q'_1}\delta_{j+1}(w) = \delta_{j-1}t_{q'_1}(w),$$

this contradicts the induction hypothesis.

Gulliksen [Gu] shows that the Poincaré series of M over R has the form $\mathcal{P}_M^R(x) = g(x)(1-x^2)^{-c}$ for some $g(x) \in \mathbf{Z}[x]$, and his finite generation result implies that the Betti numbers are eventually given by two polynomials of the same degree. Avramov [Av, Theorem 4.1] showed that they have the same leading coefficient. We can make this very explicit.

Corollary 5.7. With notation and hypotheses as in 5.1, if in addition S is local and the higher matrix factorization (d, h) is minimal, then:

(1) The Poincaré series of M over R is

$$\mathcal{P}_{M}^{R}(x) = \sum_{1 \le p \le c} \frac{1}{(1 - x^{2})^{c - p + 1}} \left(x \operatorname{rank} \left(B_{1}(p) \right) + \operatorname{rank} \left(B_{0}(p) \right) \right).$$

(2) The Betti numbers of M over R are given by the following two polynomials in z:

$$b_{2z}^R(M) = \sum_{1 \le p \le c} \binom{c-p+z}{c-p} \operatorname{rank} (B_0(p))$$
$$b_{2z+1}^R(M) = \sum_{1 \le p \le c} \binom{c-p+z}{c-p} \operatorname{rank} (B_1(p)).$$

PROOF: For (2), recall that the Hilbert function of $k[Z_p, \ldots, Z_c]$ is $g_p(z) = \binom{c-p+z}{c-p+1}$.

Recall that the *complexity* of an R-module N is defined to be $\operatorname{cx}_R(N) = \inf\{q \geq 0 \mid \text{there exists a } w \in \mathbf{R} \text{ such that } b_i^R(N) \leq wi^{q-1} \text{ for } i \gg 0 \}.$ If the complexity of N is μ then, as noted above,

$$\dim_k \operatorname{Ext}_R^{2i}(N,k) = (\beta/(\mu-1)!)i^{\mu-1} + O(i^{\mu-2})$$

for $i \gg 0$. Following [AB, 7.3] β is called the Betti degree of N and denoted Bdeg(N); this is the multiplicity of the module $Ext_R^{even}(N,L)$, which is equal to the multiplicity of the module $\operatorname{Ext}_{R}^{odd}(N, L)$.

Corollary 5.8. With notation and hypotheses as in 5.1, suppose in addition that S is local. Suppose that (d,h) is a minimal higher matrix factorization, and set

$$\gamma = \min\{ p \, | \, B_1(p) \neq 0 \}.$$

The complexity of M := M(c) is

$$\operatorname{cx}_R M = c - \gamma + 1.$$

Moreover, $B_0(p) = 0$ for $p < \gamma$, and the Betti degree of M is

$$Bdeg(M) = rank (B_1(\gamma)) = rank (B_0(\gamma)).$$

If in addition S is Cohen-Macaulay, then rank $(B_1(p)) > 0$ for every $\gamma \leq p \leq c$.

PROOF: By Corollary 3.12, $B_1(p) = 0$ implies that $B_0(p) = 0$. Hence the Betti degree of N is equal to min{ $p \mid B_1(p) \neq 0$ } and $B_0(p) = 0$ for $p < \gamma$.

The equality rank $(B_1(\gamma)) = \operatorname{rank}(B_0(\gamma))$ follows since $M(\gamma)$ is annihilated by f_{γ} and has minimal free resolution $B_1(\gamma) \xrightarrow{b_{\gamma}} B_0(\gamma)$ over $S/(f_1, \ldots, f_{\gamma-1})$. Corollary 3.14 implies that rank $(B_1(p)) > 0$ for every $\gamma \leq p \leq c$, when S is

Cohen-Macaulay.

6. Resolutions over intermediate rings

Using a slight extension of the definition of a higher matrix factorization we can describe the resolutions of the modules M(p) over any of the rings R(q) with q < p.

Definition 6.1. A generalized matrix factorization over a ring S with respect to a regular sequence $f_1, \ldots, f_c \in S$ is a pair of maps (d, h) satisfying the definition of a higher matrix factorization except that we drop the assumption that A(0) = 0, so that we have a map of free modules $A_1(0) \xrightarrow{b_0} A_0(1)$. We do not require the existence of a map h_0 .

Construction 6.2. Let (d,h) be a generalized matrix factorization with respect to a regular sequence f_1, \ldots, f_c in a ring S. Using notation as in 2.1, we choose splittings $A_s(p) = A_s(p-1) \oplus B_s(p)$ for s = 0, 1, and write ψ_p for the component of d_p mapping $B_1(p)$ to $A_0(p-1)$.

- Let **V** be a free resolution of the module $\operatorname{Coker}(b_0)$ over S, and set $\mathbf{Q}(0) := \mathbf{V}$.
- Let

$$\Psi_1: \mathbf{B}(1)[-1] \longrightarrow \mathbf{Q}(0)$$

be the map of complexes induced by $\psi_1: B_1(1) \longrightarrow A_0(0)$, and set

$$\mathbf{Q}(1) = \mathbf{Cone}(\Psi_1).$$

• For $p \geq 2$, suppose that an S-free resolution $\mathbf{Q}(p-1)$ of M(p-1) with first term $Q_0(p-1) = A_0(p-1)$ has been constructed. Let

$$\psi_p': \mathbf{B}(p)[-1] \longrightarrow \mathbf{L}(p-1)$$

be the map of complexes induced by $\psi_p: B_1(p) \longrightarrow A_0(p-1)$, and let

$$\Psi_p: \mathbf{K}(f_1,\ldots,f_{p-1})\otimes \mathbf{B}(p)[-1] \longrightarrow \mathbf{Q}(p-1)$$

be an (f_1, \ldots, f_{p-1}) -Koszul extension. Set $\mathbf{Q}(p) = \mathbf{Cone}(\Psi_p)$.

The proof of Theorem 3.4 can be applied in this situation and yields the following result.

Proposition 6.3. Let (d,h) be a generalized matrix factorization over a ring S, and let \mathbf{V} be a free resolution of the module $\operatorname{Coker}(b_0)$ over S. For each p, the complex $\mathbf{Q}(p)$, constructed in 6.2, is an S-free resolution of the module M(p). If the ring S is local then the resulting free resolution is minimal if and only if (d,h) and \mathbf{V} are minimal.

Theorem 6.4. Let (d,h) be a higher matrix factorization. Fix a number $1 \le j \le c-1$. Let $\mathbf{T}(j)$ be the free resolution of M(j) over the ring $R(j) = S/(f_1, \ldots, f_j)$ given by Theorem 5.1. Let (d',h') be the generalized matrix factorization over the ring R(j) with

$$A_s(0) = R(j) \otimes \left(\bigoplus_{1 \leq q \leq j} A_s(q) \right) \quad and \quad d'_0 = R(j) \otimes d_j,$$

for $p > j$, $A_s(p)' = R(j) \otimes A_s(p+j)$ and $d'_p = R(j) \otimes d_{p+j}$,

for s = 0, 1 and maps induces by (d, h). Then M'(0) = M(j).

- (1) Construction 6.2, starting from the R(j) free resolution $\mathbf{Q}(0) := \mathbf{T}(j)$ of M'(0) = M(j), produces a free resolution $\mathbf{Q}(c-j)$ of M over R(j).
- (2) If S is local and (d, h) is minimal, then the resolution $\mathbf{Q}(c j)$ is minimal. In that case, the Poincaré series of M over R(j) is

$$\mathcal{P}_{M}^{R(j)}(x) = \left(\sum_{1 \le p \le j} \frac{1}{(1 - x^{2})^{p - j - 1}} \left(x \operatorname{rank}(B_{1}(p)) + \operatorname{rank}(B_{0}(p))\right)\right)$$
$$\left(\sum_{j + 1 \le p \le c} (1 + x)^{p - j - 1} \left(x \operatorname{rank}(B_{1}(p)) + \operatorname{rank}(B_{0}(p))\right)\right).$$

PROOF: First, we apply Theorem 5.1, which gives the resolution $\mathbf{T}(j)$ of M(j) over the ring R(j). Then we apply Proposition 6.3.

7. Pre-stable Syzygies and Generic CI Operators

Our goal in this section and Section 9 is to show that every sufficiently high syzygy over a complete intersection is an HMF module. In this section we introduce the concepts of *pre-stable syzygy* and *stable syzygy*. We will see that any sufficiently high syzygy in a minimal free resolution over a local complete intersection ring is a stable syzygy. In Section 9 we will show that a pre-stable syzygy is an module.

Definition 7.1. Suppose that f_1, \ldots, f_c is a regular sequence in a local ring S, and set $R = S/(f_1, \ldots, f_c)$. We define the concept of a pre-stable syzygy recursively: We say that an R-module M is a pre-stable syzygy with respect to f_1, \ldots, f_c if either c = 0 and M = 0, or $c \ge 1$ and the following conditions are satisfied:

- (1) There exists a minimal R-free resolution (\mathbf{F}, δ) of an R-module of finite projective dimension over S with a surjective CI operator t_c on \mathbf{F} and such that $M = \text{Ker}(\delta_1)$;
- (2) If $\widetilde{\delta}_1$ is a lifting of δ_1 to $\widetilde{R} := S/(f_1, \dots, f_{c-1})$, then $\widetilde{M} := \operatorname{Ker}(\widetilde{\delta}_1)$ is a pre-stable syzygy with respect to f_1, \dots, f_{c-1} .

We say that a pre-stable syzygy is stable if the module resolved by \mathbf{F} in Condition (1) in 7.1 is maximal Cohen-Macaulay and the module \widetilde{M} in Condition (2) is a stable syzygy.

Remark 7.2. The property of being pre-stable is independent of choices: Condition (1) of the definition is independent of the choice of t_c because t_c is uniquely defined up to homotopy, and \mathbf{F} is assumed minimal. Condition (2) is independent of the choice of the lifting of δ_1 because, if we write L for the module resolved by \mathbf{F} , then $\text{Ker}(\widetilde{\delta}_1)$ is the second syzygy of L over \widetilde{R} by Propositions 4.8 and 4.12.

Note that if M is a pre-stable syzygy, then by (1) it follows that M has finite projective dimension over S.

The property described in Definition 7.1 is preserved under taking syzygies:

Proposition 7.3. Suppose that f_1, \ldots, f_c is a regular sequence in a local ring S, and set $R = S/(f_1, \ldots, f_c)$. If M is a pre-stable syzygy over R, then $\operatorname{Syz}_1^R(M)$ is pre-stable as well. If M is a stable syzygy over R, then so is $\operatorname{Syz}_1^R(M)$.

PROOF: Let (\mathbf{F}, δ) be a minimal R-free resolution of a module L such that $M = \operatorname{Ker}(\delta_1)$ and the conditions in Definition 7.1 are satisfied. Lifting \mathbf{F} to $\widetilde{\mathbf{F}}$ over $\widetilde{R} := S/(f_1, \ldots, f_{c-1})$ and using the hypothesis that S is local, we see that the lifted CI operator \widetilde{t}_c is surjective on \widetilde{F} . By Propositions 4.8 and 4.12, $\widetilde{\mathbf{G}} := \operatorname{Ker}(\widetilde{t}_c)$ is the minimal free resolution of the module L over \widetilde{R} .

Let $M' = \operatorname{Syz}_1^R(M)$ and let $L' = \operatorname{Syz}_1^R(L)$, so that $\mathbf{F}' = \mathbf{F}_{\geq 1}[-1]$ is the minimal free resolution of L'. Clearly $t_c\big|_{\mathbf{F}'}$ is surjective. The shifted truncation $\widetilde{\mathbf{F}}' := \widetilde{\mathbf{F}}_{\geq 1}[-1]$ is a lifting of \mathbf{F}' , and $\widetilde{\mathbf{G}}' := \operatorname{Ker}(\widetilde{t_c}\big|_{\widetilde{\mathbf{F}}'})$ is a minimal free resolution of L' over \widetilde{R} . The complex $\widetilde{\mathbf{G}}'_{\geq 2}$ agrees (up to the sign of the differential) with

$$\widetilde{\mathbf{G}}[-1]_{>2}$$
:

(7.4)
$$\widetilde{\mathbf{G}}: \qquad \dots \longrightarrow \widetilde{G}_4 \longrightarrow \widetilde{G}_3 \longrightarrow \widetilde{G}_2 \longrightarrow \widetilde{F}_1 \xrightarrow{\delta_1} \widetilde{F}_0$$
$$\widetilde{\mathbf{G}}': \qquad \dots \longrightarrow \widetilde{G}_4 \longrightarrow \widetilde{G}_3 \longrightarrow \widetilde{F}_2 \xrightarrow{\delta_2} \widetilde{F}_1,$$

Thus $\operatorname{Ker}(\widetilde{\delta}_2) = \operatorname{Syz}_1^{\widetilde{R}}(\operatorname{Ker}(\widetilde{\delta}_1))$. Since $\operatorname{Ker}(\widetilde{\delta}_1)$ is a pre-stable syzygy, we can apply the induction hypothesis to conclude that $Ker(\tilde{\delta}_2)$ is pre-stable.

The last statement in the proposition follows from the observation that if L is a maximal Cohen-Macaulay R-module, then so is L'.

The next result shows that in the codimension 1 case, pre-stable syzygies are the same as codimension 1 matrix factorizations.

Proposition 7.5. Let $f \in S$ be a non-zerodivisor in a local ring and set R = S/(f). The following conditions on an R-module M are equivalent:

- (1) M is a pre-stable syzygy with respect to f.
- (2) M has projective dimension 1 as an S-module.
- (3) The minimal R-free resolution of M comes from a codimension 1 matrix factorization of f over S.

PROOF: $(1) \Rightarrow (2)$: Let **F** be a minimal free resolution satisfying condition (1)in Definition 7.1. By Proposition 7.3 and its proof and notation, $\operatorname{Syz}_2^R(M)$ is a pre-stable syzygy, and thus the free resolution

$$\widetilde{\mathbf{G}}': \ldots \longrightarrow \widetilde{G}_4 \longrightarrow \widetilde{F}_3 \longrightarrow \widetilde{F}_2$$

(which is the kernel of the lifting of the CI operator t_c on the minimal free resolution $\mathbf{F}_{\geq 2}$ of M) is zero in degrees ≥ 4 . Since \mathbf{G}' is the minimal free resolution (up to a shift) of M over S, the projective dimension of M over S is 1.

