Optimal Robust Double Auctions

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Abstract

In a stylized exchange economy with single, continuous good and quasi-linear utilities, we propose a novel double-auction format featuring two (forward and reverse) clock auctions, Vickrey-style payments, and carefully designed per-unit taxes. In the spirit of Ausubel (2004), we define a sincere ex-post perfect equilibrium of the game and show that, under a certain disclosure policy, it is the only one surviving iterated elimination of weakly dominanted strategies. Furthermore, we show how the clocks can be adjusted dynamically to maximize disclosure. Finally, we show that, with private values, the auction can implement an ex-ante optimal mechanism under the ex-post constraints. The associated tax is private and depends on the price clock, not the number of bidders. Further tractability is achieved given quadratic utilities, allowing for comparisons with other mechanisms.

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1 Introduction

Make later...

2 Literature

Our paper is linked to three strands of literature: robust mechanisms, optimal mechanisms, and practical auction rules.

The first strand is the classical literature on optimal mechanism design. The concept of virtualization, necessary for optimality, was developed independently by Mussa and Rosen (1978) and Myerson (1981). It was later generalized, among others, by Wilson (1985), Gresik and Satterthwaite (1989), Maskin and Riley (2000) and Lu and Robert (2001), to be used for two-sided and multi-unit environments.[1]

The second strand is the design of robust mechanisms. The concept of robust implementation is in the sense of Wilson (1987), Bergemann and Morris (2005), and Chung and Ely (2007), meaning that the mechanism should work for all information structures, distributions, and beliefs. Furthermore, in the context of optimality, we can distinguish three approaches to robustness. The first classical approach is finding a mechanism with given properties, assuming the type distribution is known. The second approach is to estimate the properties of the distribution in a static environment, see Kojima and Yamashita (2017), or estimate it on the fly, see Loertscher and Marx (2020) and Loertscher and Mezzetti (2021). The third approach is to consider the worst-case, relative to the maximized objective, scenario, see Brooks and Du (2021) and Suzdaltsev (2022). Our paper belongs to the first approach, which can be justified by saying that the distribution can always be estimated using a small randomly sampled group of agents, which will be asymptotically negligible.

The third strand is the design of simple mechanisms when optimal mechanisms are impractical. For example, in Hart and Nisan (2017), it was argued that simple mechanisms for selling two goods could achieve a guaranteed fraction of the optimal revenue. In Andreyanov and Sadzik (2021), two families of simple mechanisms (σ-VCG and δ-VCG) were suggested for an exchange environment with multi-unit demands. In this paper, we give the means to compare them to the optimal mechanism and find that they often capture a

[1]We add to this body of literature a non-linear utility and a small observation, see Lemma 2, that circumvents the non-monotonicity of virtual type.

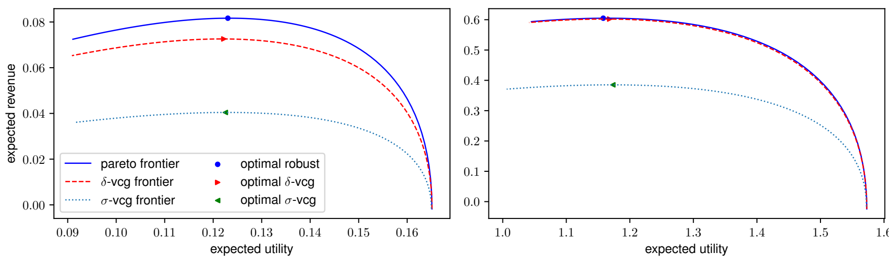

Figure 1: Welfare comparison of σ-VCG and δ-VCG mechanisms to the Pareto frontier, for $n = 100$ agents with quadratic utility and either uniform[-1,1] (left figure) or logistic (right figure) type distribution.

significant portion of optimal revenue.

Our numerical exercises contribute to the long ongoing debate over the efficiency-revenue tradeoff in two-sided markets with private information on both sides. One of the oldest results in this area is the impossibility of budget surplus under efficient trade by Myerson and Satterthwaite (1983), meaning that full ex-post efficiency is very costly in revenue. Another argument was made by Gresik and Satterthwaite (1989) that optimal mechanisms converge to efficiency at a quadratic rate, and in Lu and Robert (2001), they converge to a simple bid-ask spread. Both results, however, rely on either unit demand or linear utility. With decreasing returns to scale, optimal mechanisms neither converge to efficiency nor to bid-ask spreads, which we confirm under quadratic utility. Furthermore, Loertscher et al. (2015) argues that the efficiency-revenue trade-off is steeper in markets with two-sided private information than those with one-sided, meaning that full optimality might be too costly in terms of efficiency. It, therefore, seems natural to find a compromise between the fully optimal and efficient mechanisms. With our quadratic-utility model, we can trace the whole Pareto frontier. Interestingly, for logistic type distribution, the simple mechanisms based on bid-ask spreads (δ-VCG) almost reach that frontier.

Our paper also contributes to the expanding literature studying uniform-price and pay-as-bid auctions, see Back and Zender (1993), Ausubel et al. (2014) and Wang and Zender (2002). One of the main takeaways is that demand reduction with multi-unit demands can severely impact auction revenues. We show that one possible solution to the problem is a combination of a per-unit tax with a bid-ask spread. However, in our numerical exercises, the latter is disproportionally more important, see Figure 1. Furthermore, Burkett and

Woodward (2020) argues that there could also be low-revenue equilibria and suggests using reserve prices. Such "collusive-seeming" equilibria also emerge in our setting, but for a different reason: the inadvertent informational spillover between the two sides of the auction.

Finally, in the domain of robust auction design with private values, our paper is most similar to McAfee (1992) in its oral double-clock design and Ausubel (2004, 2006) in the clinching design of the payments. However, to our best knowledge, we are the first to characterize the optimal tax in the robust setting and to show that the price path can be optimized with respect to informational spillovers - a property unique to robust double auctions.

3 Dynamic auction

Our auction can be thought of as two copies: *forward* (i.e., ascending, buyers') and *reverse* (i.e., descending, sellers'); of the efficient dynamic auction of Ausubel (2004), with carefully crafted per-unit taxes on top of the baseline Vickrey-style payments. These additional payments are necessary to implement mechanisms other than efficient, for example, revenue-maximizing ones.

Our auction resembles the recent Incentive auction used for spectrum bandwidth reallocation in its double-clock design. However, the payment rule is very different.

Forward, reverse auctions and clinching

Two clock auctions run either continuously or in rounds. To distinguish between them, we will use superscript $+$ for the forward auction and $-$ for the reverse. We denote the *clock prices* in these auctions as p^+ and p^-.

Each player i participates in both auctions and, at any given pair of clock prices, submits a demand q_i in the forward auction and q_i^- in the reverse auction. To be precise, in each auction round, the auctioneer first announces the clock price (or a range of clock prices). Then bidders simultaneously and independently from each other respond with quantities (demand schedules).

The forward auction starts at a low price p_0^+ and gradually raises it. Likewise, the reverse auction starts at a high price p_0^- and gradually lowers it. The forward auction stops when the total demand becomes non-positive, while the reverse auction stops when the total demand becomes non-negative. We will refer to this pair of, possibly different, final clock prices as the *stop-off prices*.

There is much freedom in how the auctioneer can move the clock prices towards each other. The exact instructions would depend on the auctioning style (discrete or continuous clocks) and also on the objectives of the auctioneer, which we will discuss later.

Following Ausubel (2004), at any clock-prices, we define *residual demands* as

$$q^+_{-i} := -\sum_{j\neq i} q^+_j, \quad q^-_{-i} := -\sum_{j\neq i} q^-_j,$$

and *clinched demands* as

$$q^+_{i,c} := \max(0, q^+_{-i}), \quad q^-_{i,c} := \min(0, q^-_{-i}),$$

in the forward and reverse auctions correspondingly.

Activity rules and payments

Buyers and sellers can submit arbitrary demands as long as they satisfy two activity rules. First, demands in both auctions are non-increasing in their respective prices, which we will refer to as *demand monotonicity*. Second, at any point, the agent's demand in the forward auction is no greater than her demand in the reverse auction, which we will refer to as *no-arbitrage*.

Similar to Ausubel (2004), payments are only made for the clinched units. However, they consist of two parts. The first part is standard - each incrementally clinched unit costs exactly the clock price at which it was clinched in the corresponding auction. The second part consists of marginal taxes $m\tau$ that depend on the current price clock and the current position in clinched demands. Namely, agent i pays $m\tau(p^+, q^+_{i,c})$ for the unit clinched in the forward auction and $m\tau(p^-, q^-_{i,c})$ in the reverse.

Thus, agent i's total payment given final allocation q will therefore be equal to

$$\int_0^q p_{-i}(x) + m\tau_i(p_{-i}(x), x)dx,$$

where $p_{-i}(\cdot)$ is the *inverse residual demand curve* facing agent i.

It is worth mentioning that agents do not have direct control over the units they clinch and the payments they make or receive. However, they can affect the stop-off price.

Clock policy

The two-sided nature of the environment requires us to decide on the order in which the price clocks will be moved.[2] We refer to the protocol for switching between the two price clocks as *clock policy*. Importantly, the clocks must always advance monotonically (forward - up, reverse - down) so the auction cannot loop indefinitely.

In the first protocol, we fully advance the price clock in one of the two auctions: forward or reverse, until it hits the stop-off price. After that, we fully advance the price clock in the opposite auction. While this is a naive approach, it is convenient for the demonstration of various equilibria that are not sincere.

We will refer to it as *simple* clock policy.

A more sophisticated approach is to choose which clock to move based on the history of revealed demands. This can be done either at discrete times or continuously, depending on the nature of the price clock. We will briefly discuss two such protocols.

Consider a natural generalization of McAfee (1992) for multi-unit demands. Namely, if $\sum_i q_i^+ > \sum_i q_i^-$ - move the forward clock. If $\sum_i q_i^+ < \sum_i q_i^-$ - move the reverse clock. Otherwise, move either clock. This price dynamics can be considered as excess demand-minimizing, which is a reasonable objective since, in equilibrium, supply should meet demand. Indeed, it moves the clocks so as to minimize $\sum_i q_i^+ - \sum_i q_i^-$.

Consider a different protocol. Namely, if the number of agents i for whom $q_i^+ > q_{-i}^-$ is greater than the number of agents j for whom $q_j^- < q_{-j}^+$ - move the forward clock. If the number of agents i for whom $q_i^+ > q_{-i}^-$ is less than for whom $q_i^- < q_{-i}^+$ - move the reverse clock. Otherwise, move either clock. It turns out that it optimizes the flow of information between the forward and reverse auction. Later in the paper, we will argue that it maximizes the stimuli to bid sincerely.

We will refer to the latter as *adaptive* clock policy.

[2] At any point, only one of the two clocks moves.

Disclosure policy

While all agents necessarily observe the two clocks p^+, p^- and the demands they have chosen, a natural question is whether the bidders should be aware of their current clinched positions or the positions of other players. We will refer to the protocol of revealing information to the bidders as the *disclosure policy*.