- $(2) \Rightarrow (3)$: If M has projective dimension 1 then M is the cokernel of a square matrix over S, and the homotopy for multiplication by f defines the matrix factorization.
- $(3) \Rightarrow (1)$: Continuing the periodic free resolution of M as an R module two steps to the right we get a minimal free resolution **F** of a module $L \cong M$ on which the CI operators are surjective, and also injective on $\mathbf{F}_{\geq 2}$. It follows that $Ker(\delta_1) = 0$ in the notation of Definition 7.1, so it is pre-stable.

We now return to the situation of Theorem 4.6: Let N be an R-module with finite projective dimension over S. We regard $E := \operatorname{Ext}_R(N,k)$ as a module over $\mathcal{R} = k[\chi_1, \dots, \chi_c]$, where χ_j have degree 2. Since we think of degrees in E as cohomological degrees, we write E[a] for the shifted module whose degree i component is $E^{i+a} = \operatorname{Ext}_R^{i+a}(N,k)$. If M is the r-th syzygy module of N then $\operatorname{Ext}_R(M,k) = \operatorname{Ext}_R^{\geq r}(N,k)[-r].$

Recall that the Castelnuovo-Mumford regularity reg E is defined as

$$\operatorname{reg} E = \max_{0 \le i \le c} \left\{ i + \left\{ \max \{ j \mid H^i_{(\chi_1, \dots, \chi_c)}(E)^j \ne 0 \} \right\} \right\}.$$

Since the generators of \mathcal{R} have degree 2, some care is necessary. Note that if $\operatorname{Ext}_{R}^{odd}(N,k) \neq 0$ then $E = \operatorname{Ext}_{R}(N,k)$ can never have regularity ≤ 0 , since it is generated in degrees ≥ 0 and the odd part cannot be generated by the even part. Thus we will often have recourse to the condition reg $\operatorname{Ext}_R(N,k) = 1$. On the other hand, many things work as usual. If we split E into even and odd parts, $E = E^{even} \oplus E^{odd}$ we have reg $E = \max(\text{reg } E^{even}, \text{reg } E^{odd})$ as usual. Also, if χ_c is a non-zerodivisor on E then $reg(E/\chi_c E) = reg E$.

Theorem 7.6. Suppose that f_1, \ldots, f_c is a regular sequence in a local ring S with infinite residue field k, and set $R = S/(f_1, \ldots, f_c)$. Let N be an R-module with finite projective dimension over S, and let \mathbf{L} be the minimal R-free resolution of N. There exists a non-empty Zariski open dense set Z of upper-triangular matrices $(\alpha_{i,j})$ with entries in k, such that for every

$$r \ge 2c - 1 + \operatorname{reg}(\operatorname{Ext}_R(N, k))$$

the syzygy module $\operatorname{Syz}_r^R(N)$ is pre-stable with respect to the regular sequence $f_1', \ldots,$ f'_c with $f'_i = f_i + \sum_{j>i} \alpha_{i,j} f_j$.

To prepare for the proof of Theorem 7.6 we will explain the property of the regular sequence f'_1, \ldots, f'_c that we will use. Recall that a sequence of elements $\chi'_c, \chi'_{c-1}, \dots, \chi'_1 \in \mathcal{R}$ is said to be an almost regular sequence on a graded module E if, for $q=c,\ldots,1$, the submodule of elements of $E/(\chi'_{q+1},\ldots,\chi'_c)E$ annihilated by χ'_q is of finite length.

We will use the following lemma with $E = \operatorname{Ext}_R(N, k)$.

Lemma 7.7. Suppose that E is a non-zero graded module of regularity ≤ 1 over $\mathcal{R} = k[\chi_1, \dots, \chi_c]$. The element χ_c is almost regular on E if and only if χ_c is a non-zerodivisor on $E^{\geq 2}[2]$ (equivalently, χ_c is a non-zerodivisor on $E^{\geq 2}$).

More generally, if we set E(c) = E and

$$E(j-1) = E(j)^{\geq 2}[2]/\chi_j E(j)^{\geq 2}[2]$$

for $j \leq c$, then the sequence χ_c, \ldots, χ_1 is almost regular on E if and only if χ_j is a non-zerodivisor on $E(j)^{\geq 2}[2]$ for every j. In that case $\operatorname{reg} E(i) \leq 1$.

PROOF: By definition the element χ_c is almost regular on E if the submodule P of E of elements annihilated by χ_c has finite length. Since $\operatorname{reg}(E) \leq 1$, all such elements must be contained in $E^{\leq 1}$. Hence, χ_c is a non-zerodivisor on $E^{\geq 2}$.

Conversely, if χ_c is a non-zerodivisor on $E^{\geq 2}$ then $P \subseteq E^{\leq 1}$ so P has finite length. Therefore, χ_c is almost regular on E.

Thus χ_c is almost regular if and only if it is a non-zerodivisor on $E^{\geq 2}$ as claimed.

If χ_c is a non-zerodivisor on $E^{\geq 2}$, then

$$\operatorname{reg}(E^{\geq 2}/\chi_c E^{\geq 2}) = \operatorname{reg}(E^{\geq 2}) \leq 3,$$

whence $\operatorname{reg}(E(c-1)) \leq 1$. By induction, $\chi_{c-1}, \ldots, \chi_1$ is an almost regular sequence on E(c-1) if and only if χ_j is a non-zerodivisor on $E(j)^{\geq 2}[2]$ for every j < c, as claimed.

The following result is a well-known consequence of the "Prime Avoidance Lemma" (see for example [Ei3, Lemma 3.3] for Prime Avoidance):

Lemma 7.8. If k is an infinite field and E is a graded module over the polynomial ring $\mathcal{R} = k[\chi_1, \ldots, \chi_c]$, then there exists a non-empty Zariski open dense set \mathcal{Y} of lower-triangular matrices $(\nu_{i,j})$ with entries in k, such that the sequence of elements χ'_c, \ldots, χ'_1 with $\chi'_i = \chi_i + \sum_{j < i} \nu_{i,j} \chi_j$ is almost regular on E.

Again let f_1, \ldots, f_c be a regular sequence in a local ring S with infinite residue field k and maximal ideal \mathbf{m} , and set $R = S/(f_1, \ldots, f_c)$. Let N be an R-module with finite projective dimension over S, and let \mathbf{L} be the minimal R-free resolution of N. Suppose we have CI operators defined by a lifting $\widetilde{\mathbf{L}}$. If we make a change of generators of (f_1, \ldots, f_c) using an invertible matrix α and $f'_i = \sum_j \alpha_{i,j} f_j$ with $\alpha_{i,j} \in S$, then the lifted CI operators on the lifting $\widetilde{\mathbf{L}}$ change as follows:

$$\widetilde{\partial}^2 = \sum_i f_i' \widetilde{t}_i' = \sum_i \left(\sum_j \alpha_{i,j} f_j \right) \widetilde{t}_i' = \sum_j f_j \left(\sum_i \alpha_{i,j} \widetilde{t}_i' \right).$$

So the CI operators corresponding to the sequence f_1, \ldots, f_c are expressed as $t_j = \sum_i \alpha_{i,j} t_i'$. Thus, if we make a change of generators of the ideal (f_1, \ldots, f_c) using a matrix α then the CI operators transform by the inverse of the transpose of α . Another way to see this is from the fact that $\mathcal{R} = k[\chi_1, \ldots, \chi_c]$ can be identified with the symmetric algebra of the dual of the vector space $(f_1, \ldots, f_c)/\mathbf{m}(f_1, \ldots, f_c)$.

In view of this observation, Lemmas 7.7 and 7.8 can be translated as follows:

Proposition 7.9. Let $f_1, \ldots, f_c \in S$ be a regular sequence in a local ring with infinite residue field k, and set $R := S/(f_1, \ldots, f_c)$. Let N be an R-module of finite projective dimension over S, and set $E := \operatorname{Ext}_R(N, k)$.

- (1) [Av, Ei1] There exists a non-empty Zariski open dense set \mathcal{Z} of upper-triangular matrices $\overline{\alpha} = (\overline{\alpha}_{i,j})$ with entries in k, such that if $\alpha = (\alpha_{i,j})$ is any matrix over S that reduces to $\overline{\alpha}$ modulo the maximal ideal of S, then the sequence f'_1, \ldots, f'_c with $f'_i = f_i + \sum_{j>i} \alpha_{i,j} f_j$ corresponds to a sequence of CI operators χ'_c, \ldots, χ'_1 that is almost regular on E.
- (2) Furthermore, for such χ'_i we have the following property. Set E(c) = E and

$$E(i-1) = E(i)^{\geq 2} [2] / \chi_i' E(i)^{\geq 2} [2]$$

for $i \leq c$. Suppose $\operatorname{reg}(E) \leq 1$. Set $\nu = (\alpha^{\vee})^{-1}$. Then χ'_c is a non-zerodivisor on $\operatorname{Ext}_R^{\geq 2}(N,k)$, and more generally $\chi'_i = \sum_j \nu_{i,j} \chi_j$ is a non-zerodivisor on $E(i)^{\geq 2}[2]$ for every i.

We say that f'_1, \ldots, f'_c with $f'_i = f_i + \sum_{j>i} \alpha_{i,j} f_j$ are generic for N if $(\alpha_{i,j}) \in \mathcal{Z}$ in the sense above.

PROOF OF THEOREM 7.6: To simplify the notation, we may begin by replacing N by its $(\operatorname{reg}(\operatorname{Ext}_R(N,k)) - 1)$ -st syzygy, and assume that $\operatorname{reg}(\operatorname{Ext}_R(N,k)) = 1$. After a general change of f_1, \ldots, f_c we may also assume, by Lemma 7.7, that χ_c, \ldots, χ_1 is an almost regular sequence on $\operatorname{Ext}_R(N,k)$. By Proposition 7.3 it suffices to treat the case r = 2c. Set $M = \operatorname{Syz}_{2c}^R(N)$.

Let (\mathbf{F}, δ) be the minimal free resolution of $N' := \operatorname{Syz}_2^R(N)$, so that $M = \operatorname{Ker}(\delta_{2c-3})$. Since N has finite projective dimension over S, the module N' also has finite projective dimension over S.

Let $(\widetilde{\mathbf{F}}, \widetilde{\delta})$ be a lifting of \mathbf{F} to $\widetilde{R} := S/(f_1, \ldots, f_{c-1})$, and let \widetilde{t}_c be the lifted CI operator. Set $(\widetilde{\mathbf{G}}, \widetilde{\delta}) = \mathrm{Ker}(\widetilde{t}_c)$. By Proposition 7.9, χ_c is a monomorphism on $\mathrm{Ext}_R(N', k) = \mathrm{Ext}_R^{\geq 2}(N, k)[2]$. Since χ_c is induced by t_c , Nakayama's Lemma implies that t_c is surjective, so in particular $\mathbf{F}_{\geq 2c-2} \longrightarrow \mathbf{F}_{\geq 2c-4}$ is surjective, as required for Condition (1) in 7.1 for c > 1.

Using Nakayama's Lemma again, we see that the lifted CI operator t_c is also an epimorphism. Propositions 4.8 and 4.12 show that $\widetilde{\mathbf{G}}$ is a minimal free resolution of N' over \widetilde{R} , and \mathbf{F} is obtained from $\widetilde{\mathbf{G}}$ by the Shamash construction 4.3. Hence

$$\operatorname{Ext}_{\widetilde{R}}(N',k) = \operatorname{Ext}_{R}(N',k)/\chi_{c}\operatorname{Ext}_{R}(N',k),$$

and therefore

$$\operatorname{Ext}_{\widetilde{R}}(N',k) = \left(\operatorname{Ext}_{R}^{\geq 2}(N,k)/\chi_{c}\operatorname{Ext}_{R}^{\geq 2}(N,k)\right)[2].$$

By Proposition 7.9 we conclude that $\operatorname{Ext}_{\widetilde{R}}^{\geq 2}(N',k)$ has regularity ≤ 1 over $k[\chi_1,\ldots,\chi_{c-1}]$.

Suppose now that c=1, so that M=N' is the second syzygy of N. In this case $\widetilde{R}=S$, and by hypothesis M=N' has finite projective dimension over S. Therefore, $\operatorname{Ext}_S(M,k)$ is a module of finite length. Since it has regularity ≤ 1 (as a module over k), it follows that it is zero except in degrees ≤ 1 , that is, the projective dimension of M over \widetilde{R} is ≤ 1 . By Proposition 7.5, M is a pre-stable syzygy.

Next suppose that c > 1. We have $\operatorname{Ker}(\widetilde{\delta}_{2c-3}) = \operatorname{Syz}_{2(c-1)}^{\widetilde{R}}(N')$, and by induction on c this is a pre-stable syzygy, verifying Condition (2) in 7.1. Thus M is a pre-stable syzygy.

Remark 7.10. Some caution is necessary if we wish to work in the graded case (see for example [Pe] for graded resolutions). Suppose that $S = k[x_1, \ldots, x_n]$ is a standard graded polynomial ring with generators x_i in degree 1. Let f_1, \ldots, f_c be a homogeneous regular sequence, and set $R = S/(f_1, \ldots, f_c)$. Let N be a finitely generated graded R-module. When all the f_i have the same degree, so that a general linear scalar combination of them is still homogeneous, then Proposition 7.9 and Theorem 7.6 hold for $E = \operatorname{Ext}_R(N, k)$ verbatim, without first localizing at the maximal ideal. However when the f_1, \ldots, f_c have distinct degrees, there may be

no homogeneous linear combination of the f_j that corresponds to an eventually surjective CI operator, as can be seen from the following example. Let $R = k[x,y]/(x^2,y^3)$ and consider the module $N = R/x \oplus R/y$. Over the local ring $S_{(x,y)}/(x^2,y^3)$ the CI operator corresponding to $x^2 + y^3$ is eventually surjective. However, the minimal R-free resolution of N is the direct sum of the free resolutions of R/x and R/y. The CI operator corresponding to x^2 vanishes on the minimal free resolution of R/y. The CI operator corresponding to y^3 vanishes on the minimal free resolution of R/x, and thus the CI operator corresponding to $y^3 + ax^3 + bx^2y$, for any a, b, does too.

8. The Box complex

Suppose that $f \in S$ is a non-zerodivisor. Given an S-free resolution of an S/(f)-module L and a homotopy for f, we will construct an S-free resolution of the second syzygy $\operatorname{Syz}_2^{S/(f)}(L)$ of L as an S/(f)-module, and also a homotopy for f on it

Box Construction 8.1. Suppose that $f \in S$ is a non-zero divisor, and that

$$\mathbf{Y}: \qquad \cdots \longrightarrow Y_4 \xrightarrow{\theta_3} Y_3 \xrightarrow{\theta_2} Y_2 \xrightarrow{\theta_1} Y_1 \xrightarrow{\theta_0} Y_0$$

is an S-free resolution of a module L annihilated by f, with homotopies $\{\theta_i: Y_i \longrightarrow Y_{i+1}\}_{i\geq 0}$ and higher homotopies $\tau_0: Y_0 \longrightarrow Y_3$ and $\tau_1: Y_1 \longrightarrow Y_4$ for f, so that $\partial_3 \tau_0 + \theta_1 \theta_0 = 0$ and $\tau_0 \partial_1 + \theta_2 \theta_1 + \partial_4 \tau_1 = 0$. We call the mapping cone

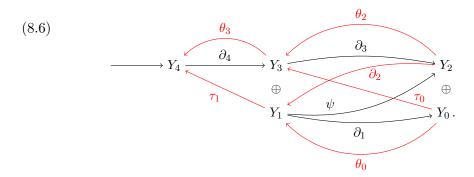
(8.3)
$$\operatorname{Box}(\mathbf{Y}): \longrightarrow Y_4 \xrightarrow{\partial_4} Y_3 \xrightarrow{\partial_3} Y_2 \\ \oplus \qquad \psi \qquad \oplus \\ Y_1 \xrightarrow{\partial_1} Y_0$$

of the map $\psi := \theta_1 : \mathbf{Y}_{\leq 1}[1] \longrightarrow \mathbf{Y}_{\geq 2}$ the box complex and denote it $Box(\mathbf{Y})$.

Box Proposition 8.4. With notation as above, the box complex Box(**Y**) is an S-free resolution of the module $\operatorname{Ker}(S/(f) \otimes Y_1 \xrightarrow{S/(f) \otimes \partial_1} S/(f) \otimes Y_0)$, the second S/(f)-syzygy of L. Moreover, the maps

(8.5)
$$\begin{pmatrix} \theta_2 & \tau_0 \\ \partial_2 & \theta_0 \end{pmatrix}, \ (\theta_3, \tau_1), \ \theta_4, \ \dots$$

give a homotopy for multiplication by f on $Box(\mathbf{Y})$ as shown in diagram (8.6):



A similar formula yields a full system of higher homotopies on Box(Y) from higher homotopies on Y, but we will not need this.

PROOF: The following straightforward computation shows that the maps in (8.5) are homotopies for f on Box(\mathbf{Y}):

$$\begin{aligned} & (8.7) \qquad \begin{pmatrix} \partial_3 & \theta_1 \\ 0 & \partial_1 \end{pmatrix} \begin{pmatrix} \theta_2 & \tau_0 \\ \partial_2 & \theta_0 \end{pmatrix} = \begin{pmatrix} \partial_3 \theta_2 + \theta_1 \partial_2 & \partial_3 \tau_0 + \theta_1 \theta_0 \\ \partial_1 \partial_2 & \partial_1 \theta_0 \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \\ & \begin{pmatrix} \partial_1 & \theta_1 \\ 0 & \partial_1 \end{pmatrix} + \begin{pmatrix} \partial_4 \theta_3 & \partial_4 \tau_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \theta_2 \partial_3 + \partial_4 \theta_3 & \theta_2 \theta_1 + \tau_0 \partial_1 + \partial_4 \tau_1 \\ \partial_2 \partial_3 & \partial_2 \theta_1 + \theta_0 \partial_1 \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \; . \end{aligned}$$

Next we will prove that $Box(\mathbf{Y})$ is a resolution. There is a short exact sequence of complexes

$$0 \longrightarrow \mathbf{Y}_{\geq 2} \longrightarrow \operatorname{Box}(\mathbf{Y}) \longrightarrow \mathbf{Y}_{\leq 1} \longrightarrow 0$$
,

so $H_i(\text{Box}(\mathbf{Y})) = \text{H}_i(\mathbf{Y}_{\geq 2}) = 0$ for $i \geq 2$ since $\mathbf{Y}_{\leq 1}$ is a two-term complex. If $(v, w) \in Y_3 \oplus Y_1$ is a cycle, then applying the homotopy maps in (8.5) we get

$$(fv, fw) = (\partial_4 \theta_3(v) + \partial_4 \tau_1(w), 0).$$

Since f is a non-zerodivisor, it follows that w=0. Thus v is a cycle in $\mathbf{Y}_{\geq 2}$, which is acyclic, so v is a boundary in $\mathbf{Y}_{\geq 2}$. Hence, the complex $\mathrm{Box}(\mathbf{Y})$ is acyclic.

To simplify notation, we write $\overline{\overline{}}$ for the functor $S/(f) \otimes -$ and set $\psi = \theta_1$. To complete the proof we will show that $H_0(\operatorname{Box}(\mathbf{Y})) = \operatorname{Ker}(\overline{\partial_1} : \overline{Y}_1 \longrightarrow \overline{Y}_0)$. Since we have a homotopy for f on \mathbf{Y} , we see that f annihilates the module resolved by \mathbf{Y} . Therefore, $H_0(\mathbf{Y}) = H_1(\overline{\mathbf{Y}})$. The complex $\overline{\operatorname{Box}(\mathbf{Y})}$ is the mapping cone $\operatorname{Cone}(\overline{\psi} \otimes S/(f))$, where $\overline{\psi} = \psi \otimes S/(f)$, so there is an exact sequence of complexes

$$0 \longrightarrow \overline{\mathbf{Y}}_{\geq 2} \longrightarrow \overline{\mathrm{Box}(\mathbf{Y})} \longrightarrow \overline{\mathbf{Y}}_{\leq 1} \longrightarrow 0$$
.

Since **Y** is a resolution, $H_0(\mathbf{Y}_{\geq 2})$ is contained in the free S-module Y_1 . Thus f is a non-zerodivisor on $H_0(\mathbf{Y}_{\geq 2})$ and $\overline{\mathbf{Y}}_{\geq 2}$ is acyclic. Therefore, the long exact sequence for the mapping cone yields

$$0 \longrightarrow H_1(\mathbf{Cone}(\overline{\psi})) \longrightarrow H_1(\overline{\mathbf{Y}}_{<1}) \stackrel{\bar{\psi}}{\longrightarrow} H_0(\overline{\mathbf{Y}}_{>2}).$$

It suffices to prove that the map induced on homology by $\overline{\psi}$ is 0. Let $u \in Y_1$ be such that $\overline{u} \in \text{Ker}(\overline{\partial}_1)$, so $\partial_1(u) = fy$ for some $y \in Y_0$. We also have $fy = \partial_1\theta_0(y)$,

so $u - \theta_0(y) \in \text{Ker}(\partial_1)$. Since **Y** is acyclic $u = \theta_0(y) + \partial_2(z)$ for some $z \in Y_2$. Applying ψ we get

$$\psi(u) = \theta_1 \theta_0(y) + \theta_1 \partial_2(z)$$

= $-\partial_3 \tau_0(y) + (fz - \partial_3 \theta_2(z))$
= $-\partial_3 (\tau_0(y) + \theta_2(z)) + fz$,

so the map induced on homology by $\overline{\psi}$ is 0 as desired.

Proposition 8.4 has a partial converse that we will use in the proof of Theorem 10.3.

Proposition 8.8. Let $f \in S$ be a non-zerodivisor and set R = S/(f). Let

$$\cdots \to Y_4 \xrightarrow{\partial_4} Y_3 \xrightarrow{\partial_3} Y_2$$

$$\oplus \psi \xrightarrow{\oplus} \oplus$$

$$Y_1 \xrightarrow{\partial_1} Y_0$$

be an S-free resolution of a module annihilated by f. Set $\theta_1 := \psi$, and with notation as in diagram (8.6), suppose that

$$\begin{pmatrix} \theta_2 & \tau_0 \\ \partial_2 & \theta_0 \end{pmatrix}$$

is the first map of a homotopy for multiplication by f on $Box(\mathbf{Y})$. If the cokernels of ∂_2 and of ∂_3 are f-torsion free, then the following complex is exact:

$$(8.9) \ldots \longrightarrow Y_4 \xrightarrow{\partial_4} Y_3 \xrightarrow{\partial_3} Y_2 \xrightarrow{\partial_2} Y_1 \xrightarrow{\partial_1} Y_0,$$

and there are homotopies for f as in (8.2).

PROOF: We first show that the sequence is a complex. The equation $\partial_3 \partial_4 = 0$ follows from our hypothesis. Let $(\theta_3, \tau_1): Y_3 \oplus Y_1 \longrightarrow Y_2$ be the next map in the homotopy for f. To show that $\partial_2 \partial_3 = 0$ and $\partial_1 \partial_2 = 0$, use the homotopy equations

$$0\theta_3 + \partial_2 \partial_3 = 0: Y_3 \longrightarrow Y_1$$
$$\partial_1 \partial_2 = 0: Y_2 \longrightarrow Y_0.$$

The equalities in (8.7) imply that $\theta_0: Y_0 \longrightarrow Y_1$, $\psi = \theta_1: Y_1 \longrightarrow Y_2$, $\theta_2: Y_2 \longrightarrow Y_3$, and $\theta_3: Y_3 \longrightarrow Y_4$ form the beginning of a homotopy for f on (8.9). Thus (8.9) becomes exact after inverting f. The exactness of (8.9) is equivalent to the statement that the induced maps $\operatorname{Coker}(\partial_3) \longrightarrow Y_1$ and $\operatorname{Coker}(\partial_2) \longrightarrow Y_2$ are monomorphisms. Since this is true after inverting f, and since the cokernels are f-torsion free by hypothesis, exactness holds before inverting f as well.

9. From Syzygies to Higher Matrix Factorizations

Higher matrix factorizations arising from pre-stable syzygies have an additional property. We introduce the concept of a pre-stable matrix factorizations, which captures that property.

Definition 9.1. A higher matrix factorization (d, h) is a pre-stable matrix factorization if, in the notation of 2.1, for each p = 1, ..., c the element f_p is a non-zerodivisor on the cokernel of the composite map

 $R(p-1)\otimes A_0(p-1)\hookrightarrow R(p-1)\otimes A_0(p) \xrightarrow{h_p} R(p-1)\otimes A_1(p) \xrightarrow{\pi_p} R(p-1)\otimes B_1(p).$ If S is Cohen-Macaulay then we say that the higher matrix factorization (d,h) is a stable matrix factorization if the cokernel of the composite map above is a maximal Cohen-Macaulay R(p-1)-module.

The advantage of stable matrix factorizations over pre-stable matrix factorizations is that if $g \in S$ is an element such that g, f_1, \ldots, f_c is a regular sequence and (d, h) is a stable matrix factorization, then $\left(S/(g) \otimes d, \ S/(g) \otimes h\right)$ is again a stable matrix factorization. We do not know of pre-stable matrix factorizations that are not stable.

Theorem 9.2. Suppose that f_1, \ldots, f_c is a regular sequence in a local ring S, and set $R = S/(f_1, \ldots, f_c)$. If M is a pre-stable syzygy over R with respect to f_1, \ldots, f_c , then M is the HMF module of a minimal pre-stable matrix factorization (d, h) such that d and h are liftings to S of the first two differentials in the minimal R-free resolution of M. If M is a stable syzygy, then (d, h) is stable as well.

Combining Theorem 9.2 and Theorem 7.6 we obtain the following more precise version of Theorem 1.4 in the introduction.

Corollary 9.3. Suppose that f_1, \ldots, f_c is a regular sequence in a local ring S with infinite residue field k, and set $R = S/(f_1, \ldots, f_c)$. Let N be an R-module with finite projective dimension over S. There exists a non-empty Zariski open dense set Z of matrices $(\alpha_{i,j})$ with entries in k such that for every

$$r \ge 2c - 1 + \operatorname{reg}(\operatorname{Ext}_R(N, k))$$

the syzygy $\operatorname{Syz}_r^R(N)$ is the module of a minimal pre-stable matrix factorization with respect to the regular sequence $\{f_i' = \sum_j \alpha_{i,j} f_j\}$.

PROOF OF THEOREM 9.2: The proof is by induction on c. If c = 0, then M = 0 so we are done.

Suppose $c \geq 1$. We use the notation of Definition 7.1. By assumption, the CI operator t_c is surjective on a minimal R-free resolution (\mathbf{F}, δ) of a module L of which M is the second syzygy. Let $(\widetilde{\mathbf{F}}, \widetilde{\delta})$ be a lifting of (\mathbf{F}, δ) to $R' = S/(f_1, \ldots, f_{c-1})$. Since S is local, the lifted CI operator $\widetilde{t}_c := (1/f_c)\widetilde{\delta}^2$ is also surjective, and we set $(\widetilde{\mathbf{G}}, \widetilde{\partial}) := \mathrm{Ker}(\widetilde{t}_c)$. By Propositions 4.12, \mathbf{F} is the result of applying the Shamash construction to $\widetilde{\mathbf{G}}$. Let $\widetilde{B}_1(c)$ and $\widetilde{B}_0(c)$ be the liftings to

R' of F_1 and F_0 respectively. By Propositions 4.8 and 4.12 the minimal R'-free resolution of L has the form

$$(9.4) \ldots \longrightarrow \widetilde{G}_4 \xrightarrow{\widetilde{\partial}_4} \widetilde{A}_1(c-1) := \widetilde{G}_3 \xrightarrow{\widetilde{\partial}_3} \widetilde{A}_0(c-1) := \widetilde{G}_2 \xrightarrow{\widetilde{\partial}_2} \widetilde{B}_1(c) \xrightarrow{\widetilde{b}} \widetilde{B}_0(c),$$
where $\widetilde{b} := \widetilde{\partial}_1, \widetilde{\partial}_2, \widetilde{\partial}_3, \widetilde{\partial}_4$ are the liftings of the differential in \mathbf{F} .