Two policies immediately come to mind. A *full-disclosure* policy informs the bidders about the most recent values of all demands in both forward and reverse auctions. A *no-disclosure* policy does not share additional information with the bidders.

The no-disclosure policy keeps the bidders "in the dark" until the auction ends. The advantage of this policy is that, between the clock switches, the bidders are effectively playing in a sealed-bid auction. They have no tools to construct a sophisticated strategy and thus, due to the Vickrey nature of payments, must play sincerely.

However, one could argue that some information can still be revealed without compromising sincerity. Namely, we will occasionally reveal to bidder i the residual demands q^+_{-i} and q^-_{-i}. The decision to reveal the latter will be based on their relative position to the true demands q^+_i and q^-_i.[3] To be precise, each bidder observes, apart from her own demands, two additional statistics:

$$q^+_{i,d} = \max(q^+_i, q^-_{-i}), \quad q^-_{i,d} = \min(q^-_i, q^+_{-i}).$$

The idea is that this knowledge bears no consequence for the bidder. Indeed, conditional on $q^+_{i,d}$ and $q^-_{i,d}$, the possible range of final allocations for bidder i is $[q^+_i, q^-_i]$, which is exactly the same as with no-disclosure.

We refer to it as *no-spoilers* disclosure policy.

Clearing rule

The *clearing rule* is a protocol for finalizing allocations and transfers when forward and reverse clock prices meet. If everyone plays continuous demands, there will be an exact market clearing. However, because demands are allowed to jump, one can end up with a

[3] It is possible to also reveal some of the individual demands of other bidders without giving out the true values of q^+_{-i} and q^-_{-i}.

mismatch of supply and demand in the auction. If such a mismatch happens, some of the most recent demands might require rationing.

One such rationing procedure was proposed in Okamoto (2018), but there are others. Any rationing procedure would suffice as long as each agent receives an allocation inside her jump of demand at the stop-off price.[4]

Structure of rounds

To summarise, our auction features forward and reverse clocks, with demand monotonicity and no-arbitrage activity rules, the adaptive clock policy, the no-spoilers disclosure policy, and any clearing rules described in the previous section.

It remains to define the structure of rounds and the domain of demands. There are two approaches: continuous and discrete. Both are stylized representations of a dynamic (oral) auction and have unique strengths and weaknesses. For the purpose of the paper, we adopt a mixed approach by modeling discrete rounds but continuous demands.

We will refer to the position of the clocks at the beginning of round k as p_k^+ and p_k^-.

The clocks start at round 0 at exogenous positions $p_0^+ < p_0^-$ and advance one at a time in discrete steps of size $(p_1^- - p_0^+)/M$. For example, these could be natural increments, such as 10$, 100$, or 1000$. Thus, the auction ends in exactly $M-1$ rounds, characterized by advancing one of the two clocks based on the clock policy.

Once a round starts, each bidder submits a demand function on the whole range between that clock's old and new position, constrained by the two activity rules. The clinched demands are then calculated in the respective range. The information is revealed at the end of the round: whether the auction is over, which clock moves next according to the clock policy, and each bidder i is informed about the latest values of $q_{i,d}^+$ and $q_{i,d}^-$ according to the disclosure policy.

In other words, the oral component of the auction happens strictly between the rounds, at discrete times, but any given round is effectively sealed-bid.

[4] It is important that the bidder is indifferent across the whole range of that jump.

4 Strategic analysis

In her section, we introduce game-theoretic primitives, such as preferences, information, strategy, and equilibrium concepts, to set the basis for strategic analysis.

We then introduce the *sincere demand* - a stylized representation of the agent's tax-adjusted preferences. If all agents reveal their sincere demands during the double auction, the outcome will be efficient in the *virtual economy* populated by agents with tax-adjusted preferences.

We make three progressively narrow statements about this outcome.

First, it is an ex-post perfect equilibrium for all clock and disclosure policies. It is neither unique nor a dominant strategy equilibrium. However, using the no-spoilers disclosure policy, we ensure that it is the only survivor of equilibrium refinement - iterated elimination of weakly dominated strategies.

Finally, we show that the adaptive clock policy, in a certain sense, maximizes disclosure under the no-spoilers disclosure policy or, equivalently, maximizes the incentives to play sincerely under full disclosure.

Primitives and solution concepts

Agent i's preferences are represented by a quasilinear utility $u_i(q_i) - t_i$, where t_i is the transfer and $u_i(q_i)$ is agent i's utility from holding q_i units of asset. Let $\mathcal{U}_i$ denote the possible utility functions of agent i. Any $u_i \in \mathcal{U}_i$ is strictly concave and continuously differentiable so that it has a strictly decreasing derivative $mu_i(q) := \frac{\partial}{\partial q} u_i(q)$ for each $q \in \mathbb{R}$. Each agent i privately observes her utility u_i at the beginning and plays the double clock auction by the rules detailed in the previous section.

Assume, without loss of generality, that at round k, the forward clock moves from position p_k^+ to p_{k+1}^+. Agent i's strategy σ_i maps $h_{i,k}$ - the entire history available to him at the beginning of the round, to a function on $[p_k^+, p_{k+1}^+]$, bound by the activity rules.

Following Ausubel (2004), we will consider two equilibrium concepts. The first one is ex-post equilibrium, extended naturally to the dynamic setting.

Definition 1. *A profile of strategies $(\sigma_i)_i$ constitutes an ex-port perfect equilibrium if for every round k, following any history $h_{i,k}$, and for every realization of the profile of $(u_i)_i$,* *the profile of continuation strategies $(\sigma_i(\cdot|k, h_{i,k}, u_i))_i$ constitutes a Nash equilibrium of the game in which $(u_i)_i$ is common knowledge.*

The second one is more subtle. It is well-known that in the Vickrey auction, the additional equilibria can be discarded by eliminating weakly dominated strategies. To achieve this feat in the dynamic setting, iterative elimination can be used. [5]

Definition 2. *A profile of strategies $(\sigma_i)_i$ is a unique survivor of iterated elimination of weakly dominated strategies if (i) there is an order of elimination such that every other strategy is weakly dominated, and (ii) there is no order of elimination, such that σ_i is weakly dominated, for some i.*

Sincere bidding

We aim to show that, in equilibrium, agents will behave similarly to a price-taking consumer. We describe such behavior by the sincere demand defined below. Note that sincere demand is not a strategy yet.

Definition 3. *The sincere demand $d_i(p)$ is defined as*

$$d_i(p) = \arg max_q \left[u_i(q) - \int_0^q m\tau_i(p, x)dx - pq \right]. \tag{1}$$

We will refer to $u_i(q) - \int_0^q m\tau_i(p, x)dx$ as *tax-adjusted utility.*

Sincere demand maximizes the agent's tax-adjusted utility, as if she was a price taker, given the wholesale price p. Thus, a market clearing price with sincere demands will achieve an efficient outcome in a fictitious economy with tax-adjusted utilities, evaluated at that price. However, these utilities are not private, as they depend on p.

For any utility u_i and tax $m\tau_i$, we wish to devise a private utility v_i that would replicate the same behavior as the tax-adjusted one, independently of the price. We will refer to these new utilities as *virtual.*

We can reverse engineer the virtual utility for agent i, up to a constant, by solving the

[5] In practice, iterative elimination here entails comparing the preferred strategy to every other strategy, conditional on all histories that are consistent with the strategies that have not been eliminated yet.

following system of first-order conditions,

$$p = mu_i(q) - m\tau_i(p, q) = mv_i(q_i).$$

In other words, $mv_i(q)$ is the graph of the set of points in the (q, p) space, where the first-order conditions are satisfied for the sincere demand. Furthermore, the v_i utilities should be strictly concave to validate the first-order approach. Thus, we introduce a joint restriction on the set of utilities $\mathcal{U}_i$ and the shape of the marginal tax.

Assumption 1. *For any i and any possible utility $u_i \in \mathcal{U}_i$,*

$$m\tau'_{i,q}(q) - mu'_{i,q}(q) > \varepsilon, \quad (1 + m\tau'_{i,q}(q,p))^{-1} > \varepsilon,$$

for all $(p, q) \in \mathbb{R} \times (\mathbb{R}\backslash\{0\})$ and some $\varepsilon > 0$.

This assumption requires that the tax be *less concave than* any utility from $\mathcal{U}_i$ at all prices, and the marginal tax can not decrease too fast in the current price. These two properties ensure that the virtual utility is, indeed, concave. To see this point, linearize the first-order condition around (p, q) to obtain

$$mv'_{i,q}(q) = -\frac{m\tau'_{i,q}(p,q) - mu'_{i,q}(q)}{1 + m\tau'_{i,p}(p,q)} > -\varepsilon^2,$$

at points of differentiability so that mv_i is strictly increasing in q. We will refer to the economy with private utilities v_i as the *virtual economy.*

Thus, a market clearing price with sincere demands will achieve an efficient (i.e., Walrasian) outcome of the virtual economy. Coupled with Vickrey-style payments, the latter will amount to an ex-post equilibrium. It remains to find a strategy that will generate sincere demand along the equilibrium path.

Such a strategy exists. It entails playing sincere demand when not constrained by either activity rule, otherwise, playing as close as possible to the sincere demand.

Definition 4. *Agent i's strategy σ_i^* is said to be the sincere strategy if, at any round k after any history of play $h_{i,k}$, her reporting plan for round k is*

$$\max(\min(d_i(p), q_i^+(p_k^+)), q_i^-(p_k^-))$$

in the forward auction, and

$$\min(\max(d_i(p), q_i^-(p_k^-), q_i^+(p_k^+)))$$

in the reverse auction.

We will refer to the profile of strategies $(\sigma_i^*)_i$ as *sincere bidding*. We are now ready to state the first main result of our paper, see Appendix A for proof.

Proposition 1. *Consider any clock policy and disclosure policy, then*

1. *Sincere bidding yields the market clearing price and allocations of the Walrasian equilibrium in the virtual economy,*
2. *Sincere bidding is an ex-post perfect equilibrium,*

for any utilities and taxes satisfying Assumption 1.

Weak dominance

It is worth noting that, with only a forward auction and no taxes, the sincere play is weakly dominant under no-disclosure, see Ausubel (2004), Theorem 1.

This is not true in the two-sided setting. The reason is that the auctioneer releases important information by merely switching between the forward and reverse auctions. This information can be used to manipulate the actions of other players in order to achieve certain results. We will use the simple clock policy to demonstrate it.

Example 1. *With the simple clock and no-disclosure policies, sincere bidding is not weakly dominant.*

Consider two players $i = 1, 2$ with sincere demands $d_1(p) = 2 - p$, $d_2(p) = 1 - 2p$ that are common knowledge (i.e., $\mathcal{U}_i$ are singleton) and no additional taxes. Let the starting prices be $p_0^+ = 0$, $p_0^- = 2$, and let the forward clock advance first, to $p_1^+ = 1$.

Under sincere bidding, the stop-off price is found by the forward clock at the end of round 0, and it equals 1. The reverse clock then moves to confirm the same stop-off price. The first and second player's total clinches amount to 1 and -1. That is, the first player is the buyer. The average prices are 0.75 and 1.5 correspondingly, see Figure 2 (left).