Since L is annihilated by f_c there exist homotopy maps $\widetilde{\theta}_0, \widetilde{\psi} := \widetilde{\theta}_1, \widetilde{\theta}_2$ and a higher homotopy $\tilde{\tau}_0$ so that on

$$(9.5) \qquad \widetilde{\widetilde{\partial}_{4}} \xrightarrow{\widetilde{G}_{3}} \widetilde{\widetilde{G}_{3}} \xrightarrow{\widetilde{\widetilde{\partial}_{2}}} \widetilde{\widetilde{\partial}_{2}} \xrightarrow{\widetilde{\partial}_{1}} \widetilde{\widetilde{\partial}_{1}} (c) \xrightarrow{\widetilde{b} = \widetilde{\partial}_{1}} \widetilde{\widetilde{\partial}_{0}} (c)$$

we have

(9.6)
$$\begin{aligned} \widetilde{\partial}_{1}\widetilde{\theta}_{0} &= f_{c}\mathrm{Id} \\ \widetilde{\partial}_{2}\widetilde{\theta}_{1} + \widetilde{\theta}_{0}\widetilde{\partial}_{1} &= f_{c}\mathrm{Id} \\ \widetilde{\partial}_{3}\widetilde{\theta}_{2} + \widetilde{\theta}_{1}\widetilde{\partial}_{2} &= f_{c}\mathrm{Id} \\ \widetilde{\partial}_{3}\widetilde{\tau}_{0} + \widetilde{\theta}_{1}\widetilde{\theta}_{0} &= 0. \end{aligned}$$

Proposition 8.4 implies that the minimal free resolution of M over R' has the form

$$(9.7) \qquad \cdots \longrightarrow \widetilde{G}_{4} \xrightarrow{\widetilde{\partial}_{4}} \widetilde{G}_{3} \xrightarrow{\widetilde{\partial}_{3}} \widetilde{G}_{2}$$

$$\oplus \qquad \qquad \oplus$$

$$\widetilde{B}_{1}(c) \xrightarrow{\widetilde{b}} \widetilde{B}_{0}(c)$$

Using this structure we change the lifting of the differential δ_3 so that

$$\widetilde{\delta}_3 = \begin{pmatrix} \widetilde{\partial}_3 & \widetilde{\psi} \\ 0 & \widetilde{b} \end{pmatrix} .$$

Note that the differential $\widetilde{\partial}$ on $\widetilde{\mathbf{G}}_{\geq 2}$ has not changed. Set $M' = \operatorname{Coker}(\widetilde{\mathbf{G}}_{\geq 2}) = \operatorname{Syz}_2^{\widetilde{R'}}(L)$. Since M' is a pre-stable syzygy, the induction hypothesis implies that M' is the HMF module of a higher matrix factorization (d',h') with respect to f_1,\ldots,f_{c-1} so that the differential $\widetilde{G}_3\longrightarrow \widetilde{G}_2$ is $\widetilde{\partial}_3=d'\otimes R'$ and the differential $\widetilde{G}_4 \longrightarrow \widetilde{G}_3$ is $\widetilde{\partial}_4 = h' \otimes R'$. Thus, there exist free S-modules $A'_1(c-1)$ and $A'_0(c-1)$ with filtrations so that

$$\widetilde{G}_3 = A'_1(c-1) \otimes R'$$
 and $\widetilde{G}_2 = A'_0(c-1) \otimes R'$.

We can now define a higher matrix factorization for M. Let $B_1(c)$ and $B_0(c)$ be free S-modules such that $B_0(c) = B_0(c) \otimes R'$ and $B_1(c) = B_1(c) \otimes R'$. For s = 0, 1,

we consider free S-modules A_1 and A_0 with filtrations such that $A_s(p) = A_s'(p)$ for $1 \le p \le c - 1$ and

$$A_s(c) = A'_s(c-1) \oplus B_s(c).$$

We define the map $d: A_1 \longrightarrow A_0$ to be

(9.8)
$$A_1(c) = A_1(c-1) \oplus B_1(c) \xrightarrow{\begin{pmatrix} d' & \psi_c \\ 0 & b_c \end{pmatrix}} A_0(c-1) \oplus B_0(c) = A_0(c)$$

where b_c and ψ_c are arbitrary lifts to S of \widetilde{b} and $\widetilde{\psi}$. For every $1 \leq p \leq c-1$, we set $h_p = h'_p$. Furthermore, we define

$$h_c: A_0(c) = A_0 \longrightarrow A_1(c) = A_1$$

to be

(9.9)
$$A_0(c) = A_0(c-1) \oplus B_0(c) \xrightarrow{\begin{pmatrix} \theta_2 & \tau_0 \\ \partial_2 & \theta_0 \end{pmatrix}} A_1(c-1) \oplus B_1(c) = A_1(c)$$

where $\theta_2, \partial_2, \theta_0, \tau_0$ are arbitrary lifts to S of $\widetilde{\theta}_2, \widetilde{\partial}_2, \widetilde{\theta}_0, \widetilde{\tau}_0$ respectively.

We must verify conditions (a) and (b) of Definition 1.2. Since (d', h') is a higher matrix factorization, we need only check

$$dh_c \equiv f_c \operatorname{Id}_{A_0(c)} \operatorname{mod}(f_1, \dots, f_{c-1}) A_0(c)$$

 $\pi_c h_c d \equiv f_c \pi_c \operatorname{mod}(f_1, \dots, f_{c-1}) B_1(c)$.

Condition (a) holds because

$$\begin{pmatrix} d' & \psi \\ 0 & b_c \end{pmatrix} \begin{pmatrix} \theta_2 & \tau_0 \\ \partial_2 & \theta_0 \end{pmatrix} = \begin{pmatrix} d'\theta_2 + \theta_1\partial_2 & d'\tau_0 + \theta_1\theta_0 \\ \partial_1\partial_2 & \partial_1\theta_0 \end{pmatrix} \equiv \begin{pmatrix} f_c & 0 \\ 0 & f_c \end{pmatrix}$$

by (9.6). Similarly, Condition (b) is verified by the computation

$$\begin{pmatrix} \theta_2 & \tau_0 \\ \partial_2 & \theta_0 \end{pmatrix} \begin{pmatrix} d' & \psi \\ 0 & b_c \end{pmatrix} = \begin{pmatrix} \theta_2 d' & \theta_2 \theta_1 + \tau_0 \partial_1 \\ \partial_2 d' & \partial_2 \theta_1 + \theta_0 \partial_1 \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & f_c \end{pmatrix}.$$

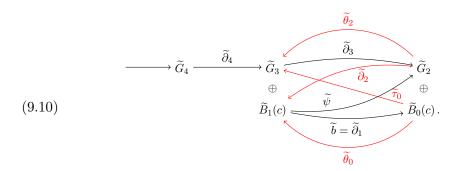
Next we show that the higher matrix factorization that we have constructed is pre-stable. Consider the complex (9.4), which is a free resolution of L over R'. It follows that

$$\operatorname{Coker}(\widetilde{A}_0(c-1) \xrightarrow{\widetilde{\partial}_2} \widetilde{B}_1(c)) \cong \operatorname{Im}(\widetilde{\partial}_1) \subset \widetilde{B}_0(c)$$

has no f_c -torsion, verifying the pre-stability condition.

It remains to show that d and h are liftings to S of the first two differentials in the minimal R-free resolution of M.

By (9.5) and Proposition 8.4 we have the following homotopies on the minimal R'-free resolution of M:



The minimal R-free resolution of M is obtained from the resolution above by applying the Shamash construction. Hence, the first two differentials are

$$R \otimes \begin{pmatrix} \widetilde{\partial}_3 & \widetilde{\psi} \\ 0 & \widetilde{b} \end{pmatrix}$$
 and $R \otimes \begin{pmatrix} \widetilde{\partial}_4 & \widetilde{\theta_2} & \tau_0 \\ 0 & \widetilde{\partial_2} & \widetilde{\theta_0} \end{pmatrix}$.

By induction hypothesis $\widetilde{\partial}_3 = R' \otimes d_{c-1}$ and $\widetilde{\partial}_4 = R' \otimes h(c-1)$. By the construction of d and h in (9.8), (9.9) we see that $R \otimes d$ and $R \otimes h$ are the first two differentials in the minimal R-free resolution of M.

Finally, we will prove that if M is a stable syzygy, then (d,h) is stable as well. The map ∂_2 is the composite map

$$A_0(p-1) \hookrightarrow A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{\pi_p} B_1(p)$$

by construction (9.9). By (9.4) it follows that if L is a maximal Cohen-Macaulay R-module, then $\operatorname{Coker}(\widetilde{\partial}_2)$ is a maximal Cohen-Macaulay R'-module, verifying the stability condition for a higher matrix factorization over R(p-1). By induction, it follows that (d,h) is stable.

Remark 9.11. In order to capture structure when minimality is not present, Definition 7.1 can be modified as follows. We extend the definition of syzygies to non-minimal free resolutions: if (\mathbf{F}, δ) is an R-free resolution of an R-module P, then we define $\operatorname{Syz}_{i,\mathbf{F}}(P) = \operatorname{Im}(\delta_i)$. Suppose that f_1, \ldots, f_c is a regular sequence in a local ring S, and set $R = S/(f_1, \ldots, f_c)$. Let (\mathbf{F}, δ) be an R-free resolution, and let $M = \operatorname{Im}(\delta_r)$ for a fixed $r \geq 2c$.

We say that M is a pre-stable syzygy in \mathbf{F} with respect to f_1, \ldots, f_c if either c = 0 and M = 0, or $c \geq 1$ and there exists a lifting $(\widetilde{\mathbf{F}}, \widetilde{\delta})$ of (F, δ) to $R' = S/(f_1, \ldots, f_{c-1})$ such that the CI operator $\widetilde{t}_c := (1/f_c)\widetilde{\delta}^2$ is surjective and, setting $(\widetilde{\mathbf{G}}, \widetilde{\partial}) := \mathrm{Ker}(\widetilde{t}_c)$, the module $\mathrm{Im}(\widetilde{\partial}_r)$ is pre-stable in $\widetilde{\mathbf{G}}_{\geq 2}$ with respect to f_1, \ldots, f_{c-1} .

With minor modifications, the proof of Theorem 9.2 yields the following result: Let \mathbf{F} be an R-free resolution. If M is a pre-stable r-th syzygy in \mathbf{F} with respect to f_1, \ldots, f_c then M is the HMF module of a pre-stable matrix factorization (d, h) such that d and h are liftings to S of the consecutive differentials δ_{r+1} and δ_{r+2} in \mathbf{F} . If \mathbf{F} is minimal then the higher matrix factorization is minimal.

We can use the concept of pre-stable syzygy and Proposition 8.8 in order to build the minimal free resolutions of the modules $\operatorname{Coker}(R(p-1) \otimes b_p)$:

Proposition 9.12. Let (d,h) be a minimal pre-stable matrix factorization for a regular sequence f_1, \ldots, f_c in a local ring S, and use the notation of 2.1. For every $p \le c$, set $R(p) = S/(f_1, \ldots, f_p)$ and $D(p) = \operatorname{Coker}(R(p-1) \otimes b_p)$. Then

$$D(p) = \operatorname{Coker}(R(p) \otimes b_p)$$
.

Let $\mathbf{T}(p)$ be the minimal R(p)-free resolution of M(p) from Construction 5.1 and Theorem 5.2. The minimal R(p-1)-free resolution of D(p) is

$$\mathbf{V}(p-1): \quad \mathbf{T}(p-1) \longrightarrow R(p-1) \otimes B_1(p) \xrightarrow{R(p-1) \otimes b_p} R(p-1) \otimes B_0(p)$$
,

where the second differential is induced by the composite map

$$\delta: A_0(p-1) \hookrightarrow A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{\pi_p} B_1(p).$$

The minimal R(p)-free resolution of D(p) is

$$\mathbf{W}(p): \quad \mathbf{T}(p) \longrightarrow R(p) \otimes B_1(p) \xrightarrow{R(p-1) \otimes b_p} R(p) \otimes B_0(p)$$

where the second differential is given by the Shamash construction applied to $\mathbf{V}(p-1)_{\leq 3}$.

PROOF: By Theorem 5.2 (using the notation in that theorem) the complex $\mathbf{T}(p)$ is an R(p)-free resolution of M(p). By Theorem 6.4, the minimal R(p-1)-free resolution of M(p) is

$$\longrightarrow T(p-1)_2 \longrightarrow T(p-1)_1 \xrightarrow{R(p-1) \otimes d_{p-1}} T(p-1)_0$$

$$\oplus \qquad \qquad \oplus$$

$$R(p-1) \otimes B_1(p) \xrightarrow{R(p-1) \otimes b_p} R(p-1) \otimes B_0(p).$$

Since f_p is a non-zerodivisor on M(p-1) by Corollary 3.11 and since the matrix factorization is pre-stable, we can apply Proposition 8.8, where the homotopies θ_i and τ_i for f_p are chosen to be the appropriate components of the map $R(p-1)\otimes h_p$. We get the minimal R(p-1)-free resolution

$$\mathbf{V}(p-1): \quad \mathbf{T}(p-1) \longrightarrow R(p-1) \otimes B_1(p) \xrightarrow{R(p-1) \otimes b_p} R(p-1) \otimes B_0(p)$$

where the second differential is induced by the composite map

$$\delta: A_0(p-1) \hookrightarrow A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{\pi_p} B_1(p)$$
.

Since we have a homotopy for f_p on $R(p-1)\otimes B_1(p)\longrightarrow R(p-1)\otimes B_0(p)$ it follows that $D(p)=\operatorname{Coker}(R(p)\otimes b_p)$.

We next apply the Shamash construction to the following diagram with homotopies:

$$\mathbf{V}(p-1)_{\leq 3}: \rightarrow A_1(p-1)' \xrightarrow{d'_{p-1}} A_0(p-1)' \xrightarrow{\partial_2} B_1(p)' \xrightarrow{\partial_1 = b'_p} B_0(p)',$$

where -' stands for $R(p-1) \otimes -$. By Proposition 4.8 we obtain an exact sequence

$$R(p) \otimes A_1(p) \longrightarrow R(p) \otimes A_0(p) \longrightarrow R(p) \otimes B_1(p) \longrightarrow R(p) \otimes B_0(p)$$
.

It is minimal since θ_0 is induced by h_p . The leftmost differential

$$R(p) \otimes A_1(p) \xrightarrow{R(p) \otimes b_p} R(p) \otimes A_0(p)$$

coincides with the first differential in $\mathbf{T}(p)$.

The following result (stated somewhat differently) and the idea of the proof are from [AGP, Theorem 7.3]. We will use it in Corollary 9.14 in order to obtain numerical information about pre-stable matrix factorizations.

Proposition 9.13. Let $f \in S$ be a non-zerodivisor in a local ring S, and let \mathbf{F} be a minimal free resolution of a nonzero module over S/(f). If the CI operator $t: F_2 \longrightarrow F_0$ corresponding to f is surjective, then $\operatorname{rank}(F_1) \geq \operatorname{rank}(F_0)$, and if equality holds then \mathbf{F} is periodic of period 2 (that is, $\operatorname{Syz}_2^{S/(f)}(L) \cong L$ where $L = H_0(\mathbf{F})$). In the latter case, the ranks of the free modules F_i are constant.