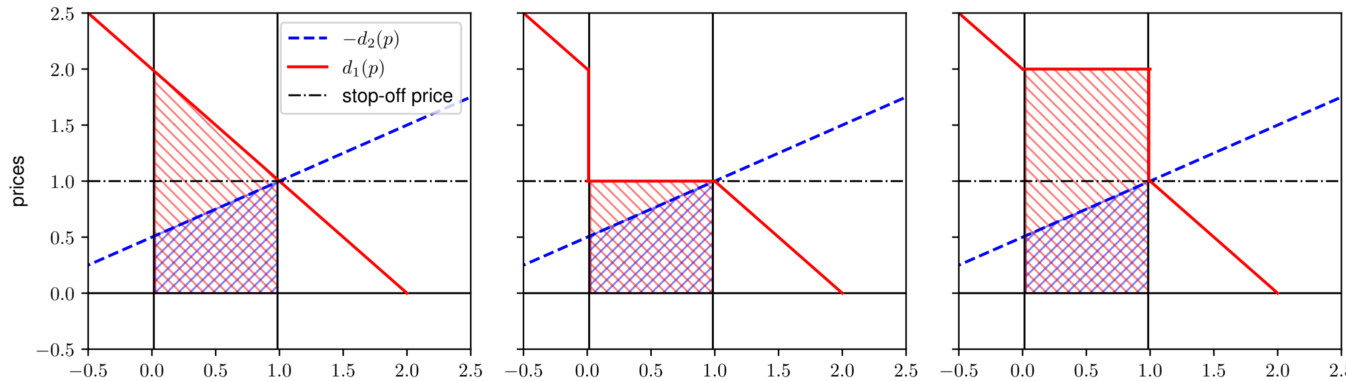

Figure 2: Payments when player 1 plays sincerely (left figure) when player 1 advances her demand late (middle figure) and early (right figure) in the reverse auction.

Consider now a modified strategy for player 1. Namely, if the stop-off price after the forward auction is less than 1, she plays sincerely in the reverse auction. Otherwise, she plays $\tilde{d}_1(p) = d_1(2) = 0$, that is, she advances her demand in the reverse auction earlier than the sincere demand prescribes.

If player 2 proceeds with bidding sincerely, she will clinch everything at the stop-off price 1, see Figure 2 (right). Her loss due to the insidious actions of the first player in the reverse auction amounts to exactly 0.5. If, however, player 2 shifts her demand to $\tilde{d}_2(p) = d_2(p) - \varepsilon$, for a small $\varepsilon > 0$, the stop-off price in the forward auction will be equal to $1 - \frac{\varepsilon}{2}$ and player 1 will then play sincerely in the reverse auction.

Thus, playing sincerely is not a dominant strategy for player 2.

Iterated elimination

Clearly, weak dominance is too strong an equilibrium concept. It is, however, possible to discard insincere strategies using iterated elimination of weakly dominated strategies. With only a forward auction and no taxes, sincere bidding was shown to be the unique survivor of such elimination, under full disclosure, see Ausubel (2004), Theorem 2.

Surprisingly, this is also not true in the two-sided setting. The reason is that the forward auction generates information that can be used in the reverse auction and vice versa. Thus, the auctioneer might have to withhold some information from the bidders.

Example 2. *With the simple clock and full-disclosure policies, there are equilibria other than sincere bidding which survive iterated elimination of weakly dominated strategies.*

To build the counterexample, consider the same setting as in Example 1.

Consider now a modified strategy for player 1. Namely, if at the end of the first round, the stop-off price turns out to be 1 with final allocations 1 and -1, player 1 submits a flat demand of size 1 in the reverse auction. Otherwise, she plays sincerely. This can be thought of as dropping the demand "later" than the sincere strategy would prescribe, see Figure 2 (middle).

Since the non-standard strategy of player 1 is in the final round, it can not be eliminated in that subgame and thus can not be eliminated iteratively.

Full support assumption

The reason why in the previous example, player 1 could deviate was that, by the end of round 0, her final allocation was known to be 1. Thus, she faced no consequence for changing her demand.

To keep the players from executing such deviations, one has to make sure that i) the information generated in the forward auction does not inform the players in the reverse auction about the potential range of allocations and vice versa, and ii) no matter what the players do, there is persisting uncertainty about the realization of the stop-off price and final allocations.

The latter is typically called a full support assumption, see Ausubel (2004).

We will model this uncertainty by letting the auctioneer participate in the auction as a shill bidder, non-strategically and without taxes. In particular, she must be able to reduce her demand at any price and by any amount that does not violate the two activity rules. [6]

Definition 5. *The double clock auction is said to satisfy the full support assumption if the auctioneer can play any demand that satisfies the activity rules.*

We are now ready to state the second main result in our paper, see Appendix A for proof.

Proposition 2. *Consider any clock policy and no-spoilers disclosure policy, then sincere*

[6] Alternatively, we can interpret it as a population of noise traders.

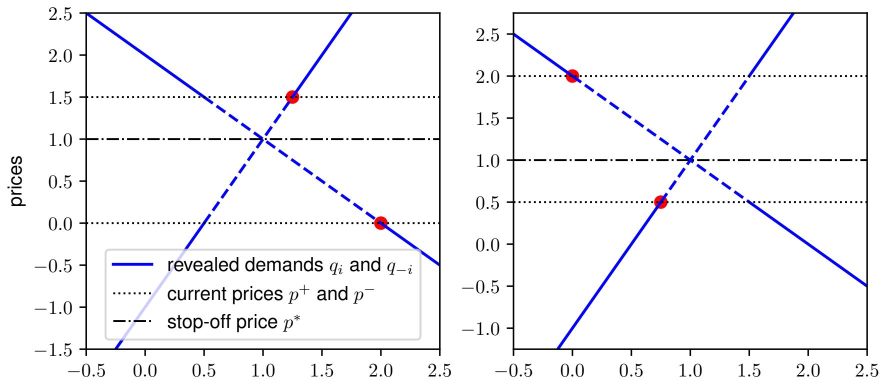

Figure 3: Informational spillover into the forward auction (left figure) and into the reverse auction (right figure).

bidding is a unique survivor of iterated elimination of weakly ex-post dominated strategies under the full support assumption.

5 Informational spillover

The multiplicity of equilibria, discussed in the previous sections, is a consequence of a general phenomenon in double auctions: the inadvertent *informational spillover* between the forward and reverse auctions.

Definition 6. *For agent i, and at clock prices $p^+ \leqslant p^-$ we say that there is informational spillover into the forward auction if $q^-_{-i}(p^-) < q^+_i(p^+)$, and into the reverse auction if $q^-_i(p^-) < q^+_{-i}(p^+)$.*

Imagine that at some point in time, the residual demand in the forward auction is ahead of agent i's sincere demand in the reverse auction, that is, $d_i(p^-) < q^+_{-i}(p^+)$, see Figure 3 (right). Then, assuming that all other agents bid sincerely, i can reveal any value between $[d_i(p^-), q^+_{-i}(p^+)]$ in the reverse auction without risking changing the stop-off price. Alternatively, she can keep her demand unchanged for the range of prices $[d_i^{-1}(p^+), p^-]$ in the reverse auction. Thus, informational spillover allows supporting equilibria that are not sincere.

One can see that the no-spoilers disclosure policy, introduced in previous sections, was designed precisely to combat informational spillover. Indeed, spillover into the forward auction happens if and only if the $q^+_{i,d} = q^+_i$ making it completely uninformative, and similarly for the reverse auction. This is why we could eliminate other equilibria under the no-spoilers policy.

The question that we want to answer is whether there exists a clock policy that, in some sense, minimizes informational spillover and thus maximizes disclosure under the no-spoilers disclosure policy.

The simple clock policy can not give us this property. Indeed, after fully advancing the clock in the forward auction, the residual demand there is no less than the sincere demand in the reverse auction for every player, and strictly so for strictly decreasing demands. Moreover, for $n = 2$, it is simply impossible to rule out informational spillover with generic demands.

However, we can try to make the number of agents that experience informational spillover as small as possible. This approach is motivated by the following lemma:

Lemma 1. *For any clock prices $p^+ \leqslant p^-$: if there is spillover into both auctions, then there is spillover for exactly one agent.*

According to this lemma, the number of agents experiencing spillovers at any point in time is far from arbitrary. Represented by a pair of numbers, it can only be one of the following: $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, $(2+, 0)$, $(0, 2+)$; where $x+$ stands for "x and more", see Figure 4 for a stylized illustration. Moreover, when the numbers are $(1, 1)$, the same agent experiences spillover on both sides.

With this structure at hand, we can show that, for any collection of well-behaved sincere demands, a price path exists with special properties. Namely, along this path, the number of agents with spillovers monotonically decreases until there is at most one such agent, and it stays that way, see Appendix B for formal proof.

To make the result sharp, we put a few technical assumptions on the sincere demands and treat them as known.

Proposition 3. *Let agents play continuous and (weakly) monotone sincere demands, and there exist a stop-off price p^* such that the market clears. Then, for any starting prices $p^+_0 \leqslant p^-_0$, there exist a (weakly) monotone path $p_+(t), p_-(t)$ connecting (p^+_0, p^-_0) with (p^*, p^*)*

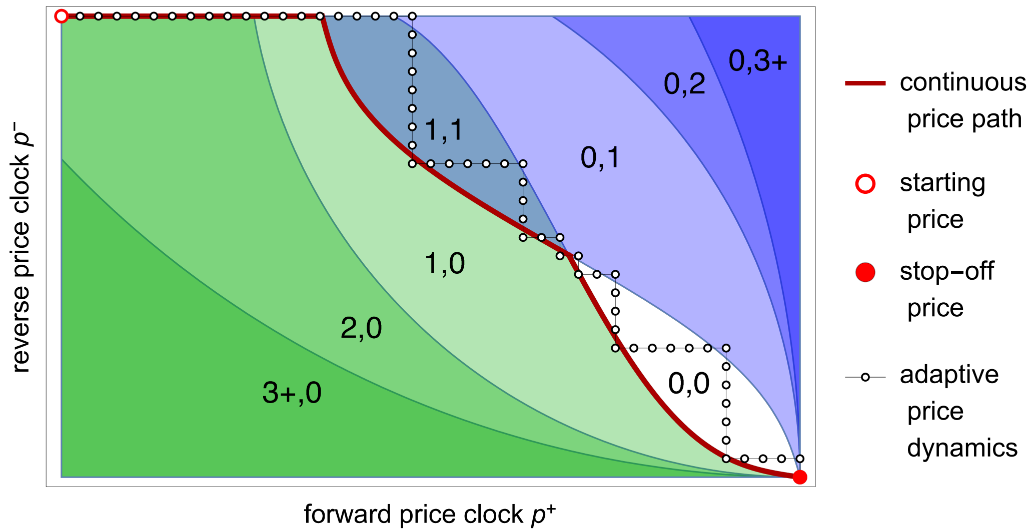

Figure 4: Illustration of the price path associated with adaptive clock policy, and the spillover-minimizing continuous price path in Proposition 3.

continuously.

The path consists of two parts. In the first part, the number of agents experiencing informational spillover decreases monotonically until there is, at most, one such agent. In the second part, there is still at most one such agent.

How does this help with the design of the auction? If we could find a realistic clock policy that mimics the aforementioned path, it could be considered superior to other clock policies, as it minimizes spillover and thus maximizes disclosure under the no-spoilers policy.

Now, recall the adaptive clock policy. If the number of agents for whom there is spillover into the forward auction is greater than the number of agents for whom there is spillover into the reverse auction, we move the forward clock. If the number of agents for whom there is spillover into the reverse auction is greater than the number of agents for whom there is spillover into the forward auction, we move the reverse clock. Otherwise, we move either clock.

In other words, adaptive price dynamics is moving the clocks in a way that balances the number of spillovers in the forward and reverse auctions, eventually localizing them to at most one agent, see Figure 4, which is exactly what we need. [7]

6 Direct mechanisms

This section complements the double clock auction with taxes by providing an *optimal* taxation function. To achieve this goal, we add more structure to the agents' preferences: single-dimensional types and single-crossing preferences. We consider a flexible class of designer objectives, which covers expected profit maximization and near-efficiency as special cases. First, we derive an optimal direct mechanism for this class of objectives. Then, we derive the taxation function that achieves the same allocation and payoffs in the sincere equilibrium of our dynamic auction.

Single-dimensional types

We model agent i's preferences as a single-dimensional, private type $\theta_i \in \mathbb{R}$. Thus, agent i's payoff with type θ_i, asset allocation $q_i \in \mathbb{R}$ and money transfer $t_i \in \mathbb{R}$, is

$$u_i(\theta_i, q_i) - t_i.$$

We will refer to the whole profile of types as θ and the profile of types other than agent i as θ_{-i}. We begin with a minimal set of assumptions that are typically used in the mechanism design literature.

Assumption 2. *θ_i is independently distributed with a strictly increasing CDF F_i, $u_i(\theta_i, q_i)$ is twice continuously differentiable and strictly single crossing ($\frac{\partial^2}{\partial\theta_i \partial q_i} u_i(\theta_i, q_i) > 0$), for all i and θ_i, q_i in the support.*

We focus on direct mechanisms with a truth-telling equilibrium, invoking the revelation principle. A direct mechanism (q, t) consists of an allocation rule $q : \mathbb{R}^n \to \mathbb{R}^n$ and a transfer rule $t : \mathbb{R}^n \to \mathbb{R}^n$. A direct mechanism must satisfy the incentive compatibility (IC) and individual rationality (IR) constraints so that the agents play a truth-telling equilibrium. In this paper, we require that both IC and IR constraints are satisfied ex-post, that is, at each type profile on the type space, as in Andreyanov and Sadzik (2021),

[7] One could say that adaptive price dynamics does more - it tries to reduce the number of spillovers to 0. However, there is no guarantee that the set of prices for which there are no spillovers is connected nor that it reaches the stop-off price. Thus, we can only guarantee to monotonically reduce the number of agents for whom spillover takes place to at most 1, but not 0.

rather than on average, as in Lu and Robert (2001). Formally, they are defined as below.

Definition 7. *A direct mechanism* (q, t) *satisfies the ex-post IC and IR constraint if it satisfies the following inequalities.*

IC: $u_i(\theta_i, q(\theta_i, \theta_{-i})) - t(\theta_i, \theta_{-i}) \geqslant u_i(\theta_i, q(\theta_i', \theta_{-i})) - t(\theta_i', \theta_{-i}),$

IR: $u_i(\theta_i, q(\theta_i, \theta_{-i})) - t(\theta_i, \theta_{-i}) \geqslant u_i(\theta_i, 0).$

for all i *and all* θ *in the support.*

Denote by $\tilde{u}_i(\theta_i, q_i)$ the net (i.e. added relative to the autarky), utility of agent i:

$$\tilde{u}_i(\theta_i, q_i) = u_i(\theta_i, q_i) - u_i(\theta_i, 0).$$

A standard mechanism-design argument tells, see e.g. Milgrom and Shannon (1994); Milgrom and Segal (2002); Sinander (2022), that under strict single crossing, a direct mechanism (q, t) is ex-post IC if and only if: $q_i(\theta_i, \theta_{-i})$ is non-decreasing in θ_i and the *envelope conditions* hold:

$$t_i(\theta_i, \theta_{-i}) = \tilde{u}_i(\theta_i, q_i(\theta_i, \theta_{-i})) - \tilde{u}_i(\theta_i', q_i(\theta_i', \theta_{-i})) - \int_{\theta_i'}^{\theta_i} \frac{\partial}{\partial \theta_i} \tilde{u}_i(x, q_i(x, \theta_{-i})) dx, \qquad (2)$$

for all i and $\theta_i, \theta_i', \theta_{-i}$ in the support.

Net surplus and worst-off-types

Another convenient way to describe an ex-post IC direct mechanism - is in terms of the agent's net equilibrium payoff $\tilde{s}_i(\theta_i, \theta_{-i})$ that we refer to as her *net surplus*:

$$\tilde{s}_i(\theta_i, \theta_{-i}) = \tilde{u}_i(\theta_i, q_i(\theta_i, \theta_{-i})) - t_i(\theta_i, \theta_{-i}) = \max_{\theta_i'} \{\tilde{u}_i(\theta_i, q_i(\theta_i', \theta_{-i})) - t_i(\theta_i', \theta_{-i})\}.$$

Furthermore, let $wot(\theta_{-i})$ denote the set of *worst-off types*, and $tet(\theta_{-i})$ denote the set of *types excluded from trade*, of agent i in a mechanism:

$$wot(\theta_{-i}) = arg \min_{\theta_i'} \tilde{s}_i(\theta_i', \theta_{-i}), \quad tet(\theta_{-i}) = \{\theta_i' : q_i(\theta_i', \theta_{-i}) = 0\}.$$

Lemma 2. *Under Assumption 2, in an ex-post IC direct mechanism* (q,t)*,*

$$tet(\theta_{-i}) \subset wot(\theta_{-i}).$$

The above lemma allows to recast the envelope conditions (2):

$$\tilde{s}_i(\theta_i,\theta_{-i}) = \inf_{\theta'} \tilde{s}_i(\theta'_i,\theta_{-i}) + \int_{\theta_i^*}^{\theta_i} \frac{\partial}{\partial\theta_i}\tilde{u}_i(x, q_i(x,\theta_{-i}))dx, \quad \forall\theta_i^* \in tet(\theta_{-i}), \tag{3}$$

for any i and θ_{-i} in the support such that $tet(\theta_{-i})$ is non-empty.

v-optimality

We are interested in a broad class of ex-post IC and IR direct mechanisms, which we will refer to as *v-optimal*.

Consider a collection of functions $v_i(\theta_i, q)$, which can be interpreted as individual contributions of each agent to a certain social utility. We wish to maximize this social utility subject to the *market clearing* constraint $\sum_{i=1}^n q_i = 0$, also ex-post, that is, satisfied for all types in the support. Additionally, we normalize each agent's payoff at the worst-off type to be equal to her payoff in the autarky.

Definition 8. *A v-optimal direct mechanism* (q,t) *maximizes*

$$\iiint\limits_{\mathbb{R}^n} \left[\sum_{i=1}^n v_i(\theta_i, q_i)\right] \prod_j dF_j(\theta_j) \tag{4}$$

subject to the constraints: $q_i(\theta_i,\theta_{-i})$ *is weakly increasing in* θ_i*, envelope conditions (2), market clearing and* $\inf_{\theta'_i} \tilde{s}(\theta'_i,\theta_{-i}) = 0$*, for all* i *and* θ_i,θ_{-i} *in the support.*

While not fully general, this formulation covers a number of important families of mechanisms. In particular, three such families have been studied before. The first family, studied in Gresik and Satterthwaite (1989), Lu and Robert (2001), in the context of Bayesian IC and IR constraints, can be informally defined via $v_i = (1-\alpha)u_i + \alpha t_i$, and can be thought of as a convex combination of efficient and profit-maximizing mechanisms. The second and third families, studied in Andreyanov and Sadzik (2021), are $v_i = u_i - \sigma q_i^2/2$ and $v_i = u_i - \delta|q|$. They can be thought of as nearly efficient mechanisms capable of balancing the budget ex-post through controlled demand reduction. By coincidence, if the utility is quadratic: $u_i(\theta_i, q) = \theta_i q - \mu q^2/2$, the second family also contains (for $\sigma = \frac{\mu}{n-2}$) the uniform-price double auction, studied, among others, in Kyle (1989) and Rostek and Weretka (2012).

We place a few technical assumptions on the auctioneer's objective v, which ensure that the v-optimal mechanism is a solution to a smooth (with a notable exception of $q = 0$) and convex optimization problem.

Assumption 3. *$v_i(\theta_i, q_i)$ is twice continuously differentiable, strictly concave in q_i and strictly single crossing, for all i, $q_i \neq 0$, and θ_i in the support.*

With a slight abuse of notation, let $mv_i(\theta_i, q_i)$ denote $\frac{\partial}{\partial q_i} v_i(\theta_i, q_i)$ at points of differentiability, and the sub-gradient of v_i otherwise. Likewise, let $mu_i(\theta_i, q_i)$ denote $\frac{\partial}{\partial q_i} u_i(\theta_i, q_i)$. We now move on to characterize v-optimal direct mechanisms.

Monotonicity of allocation

As is common in the literature, we first attempt to solve a relaxed problem - where we drop the monotonicity constraint so that we can solve for the allocation q, pointwise, and then check if it is monotone. A version of the Kuhn-Tucker theorem ensures that the Lagrangian method applies in the relaxed problem, yielding the first order conditions:

$$p(\theta) \in mv_i(\theta_i, q_i(\theta)), \qquad \sum_{i=1}^{n} q_i(\theta) = 0, \tag{5}$$

where $p(\theta) \in \mathbb{R}$ is the Lagrange multiplier.[8] By strict concavity of the v_i functions, $q_i(\theta)$ is single-valued.

Below we verify that the solution to the relaxed problem is indeed monotone. For convenience, let $mv'_{i,\theta}, mv'_{i,q}$ denote $\frac{\partial mv_i}{\partial \theta_i}(\theta_i, q_i)$, $\frac{\partial mv_i}{\partial q_i}(\theta_i, q_i)$ respectively. For any $q \neq 0$, we may linearize (5) around (p, q) as below

$$mv'_{i,\theta} + mv'_{i,q} \cdot \frac{\partial}{\partial \theta_i} q_i(\theta) = \frac{\partial}{\partial \theta_i} p(\theta), \quad mv'_{j,q} \cdot \frac{\partial}{\partial \theta_i} q_j(\theta) = \frac{\partial}{\partial \theta_i} p(\theta), \quad j \neq i. \tag{6}$$

[8] For example, Theorem 1 and 2 in Luenberger 1969, 217p and 221p.