PROOF: We lift the first two steps of \mathbf{F} to S as $\widetilde{F}_2 \xrightarrow{\widetilde{\delta}_2} \widetilde{F}_1 \xrightarrow{\widetilde{\delta}_1} \widetilde{F}_0$, so that $\widetilde{\delta}_1 \widetilde{\delta}_2 = f\widetilde{t}$. Since t is surjective and f is in the maximal ideal, \widetilde{t} is surjective. Thus the image of $\widetilde{\delta}_1$ contains $f\widetilde{F}_0$, and it follows that $\operatorname{rank}(\widetilde{\delta}_1) = \operatorname{rank}(\widetilde{F}_0)$. In particular, $\operatorname{rank}(F_1) \geq \operatorname{rank}(F_0)$. In case of equality $\widetilde{\delta}_1$ is a monomorphism, and we can factor the multiplication by f on \widetilde{F}_0 as $\widetilde{\delta}_1\widetilde{u}_1$ for some u_1 —a matrix factorization of f. Thus the cokernel of δ_1 is resolved by the periodic resolution coming from this matrix factorization, so \mathbf{F} is periodic. Then the ranks of the free modules F_i are constant by [Ei1, Proposition 5.3].

Using Proposition 9.13, we get a stronger version of Corollary 3.14 for prestable matrix factorizations.

Corollary 9.14. Let (d,h) be a minimal pre-stable matrix factorization, and use the notation of 2.1. Let γ be the minimal number such that $A(\gamma) \neq 0$. Then

$$\operatorname{cx}_{R}(M) = c - \gamma + 1 \ and$$

$$\operatorname{rank}(B_{1}(p)) = \operatorname{rank}(B_{0}(p)) = 0 \quad \text{for every } 1 \leq p \leq \gamma - 1$$

$$\operatorname{rank}(B_{1}(\gamma)) = \operatorname{rank}(B_{0}(\gamma)) > 0$$

$$\operatorname{rank}(B_{1}(p)) > \operatorname{rank}(B_{0}(p)) > 0 \quad \text{for every } \gamma + 1 \leq p \leq c.$$

The multiplicity of Ext^{even} (equal to the multiplicity of Ext^{odd} and called the Betti degree) is the size of the hypersurface matrix factorization that is the top non-zero part of the higher matrix factorization (d, h).

For every $p \leq \gamma - 1$, the projective dimension of M over R(p) is finite and we have the equality of Poincaré series

$$\mathcal{P}_M^{R(p)}(x) = (1+x)^p \, \mathcal{P}_M^S(x) \,.$$

PROOF: By Corollary 5.7(2) it follows that $\operatorname{cx}_R(M) = c - \gamma + 1$. The definition of a higher matrix factorization shows that (d_{γ}, h_{γ}) is a codimension 1 matrix factorization, and hence $\operatorname{rank}(B_1(\gamma)) = \operatorname{rank}(B_0(\gamma)) > 0$.

For $\gamma+1 \leq p \leq c$, we apply Proposition 9.12. Since $\mathbf{V}(p-1)$ is a free resolution, it follows that $B_0(p)=0$ if $B_1(p)=0$. But $B_0(p)=B_1(p)=0$ implies that M(p)=M(p-1) which contradicts to the fact that f_p annihilates M(p) and f_p is a non-zerodivisor on M(p-1) by Corollary 3.11. Hence, $B_1(p) \neq 0$. Since the map h_p is minimal and the free resolution $\mathbf{V}(p-1)$ is minimal, it follows that $B_0(p) \neq 0$. The inequality rank $(B_1(p)) > \operatorname{rank}(B_0(p))$ follows from Proposition 9.13 since the free resolution $\mathbf{W}(p)$ has a surjective CI operator.

It follows at once that the higher matrix factorization in Example 3.15 is not pre-stable. But in fact Corollary 9.14 implies stronger restrictions on the Betti numbers in the finite resolution of modules that are pre-stable syzygies:

Corollary 9.15. If M is a pre-stable syzygy of complexity ζ with respect to the regular sequence f_1, \ldots, f_c in a local ring S and $b_i^S(M)$ denotes the i-th Betti number of M as an S-module, then

$$b_0^S(M) \ge \zeta$$

$$b_1^S(M) \ge (c - \zeta + 1)b_0^S(M) + \frac{\zeta(\zeta + 1)}{2} - 1$$

PROOF: Set $\gamma = c - \zeta + 1$. By Theorem 3.4, Theorem 5.2, and Corollary 9.14 we get

$$b_0^S(M) = b_0^{R(\gamma)}(M) = \sum_{p=\gamma}^c \text{rank } B_0(p) \ge \zeta$$

and

$$b_1^S(M) = b_1^{R(\gamma)}(M) + (c - \zeta)b_0^{R(\gamma)}(M) = b_1^{R(\gamma)}(M) + (c - \zeta)b_0^S(M)$$

$$b_1^{R(\gamma)}(M) = \sum_{p=\gamma}^{c} \operatorname{rank} B_1(p) + \sum_{p=\gamma}^{c-1} (c - p)\operatorname{rank} B_0(p)$$

$$\geq \left(c - \gamma + 1 - 1 + \sum_{p=\gamma}^{c} \operatorname{rank} B_0(p)\right) + \sum_{p=\gamma}^{c-1} (c - p)\operatorname{rank} B_0(p)$$

$$= \zeta - 1 + b_0^S(M) + \sum_{p=\gamma}^{c-1} (c - p)\operatorname{rank} B_0(p).$$

Therefore,

$$b_1^S(M) = (c - \zeta + 1)b_0^S(M) + \zeta - 1 + \sum_{p=\gamma}^{c-1} (c - p) \operatorname{rank} B_0(p)$$

$$\geq (c - \zeta + 1)b_0^S(M) + \zeta - 1 + {\zeta \choose 2}$$

$$= (c - \zeta + 1)b_0^S(M) + \frac{\zeta(\zeta + 1)}{2} - 1.$$

For example, a pre-stable syzygy module of complexity ≥ 2 cannot be cyclic and cannot have $b_1^S(M) = b_0^S(M) + 1$.

We close this section by a remark on the graded case. We use the formulas in 9.16 to study quadratic complete intersections in [EPS2].

Corollary 9.16. Let k be an infinite field, $S = k[x_1, ..., x_n]$ be standard graded with $\deg(x_i) = 1$ for each i, and I be an ideal generated by a regular sequence of c homogeneous elements of the same degree q. Set R = S/I, and suppose that N is a finitely generated graded R-module. Let $f_1, ..., f_c$ be a generic for N regular sequence of forms minimally generating I. If M is a sufficiently high graded syzygy of N over R, then M is the module of a minimal higher matrix factorization (d,h) with respect to $f_1, ..., f_c$; it involves modules $B_s(p)$ for s = 0, 1 and $1 \le p \le c$. Denote $b_{i,j}^R(M) = \dim \left(\operatorname{Tor}_i^R(M,k)_j \right)$ the graded Betti numbers of M over R. The graded Poincaré series $\mathcal{P}_M^R(x,z) = \sum_{i>0} b_{i,j}^R(M) x^i z^j$ of M over R is

(9.17)
$$\mathcal{P}_{M}^{R}(x,z) = \sum_{1 \le p \le c} \frac{1}{(1 - x^{2}z^{q})^{c-p+1}} (x \, m_{p;1}(z) + m_{p;0}(z)),$$

where for each s = 0, 1 and $1 \le p \le c$ we use the polynomial

$$m_{p;s}(z) := \sum_{j\geq 0} b_{s,j}^S (B_s(p)) z^j$$

such that its coefficient $b_{s,j}^S(B_s(p))$ is the number of minimal generators of degree j of the S-free module $B_s(p)$.

PROOF: By Remark 7.10, it follows that Corollary 9.3 holds verbatim, without first localizing at the maximal ideal. Note that (9.17) is a refined version of the formula in Corollary 5.7(1). The CI operators t_i on the minimal R-free resolution \mathbf{T} , constructed in 5.1, of the HMF module M can be taken homogeneous. Since they are projections by Proposition 5.3(2), it follows that they have degree 0. The lifted (to S) CI operators \widetilde{t}_i satisfy

$$\widetilde{d}^2 = f_1 \widetilde{t}_1 + \dots + f_c \widetilde{t}_c.$$

Therefore, $deg(t_i) = -q$ for every i.

10. Stable Syzygies in the Local Gorenstein case

In this section S will denote a local Gorenstein ring. We write f_1, \ldots, f_c for a regular sequence in S and $R = S/(f_1, \ldots, f_c)$. Thus R is also a Gorenstein ring. In this setting matters are simplified by the fact that a maximal Cohen-Macaulay module is, in a canonical way, an m-th syzygy for any m.

When M is a maximal Cohen-Macaulay S-module we let $\operatorname{Cosyz}_{j}^{S}(M)$ be the dual of the j-th syzygy of $M^{*} := \operatorname{Hom}_{S}(M, S)$. When we speak of syzygies or cosyzygies, we will implicitly suppose that they are taken with respect to a minimal resolution. The following result is well-known.

Cosyzygy Lemma 10.1. Let S be a local Gorenstein ring.

- (1) If M is a maximal Cohen-Macaulay S-module, then M^* is a maximal Cohen-Macaulay S-module, M is reflexive, and $\operatorname{Ext}_S^i(M,S)=0$ for all i>0.
- (2) If M is the first syzygy module in a minimal free resolution of a maximal Cohen-Macaulay S-module, then M has no free summands.
- (3) If M is a maximal Cohen-Macaulay module without free summands, then

$$M \cong \operatorname{Syz}_j^S(\operatorname{Cosyz}_j^S(M)) \cong \operatorname{Cosyz}_j^S(\operatorname{Syz}_j^S(M))$$

for every $j \geq 0$, and $N := \operatorname{Cosyz}_{j}^{S}(M)$ is the unique maximal Cohen-Macaulay S-module N without free summands such that M is isomorphic to $\operatorname{Syz}_{j}^{S}(N)$.

PROOF SKETCH: After replacing S by its completion we may choose a regular local ring $S' \subseteq S$ over which S is finite, and we have $\operatorname{Ext}_S(M,S) = \operatorname{Ext}_{S'}(M,S')$, and M is free over S'. Part (2) is obvious over an artinian ring, and the general case follows by factoring out a maximal regular sequence. The first statement of (3) follows from the vanishing of Ext, and the second part follows from the first.

When M is a maximal Cohen-Macaulay module over the Gorenstein ring S, we define the Tate resolution of M to be the doubly infinite free complex \mathbf{T} without homology that results from splicing the minimal free resolution of M with the dual of the minimal free resolution of M^* . If N is also an S-module then the stable

Ext is by definition the collection of functors $\widehat{\operatorname{Ext}}^{j}(M,N)$, the j-th homology of $\operatorname{Hom}(\mathbf{T}, N)$; here j can be any integer.

Let f_1, \ldots, f_c be a regular sequence in a Gorenstein local ring S with maximal ideal **m** and residue field k. Set $R = S/(f_1, \ldots, f_c)$. Let M be a maximal Cohen-Macaulay R-module with no free summands and finite projective dimension over S. If **T** is the Tate resolution of M over R, the CI operators corresponding to f_1, \ldots, f_c are defined on all of **T**, so that $\widehat{\operatorname{Ext}}_R(M, k) := \bigoplus_i \widehat{\operatorname{Ext}}_R^i(M, k)$ becomes a graded module over the ring $\mathcal{R} = k[\chi_1, \ldots, \chi_c]$. Then

$$\widehat{\operatorname{Ext}}_R^{\geq j}(M,k) = \operatorname{Ext}_R(\operatorname{Cosyz}_j^R(M),k)[j]$$

is a finitely generated module over \mathcal{R} for any integer j. In this case the definition of a stable syzygy (Definition 7.1) takes a particularly canonical form:

Proposition 10.2. With hypotheses as above, M is stable with respect to f_1, \ldots, f_c if and only if either c=0 and M=0, or the following two conditions are satisfied:

- (1) χ_c is a non-zerodivisor on Ext_R^{≥-2}(M, k).
 (2) Syz₂^{R'}(Cosyz₂^R(M)) is a stable syzygy with respect to f₁,..., f_{c-1} ∈ S, where R' = S/(f₁,..., f_{c-1}).

PROOF: $\widehat{\operatorname{Ext}}_R^{\geq -2}(M,k)$ is, up to a shift in grading, the same as $\operatorname{Ext}_R(\operatorname{Cosyz}_2^R(M),k)$, and $\operatorname{Cosyz}_2^R(M)$ is the only maximal Cohen-Macaulay module of which M could be the second syzygy.

We will show that stable syzygies all come from stable matrix factorizations.

Theorem 10.3. Let f_1, \ldots, f_c be a regular sequence in a Gorenstein local ring S, and set $R = S/(f_1, \ldots, f_c)$. An R-module M is a stable syzygy if and only if it is the module of a minimal stable matrix factorization with respect to f_1, \ldots, f_c .

We postpone the proof to give a necessary homological construction:

Proposition 10.4. Let f_1, \ldots, f_c be a regular sequence in a Gorenstein local ring S, and set $R = S/(f_1, \ldots, f_c)$. Let M be the HMF module of a minimal stable $matrix\ factorization\ (d,h).\ Then$

$$\operatorname{Cosyz}_{2}^{R(p)}M(p) = \operatorname{Coker}(R(p) \otimes b_{p}) = \operatorname{Coker}(R(p-1) \otimes b_{p}).$$

In the notation of Proposition 9.12, the minimal R(p-1)-free resolution of the module $\operatorname{Cosyz}_{2}^{R(p)}M(p)$ is $\mathbf{V}(p-1)$, and the minimal R(p)-free resolution of the module $\operatorname{Cosyz}_{2}^{R(p)}M(p)$ is $\mathbf{W}(p)$.

PROOF: We apply Proposition 9.12. As the higher matrix factorization is stable, we conclude that the depth of the R(p-1)-module $\operatorname{Coker}(R(p-1)\otimes b_p)$ is one less than that of a maximal Cohen-Macaulay R(p-1)-module. Therefore, it is a maximal Cohen-Macaulay R(p)-module. The free resolution **W** implies that $\operatorname{Cosyz}_{2}^{R(p)}M(p) = \operatorname{Coker}(R(p) \otimes b_{p}).$ **Corollary 10.5.** Let f_1, \ldots, f_c be a regular sequence in a Gorenstein local ring S, and set $R = S/(f_1, \ldots, f_c)$. If M is the module of a minimal stable matrix factorization with respect to f_1, \ldots, f_c , then

$$M(p-1) \cong \operatorname{Syz}_2^{R(p-1)} \left(\operatorname{Cosyz}_2^{R(p)} \left(M(p) \right) \right).$$

PROOF: For each p = 1, ..., c, by Proposition 10.4 we have

$$M(p-1) = \operatorname{Coker}(R(p-1) \otimes d_{p-1}) = \operatorname{Syz}_{2}^{R(p-1)} \left(\operatorname{Coker}(R(p-1) \otimes b_{p}) \right)$$
$$= \operatorname{Syz}_{2}^{R(p-1)} \left(\operatorname{Cosyz}_{2}^{R(p)} \left(\operatorname{Coker}(R(p) \otimes d_{p}) \right) \right)$$
$$= \operatorname{Syz}_{2}^{R(p-1)} \left(\operatorname{Cosyz}_{2}^{R(p)} \left(M(p) \right) \right)$$

where as usual $d_p: A_1(p) \longrightarrow A_0(p)$ denotes the restriction of $d: A_1 \longrightarrow A_0$. \square

PROOF OF THEOREM 10.3: Theorem 9.2 shows that a stable syzygy yields a stable matrix factorization.