We can then solve for the slopes of allocation and price using market clearing

$$\frac{\partial}{\partial\theta_i}p(\theta) = \frac{mv'_{i,\theta}}{mv'_{i,q}}(\sum \frac{1}{mv'_{k,q}})^{-1}, \quad \frac{\partial}{\partial\theta_i}q_j(\theta) = \frac{mv'_{i,\theta}}{mv'_{j,q}}\left(\frac{1/mv'_{j,q}}{\sum 1/mv'_{k,q}} - \mathbb{I}(j=i)\right). \tag{7}$$

Clearly, under strict single crossing and strict concavity of the v_i functions, the allocation of any agent is strictly increasing in her type.[9] We would like to further strengthen this property by uniformly bounding the slopes of mv_i.

Assumption 4. *$mv'_{i,\theta} > \varepsilon$, and $-1/mv'_{i,q} > \delta$, for all i and $\theta_i, \theta_{-i}, q_i$ in the support, and some $\varepsilon, \delta > 0$.*

We can therefore bound the slope of the allocation function from below:

$$\frac{\partial}{\partial\theta_i}q_i(\theta) = mv'_{i,\theta} \cdot \frac{(-1/mv'_{i,q}) \cdot (\sum_{k\neq i} -1/mv'_{k,q})}{(-1/mv'_{i,q}) + (\sum_{k\neq i} -1/mv'_{k,q})} \geqslant \frac{n-1}{n}\varepsilon\delta. \tag{8}$$

Consequently, one can invert the allocation function with respect to own type everywhere except $q = 0$. We will refer to it as an *inverse allocation function* $q_i^{-1}(x, \theta_{-i})$, defined on the $(\mathbb{R}\backslash\{0\}) \times \mathbb{R}^{n-1}$ domain.

Existence of the type excluded from trade

Before we proceed to the first main result of this section, there is one technical observation that we need to make. Namely, we would like that, in a v-optimal mechanism, there exists a type excluded from trade, that is, $tet(\theta_{-i})$ is nonempty. Together with the *taxation principle*, see Rochet (1985), this will allows to once again recast the envelope conditions using the inverse allocation function.

Lemma 3. *Under Assumptions 2 to 4, in a v-optimal mechanism (q,t), $tet(\theta_{-i})$ is nonempty, and the transfers can be written as:*

$$\tilde{t}_i(q, \theta_{-i}) = \int_0^q mu_i(q_i^{-1}(x, \theta_{-i}), x)dx, \tag{9}$$

for all i and θ_{-i} in the support.

[9] The allocation function is also strictly decreasing in types of others, and the market clearing price is strictly increasing in all types.

For the exposition, we provide two alternative versions of Assumption 4 that would ensure the existence of types excluded from trade, see Section C.3. The first version requires that all v_i, F_i are identical, and can be used with compact support. The second version requires that for any $p \in \mathbb{R}$ there exists a type θ_i in the support such that $p \in mv_i(\theta_i, 0)$.

Taxation scheme

We are now ready to derive the taxation scheme associated with the auction described in Section 3, which would match the one in our v-optimal mechanism. According to the rules of the auction, the payments consist of two parts: the Vickrey-style payments and the integrated (along the residual supply curve) marginal taxes

$$\tilde{\tilde{t}}_i(q, \theta_{-i}) = \int_0^q m\tau(p_{-i}(x), \theta_{-i}) + p_{-i}(x)dx, \tag{10}$$

where $p_{-i}(x)$ is the residual supply curve facing agent i.

If we could set the marginal tax equal to the wedge between mu_i and mv_i at the desired allocation, the agents would essentially perceive v_i as their true utility. It only remains to do it for every realization of types.

Definition 9. *Set the marginal tax $m\tau_i(p, q) = x$, where $(x, \hat{\theta})$ solves*

$$\begin{cases} x = mu_i(\hat{\theta}, q) - mv_i(\hat{\theta}, q), \\ p = mv_i(\hat{\theta}, q), \end{cases} \tag{11}$$

for all p, q in the support.

We refer to the solution $\hat{\theta}_i(p, q)$ to the system of equations (11), as the *fixed-point type*. It is correctly defined on the $\mathbb{R} \times (\mathbb{R}\backslash\{0\})$ domain, and so is the marginal tax.

We are ready to formulate our second main result.

Proposition 4. *Under Assumptions 2 to 4, the sincere equilibrium in the double clock auction with the marginal tax $m\tau(p, q)$ defined by (11) achieves the same allocation and transfer as in the v-optimal mechanism.*

The proof proceeds by observing, quite mechanically, the equivalence between the transfers $\tilde{t}_i(\theta_i, q_i)$ and $\tilde{\tilde{t}}_i(\theta_i, q_i)$, see Appendix C.

Finally, by linearizing (11) around (p, q), we can derive the slopes of the fixed-point type and the marginal tax

$$\hat{\theta}'_p = 1/mv'_\theta, \quad \hat{\theta}'_q = -mv'_q/mv'_\theta,$$
$$m\tau'_p = mu'_q\hat{\theta}'_p - 1, \quad m\tau'_q = mu'_\theta\hat{\theta}'_q + mu'_q.$$

Corollary 1. *Under Assumptions 2 to 4, the marginal tax $m\tau_i$ is continuously differentiable for all $q \neq 0$ and satisfies*

$$m\tau'_q - mu'_q > 0, \quad m\tau'_p + 1 > 0.$$

In other words, the (integrated) tax is less concave than the utility, and the marginal tax can not respond to the change in price too fast.

7 Revenue maximization

We now move on to the special case of interest, which is revenue maximization. Ignoring the monotonicity constraint, we will attempt to maximize the average transfer

$$\iint\limits_{\mathbb{R}^{n-1}} \sum_{i=1}^{n} \left[\int_{\mathbb{R}} \left(\tilde{u}_i(\theta_i, q_i) - \tilde{s}_i(\theta_i, \theta_{-i}) \right) dF_i(\theta_i) \right] \prod_{j \neq i} dF_j(\theta_j) \tag{12}$$

subject to market clearing and envelope conditions. Naturally, in the revenue-maximizing mechanism, any leftover surplus can be extracted via translation of monetary transfers, therefore, $\inf_{\theta'_i} \tilde{s}_i(\theta'_i, \theta_{-i}) = 0$.

Before we proceed with a classic Myersonian trick, there is one more assumption that we have to make, related to the integrability of net surplus, which is necessary for integration by parts on the whole real line. [10]

Assumption 5. $\sum_{i=1}^{n} \tilde{u}_i(\theta_i, q_i) \leqslant C(\theta_i)$ *for any* $q : \sum_{i=1}^{n} q_i = 0$ *and some function* $C(x)$*, such that* $\int C(x) dF_i(x) < \infty$*.*

Although this assumption is very weak, from it follows that the expected net surplus in

[10] Riemann if F is continuous, or, more generally, Stiltjes

the exchange economy is finite. To see the importance of this observation, note that even with simple quadratic models as in Section 8, the utility is not bounded on $\mathbb{R}$, and thus the expected net surplus is not obviously bounded.

Lemma 4. *Under Assumptions 2, 3 and 5:*

$$\int \tilde{s}_i(z, \theta_{-i}) dF_i(z) < \infty$$

for all θ_{-i} in the support.

With this at hand, we split the integral of the net surplus at the type excluded from trade (which is guaranteed to exist) and apply integration by parts to each of the two halves. This gives us the following equivalence

$$\int_{\mathbb{R}} \left(\tilde{u}_i(\theta_i, q_i) - \tilde{s}_i(\theta_i, \theta_{-i})\right) dF_i(\theta_i) = \int_{\mathbb{R}} J_i(\theta_i, q_i) dF_i(\theta_i) \tag{13}$$

$$J_i(\theta_i, q_i) = \tilde{u}(\theta_i, q_i) - \frac{\mathbb{I}(q_i > 0) - F(\theta_i)}{f(\theta_i)} \frac{\partial}{\partial \theta} \tilde{u}(\theta_i, q). \tag{14}$$

We will refer to J_i as the *virtual utility.*

It is worth noting that the virtual utility, in this particular form, is continuous in both allocation and type. Indeed, the only potential source of discontinuity is the indicator function $\mathbb{I}(q_i > 0)$ multiplied by $\frac{\partial}{\partial \theta} \tilde{u}(\theta_i, q)$, which is zero at $q = 0$. Thus, there is no jump at $q_i = 0$. Instead, there is a concave kink. Had we not used Lemma 2, the virtual utility would be discontinuous in type.

It remains to check whether the premise of the Proposition 4 is satisfied so that we can also claim the implementation of the profit-maximizing mechanism. One simple way to achieve this - is to put more assumptions on the true utilities u_i.

Assumption 6. *F_i is log-concave and both $-mu'_{i,q} \cdot sgn(q_i(\theta))$ and $mu'_{i,\theta} \cdot sgn(q_i(\theta))$ are increasing in θ_i, for all i and θ in the support, such that $q_i(\theta) \neq 0$.*

This assumption guarantees that Assumption 3 is satisfied for $v_i = J_i$.

This leads to the following proposition.

Proposition 5. *Under Assumptions 2, 3 and 5, the profit-maximizing mechanism is v-*

optimal with v_i equal to the virtual utility J_i.

See Appendix D for formal proof.

Corollary 2. *Under Assumptions 2, 5 and 6, the virtual utility J_i is twice continuously differentiable, strictly concave in q_i and strictly single crossing, for all i, $q_i \neq 0$, and θ_i in the support.*

Likewise, even stronger restrictions on the utility u_i can guarantee that Assumption 4, or its alternative versions, are satisfied for $v_i = J_i$.

8 Symmetric quadratic model

This section illustrates our methodology in a symmetric model where each agent i has the following quadratic utility function.

$$u_i(\theta_i, q_i) = \theta_i q_i - \frac{\mu}{2} q_i^2,$$

for some known $\mu > 0$. We consider two log-concave distributions of private types θ_i: uniform on the $[-1, 1]$ interval and logistic (i.e., with full support). The above specification provides additional tractability and allows for comparison across different mechanisms.

8.1 Pareto frontier

We now solve for the mechanism that maximizes a linear combination of expected revenue and efficiency, in other words, finds the Pareto frontier. Following the arguments in Section 6, we have to maximize $\sum_i J_{\alpha,i}(\theta_i, q_i)$ over q, subject to the market clearing constraint $\sum q_i = 0$, where

$$J_\alpha(\theta_i, q_i) = q_i \left[\varphi_\alpha(\theta_i) \cdot \mathbb{I}(q_i > 0) + \psi_\alpha(\theta_i) \cdot \mathbb{I}(q_i \leqslant 0)\right] - \frac{\mu}{2} q_i^2,$$

pointwise, where $\varphi_\alpha(\theta) = \theta_i - \alpha \frac{1-F(\theta_i)}{f(\theta_i)}$ and $\psi_\alpha(\theta) = \theta + \alpha \frac{F(\theta_i)}{f(\theta_i)}$ are monotone functions, as long as F is log-concave.

To identify the optimal allocation, we must find a Lagrange multiplier $p(\theta)$ such that the market clears and the first-order conditions hold. This leads to the following solution

$$\begin{aligned} d_i(p|\theta_i) &= \mu^{-1}\left[(\varphi(\theta_i) - p)\cdot \mathbb{I}(q>0) + (\psi(\theta_i) - p)\cdot \mathbb{I}(q<0)\right] = \\ &= \mu^{-1}\left[\min(0,\ \psi(\theta_i) - p) + \max(0,\ \varphi(\theta_i) - p)\right], \end{aligned}$$

for each agent i, which will also be her sincere demand in the auction implementation. The Lagrange multiplier $p(\theta)$ is then the root of $\sum_{i=1}^{n} d_i(p|\theta_i)/n$, that is, the average sincere demand.

Finally, the marginal tax $m\tau$ and the fixed-point type $\hat{\theta}$ solve the system of equations (11) and thus

$$\hat{\theta}(p,q) = \varphi_\alpha^{-1}(\mu q + p)\cdot \mathbb{I}(q>0) + \psi_\alpha^{-1}(\mu q + p)\cdot \mathbb{I}(q<0) \tag{15}$$

$$m\tau(p,q) = \hat{\theta}(p,q) - (\mu q + p) \tag{16}$$

Proposition 6. *In the symmetric quadratic model with a log-concave distribution F, the optimal mechanism is implemented via marginal taxes* (16)*.*

Since the worst-off types are in the interior of the type space, the transfers can be formally written out, conditional on the value of the Lagrange multiplier.

$$t_i(q_i|\theta_{-i}) = \begin{cases} \int_0^q \left(\varphi_\alpha^{-1}(\mu z + p_{-i}(z|\theta_{-i})) - \mu z\right) dz, & q>0 \\ \int_0^q \left(\psi_\alpha^{-1}(\mu z + p_{-i}(z|\theta_{-i})) - \mu z\right) dz, & q<0 \end{cases}, \tag{17}$$

where $p_{-i}(z|\theta_{-i})$ is the inverse residual supply curve. Despite a relatively simple mechanism implementation, its explicit characterization is rather difficult in a finite economy, even for standard distributions.

8.1.1 Uniform distribution

When the distribution is uniform, $\varphi_1(\theta) = 2\theta - 1$, $\psi_1(\theta) = 2\theta + 1$, thus

$$d_i(p|\theta_i) = \mu^{-1}\left[\min(0,\ 2\theta_i + 1 - p) + \max(0,\ 2\theta_i - 1 - p)\right],$$

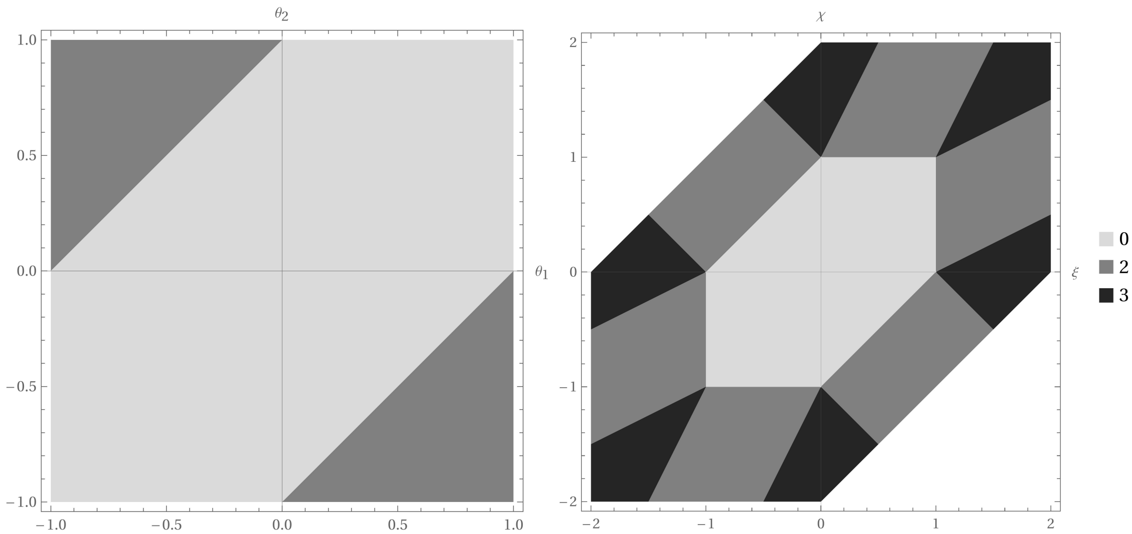

Figure 5: Exclusion region for two (left) players and three (right) players for a uniform[-1,1] distribution of types. The latter is in the coordinates $\xi = \theta_1 - \theta_3$, $\chi = \theta_2 - \theta_3$, independent from the value of θ_3.

and the marginal tax is

$$m\tau(p, q) = \frac{-\mu q - p - 1}{2} + \mathbb{I}(q > 0).$$

Note that the number of agents excluded from trade depends on the location of the root of the average demand curve and thus can not be easily characterized.

For example, for just $n = 3$ agents, the exclusion region follows an elaborate pattern, see Figure 5. When all three types are close to each other (a light grey area), nobody is trading. Next, with two significantly opposing types and a third in the middle (a dark grey area), only opposing types are trading with each other. Finally, when two types oppose the third, all three players are trading (black area).

When the number of players grows, the pattern becomes more complicated. However, the root of the average demand curve will converge in the probability limit, which is equal to 0. Thus, the limit exclusion region will be simply $\theta_i \in [-1/2, 1/2]$.

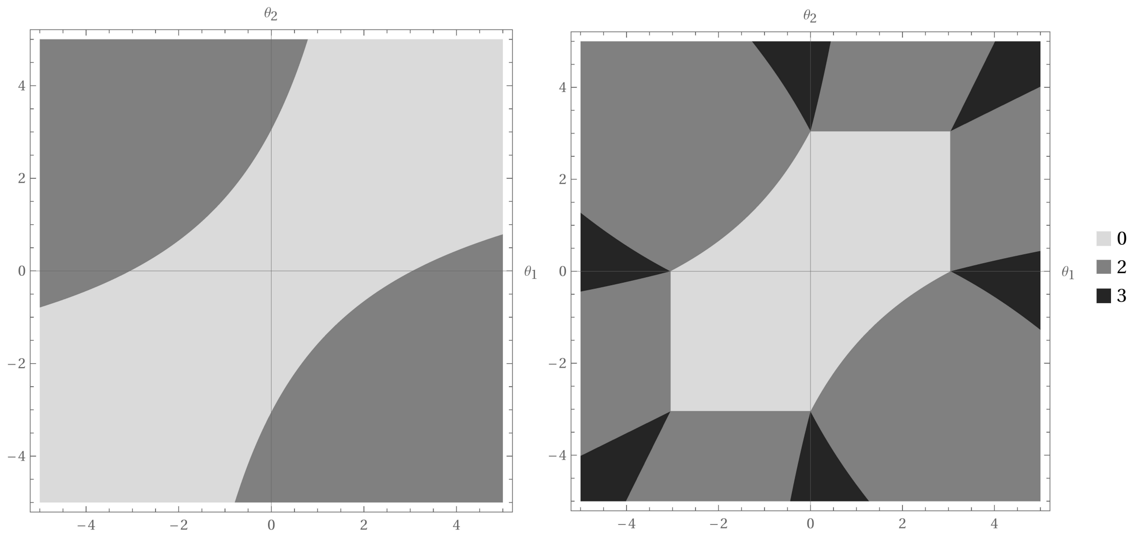

Figure 6: Exclusion region for two (left figure) players and three (right figure) players for a logistic distribution of types. The latter is in the coordinates θ_1, θ_2 for a fixed $\theta_3 = 0$.

8.1.2 Logistic distribution

For a logistic distribution, $\varphi_1(\theta) = \theta - 1 - e^{-\theta}$, $\psi_1(\theta) = \theta + 1 + e^{\theta}$, thus

$$d_i(p|\theta_i) = \mu^{-1}\left[\min(0,\ \theta_i + 1 + e^{\theta} - p) + \max(0,\ \theta_i - 1 - e^{-\theta} - p)\right],$$

and the marginal tax is

$$m\tau(p,q) = sgn(q) \cdot \left[1 + \omega(e^{-1-sgn(q)\cdot(\mu q+p)})\right].$$

The exclusion region for $n = 3$ follows a pattern similar to that of the uniform distribution, see Figure 6. In the limiting economy, again, the root of the average demand curve will be equal to 0. Thus the limit exclusion region is simply $\theta_i \in [-1 - \omega(1/e), 1 + \omega(1/e)]$, where $\omega(z)$ is the product-logarithm function.

8.2 σVCG mechanisms

Our first benchmark is smooth nearly-efficient mechanisms in Andreyanov and Sadzik (2021) called σ-VCG mechanisms, which can be thought of as an attempt to control demand reduction explicitly.

One way to define this mechanism is the maximizer of $\sum_i J_{\sigma,i}(\theta_i, q_i)$ over q, subject to the market clearing constraint $\sum q_i = 0$, where

$$J_{\sigma,i}(\theta_i, q_i) = \theta_i q_i - \frac{\mu + \sigma}{2} q_i^2.$$

The ex-post allocation and transfer in this mechanism can be derived:

$$q_i = \frac{n-1}{n} \frac{\theta_i - \bar{\theta}_{-i}}{\mu + \sigma}, \quad t_i(q_i) = \bar{\theta}_{-i} q + \frac{\mu + n\sigma}{2(n-1)} q^2,$$

where $\bar{\theta}_{-i} = \frac{1}{n-1} \sum_{j \neq i} \theta_j$ is the average type other than agent i's type.

Since both transfer and utility are quadratic in types, we can compute their expected values given the variance of the type distribution:

$$\mathbb{E} t_i = \frac{(n-2)\sigma - \mu}{2n(\mu + \sigma)^2} \mathbb{V}\theta_i, \quad \mathbb{E} u_i = \frac{(n-1)(\mu + 2\sigma)}{2n(\mu + \sigma)^2} \mathbb{V}\theta_i,$$

since $\mathbb{E}\bar{\theta}_{-i} = \mathbb{E}\theta_i$, $\mathbb{E}(\bar{\theta}_{-i})^2 = \frac{\mathbb{E}\theta_i^2 + (n-2)(\mathbb{E}\theta_i)^2}{n-1}$ and $\mathbb{V}\theta = \mathbb{E}\theta_i^2 - (\mathbb{E}\theta_i)^2$. Naturally, for a uniform-price double auction ($\sigma = \frac{\mu}{n-2}$), the expected payment is equal to zero, while for the efficient mechanism ($\sigma = 0$), it is negative.

Finally, the maximum expected transfer over σ-VCG mechanisms is attained at $\sigma = \frac{n\mu}{n-2}$, and is equal to $\frac{(n-2)^2}{n(n-1)} \cdot \frac{\mathbb{V}\theta}{8\mu}$, while the utility is equal to $\frac{(n-2)(3n-2)}{n(n-1)} \cdot \frac{\mathbb{V}\theta}{8\mu}$.