Conversely, let M be the module of a minimal stable matrix factorization (d,h). Use notation as in 2.1. By Proposition 10.4 and in its notation, $\mathbf{W}(p)$ is the minimal R-free resolution of $\operatorname{Cosyz}_2^{R(p)}(M(p)) = \operatorname{Coker}(R(p) \otimes b_p)$. We have a surjective CI operator t_c on $\mathbf{W}(p)$ because on the one hand, we have it on $\mathbf{T}(p)$ and on the other hand $\mathbf{W}(p)_{\leq 3}$ is given by the Shamash construction so we have a surjective standard CI operator on $\mathbf{W}(p)_{\leq 3}$. Furthermore, the standard lifting of $\mathbf{W}(p)$ to R(p-1) starts with $\mathbf{V}(p-1)_{\leq 1}$, so in the notation of Definition 7.1 we get $\operatorname{Ker}(\widetilde{\delta}_1) = M(p-1)$, which is stable by induction hypothesis.

Corollary 10.6. Let f_1, \ldots, f_c be a regular sequence in a Gorenstein local ring S, and set $R = S/(f_1, \ldots, f_c)$. Let M be a stable syzygy with a minimal stable matrix factorization (d, h). For every $p = 1, \ldots, c$ we have

$$(\operatorname{Syz}_{1}^{R(p)}(M(p)))(p-1) = \operatorname{Syz}_{1}^{R(p-1)}(M(p-1)).$$

PROOF: By induction, it will suffice to prove this assertion for M = M(c).

The syzygy module $\operatorname{Syz}_1^R(M)$ is stable by Proposition 7.3. Recall the proof of Proposition 7.3 with $L = \operatorname{Cosyz}_2^R(M)$. The first and last equalities below are from Corollary 10.5, and then we apply (7.4) to get

$$(\operatorname{Syz}_{1}^{R}(M))(c-1) = \operatorname{Syz}_{2}^{R(c-1)} (\operatorname{Cosyz}_{2}^{R}(\operatorname{Syz}_{1}^{R}(M)))$$

$$= \operatorname{Im}(\widetilde{\delta}_{3}) = \operatorname{Syz}_{3}^{R(c-1)} (\operatorname{Cosyz}_{2}^{R}(M))$$

$$= \operatorname{Syz}_{1}^{R(c-1)} (\operatorname{Syz}_{2}^{R(c-1)} (\operatorname{Cosyz}_{2}^{R}(M)))$$

$$= \operatorname{Syz}_{1}^{R(c-1)} (M(c-1)) .$$

Recall that if E is a graded \mathcal{R} -module then we define the S2-ification of E, written S2(E), by the formula

$$S2(E) = \bigoplus_{j \in \mathbf{Z}} H^0(\widetilde{E}(j))$$

where \widetilde{E} denotes the coherent sheaf on projective space associated to E.

Proposition 10.7. Suppose that R = S/I, where S is a regular local ring and I is generated by a regular sequence, and let M be maximal Cohen-Macaulay Rmodule.

- (1) If M is a stable syzygy then M has no free summand. (2) Set $E := \widehat{\operatorname{Ext}}_R^{\geq -2}(M,k)$. If M is a stable syzygy, then $\operatorname{reg} E = -1$, and E coincides with S2(E) in degrees

We could restate the last condition of (2) in terms of local cohomology by saying that $H^1_{\mathcal{R}_+}(E)$ is 0 in degree ≥ -2 .

PROOF: (1): This follows at once from part (2) of Lemma 10.1.

(2): We do induction on c. If c=1 then E is free and generated in degrees -2and -1, so the result is obvious, and we may suppose c > 1.

From Proposition 10.2 we see that χ_c is a non-zerodivisor on E, so

$$reg(E) = reg(E/\chi_c E),$$

and Corollary 4.14 shows $(E/\chi_c E)^{\geq 0} = \widehat{\operatorname{Ext}}_{R'}^{\geq 0}(M',k)$, where $M' = \operatorname{Syz}_2^{R'}(\operatorname{Cosyz}_2^{R})$

Since M' is stable, χ_{c-1} is a non-zerodivisor on $E' := \widehat{\operatorname{Ext}}_{R'}^{\geq -2}(M',k)$, and thus also on $E'^{\geq 0} = (E/\chi_c E)^{\geq 0}$, so

$$H^0_{(\chi_1,\dots,\chi_c)}((E/\chi_c E)^{\geq 0}) = 0.$$

Since the modules E', $E'^{\geq 0}$ and $E/\chi_c E$ differ by modules of finite length, they have the same i-th local cohomology for $i \geq 1$. By induction, reg(E') = -1, so $\operatorname{reg}(E/\chi_c E) = -1$ as well, proving that $\operatorname{reg} E = -1$.

Finally we show that E agrees with S2(E) in degrees ≥ -2 . Since χ_c is a non-zerodivisor on E, we see that E is a submodule of $F := S2(E)^{\geq -2}$. Because $\operatorname{reg} E = -1$ the natural map $E \longrightarrow \operatorname{S2}(E)$ is surjective in degrees ≥ -1 .

Thus we need only prove that $E \longrightarrow S2(E)$ is surjective in degree -2. By induction, $E^{\geq 0}/\chi_c E = \operatorname{Ext}^{\geq 0}(M',k)$ has depth at least 1. But from the exact sequence

$$0 \longrightarrow \chi_c F/\chi_c E \longrightarrow E^{\geq 0}/\chi_c E \longrightarrow E^{\geq 0}/\chi_c F \longrightarrow 0$$

we see that the module of finite length $\chi_c F/\chi_c E$ is contained in $E^{\geq 0}/\chi_c E$, so $\chi_c F/\chi_c E=0$. Since χ_c is a non-zerodivisor on E, and thus also on F, this implies that F/E = 0 as well.

11. Syzygies over Intermediate Rings

In this section we suppose that S is a Gorenstein ring. Let $I \subset S$ be an ideal generated by a regular sequence, and set R = S/I. Let N' be a finitely generated R-module of finite projective dimension over S. If $M = \operatorname{Syz}_i^R(N')$ is a sufficiently high syzygy, then by Theorem 7.6 and Theorem 9.2 M comes from a higher matrix factorization with respect to a generic choice of generators f_1, \ldots, f_c for the ideal I. Set $R(p) = S/(f_1, ..., f_p)$. The following result identifies the HMF module M(p) with the module $Syz_i^{R(p)}(N)$, where we have chosen $N = Cosyz_{c+1}^R(M)$.

Theorem 11.1. Let f_1, \ldots, f_c be a regular sequence in a local Gorenstein ring S. Set $R(p) = S/(f_1, \ldots, f_p)$ and R = R(c). Suppose that M is a stable syzygy with stable matrix factorization (d, h) with respect to f_1, \ldots, f_c . Let $N = \operatorname{Cosyz}_{c+1}^R(M)$, and set M(0) = 0.

(1) With notation as in 2.1,

$$\operatorname{Syz}_{c+1}^{R(p)}(N) \cong M(p) \text{ for } p \geq 0.$$

(2) We write ν_p for the map

$$R(p) \otimes \operatorname{Syz}_{i}^{R(p-1)}(N) \xrightarrow{\nu_{p}} \operatorname{Syz}_{i}^{R(p)}(N),$$

induced by the comparison map from the minimal R(p-1)-free resolution of N to the minimal R(p)-free resolution of N inducing the identity map on N (this comparison map is unique up to homotopy). For each p, there is a short exact sequence

$$0 \longrightarrow R(p) \otimes M(p-1) \xrightarrow{\nu_p} M(p) \longrightarrow \operatorname{Cosyz}_2 M(p) \longrightarrow 0.$$

For the proof of Theorem 11.1 we will make use of the following well-known result. For the reader's convenience we sketch the proof. Write $\operatorname{mod}(R)$ for the category of finitely generated R-modules and $\operatorname{\mathbf{\underline{MCM}}}(R(p))$ for the stable category of maximal Cohen-Macaulay R(p)-modules, where the morphisms are morphisms in $\operatorname{mod}(R(p))$ modulo those that factor through projectives. We say that S-modules M, M' have a common syzygy if there exists a j such that $\operatorname{Syz}_j^S(M) \cong \operatorname{Syz}_j^S(M')$ in $\operatorname{\mathbf{\underline{MCM}}}(S)$.

Lemma 11.2. Suppose that S is a Gorenstein ring and that M, M' are S-modules.

(1) If N, N' are S-modules and there are exact sequences

$$0 \longrightarrow M \longrightarrow P_r \longrightarrow \cdots \longrightarrow P_0 \longrightarrow N \longrightarrow 0,$$

$$0 \longrightarrow M' \longrightarrow P'_r \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow N' \longrightarrow 0$$

such that each P_i and each P'_i is a module of finite projective dimension over S, then M and M' have a common syzygy if and only if N and N' have a common syzygy.

- (2) If M and M' have a common syzygy and are both maximal Cohen-Macaulay S-modules then $M \cong M'$ in $\underline{\mathbf{MCM}}(S)$.
- (3) If $M \cong M'$ in $\underline{\mathbf{MCM}}(S)$, the ring S is local, and both M and M' are maximal Cohen-Macaulay S-modules without free summands, then $M \cong M'$ as S-modules.

PROOF: (1): It suffices to do the case r = 0. Let $N_1 = \text{Ker}(P_0 \longrightarrow N)$, and let **V** be a free resolution of N_1 . The mapping cone of a map from **V** to a finite resolution of P_0 is a free resolution of N, so that for $i \gg 0$ we have $\text{Syz}_i^S(N) \cong \text{Syz}_{i-1}(N_1)$

in $\mathbf{MCM}(S)$. By induction, for $i \gg 0$ the (i-1-r)-th syzygy of M agrees with the i-th syzygy of N, and the same is true for M' and N'.

- (2): If $\operatorname{Syz}_{i}^{S}(M) \cong \operatorname{Syz}_{i}^{S}(M') \cong N$, then $M \cong \operatorname{Cosyz}_{i}^{S}(N) \cong M'$ in $\operatorname{\mathbf{\underline{MCM}}}(S)$.
- (3): Let $M \xrightarrow{\alpha} M' \xrightarrow{\beta} M$ be inverse isomorphisms in $\underline{\mathbf{MCM}}(S)$. This means that $\beta \alpha = \mathrm{Id}_M + \phi \varphi$, where $M \xrightarrow{\varphi} F \xrightarrow{\phi} M$ for some free module F. Since S is local and M has no free summand, φ must have image inside the maximal ideal times F, and thus $\phi \varphi$ has image inside the maximal ideal times M. By Nakayama's Lemma, $\beta \alpha$ is an epimorphism, and it follows that $\beta \alpha$ is an isomorphism. Since the same goes for $\alpha \beta$, we see that $M \cong M'$.

Proof of Theorem 11.1:

(1): By Corollary 3.11 M(p) is a maximal Cohen-Macaulay R(p)-module, and by Corollary 3.12 it has no free summand. In particular, $N = \operatorname{Cosyz}_{c+1}^R(M)$ is well-defined and has no free summands. It follows that $\operatorname{Syz}_{c+1}^{R(p)}(N)$ is a maximal Cohen-Macaulay R(p)-module and by the Cosyzygy Lemma 10.1 it has no free summands. By Lemma 11.2(3), it suffices to show that the maximal Cohen-Macaulay R(p)-modules M(p) and $\operatorname{Syz}_{c+1}^{R(p)}(N)$ have a syzygy in common over R(p). We will do this by showing that each of these modules has an R(p)-syzygy in common with M.

Observe that R has finite projective dimension over R(p). Lemma 11.2(1) implies that, indeed, $M = \operatorname{Syz}_{c+1}^R(N)$ and $\operatorname{Syz}_{c+1}^{R(p)}(N)$ have a common syzygy over R(p).

We next compare M = M(c) with M(p). When p > q the module R(p) has finite projective dimension over R(q). By Corollary 10.5,

$$M(p-1) = \operatorname{Syz}_{2}^{R(p-1)} \left(\operatorname{Cosyz}_{2}^{R(p)} (M(p)) \right).$$

Applying Lemma 11.2(1) to an R(p-1)-free resolution of $\operatorname{Cosyz}_2^{R(p)}(M(p))$ and to an R(p)-free resolution of $\operatorname{Cosyz}_2^{R(p)}(M(p))$, we conclude that M(p-1) and M(p) have a common syzygy over each ring R(q) with $q \leq p-1$.

(2): For each p, let $\mathbf{T}(p)$ be the minimal R(p)-free resolution of M(p) and let $\mathbf{W}(p)$ be the minimal R(p)-free resolution of $\operatorname{Cosyz}_2^{R(p)}(M(p))$. See also Proposition 10.4. Since M(p-1) is a maximal Cohen-Macaulay R(p-1)-module by Corollary 3.11, the minimal free resolution of $R(p) \otimes M(p-1)$ as an R(p)-module is $R(p) \otimes \mathbf{T}(p-1)$.

Since M(p) is a stable syzygy, the CI operator t_p is surjective on $\mathbf{W}(p)$. Take a lifting \widetilde{t}_p acting on a lifting of $\mathbf{W}(p)$ to R(p-1). The kernel of \widetilde{t}_p is a minimal R(p-1)-free resolution $\widetilde{\mathbf{G}}$ of $\operatorname{Cosyz}_2^{R(p)}(M(p))$. By Corollary 10.5, $\mathbf{T}(p-1)$ is isomorphic to $\widetilde{\mathbf{G}}_{\geq 2}[-2]$. Thus we have a short exact sequence of minimal free resolutions

$$0 \longrightarrow R(p) \otimes \mathbf{T}(p-1) \longrightarrow \mathbf{T}(p) \xrightarrow{t_p} \mathbf{W}(p)[-2] \longrightarrow 0,$$

and this induces the desired short exact sequence of modules.

The last claim in the theorem follows from Corollary 10.6.

Corollary 11.3. With hypotheses as in Theorem 11.1, let M be a stable syzygy with respect to f_1, \ldots, f_c , with stable matrix factorization (d, h). If we denote the codimension 1 part of (d, h) by (d_1, h_1) , then the codimension 1 part of the higher matrix factorization of $\operatorname{Syz}_1^R(M)$ is (h_1, d_1) .

PROOF: If (d_1, h_1) is non-trivial, then the minimal R(1)-free resolution of $M(1) = R(1) \otimes d_1$ is periodic of the form

$$\cdots \xrightarrow{d_1} F_4 \xrightarrow{h_1} F_3 \xrightarrow{d_1} F_2 \xrightarrow{h_1} F_1 \xrightarrow{d_1} F_0.$$

Theorem 11.4. Suppose that f_1, \ldots, f_c is a regular sequence in a Gorenstein local ring S, and set $R = S/(f_1, \ldots, f_c)$. Suppose that N is an R-module of finite projective dimension over S. Assume that f_1, \ldots, f_c are generic with respect to N. Denote $\gamma := c - cx_R(N) + 1$, where $cx_R(N)$ is the complexity of N (see Corollary 5.8). Then:

- (1) The projective dimension of N over $R(p) = S/(f_1, ..., f_p)$ is finite for $p < \gamma$.
- (2) Choose a $j \geq 1$ large enough so that $M := \operatorname{Syz}_j^R(N)$ is a stable syzygy and $\operatorname{Syz}_j^{R(p)}(N)$ is a maximal Cohen-Macaulay R(p)-module for every $p \leq \gamma$. The hypersurface matrix factorization for the periodic part of the minimal free resolution of N over $S/(f_1, \ldots, f_{\gamma})$ is isomorphic to the top non-zero part of the higher matrix factorization of M.

A version of (1) is proved in [Av, Theorem 3.9], [AGP, 5.8 and 5.9].

PROOF: Choose M as in (2). By Corollary 5.8, M(p) = 0 for $p < \gamma$. Apply Proposition 11.1 for $p \le \gamma$. The case $p = \gamma$ establishes (2).

Remark 11.5. In particular, the above theorem shows that the codimension 1 matrix factorization that is obtained from a high $S/(f_1)$ -syzygy of N agrees with the codimension 1 part of the higher matrix factorization for M over R, and both codimension 1 matrix factorizations are trivial if the complexity of N is < c, where M is a sufficiently high syzygy of N over R.

12. Functoriality

From Theorem 11.1 it follows immediately that the higher matrix factorization construction induces functors on the stable module category. In this section we make the result explicit.

Let $R(p) = S/(f_1, \ldots, f_p)$. If $i > \dim R$ then the modules $\operatorname{Syz}_i^{R(p)}(N)$ are maximal Cohen-Macaulay R(p)-modules. We define functors

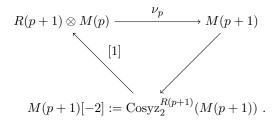
$$\mathcal{F}_i : \operatorname{mod}(R) \longrightarrow \prod_{\substack{p \ 52}} \operatorname{Mor}(\underline{\mathbf{MCM}}(R(p)))$$

taking N to the collection of morphisms

$$R(p) \otimes \operatorname{Syz}_i^{R(p-1)}(N) \xrightarrow{\nu_p} \operatorname{Syz}_i^{R(p)}(N),$$

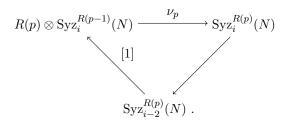
where ν_p is the comparison map defined in Theorem 11.1. The map ν_p is unique up to homotopy, and thus yields a well-defined morphism in $\underline{\mathbf{MCM}}(R(p))$.

Corollary 12.1. With assumptions and notation as in Theorem 11.1, for each $p = 1 \dots c - 1$ there exists a triangle in $\underline{\mathbf{MCM}}(R(p+1))$ of the form



If M' is a first syzygy of M, then the corresponding triangles for M' are obtained from the triangles for M by applying the shift (equivalently, taking first syzygy) operator to each M(p).

We remark that Theorem 11.1 implies that for $i \geq c+3$ we get a triangle



Let $\mathbf{MF}(f_1,\ldots,f_c)$ be the full subcategory of $\underline{\mathbf{MCM}}(R)$ whose objects are stable equivalence classes of maximal Cohen-Macaulay modules that are stable syzygies with respect to f_1,\ldots,f_c . We get a functor $\mathcal{F}:\mathbf{MF}(f_1,\ldots,f_c)\longrightarrow \mathcal{C}$, where an object \mathbf{M} of \mathcal{C} is a collection of objects $M(p)\in \underline{\mathbf{MCM}}(R(p))$ for $p=1,\ldots,c$ that fit into triangles as in Corollary 12.1 in $\underline{\mathbf{MCM}}(R(p+1))$ and whose morphisms $\mathbf{M}=\{M(p)\}\longrightarrow \mathbf{M}'=\{M'(p)\}$ are collections of morphisms $\{(M(p)\longrightarrow M'(p))\in \underline{\mathbf{MCM}}(R(p))\}$ that commute with the morphisms in the triangles. Furthermore, if M' is the first syzygy of M, then $\mathcal{F}(M')$ is obtained from $\mathcal{F}(M)$ by applying the shift (equivalently, taking first syzygy) operator in $\underline{\mathbf{MCM}}(R(p))$ to each M(p) and to each triangle.

13. Morphisms

In this section we introduce the concept of an HMF morphism (a morphism of higher matrix factorizations) so that it preserves the structures described in Definition 1.2, and then we show that any homomorphism of matrix factorization modules induces an HMF morphism.

Definition 13.1. A morphism of matrix factorizations or HMF morphism α : $(d,h) \longrightarrow (d',h')$ is a triple of homomorphisms of free modules

$$\alpha_0: A_0 \longrightarrow A'_0$$

$$\alpha_1: A_1 \longrightarrow A'_1$$

$$\alpha_2: \bigoplus_{p \le c} A_0(p) \longrightarrow \bigoplus_{p \le c} A'_0(p)$$

such that, for each p:

- (a) $\alpha_s(A_s(p)) \subseteq A_s'(p)$ for s = 0, 1. We write $\alpha_s(p)$ for the restriction of α_s to $A_s(p)$.
- (b) $\alpha_2 (\bigoplus_{q \leq p} A_0(q)) \subseteq \bigoplus_{q \leq p} A'_0(q)$, and the component $A_0(p) \longrightarrow A'_0(p)$ of α_2 is $\alpha_0(p)$. We write $\alpha_2(p)$ for the restriction of α_2 to $\bigoplus_{q \leq p} A_0(q)$.
- (c) The diagram

$$\bigoplus_{q \leq p} A_0(q) \xrightarrow{h} A_1(p) \xrightarrow{d_p} A_0(p)$$

$$\alpha_2(p) \downarrow \qquad \qquad \downarrow \alpha_1(p) \qquad \qquad \downarrow \alpha_0(p)$$

$$\bigoplus_{q \leq p} A'_0(q) \xrightarrow{h'} A'_1(p) \xrightarrow{d'_p} A'_0(p)$$

commutes modulo (f_1, \ldots, f_{p-1}) .

Theorem 13.2. Suppose that f_1, \ldots, f_c is a regular sequence in a Gorenstein local ring S, and set $R = S/(f_1, \ldots, f_c)$. Let M and M' be stable syzygies over R, and suppose $\zeta : M \longrightarrow M'$ is a morphisms of R-modules. With notation as in 2.1, let M and M' be HMF modules of stable matrix factorizations (d,h) and (d',h'), respectively. There exists an HMF morphism

$$\alpha:(d,h)\longrightarrow(d',h')$$

such that the map induced on

$$M = \operatorname{Coker}(R \otimes d) \longrightarrow \operatorname{Coker}(R \otimes d') = M'$$

is ζ .

We first establish a strong functoriality statement for the Shamash construction. Suppose that \mathbf{G} and \mathbf{G}' are S-free resolutions of S-modules M and M' annihilated by a non-zerodivisor f, and $\zeta: M \longrightarrow M'$ is any homomorphism. If we choose systems of higher homotopies σ and σ' for f on \mathbf{G} and \mathbf{G}' respectively,

then the Shamash construction yields resolutions $\operatorname{Sh}(\mathbf{G}, \sigma)$ and $\operatorname{Sh}(\mathbf{G}', \sigma')$ of M and M over R = S/(f), and thus there is a morphism of complexes

$$\widetilde{\phi}: \operatorname{Sh}(\mathbf{G}, \sigma) \longrightarrow \operatorname{Sh}(\mathbf{G}', \sigma')$$

covering ζ . To prove the Theorem we need more: a morphism defined over S that commutes with the maps in the "standard liftings" $\widetilde{\operatorname{Sh}}(\mathbf{G},\sigma)$ and $\widetilde{\operatorname{Sh}}(\mathbf{G}',\sigma')$ (see Construction 4.7) and respects the natural filtrations of these modules. The following statement provides the required morphism.

Lemma 13.3. Let S be a commutative ring, and let $\varphi_0 : (\mathbf{G}, d) \longrightarrow (\mathbf{G}', d')$ be a map of S-free resolutions of modules annihilated by an element f. Given systems of higher homotopies σ_j and σ'_j on \mathbf{G} and \mathbf{G}' , respectively, there exists a system of maps φ_j of degree 2j from the underlying free module of \mathbf{G} to that of \mathbf{G}' such that, for every index m,

$$\sum_{i+j=m} (\sigma_i' \varphi_j - \varphi_j \sigma_i) = 0.$$

We say that $\{\varphi_j\}$ is a system of homotopy comparison maps if they satisfy the conditions in the lemma above.

Recall that a map of free complexes $\lambda: \mathbf{U} \longrightarrow \mathbf{W}[-a]$ is a homotopy for a map $\rho: \mathbf{U} \longrightarrow \mathbf{W}[-a+1]$ if $\delta\lambda - (-1)^a\lambda\partial = \rho$, where ∂ and δ are the differentials in \mathbf{U} and \mathbf{W} respectively. Since in Lemma 13.3 σ_0 and σ'_0 are the differentials d and d', the equation above in Lemma 13.3 says that, for each m, the map φ_m is a homotopy for the sum

$$-\sum_{\substack{i+j=m\\i>0,i>0}} (\sigma'_i \varphi_j - \varphi_j \sigma_i).$$

PROOF: The desired condition on φ_0 is equivalent to the given hypothesis that φ_0 is a map of complexes. We proceed by induction on m > 0 and on homological degree to prove the existence of φ_m . The desired condition can be written as

$$d'\varphi_m = -\sum_{\substack{i+j=m\\i\neq 0}} \sigma_i'\varphi_j + \sum_{\substack{i+j=m\\i\neq 0}} \varphi_j\sigma_i.$$

Since **G** is a free resolution, it suffices to show that the right-hand side is annihilated by d'. Indeed,

$$\begin{split} &-\sum_{\substack{i+j=m\\i\neq 0}}(d'\sigma_i')\varphi_j+\sum_{\substack{i+j=m\\i\neq 0}}(d'\varphi_j)\sigma_i\\ &=\sum_{\substack{i+j=m\\i\neq 0}}\sum_{\substack{v+w=i\\v\neq 0}}\sigma_v'\sigma_w'\varphi_j-f\varphi_{m-1}-\sum_{\substack{i+j=m\\q\neq 0}}\sum_{\substack{q+u=j\\q\neq 0}}\sigma_q'\varphi_u\sigma_i+\sum_{\substack{i+j=m\\u+q=j}}\sum_{\substack{u+q=j\\u+q=j}}\varphi_u\sigma_q\sigma_i\\ &=\sum_{\substack{v+w+j=m\\v\neq 0}}\sigma_v'\sigma_w'\varphi_j-f\varphi_{m-1}-\sum_{\substack{i+q+u=m\\q\neq 0}}\sigma_q'\varphi_u\sigma_i+\sum_{\substack{i+u+q=m\\q\neq 0}}\varphi_u\sigma_q\sigma_i\,, \end{split}$$

where the first equality holds by (3) in 4.1 and by the induction hypothesis. Reindexing the first summand by v = q, w = i and j = u we get

$$\sum_{\substack{q+i+u=m\\q\neq 0}} \sigma'_q \sigma'_i \varphi_u - f \varphi_{m-1} - \sum_{\substack{i+q+u=m\\q\neq 0}} \sigma'_q \varphi_u \sigma_i + \sum_{\substack{i+u+q=m\\q\neq 0}} \varphi_u \sigma_q \sigma_i$$

$$= -f \varphi_{m-1} + \sum_{\substack{q\neq 0}} \sigma'_u \left(\sum_{\substack{i+u=m-q\\i+u=m-q}} \sigma'_i \varphi_u - \varphi_u \sigma_i \right) + \sum_{\substack{u}} \varphi_u \left(\sum_{\substack{q+i=m-u\\q\neq 0}} \sigma_q \sigma_i \right)$$

$$= -f \varphi_{m-1} + 0 + 0 + \varphi_{m-1} f = 0,$$

where the last equality holds by (3) in 4.1 and by induction hypothesis. \Box

The next result reinterprets the conditions of Lemma 13.3 as defining a map between liftings of Shamash resolutions.

Proposition 13.4. Let S be a commutative ring, and let G and G' be S-free resolutions with systems of higher homotopies $\sigma = {\sigma_j}$ and $\sigma' = {\sigma'_j}$ for $f \in S$, respectively. Suppose that ${\{\varphi_j\}}$ is a system of homotopy comparison maps for σ and σ' . We use the standard lifting of the Shamash resolution defined in 4.7, and the notation established there. Denote by $\widetilde{\varphi}$ the map with components

$$\varphi_i: y^{(v)}G_j \longrightarrow y^{(v-i)}G'_{j+2i}$$

from the underlying graded free S-module of the standard lifting $\widetilde{Sh}(\mathbf{G}, \sigma)$ of the Shamash resolution $Sh(\mathbf{G}, \sigma)$, to the underlying graded free S-module of the standard lifting $\widetilde{Sh}(\mathbf{G}', \sigma')$ of the Shamash resolution $Sh(\mathbf{G}', \sigma')$. The maps $\widetilde{\varphi}$ satisfy $\widetilde{\delta'}\widetilde{\varphi} = \widetilde{\varphi}\widetilde{\delta}$, where $\widetilde{\delta}$ and $\widetilde{\delta'}$ are the standard liftings of the differentials defined in 4.7.