8.3 δVCG mechanisms

Our second benchmark is non-smooth nearly-efficient mechanisms in Andreyanov and Sadzik (2021), which can be thought of as a bid-ask spread of size 2δ, which we refer to as δ-VCG mechanisms.

One way to define this mechanism is the maximizer of $\sum_i J_{\delta,i}(\theta_i, q_i)$ over q, subject to the market clearing constraint $\sum q_i = 0$, where

$$J_{\delta,i}(\theta_i, q_i) = q_i \left[\varphi_\delta(\theta_i) \cdot \mathbb{I}(q_i > 0) + \psi_\delta(\theta_i) \cdot \mathbb{I}(q_i \leqslant 0)\right] - \frac{\mu}{2} q_i^2,$$

pointwise, where $\varphi_\delta(\theta) = \theta_i - \delta$ and $\psi_\delta(\theta) = \theta + \delta$. The rest of the algorithm is identical to the one used for revenue maximization, so we have to rely on Monte Carlo simulations for finite economies.

In the limit economy, however, there is no supply reduction, so the agent's demand is equal to $(\theta_i - \delta)/\mu$ if he turns out to be a buyer, and $(\theta_i + \delta)/\mu$ if he turns out to be a seller. Moreover, for symmetric distributions, the limit of the equilibrium price will be equal to 0, so buyers will pay a per-unit price of δ, while sellers will get a per-unit price of $-\delta$. Thus, we can compute the expected payment and utility:

$$\mathbb{E}t_i = 2\delta \int_{\delta}^{F^{-1}(1)} \left[\frac{x-\delta}{\mu}\right] dF(x)$$
$$\mathbb{E}u_i = 2 \int_{\delta}^{F^{-1}(1)} \left[x\frac{x-\delta}{\mu} - \frac{\mu}{2}(\frac{x-\delta}{\mu})^2\right] dF(x)$$

which can be easily maximized over δ, for any given distribution.

9 Conclusion

We have studied an optimal robust mechanism in a double-auction environment similar to Lu and Robert (2001). While the direct mechanism is rather complicated, the associated implementation is simple - two Ausubel auctions: forward and reverse, with the price clocks running towards each other, much like in the celebrated Incentive Auction.

The clock nature of the auction can be thought of as a means to solve for the Lagrangean multiplier in the optimization problem, where the sum of virtual utilities is maximized subject to market clearing constraints, as long as these virtual utilities are concave. Moreover, the task of finding the worst-off types necessary for explicitly characterizing the direct mechanism is implicitly solved in the equilibrium of the auction.

The virtualization of utilities in a model with single-dimensional types is achieved by introducing a marginal tax that depends on the player's current (clinched) position and the clock price but not on other players' positions. A hallmark feature of this tax is that it combines a relatively standard exclusion of the weakest traders with a tax scheme that can be explicitly computed.

The equilibrium is sincere, similar to Ausubel (2004), but players submit demands as if their utilities were replaced with virtual utilities. However, the two-sided nature of the auction poses new challenges. In particular, the informational spillover between the forward and reverse auctions makes it impossible to eliminate all non-sincere equilibria if the auction is fully transparent. However, it is possible to move the clocks to minimize this spillover.

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Appendix A Proofs for Section 4

A.1 Proof of Proposition 1

Observe first that, in every subgame, along the conjectured equilibrium path, the revealed demands in the sincere ex-post equilibrium coincide with the sincere demands $\{d_i(p)\}_{i=1}^n$ which, together with the market-clearing condition fully characterize the outcome (price and allocation) of the game.

We will now prove that *the market clearing price and allocation under sincere bidding coincide with the Walrasian equilibrium in the virtual economy.*

Suppose that players bid by the sincere strategy in the clock auction. Then auction outcomes are characterized by the first-order condition for the sincere demands

$$p = mu(q_i) - m\tau(p, q_i), \quad i = 1, \ldots, n.$$

By definition of mv_i, the first-order condition above can also be expressed as

$$p = mv_i(q_i), \quad i = 1, \ldots, n,$$

which are the first order conditions for Walrasian demands in the economy with utilities v_i. Since the second-order conditions in both cases are satisfied by Assumption 1, the first-order conditions show that the market clearing price and allocation in both equilibria are the same.

We will now prove that *sincere bidding is an ex-post perfect equilibrium.* The proof considers two cases regarding the prior history of play in the clock auction.

First, we examine agents' incentives on the equilibrium path of play, where the demands revealed before the subgame was all sincere.

Assuming that all but player j continue to play sincerely, bidder j's payoff is path-independent in the following sense. At any counterfactual allocation q, the stop-off price $p_{-j}(q)$ is uniquely defined by the sincere demands of other players. Moreover, her payment to the auctioneer is equal to the marginal tax $m\tau_i(p_{-j}(x), x)$ plus $p_{-j}(x)$, integrated over

$x \in [0, q]$. The optimality condition is, therefore

$$\begin{cases} mu_j(q) = m\tau_j(p, q) + p \\ p = mv_i(q_i), \quad i = 1, \dots, n, \quad i \neq j \end{cases}$$

which yields $p = mv_i(q_j)$ by the definition of the v_i functions. Player j's payoff is aligned with the social surplus in the virtual economy, which is maximized by playing sincerely. Thus, sincere play is a Nash equilibrium of the subgame.

Second, we consider incentives off the equilibrium path, where traders reported non-sincerely before this subgame.

Assuming that all but player j continue to play sincerely, bidder j's payoff is path-independent, but her actions are constrained by the demands revealed before the subgame.

These payoffs are monotonically decreasing in the distance from the conjectured allocation. Thus, she finds it optimal to play as close to the sincere demand as possible. Therefore, sincere play is a Nash equilibrium of the subgame.

A.2 Proof of Proposition 2

Observe first that, in every subgame, and independently of the history of play, the no-spoilers policy with the full support assumption implies that, from the bidder's perspective, the stop-off price can be anywhere in the currently played range of prices.

Consequently, if the reported demand differs from the sincere at any price p^{ast} in this range, the auctioneer has a strategy that makes p^{ast} the stop-off price. The payoff of the bidder would then be strictly smaller than if she was playing sincerely.

We will now prove that *there is an order of elimination that yields sincere bidding.*

The proof is by induction over the auction rounds, counting from the final round.

Assume that, in the final round, the strategy of agent i differs from her sincere strategy at a price p^*. Since the stop-off price can be anywhere, this strategy is dominated in the event that the stop-off price is equal to exactly p^*. Thus, all strategies other than sincere ones are weakly dominated in the final round.

Consider a non-final round, and suppose we eliminated insincere bidding for all subsequent rounds. The same argument then applies to the range of prices in the current round as for the final round.

Thus, all strategies other than sincere ones have been eliminated.

We will now prove that *no order of elimination eliminates the sincere strategy.*

Assume that there exists an order of elimination that eliminates a sincere strategy. Consider the earliest instance in this order and the first round (starting from the end of the auction) of such elimination. By construction, in the current round and in the rounds that follow, no sincere strategies have been eliminated yet.[11]

Consider now the sincere strategy in this subgame. Since the stop-off price can be anywhere, the sincere strategy dominates all remaining strategies again. Thus, the sincere strategy can not be dominated in this subgame.

Thus, no order of elimination eliminates the sincere strategy.

Appendix B Proofs for Section 5

B.1 Proof of Lemma 1

Proof. To the contrary, assume that at some prices $p^+ \leqslant p^-$, there is informational spillover into both auctions, and, at the same time, there is informational spillover for more than one agent. This means that there exist two agents $i \neq j$ such that:

$$q^-_{-i}(p^-) < q^+_i(p^+), \quad q^-_j(p^-) < q^+_{-j}(p^+).$$

Using the definition of the residual demands, we can pair these inequalities:

$$-\sum_{k\neq i,j} q^-_k(p^-) < q^+_i(p^+) + q^-_j(p^-) < -\sum_{k\neq i,j} q^+_k(p^+)$$

which contradicts the fact that $q^-_k(p^-) \leqslant q^+_k(p^+)$ for all k. □

[11] If sincere bidding constitutes an ex-post equilibrium in the subgame and can not be eliminated strictly, but still might be eliminated weakly.

B.2 Proof of Proposition 3

Proof. Consider the domain of prices $(p^+, p^-) \in [p_0^+, p^*] \times [p^*, p_0^-]$ and denote the subset of prices that have x spillovers into the forward and y spillovers into the reverse auctions by $S_{x,y}$.

Observe first that Lemma 1 implies that $S_{1+,2+} = \varnothing$ and $S_{2+,1+} = \varnothing$. Thus, any point in the price domain belongs to either $S_{0,1+}$, $S_{1+,0}$, $S_{1,1}$ or $S_{0,0}$.

The price path connecting (p_0^+, p_0^-) with (p^*, p^*) will have two parts. The first part is a straight line, and the second part goes along the boundary of either $S_{0,1+}$ or $S_{1+,0}$, see Figure 4. To construct the price path, consider three cases.

Case 1: If the starting prices are in $S_{1+,0}$, we first advance the forward clock till it reaches the boundary of $S_{1+,0}$. After that we move along the path $(\tilde{p}^+(p), p)$ where

$$\tilde{p}^+(p) = \sup_{x \in [p_0^-, p^*]} x : (x, p) \in S_{1+,0}.$$

Case 2: If the starting prices are in $S_{0,1+}$, we first advance the reverse clock till it reaches the boundary of $S_{0,1+}$. After that we move along the path $(p, \tilde{p}^-(p))$ where

$$\tilde{p}^-(p) = \inf_{x \in [p^*, p_0^+]} x : (p, x) \in S_{0,1+}.$$

Case 3: If the starting prices (p_0^+, p_0^-) are in $S_{1,1}$, any of the aforementioned trajectories will work. Finally, the starting prices can not be in $S_{0,0}$ by assumption.

We argue that along the first part of the trajectory, the number of agents experiencing spillovers is weakly decreasing. Indeed, on the one hand, advancing the forward (reverse) clock does not increase the number of spillovers in the forward (reverse) auction. On the other hand, the number of spillovers in the reverse (forward) auction is fixed at 0 by construction.

The function $\tilde{p}^+(.)$ does not have to be continuous. However, if it is monotone, we can connect the (at most countably many) points of discontinuity to obtain a monotone and continuous path $p^+(t), p^-(t)$. It remains to show that $\tilde{p}^+(.)$ is weakly monotone and that, along this path, the number of agents that experience spillover is at most one.

Monotonicity: Assume that $\tilde{p}^+(p_1^-) = p_1^+$, that is, (p_1^+, p_1^-) belongs to the closure of $S_{1+,0}$.

Now, pick any $p_2^- < p_1^-$. When the clock prices move from (p_1^+, p_1^-) to (p_1^+, p_2^-), the number of spillovers in the reverse auction can not increase, while the number of spillovers in the forward auction is already at 0. Thus, (p_1^+, p_2^-) belongs to the closure of $S_{1+,0}$ as well, thus $\tilde{p}^+(p_2^-) \geqslant p_1^+$. Consequently, $\tilde{p}^+(p)$ is weakly monotone.