PROOF: Fix a and v. We must show that the diagram

$$y^{(a)}G_{v} \xrightarrow{\widetilde{\delta}} \bigoplus_{0 \leq i \leq a} y^{(a-i)}G_{v+2i-1}$$

$$\downarrow \widetilde{\varphi} \qquad \qquad \downarrow \widetilde{\varphi}$$

$$\bigoplus_{0 \leq j \leq a} y^{(a-j)}G'_{v+2j} \xrightarrow{\widetilde{\delta'}} \bigoplus_{0 \leq i \leq a-j} y^{(a-i-j)}G'_{v+2i+2j-1}.$$

commutes. Fix $0 \le q \le a$. The map $\widetilde{\delta'}\widetilde{\varphi} - \widetilde{\varphi}\widetilde{\delta}$ from $y^{(a)}G_v$ to $y^{(q)}G'_{v+2a-2q-1}$ is equal to $\sum_{i+j=a-q} (\sigma'_i \varphi_j - \varphi_j \sigma_i)$, which vanishes by Lemma 13.3.

Remark 13.5. A simple modification of the proof of Lemma 13.3 shows that systems of homotopy comparison maps also exist in the context of systems of higher homotopies for a regular sequence f_1, \ldots, f_c , not just in the case c = 1 as above, and one can interpret this in terms of Shamash resolutions as in Proposition 13.4 as well, but we do not need these refinements.

PROOF OF THEOREM 13.2: The result is immediate for c=1, so we proceed by induction on c>1. Let $\widetilde{R}=S/(f_1,\ldots,f_{c-1})$. To simplify the notation, we will write $\widetilde{-}$ for $\widetilde{R}\otimes_S-$, and $\overline{-}$ for $R\otimes-$. We will make use of our standard notation 2.1. Since (d,h) is stable we can extend the map \overline{d} to a complex

$$\overline{A}_1(c) \longrightarrow \overline{A}_0(c) \longrightarrow \overline{B}_1(c) \longrightarrow \overline{B}_0(c)$$

that is the beginning of an R-free resolution \mathbf{F} of $\operatorname{Cosyz}_2^R(M)$, and there is a similar complex that is the beginning of the R-free resolution \mathbf{F}' of $\operatorname{Cosyz}_2^R(M')$. By stability these cosyzygy modules are maximal Cohen-Macaulay modules, so dualizing these complexes we may use $\zeta(c) := \zeta : M \longrightarrow M'$ to induce maps

$$\eta: \operatorname{Cosyz}_2^R(M) \longrightarrow \operatorname{Cosyz}_2^R(M')$$

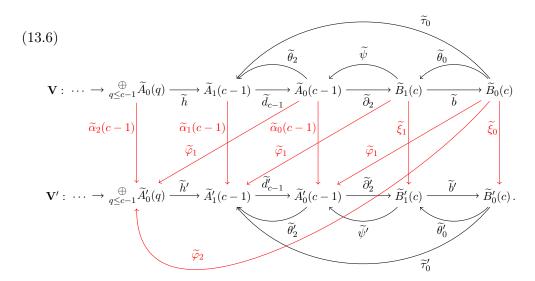
 $\lambda: \mathbf{F} \longrightarrow \mathbf{F}'.$

Moving to \widetilde{R} , we have

$$M(c-1) = \operatorname{Coker} \widetilde{d}(c-1) = \operatorname{Syz}_{2}^{\widetilde{R}}(\operatorname{Cosyz}_{2}^{R}(M))$$

$$M'(c-1) = \operatorname{Coker} \widetilde{d}'(c-1) = \operatorname{Syz}_{2}^{\widetilde{R}}(\operatorname{Cosyz}_{2}^{R}(M')).$$

We will use the notation and the construction in the proof of Theorem 9.2, where we produced an \widetilde{R} -free resolution \mathbf{V} of $\operatorname{Cosyz}_2^R(M)$, and various homotopies on it. Of course we have a similar resolution \mathbf{V}' of $\operatorname{Cosyz}_2^R(M)$. See the diagram:



The map η induces $\widetilde{\xi}: \mathbf{V} \longrightarrow \mathbf{V}'$, which in turn induces a map

$$\zeta(c-1): M(c-1) \longrightarrow M'(c-1).$$

See diagram (13.6).

By induction, the map $\zeta(c-1)$ is induced by an HMF morphism with components

$$\alpha_s(c-1): A_s(c-1) \longrightarrow A'_s(c-1)$$

for s = 0, 1 and

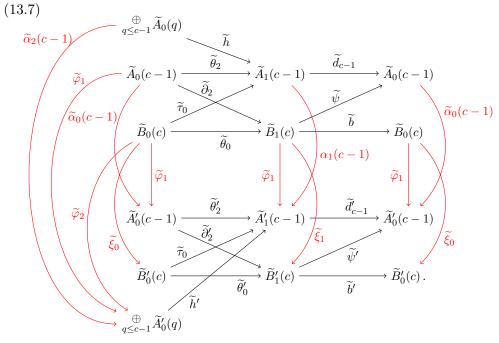
$$\alpha_2(c-1): \bigoplus_{q \leq c-1} \widetilde{A}_0(q) \longrightarrow \bigoplus_{q \leq c-1} \widetilde{A}_0(q)'.$$

By the conditions in 13.1, it follows that the first two squares on the left are commutative; clearly, the last square on the right is commutative as well. Since $\widetilde{\alpha}_0(c-1)$ induces the same map on M(c-1) as $\widetilde{\xi}$, the remaining square commutes. Therefore we can apply Lemma 13.3 and conclude that there exists a system of homotopy comparison maps, the first few of which are shown as $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ in diagram (13.6).

With notation as in (9.8) and (9.9) we may write the first two steps of the minimal \widetilde{R} -free resolution \mathbf{U} of M in the form given by the top three rows of diagram (13.7), where we have used a splitting

$$A_0(c) = A_0(c-1) \oplus B_0(c)$$

to split the left-hand term $\bigoplus_{q \leq c} A_0(q)$ into three parts, $\bigoplus_{q \leq c-1} A_0(q)$, $A_0(c-1)$, and $B_0(c)$; and similarly for M' and the bottom three rows. Straightforward computations using the definition of the system of homotopy comparison maps shows that diagram (13.7) commutes.



Next, we will construct the maps α_i . We construct α_0 by extending the map $\alpha_0(c-1)$ already defined over S by taking $\alpha_0|_{B_0(c)}$ to have as components arbitrary liftings to S of $\widetilde{\xi}_0$ and $\widetilde{\varphi}_1$. Similarly we take α_1 to be the extension of $\alpha_1(c-1)$ that has arbitrary liftings of $\widetilde{\xi}_1$ and $\widetilde{\varphi}_1$ as components. Finally, we take α_2 to agree with $\alpha_2(c-1)$ on $\bigoplus_{q \leq c-1} A_0(q)$ and on the summand $A_0(c) = A_0(c-1) \oplus B_0(c)$, to be the map given by

$$\alpha_0(c-1): A_0(c-1) \longrightarrow A'_0(c-1)$$

and arbitrary liftings

$$\varphi_1: A_0(c-1) \longrightarrow \bigoplus_{q \le c-1} A_0'(q) \qquad \qquad \varphi_1: B_0(c) \longrightarrow A_0'(c-1)$$

$$\xi_0: B_0(c) \longrightarrow B_0'(c) \qquad \qquad \varphi_2: B_0(c) \longrightarrow \bigoplus_{q \le c-1} A_0'(q)$$

to S of $\widetilde{\varphi}_1, \widetilde{\varphi}_2$ and $\widetilde{\xi}_0$.

It remains to show that $\overline{\alpha}_0 = R \otimes_S \alpha_0$ induces $\zeta : M \longrightarrow M'$.

By Proposition 10.4 the minimal R-free resolutions \mathbf{F} and \mathbf{F}' of $\operatorname{Cosyz}_2^R(M)$ and $\operatorname{Cosyz}_2^R(M')$ have the form given in the following diagram.

$$\mathbf{F}: \quad \cdots \longrightarrow \overline{B}_0(c) \oplus \overline{A}_0(c-1) \xrightarrow{\overline{\theta}_0} \overline{B}_1(c) \xrightarrow{\overline{b}} \overline{B}_0(c)$$

$$\overline{\alpha}_0 \downarrow \lambda_2 \qquad \qquad \downarrow \lambda_1 = \overline{\xi}_1 \qquad \qquad \downarrow \lambda_0 = \overline{\xi}_0$$

$$\mathbf{F}': \quad \cdots \longrightarrow \overline{B}'_0(c) \oplus \overline{A}'_0(c-1) \xrightarrow{\overline{\theta}'_0, \overline{\partial}'_2} \overline{B}'_1(c) \xrightarrow{\overline{b}'} \overline{B}'_0(c),$$

By definition the map of complexes $\lambda: \mathbf{F} \longrightarrow \mathbf{F}'$ induces $\zeta: M \longrightarrow M'$. Using Lemma 13.3, we see that the left-hand square of the diagram also commutes if we replace λ_2 with α_0 , and thus these two maps induce the same map $M \longrightarrow M'$, concluding the proof.

14. Strong Higher Matrix Factorizations

We introduce a stronger version of Definition 1.2 in which we require that the map h is part of a homotopy. In Theorem 14.2 we show that an HMF module always has a strong matrix factorization.

Definition 14.1. Let (d, h) be a higher matrix factorization and $M = \operatorname{Coker}(d \otimes R)$ be its module. We say that (d, h) is a *strong matrix factorization* for M if for each p, the map h_p can be extended to a homotopy $h(f_p)$ for f_p at $L(p)_0 = A_0(p)$ on the S-free resolution $\mathbf{L}(p)$ of $M(p) = \operatorname{Coker}(d_p \otimes S/(f_1, \ldots, f_p))$ constructed in 3.3.

For example, in the codimension 2 case in Example 3.6, the third equation in (3.7) shows that a higher matrix factorization satisfies

$$\rho dh_2 \equiv f_2 \rho \, \operatorname{mod}(f_1 B_0(1)) \,,$$

where we denote ρ the projection from A_0 to $B_0(1)$. A strong matrix factorization must satisfy the stronger condition

$$\rho dh_2 = f_2 \rho$$
.

Theorem 14.2.

- (1) If (d,h) is a strong matrix factorization, then it is a higher matrix factorization.
- (2) Let (d, h') be a higher matrix factorization. There exists a strong matrix factorization (d, h) with the same filtrations $0 \subseteq A_s(1) \subseteq \cdots \subseteq A_s(c) = A_s$ for s = 0, 1 as (d, h'). Note that (d, h) and (d, h') have the same higher matrix factorization module $M := \operatorname{Coker}(d \otimes R)$. If the ring S is local and the higher matrix factorization (d, h') is minimal, then (d, h) is minimal as well.

PROOF: We will use the following notation: if φ is a map of modules

$$\bigoplus_{1 \leq i \leq s} P_i \longrightarrow \bigoplus_{1 \leq j \leq s} Q_j ,$$

then we denote $\varphi_{P_i \longrightarrow Q_j}$ the projection of $\varphi|_{P_i}$ on Q_j and call it the *component of* φ from P_i to Q_j .

Consider the finite S-free resolution \mathbf{L} of M constructed in 3.3, and use the notation in 3.5 and in 14.1.

(1): We have to show that conditions (a) and (b) in Definition 1.2 are satisfied by d and h. First, we consider (a). For a fixed p, the map $h(f_p)$ has components

$$h_p: \ A_1(p) \longleftarrow A_0(p)$$

$$h(f_p)_{e_iB_0(w) \leftarrow B_0(v)}: \ e_iB_0(w) \leftarrow B_0(v) \ \text{for} \ i < w \ .$$

For every $q \leq p$ we have

(14.3)
$$d_p h_p|_{B_0(q)} + \sum_{1 \le i < w \le p} f_i h(f_p)_{e_i B_0(w) \leftarrow B_0(q)} = f_p \operatorname{Id}_{B_0(q)}.$$

This condition is stronger than condition (a) in Definition 1.2.

We will prove that (b) holds. Fix p, and denote ∂ the differential in $\mathbf{L}(p)$. Let $\sigma(p)$ be a homotopy for f_p on $\mathbf{L}(p)$ that extends $h(f_p)$. The second differential in $\mathbf{L}(p)$ is mapping

$$L(p)_{2} = (\bigoplus_{i < q \le p} e_{i}B_{1}(q)) \oplus (\bigoplus_{j < i < q \le p} e_{i}e_{j}B_{0}(q))$$

$$\downarrow$$

$$L(p)_{1} = (\bigoplus_{q < p} B_{1}(q)) \oplus (\bigoplus_{i < q < p} e_{i}B_{0}(q)).$$

By Remark 3.5 the only components of the differential that land in $B_1(p)$ are

$$f_i: e_i B_1(p) \longrightarrow B_1(p)$$
 for $i < p$.

Therefore,

$$\begin{split} f_p \mathrm{Id}_{B_1(p)} &= \pi_p \big(f_p \mathrm{Id}_{A_1(p)} \big) = \pi_p \big(\sigma(f_p) \partial + \partial \sigma(f_p) \big) \big|_{A_1(p)} \\ &= \pi_p \sigma(f_p) \partial \big|_{A_1(p)} + \pi_p \partial \sigma(f_p) \big|_{A_1(p)} \\ &= \pi_p h_p d_p + \sum_{1 \le i < p} f_i \sigma(f_p)_{e_i B_1(p) \leftarrow A_1(p)} \\ &\equiv \pi_p h_p d_p \ \mathrm{mod}(f_1, \dots, f_{p-1}) B_1(p) \,. \end{split}$$

(2): For each p, there exists a homotopy $\sigma(f_p)$ for f_p on the free resolution $\mathbf{L}(p)$ since the module M(p) is annihilated by f_p . Let $h: A_0(p) \longrightarrow A_1(p)$ be the components of $\sigma(f_p)$ from $A_0(p)$ to $A_1(p)$.

Suppose that S is local and the higher matrix factorization (d, h') is minimal. We will prove that the map h is minimal. By Theorem 5.2, (d, h') yields a minimal R-free resolution \mathbf{T}' of M. By (1), (d, h) is a higher matrix factorization, and so by Theorem 5.2 it yields an R-free resolution \mathbf{T} of M over R. Both resolutions have the same ranks of the corresponding free modules in them because the free modules in the filtrations of (d, h') and (d, h) have the same ranks. Therefore, the resolution \mathbf{T} is minimal as well. The second differential in \mathbf{T} is $\overline{h} = h \otimes R$. Hence, the map h is minimal.

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