Finally, observe that $S_{1+,0}$ does not intersect with $S_{0,2}$ by Lemma 1. Thus, it can only share a boundary with $S_{1,1}$, $S_{0,1}$ and $S_{0,0}$. In either case, the number of agents experiencing spillovers is at most one.

□

Appendix C Proofs for Section 6

C.1 Proof of Lemma 2

Fix θ_{-i} and consider two mutually exclusive cases. Suppose first that the set of types excluded from trade is empty. Then the claim holds trivially.

Suppose that it is not empty. Let $\hat{\theta}_i$ be a type excluded from trade. By definition of net utility, $\frac{\partial}{\partial \theta_i}\tilde{u}_i(\hat{\theta}_i, q_i(\hat{\theta}_i, \theta_{-i})) = 0$.

Next, the net surplus functions $\tilde{s}_i$ are absolutely continuous, a.e. differentiable and

$$\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i-, \theta_{-i}) \leqslant \frac{\partial}{\partial \theta_i}\tilde{u}_i(\hat{\theta}_i, q_i(\hat{\theta}_i, \theta_{-i})) \leqslant \frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i+, \theta_{-i}),$$

where $\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i-, \theta_{-i})$ and $\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i+, \theta_{-i})$ are left-hand and right-hand partial derivatives respectively, see Theorems 1,2 in Milgrom and Segal (2002).

Next, at points of differentiability we can write:

$$\frac{\partial}{\partial \theta_i}\tilde{s}_i(\theta) = \frac{\partial}{\partial \theta_i}\tilde{u}_i(\theta_i, q_i(\theta)) = \int_0^{q(\theta)} \frac{\partial}{\partial \theta_i \partial q_i} u_i(\theta, x)dx,$$

thus $\tilde{s}_i$ is convex in θ_i by monotonicity of q_i in θ_i and single-crossing of u_i.

Finally, since $[\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i-, \theta_{-i}), \frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i+, \theta_{-i})]$ contains 0 at the type excluded from trade, by the necessary first-order conditions, $\hat{\theta}_i$ is also the worst-off type.

C.2 Proof of Lemma 3

Equation (8) shows that $q_i(\theta_i, \theta_{-i})$ is continuous in θ_i and bounds it's slope away from zero. Thus, $q_i(\theta_i, \theta_{-i})$ is guaranteed to cross 0 at some type $\theta_i \in \mathbb{R}$, in other words, $tet(\theta_{-i})$ is non-empty, for any θ_{-i} in the support. Next, by formula (3)

$$t_i(\theta) = \tilde{u}(\theta_i, q(\theta)) - \tilde{s}(\theta_i, q(\theta)) =$$
$$= \tilde{u}(\theta_i, q(\theta)) - \inf_{\theta_i} \tilde{s}_i(\theta_i, \theta_{-i}) - \int_{\theta*}^{\theta_i} \frac{\partial}{\partial \theta} \tilde{u}(x, q(x, \theta_{-i}))dx$$

where $\theta^* \in tet(\theta_{-i})$. Recalling that, in a v-optimal mechanism, $\inf_{\theta'} \tilde{s}_i(\theta_i', \theta_{-i}) = 0$

$$t_i(\theta) = \int_{\theta*}^{\theta_i} \frac{d}{dx} \tilde{u}(x, q(x, \theta_{-i}))dx - \int_{\theta*}^{\theta_i} \frac{\partial}{\partial \theta} \tilde{u}(x, q(x, \theta_{-i}))dx =$$
$$= \int_{\theta*}^{\theta_i} \frac{\partial}{\partial q} \tilde{u}(x, q_i(x, \theta_{-i}))dq_i(x, \theta_{-i}) = \int_{\theta*}^{\theta_i} mu_i(x, q_i(x, \theta_{-i}))dq_i(x, \theta_{-i}).$$

Finally, we get formula (9) via monotone change of variables from x to $q_i^{-1}(x, \theta_{-i})$.

C.3 Proofs of Lemma 3 with alternative versions of Assumption 4

Version 1: v_i are identical, F_i are identical.

Proof. To the contrary, assume that for some realization of types θ_{-i}, trader i only trades strictly positive quantities. Define a type $z = \min_{j \neq i} \theta_j$, and observe that it belongs to the support of each agent. Consequently, we can say that $q_i(z, \theta_{-i}) > 0$.

Furthermore, the allocation can not decrease if we lower the types of traders $j \neq i$. Consequently, we can say that $q_i(z, \ldots, z) > 0$. But this can not be true because any $p \in mv_i(z, 0)$ solves the first-order conditions in the symmetric case. □

Version 2: for any i and $p \in \mathbb{R}$, there exist a type z in the support such that $p \in mv_i(z, 0)$.

Proof. Pick a trader i, and fix a profile of types θ_{-i}. Next, consider the economy without trader i, that is, solve a system of first-order conditions

$$mv_j(\theta_j, \tilde{q}_j) = \tilde{p}, \quad \forall j \neq i, \quad \sum_{j \neq i} \tilde{q}_j = 0.$$

This solution exists for some $\tilde{p}$.

Next, pick a type z in the support, such that $\tilde{p} = mv_i(z, 0)$. By construction, i is excluded from trade in the original economy with the profile of types (z, θ_{-i}). $\square$

C.4 Proof of Proposition 4

We want to prove that agents face the same menus $t_i(q)$ in both the auction and the optimal mechanism. For that, it suffices to show that the integrand in (9) coincides with the one in (10) for any $q(\theta) \neq 0$.

Using the left-hand side of (11) we first write that

$$m\tau(p_{-i}(x), x) + p_{-i}(x) = mu_i(\hat{\theta}_i(p_{-i}(x), x), x).$$

Second, we combine the right-hand side of (11) with the definition of the residual supply curve

$$\begin{cases} p_{-i}(x) = mv_i(\hat{\theta}_i(p_{-i}(x), x), x) \\ p_{-i}(x) = mv_j(\theta_j, q_j), \quad \forall j \neq i \\ x + \sum_{j \neq i} q_j = 0. \end{cases}$$

The latter can be recognized as the system of first-order conditions for the optimal mechanism, given that x is the allocation of agent i and θ_{-i} are the types of others. Thus $\hat{\theta}_i(p_{-i}(x), x)$ and $q^{-1}(x, \theta_{-i})$ coincide and, therefore,

$$mu_i(\hat{\theta}_i(p_{-i}(x), x), x) = mu_i(q^{-1}(x, \theta_{-i}), x),$$

which completes the proof.

Appendix D Proofs for Section 7

D.1 Proof of Lemma 4

The boundedness of the expected net surplus comes from the fact that, on the one hand, the net surplus is nonnegative by IR, and on the other hand, the sum of net surpluses can not exceed the sum of net utilities at the efficient allocation

$$\tilde{s}_i(\theta_i, \theta_{-i}) \geqslant 0, \quad \sum \tilde{s}_i(\theta_i, \theta_{-i}) \leqslant C(\theta_i),$$

therefore $\tilde{s}_i(\theta_i, \theta_{-i}) \leqslant C(\theta_i)$ for any θ in the support, and thus $\int \tilde{s}_i(z, \theta_{-i}) dF_i(z)$ is majorized by $\int C(z) dF_i(z) < \infty$.

D.2 Proof of Proposition 5

Recall that our objective is

$$\iint\limits_{\mathbb{R}^{n-1}} \sum_{i=1}^{n} \left[\int_{\mathbb{R}} \left(\tilde{u}_i(\theta_i, q_i) - \tilde{s}_i(\theta_i, \theta_{-i}) \right) \right) dF_i(\theta_i) \right] \prod_{j \neq i} dF_j(\theta_j), \tag{18}$$

where $\tilde{s}_i(\theta_i, \theta_{-i}) = \int_{\theta^*}^{\theta_i} \tilde{u}_1'(x, q(x, \theta_{-i})) dx$ and $\theta^* \in tet(\theta_{-i})$, that is, θ^* is one of the types excluded from trade by Lemma 2, and also one of the worst-off types, that is, $\tilde{s}_i(\theta_i^*, \theta_{-i}) = 0$. Since $\tilde{s}_i$ integrable by Lemma 4 and θ^* is finite by Lemma 3, we can use the following representation:

$$\begin{aligned} \int_{\mathbb{R}} \tilde{s}_i(\theta_i, \theta_{-i}) dF(\theta_i) &= \lim_{N \to \infty} \int_{-N}^{N} \tilde{s}_i(\theta_i, \theta_{-i}) dF_i(\theta_i) \\ &= \int_{-N}^{\theta^*} \tilde{s}(\theta_i, \theta_{-i}) dF(\theta_i) + \int_{\theta^*}^{N} \tilde{s}(\theta_i, \theta_{-i}) d(F(\theta_i) - 1) + o(N; \theta_{-i}), \end{aligned}$$

for any N sufficiently large. The remainder term vanishes in the limit, $\lim_{N \to \infty} o(N; \theta_{-i}) = 0$ due to the definition of improper integral. Integrating the first two terms by parts, we get that

$$\begin{aligned} \int_{\mathbb{R}} \tilde{s}(\theta_i, \theta_{-i}) dF(\theta_i) &= \int_{-N}^{\theta^*} \tilde{u}_1'(\theta_i, q(\theta_i, \theta_{-i})) F(\theta_i) d\theta_i \\ &+ \int_{\theta^*}^{N} \tilde{u}_1'(\theta_i, q(\theta_i, \theta_{-i}))(F(\theta_i) - 1) d\theta_i + o(N; \theta_{-i}) \\ &= \int_{\mathbb{R}} \frac{\mathbb{I}(q_i > 0) - F(\theta_i)}{f(\theta_i)} \tilde{u}_1'(\theta_i, q) dF(\theta_i) \end{aligned}$$

Note that $\mathbb{I}(\theta_i > \theta_i^*) = \mathbb{I}(q_i > 0)$ since q_i is monotone in θ_i and θ_i^* is a type excluded from trade. Plugging it into (18) gives us the virtual value J_i.

We next need to show that the virtual value J is concave and single-crossing to use the first-order approach.

$$
\begin{aligned}
\frac{\partial^2 J}{\partial\theta\partial q} &= \frac{\partial^2 \tilde{u}}{\partial\theta\partial q} - \frac{\partial}{\partial\theta}\left(\frac{\mathbb{I}(q_i > 0) - F(\theta_i)}{f(\theta_i)}\right)\frac{\partial^2 \tilde{u}}{\partial\theta\partial q} - \frac{\mathbb{I}(q_i > 0) - F(\theta_i)}{f(\theta_i)}\frac{\partial^3}{\partial\theta^2\partial q}\tilde{u} > 0, \\
\frac{\partial^2 J}{\partial q^2} &= \frac{\partial^2 \tilde{u}}{\partial q^2} - \frac{\mathbb{I}(q_i > 0) - F(\theta_i)}{f(\theta_i)}\frac{\partial^3}{\partial\theta\partial q^2}\tilde{u} < 0.
\end{aligned}
$$

Both properties are guaranteed by Assumption 6.